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ON ESSENTIAL RING EMBEDDINGS AND THE EPIMORPHIC HULL OF $C(X)$

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Abstract. Storrer introduced the epimorphic hull of a commutative semiprime ring R and showed that it is (up to isomorphism) the unique essential epic von Neumann regular extension of R. In the case when $R = C(X)$ with X a Tychonoff space, we show that the embedding induced by a dense subspace of X is always essential. This simplifies the search for spaces whose epimorphic hull is a full ring of continuous functions, and allows us to obtain new examples where this occurs. The main theorem comes close to a characterisation of this phenomenon.

1. Introduction

Suppose that R is a commutative semiprime ring with identity and that $Q(R)$ is its maximal ring of quotients. It is well-known that $Q(R)$ is von Neumann regular [Lambek (1966), 2.4, Prop 1]. Suppose that $E(R)$ is the intersection of all the von Neumann regular rings lying between R and $Q(R)$. The ring $E(R)$ was studied intensively by Storrer who showed that it is an epimorphic extension of R in the category $\mathcal{CR}$ of commutative rings with identity, that it is von Neumann regular, that it contains $Q_{cl}(R)$ the classical ring of quotients of R and that it is an essential extension of R in $\mathcal{CR}$. Storrer called $E(R)$ the epimorphic hull of R and showed that it is the only ring with these properties (up to isomorphism over R). The fact that $E(R)$ consists of all polynomials in the quasi-inverses of elements from R with coefficients from R was mentioned in [Raphael (1999), Section 3 and remark 2].

The article [Raphael & Woods (2000)] studied the epimorphic hull of $C(X)$. (It was denoted $H(X)$ to avoid confusion with existing topological terminology.) Like $Q(X)$, $H(X)$ is a lattice under the partial order on $Q(X)$. In the terms of [Henriksen, Johnson (1961)] it is a ϕ-algebra and this raises the natural question of when $H(X)$ is isomorphic to a $C(Y)$, for a (necessarily) P-space Y. Hager [Hager (1969)] had already determined (in the absence of measurable cardinals) when the ϕ-algebra $Q(X)$ was isomorphic to a $C(Y)$.

In this paper we view $C(X)$ exclusively as an object in $\mathcal{CR}$, the category of commutative rings. Epimorphisms in $\mathcal{CR}$ that are induced by topological embeddings have been studied in [Barr, Burgess, & Raphael (2003)], [Barr, Raphael, & Woods (2004)], and in

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[Barr, Kennison, & Raphael (2004)]. We recall some notions. An inclusion $X \to Y$ is called a **$\mathcal{CR}$-epic embedding** if the map $C(Y) \to C(X)$ induced by restriction is an epimorphism in $\mathcal{CR}$. The space X is called **absolute $\mathcal{CR}$-epic** if every embedding of X into any space Y is $\mathcal{CR}$-epic.

1.1. Notation. All spaces considered in this paper are assumed to be Tychonoff (completely regular Hausdorff) and all functions, unless explicitly stated otherwise, are assumed continuous. We denote by υX the Hewitt realcompactification of a Tychonoff space X; see Chapter 8 of [Gillman & Jerison (1960)] or 5.5(c) and 5.10 of [Porter & Woods (1988)] for background. We denote by gX the intersection of the dense cozero-sets of X; see [Raphael & Woods (2000)] and [Raphael & Woods (2004)] for some elementary properties of gX.

2. Essentiality in $\mathcal{CR}$ and topological embeddings

Recall that in any category a monic α is called **essential** if for all β, $\beta \circ \alpha$ monic implies β is monic. The classical example of essential extensions was in module theory but there have been other instances of interest. It is easy to verify that a ring extension of a field is essential in the category $\mathcal{CR}$ if and only if the extension is a field. Thus, another way of saying that a field extension L of K is algebraic is to say that each ring between L and K is an essential extension of K. These ideas were pursued in [Raphael (1970)] to get an algebraic closure for commutative semiprime rings.

Given the unlikelihood of a monic being essential in $\mathcal{CR}$, the following result is somewhat surprising:

2.1. Theorem. *Let X be a subspace of Y. If $k \in C(X)$, $k \neq \mathbf{0}$, then there exist functions $h \in C(X)$, $m \in C(Y)$ such that $kh = m|X \neq \mathbf{0}$. In particular if X is dense in Y, then the induced monic $C(Y) \to C(X)$ is essential in $\mathcal{CR}$.*

Proof. Take $k \in C(X)$, $k \neq \mathbf{0}$. There is a closed subset $A \subseteq Y$ such that $A \cap X = Z(k)$. Take $p \in \mathrm{coz}(k)$. By complete regularity there is a $t \in C(Y)$, $\mathbf{0} \leq t \leq \mathbf{1}$ so that $t(A) = 0$ and $t(p) = 1$. There is a function $m \in C(Y)$ such that $Z(m) = t^{-1}[0, 1/2]$. Now $Z[m|X]$ is an X-neighbourhood of $Z(k)$, so by [Gillman & Jerison (1960), 1D.1], $m|X = kh$ for some $h \in C(X)$. Also $m|X \neq \mathbf{0}$ since $p \notin Z(m)$. ∎

2.2. Remark. A space X is called a **P-space** if each zero-set is open and is called an **almost-P-space** if each non-empty zero-set has non-empty interior (cf [Levy (1977)]). By X_δ we will mean the (generally stronger) topology on X that is generated by declaring that all zero-sets of X are open. The space X_δ is a P-space. The same topology is generated by asking that all G_δ sets in X are open. In [Raphael & Woods (2000), 4.5] it was shown that it is precisely for the almost-P-spaces X, that the embedding $C(X) \to C(X_\delta)$ is a ring of quotients. An easy argument as in [Raphael & Woods (2000), 4.3] shows that this is equivalent to the (weaker) demand that $C(X) \to C(X_\delta)$ be essential in $\mathcal{CR}$.

3. Applications to the epimorphic hull of $C(X)$

The main goal of [Raphael & Woods (2000)] was to characterize spaces X for which the epimorphic hull $H(X)$ of $C(X)$ is a ring of continuous functions. The following (almost) completes this goal. It generalizes [Raphael & Woods (2000), 5.16 and 5.17] and provides additional examples of spaces where the epimorphic hull is a ring of functions. In [Raphael & Woods (2000), section 5] an algebra $G(X)$ was defined for an arbitrary Tychonoff space X. The ring $G(X)$ is von Neumann regular and is an epimorphic extension of $C(X)$ in $\mathcal{CR}$. A space X is called RG [Henriksen, Raphael, & Woods (2001), 1.2] if $G(X) = C(X_\delta)$. It is clear that X is RG if and only if the embedding $C(X) \to C(X_\delta)$ is epic in $\mathcal{CR}$.

3.1. Theorem.

(i) *If K densely and epically contains an almost-Pspace X that is RG , then $H(K)$ is a ring of functions, namely $C(X_\delta)$.*

(ii) *(partial converse) if $H(K)$ is a ring of functions then K densely contains an almost-P-space that is RG.*

Proof.

(i) Let X be a dense absolute $\mathcal{CR}$-epic almost-P, RG subspace of K. The induced map $C(K) \to C(X)$ is essential by 2.1 and it is epi by assumption. The map $C(X) \to C(X_\delta)$ is essential since X is almost-P and is epi because X is RG. Thus $C(X_\delta) = H(K)$ by [Storrer (1968), 11.2].

(ii) If $H(K)$ is a ring of functions gK is a dense almost-P-space by [Raphael & Woods (2000), 4.9 and 4.11]. Also by [Raphael & Woods (2000), 4.12] $H(K) = C((gK)_\delta)$. Now the epi from $C(K)$ into $C((gK)_\delta)$ factors through $C(gK)$, so $C(gK) \to C((gK)_\delta)$ is epi which makes gK an RG space.

∎

3.2. Remark. Example 6.4 of [Storrer (1968)] shows that gK can be a dense almost-P-subspace of K without being $\mathcal{CR}$-epic in K.

3.3. Corollary. *If K densely contains an almost-P, absolute $\mathcal{CR}$-epic, RG space X then $H(K) = C(X_\delta)$. In particular this holds if X is a Lindelöf P-space or an RG space that is almost-P, locally compact and σ-compact. A non realcompact example is given by an almost compact, almost-P-space that is RG but not compact .*

3.4. Remark. There are many examples of Lindelöf P-spaces — an example without isolated points is given in [Henriksen, Raphael, & Woods (2001), 3.11]. The σ-compact space T of [Raphael & Woods (2000), 6.2] is absolute $\mathcal{CR}$-epic almost-$P$and RG, but not a P-space. One can also get a realcompact candidate for the space X of the corollary that is almost-Pabsolute $\mathcal{CR}$-epic, RG but neither a Lindelöf P-space space, nor a locally compact σ-compact space via a product as follows: let D be an uncountable discrete set, let $L = D \cup \{p\}$ be its one-point Lindelofization, and let $D^* = D \cup \{\omega\}$ be its one-point compactification. Then the product $X = L \times D^*$ clearly fails to be a P-space, or a σ-compact space. It also fails to be locally compact at the points of $L \times \omega$. However X is an RG-space by [Henriksen, Raphael, & Woods (2001), 2.11(a)], it is absolute $\mathcal{CR}$-epic by [Barr, Kennison, & Raphael (2004), 3.4 and 4.6] and it is almost-P because any zero-set of D^* that contains ω is a co-countable subset of D^*, and hence contains uncountably many isolated points.

For the third example recall that a space X is called **almost compact** if $|\beta X \setminus X| \leq 1$ [Gillman & Jerison (1960), 6J]. Almost compact spaces are easily seen to be absolute $\mathcal{CR}$-epic [Barr, Raphael, & Woods (2004)]. The existence of non-compact, almost compact, almost-P RG-spaces was established in [Hrusak, Raphael, & Woods (2004), Theorem 6].

3.5. Remark. In contrast to the results for P-spaces, there are many Lindelöf almost-P-spaces that are not absolute $\mathcal{CR}$-epic. Recent unpublished work in connection with [Barr, Kennison, & Raphael (2004)] showed that any countable Tychonoff space is homeomorphic to a closed subspace of a Lindelöf almost-P-space and that the latter is absolute $\mathcal{CR}$-epic if the former is. Thus there is, for example, a Lindelöf almost-P-space containing the rationals and it is not absolute $\mathcal{CR}$-epic.

3.6. Lemma. *Suppose that K is a space that contains a dense z-embedded almost-P-space X. Then $C(X)$ and $C(gK)$ are naturally isomorphic. If K is realcompact then $gK = \upsilon X$. Furthermore $C(X)$ contains the classical ring of quotients $Q_{cl}(K)$ of $C(K)$ and the two rings have the same idempotents.*

Proof. As noted in [Raphael & Woods (2000), 4.9] $X \subset gK$ since gK is the largest almost-P subspace of K. Since X is z-embedded in K [Blair & Hager (1974), 2.4] applies and each function on X can be extended to a countable intersection of dense cozero-sets of K. Thus each function on X extends to gK because it is the intersection of all the dense cozero-sets of K. So X is C-embedded in gK and by the denseness of X in gK $C(X)$ and $C(gK)$ are naturally isomorphic. In the case when K is realcompact, so are its cozero sets and their intersection gK. Thus $gK = \upsilon X$.

It is well-known (cf [Fine, Gillman, & Lambek (1965)]) that $Q_{cl}(K)$ is the ring of (equivalence classes) of functions on the dense cozero-sets of K. Thus each function in $Q_{cl}(K)$ is defined on gX by restriction and $Q_{cl}(K)$ is a subring of $C(gK)$. Finally let e be an idempotent in $C(X)$ whose zero-set is B and cozero-set is A. Both A and B are cozero-sets in X and since X is z-embedded in K there are cozero-sets C and D of K such that $A = X \cap C$ and $B = X \cap D$. Since X is dense in K, C and D are disjoint. If one defines a function to be 1 on C and 0 on D, then one has an extension of e to an idempotent defined on a dense cozero-set of K, an element of $Q_{cl}(K)$. ∎

Now we consider the case where X is a P-space.

3.7. Theorem. *Suppose that K is a space that has a dense $\mathcal{CR}$-epic P-subspace X. Then gK is a P-space and the rings $H(K)$, $C(gK)$ and $Q_{cl}(K)$ coincide. In particular this holds if X is a dense Lindelöf P-subspace of K.*

Proof. Since X is a $\mathcal{CR}$-epic P-subspace of K it is z-embedded in K by [Barr, Raphael, & Woods (2004), Lemma 5.1]. By theorem 3.1 and lemma 3.6 we have $C(K) \to Q_{cl}(K) \to C(gK) = C(X) = H(K)$. Thus gK is a P-space. It is easy to see that $H(K) = Q_{cl}(K)$ because algebraically one gets $H(K)$ by adjoining to $C(K)$ the idempotents of $H(K)$ and then taking the classical ring of quotients of the resulting ring. By lemma 3.6 the ring $Q_{cl}(K)$ already contains these idempotents and is its own classical ring of quotients. ∎

3.8. Remark. While on the topic of epimorphic hulls we note that [Raphael & Woods (2000), Lemma 5.14] holds without assuming the realcompactness of X. As a consequence Lemma 5.15 and Theorems 5.16, and 5.17 of [Raphael & Woods (2000)] also hold without the assumption of realcompactness. This assertion follows once one has established the following:

3.9. Claim. *Let T be Lindelöf and almost-P. If T is dense in the Tychonoff space X, then $gX = T$.*

Proof. Since T is a dense almost-P-subspace, T lies in gX by [Raphael & Woods (2000), 4.9]. Thus it lies in every dense cozero-set of X. Conversely, if $p \in X - T$, for each $x \in T$ there is a cozero-set of X that contains x but not p. These cover T. As T is Lindelöf there is a countable subfamily that covers T with union $W(p)$, say. Then $W(p)$ is a dense cozero-set of X not containing p and thus $p \notin gX$. It follows that $T = gX$. Note that T is realcompact because it is Lindelof. ∎

We note that [Henriksen, Raphael, & Woods (2001), Cor 4.12] is missing the assumption that X and Y are realcompact.

4. Some examples

1. The condition in part (ii) of theorem 3.1 is not sufficient to imply that the embedding $gK \to K$ is $\mathcal{CR}$-epic, i.e. this can fail to occur even if K has a dense almost-P subspace that is RG. A compact example is provided by the Alexandroff double $K = A(M)$ of [Raphael & Woods (2000), Example 6.4]. As shown in the example $H(K)$ is not a ring of functions. Hence by part (i) of theorem 3.1 the embedding $gK \to K$ is not $\mathcal{CR}$-epic. Interestingly, gK is a P-space in this case.

2. There exists a Lindelof almost-P space that is scattered of finite Cantor Bendixon index and not absolute $\mathcal{CR}$-epic. An example can be constructed as follows. Let Y be a metric space that is countable, scattered, of finite Cantor Bendixon index, and not locally compact. (An easy example is given in [Barr, Burgess, & Raphael (2003)]). Note that since Y is not locally compact it is not absolute $\mathcal{CR}$-epic by [Barr, Raphael, & Woods (2004), 2.27]. Now let X be the space mentioned in Remark 3.5 constructed on the (set) product of Y and the one point Lindelofization of an uncountable discrete set. The space X is almost-P and Lindelof scattered of finite Cantor Bendixon index. It follows by [Henriksen, Raphael, & Woods (2001), 2.12] that X is also RG. However by [Barr,

Kennison, & Raphael (2004), 6.4] X is not absolute $\mathcal{CR}$-epic because it has a closed subspace that is not absolute $\mathcal{CR}$-epic, namely Y.

3. The space X of example 2 provides a "smaller" example of the phenomenon discussed in example 1. Since X is not absolute $\mathcal{CR}$-epic it has a compactifcation K for which the embedding $X \to K$ is not $\mathcal{CR}$-epic. And by Claim 3.9 $gK = X$, an almostPRG-space.

5. Question

If $H(K)$ is a ring of functions then must the map $C(K) \to C(gK)$ be epic? If so, then the condition in part (i) of theorem 3.1 is necessary and sufficient. There is a counterexample if the space X of example 2 is dense and not $\mathcal{CR}$-epic in an ambient *RG*-space K.

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