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CLASSIFICATION OF CONCRETE GEOMETRICAL CATEGORIES

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ABSTRACT. A precise concept of concrete geometrical category is introduced in an axiomatic way. To any algebra L for an many-sorted infinitary algebraic theory $\mathbf{T}$ is associated a concrete geometrical category $\mathbf{Geo}(L)$, the so-called classifying concrete geometrical category of L, satisfying a universal property. The terminology "geometrical" is justified firstly for $\mathbf{Geo}(L)$ and secondly for any concrete geometrical category by proving that they are all classifying ones. The legitimate category **CGC** of concrete geometrical categories is build up and proved to be the dual of the legitimate category **TGC** of topological geometrical categories.

1. Introduction

The aim of the work is the unification of geometrical structures and the introduction of a precise concept of geometrical categories. We set bounds to our ambition by dealing only with geometrical structures which are carried by sets and lead to categories which are concrete over **Set**. The objects of these geometrical categories are sets equipped with a geometrical structure, called geometrical spaces, and their morphisms are maps preserving the structure, called geometrical maps.

We start with an axiomatic definition of concrete geometrical categories which implies that they are complete, cocomplete, wellpowered, cowellpowered, coregular, with enough injective objects and free objects, and enjoy the transportability of structures along bijections. Morphisms of concrete geometrical categories are defined in a natural way and are the morphisms of the category **CGC** of concrete geometrical categories. Not only this category is legitimate i.e. it has small hom sets but, moreover, it is cocomplete and coregular. Its regular monomorphisms are precisely the full embeddings of concrete geometrical categories ; they are represented by full geometrical subcategories and characterized as full limit-completions of sets of injective objects.

We construct, for any algebra L for an many-sorted infinitary algebraic theory $\mathbf{T}$, a concrete geometrical category $\mathbf{Geo}(L)$, the so-called classifying concrete geometrical category of L, which contains a canonical injective algebra object A carried by L, the so-called generic injective algebra carried by L, which satisfies the following universal property: for any concrete geometrical category $\mathbf{X}$ and any injective algebra object B of $\mathbf{X}$ carried by L, there exists a unique morphism of concrete geometrical categories

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$R : \mathbf{Geo}(L) \longrightarrow \mathbf{X}$ such that the algebra object $R(A)$ is identical to B. The objects of $\mathbf{Geo}(L)$ are, up to isomorphisms, precisely the sets of solutions with values in L of systems of $\mathbf{T}$-algebraic equations in many variables. According to the general idea of algebraic geometry, these objects are geometrical spaces: if one thinks of L as being a generalized affine line, any power of L as being a generalized affine space whose elements are points, then any object of $\mathbf{Geo}(L)$ is thought as being a locus of points satisfying algebraic equations. In this way, we justify the terminology "geometrical" not only for $\mathbf{Geo}(L)$ but indeed for any concrete geometrical category by proving that they are all classifying concrete geometrical categories.

A special class of concrete geometrical categories is that of topological geometrical categories introduced previously in [5]. With their natural morphisms, they form the legitimate category $\mathbf{TGC}$. The relation between the two categories $\mathbf{CGC}$ and $\mathbf{TGC}$ is not simple. On the one hand, there is a canonical natural embedding of categories $\mathcal{J} : \mathbf{TGC}^{\mathbf{op}} \to \mathbf{CGC}$, but on the other hand, there is a duality of categories $\mathcal{A} : \mathbf{TGC}^{\mathbf{op}} \to \mathbf{CGC}$. The functor $\mathcal{A}$ assigns to a topological geometrical category $\mathbf{X}$ its full subcategory $\mathbf{AX}$ of algebraic objects, and its quasi-inverse assigns to a concrete geometrical category $\mathbf{X}$ its topological geometrical completion $\mathbf{EX}$.

We end with a long list of examples of concrete geometrical categories classified by algebras.

2. Concrete geometrical categories

Let us recall that a concrete category over $\mathbf{Set}$ is a category $\mathbf{X}$ equipped with a faithful forgetful functor $U : \mathbf{X} \to \mathbf{Set}$ ([1], Definition 5.1.), that it is said to be uniquely transportable provided that, for any object X of $\mathbf{X}$, any set E and any bijection $g : U(X) \to E$, there exists a unique isomorphism $f : X \to Y$ of $\mathbf{X}$ such that $U(f) = g$ ([1], Definition 5.28.), and that it is said to be concretely complete provided it has small limits preserved by U ([1], Definition 13.12). Let us say that an object X of $\mathbf{X}$ is injective if it is injective with respect to the class of regular monomorphisms of $\mathbf{X}$ ([1], Definition 9.22), that is, if $\mathrm{Hom}_{\mathbf{X}}(f, X)$ is surjective for any regular monomorphism f of $\mathbf{X}$, and that the category $\mathbf{X}$ has enough injective objects provided that any object is a regular subobject of an injective one. Let us recall that a set $\mathfrak{X}$ of objects of $\mathbf{X}$ is called a strong cogenerating set provided that any morphism f of $\mathbf{X}$ such that $\mathrm{Hom}_{\mathbf{X}}(f, X)$ is bijective for any $X \in \mathfrak{X}$ is an isomorphism.

2.1. Definition. *A concrete geometrical category is a concrete category over* $\mathbf{Set}$ *which is concretely complete, uniquely transportable, and has a strong cogenerating set of injective objects.*

2.2. Proposition. *A concrete geometrical category is amnestic, fibre-small, wellpowered, cowellpowered, cocomplete, coregular, and has enough injective objects and free objects.*

Proof. Let $(\mathbf{X}, U)$ be a concrete geometrical category with a strong cogenerating small family of injective objects $(X_i)_{i \in I}$. According to ([1], Proposition 5.29), $\mathbf{X}$ is amnestic i.e. any isomorphism of $\mathbf{X}$ whose underlying map is an identity is itself an identity. Let us prove that $\mathbf{X}$ has an (epi, regular mono)-factorization structure. Let $f : X \to Y$ be a morphism, $g : Z \to Y$ the collective equalizer of the set of pairs of morphisms of the form $(u, v) : Y \rightrightarrows X_i$ with $i \in I$ coequalizing f, and $h : X \to Z$ the morphism such that $gh = f$. Then g is a regular monomorphism and h is an epimorphism since, for any pair of morphisms $(v, w) : Z \rightrightarrows X_i$ coequalizing h, there exists a pair of morphisms $(v', w') : Y \rightrightarrows X_i$ such that $v'g = v$ and $w'g = w$, hence we have $v'f = v'gh = vh = wh = w'gh = w'f$, thus $v'g = w'g$ and $v = w$. It follows that regular monomorphisms are identical to extremal monomorphisms and stable under composition. Moreover a morphism f such that $\mathrm{Hom}_{\mathbf{X}}(f, X_i)$ is surjective for any $i \in I$ factors in the form $f = gh$ where h is an epimorphism such that $\mathrm{Hom}_{\mathbf{X}}(h, X_i)$ is bijective, hence h is an isomorphism and g is a regular monomorphism. Consequently f is a regular monomorphism if and only if the maps $\mathrm{Hom}_{\mathbf{X}}(f, X_i)$ are surjective for any $i \in I$, and therefore regular monomorphisms are co-universal i.e stable under pushout along any morphism. As a result, the category $\mathbf{X}$ is coregular [1, 14.E]. For any object X of $\mathbf{X}$, the set of morphisms of the form $X \to X_i$ $(i \in I)$ defines a morphism $f : X \to \Pi_{i \in I} X_i^{\mathrm{Hom}_{\mathbf{X}}(X, X_i)}$ which is a regular monomorphism since $\mathrm{Hom}_{\mathbf{X}}(f, X_i)$ is surjective for any $i \in I$. Since any X_i is an injective object, $\Pi_{i \in I} X_i^{\mathrm{Hom}_{\mathbf{X}}(X, X_i)}$ is injective, and X is a regular subobject of an injective object. Consequently $\mathbf{X}$ has enough injective objects. Let us prove that $\mathbf{X}$ is fibre-small. Let E be a set, $\mathbf{X}(E)$ the fibre of $\mathbf{X}$ at E, $E_i = U(X_i)$ for any $i \in I$, $\mathcal{E} = \Pi_{i \in I} \mathcal{P}(E_i^E)$ the product of the power sets $\mathcal{P}(E_i^E)$ and $\varphi : \mathbf{X}(E) \to \mathcal{E}$ the map defined by $\varphi(X) = (\varphi_i(X))_{i \in I} = (\{U(f) : f : X \to X_i\})_{i \in I}$. Let us prove that φ is injective. Let $X, X' \in \mathbf{X}(E)$ such that $\varphi(X) = \varphi(X')$. According to what we have seen above, we get two regular monomorphisms , $f : X \to \Pi_{i \in I} X_i^{\mathrm{Hom}_{\mathbf{X}}(X, X_i)} \simeq \Pi_{i \in I} X_i^{\varphi_i(X)}$, $f' : X' \to \Pi_{i \in I} X_i^{\mathrm{Hom}_{\mathbf{X}}(X', X_i)} \simeq \Pi_{i \in I} X_i^{\varphi_i(X')}$ having the same underlying map $U(f) : E \to \Pi_{i \in I} E_i^{\varphi_i(X)}$ and same codomain. Then we have $X \simeq X'$ in $\mathbf{X}(E)$, and $X = X'$ since $\mathbf{X}$ is amnestic. As a result $\mathbf{X}$ is fibre-small. Let X be an object of $\mathbf{X}$ and $E = U(X)$. For a monomorphism $f : Y \to X$, the injective map $U(f) : U(Y) \to E$ induces a bijection $U(Y) \simeq F$ between $U(Y)$ and the image F of $U(f)$ thus, since $\mathbf{X}$ is uniquely transportable, there exists a subobject $f' : Y' \to X$ of X isomorphic to f such that $U(Y') = F$ and $U(f')$ is the insertion of F in E. It follows that the class of subobjects of X can be embedded into the set $\bigcup_{F \in \mathcal{P}(E)} \mathbf{X}(F)$, so that it is a set. Consequently $\mathbf{X}$ is wellpowered. Then, according to ([12], Proposition 16.4.8), $\mathbf{X}$ is cocomplete. According to ([12], Proposition 16.4.7) the forgetful functor $U : \mathbf{X} \to \mathbf{Set}$ has a left adjoint i.e. $\mathbf{X}$ has enough free objects. Let X be an object of $\mathbf{X}$ and $\mathfrak{F}$ the set of morphisms of the form $f : X \to \Pi_{i \in I} X_i^{E_i}$ with $E_i \subset \mathrm{Hom}_{\mathbf{X}}(X, X_i)$ for any $i \in I$. For any epimorphism $f : X \to Y$, there is a regular monomorphism of the form $g : Y \to \Pi_{i \in I} X_i^{\mathrm{Hom}_{\mathbf{X}}(Y, X_i)} \simeq \Pi_{i \in I} X_i^{E_i}$. Thus any quotient object of X arises in the coregular factorization of some morphism of $\mathfrak{F}$, hence they form a set and $\mathbf{X}$ is cowellpowered. ∎

2.3. DEFINITION. *A morphism of concrete geometrical categories is a concrete functor* $R : \mathbf{X} \to \mathbf{Y}$ *between two concrete geometrical categories which is continuous and preserves injective objects.*

2.4. PROPOSITION. *For a concrete functor* $R : \mathbf{X} \to \mathbf{Y}$ *between two concrete geometrical categories, the following assertions are equivalent:*

(i). R is morphism of concrete geometrical categories.

(ii). R has a left adjoint preserving regular monomorphisms.

PROOF. (i) $\Rightarrow$ (ii): The functor R has a left adjoint S according to ([12], Proposition 16.4.7). Let $f : Y \to Z$ be a regular monomorphism of $\mathbf{Y}$ and $S(f) = me$ the coregular factorization of $S(f)$ in $\mathbf{X}$ where $e : S(Y) \to X$ is an epimorphism and $m : X \to S(Z)$ a regular monomorphism. For any injective object T of $\mathbf{X}$, $R(T)$ is injective in $\mathbf{Y}$, thus the map $\mathrm{Hom}_{\mathbf{X}}(S(f), T) \simeq \mathrm{Hom}_{\mathbf{Y}}(f, R(T))$ is surjective, hence the map $\mathrm{Hom}_{\mathbf{X}}(e, T)$ is surjective and indeed bijective. Since injective objects form a strong cogenerating class in $\mathbf{X}$, e is an isomorphism and $S(f)$ is a regular monomorphism.

(ii) $\Rightarrow$ (i): The functor R is continuous since it has a left adjoint S. For an injective object X of $\mathbf{X}$ and any regular monomorphism f of $\mathbf{Y}$, $S(f)$ is a regular monomorphism, thus $\mathrm{Hom}_{\mathbf{Y}}(f, R(X)) \simeq \mathrm{Hom}_{\mathbf{X}}(S(f), X)$ is surjective, hence $R(X)$ is injective. ∎

2.5. PROPOSITION. *Concrete geometrical categories and their morphisms form a legitimate category* **CGC**.

PROOF. It is enough to prove that the class $\mathcal{M}$ of morphisms $R : \mathbf{X} \to \mathbf{Y}$ between two concrete geometrical categories $\mathbf{X}$ and $\mathbf{Y}$ is a set. Let $(X_i)_{i\in I}$ be a small strong cogenerating set of injective objects of $\mathbf{X}$, $\mathcal{Y}$ the set of objects of $\mathbf{Y}$ whose underlying set is the underlying set of some X_i ($i \in I$), and $\varphi : \mathcal{M} \to \mathcal{Y}^I$ the map defined by $\varphi(R) = (R(X_i))_{i\in I}$. Let $R, T \in \mathcal{M}$ be such that $\varphi(R) = \varphi(T)$. For any object X of $\mathbf{X}$, the canonical morphism $f : X \longrightarrow \Pi_{i\in I} X_i^{\mathrm{Hom}_{\mathbf{X}}(X,X_i)}$ is a regular monomorphism according to the proof of Proposition 2.2, hence both morphisms $R(f) : R(X) \to \Pi_{i\in I} R(X_i)^{\mathrm{Hom}_{\mathbf{X}}(X,X_i)}$ and $T(f) : T(X) \to \Pi_{i\in I} T(X_i)^{\mathrm{Hom}_{\mathbf{X}}(X,X_i)}$ are regular monomorphisms of $\mathbf{Y}$ having the same codomain and same underlying map, and thus are identical, so that $R(X) = T(X)$. Consequently $R = T$, φ is injective and $\mathcal{M}$ is a set. ∎

3. Full geometrical subcategories.

3.1. DEFINITION. *A full geometrical subcategory of a concrete geometrical category* $\mathbf{X}$ *is a full subcategory* $\mathbf{Y}$ *of* $\mathbf{X}$ *such that the insertion functor* $\mathbf{Y} \to \mathbf{X}$ *is a morphism of concrete geometrical categories.*

3.2. Theorem. *For a full subcategory* $\mathbf{Y}$ *of a concrete geometrical category* $\mathbf{X}$*, the following assertions are equivalent:*

(i). $\mathbf{Y}$ *is a full geometrical subcategory of* $\mathbf{X}$

(ii). $\mathbf{Y}$ *is a full limit-completion of a set of injective objects of* $\mathbf{X}$.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{Y}$ be a strong cogenerating set of injective objects of $\mathbf{Y}$. It is a set of injective objects of $\mathbf{X}$, according to Definition 2.3 Let $\mathcal{P}$ be the class of objects of $\mathbf{X}$ which are products of small families of objects of $\mathcal{Y}$. According to the proof of Proposition 2.2, any object of $\mathbf{Y}$ is a regular subobject of an object of $\mathcal{P}$, and thus is an equalizer of a pair of morphisms between two objects of $\mathcal{P}$. It follows that any full limit-closed subcategory of $\mathbf{X}$ containing $\mathcal{Y}$ contains $\mathbf{Y}$. Since $\mathbf{Y}$ is limit-closed in $\mathbf{X}$, it is a limit-completion of $\mathcal{Y}$ in $\mathbf{X}$.

(ii) $\Rightarrow$ (i): Let $\mathcal{Y}$ be a set of injective objects of $\mathbf{X}$. Let $\mathcal{P}$ be the class of objects of $\mathbf{X}$ which are products of small families of objects of $\mathcal{Y}$, and $\mathcal{M}$ the class of morphisms of $\mathbf{X}$ which are equalizer of some parallel pair of morphisms whose codomain belongs to $\mathcal{P}$. Then $\mathcal{M}$ is a class of regular monomorphisms of $\mathbf{X}$ closed under products, intersections and pullback along any morphism. Let us prove that $\mathcal{M}$ is closed under composition. Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of $\mathcal{M}$, respective equalizers of some pair of morphisms $(u, v) : Y \rightrightarrows R$, $(r, s) : Z \rightrightarrows S$ with $R, S \in \mathcal{P}$. Since R is injective, there exists a pair of morphisms $(u', v') : Z \rightrightarrows R$ such that $u'g = u$ and $v'g = v$. Let us prove that gf is the equalizer of the pair of morphisms $((u', r), (v', s)) : Z \rightrightarrows R \times S$. If $t : T \to Z$ is a morphism equalizing $((u', r), (v', s))$, then t equalizes (r, s) hence factors through g in the form $t = gh$ where $h : T \to Y$ is a morphism equalizing (u, v) since $uh = u'gh = u't = v't = v'gh = vh$, and thus h factors through f in the form $h = fw$, so that the morphism $t = gh = gfw$ factors uniquely through gf. As a result $\mathcal{M}$ is stable under composition. Let $\mathbf{Y}$ be the full subcategory of $\mathbf{X}$ whose objects are those X such that there exists some morphism $f : X \to Y$ of $\mathcal{M}$ whose codomain Y belongs to $\mathcal{P}$. Then $\mathbf{Y}$ is closed in $\mathbf{X}$ under products. It is also closed under equalizers since for any pair of morphisms $(f, g) : X \rightrightarrows Y$ of $\mathbf{Y}$ there are objects X' and Y' in $\mathcal{P}$ and morphisms $u : X \to X'$ and $v : Y \to Y'$ in $\mathcal{M}$, so that the equalizer $h : Z \to X$ of (f, g) in $\mathbf{X}$ is the equalizer of (vf, vg), thus $h \in \mathcal{M}$, $uh \in \mathcal{M}$, $Z \in \mathbf{Y}$, and h is the equalizer of (f, g) in $\mathbf{Y}$. It follows that $\mathbf{Y}$ is limit-closed in $\mathbf{X}$ and thus is a concretely complete concrete category over $\mathbf{S}et$. Moreover $\mathcal{Y}$ is a set of injective objects of $\mathbf{Y}$ which is a regular cogenerating set of $\mathbf{Y}$, hence which is a strong cogenerating set in $\mathbf{Y}$. As a result, $\mathbf{Y}$ is a concrete geometrical category. Let Y be an injective object of $\mathbf{Y}$, and $m : Y \to Z$ a morphism in $\mathcal{M}$ with $Z \in \mathcal{P}$. Since m is a regular monomorphism of $\mathbf{Y}$, Y is a split subobject of Z. Because Z is injective in $\mathbf{X}$, Y is injective in $\mathbf{X}$.Consequently the insertion functor $\mathbf{Y} \to \mathbf{X}$ as a morphism of concrete geometrical categories, so that $\mathbf{Y}$ is a full geometrical subcategory of $\mathbf{X}$. ∎

Proof. In the second part of the proof of Theorem 3.2, the property that $\mathbf{X}$ has a strong cogenerating set of injective objects has not been used. Therefore, by abuse of language, one can speak of a full geometrical subcategory of $\mathbf{X}$ even in the case when $\mathbf{X}$ has no strong cogenerating set of injective objects, provided that the other properties are fulfilled. Then, the limit-completion of a set of injective objects $\mathcal{Y}$ of $\mathbf{X}$ will be called the *full geometrical subcategory* of $\mathbf{X}$ *generated* by $\mathcal{Y}$.

4. Topological geometrical categories

According to ([5], Definition 3.1.), a topological geometrical category is a concrete category over **Set** which is topological i.e. creates initial or final structures, and has an initially dense set of initially injective objects.

4.1. THEOREM. *Any topological geometrical category is a concrete geometrical category.*

PROOF. Let $\mathbf{X}$ be a topological geometrical category with an initially dense set of initially injective objects $\mathfrak{X}$. Since regular monomorphisms of $\mathbf{X}$ are embeddings hence are initial morphisms, the objects of $\mathfrak{X}$ are injective. Let T be the indiscrete object of $\mathbf{X}$ on the set $\{0, 1\}$. It is an injective object since, for any regular monomorphism f of $\mathbf{X}$, $U(f)$ is injective, thus $\mathrm{Hom}_{\mathbf{X}}(f, T) \simeq \mathrm{Hom}_{\mathbf{Set}}(U(f), \{0, 1\})$ is surjective. Let us prove that $\mathfrak{X} \cup \{T\}$ is an initially dense set in $\mathbf{X}$. Let $f : Y \to Z$ be a morphism of $\mathbf{X}$ such that $\mathrm{Hom}_{\mathbf{X}}(f, X)$ is bijective for any $X \in \mathfrak{X} \cup \{T\}$. Then $\mathrm{Hom}_{\mathbf{Set}}(U(f), \{0, 1\}) \simeq \mathrm{Hom}_{\mathbf{X}}(f, T)$ is bijective and, since $\{0, 1\}$ is a strong cogenerator in **Set**, $U(f)$ is bijective. Since the set of morphisms $g : Y \to X$ $(X \in \mathfrak{X})$ is initial and each of them factors through f, the morphism f is initial, hence is an isomorphism. As a result, $\mathfrak{X} \cup \{T\}$ is strong cogenerating set of injective objects of $\mathbf{X}$, and $\mathbf{X}$ is a concrete geometrical category. ∎

4.2. **The embedding $\mathcal{T} : \mathbf{TGC}^{\mathbf{op}} \to \mathbf{CGC}$.**

According to ([5], Definition 4.1.), a morphism of topological geometrical categories is a concrete functor $S : \mathbf{X} \to \mathbf{Y}$ between two topological geometrical categories which preserves initial morphisms and final families of morphisms. They are the morphisms of the legitimate category **TGC**. According to the Galois correspondence Theorem ([1], Theorem 21.24), the functor S has a unique concrete right adjoint $R : \mathbf{Y} \to \mathbf{X}$ which is a morphism of concrete geometrical categories, since its left adjoint S preserves regular monomorphisms. Then we get the embedding of categories $\mathcal{J} : \mathbf{TGC}^{\mathbf{op}} \longrightarrow \mathbf{CGC}$ defined by $\mathcal{J}(\mathbf{X}) = \mathbf{X}$ and $\mathcal{J}(S) = R$.

5. Algebraic objects

Let $\mathbf{X}$ be a topological geometrical category.

5.1. DEFINITION. *The category* $\mathbf{AX}$ *of algebraic objects of a topological geometrical category* $\mathbf{X}$ *is the full limit-completion of the class of initially injective objects of* $\mathbf{X}$.

5.2. THEOREM. $\mathbf{AX}$ *is a full geometrical subcategory of* $\mathbf{X}$.

Proof. Let $\mathfrak{X}$ be an initially dense set of initially injective objects of $\mathbf{X}$. According to Theorem 3.2 the full limit-completion of $\mathfrak{X}$ in $\mathbf{X}$ is a full geometrical subcategory $\mathbf{Y}$ of $\mathbf{X}$. Let Y be an initially injective object of $\mathbf{X}$. The canonical morphism $f : Y \longrightarrow \Pi_{X\in\mathfrak{X}} X^{\mathrm{Hom}_{\mathbf{X}}(Y,X)}$ is initial, thus factorizes the identity of Y, thus is a split monomorphism whose codomain belongs to $\mathbf{Y}$, hence Y belongs to $\mathbf{Y}$. It follows that $\mathbf{Y}$ is the full limit-completion of the class of initially injective objects of $\mathbf{X}$. ■

5.3. Proposition. *If $Z : \mathbf{X} \longrightarrow \mathbf{AX}$ is the reflector and η its unit, then:*

1. *η_X is an initial morphism for any X in $\mathbf{X}$,*
2. *a morphism f of $\mathbf{X}$ is initial if and only if $Z(f)$ is a regular monomorphism of $\mathbf{AX}$.*
3. *an object X of $\mathbf{X}$ is initially injective if and only if it is an injective object of $\mathbf{AX}$.*

Proof.

1. Let $\mathfrak{X}$ be an initially dense set of initially injective objects of $\mathbf{X}$. The canonical morphism $f : X \longrightarrow \Pi_{Y\in\mathfrak{X}} Y^{\mathrm{Hom}_{\mathbf{X}}(X,Y)}$ is an initial morphism of $\mathbf{X}$ which factors through η_X. Thus η_X is initial.
2. Let $f : X \longrightarrow Y$ be an initial morphism of $\mathbf{X}$ and $Z(f) = me$ the coregular factorization of $Z(f)$ in $\mathbf{AX}$, where e is an epimorphism and m a regular monomorphism. For any object $T \in \mathfrak{X}$, the map $\mathrm{Hom}_{\mathbf{AX}}(Z(f), T) \simeq \mathrm{Hom}_{\mathbf{X}}(f, T)$ is surjective, thus the map $\mathrm{Hom}_{\mathbf{AX}}(e, T)$ is surjective and indeed bijective. Since $\mathfrak{X}$ is a strong cogenerating set of objects of $\mathbf{AX}$, e is an isomorphism, hence $Z(f)$ is a regular monomorphism of $\mathbf{AX}$. Conversely if $Z(f)$ is a regular monomorphism of $\mathbf{AX}$, it is a regular monomorphism of $\mathbf{X}$, thus an initial morphism of $\mathbf{X}$, hence $\eta_Y f = Z(f)\eta_X$ is initial in $\mathbf{X}$ and f is an initial morphism.
3. An initially injective object of $\mathbf{X}$ belongs to $\mathbf{AX}$ by definition and is injective in $\mathbf{X}$ and in $\mathbf{AX}$. Conversely if X is an injective object of $\mathbf{AX}$ then, for any initial morphism f of $\mathbf{X}$, $Z(f)$ is a regular monomorphism of $\mathbf{AX}$, hence the map $\mathrm{Hom}_{\mathbf{X}}(f, X) \simeq \mathrm{Hom}_{\mathbf{AX}}(Z(f), X)$ is surjective, thus X is initially injective in $\mathbf{X}$. ■

5.4. **The functor $\mathcal{A} : \mathbf{TGC}^{\mathbf{op}} \longrightarrow \mathbf{CGC}$.**

Let $G : \mathbf{X} \longrightarrow \mathbf{Y}$ be a morphism of topological geometrical categories. Its concrete right adjoint H is continuous and preserves algebraic objects and induces a continuous concrete functor $\mathcal{A}(G) : \mathbf{AY} \longrightarrow \mathbf{AX}$. If Y is an injective object of $\mathbf{AY}$, it is an initially injective object of $\mathbf{Y}$ (Proposition 5.3), thus $H(Y)$ is an initially injective object of $\mathbf{X}$ and $\mathcal{A}(G)(Y)$ an injective object of $\mathbf{AX}$. Therefore $\mathcal{A}(G) : \mathbf{AY} \longrightarrow \mathbf{AX}$ is a morphism of concrete geometrical categories. It follows a functor $\mathcal{A} : \mathbf{TGC}^{\mathbf{op}} \longrightarrow \mathbf{CGC}$ defined by $\mathcal{A}(\mathbf{X}) = \mathbf{AX}$ and $\mathcal{A}(G)$ just defined.

5.5. Proposition. . *The functor $\mathcal{A} : \mathbf{TGC}^{\mathbf{op}} \longrightarrow \mathbf{CGC}$ is full and faithful.*

Proof.

(a). Let $G, G' : \mathbf{X} \rightrightarrows \mathbf{Y}$ be a pair of morphisms of topological geometrical categories such that $\mathcal{A}(G) = \mathcal{A}(G')$. Let H, H' be the respective concrete right adjoints to G, G' and Y an object of $\mathbf{Y}$. Since $\mathbf{AY}$ contains all initially injective objects of $\mathbf{Y}$, the family of morphisms $f : Y \longrightarrow Z$ (Z in $\mathbf{AY}$) is initial in $\mathbf{Y}$, thus both the families of morphisms $H(f) : H(Y) \longrightarrow H(Z)$ (Z in $\mathbf{AY}$) and $H'(f) : H'(Y) \longrightarrow H'(Z)$ (Z in $\mathbf{AY}$) are initial in $\mathbf{X}$. Since they have the same families of underlying maps and $H(Z) = H'(Z)$ for any Z in $\mathbf{AY}$, they are identical, hence $H(Y) = H'(Y)$. As a result $H = H'$, $G = G'$, and $\mathcal{A}$ is faithful.

(b). Let $\mathbf{X}$, $\mathbf{Y}$ be a pair of topological geometrical categories with respective underlying functors U, V and categories of algebraic objects $\mathbf{AX}$, $\mathbf{AY}$ whose reflectors are both denoted by Z and units by η. Let $R : \mathbf{AY} \longrightarrow \mathbf{AX}$ be a morphism of concrete geometrical categories with left adjoint S and unit θ. For any object X of $\mathbf{X}$, let $\varphi_X : G(X) \longrightarrow S(Z(X))$ be the initial morphism of $\mathbf{Y}$ whose underlying map is $V(\varphi_X) = U(\theta_{Z(X)}\eta_X)$. For any morphism $f : X \longrightarrow X'$ of $\mathbf{X}$, let $G(f) : G(X) \longrightarrow G(X')$ be the morphism of $\mathbf{Y}$ such that $\varphi_{X'}G(f) = S(Z(f))\varphi_X$ and $V(G(f)) = U(f)$. We get a concrete functor $G : \mathbf{X} \longrightarrow \mathbf{Y}$ and are going to prove that it is a morphism of $\mathbf{TGC}$ such that $\mathcal{A}(G) = R$.

(c). The functor G preserves initial morphisms since, for any initial morphism $f : X \longrightarrow X'$ of $\mathbf{X}$, $Z(f)$ is a regular monomorphism of $\mathbf{AX}$ (Proposition 5.3, $S(Z(f))$ is a regular monomorphism of $\mathbf{AY}$ (Proposition 2.4) thus is an initial morphism of $\mathbf{Y}$, $\varphi_{X'}G(f) = S(Z(X))\varphi_X$ is an initial morphism of $\mathbf{Y}$, hence $G(f)$ is an initial morphism of $\mathbf{Y}$.

(d). For any object Y of $\mathbf{Y}$, let $\psi_Y : H(Y) \longrightarrow R(Z(Y))$ be the initial morphism of $\mathbf{X}$ whose underlying map is $U(\psi_Y) = V(\eta_Y)$. For any morphism $g : Y \longrightarrow Y'$ of $\mathbf{Y}$, let $H(g) : H(Y) \longrightarrow H(Y')$ be the morphism such that $\psi_{Y'}H(g) = (R(Z(g)))\psi_Y$ and $U(H(g)) = V(g)$. We get a concrete functor $H : \mathbf{X} \longrightarrow \mathbf{Y}$ and are going to prove that it a concrete right adjoint to G or equivalently that (G, H) is a Galois correspondence ([1], Theorem 21.24).

(e). The functor H induces the functor R since for any object Y of $\mathbf{AY}$, the morphism η_Y is an identity, $V(\eta_Y) = U(\psi_Y)$ is an identity, ψ_Y is an identity and $H(Y) = R(Y)$. The functor H preserves initial morphism since, for any initial morphism $g : Y \longrightarrow Y'$ of $\mathbf{Y}$, $Z(g)$ is a regular monomorphism of $\mathbf{AY}$ (Proposition 5.3), $R(Z(g))$ is a regular monomorphism of $\mathbf{AX}$ (Proposition 2.4) thus is an initial morphism of $\mathbf{X}$, $\psi_{Y'}H(g) = R(Z(g))\psi_Y$ is an initial morphism of $\mathbf{X}$, hence $H(g)$ is an initial morphism of $\mathbf{X}$.

(f). Let X be an object of $\mathbf{X}$. The morphisms $H(\varphi_X) : H(G(X)) \longrightarrow H(S(Z(X)))$ and $\theta_{Z(X)}\eta_X : X \longrightarrow R(S(Z(X)))$ have the same codomain and the same underlying map $U(H(\varphi_X)) = V(\varphi_X) = U(\theta_{Z(X)}\eta_X)$. Since $H(\varphi_X)$ is initial, we have $X \leqslant H(G(X))$ in the fibre of $\mathbf{X}$ at $U(X)$.

(g). Let Y be an object of $\mathbf{Y}$. The morphism $\psi_Y : H(Y) \longrightarrow R(Z(Y))$ factors in the form $\psi_Y = f\eta_{H(Y)}$ where $f : Z(H(Y)) \longrightarrow R(Z(Y))$ factors in the form $f = R(g)\theta_{Z(H(Y))}$ where $g : S(Z(H(Y))) \longrightarrow Z(Y)$. Then we get a morphism $g\varphi_{H(Y)} : G(H(Y)) \longrightarrow Z(Y)$ whose underlying map is

$$\begin{aligned} V(g\varphi_{H(Y)}) &= V(g)V(\varphi_{H(Y)}) = U(R(g))\, U\left(\theta_{Z(H(Y))}\right) \\ &= U\left(R(g)\theta_{Z(H(Y))}\right) = U(f\eta_{H(Y)}) = U(\psi_Y) = V(\eta_Y). \end{aligned}$$

Since η_Y is initial, we have $G(H(Y)) \leq Y$ in the fibre of $\mathbf{Y}$ at $V(Y)$. As a result, (G, H) is a Galois correspondence between $\mathbf{X}$ and $\mathbf{Y}$ ([1], Definition 6.25), the functor G has a concrete right adjoint H and preserves final families of morphisms ([1], Theorem 21.24.), hence is a morphism of **TGC** such that $\mathcal{A}(G) = R$. Therefore $\mathcal{A}$ is full. ∎

6. Topological geometrical completion

Let $\mathbf{X}$ be a concrete geometrical category with forgetful functor U, free functor F, unit η and counit μ. Let $\mathbf{EX}$ be the category whose objects are the ordered pairs (E, ε) of a set E and a quotient object ε of $F(E)$ in $\mathbf{X}$, and whose morphisms $(E, \varepsilon) \longrightarrow (E', \varepsilon')$ are the maps $f : E \longrightarrow E'$ such that there exist some representative $q : F(E) \longrightarrow X$ of ε, $q' : F(E') \longrightarrow X'$ of ε' and some morphism $u : X \longrightarrow X'$ of $\mathbf{X}$ such that $uq = q'F(f)$.

6.1. Theorem. $\mathbf{EX}$ *is a topological geometrical category whose category of algebraic objects is isomorphic to* $\mathbf{X}$.

Proof.

(a). $\mathbf{EX}$ is a concrete category over **Set** whose forgetful functor $V : \mathbf{EX} \longrightarrow \mathbf{Set}$ is defined by $V(E, \varepsilon) = E$ and $V(f) = f$. Let $f : (E, \varepsilon) \longrightarrow (E', \varepsilon')$ be an isomorphism of $\mathbf{EX}$ such that $V(f)$ is an identity. Then $E = E'$, $F(E) = F(E')$, and if ε, ε' are represented by $q : F(E) \longrightarrow X$, $q' : F(E') \longrightarrow X'$ respectively, there exist $u : X \longrightarrow X'$ and $v : X' \longrightarrow X$ such that $uq = q'$ and $vq' = q$. Then $q \sim q'$, $\varepsilon = \varepsilon'$, $(E, \varepsilon) = (E', \varepsilon')$, and f is the identity of (E, ε). As a result, the functor V is amnestic ([1], Definition 5.4).

(b). Let us prove that $\mathbf{EX}$ has initial structures. For any morphism $f : X \longrightarrow Y$ of $\mathbf{X}$ and any quotient object ε of Y represented by $q : Y \longrightarrow Q$, we denote by $f^*(\varepsilon)$ the quotient object of X represented by the epimorphism $p : X \longrightarrow P$ arising in the coregular factorization $mp = qf$ of qf in $\mathbf{X}$. Let $(E_\lambda, \varepsilon_\lambda)_{\lambda\in\Lambda}$ be a family of objects of $\mathbf{EX}$ and $(f_\lambda : E \longrightarrow E_\lambda)_{\lambda\in\Lambda}$ a family of maps. Let $\varepsilon = \bigvee_{\lambda\in\Lambda} F(f_\lambda)^*(\varepsilon_\lambda)$ be the co-union, in the complete lattice $Quot(F(E))$ of quotient objects of $F(E)$ in

$\mathbf{X}$, of the family $(F(f_\lambda)^*(\varepsilon_\lambda))_{\lambda\in\Lambda}$. Then we get an object (E,ε) of $\mathbf{EX}$ together with a family of morphisms $(f_\lambda : (E,\varepsilon) \longrightarrow (E_\lambda,\varepsilon_\lambda))_{\lambda\in\Lambda}$ of $\mathbf{EX}$. Let $(g_\lambda : (D,\delta) \longrightarrow (E_\lambda,\varepsilon_\lambda))_{\lambda\in\Lambda}$ be a family of morphisms of $\mathbf{EX}$ and $m : D \longrightarrow E$ a map such that $f_\lambda m = g_\lambda$ for any $\lambda \in \Lambda$. The direct image of δ along $F(m)$ is bigger in $Quot(F(E))$ than any $F(f_\lambda)^*(\varepsilon_\lambda)$ with $\lambda \in \Lambda$, thus is bigger than ε. Hence m is a morphism $(D,\delta) \longrightarrow (E,\varepsilon)$. As a result $(f_\lambda : (E,\varepsilon) \longrightarrow (E_\lambda,\varepsilon_\lambda))_{\lambda\in\Lambda}$ is an initial family of morphisms and $\mathbf{EX}$ a topological category over $\mathbf{Set}$.

(c). Let $J : \mathbf{X} \longrightarrow \mathbf{EX}$ be the functor defined by $J(X) = (U(X), \mu_X)$ and $J(f) = U(f)$. It is an embedding of categories. It is full since for any objects X, Y of $\mathbf{X}$ and any morphism $g : (U(X), \mu_X) \longrightarrow (U(Y), \mu_Y)$, there exists a morphism $f : X \longrightarrow Y$ of $\mathbf{X}$ such that $g = U(f) = J(f)$. The functor J has a left adjoint S which assigns to an object (E,ε) an object X such that ε is represented by $q : F(E) \longrightarrow X$ and to a morphism $f : (E,\varepsilon) \longrightarrow (E',\varepsilon')$ the morphism $u : X \longrightarrow X'$ such that $uq = q'F(f)$ where $q' : F(E') \longrightarrow X'$ represents ε'. It follows that J induces an isomorphism of concrete categories between $\mathbf{X}$ and the full reflective subcategory $\mathbf{Y} = J(\mathbf{X})$ of $\mathbf{EX}$.

(d). According to the description of initial structures given in **b**), we see that a morphism f of $\mathbf{EX}$ is initial if and only if the morphism $S(f)$ is a regular monomorphism of $\mathbf{X}$. It follows that an object X of $\mathbf{X}$ is injective in $\mathbf{X}$ if and only if $J(X)$ is initially injective in $\mathbf{EX}$. Let $(X_i)_{i\in I}$ be a small strong cogenerating family of injective objects of $\mathbf{X}$. Then $(J(X_i))_{i\in I}$ is a small family of initially injective objects of $\mathbf{EX}$. Let (E,ε) be an object of $\mathbf{EX}$. The unit morphism $\nu : (E,\varepsilon) \longrightarrow J(S(E,\varepsilon))$ is initial, there exists a regular monomorphism of $\mathbf{X}$ of the form $f : S(E,\varepsilon) \longrightarrow \Pi_{i\in I} X_i^{n_i}$, thus there exists an initial morphism of $\mathbf{EX}$ of the form $J(f)\nu : (E,\varepsilon) \longrightarrow \Pi_{i\in I} J(X_i)^{n_i}$. Therefore the family of objects $(J(X_i))_{i\in I}$ is initially dense in $\mathbf{EX}$. As a result, $\mathbf{EX}$ is a topological geometrical category.

(e). The subcategory $\mathbf{Y}$ of $\mathbf{EX}$ being the full limit-completion of $(J(X_i))_{i\in I}$ is also the full limit-completion of the class of initially injective objects of $\mathbf{EX}$, hence is the category of algebraic objects of $\mathbf{EX}$. ■

6.2. Definition. $\mathbf{EX}$ *is called the topological geometrical completion of* $\mathbf{X}$.

6.3. Theorem. *The functor* $\mathcal{A} : \mathbf{TGC}^{\mathbf{op}} \longrightarrow \mathbf{CGC}$ *is an equivalence of categories.*

Proof. Follows from Proposition 5.5. and Theorem 6.1. ■

6.4. Corollary. *The category* $\mathbf{CGC}$ *is cocomplete and coregular.*

Proof. Follows from ([5], Theorem 4.5.) ■

6.5. PROPOSITION. *For a morphism of concrete geometrical categories $R : \mathbf{X} \longrightarrow \mathbf{Y}$, the following assertions are equivalent:*

(i). R is full.

(ii). R is a full embedding.

(iii). R is a regular monomorphism in **CGC**.

PROOF. (i) $\Longrightarrow$ (ii). Follows from ([1], Proposition 5.10.).

(ii) $\Longrightarrow$ (iii). Let G be a left adjoint to R such that $GR = 1_{\mathbf{X}}$ (Proposition 2.4). Let us identify $\mathbf{X}$ (resp. $\mathbf{Y}$) with a subcategory of $\mathbf{EX}$ (resp. $\mathbf{EY}$). According to Theorem 6.3, the functor G extends to a morphism of topological geometrical categories $T : \mathbf{EY} \longrightarrow \mathbf{EX}$ such that $\mathcal{A}(T) = R$. Let E be an object of $\mathbf{EX}$. There exists an object X of $\mathbf{X}$ together with an initial morphism $f : E \longrightarrow X$ of $\mathbf{EX}$. Let $Y = R(X)$ and $g : F \longrightarrow Y$ be the initial lift in $\mathbf{EY}$ of the map $U(f) : U(E) \longrightarrow V(Y)$. Since T preserves initial morphisms, $T(g) : T(F) \longrightarrow T(Y) = GR(X) = X$ is an initial morphism of $\mathbf{X}$ having the same underlying map as the initial morphism $f : E \longrightarrow X$, thus we have $T(F) = E$. Therefore the functor T is surjective on objects, and thus surjective, being part of a Galois correspondence. According to ([5], Corollary 4.6) T is a regular epimorphism in the category **TGC** and, according to Theorem 6.3, $R = \mathcal{A}(T)$ is a regular monomorphism in the category **CGC**.

(iii) $\Longrightarrow$ (i). According to Theorem 6.3, the morphism R is of the form $R = \mathcal{A}(T)$ where T a regular epimorphism of **TGC** whose right adjoint M induces R. Then $TM = 1_{\mathbf{EX}}$, M is full, hence R is full. ■

6.6. COROLLARY. *Regular subobjects in the category* **CGC** *are precisely represented by full geometrical subcategories.*

7. Classifying concrete geometrical categories

Let $L = (L_i)_{i \in I}$ be an algebra for an I-sorted infinitary algebraic theory $\mathbf{T}$.

7.1. **Affine sets over** L.

According to ([5], Definition 2.1.), an *affine set* over L is a set X equipped with *a* $\mathbf{T}$-subalgebra $A(X)$ of the power $\mathbf{T}$-algebra $L^X = (L_i^X)_{i \in I}$. They are the objects of the topological geometrical category $\mathbf{A}f\,\mathbf{Set}(L)$ whose morphisms $(X, A(X)) \to (Y, A(Y))$ are the maps $f : X \to Y$ such that $vf \in A_i(X)$ for any $i \in I$ and $v \in A_i(Y)$ [5]. In particular for an I-indexed family of sets $(n_i)_{i \in I}$, $E = \Pi_{i \in I} L_i^{n_i}$ is an affine set over L called *the affine space* of type $(n_i)_{i \in I}$ over L. An *algebraic equation* of type $(n_i)_{i \in I}$ of $\mathbf{T}$ is a pair of operations $(\omega, \mu) : (n_i)_{i \in I} \rightrightarrows j$ of type $((n_i)_{i \in I}, j)$ of $\mathbf{T}$, and an *algebraic system* of type $(n_i)_{i \in I}$ of $\mathbf{T}$ is a set S of algebraic equations of type $(n_i)_{i \in I}$ of $\mathbf{T}$. A *solution* of S with values in L is an element $x \in E = \Pi_{i \in I} L_i^{n_i}$ such that $L(\omega)(x) = L(\mu)(x)$ for any $(\omega, \mu) \in S$. The set $Z(S)$ of solutions of S with values in L is an affine subset of the affine space E, called an *algebraic subset of* E.

7.2. Definition. *A geometrical space over L is an affine set over L isomorphic to some algebraic subset of some affine space over L. They are the objects of the full subcategory $\mathbf{Geo}(L)$ of $\mathbf{A}f\mathbf{Set}(L)$.*

7.3. Theorem. *$\mathbf{Geo}(L)$ is a concrete geometrical category identical to the category of algebraic objects of $\mathbf{A}f\mathbf{Set}(L)$.*

Proof. Let $X \simeq Z(S)$ be a geometrical space over L. For all $(\omega,\mu) : (n_i)_{i\in I} \rightrightarrows j$ belonging to S, the pair of maps $(L(\omega), L(\mu)) : \Pi_{i\in I} L_i^{n_i} \rightrightarrows L_j$ is a pair of morphisms of $\mathbf{A}f\mathbf{Set}(L)$ whose collective equalizer is $Z(S) \simeq X$. According to the proof of ([5], Theorem 3.2), each object L_i is initially injective in $\mathbf{A}f\mathbf{Set}(L)$, thus $\Pi_{i\in I} L_i^{n_i}$ is initially injective, and $X \simeq Z(S)$ is an algebraic object in $\mathbf{A}f\mathbf{Set}(L)$. Conversely let X be an algebraic object of $\mathbf{A}f\mathbf{Set}(L)$. Since $\{L_i\}_{i\in I}$ is a strong cogenerating set of objects in the category of algebraic objects of $\mathbf{A}f\mathbf{Set}(L)$, there exist two affine spaces $E = \Pi_{i\in I} L_i^{n_i}$ and $F = \Pi_{j\in I} L_j^{m_j}$ and a pair of morphisms $(f,g) : E \rightrightarrows F$ whose equalizer is X (Proof of Theorem 3.2). Then f,g are of the form $f = (f_{jm})$ and $g = (g_{jm})$ where $(j,m) \in \coprod_{j\in I} m_j$ and each f_{jm}, g_{jm} is of the form $f_{jm} = L(\omega_{jm})$, $g_{jm} = L(\mu_{jm})$ where $(\omega_{jm}, \mu_{jm}) : (n_i)_{i\in I} \rightrightarrows j$ is an algebraic equation of $\mathbf{T}$. Then $X \simeq Z(S)$ with $S = \{(\omega_{jm}, \mu_{jm}) : (j,m) \in \coprod_{j\in I} m_j\}$ is a geometrical space over L. ∎

Let $(\mathbf{X}, U)$ be concrete geometrical category.

7.4. Definition. *An algebra of $\mathbf{X}$ carried by L is a $\mathbf{T}$-algebra $A = (A_i)_{i\in I}$ of $\mathbf{X}$ such that the $\mathbf{T}$-algebra $U(A) = (U(A_i))_{i\in I}$ of $\mathbf{Set}$ is identical to L. It is injective if any object A_i is injective in $\mathbf{X}$.*

7.5. Examples. $L = (L_i)_{i\in I}$ is an injective algebra of $\mathbf{Geo}(L)$ carried by L and, for any morphism of concrete geometrical categories $R : \mathbf{Geo}(L) \longrightarrow \mathbf{X}$, $R(L) = (R(L_i))_{i\in I}$ is an injective algebra of $\mathbf{X}$ carried by L.

7.6. Theorem. *For any algebra L, there exists a concrete geometrical category $\mathbf{G}eo(L)$ equipped with an injective algebra A carried by L such that, for any concrete geometrical category $\mathbf{X}$ and any injective algebra B of $\mathbf{X}$ carried by L, there exists a unique morphism of concrete geometrical categories $R : \mathbf{Geo}(L) \longrightarrow \mathbf{X}$ such that the algebra $R(A)$ is identical to B.*

Proof. Let us prove that the previously defined category $\mathbf{Geo}(L)$ equipped with the injective algebra $A = L$ carried by L satisfies the property. Let $B = (B_i)_{i\in I}$ be an injective algebra carried by L of a concrete geometrical category $(\mathbf{X}, U)$. The two functors $(\mathrm{Hom}_{\mathbf{X}}(-, B_i))_{i\in I}$, $(\mathrm{Hom}_{\mathbf{Set}}(U(-), L_i))_{i\in I} : \mathbf{X}^{\mathbf{op}} \rightrightarrows \mathbf{Set}^I$ and the natural transformation $(\alpha_i)_{i\in I} : (\mathrm{Hom}_{\mathbf{X}}(-, B_i))_{i\in I} \longrightarrow (\mathrm{Hom}_{\mathbf{Set}}(U(-), L_i))_{i\in I}$ defined by $\alpha_{iX}(f) = U(f)$ for any $f : X \longrightarrow B_i$ in $\mathbf{X}$, lift to the functors $H, L^{U(-)} : \mathbf{X}^{\mathbf{op}} \longrightarrow \mathbf{Alg}(\mathbf{T})$ and natural transformation $\alpha : H \longrightarrow L^{U(-)}$, respectively. Let $A(U(-)) \longrightarrow L^{U(-)}$ be the image of the pointwise injective natural transformation α. We get a concrete functor $G : \mathbf{X} \longrightarrow \mathbf{A}f\mathbf{Set}(L)$ defined by $G(-) = (U(-), A(U(-)))$ such that $G(B_i) = L_i$ for any $i \in I$. Therefore if $S : \mathbf{A}f\mathbf{Set}(L) \longrightarrow \mathbf{Geo}(L)$ denotes the reflector, we get a functor $SG : \mathbf{X} \longrightarrow \mathbf{Geo}(L)$

such that $SG(B_i) = L_i$ for any $i \in I$. Let us prove that SG has a right adjoint R. For any $i \in I$, $R(L_i) = B_i$ is a co-universal object from SG to L_i since, for any object X of $\mathbf{X}$, we have $\mathrm{Hom}_{\mathbf{X}}(X, B_i) \simeq \mathrm{Hom}_{\mathbf{A}f\mathbf{Set}(L)}(G(X), L_i) \simeq \mathrm{Hom}_{\mathbf{Geo}(L)}(S(GX)), L_i)$. It follows that $R(\Pi_{i\in I} L_i^{n_i}) = \Pi_{i\in I} B_i^{n_i}$ is a co-universal object from SG to $\Pi_{i\in I} L_i^{n_i}$, for any I-indexed family of sets $(n_i)_{i\in I}$. Since any object Y of $\mathbf{Geo}(L)$ is an equalizer object of a pair of morphisms between affine spaces over L of the form $\Pi_{i\in I} L_i^{n_i}$, it follows that there exists a co-universal object $R(Y)$ from SG to Y, defining the right adjoint functor $R : \mathbf{Geo}(L) \longrightarrow \mathbf{X}$, which can be chosen to be a concrete functor such that $R(A) = (R(L_i))_{i\in I}$ is the algebra B. Moreover R preserves injective objects because they are in $\mathbf{Geo}(L)$ split subobjects of affine spaces over L. Then R is a morphism of concrete geometrical categories, easily seen to be the unique possible one which satisfies $R(A) = B$. ∎

7.7. Notation. The category $\mathbf{Geo}(L)$, uniquely defined up to a unique isomorphism, is called the *classifying concrete geometrical category of* L, and its algebra A is called the *generic injective algebra carried by* L.

7.8. Theorem. *Any concrete geometrical category is the classifying concrete geometrical category of some algebra* L

Proof. Let $\mathbf{X}$ be a concrete geometrical category. According to Theorem 6.1. the topological geometrical completion $\mathbf{EX}$ of $\mathbf{X}$ is a topological geometrical category whose category of algebraic objects is isomorphic to $\mathbf{X}$. According to ([5], Theorem 3.3), the concrete category $\mathbf{EX}$ is isomorphic to a concrete category of the form $\mathbf{A}f\mathbf{Set}(L)$ for some algebra L. Thus $\mathbf{X}$ is isomorphic to the category of algebraic objects of $\mathbf{A}f\mathbf{Set}(L)$ which is, according to Theorem 7.3, identical to the category $\mathbf{Geo}(L)$. According to the proof of Theorem 7.6, $\mathbf{X}$ is the classifying concrete geometrical category of the algebra L. ∎

8. Examples of concrete geometrical categories

The following concrete geometrical categories are classified by algebras. Any algebra carried by $\{0\}, \{0,1\}, \{0,1,2\}, \ldots$ will be denoted by **1, 2, 3, ...** respectively. Their algebraic structures may be different and not explicitely described. For example the three elements grid $\{0,1,2\}$ [2] is denoted by **3**, but other algebraic structures on $\{0,1,2\}$ are also denoted by **3**.

The category **Sgl** of singleton sets is the concrete geometrical category classifying the singleton algebra **1**. It is the initial concrete geometrical category.

For any set E, the category $\mathbf{Exp}(E)$ is the concrete geometrical category classifying the set E : It is the uniquely transportable modification ([1], 5.36) of the concrete category whose objects are the sets E^I for any set I, and whose morphisms $E^I \longrightarrow E^J$ are the maps of the form E^α for any map $\alpha : J \longrightarrow I$.

The category **Spec** of spectral spaces is the concrete geometrical category classifying the bounded lattice **2**: Following Hochster [7] a spectral space is a topological To-space whose compact open sets form an open basis closed under finite intersections and whose irreducible closed sets have a generic point. With proper continuous maps, they form the category **Spec** which has the Sierpinski space S as a strong injective cogenerator.

The category **BoolSp** of boolean (or Stone) spaces is the concrete geometrical category classifying the boolean algebra **2**.

The category **Sob** of sober spaces is the concrete geometrical category classifying the frame **2** [9].

The category **Ord** of ordered sets (i.e posets) is the concrete geometrical category classifying the completely distributive complete lattice **2**.

The category **Set** of sets is the concrete geometrical category classifying the complete atomic boolean algebra **2**

The category **PtSet** of pointed sets is the concrete geometrical category classifying the completely distributive complete and conditionally cocomplete lattice **2**.

The category **Sp** of spaces has as objects the sets X equipped with a set $S(X)$ of subsets of X, and as morphisms the maps $f : X \longrightarrow Y$ such that $f^{-1}(Y') \in S(X)$ for any $Y' \in S(Y)$. It is the concrete geometrical category classifying the algebra $(\mathbf{2}, \mathbf{2})$ for a 2-sorted algebraic theory **T** whose first sort is the theory of sets and the second sort is that of complete atomic boolean algebras. Notice that there exists no 1-sorted algebra L such that **Sp** is the classifying concrete geometrical category of L.

The category $\mathbf{Sp}_\wedge$ (resp. $\mathbf{Sp}_\vee, \mathbf{Sp}_\Diamond, \mathbf{Sp}_c$) of meet (resp. join, lattice, complemented) spaces is the full subcategory of **Sp** whose objects are the spaces X such that $S(X)$ is closed under binary meets (resp. binary joins, binary meets and joins, complements). They are the concrete geometrical categories classifying the algebra $(\mathbf{2}, \mathbf{2})$ for a 2-sorted algebraic theory whose first sort is that of meet semilattices (resp. join semilattices, lattices, S_2-sets) and whose second sort is that of complete atomic boolean algebras.

The category **Top** of topological spaces is the concrete geometrical category classifying the grid **3** [2].

The category **Ext** of exterior topological spaces [6] is a concrete geometrical category classifying an algebra of the form **3**.

The category **PrTop** (resp. **Neigh**) ([1], 5N) of pretopological (resp. neighborhood) spaces is a concrete geometrical category classifying an algebra of the form **3**.

The category **Mes** of measurable spaces and maps is a concrete geometrical category classifying an algebra of the form **3**.

The category **PrCl** of preclosure spaces has as objects, the sets X equipped with an order preserving inflating map $c : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ and as morphisms, the maps $f : X \longrightarrow Y$ such that $f(c(A)) \subset c(f(A))$ for any $A \in \mathcal{P}(X)$. It is a concrete geometrical category classifying an algebra of the form **3**.

The categories $\mathbf{PrCl}_o, \mathbf{PrCl}_{o\vee}, \mathbf{Cl}, \ldots$ of preclosure spaces X such that $c(\varnothing) = \varnothing$, $c(A \bigcup B) = c(A) \bigcup c(B)$, $c(\bigcap_{i\in I} Ai) = \bigcap_{i\in I} c(Ai), \ldots$, respectively, are concrete geometrical categories classifying algebras of the form **3**.

The categories $\mathbf{Sp}_1, \mathbf{Sp}_0, \mathbf{Sp}_{1\wedge}, \mathbf{Sp}_{0\vee}, \ldots$ are the full subcategories of **Sp** whose objects are the spaces X such that $S(X)$ contains X, $\varnothing$, is stable under finite meets, joint, $\ldots$, respectively, are concrete geometrical categories classifying algebras of the form **3**.

The category **SubSet** of sets equipped with a subset and maps inducing a map of subsets, is a concrete geometrical category classifying an algebra of the form **3**.

The categories **RRel**, **RSRel**, **PrOrd**, **Equ** of sets equipped with a reflexive, reflexive symmetric, preorder, equivalence relation, respectively, and relation preserving maps, are concrete geometrical categories classifying algebras of the form **3**.

The categories **Rel**, **SRel**, **TRel**, **STRel** of sets equipped with a relation, a symmetric relation, a transitive relation, a symmetric transitive relation, respectively, are concrete geometrical categories classifying algebras of the form **4**.

The category **RTerRel** (resp. **SRTerRel**) of sets equipped with a reflexive (resp. reflexive symmetric) ternary relation is a concrete geometrical category classifying an algebra of the form **4**.

The category **STerRel** of sets equipped with a symmetric ternary relation is a concrete geometrical category classifying an algebra of the form **6**.

The category **TerRel** of sets equipped with a ternary relation is a concrete geometrical category classifying an algebra of the form **10**.

The category $\mathbf{Set}^G$ of left actions of a group G is a concrete geometrical category classifying the G-complete atomic boolean algebra $\mathcal{P}(G)$.

The category **HCompAb** of compact Hausdorff abelian groups is the concrete geometrical category classifying the abelian group $\mathbf{R}/2\pi\mathbf{Z}$.

The category **Ab** of abelian groups is the concrete geometrical category classifying the compact Hausdorff abelian group $\mathbf{R}/2\pi\mathbf{Z}$.

The category **HComp** of compact Hausdorff spaces is the concrete geometrical category classifying the unit interval algebra $I = \{x \in \mathbf{R} : /x/ \leqslant 1\}$ for an algebraic theory described by Isbell in [8].

The category $\mathbf{LCVct}(K)$ of linearly compact vector spaces over a commutative field K [3] is the concrete geometrical category classifying the K-vector space K.

The category $\mathbf{Vct}(K)$ of K-vector spaces is the concrete geometrical category classifying the linearly compact K-vector space K.

The category $\mathbf{LCAf}(K)$ of linearly compact affine spaces over K is the concrete geometrical category classifying the pointed K-vector space K with point 1.

The category $\mathbf{Af}(K)$ of affine spaces over K is the concrete geometrical category classifying the pointed linearly compact K-vector space K with point 1.

For any real closed field R, the category $\mathbf{Geo}(R)$ of geometrical spaces over R is a concrete geometrical category described as follows. Let $\mathbf{T}$ be the 1-sorted algebraic theory of formally real algebras over R, defined as being commutative algebras over R with unit in which any element of the form $1+x_1^2+\ldots+x_n^2$ is invertible. Then R is a $\mathbf{T}$-algebra defining the classifying concrete goemetrical category $\mathbf{Geo}(R)$ of R. The objects of $\mathbf{Geo}(R)$ are precisely the algebraic subsets of the linearly compact affine spaces over R. Let us recall that any linearly compact affine space over R is isomorphic to some space R^I and that an algebraic subset of R^I is the set of solutions of a system of algebraic equations of the form $P((x_i)_{i\in I}) = 0$ where $P \in R[X_i]_{i\in I}$. The morphisms of $\mathbf{Geo}(R)$ are the rational maps definable by families of rational fractions with coefficients in R. The category $\mathbf{Geo}(R)$ contains in particular all finitely dimensional affine or projective spaces over R and their algebraic subsets. For example, if V is a finitely dimensional vector space over R, then $V_* = V \setminus \{0\}$ is an object of $\mathbf{Geo}(R)$ on which acts the group object R^*, giving rize to a coequalizer diagram $R^* \times V_* \rightrightarrows V_* \longrightarrow P(V)$ in $\mathbf{Geo}(L)$ which gives rize to the projective space $P(V)$.

For any non algebraically closed commutative field K, the category $\mathbf{Geo}(K)$ of geometrical spaces over K is a concrete geometrical category described in a similar way as in the previous example, by substituting the notion of formally rational algebra over K to the notion of formally real algebra [10].

For any commutative field extension L of K, the category $\mathbf{Geo}(L/K)$ of geometrical spaces over L/K is the concrete geometrical category classifying the commutative algebra L over K. It objects are precisely the algebraic subsets of the linearly compact affine spaces over L defined by systems of algebraic equations with coefficients in K and its morphisms are maps definable by families of polynomials with coefficients in K. Whenever L/K is algebraic, the Galois group $Gal(L/K)$ is the free object of $\mathbf{Geo}(L)$ generated by one element. One can take for L, an algebraic closure of K, a separable closure of K, a Galois extension of K, etc... The classical Galois theory of algebraic equations lives entirely in such concrete geometrical categories.

The category **Dio** of diophantian spaces is the concrete geometrical category classifying the unitary ring $\mathbf{Z}$: its objects are, up to isomorphisms, precisely the subsets of the spaces $\mathbf{Z}^I$ which are the sets of solutions of systems of diophantian equations in unknown $(x_i)_{i\in I}$, and its morphisms are the maps definable by families of polynomials with coefficients in $\mathbf{Z}$.

The category $\mathbf{Geo}(C^\infty)$ of C^∞-spaces is the concrete geometrical category classifying the C^∞-ring $\mathbf{R}$. [11]: its objects are, up to isomorphisms, precisely the closed subsets of the spaces $\mathbf{R}^I$ and its morphisms are the smooth maps. This category contains, as a full subcategory, the category of smooth manifolds with countable basis [11].

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