Reprints in Theory and Applications of Categories, No. 25, 2016, pp. 1–16.

THE BICATEGORY OF TOPOI AND SPECTRA

J. C. COLE

Author's note. The present appearance of this paper is largely due to Olivia Caramello's tracking down a citation of Michel Coste which refers to this paper as "to appear...". This "reprint" is in fact the first time it has been published – after more than 35 years! My apologies for lateness therefore go to Michel, and my thanks to Olivia! Thanks also to Anna Carla Russo who did the typesetting, and to Tim Porter who remembered how to contact me.

The "spectra" referred to in the title are right adjoints to forgetful functors between categories of topoi-with-structure. Examples are the local-rings spectrum of a ringed topos, the etale spectrum of local-ringed topos, and many others besides. The general idea is to solve a universal problem which has no solution in the ambient set theory, but does have a solution when we allow a change of topos. The remarkable fact is that the general theorems may be proved abstractly from no more than the fact that **Topoi** is finitely complete, in a sense appropriate to bicategories.

1 Bicategories

1.1 A *2-category* is a Cat-enriched category: it has hom-categories (rather than hom-sets) and composition is functorial, so than the composite of a diagram

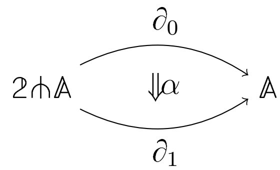
$$\mathbb{A} \xrightarrow{\;f\;} \mathbb{B} \;\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow\alpha}}\; \mathbb{C} \xrightarrow{\;g\;} \mathbb{D}$$

denoted $f * \alpha * g$ is unambiguously defined.

In a 2-category $\mathcal{A}$, as well as the (ordinary) finite limits obtained from a terminal object and pullbacks, we consider limits of diagrams having 2-cells.

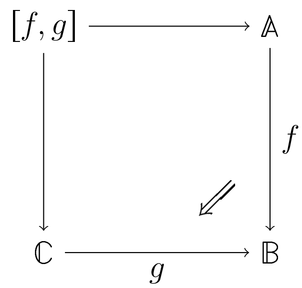
1.2 For each $\mathbb{A}$, the *cotensor* with $\mathbb{2}$ of $\mathbb{A}$ is a diagram

Typeset by Anna Carla Russo
Received by the editors 2015-09-28.
Transmitted by Martin Hyland, Robert Paré, Robert Rosebrugh. Reprint published on 2016-01-03.
2010 Mathematics Subject Classification: 03G30, 18B25, 18C10, 18D05, 18D25, 18F10, 18F99.
Key words and phrases: topos, bicategory, 2-limits, classifying topos, geometric theory, factorization systems, spectrum.



for which ∂_0, ∂_1 and α induce an isomorphism of hom-categories, $\mathcal{A}(\mathbb{X}, \mathbb{2} \pitchfork \mathbb{A}) = \mathcal{A}(\mathbb{X}, \mathbb{A})^{\mathbb{2}}$, natural in $\mathbb{X}$, where the right-hand category is the (usual) category of morphisms. Thus $\phi : f \to g : \mathbb{X} \to \mathbb{A}$ induces a unique $\ulcorner \phi \urcorner : \mathbb{X} \to \mathbb{2} \pitchfork \mathbb{A}$ such that $\ulcorner \phi \urcorner * \alpha = \phi$, and with 2-cells $\ulcorner \phi \urcorner \Rightarrow \ulcorner \psi \urcorner$ being induced by commuting squares of 2-cells over $\mathbb{A}$.

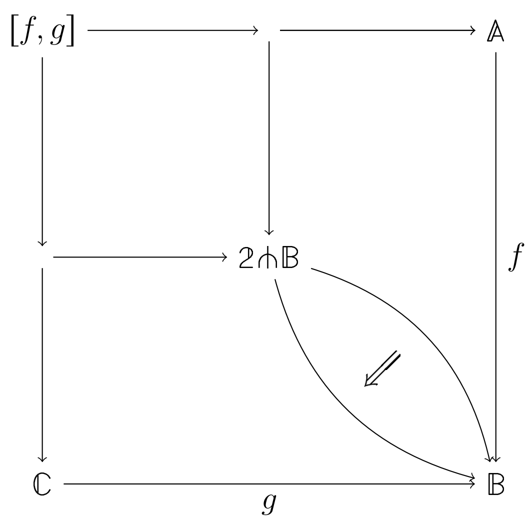
1.3 A *comma-object* $[f, g]$ for a pair of 1-cells with common codomain is a square



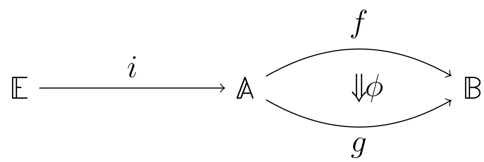
with the universal property $\mathcal{A}(\mathbb{X}, [f, g]) \cong \mathcal{A}(\mathbb{X}, g)$, naturally in $\mathbb{X}$, where the right-hand category is the usual comma-category of composition functors.

In the presence of pullbacks, comma-objects may be constructed from cotensors with $\mathbb{2}$, simply by pulling back ∂_0 and ∂_1 along f and g respectively:

1.4



1.5 An *identifier* is a diagram



with the universal property that $h : \mathbb{X} \to \mathbb{A}$ factors (uniquely) through $\mathbb{E} \to \mathbb{A}$ if and only if $h * \phi$ is the identity 2-cell and with the obvious condition for 2-cells.

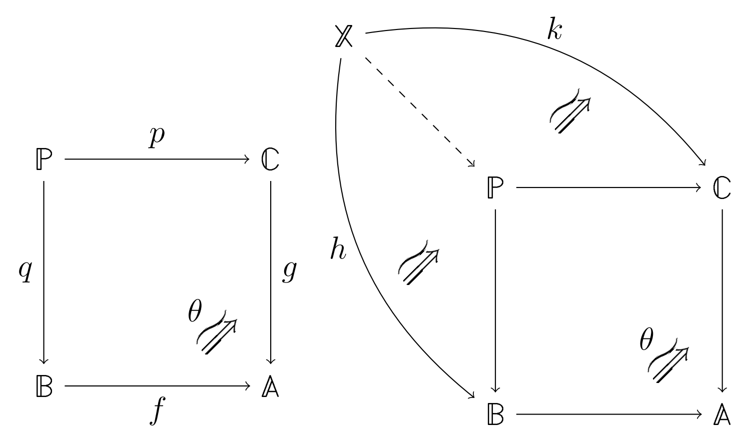
Notice that the identity 2-cell $\mathbb{B} \to \mathbb{B}$ induces a "diagonal" map $I : \mathbb{B} \to \mathbb{2}\pitchfork\mathbb{B}$. It is not hard to see that I, ∂_0 and ∂_1 make $\mathbb{2}\pitchfork\mathbb{B}$ into a category-object; furthermore, we have adjointness $\partial_1 \vdash I \vdash \partial_0$. The identifier of ϕ may be constructed simply by pulling back $\ulcorner\phi\urcorner : \mathbb{A} \to \mathbb{2}\pitchfork\mathbb{B}$ along $I : \mathbb{B} \to \mathbb{2}\pitchfork\mathbb{B}$.

1.6 We say that a 2-category $\mathcal{A}$ is *finitely complete* if it has a terminal object, pullbacks and cotensors with $\mathbb{2}$.

1.7 A *bicategory* (Benabou [2]) has a composition of 1-cells which is associative and unitary only up to coherent isomorphism (example: a monoidal category is a bicategory with only one object): composition is pseudo-functorial.

To translate 2-category notions into the corresponding bicategory Notions, it is hence necessary to replace equality of 1-cells by isomorphisms. In particular, limits defined by an isomorphism of hom-categories must now be replaced by Limits, defined by the corresponding equivalence of hom-categories. Unique existence of a 1-cell is replaced by existence, unique to unique isomorphism, and so on. We follow Grothendieck [1] in distinguishing bicategory Limits from 2-category limits by the use of an initial capital letter.

1.8 A *Pullback* is a square



such that for each $h, k, \lambda : h.f \overset{\sim}{\Rightarrow} k.g$, there is $\ell : \mathbb{X} \to \mathbb{P}$, unique up to unique isomorphism, together with $\kappa : h \overset{\sim}{\Rightarrow} \ell.q$, $\mu : \ell.p \overset{\sim}{\Rightarrow} k$ such that $\lambda = (\kappa * f).(\ell * \theta).(\mu * g)$; further, 2-cells

$h \Rightarrow h'$ and $k \Rightarrow k'$ commuting with λ and λ' induce $\ell \Rightarrow \ell'$; in short, there is a natural equivalence

$$\mathcal{A}(\mathbb{X}, \mathbb{P}) \cong \mathcal{A}(\mathbb{X}, \mathbb{B}) \times_{\mathcal{A}(\mathbb{X},\mathbb{A})} \mathcal{A}(\mathbb{X}, \mathbb{C})$$

where the right-hand category has as objects triples (h, λ, k) with $\lambda : h.f \stackrel{\sim}{\Rightarrow} k.g$.

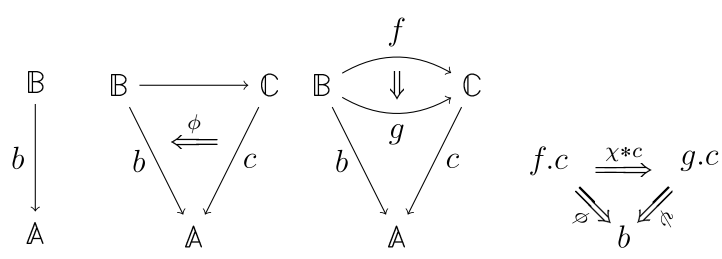
Here "natural equivalence" in $\mathbb{X}$ means (because of the associativity isomorphisms for composition) that the naturality squares commute up to an isomorphism satisfying the obvious "pasting" condition for composites $\mathbb{X} \to \mathbb{Y} \to \mathbb{Z}$.

1.9 Other Limits are similarly defined and we say that a bicategory is *finitely Complete* if it has a Terminal object, Pullbacks and coTensors with $\mathbb{2}$, whence also Comma-objects and Inverters (corresponding to identifier). A morphism of bicategories can only be a pseudo-functor and, as we would expect, Limits are pseudo-functorial once they have been chosen (they are, of course, unique up to equivalence).

1.10 A pair of pseudo-functors $U : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$ is *Adjoint*, in symbol $F \vdash U$, if there is an equivalence $\mathcal{A}(F(\mathbb{B}), \mathbb{A}) \simeq \mathcal{B}(\mathbb{B}, U(\mathbb{A}))$, natural in $\mathbb{A}$ and $\mathbb{B}$ in the same sense as for 1.8. Equivalently, for each $\mathbb{B}$, there is $\eta : \mathbb{B} \Rightarrow U(F(\mathbb{B}))$ such that for every $h : \mathbb{B} \to U(\mathbb{A})$, there is $\bar{h} : F(\mathbb{B}) \to \mathbb{A}$, unique up to isomorphism, with $\varepsilon : h : \stackrel{\sim}{\Rightarrow} \eta.U(\bar{h})$; a 2-cell $h \Rightarrow h'$ induces $\bar{h} \Rightarrow \bar{h}'$ commuting with ε and ε'.

1.11 We define the *comma-bicategory* $\mathcal{A}//\mathbb{A}$, for $\mathbb{A} \in \mathcal{A}$, as follows:

- an object $b \in \mathcal{A}//\mathbb{A}$ is a 1-cell $b : \mathbb{B} \to \mathbb{A}$ in $\mathcal{A}$;
- a 1-cell $(f, \phi) : b \to c$ in $\mathcal{A}//\mathbb{A}$ is $f : \mathbb{B} \to \mathbb{C}$ and $\phi : f.c \Rightarrow b$ in $\mathcal{A}$,
- a 2-cell $\chi : (f, \phi) \Rightarrow (g, \psi)$ in $\mathcal{A}//\mathbb{A}$ is a 2-cell $\chi : f \to g$ such that $(\chi * c).\psi = \phi$ in $\mathcal{A}$.



object 1-cell 2-cell

1.12 Example *If* $u : \mathbb{B} \to \mathbb{A}$ *induces the obvious* $U : \mathcal{A}//\mathbb{B} \to \mathcal{A}//\mathbb{A}$, *then* $[-, u]$ *is right Adjoint to* U *and this defines Comma operation.*

1.13 We recall also that a pair of 1-cells $u : \mathbb{B} \to \mathbb{A}$, $f : \mathbb{A} \to \mathbb{B}$ is *adjoint*, in symbol $f \vdash u$, if there are 2-cells $\eta : 1_{\mathbb{A}} \Rightarrow f.u$ and $\varepsilon : u.f \Rightarrow 1_{\mathbb{B}}$ satisfying the usual equations:

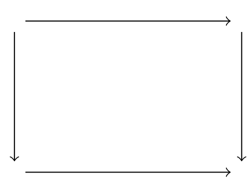
$$(\eta * f).(f * \varepsilon) = 1_f$$
$$(u * \eta).(\varepsilon * u) = 1_u.$$

Equivalently, for each $\mathbb{X}$, $\mathcal{A}(\mathbb{X}, f) \vdash \mathcal{A}(\mathbb{X}, u)$, or, again, for each $\mathbb{Y}$, $\mathcal{A}(u, \mathbb{Y}) \vdash \mathcal{A}(f, \mathbb{Y})$, the (ordinary) adjunction transformations being natural up to isomorphism in $\mathbb{X}$ or $\mathbb{Y}$. A 1-cell $f : \mathbb{A} \to \mathbb{B}$ in $\mathcal{A}$ is *fully-faithful* if $\mathcal{A}(\mathbb{X}, f)$ is a fully-faithful functor for each $\mathbb{X}$; $f \vdash u$ is a *reflection*, and f the *reflector*, if u is fully-faithful. Equivalently, the end adjunction ε is an isomorphism.

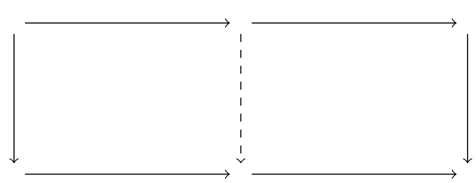
1.14 Lemma *The Pullback of a reflector (coreflector) is a reflector (coreflector).*

Proof Since $\varepsilon : u.f \Rightarrow 1_{\mathbb{B}}$ is an isomorphism, so is its Pullback along $g : \mathbb{C} \to \mathbb{A}$, $\bar{\varepsilon} : \bar{u}.\bar{f} \Rightarrow 1_{\bar{\mathbb{B}}}$. But $\eta : 1_{\mathbb{A}} \Rightarrow f.u$ Pulls back to $\bar{\eta} : 1_{\bar{\mathbb{A}}} \Rightarrow \bar{f}.\bar{u}$, satisfying the relevant equations. ∎

We consider $\mathcal{E}$-$\mathcal{M}$ factorisation system on an ordinary category $\mathbb{A}$. If $\mathcal{M}$ is a class of maps of $\mathbb{A}$ containing isomorphisms closed under composition, we say that $\mathcal{M}$ *gives best factorisations* if every map $A \to B$ in $\mathbb{A}$ has factorisation $A \to C \to B$ with $C \to B$ in $\mathcal{M}$, such that for any other such factorisation $A \to C' \to B$ with $C' \to B$ in $\mathcal{M}$ there is a unique $C \to C'$ in $\mathcal{M}$ making both triangles commute. We say that a factorisation is *functorial* if a commuting square factors into commuting squares:



factors with



commuting (for $\mathcal{E}$-$\mathcal{M}$ factorisation, this is equivalent to the usual diagonal property).

1.15 Proposition *For a category $\mathbb{A}$ the following data are equivalent:*

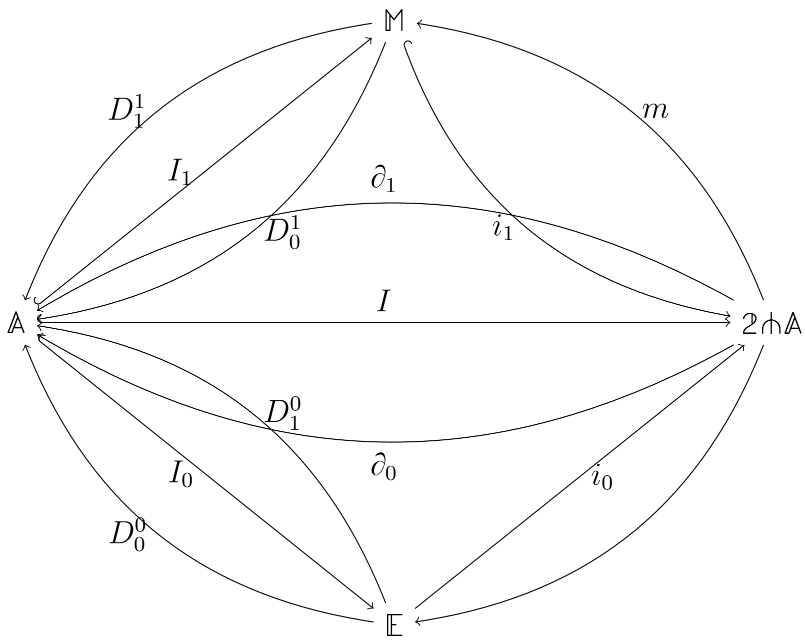
(a) *there is a class $\mathcal{M}$ of maps giving functorial best factorisations;*

(b) *there is a class $\mathcal{E}$ of maps giving functorial co-best factorisations;*

(c) *there is a functorial $\mathcal{E}$-$\mathcal{M}$ factorisation;*

(d) *for each category* $\mathbb{X}$*, a factorisation of type (a), (b) or (c) on Cat*$(\mathbb{X},\mathbb{A})$ *such that for each* $f:\mathbb{Y}\to\mathbb{X}$*, if* $\alpha.\beta$ *is a factored map in Cat*$(\mathbb{X},\mathbb{A})$*, then* $(f*\alpha).(f*\beta)$ *is a factorisation in Cat*$(\mathbb{Y},\mathbb{A})$ *of* $f*(\alpha.\beta)$*;*

(e) *the diagram in Cat:*



in which $I_1.i_1 = I = I_0.i_0$*, each functor is left adjoint to the one immediately below it,* $(\mathbb{A}, I_0, I_1)$ *is the Pullback of* (i_0, i_1) *and* $(2\pitchfork\mathbb{A}, m, e)$ *is the Pullback of* (D_0^1, D_1^0)*.*

Proof (*Sketch*) Given $\mathcal{M}$, define $\mathcal{E}$ to be the class of maps whose "best $\mathcal{M}$-factor" is an isomorphism, and conversely. This establish the equivalence of the first three. For (d), take those natural transformations whose components lie in $\mathcal{M}$ or $\mathcal{E}$ and, conversely, take $\mathbb{X} = \mathbb{1}$. Finally, for (e), let $\mathbb{M}$ and $\mathbb{E}$ be the full subcategories of $\mathbb{A}^2$ consisting of those maps which are in $\mathcal{M}$ or $\mathcal{E}$. The functors m and e are "best $\mathcal{M}$-factor" (resp. $\mathcal{E}$-factor). ∎

We take (d) to be the definition in an arbitrary bicategory of *$\mathcal{E}$-$\mathcal{M}$ factorisation on* $\mathbb{A}$ and say that it is *representable* if the diagram (e), above, exists.

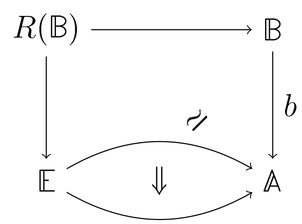
1.16 Proposition *If $\mathcal{A}$ is finitely Complete, then any $\mathcal{E}$-$\mathcal{M}$ factorisation on an object is representable.*

Proof Factorise the universal 2-cell $\alpha : \partial_0 \Rightarrow \partial_1 : 2\pitchfork\mathbb{A} \to \mathbb{A}$ into $\eta : \partial_0 \Rightarrow d$ and $\mu : d \Rightarrow \partial_1$. Define $\mathbb{M}$ to be the Inverter of η and $\mathbb{E}$ the Inverter of μ. Since a map is in $\mathcal{M}$ (resp. in $\mathcal{E}$) if and only if its best $\mathcal{E}$-(resp. $\mathcal{M}$)-factor is an isomorphism, it is clear that $\phi : f \Rightarrow g : \mathbb{X} \to \mathbb{A}$ is in $\mathcal{M}_{\mathbb{X}}$ (resp. in $\mathcal{E}_{\mathbb{X}}$) if and only if $\ulcorner\phi\urcorner : \mathbb{X} \to 2\pitchfork\mathbb{A}$ factor through $\mathbb{M}$ (resp. $\mathbb{E}$). The rest of the diagram follows immediately. ∎

Given a class $\mathcal{M}$ of 2-cells over $\mathbb{A}$ that contains isomorphisms, is closed under composition and satisfies $\alpha \in \mathcal{M}$ implies $f * \alpha \in \mathcal{M}$, it is clear how we may modify the definition of the comma-bicategory by allowing only 2-cells of $\mathcal{M}$ to appear, giving a bicategory $\mathcal{M}\text{-}\mathcal{A}//\mathbb{A}$.

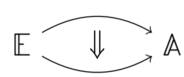
1.17 PROPOSITION *If $\mathcal{A}$ is finitely Complete, the obvious $\mathcal{M}\text{-}\mathcal{A}//\mathbb{A} \to \mathcal{A}//\mathbb{A}$ has right Adjoint if and only if $\mathcal{M}$ forms a $\mathcal{E}\text{-}\mathcal{M}$ factorisation on $\mathbb{A}$.*

PROOF Given an $\mathcal{E}\text{-}\mathcal{M}$ factorisation, it is representable by Proposition 1.16. We define the required right Adjoint by taking $R(b : \mathbb{B} \to \mathbb{A})$ to be the Pullback of $D^0_0 : \mathbb{E} \to \mathbb{A}$ along b with structure-map $R(\mathbb{E}) \to \mathbb{E} \xrightarrow{D^0_1} \mathbb{A}$. The end adjunction id the map in $\mathcal{A}//\mathbb{A}$

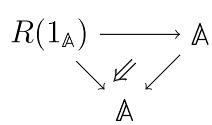


where the 2-cell is the universal $\mathcal{E}$-map whence we see that the end adjunction is the universal $\mathcal{E}$-map with domain b. The universal property emerges immediately from the definition (d) of factorisation.

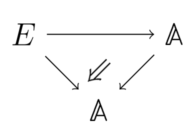
Conversely, given the right Adjoint R, define



by taking $D^0_1 : \mathbb{E} \to \mathbb{A}$ to be $R(\mathbb{A} \to \mathbb{A})$ and the adjunction to be the universal $\mathcal{E}$-map:



is



The universal property of the end adjunction leads directly to the best factorisation property of the class of maps represented by $\mathbb{E}$ whose orthogonal $\mathcal{M}$-class is that original given. ∎

Notice in particular that since $\mathbb{A}$ is a coreflective subobject of $\mathbb{E}$, the construction of R shows that $R(\mathbb{B})$ contains $\mathbb{B}$ as coreflective subobject by Lemma 1.14. The inclusion "classifies" the identity-map of b as an $\mathcal{E}$-map.

Finally, and not very elegantly, we combine Example 1.12, Proposition 1.17 and a restricted class of 2-cells. Suppose $\mathcal{M}$ is a class of 2-cells closed under composition etc, so that $\mathcal{M}\text{-}\mathcal{A}//\mathbb{A}$ is defined and similarly $\mathcal{N}\text{-}\mathcal{A}//\mathbb{B}$. Suppose we are given $u : \mathbb{B} \to \mathbb{A}$ such that if $\alpha \in \mathcal{N}$ then $\alpha * u \in \mathcal{M}$. Then we obtain a pseudo-functor $\mathcal{N}\text{-}\mathcal{A}//\mathbb{B} \to \mathcal{M}\text{-}\mathcal{A}//\mathbb{A}$. In this situation we say that $\mathcal{N}$ forms a u-$\mathcal{M}$-factorisation if every 2-cell $\alpha : f \Rightarrow g.u$ has a best factorisation $f \Rightarrow h.u$, $h \Rightarrow g$, with $h \Rightarrow g \in \mathcal{N}$ such that for any other such factorisation $f \Rightarrow h'.u \Rightarrow g.u$, there is a unique $h \Rightarrow h'$, making both triangles commute and such that the best factorisation are stable under compositions with 1-cells as for Proposition 1.15(d).

1.18 Proposition *If $\mathcal{A}$ is finitely Complete and $\mathcal{M}$ is representable, then*

$$\mathcal{N}\text{-}\mathcal{A}//\mathbb{B} \to \mathcal{M}\text{-}\mathcal{A}//\mathbb{A}$$

has a right Adjoint if and only if $\mathcal{N}$ forms a u-$\mathcal{M}$-factorisation.

Proof If $\mathcal{N}$ forms a factorisation, define $\mathbb{E} \to [\mathbb{A}, u]$ to represent the $\mathcal{N}$-extremal $\mathcal{M}$-maps (those whose best $\mathcal{N}$-factor is an isomorphism) by Inverting the best $\mathcal{N}$-factor of the universal 2-cell obtained by Pulling back

$$\mathbb{M} \longrightarrow 2 \pitchfork \mathbb{A} \xrightarrow{\partial_1} \mathbb{A}$$

along $u : \mathbb{B} \to \mathbb{A}$. Pull the "domain" map $\mathbb{E} \to \mathbb{A}$ along $c : \mathbb{C} \to \mathbb{A}$ to define $R(c : \mathbb{C} \to \mathbb{A})$, right Adjoint to the given forgetful functor. Conversely, given the right Adjoint R, define the universal $\mathcal{N}$-extremal $\mathcal{M}$-map to be the end adjunction for $R(\mathbb{A} \to \mathbb{A})$ and proceed as in Proposition 1.17.

$$\begin{array}{ccc} R(1_{\mathbb{A}}) & \longrightarrow & \mathbb{A} \\ \searrow & \Leftarrow & \swarrow \\ & \mathbb{A} & \end{array} \qquad \text{is} \qquad \begin{array}{ccc} E & \longrightarrow & \mathbb{A} \\ \searrow & \Leftarrow & \swarrow \\ & \mathbb{A} & \end{array}$$

∎

2 Limits in Topoi

We consider two 2-categories and a bicategory. **Lex** is the 2-category of finitely complete (small) categories, left exact functors and natural transformations. **LexSite** is the 2-category of finitely complete (small) categories equipped with a Grothendieck topology, left exact cover-preserving functors and natural transformations. **Topoi** is the bicategory of cocomplete topoi (i.e. Sets-topoi), geometric morphisms and natural transformations between the inverse image functors (with composition defined whichever way you prefer). While it is true that **Topoi** may be made into a 2-category, we choose not to. Each way of defining compositions associative up to equality has its disadvantages and none seems canonical. The real point is that **Topoi** has Limits, rather than limits. "Straightening out" all the canonical isomorphisms seems an insuperable task and is probably not worth it: it seems that the cheapest way of handing the difficulties is to put them in at the start.

Since **Lex** is "monadic" over **Cat** (in a sense we leave to the experts to make precise), it is clear that (strict) limits may be constructed at the underlying category level. What is a little mysterious is the fact that many of these limits in **Lex** turn out to be coLimits (in the "underlying" bicategory).

2.1 LEMMA $\mathbb{1}$ *is coterminal in* **Lex**.

PROOF The canonical map $\mathbb{A} \to \mathbb{1}$ has a right adjoint $\mathbb{1} \to \mathbb{A}$ (the terminal object of $\mathbb{A}$) which is unique among the left-exact functors ∎

2.2 LEMMA $\mathbb{A} \times \mathbb{B}$ *is the coProduct in* **Lex**.

PROOF The projections have right adjoints $\mathbb{A} \mapsto (A, 1)$ and $B \mapsto (1, B)$ which give injections. Given $h : \mathbb{A} \to \mathbb{X}$ and $k : \mathbb{B} \to \mathbb{X}$, define $\ell : \mathbb{A} \times \mathbb{B} \to \mathbb{X}$ by $\ell(A, B) = h(A) \times k(B)$, the product in $\mathbb{X}$. ∎

2.3 LEMMA *The cotensor* $\mathbb{2} \pitchfork \mathbb{A}$ *(the category* $\mathbb{A}^2$*) is also the Tensor* $\mathbb{2} \otimes \mathbb{A}$ *in* **Lex**.

PROOF Again, the projections have right adjoints $\delta_0 : A \mapsto (A \to A)$ and $\delta_1 : A \mapsto (A \to 1)$ for injections with the obvious 2-cell. Given $\alpha : f \Rightarrow g : \mathbb{A} \to \mathbb{X}$, define $\ulcorner \alpha \urcorner : \mathbb{A}^2 \to \mathbb{X}$ by taking $\ulcorner \alpha \urcorner (a : A_1 \to A_2)$ to be the pullback in $\mathbb{X}$ of $g(a)$ along α_{A_2}. ∎

2.4 LEMMA *The comma-category* $(\mathbb{B}, f)$ *for* $f : \mathbb{A} \to \mathbb{B}$ *in* **Lex** *is also the coComma object* $< \mathbb{A}, f >$.

PROOF Just as in Lemma 2.3 but with more letters. ∎

Thus **Lex** is almost finitely coComplete. [It remains to provide coInverters, for which "monadicity" over **Cat** might prove useful.] Since an intersection of topologies is a topology, we may always find the "least topology such that . . . ". In an appropriate sense, the forgetful **LexSite** → **Lex** is an initial structure functor which we use to lift coLimits from **Lex** to **LexSite**.

2.5 LEMMA *The functor* **LexSite** → **Lex** *creates coLimits.*

PROOF Given a diagram $\mathcal{D}$ in **LexSite** having a coLimit in **Lex**, we simply provide the coLimit with the least topology for which the injections preserves coverings: the smallest containing the images under injection of coverings in the diagram. Then a map out of the coLimit preserves coverings if and only if its composites with the injections all do, so we are finished. ∎

2.6 REMARK *Suppose that* $f : \mathbb{A} \to \mathbb{B}$ *is a topos-map. Then the comma-category* $(\mathbb{B}, f_*) \cong (f^*, \mathbb{A})$ *since* $f^* \dashv f_*$ *and it satisfies the coComma property for left-exact functors. In fact, it is the coComma object in the bicategory* **Topoi** *where the inverse-image functors being given by the comma-property and the direct-images is provided by the coComma property. Thus* **Topoi** *has coComma objects of the form* $< f, \mathbb{A} >$. *With an arbitrary left-exact functor in place of* f_**, this construction is well-known Artin glueing, [1], [9].*

We turn now to **Topoi** and recall that we have pseudo functors

$$\hat{()} : \mathbf{Lex}^{op} \to \mathbf{Topoi}$$
$$\widetilde{()} : \mathbf{LexSite}^{op} \to \mathbf{Topoi}$$

called "presheaves (resp. sheaves) on $(-)$"; the "op" indicates that the 1-cells are reversed but 2-cells retain their direction. On 1-cells, the direct-image functors are induced by composition and inverse-image functor is the left Kan extension, left exact because the original 1-cell is.

We state without proof the classification theorem ([1],[3]).

2.7 THEOREM

(a) $\mathbf{Topoi}(\mathbf{E}, \hat{\mathbb{A}}) \simeq \mathbf{Lex}(\mathbb{A}, \mathbf{E})$

(b) $\mathbf{Topoi}(\mathbf{E}, \widetilde{\mathbb{A}}) \simeq \mathbf{LexSite}(\mathbb{A}, \mathbf{E})$, *naturally in* $\mathbf{E}$ *and* $\mathbb{A}$.

Note the abuse of language whereby we have treated the (large) underlying category of a topos as an object of **Lex** or, with its canonical topology, of **LexSite**.

2.8 COROLLARY $\hat{()}, \widetilde{()}$ *take coLimits to Limits.*

PROOF The usual argument for adjoint functors also works for this partial Adjointness of pseudo-functors: if $\mathcal{D}$ is a diagram in **Lex** having a coLimits $\mathbb{L}$ and $\hat{\mathcal{D}}^{op}$ is the corresponding diagram of topoi, then

$$\begin{aligned} \mathbf{Topoi}(\mathbb{E}, \hat{\mathcal{D}}^{op}) &\simeq \mathbf{Lex}(\mathcal{D}, \mathbf{E}) \text{ by Theorem 2.7(a)} \\ &\simeq \mathbf{Lex}(\mathbb{L}, \mathbf{E}) \text{ by the definition of coLimit} \\ &\simeq \mathbf{Topoi}(\mathbf{E}, \hat{\mathbb{L}}) \text{ by Theorem 2.7(a)} \end{aligned}$$

whence $\hat{\mathbb{L}}$ is the Limit in **Topoi**. A similar argument works for sites. ■

Recall that a topos is a Grothendieck topos if (it is cocomplete and) it has a (small) set of generators. We denote by **GrTopoi** the full subcategory of Grothendieck topoi. By relativising these notions to an arbitrary elementary topos playing the role of Sets, Diaconescu arrives at the notion of a bounded topos-map $\mathbf{E} \to \mathbf{F}$, one for which $\mathbf{E}$ has an $\mathbf{F}$-object of generators and by relativising the classification theorem, obtains [3]:

2.9 The Pullback of a topos-map along a bounded topos-map exists. ■

It is easy to show that any map $\widetilde{\mathbb{A}} \to \mathbf{E}$ is bounded and the Giraud theorem characterises Grothendieck topoi as those of the form $\widetilde{\mathbb{A}}$ for some (not unique) site $\mathbb{A}$.

Combining 2.9 with the results above, we obtain:

2.10 PROPOSITION **GrTopoi** *is finitely Complete.* **Topoi** *has a terminal object, the Pullback* $\mathbf{A} \times_{\mathbf{C}} \mathbf{B}$ *exists if one of* $\mathbf{A}$ *and* $\mathbf{B}$ *is bounded over* $\mathbf{C}$*, the Comma-topos* $[f, g]$ *exists if one of* f *and* g *has Grothendieck domain and codomain, and the Inverter of* $\alpha : f \Rightarrow g : \mathbf{A} \to \mathbf{B}$ *exists if* $\mathbf{B}$ *is Grothendieck.*

PROOF Sets is Terminal, Pullbacks along maps between Grothendieck topoi exist and coTensors with $\mathbb{2}$ are obtained from Tensor-sites. The rest are constructed from these. ■

We are thus in a situation where the results of §1 apply.

Needless to say, Inverters may also be constructed as largest sheaf subtopos for which the components α are bidense. Conversely, it is not hard to show that every sheaf embedding is an Inverter by using the relativised version of 2.7(a). $Sh_j(\mathbf{E}) \to \mathbf{E}$ is the Inverter of "1"⇒"J": $\mathbf{E} \to \mathbf{E}^{\mathbf{\Omega}^{op}}$, where "1" and "$J$" are induced by the flat discrete fibrations $1 \to \mathbf{\Omega}^{op}$ and $\mathbf{J}^{op} \to \mathbf{\Omega}^{op}$ (this is essentially due to Johnstone [6]).

Adjoint 1-cells in **Topoi** are just what one would expect.

2.11 If $f : \mathbf{E} \to \mathbf{F}$ and $g : \mathbf{F} \to \mathbf{E}$ are 1-cells in **Topoi**, then $f \dashv g$ if and only if $f^* \simeq g^*$ if and only if $g^* \dashv f^*$ if and only if $g_* \dashv f_*$.

Proof The equivalence of the last three is immediate from uniqueness of adjoints. The equivalence with the first is shown simply by unwinding the equational definition of adjointness for the 1- and 2-cells of **Topoi**. ∎

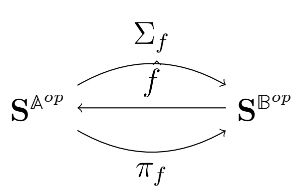
2.12 A 1-cell in **Topoi** is fully-faithful if and only if it is equivalent to a sheaf embedding.

3 Examples

We take the view that "every Grothendieck topos classifies something" (namely, the left-exact, cover-preserving functors - this may be given a first-order syntactic form, albeit with possibly infinitary disjunctions).

If $\mathbf{T}$ is the classifying topos for the theory $\mathcal{T}$, i.e. $\mathbf{Topoi}(\mathbf{E}, \mathbf{T}) \simeq \mathcal{T}$-models$(\mathbf{E})$, naturally in $\mathbf{E}$, then the inverse-image $f^*(M)$ of a model by a map of topoi is again a model. Similarly, a map $u : \mathbf{T}_2 \to \mathbf{T}_1$ induces a "forgetful" functor $\mathcal{T}_1$-models$(\mathbf{E}) \to \mathcal{T}_2$-models$(\mathbf{E})$ by composition. A 2-cell $\alpha : f \Rightarrow g : \mathbf{E} \to \mathbf{T}$ is interpreted as a $\mathcal{T}$-model homomorphism whence we see immediately that $\mathbb{2} \pitchfork \mathbf{T}$ is the $\mathcal{T}$-homomorphism-classifier. Thus the model theory of topoi coextensive with the study of the bicategory-structure of **Topoi**. We shall usually identify a $\mathcal{T}$-model M in $\mathbf{E}$ with its classifying map $M : \mathbf{E} \to \mathbf{T}$, hoping that this simplifies life for the reader rather than confusing him.

In this light, we examine an example of adjoint topos-maps. A left-exact functor $f : \mathbb{A} \to \mathbb{B}$ induces three functors, $\hat{f}, \Sigma_f$ and π_f, where Σ_f and π_f are the left and the right Kan extensions.



$$\Sigma_f \dashv \hat{f} \dashv \pi_f$$

We have already identified $\Sigma_f \dashv \hat{f}$ as the topos-map $\hat{f} : \hat{\mathbb{B}} \to \hat{\mathbb{A}}$. But since $\hat{f}$ is left exact, $\hat{f} \dashv \pi_f$ is also a topos-map $f^\# : \hat{\mathbb{A}} \to \hat{\mathbb{B}}$ and from 2.11 we know that $f^\# \dashv \hat{f}$.

Now we may consider $\mathbb{A}$ and $\mathbb{B}$ to be the duals of categories of finitely presented algebras, thus thinking of $\mathbb{A}$ and $\mathbb{B}$ themselves as algebraic theories, with f an interpretation. Then $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ are the $\mathbb{A}$ and $\mathbb{B}$-algebras classifiers, $\hat{f}$ represents the "forgetful" $\mathbb{B}$-alg$(\mathbf{E}) \to \mathbb{A}$-alg$(\mathbf{E})$ and $f^{\#}$ represents its left adjoint, the relatively free-functor. For example, if $f^{op} : \mathbb{A}^{op} \to \mathbb{B}^{op}$ is the abelianisation functor from finitely presented groups to finitely presented abelian groups, we obtain the abelian-group classifier as a reflective sub-topos of the group classifier. We mention a further point of interest for this example. If the interpretation is finitary - involves the imposition of finitely many new axioms - as for the example of groups and abelian groups - in the sense that if B is a finitely presented $\mathbb{B}$-algebra then $U(B)$ is finitely-presented as an $\mathbb{A}$-algebra, then the forgetful functor restricts to $\mathbb{B}^{op} \to \mathbb{A}^{op}$, providing a left adjoint g to f. We obtain from this a fourth functor, $\Sigma_g \dashv \Sigma_f$ between the classifying topoi, so that the "forgetful" map $\hat{f}$ is actually an essential topos-map.

A more geometric example of adjoint topos-map is furnished by the relationship (given in [1]) between the "gros" topos of a space and its "ordinary" topos. In fact, since the map of sites $i : Open(X) \to Spaces/X$ is cover-reflecting, it induces not only the "restriction" map $I : TOP(X) \to Sh(X)$ but also the left adjoint inclusion $Sh(X) \to TOP(X)$, so that $Sh(X)$ is a coreflective sub-topos of $TOP(X)$. The remark that these topoi are therefore cohomologically equivalent applies equally to other coreflective situations. For example, the Zariski topos, Zar, sheaves on affine schemes of finite type, may (by the Lemma de Comparaison of [1]) equally be constructed as sheaves on the category of schemes. The Zariski topology is less fine than the canonical and so, using the Yoneda functor, we may consider a scheme X both as a ringed space and as an object of Zar. Essentially the same argument as for the gros topos shows that $Sh(X)$ is a coreflective subtopos of Zar/X.

Suppose $\mathcal{D}$ is a finite diagram-type and $\mathbf{T}$ a classifying topos - the object-classifier, pour fixer les ides. We may form $\mathcal{D}\pitchfork\mathbf{T}$, the $\mathcal{D}$-diagram classifier (of $\mathcal{T}$-models) by taking Pullbacks and Comma-objects according to the recipe by which $\mathcal{D}$ is made up from nodes, arrows and commutative relations. Thus for example the classifier for diagrams $\cdot \to \cdot \leftarrow$ is obtained by Pulling back $\partial_1 : \mathbb{2}\pitchfork\mathbf{T} \to \mathbf{T}$ along itself; the commuting-square-classifier is got by Pulling ∂_1 along ∂_0 to obtain $\mathbb{3}\pitchfork\mathbf{T}$ and then Pulling the "composition" map $\mathbb{3}\pitchfork\mathbf{T} \to \mathbb{2}\pitchfork\mathbf{T}$ back along itself.

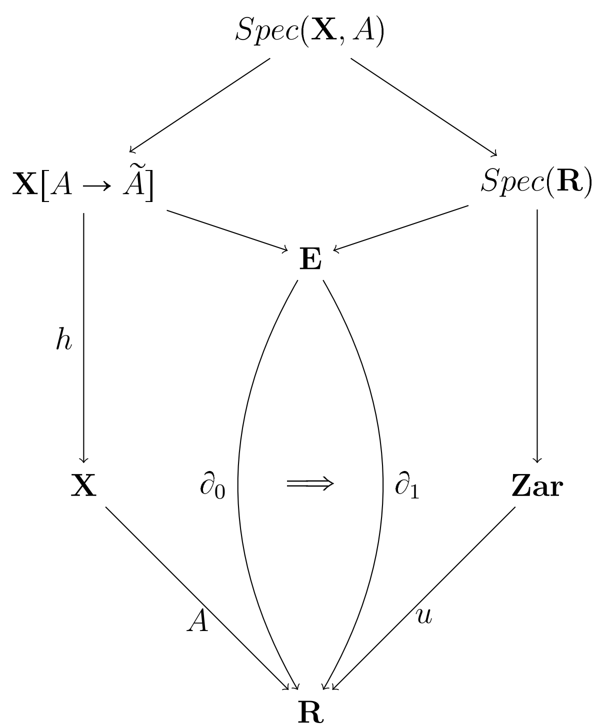
Just as $\mathbb{2}\pitchfork\mathbf{T}$ is the $\mathcal{T}$-morphism classifier, so, for $u : \mathbf{T}_2 \to \mathbf{T}_1$, the Comma-topos $[\mathbf{T}_1, u]$ classifies $\mathcal{T}_1$-morphisms $A \to u(B)$, where A is a $\mathcal{T}_1$-model and B is a $\mathcal{T}_2$-model. Recall that Lemmas 2.4 and 2.5 construct a site of definition of $[\mathbf{T}_1, u]$ from a site-map defining u. The sites defining the spectra of Hakim [5] are of closely related form, with a finer topology. Given a particular model $A : \mathbf{E} \to \mathbf{T}_1$, the *Comma-topos* $[A, \mathbf{T}_1]$ *classifies* "$\mathcal{T}_1$-maps with domain A": given $f : \mathbf{F} \to \mathbf{E}$, maps $\mathbf{F} \to [A, \mathbf{T}_1]$ over $\mathbf{E}$ correspond to $\mathcal{T}_1$-maps $f^*(A) \to (-)$ where $(-)$ is any $\mathcal{T}_1$-model in $\mathbf{F}$. Similarly $[u, A]$ classifies $\mathcal{T}_1$-maps $u(-) \to A$ and $[A, u]$ classifies maps $A \to u(-)$.

Clearly various "epi-mono" factorisations of $\mathcal{T}$-maps give rise to applications of 1.17. It is easy to see that **Topoi**$//\mathbf{T}$ is the category of $\mathcal{T}$-modelled topoi defined in the same way as the usual category of ringed topoi. For the category of local-ringed topoi, however, we must insist that all ring-homomorphisms be local, i.e. reflect the units (invertible elements), whence, in our previous notation, *Loc*-**Topoi**//Zar is the category of local-ringed topoi. We see from §1 that the existence of a right Adjoint (the spectrum of Hakim [5]) to the forgetful *Loc*-**Topoi**//Zar → **Topoi**//**R** (**R** the ring classifier) is equivalent to the fact that a ring-homomorphism $A \to L$ with L a local ring has a best factorisation $A \to F \to L$ with $F \to L$ a local map; the associated extremal maps $A \to F$ are the localisations, obtained by pulling back the units of L to A, and forming the ring-of-fractions to invert this "prime co-ideal", giving the local ring F (Tierney [8]).

It is worth unravelling the proof of the relevant version of Proposition 1.18 for this case. There is an underlying factorisation of ring-homomorphisms (not just those with local codomain), namely, with $\mathcal{M}$ = {unit-reflecting maps} and $\mathcal{E}$ = {ring-of-fractions maps} (of the form $A \to A[S^{-1}]$ for some multiplicatively closed subobject S of A). We factorise the universal ring-homomorphism

$$2 \pitchfork \mathbf{R} \overset{\Downarrow}{\rightrightarrows} \mathbf{R}$$

and invert its $\mathcal{M}$-part to obtain the fractions-map-classifier **E**. Now Pull back the "codomain" map along $u : Zar \to \mathbf{R}$ to obtain the localisation-classifier (the spectrum of the universal ring) and finally Pull the "domain" map back along a given ring $A : \mathbf{X} \to \mathbf{R}$ to obtain $Spec(\mathbf{X}, A)$. Notice that these steps all commute with each other: we may factorise and Pull back in any convenient order.



In particular $\mathbf{X}[A \to \widetilde{A}]$ is the classifier for "fractions-maps with domain A". It has a map to the topos $\mathbf{X}$ so we may imagine $\mathbf{X}[A \to \widetilde{A}]$ as being a topos of sheaves with values in $\mathbf{X}$, the direct-image functor h_* being thought of as "global sections", the inverse-image h^* being "constant sheaf".

The universal fractions-map

$$\mathbf{X}[A \to \widetilde{A}] \longrightarrow \mathbf{E} \underset{}{\overset{}{\rightrightarrows}} \mathbf{R} \quad (\Downarrow)$$

then looks like a fractions-map $h^*(A) \to \widetilde{A}$ of rings in $\mathbf{X}[A \to \widetilde{A}]$ corresponding by adjointness to $A \to h_*(A)$. Thus A is represented in the "global sections of a sheaf" (when $\mathbf{X}$ =Sets, this is literally true). But recall that since $\mathbf{R}$ is a coreflective subtopos of $\mathbf{E}$, by Lemma 1.14, $\mathbf{X}$ is a coreflection subtopos of $\mathbf{X}[A \to \widetilde{A}]$, whence by Proposition 2.11, we see that the functor h_* is actually the inverse-image functor of the inclusion $\mathbf{X} \to \mathbf{X}[A \to \widetilde{A}]$. The front adjunction isomorphism then gives immediately that $A \to h_*(\widetilde{A})$ is an isomorphism since the inclusion classifies $A \to A$ as fractions-map. This argument shows that for "spectra" of the kind given by Proposition 1.17, the "representation of A in a sheaf" is always an isomorphism $A \tilde{\to} h_*(A)$. However, for most purpose, this is not enough: we "force" the codomain $\widetilde{A}$ to be a model of a richer theory (local rings in this case), by Pulling back ∂_1 (along $\mathbf{Zar} \to \mathbf{R}$) which obstructs the argument. This author suspects that further progress will involve considering the Beck condition for Pullbacks of coherent topoi.

Another example is furnished by the étale spectrum of a local-ringed topos (Hakim [5]). Joyal and Wraith have determined that Hakim's strictly local rings are those local rings A which are "separably closed" in the following sense. If a polynomial $f(t) \in A[t]$ is monic (i.e. has leading coefficient 1), consider $D(f)(t) = t^n - \prod_{i=1}^{n}(t - f'(\alpha_i))$, where $\alpha_1, \ldots, \alpha_n$ are the roots of f (in some hypothetical extension of A), and f' is the formal derivative of f. Since $D(f)$ is symmetric in the α_i's, it has coefficients lying in A (Newton's theorem on symmetric polynomials), whence we have a purely combinatorial procedure for defining $D(f)(t)$ without reference to any roots. Classically $D(f) \equiv 0$ if and only if all the roots of f are repeated roots. The axiom for a local ring to be strictly local says: $D(f)(t)$ has an invertible coefficient implies $\exists a \in A : f(a) = 0$ and $f'(a)$ is invertible. Hakim considers local homomorphisms between strictly local rings and constructs a "spectrum" to "strictify" a local ring, universally, of which the étale topos of a scheme is an example. Wraith has (tentatively) identified the extremal maps for the best factorisation of a local map $A \to S$ into $A \to T \to S$, with S and T strictly local, as being those maps $\varphi : A \to T$ for which every $t \in T$ satisfies a polynomial equation $(\varphi(f))(t) = 0$ with $(\varphi(f))'(t)$ invertible, for f a monic polynomial over A (T is "separably integral" over A) and T is strictly local. Such a factorisation, stable under inverse-image functors, is equivalent to Hakim's construction of a right Adjoint to the forgetful Loc-**Topoi**$//StrZar \to$ **Loc-Topoi**$//Zar$ by 1.18.

In similar vein, it is conjectured that the crystalline topos of a scheme will be associated with a universal extremal "extension of A by a nil-ideal with divided power structure" $I \hookrightarrow B \to A$ (plus further structure whose details are here irrelevant).

An unfamiliar application is to ordered sets. An order-preserving map $P \to L$ from an ordered set to a linearly ordered set has a best factorisation whose second factor is order-reflecting ($f(x) < f(y)$ implies $x < y$) between linear orderings: pull back the ordering of L to P and quotient by the antisymmetry law. Hence there is a right Adjoint to the forgetful $OrdRefl$-$\mathbf{Topoi}//\mathbf{L} \to \mathbf{Topoi}//\mathbf{P}$, where $\mathbf{L}$ and $\mathbf{P}$ classify respectively linear and partial orderings.

As a final example, we construct a spectrum for ordered rings, for which the "Zariski topology" would better be called the Euclidean topology. An ordered ring in this case means a ring with a predicate $P(x)$ (read "x is positive") satisfying $\neg P(0), P(1), P(x) \wedge P(y)$ implies $P(x+y) \wedge P(xy)$. Say that a ring A is *linear* if in addition $P(x+y)$ implies $P(x) \vee P(y)$, and $P(xy)$ implies $P(x) \vee P(-x)$. Call A *full* if $P(x)$ implies $\exists y(xy = 1)$. Since the positive elements are multiplicatively closed, any ordered ring may be made full by taking fractions. Say that a linear full ordered ring is local (it is local in the usual sense). A map of local ordered rings (a homomorphism preserving positivity) is local if and only if it reflects positivity, i.e. it reflects the ordering. To factorise a map $A \to L$ from an ordered ring to a local ordered ring, proceed as above to linearise and then add fullness: pull back the ordering from L to A and make A full with respect to this finer ordering. Extremal maps are localisations in the ordinary sense, thought of primarily as linearisations of the ordering. This leads to spectrum, right Adjoint to *Loc*-$\mathbf{Topoi}//\mathbf{OrdZar} \to \mathbf{Topoi}//\mathbf{OrdR}$. Closer analysis (private communication with M. P. Fourman) reveals that a base of "open sets" of this spectrum is of the form $\{\{x : f(x) > 0\} : f \in A\}$ whereas the Zariski base is of the form $\{\{x : f(x) \neq 0\} : f \in A\}$, x ranging over the "points" of the spectrum (it is indeed "spatial" over its domain), in the sense that it is generated by its subobjects of 1; and when Zorn's Lemma holds in the domain topos, it has enough points, so that it is spatial in the strong sense).

References

[1] M. Artin, A. Grothendieck and J. L. Verdier, *Theorie des topos et cohomologie etale des schemas*, Springer Lecture Notes in Mathematics, Vol. 305 (1973).

[2] J. Benabou, *An Introduction to Bicategories*, Reports of the Midwest Category Seminar, Springer Lecture Notes in Mathematics, Vol. 47 (1967).

[3] R. Diaconescu, *Change of base for toposes with generators*, Journal of Pure and Applied Algebra, Vol. 6, Issue 3, pp.191-218 (1975).

[4] J. W. Gray, *Formal Category Theory: Adjointness for 2-categories*, Springer Lecture Notes in Mathematics, Vol. 391 (1974).

[5] M. Hakim, *Topos Annelés et schemas relatifs*, Springer, Ergebnisse, Vol. 64 (1972).

[6] P. Johnstone, *Internal Category Theory*, Thesis, Cambridge (1973).

[7] G. M. Kelly, *Category Seminar*, Springer Lecture Notes in Mathematics, Vol. 420 (1974).

[8] M. Tierney, *On the Spectrum of a ringed topos*, Algebra, Topology and Cathegory Theory: a collection of papers in honor of Samuel Eilenberg (ed. A. Heller and M. Tierney), Academic Press, pp. 189-210 (1976).

[9] G. C. Wraith, *Artin Glueing*, Journal of Pure and Applied Algebra, Vol. 4, Issue 3, pp. 345-348 (1974).

[10] F. W. Lawvere, C. Maurer and G. C. Wraith, *Model Theory and Topoi*, Springer Lecture Notes, Vol. 445 (1975).

This article may be accessed at `http://www.tac.mta.ca/tac/reprints`