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TOPOSES GENERATED BY CODISCRETE OBJECTS IN COMBINATORIAL TOPOLOGY AND FUNCTIONAL ANALYSIS

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Author's commentary

The study of particular toposes (such as Mike Roy's Ball Complexes [5]) gives experience that develops spatial intuitions, which are useful also for applications beyond the purely geometric. Classical examples were G-sets and simplicial sets, as well as the basic 2-valued representations of Boolean logic. George W. Mackey's bornological sets have become fundamental in functional analysis; the category of Banach spaces is fully embedded among its vector space objects. That illustrates how the appropriate topos will serve as the 'cohesive' background for algebraic and other structures. Some of these examples reveal the very special condition that occasionally holds for an abstract grounding $\mathcal{E} \longrightarrow \mathcal{S}$, where the one category $\mathcal{S}$ is embedded in two ways, left and right adjoint to the grounding; these are conveniently thought of as 'discrete' and 'co-discrete'. The special condition studied here is that the general objects of $\mathcal{E}$ are 'generated' by the co-discrete ones (called 'bounded' parts in the bornological case).

I hope that these and related considerations will clarify the disputed connection between basic structures and the architecture of Mathematics. (Even Notre Dame will be recreated by careful scientific and artistic efforts and collaboration.)

The paper below is the summary of the Notes for the Colloquium lectures at North Ryde, NSW, Australia (1988); the subject was treated in lectures in Wisconsin (1989) and at the Seminario Matematico e Fisico, in Milano, Italy (1992). It presents two toposes 'of spaces' and outlines their application in Combinatorial Topology and Functional Analysis. The classifier of non-trivial Boolean algebras is one of the toposes that partially motivates work by Grandis [2] and by Rosicky-Tholen [4], whereas the 'bornological topos' is briefly introduced in [3, Section 7] and has influenced work by Español-Lambán [1].

There are several hard copies of the original typewritten paper around the world but the present latexed version was produced by F. Marmolejo. We hope that its publication in TAC-Reprints will motivate others to work on the ideas that are still unexplored.

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Toposes generated by Codiscrete Objects in Combinatorial Topology and Functional Analysis

(20 December 1989)

For an $\mathcal{S}$-based topos $\mathcal{X} \xrightarrow{p} \mathcal{S}$, the adjointness

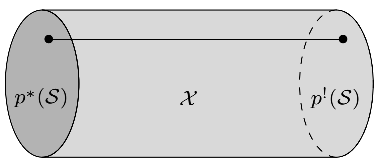
$$\frac{p^*S \longrightarrow X}{S \longrightarrow p_*X}$$

justifies calling p^*S a *discrete* space if we think of p_*X as the abstract set of *points* of the general "space" (= cohesive set) X.[1] $\mathcal{X}$ is called *connected* if p^* is full and faithful, *essential* (or locally connected) if p^* has a further left adjoint $p_!$, and *local* if p_* has a further right adjoint $p^!$. A locally connected topos is connected iff $p_!(1) = 1$, for $p_!$ has the meaning "components" ["orbits" in the case of G-sets].

We always require that p^* is left exact, so for a connected $\mathcal{X}$ the discrete spaces form a full subcategory of $\mathcal{X}$ which is equivalent to $\mathcal{S}$ and closed under arbitrary colimits and finite limits; it will be closed under infinite products as well iff $\mathcal{X}$ is locally connected. For local $\mathcal{X}$ the adjointness

$$\frac{X \longrightarrow p^!S}{p_*X \longrightarrow S}$$

justifies (from the "gros" perspective) calling $p^!S$ a *codiscrete* (indiscrete, chaotic) space. For a connected local $\mathcal{X}$, the codiscrete spaces thus form a second full subcategory of $\mathcal{X}$ which is equivalent to $\mathcal{S}$, closed this time under arbitrary limits but few colimits. The "adjoint cylinder" with base $\mathcal{S}$,



formed by discrete and codiscrete with each horizontal fiber consisting of all spaces in $\mathcal{X}$ with a given set of points, forms a "unity and identity of opposites".

[1] This terminology is appropriate for "gros" $\mathcal{X}$; in the "petit" case we would speak instead of *constant sheaves* and the set of *global sections* of the (variable) set X.

Now the usual petit toposes, such as $\mathcal{S}$-sheaves on a locale or $\mathcal{S}^G$ for G a group, all have the property that they are generated by "empty" objects, that is by the U such that $p_*U = 0$. Let us by contrast consider some connected local toposes which are generated by *codiscrete* objects. In general (when we have a set $\mathcal{U}$ of generators in mind) a map $U \xrightarrow{x} X$ from a generator to an arbitrary object may be thought of as a singular figure in X of the form U (and the category $\mathcal{U}/X$ as specifying incidence relations between these). The form of a space such as $p^!(3)$ may be imagined as a *blob* with three points, totally different from the discrete space $p^*(3)$ in that it has total cohesiveness as opposed to total lack of cohesiveness. There are typically few maps indeed (see the examples below for more precise statements) from a codiscrete space to a discrete one, but by contrast there is a canonical inclusion $p^*S \to p^!S$ (which induces an isomorphism when we apply p_* to it); this map concretizes the contradictory property of "Kardinalen als Mengen": the points have no distinguishing properties yet are completely distinguished. The "geometrical" structure of any space X in such a chaotically-generated connected local topos lies primarily in this: given a family $p^*(S) \xrightarrow{x} X$ of points of X, there may or may not be blobs $\overline{x}$ in X with these as vertices

$$\begin{array}{ccc} p^*(S) & \xhookrightarrow{\text{canon}} & p^!(S) \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle \overline{x}} & \\ & X & \end{array}$$

Those X for which there is at most one blob with given vertices form a subcategory $\mathcal{X}_1 \subset \mathcal{X}$ which is cartesian closed but not a topos.

1. Simplicial schemes in the Boolean algebra classifier

If $\mathbb{S}$ is the category of non-empty finite sets, then the functor category $\mathcal{X} = \mathcal{S}^{\mathbb{S}^{\text{op}}}$ is a topos which is local since

$$p^!(S)(n) = S^n \qquad \text{for } S \in \mathcal{S}, n \in \mathbb{S}$$

is right adjoint to $p_*(X) = X(1)$, and chaotically generated since for any such functor category the Yoneda embedding generates, but here the Yoneda embedding is just a restriction of $p^!$

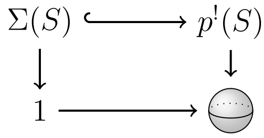
$$\begin{array}{ccc} \mathbb{S} & \xhookrightarrow{\text{Yoneda}} & \mathcal{S}^{\mathbb{S}^{\text{op}}}. \\ & \searrow \quad \nearrow{\scriptstyle p^!} & \\ & \mathcal{S} & \end{array}$$

The maps from a codiscrete to a discrete are *constant* in this topos.

This topos $\mathcal{X} = \mathcal{S}^{\mathbb{S}^{\mathrm{op}}}$ has a rich mathematical content. The cartesian-closed subcategory $\mathcal{X}_1$ is the classical category of simplicial schemes, and indeed we can define a "geometric realization" (on even the "singular" objects of $\mathcal{X}$) as the left Kan extension

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\Delta} & \mathbf{top} \\ \downarrow & \nearrow & \\ \mathcal{X} & & \end{array}$$

of $\Delta(n)$ = the underlying topological space ($n-1$-dimensional) of the free convex set on n generators. Each $p^!(S)$ has a subobject $\Sigma(S)(n) = \{n \xrightarrow{x} S \mid x \text{ not surjective}\}$ for which the pushout in $\mathcal{X}$



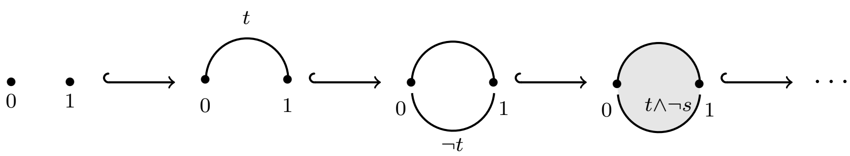
has (for $S < \infty$) a sphere as realization.

Besides $\mathcal{X}_1$, there is in this $\mathcal{X}$ another cartesian-closed reflective subcategory which is not a topos, namely the category **Gp** of all groupoids (small categories in which all morphisms are isomorphisms). This is achieved by associating to any nonempty finite set n the groupoid $\underline{n}$ with n objects and exactly one morphism $i \to j$ for any ordered pair $\langle i, j\rangle$ of these objects; then $\mathbf{Gp} \hookrightarrow \mathcal{X}$ assigns to any groupoid G the functor whose typical value is

$$G(n) = \mathbf{Gp}(\underline{n}, G)$$

the set of all functors from $\underline{n}$ to G. The embedding into $\mathcal{X}$ is full because the incidence relations between $G(2)$ and $G(3)$ determine the multiplication table of any G. The left adjoint of the embedding is the Poincaré functor π_1.

The above geometric realization, with $p^!(2)$ realized as the unit interval I, is not left exact nor does it even preserve finite cartesian products, for I has no continuous Boolean algebra structure yet $p^!(2)$ is obviously a Boolean algebra in any local topos. On the other hand I does have a continuous distributive lattice structure (max / min) and hence can be embedded in a continuous Boolean algebra I_∞. This construction can be done in stages



where hemispheres contracting (at odd stages) previous lower-dimensional spheres are logically negated (at even stages) to become spheres. The direct limit is the (weak) infinite-dimensional sphere, which is contractible. Its topology can be extended to the whole ring $\mathcal{R}$ of step functions on I; in any commutative $\mathbb{R}$-algebra, the Boolean algebra $\{f \mid f^2 = f\}$ is the "sphere" $\{f \mid (f - \frac{1}{2})^2 = \frac{1}{2}\}$. Thus the infinite-dimensional sphere I_∞ is revealed to have a Boolean algebra structure with $\wedge$, $\vee$ continuous and with logical negation $\neg$ identified with the geometric antipodal map; but it is also contractible, contrasting with the fact that a *compact* Boolean algebra must be prodiscrete.

For *most* purposes of algebraic topology and functional analysis, the category **top** would much better be replaced by the topos $\mathcal{T}$ described by Johnstone [**?**] where in particular the above construction can also be carried out. For *some* purposes of homotopy theory, the infinite sphere I_∞ might as well replace the interval as path parameterizer (picturing I_∞ as a fattened interval with lots of back-tracking suggest "Zitterbewegung"). Thus another geometric realization of our topos is the left Kan extension

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\quad I_\infty \quad} & \mathcal{T} \\ \big\downarrow & \nearrow_{\text{- - -}} & \\ \mathcal{S}^{\mathbb{S}^{\text{op}}} & & \end{array}$$

This one is in fact left exact, and for a general reason, which we now will make explicit.

The topos $\mathcal{X} = \mathcal{S}^{\mathbb{S}^{\text{op}}}$ is characterized as the Boolean-algebra classifier among all $\mathcal{S}$-toposes, with $p^!(2)$ the generic Boolean algebra. That is, if B is any non trivial Boolean algebra in any $\mathcal{S}$-topos $\mathcal{E}$, then there is a unique functor $\mathcal{E} \xrightarrow{\beta} \mathcal{X}$ with a left-exact left adjoint β^* for which $\beta^*(p^!2) = B$. For such β^* must be Kan extensions of their Yoneda restrictions $\mathbb{S} \to \mathcal{E}$, but left-exact functors from $\mathbb{S}$ "are" just Boolean algebras because any nonempty finite set is a retract of a power of 2.

Fractional exponents also appear in $\mathcal{X} = \mathcal{S}^{\mathbb{S}^{\text{op}}}$, where the resulting "combinatorial Lagrangians" clarify a step in the construction of Eilenberg-Mac Lane spaces. Here by "fractional exponents" we refer to the possibility (excluded for $\mathcal{S}$) that certain objects D may have the property that $(\ )^D$ (itself a right adjoint) has a further right adjoint $(\ )^{\frac{1}{D}}$. Such objects must be connected, i.e. $(X_1 + X_2)^D = X_1^D + X_2^D$ for all X_1, X_2, and have even stronger properties; as a class they are closed under finite products. Such objects D in a topos $\mathcal{X}$ could be called *local* objects for (as Freyd showed), the canonical geometric morphism $\mathcal{X}/D \xrightarrow{\Pi} \mathcal{X}$ is local. The local objects in the Boolean algebra classifier $\mathcal{S}^{\mathbb{S}^{\text{op}}}$ are just the finite codiscrete ones, for if D is represented by $p \in \mathbb{S}$, then for all Y, we can define

$$Y^{\frac{1}{D}}(n) = Y(n^p) \qquad n \in \mathbb{S}$$

Usually $(\ )^{\frac{1}{D}}$ does *not* have a still further right adjoint, as it does in this exceptional case.

The term "Lagrangian" above alludes to the situation in synthetic differential geometry where there is a local object D such that for any smooth space X, X^D is its tangent bundle. Maps $X^D \to R$ are often called Lagrangians on X and sometimes thought of as an extended kind of function on X itself. The latter point of view receives precise justification in the adjointness

$$\frac{X^D \longrightarrow R}{X \longrightarrow R^{\frac{1}{D}}}$$

wherein, since $(\ )^{\frac{1}{D}}$ preserves products, $R^{\frac{1}{D}}$ will be a ring if R is. The connection with Eilenberg-Mac Lane spaces, on the other hand, may be sketched as follows: Since $D_n = p^!(n+1)$ is a generic n-simplex, a map

$$X^{D_n} \to R$$

is an n-cochain on X. In a topos such as our $\mathcal{S}^{\mathbb{S}^{\mathrm{op}}}$, D_n is a local object, so these cochains may be considered equivalently as maps from X itself to $R^{\frac{1}{D_n}}$; the inclusions $D_n \hookrightarrow D_{n+1}$ induce maps whose alternating sum has a kernel

$$K(R,n) \hookrightarrow R^{\frac{1}{D_n}} \rightrightarrows R^{\frac{1}{D_{n+1}}}$$

so that maps $X \to K(R,n)$ are cocycles. We only need to verify that between a pair of cocycles the homotopy relation reduces to the cohomology relation, in order to see that $K(R,n)$ homotopically represents $H^n(X,R)$.

Another virtue of $\mathcal{S}^{\mathbb{S}^{\mathrm{op}}}$ as an example is that we can explicitly determine all its subtopos: they are in fact all again presheaf toposes $\mathcal{S}^{{\mathbb{S}_p}^{\mathrm{op}}}$, where $\mathbb{S}_p$ is the category of non empty sets of cardinality $\leq p$. The inclusion geometric morphisms are the right Kan extensions p_* of the inclusions $\mathbb{S}_p \xhookrightarrow{p} \mathbb{S}$; since there are also the left Kan extensions $p_!$, all subtoposes are essential in this case. The "canonical" topology, defined in general to determine the smallest subtopos whose inclusion p_* includes the Yoneda embedding, in our case of codiscrete generation reduces to the same as the double negation topology, whose sheaves are just $\mathcal{S}$. However, the truth-value object $\Omega_{\neg\neg}$ of $\mathcal{S}$ is (though Boolean) the *connected* object $p^!(2)$ of $\mathcal{S}^{\mathbb{S}^{\mathrm{op}}}$. The "Aufhebung of double negation" exists in this case: the smallest essential subtopos such that the p-skeleton $p_!p^*X$ of any X has the same components as X itself is given by $p = 2$, i.e. the topos $\mathcal{S}^{{\mathbb{S}_2}^{\mathrm{op}}}$ of reversible graphs.

2. Bornology as a topos and Abelian Categories in Functional Analysis

A functor which preserves finite limits, finite coproducts and coequalizers of equivalence relations may still fail to preserve arbitrary coequalizers, (even a functor between very good categories) “because” the passage from coequalizer data to the equivalence relation with the same coequalizer involves countable coproducts. The minimal counterexample which illustrates this is also a basis of much of functional analysis, and also provides example of codiscrete generation of a topos which is not a presheaf topos.

Let $\mathbb{C}$ denote the category of countable sets (or quasi-equivalently, the monoid of all endomaps of the set of natural numbers). This is a “strictly distributive” category in the sense that pullback and coproduct provide equivalences of categories

$$\mathbb{C}/(A+B) \xrightarrow{\sim} \mathbb{C}/A \times \mathbb{C}/B$$
$$\mathbb{C}/0 \xrightarrow{\sim} 1$$

for A, B in $\mathbb{C}$. For any such category, the category $\mathcal{G}(\mathbb{C})$ of all product-preserving functors $\mathbb{C}^{\mathrm{op}} \xrightarrow{X} \mathcal{S}$, i.e.

$$X(A+B) \xrightarrow{\sim} X(A) \times X(B)$$
$$X(0) \xrightarrow{\sim} 1$$

is actually a *topos*, namely such X are the sheaves for the “finite disjoint cover” topology. Swan pointed out [2] that the associated sheaf functor $\mathcal{S}^{\mathbb{C}^{\mathrm{op}}} \to \mathcal{G}(\mathbb{C})$ (left adjoint to the inclusion) can be computed as a single colimit over covers in this case, in contrast to the twice-iterated colimit required for more general topologies. By construction the Yoneda embedding $\mathbb{C} \to \mathcal{G}(\mathbb{C})$ preserves finite limits and finite coproducts; moreover, for our specific example of countable sets, it preserves coequalizers of equivalence relations as well since by the axiom of choice surjections in $\mathbb{C}$ split. (Another important example of a strictly-distributive category is $\mathbb{A}_K^{\mathrm{op}}$, where $\mathbb{A}_K$ is the category of finitely-presented commutative algebras over a field K, as was pointed out by Gaeta in the foundation of algebraic geometry).

Now the coequalizer in $\mathbb{C}$ of

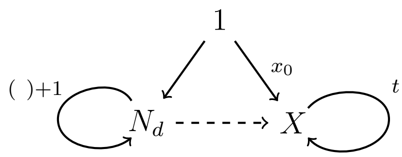
$$N \underset{\mathrm{id}}{\overset{(\ )+1}{\rightrightarrows}} N \longrightarrow ?$$

is 1, yet for the corresponding codiscrete/representable objects

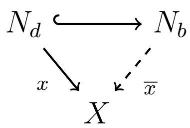
$$N_b \rightrightarrows N_b \longrightarrow ?$$

in $\mathcal{G}(\mathbb{C})$, the coequalizer is *not* the terminal object (though it has only one *point*); it has a very rich structure reminiscent of the Frechet filter.

The topos $\mathcal{B} = \mathcal{G}(\mathbb{C})$ for $\mathbb{C}$ = countable sets deserves to be called the *bornological* topos, and in particular the codiscrete natural numbers N_b could also be considered as the "bounded" natural numbers; a figure $N_b \xrightarrow{x} X$ in an arbitrary $X \in \mathcal{B}$ may be called a *bounded sequence in* X. It is still the *discrete* N_d which (as in any topos) satisfies the universal property of recursion



for any object X and map t. As "identical opposites", N_d and N_b have isomorphic endomorphism monoids and there is a canonical inclusion $N_d \hookrightarrow N_b$; however, they are very different for all maps $N_b \to N_d$ have *finite* image, as befits "bounded sequences of natural numbers". Since $N_d = \sum 1$, a map $N_d \to X$ is an arbitrary sequence of points of X; such may or may not be "bounded", i.e. admit



Of course morphisms $X \to Y$ are automatically bornological, i.e. take bounded sequences to such. Those X for which there is always at most one $\overline{x}$ for a given x in the above form a full reflective cartesian closed subcategory $\mathcal{B}_1$ of our topos $\mathcal{B}$, which is closely related to traditional functional analysis, as explained below; however, $\mathcal{B}$ as a topos has much better exactness properties (such as epi & mono $\Rightarrow$ iso) as well as permitting spatialization of many more concepts and constructions (such at Ω^X).

The usual category **bor** has as objects sets equipped with a class of "bounded" subsets closed under finite union and passage to smaller subsets, and as morphisms maps which *covariantly* preserve bounded subsets. This last feature makes it easy to see that **bor** has function spaces, i.e. is cartesian closed. Most examples arising in analysis have the further property that a subset is bounded if every countable subset of it is bounded; call this full subcategory $\mathbf{bor}_\omega$. Then there is an obvious full embedding $\mathbf{bor}_\omega \to \mathcal{B}$, which generates.

Although simpler (in that it is covariant) than topology, bornology in itself has at first glance the air of being too abstract to be of direct mathematical value. However, that changes when we consider algebraic structures in the category; the category $\mathbf{Ab}(\mathcal{B})$ of abelian groups in bornology contains the category of Banach spaces as a *full* subcategory, as is historically enshrined in the term "bounded linear transformation". The famous "uniform boundedness" theorems have the form: let $H(X, Y) \subset Y^X$ be a part compatible with certain algebraic structures with which X, Y are endowed, and let $|X| \hookrightarrow X$ be the discrete part of X. Then any sequence bounded in $H(|X|, Y)$ is bounded in $H(X, Y)$, i.e. any commutative square (with canonical vertical arrows) has a diagonal

$$\begin{array}{ccc} N_d & \longrightarrow & H(X,Y) \\ \downarrow & \nearrow & \downarrow \\ N_b & \longrightarrow & H(|X|,Y) \end{array}$$

The real number object $\mathbb{R}_{\mathcal{B}}$ of the topos $\mathcal{B}$ is a ring so there is the abelian (even Grothendieck AB 5) category $\mathbb{R}_{\mathcal{B}}$-$\mathbf{Lin}(\mathcal{B})$ of all $\mathbb{R}_{\mathcal{B}}$ modules in $\mathcal{B}$. Similarly there is $\mathbb{R}_{\mathcal{T}}$-$\mathbf{Lin}(\mathcal{T})$ where $\mathcal{T}$ is the Johnstone topological topos mentioned earlier. There are two prejudices associated with the usual approach to topology which it would be good for analysis to dispel. One is that epi-monos need not have continuous inverses so that toposes and abelian categories could not directly reflect topology or functional analysis. However, the actual meaning of "epimorphism" in $\mathcal{B}$ and $\mathbb{R}_{\mathcal{B}}$-$\mathbf{Lin}(\mathcal{B})$ (as well as in $\mathcal{T}$ and $\mathbb{R}_{\mathcal{T}}$-$\mathbf{Lin}(\mathcal{T})$) is (not only surjective on points but even) locally surjective on bounded sequences (or on convergent sequences). The other (fostered by some treatises on topological vector spaces) is that an infinite morass of counter examples refutes any reasonable conjecture about the behavior of dual spaces or function spaces. However, that is much ameliorated by the covariant "figure" concept of geometrical/topological structure (with contravariant notions such as open set or real continuous function *derived* rather than fundamental) and even more by concentration on *sequential* convergence, as in $\mathcal{T}$. Work of Kolmogorov in the 1930's (further developed in Mackey's thesis) indicates a pair of adjoint functors between the two abelian categories

$$\mathbb{R}_{\mathcal{T}}\text{-}\mathbf{Lin}(\mathcal{T}) \rightleftarrows \mathbb{R}_{\mathcal{B}}\text{-}\mathbf{Lin}(\mathcal{B})$$

which may be roughly described as Hom and $\otimes$ with the space c_0. The fundamental lemma of functional analysis should say that this is very nearly an equivalence of categories (even though $\mathcal{T}$ and $\mathcal{B}$ themselves are quite different—for example, $\mathcal{T}$ is *not* "generated by codiscrete objects").

What is the role of $\mathcal{B}$ as a classifying topos? That is, for a geometric morphism $\mathcal{X} \xrightarrow{f} \mathcal{B}$, what structure does $f^*(N_b)$ have? It is a non-standard model of the following sort of arithmetic: All functions $N^k \to N$ are interpreted as operations, and all positive truths involving $\exists$ and only *finite* disjunctions are valid for the interpretation.

The construction of $\mathcal{B}$ admits a generalization which may be of set-theoretic interest. Let $\mathbb{L}$ be the category of sets of cardinality $\leq \lambda$ where λ is a "large" cardinal, and let $\mathcal{L} \hookrightarrow \mathcal{S}^{\mathbb{L}^{\mathrm{op}}}$ be the subcategory of those X for which

$$X\left(\sum_{i \in I} c_i\right) = \prod_{i \in I} X(c_i)$$

for $c_i \in \mathbb{L}$, $\mathrm{card}(I) < \lambda$. Then $\mathcal{L}$ is a topos and (for $\lambda > \omega$) the Yoneda embedding preserves all colimits (including coequalizers) of size $< \lambda$ (λ should at least be regular in order that these colimits exist in $\mathbb{L}$.) Then $\mathcal{L}$ is quite different from $\mathcal{S}$, yet the discrete and codiscrete embeddings $\mathcal{S} \rightrightarrows \mathcal{L}$ *agree* on sets smaller than λ. Then the "oppositeness" in this unity $\mathcal{L}$ of identical opposites $\mathcal{S}$ is only revealed at λ and above where no map $\lambda_b \to \lambda_d$ is surjective.

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