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PATHS IN DOUBLE CATEGORIES

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Abstract. Two constructions of paths in double categories are studied, providing algebraic versions of the homotopy groupoid of a space. Universal properties of these constructions are presented. The first is seen as the codomain of the universal oplax morphism of double categories and the second, which is a quotient of the first, gives the universal normal oplax morphism. Normality forces an equivalence relation on cells, a special case of which was seen before in the free adjoint construction. These constructions are the object part of 2-comonads which are shown to be oplax idempotent. The coalgebras for these comonads turn out to be Leinster's **fc**-multicategories, with representable identities in the second case.

Introduction

In [DPP1] and [DPP2] we studied the 2-category $\Pi_2\mathbf{A}$ obtained by freely adjoining right adjoints to each arrow of a category $\mathbf{A}$ considered as a locally discrete 2-category. The 2-category $\Pi_2\mathbf{A}$ was conceived as a more informative 2-dimensional version of $\Pi_1\mathbf{A}$, which is obtained from $\mathbf{A}$ by freely inverting all arrows, and indeed, $\Pi_1\mathbf{A}$ can be obtained by applying Π_0 locally in $\Pi_2\mathbf{A}$. Morphisms of $\Pi_2\mathbf{A}$ are zig-zag paths of arrows of $\mathbf{A}$ which may be thought of as paths of spans in $\mathbf{A}$. The cells of $\Pi_2\mathbf{A}$ are a bit more complicated, being equivalence classes of certain diagrams, which we call fences and which formally look like directed homotopies between the paths.

In [DPP3] we studied several universal properties of the Span construction applied to categories with pullbacks, each expressing the sense in which Span$\,(\mathbf{A})$ is the result of adjoining right adjoints for the arrows of $\mathbf{A}$.

It should be instructive to break the construction of $\Pi_2\mathbf{A}$ into two steps, first applying the span construction and then taking paths. Each construction is interesting in its own right. In the sequel to this paper [DPP4] we generalise the span construction to apply to categories without pullbacks and even 2-categories. We give there its universal properties and relate it to the Π_2 construction. In that paper we will also see the advantages of using double categories and even Leinster's **fc**-multicategories rather than bicategories.

This paper is concerned with the "pathology" of double categories. After a prologue in which we review two path constructions for mere categories we begin our study of paths in double categories. The first, simpler, construction gives the *universal oplax morphism*

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of double categories. (The universal oplax morphism for 2-categories will be derived from this in [DPP4].) The resulting comonad in double categories is in fact an oplax idempotent comonad (also known as KZ-comonad). The Kleisli morphisms are the oplax morphisms of double categories, which is not surprising. What is perhaps more surprising is that the Eilenberg-Moore coalgebras turn out to be exactly Leinster's **fc**-multicategories, so that these are precisely the structures corresponding to oplax morphisms and we call them oplax double categories.

Working with oplax morphisms, we rapidly find ourselves wanting more structure, and in order to get any significant results we need the morphisms to be at least normal. Accordingly, our second path construction gives the *universal oplax normal* morphism of double categories. Here is where the equivalence relation on fences in the Π_2 construction comes in. The Kleisli morphisms for this second path construction are, not surprisingly, the oplax normal morphisms of double categories. We identify an interesting class of **fc**-multicategories that constitute the Eilenberg-Moore coalgebras for this comonad.

Both of these constructions are like the fundamental groupoid of a topological space. Further study of this analogy should prove fruitful.

0. Prologue

Let **Cat** be the category of small categories and **Gph** the category of directed multigraphs. The forgetful functor $U\colon \mathbf{Cat} \to \mathbf{Gph}$ has a left adjoint F which takes a graph to the category whose objects are nodes and whose morphisms are compatible paths of edges

$$\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \xrightarrow{f_3} \cdots \xrightarrow{f_n} \bullet$$

for $n \geqslant 0$. Composition is given by concatenation and identities are paths of length zero. Thus we get a monad $(\ )^*$ on **Gph** whose Eilenberg-Moore algebras are small categories; *i.e.*, U is monadic. This of course is well known. What is perhaps less well known is that, in the spirit of Barr's paper [Ba], F is comonadic. Thus, a graph may be considered as a category equipped with a costructure.

It will be instructive for us to work out in detail how this works. The triple $(FU, \varepsilon, F\eta U)$ is a comonad on **Cat** which we shall denote by (Path, E, D). Thus, for a category $\mathbf{A}$, $\text{Path}\,(\mathbf{A})$ is the category with the same objects as $\mathbf{A}$ but whose morphisms are paths

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$$

of composable morphisms of $\mathbf{A}$, denoted by $\langle f_i \rangle_{i\in\{1,\ldots,n\}}$. Both E and D are the identity on objects. The counit $E\colon \text{Path}\,(\mathbf{A}) \to \mathbf{A}$ is given by $E(\langle f_i \rangle) = f_n \circ f_{n-1} \circ \cdots \circ f_1$, which we will also denote by $\prod_{i=1}^{n} f_i$. The comultiplication $D\colon \text{Path}\,(\mathbf{A}) \to \text{Path}\,\text{Path}\,(\mathbf{A})$ takes a path $\langle f_i \rangle_{i\in\{1,\ldots,n\}}$ to the path $\langle \langle f_i \rangle \rangle_{i\in\{1,\ldots,n\}}$, *i.e.*, the path of paths of length n where each inside path has length one.

A coalgebra, then, is a category $\mathbf{A}$ with a comultiplication (coaction) $G\colon \mathbf{A} \to \mathrm{Path}\,(\mathbf{A})$. The counit law

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{G} & \mathrm{Path}\,(\mathbf{A}) \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle E} \\ & & \mathbf{A} \end{array}$$

says that G is the identity on objects and factorises every morphism into a string of compatible morphisms. While this may appear to require rather arbitrary choices, in fact the coassociativity law removes the arbitrariness.

The coassociativity law of a coalgebra requires the following diagram to commute:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\quad G \quad} & \mathrm{Path}(\mathbf{A}) \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \mathrm{Path}\,(G)} \\ \mathrm{Path}\,(\mathbf{A}) & \xrightarrow[\quad D \quad]{} & \mathrm{Path}\ \mathrm{Path}(\mathbf{A}). \end{array}$$

In detail, this says that for every $f\colon A \to B$, factored by G into $\langle f_i \rangle$, the path of paths

$$\big\langle \langle f_{ij} \rangle_{j\in\{1,\ldots,m_i\}} \big\rangle_{i\in\{1,\ldots,n\}}$$

obtained by applying G to each of the f_i is equal to $\langle\langle f_i\rangle\rangle_{i\in\{1,\ldots,n\}}$. Thus, each path $\langle f_{ij}\rangle_{j\in\{1,\ldots,m_i\}}$ has length one and $f_{ij} = f_i$; and G applied to f_i returns f_i itself. Now, these f_i cannot be identity arrows because, as a functor, G takes an identity arrow to a path of length zero. Moreover, the f_i cannot be factored non-trivially into two arrows, since if $f_i = gh$ with both g and h non-identity arrows then $G(f_i) = G(g)G(h)$ would have length greater than one. We will call an arrow $f \neq 1$ *prime* if it cannot be factored in a non-trivial way, *i.e.*, $f = gh$ implies $g = 1$ or $h = 1$. The factorisation G is thus a *prime* factorisation, and it is unique because $Gg = \langle g \rangle$ for any prime g. Indeed, if $f = f_n \circ f_{n-1} \circ \cdots \circ f_1 = g_m \circ g_{m-1} \circ \cdots \circ g_1$ where all the f_i and g_j are prime, then

$$\begin{aligned} \langle f_i \rangle_{i\in\{1,\ldots,n\}} &= \langle f_n \rangle \circ \langle f_{n-1} \rangle \circ \cdots \circ \langle f_1 \rangle \\ &= G(f_n) \circ G(f_{n-1}) \circ \cdots \circ G(f_1) \\ &= G(g_m) \circ G(g_{m-1}) \circ \cdots \circ G(g_1) \\ &= \langle g_m \rangle \circ \langle g_{m-1} \rangle \circ \cdots \circ \langle g_1 \rangle \\ &= \langle g_j \rangle_{j\in\{1,\ldots,m\}}, \end{aligned}$$

so $m = n$ and $f_i = g_i$ for all $i \in \{1, \ldots, n\}$. It follows that $\mathbf{A}$ is the free category on the graph whose nodes are the objects of $\mathbf{A}$ and whose edges are the prime arrows of $\mathbf{A}$.

Given two coalgebras $(\mathbf{A}, G)$ and $(\mathbf{B}, H)$, for any homomorphism

$$\Phi\colon (\mathbf{A}, G) \to (\mathbf{B}, H)$$

the following square commutes

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\quad\Phi\quad} & \mathbf{B} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle H} \\ \mathrm{Path}\,(\mathbf{A}) & \xrightarrow[\mathrm{Path}\,(\Phi)]{} & \mathrm{Path}\,(\mathbf{B}). \end{array}$$

So, for any $f\colon A \to B$, with G-factorisation $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$, the H-factorisation of $\Phi(f)$ is given by $\Phi(f_n) \circ \Phi(f_{n-1}) \circ \cdots \circ \Phi(f_1)$. As primes are those arrows whose factorisation has length one, Φ preserves primes and so restricts to the prime graphs. That is to say, every coalgebra morphism comes from a morphism of graphs. We have thus established an equivalence of categories

$$\mathbf{Gph} \simeq \mathbf{Cat}_{\mathrm{Path}}. \tag{1}$$

0.1. Remarks.

1. Although Path is a natural comonad on **Cat**, it does not preserve natural transformations, so it is not a 2-functor. Given a natural transformation $t\colon \Phi \to \Phi'$, one might be tempted to define $\mathrm{Path}\,(t)\colon \mathrm{Path}\,(\Phi) \to \mathrm{Path}\,(\Phi')$ at A to be the path of length one, $\langle t(A)\rangle\colon \Phi A \to \Phi' A$, but the naturality squares won't commute, except in the most trivial cases, because in $\mathrm{Path}\,(\mathbf{A})$ nothing commutes non-trivially.

2. For any category $\mathbf{A}$ there is at most one coalgebra structure on it, a rather strong property usually associated with idempotent (co)monads. We will be able to explain better why this is the case once we have introduced the double category version $\mathbb{P}\mathrm{ath}$ of this construction.

3. The Kleisli category of Path has small categories as its objects but its morphisms are 'functors' which preserve neither composition nor identities, pretty poor morphisms indeed.

4. We could have used the comonadicity theorem to get the above equivalence (1). Indeed, **Gph** has all equalisers and F reflects isomorphisms and preserves equalisers, as is easily checked.

We wish to examine a similar construction over the category **Rgph** of reflexive graphs. A reflexive graph is a graph as above with the specification of a distinguished loop $1_A\colon A \to A$ for each vertex A. It is thus determined by a diagram of the form $E \overset{s}{\underset{t}{\substack{\longrightarrow \\ \xleftarrow{\;i\;} \\ \longrightarrow}}} V$ in **Set** with $si = ti = 1_V$. Again there is a forgetful functor $U\colon \mathbf{Cat} \to \mathbf{Rgph}$ which has a left adjoint F. For a reflexive graph $\mathbf{G}$, $F(\mathbf{G})$ is the category whose objects are the nodes of $\mathbf{G}$ and whose morphisms are equivalence classes of directed paths of edges, two paths being equivalent if one can be obtained from the other by inserting and/or deleting identity loops. Each equivalence class has a canonical representative, namely the path without any identities. Equivalently, $F(\mathbf{G})$ could have been defined as having morphisms which are paths without identities. This result is useful for some computations, but should be treated with some caution as it does not generalise for graphs in an arbitrary topos.

Once again, it is well-known that **Cat** is monadic over **Rgph**. Indeed, an algebra structure $UF\mathbf{G} \to \mathbf{G}$ on a reflexive graph $\mathbf{G}$ equips it with a family of n-fold composition operations defined on compatible paths of non-identity arrows. As this is a morphism of reflexive graphs, the empty path at A is sent to 1_A. Moreover, the unit law says that paths of length one compose to themselves, and the associative law says that for a path of paths, we have

$$(f_{k_n} \circ f_{k_n - 1} \circ \cdots \circ f_{l_n}) \circ \cdots \circ (f_{k_1} \circ \cdots \circ f_2 \circ f_1) = f_{k_n} \circ \cdots \circ f_2 \circ f_1.$$

Clearly, such a family of composition operations is determined by an associative binary composition operation, and we know how to compose with identities.

However, it is the comonad induced on **Cat** we are interested in. We denote it by $(\mathrm{Path}_*, E_*, D_*)$ with counit E_* and comultiplication D_*. For a category $\mathbf{A}$, $\mathrm{Path}_*(\mathbf{A})$ has the same objects as $\mathbf{A}$ and has as morphisms compatible paths of non-identity morphisms. For a functor $\Phi\colon \mathbf{A} \to \mathbf{B}$, $\mathrm{Path}_*(\Phi)$ is the same as Φ on objects; on paths Φ is applied to each component and identities are deleted, so that the length of $\Phi(\langle f_i \rangle)$ might be less than the length of $\langle f_i \rangle$, even zero. Just as for Path, E_* takes a path to its composite in $\mathbf{A}$, and D_* takes a path to the path of paths whose components are paths of length one (for example, $D_*(\langle f_1, f_2, f_3 \rangle) = \langle \langle f_1 \rangle, \langle f_2 \rangle, \langle f_3 \rangle \rangle$).

As before, a coalgebra structure $G\colon \mathbf{A} \to \mathrm{Path}_*(\mathbf{A})$ is the identity on objects and gives a factorisation of each morphism into n non-identity morphisms, each of which G does not factor further. We now construct the reflexive graph $\mathrm{Prime}_*(\mathbf{A})$ consisting of those arrows of $\mathbf{A}$ such that $\mathrm{length}\,(G(f)) \leqslant 1$. Note that any functor $\Phi\colon \mathbf{A} \to \mathbf{B}$ restricts to a reflexive graph morphism

$$\mathrm{Prime}_*(\Phi)\colon \mathrm{Prime}_*(\mathbf{A}) \to \mathrm{Prime}_*(\mathbf{B}).$$

Thus we get a functor

$$\mathrm{Prime}_*\colon \mathbf{Cat}_{\mathrm{Path}_*} \to \mathbf{Rgph}.$$

Of course, the free functor $F\colon \mathbf{Rgph} \to \mathbf{Cat}$ factors through $\mathbf{Cat}_{\mathrm{Path}_*}$ which gives a weak inverse for Prime_*, thus demonstrating the equivalence

$$\mathbf{Rgph} \simeq \mathbf{Cat}_{\mathrm{Path}_*}. \tag{2}$$

0.2. Remarks. Here, Kleisli morphisms are 'functors' which don't preserve composition but do preserve identities; Path_* is still not a 2-functor; and coalgebra structures relative to UF are again unique.

1. $\mathbb{P}$ath

1.1. The Construction. We now wish to examine similar constructions for double categories. Rather than starting from graphs and studying the monad generated by the free category construction, we instead concentrate on the comonad of paths and then determine what the corresponding notion of 'graph' is by considering the Eilenberg-Moore coalgebras. There are several possible definitions for the notion of 2-cells of paths. Some are not very interesting, whereas others are dual forms of the notion we will introduce in this section. We have been guided in our definition by our desire for a deeper understanding of the Π_2 construction and its relationship to spans. The main results of this section, Theorems 1.17 and 1.21 below, justify our choices.

Let $\mathbb{A}$ be a double category (see, for example, [E] or [DP1]). We construct a new double category $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ with the same objects and vertical arrows as in $\mathbb{A}$. A horizontal arrow in $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ is a path

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \longrightarrow \cdots \longrightarrow A_{m-1} \xrightarrow{f_m} A_m \tag{3}$$

of horizontal arrows in $\mathbb{A}$, with $m \geqslant 0$. Let

$$B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2 \longrightarrow \cdots \longrightarrow B_{n-1} \xrightarrow{g_n} B_n$$

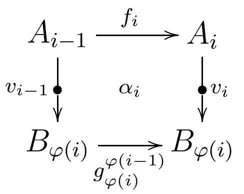
be another path. For $i \leqslant j$, define $g^i_j \colon B_i \to B_j$ by

$$g^i_j = \begin{cases} g_j g_{j-1} \cdots g_{i+1} & \text{if} \quad i < j \\ 1_{B_i} & \text{if} \quad i = j. \end{cases}$$

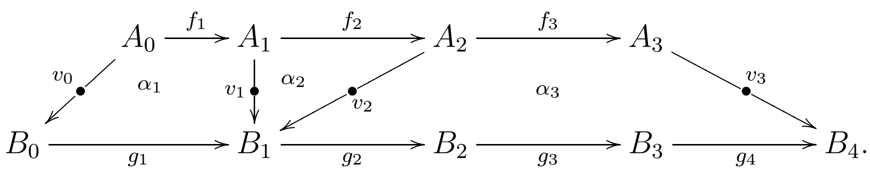
Then a double cell from $\langle f_i \rangle$ to $\langle g_j \rangle$ is a triple $(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$ where

$$\varphi \colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$$

is an order preserving function with $\varphi(0) = 0$ and $\varphi(m) = n$, the $v_i \colon A_i \longrightarrow\!\!\!\!\!\bullet\;\; B_{\varphi(i)}$ are vertical arrows, and



are cells of $\mathbb{A}$. A typical cell might look like



(4)

Horizontal composition of both arrows and cells is by concatenation. Vertical composition of arrows is as in $\mathbb{A}$. To define vertical composition of cells, extend the notation as follows, for $i \leqslant j$,

$$\alpha^i_j = \begin{cases} \alpha_j \alpha_{j-1} \cdots \alpha_{i+1} & \text{if} \quad i < j \\ 1_{v_i} & \text{if} \quad i = j \end{cases},$$

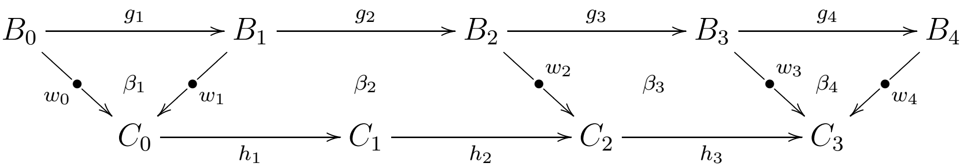
so

$$\begin{array}{ccc} A_i & \xrightarrow{f_j^i} & A_i \\ {\scriptstyle v_i}\downarrow\bullet & \alpha_j^i & \bullet\downarrow{\scriptstyle v_j} \\ B_{\varphi(i)} & \xrightarrow[g_{\varphi(j)}^{\varphi(i)}]{} & B_{\varphi(j)}. \end{array}$$

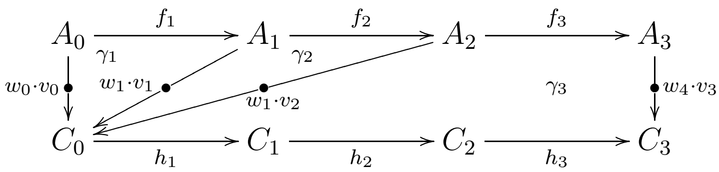
If $(\psi, \langle w_j \rangle, \langle \beta_j \rangle)$ is a cell $\langle g_j \rangle \to \langle h_k \rangle$, then the vertical composition is defined by

$$(\psi, \langle w_j \rangle, \langle \beta_j \rangle) \cdot (\varphi, \langle v_i \rangle, \langle \alpha_i \rangle) = (\psi\varphi, \langle w_{\varphi(i)} \cdot v_i \rangle, \langle \beta_{\varphi(i)}^{\varphi(i-1)} \cdot \alpha_i \rangle).$$

An example will make this transparent. The composition of



with the cell in (4) is



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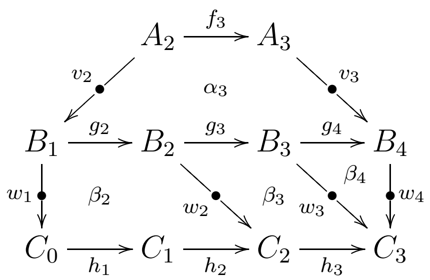
where γ_3 is, *e.g.*, the pasting of



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It is straightforward to check that $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ is a (strict) double category.

1.2. Example. Let $\mathbb{1}$ be the terminal double category. Then $\mathbb{P}\mathrm{ath}\,\mathbb{1}$ has just one object $*$ and one vertical arrow, 1_*. Horizontal arrows are in bijective correspondence with non-empty ordinals, $[m] = \{0, 1, \ldots, m\}$ and cells

$$\begin{array}{ccc} * & \xrightarrow{[m]} & * \\ \| & \varphi & \| \\ * & \xrightarrow[{[n]}]{} & * \end{array}$$

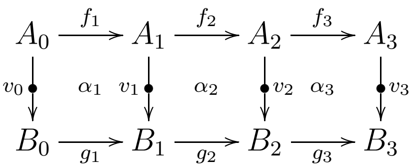
with order preserving functions $\varphi\colon [m] \to [n]$ such that $\varphi(0) = 0$ and $\varphi(m) = n$. Horizontal composition is given by $[m'] \otimes [m] = [m' + m]$, the ordinal sum with merged endpoints; this extends to functions in the obvious way:

$$(\varphi' \otimes \varphi)(i) = \begin{cases} \varphi'(i) & \text{if } i \leqslant m' \\ m' + \varphi(i) & \text{otherwise.} \end{cases}$$

1.3. Remark. $\mathbb{P}\mathrm{ath}\,\mathbb{1}$ is really a 2-category, since all vertical arrows are identities. It is even a strict monoidal category since there is only one object. As such it is equivalent to $\mathbb{\Delta}^{\mathrm{op}}$ where $\mathbb{\Delta}$ is the category of finite ordinals including zero, with ordinal sum as $\otimes$. The equivalence $\mathbb{\Delta}^{\mathrm{op}} \to \mathbb{P}\mathrm{ath}\,\mathbb{1}$ is as follows: the ordinal $m = \{0, 1, 2, \ldots, m-1\}$ is mapped to $[m]$. An order preserving function $f\colon m \to n$ can be extended to $[f]\colon [m] \to [n]$ by defining $[f](m) = n$. The function $[f]$ preserves the top element and therefore has a left adjoint f^* which of course preserves 0. But the left adjoint of a morphism $g\colon [m] \to [n]$ preserves the top element if and only if $g(i) = n$ precisely when $i = m$. So f^* also preserves the top. The equivalence $\mathbb{\Delta}^{\mathrm{op}} \to \mathbb{P}\mathrm{ath}\,\mathbb{1}$ is given by sending f to f^*.

1.4. Definition.

1. *A 2-cell* $(\varphi, \langle v_i\rangle, \langle \alpha_i\rangle)$ *is called* neat *if* φ *is an identity, for example*



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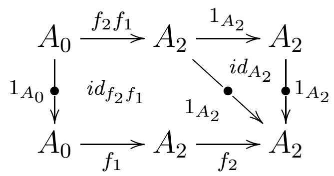
2. *A 2-cell* $(\varphi, \langle v_i\rangle, \langle \alpha_i\rangle)$ *is called* cartesian *if the* v_i *and* α_i *are all vertical identities, for example*



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3. *A factorisation* $(\varphi, \langle v_i\rangle, \langle \alpha_i\rangle) = (\varphi^2, \langle v_i^2\rangle, \langle \alpha_i^2\rangle)(\varphi^1, \langle v_i^1\rangle, \langle \alpha_i^1\rangle)$ *is called* neat-cartesian *if* $(\varphi^1, \langle v_i^1\rangle, \langle \alpha_i^1\rangle)$ *is neat and* $(\varphi^2, \langle v_i^2\rangle, \langle \alpha_i^2\rangle)$ *is cartesian.*

1.5. Proposition. *Every 2-cell* $(\varphi, \langle v_i\rangle, \langle \alpha_i\rangle)$ *has a unique neat-cartesian factorisation.*

Proof. One can easily check that

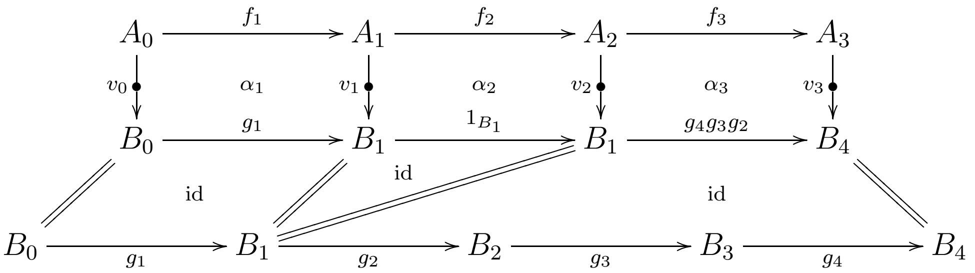
$$(\varphi, \langle \mathrm{id}_{B_i}\rangle, \langle \mathrm{id}_{g^i_{i+1}}\rangle)(1_{[m]}, \langle v_i\rangle, \langle \alpha_i\rangle).$$

is a neat-cartesian factorisation. To prove uniqueness, let

$$(\varphi, \langle v_i\rangle, \langle \alpha_i\rangle) = (\nu, \langle w_i\rangle, \langle \beta_i\rangle)(\psi, \langle u_i\rangle, \gamma_i),$$

where $(\psi, \langle u_i \rangle, \langle \gamma_i \rangle)$ is neat and $(\nu, \langle w_i \rangle, \beta_i)$ is cartesian. Since $(\nu, \langle w_i \rangle, \beta_i)$ is cartesian, the w_i are identity arrows and the β_i are identity cells, so in order to compose to $(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$, we must have $u_i = v_i$ and $\gamma_i = \alpha_i$ for all i. Since $(\psi, \langle u_i \rangle, \langle \gamma_i \rangle)$ is neat, it follows that the horizontal arrows in its codomain must be of the form g^i_{i+1}. Since $\psi = \mathrm{id}$, it follows that $\nu = \phi$, and $(\nu, \langle w_i \rangle, \beta_i) = (\varphi, \langle \mathrm{id}_{B_i} \rangle, \langle \mathrm{id}_{g^i_{i+1}} \rangle)$ and $(\psi, \langle u_i \rangle, \gamma_i) = (1_{[m]}, \langle v_i \rangle, \langle \alpha_i \rangle)$. ∎

1.6. Example. The 2-cell $(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$ depicted in (4) above factors as



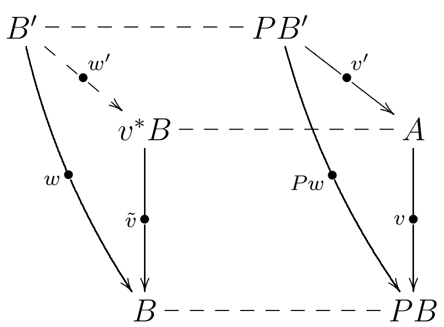
We recall that a double category $\mathbb{A}$ can be viewed as a category object

$$\mathbf{A}_2 \;\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\text{Comp}}}}\; \mathbf{A}_1 \;\overset{\partial_0}{\underset{\partial_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}}}}\; \mathbf{A}_0$$

in **Cat**, the category of categories. In this notation, we will assume that $\mathbf{A}_0$ is the category of objects and vertical arrows, $\mathbf{A}_1$ is the category with horizontal arrows as objects and double cells as morphisms, and $\mathbf{A}_2$ is the category with composable pairs of horizontal arrows as objects and composable pairs of double cells as morphisms. The functors ∂_0 and ∂_1 give the horizontal domain and codomain of a double cell. So the vertical composition of double cells is defined by the composition in $\mathbf{A}_1$ whereas the horizontal composition is defined by the functor Comp: $\mathbf{A}_2 \to \mathbf{A}_1$.

1.7. Definition. *A double functor $P\colon \mathbb{B} \to \mathbb{A}$ is a* vertical fibration *if both functors $P_0\colon \mathbf{B}_0 \to \mathbf{A}_0$ and $P_1\colon \mathbf{B}_1 \to \mathbf{A}_1$ are fibrations in the usual sense and the functors $\partial_0, \partial_1\colon \mathbf{B}_1 \to \mathbf{B}_0$, Id: $\mathbf{B}_0 \to \mathbf{B}_1$, Comp: $\mathbf{B}_2 \to \mathbf{B}_1$, are cartesian over the corresponding functors for $\mathbb{A}$. In elementary terms this means:*

(VF1) *For every object B in $\mathbb{B}$ and vertical arrow $v\colon A \longrightarrow PB$ there is a vertical arrow $\tilde{v}\colon v^*B \longrightarrow B$ such that $P(\tilde{v}) = v$ and for any $w\colon B' \longrightarrow B$ and factorisation $P(w) = v \cdot v'$ there is a unique w' such that $P(w') = v'$ and $w = \tilde{v} \cdot w'$. In a diagram,*



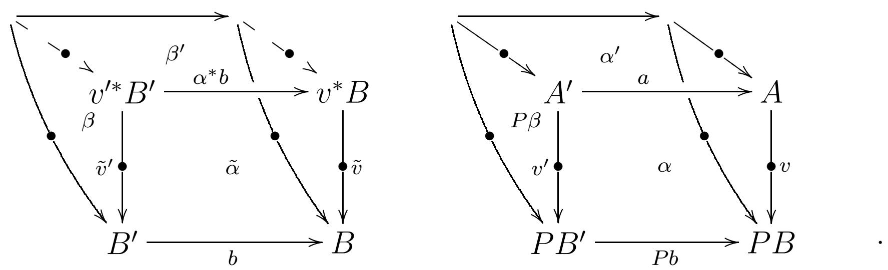
(VF2) *For every horizontal arrow* $b\colon B' \to B$ *in* $\mathbb{B}$ *and cell*

$$\begin{array}{ccc} A' & \xrightarrow{a} & A \\ {\scriptstyle v'}\downarrow & \alpha & \downarrow{\scriptstyle v} \\ PB' & \xrightarrow[Pb]{} & PB \end{array}$$

in $\mathbb{A}$*, there exists a cell*

$$\begin{array}{ccc} v'^*B' & \xrightarrow{\alpha^*b} & v^*B \\ {\scriptstyle \tilde{v}'}\downarrow & \tilde{\alpha} & \downarrow{\scriptstyle \tilde{v}} \\ B' & \xrightarrow[b]{} & B \end{array}$$

in $\mathbb{B}$*, such that* $P(\tilde{\alpha}) = \alpha$*, and for any* β *and factorisation* $P\beta = \alpha \cdot \alpha'$ *there exists a unique* β' *such that* $P\beta' = \alpha'$ *and* $\tilde{\alpha} \cdot \beta' = \beta$*, as depicted in*



(VF3) *For any object* B *and vertical arrow* $v\colon A \to PB$*, the cell*

$$\begin{array}{ccc} v^*B & \xrightarrow{1_{v^*B}} & v^*B \\ {\scriptstyle \tilde{v}}\downarrow & 1_{\tilde{v}} & \downarrow{\scriptstyle \tilde{v}} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

is cartesian over

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle v}\downarrow & 1_v & \downarrow{\scriptstyle v} \\ PB & \xrightarrow[1_{PB}]{} & PB, \end{array}$$

i.e., $\tilde{1}_v = 1_{\tilde{v}}$.

(VF4) *For any path of horizontal arrows* $B'' \xrightarrow{b'} B' \xrightarrow{b} B$ *with a path of cells*

$$\begin{array}{ccccc} A'' & \xrightarrow{a'} & A' & \xrightarrow{a} & A \\ {\scriptstyle v''}\downarrow & \alpha' & \downarrow{\scriptstyle v'} & \alpha & \downarrow{\scriptstyle v} \\ PB'' & \xrightarrow[Pb']{} & PB' & \xrightarrow[Pb]{} & PB, \end{array}$$

the composite

$$\begin{array}{ccccc} v''^*B'' & \xrightarrow{\alpha'^*b'} & v'^*B' & \xrightarrow{\alpha'^*b} & v^*B' \\ \tilde{v}'' \big\downarrow & \tilde{\alpha}' & \big\downarrow \tilde{v}' & \tilde{\alpha} & \big\downarrow \tilde{v} \\ B'' & \xrightarrow[b']{} & B' & \xrightarrow[b]{} & B \end{array}$$

is cartesian over $\alpha\alpha'$*, i.e.,* $(\alpha\alpha')^*(bb') = (\alpha^* b)(\alpha'^* b')$*.*

1.8. Proposition. *The projection double functor* $\mathbb{P}\mathrm{ath}\,\mathbb{A} \to \mathbb{P}\mathrm{ath}\,\mathbb{1}$ *is a vertical fibration.*

Proof. The zero dimensional condition (VF1) is trivial as $\mathbb{P}\mathrm{ath}\,\mathbb{1}$ only has identity vertical arrows.

Given a path $B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} B_n$ and a function $\theta\colon [p] \to [n]$, $\theta^*\langle g_i \rangle$ is the path $\left\langle g^{\theta(j-1)}_{\theta(j)} \right\rangle$,

$$B_0 = B_{\theta(0)} \xrightarrow{g^{\theta(0)}_{\theta(1)}} B_{\theta(1)} \xrightarrow{g^{\theta(1)}_{\theta(2)}} \cdots\cdots \xrightarrow{g^{\theta(p-1)}_{\theta(p)}} B_{\theta(p)} = B_n,$$

and the cartesian cell is $(\theta, \langle \mathrm{id}_{B_{\theta(j)}} \rangle, \langle \mathrm{id}_{g^{\theta(j-1)}_{\theta(j)}} \rangle)$,

$$\begin{array}{ccccccc} & B_{\theta(j-1)} & & \xrightarrow{g^{\theta(j-1)}_{\theta(j)}} & & B_{\theta(j)} & \\ /\!/ & & & \mathrm{id} & & & \backslash\!\backslash \\ B_{\theta(j-1)} & \longrightarrow & B_{\theta(j-1)+1} & \longrightarrow & \cdots\cdots & \longrightarrow & B_{\theta(j)}. \end{array}$$

Given another cell

$$\begin{array}{ccc} A_0 & \xrightarrow{\langle f_i \rangle} & A_m \\ \big\downarrow & (\varphi, \langle v_i \rangle, \langle \alpha_i \rangle) & \big\downarrow \\ B_0 & \xrightarrow[\langle g_j \rangle]{} & B_n, \end{array}$$

and a factorisation

$$\begin{array}{ccc} [m] & \xrightarrow{\varphi} & [n] \\ & {\scriptstyle\psi} \searrow \quad \nearrow {\scriptstyle\theta} & \\ & [p] & \end{array}$$

there is a unique cell

$$\begin{array}{ccc} A_0 & \xrightarrow{\langle f_i \rangle} & A_m \\ \big\downarrow & (\psi, \langle \bar{v}_i \rangle, \langle \beta_i \rangle) & \big\downarrow \\ B_{\theta(0)} & \xrightarrow[\langle g^{\theta(j-1)}_{\theta(j)} \rangle]{} & B_{\theta(n)} \end{array}$$

such that $(\psi, \langle \bar{v}_i \rangle, \langle \beta_i \rangle)(\theta, \langle \mathrm{id}_{B_{\theta(j)}} \rangle, \langle \mathrm{id}_{g^{\theta(j-1)}_{\theta(j)}} \rangle) = (\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$, namely $\bar{v}_i = v_i$ and $\beta_i = \alpha_i$. Conditions (VF3) and (VF4) follow from the definition of θ^*. ∎

1.9. Oplax morphisms. In order to express the universal property of $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ we need the notion of *oplax morphism* of double categories. Recall [GP] that $F\colon \mathbb{A} \to \mathbb{B}$ is an oplax morphism if it assigns to objects, vertical and horizontal arrows, and cells of $\mathbb{A}$ corresponding elements of $\mathbb{B}$, preserving domains and codomains

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & A_2 \\ {\scriptstyle v_1}\downarrow & \alpha & \downarrow{\scriptstyle v_2} \\ A_3 & \xrightarrow[f_3]{} & A_4 \end{array} \quad \mapsto \quad \begin{array}{ccc} FA_1 & \xrightarrow{Ff_1} & FA_2 \\ {\scriptstyle Fv_1}\downarrow & F\alpha & \downarrow{\scriptstyle Fv_2} \\ FA_3 & \xrightarrow[Ff_3]{} & FA_4, \end{array}$$

and also preserving vertical identities and vertical composition of arrows and cells. The oplax structure of F describes its effect on the horizontal structure of $\mathbb{A}$. For every object A of $\mathbb{A}$ there is given a cell

$$\begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ \| & \varphi_A & \| \\ FA & \xrightarrow[1_{FA}]{} & FA, \end{array}$$

and for every composable pair of horizontal arrows $A \xrightarrow{f} A' \xrightarrow{f'} A''$ a cell

$$\begin{array}{ccccc} FA & & \xrightarrow{F(f'f)} & & FA'' \\ \| & & \varphi_{f',f} & & \| \\ FA & \xrightarrow[Ff]{} & FA' & \xrightarrow[Ff']{} & FA''. \end{array}$$

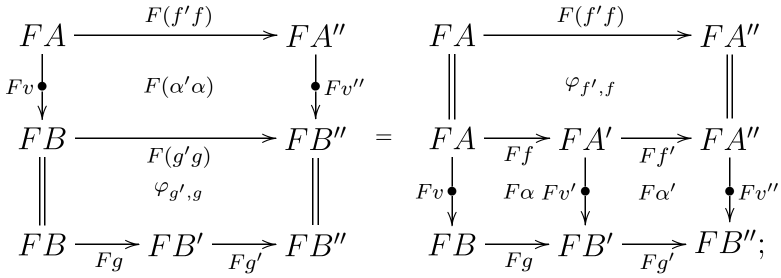
These are required to satisfy:

(OL1; vertical naturality) For every vertical arrow $v\colon A \to B$

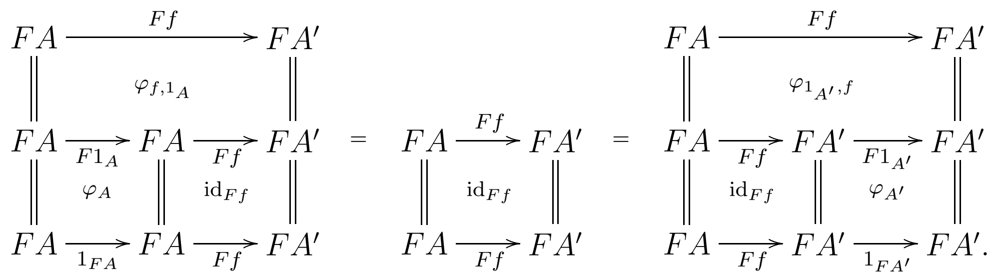
$$\begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ {\scriptstyle Fv}\downarrow & F1_v & \downarrow{\scriptstyle Fv} \\ FB & \xrightarrow[F1_B]{} & FB \\ \| & \varphi_B & \| \\ FB & \xrightarrow[1_{FB}]{} & FB \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ \| & \varphi_A & \| \\ FA & \xrightarrow[1_{FA}]{} & FA \\ {\scriptstyle Fv}\downarrow & 1_{Fv} & \downarrow{\scriptstyle Fv} \\ FB & \xrightarrow[1_{FB}]{} & FB; \end{array}$$

(OL2; horizontal naturality) For every pair of composable cells

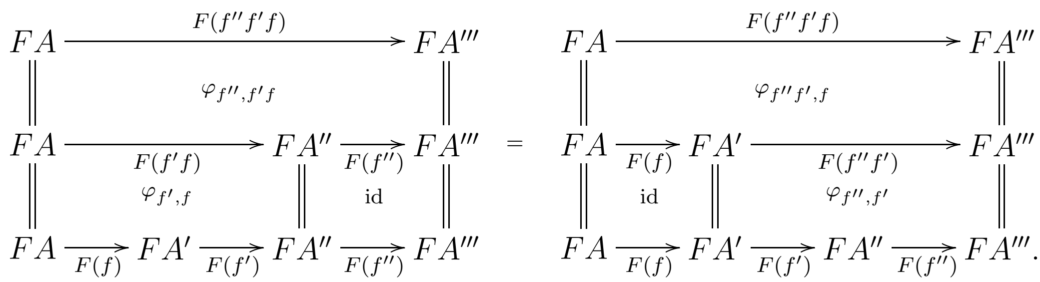
$$\begin{array}{ccccc} A & \xrightarrow{f} & A' & \xrightarrow{f'} & A'' \\ {\scriptstyle v}\downarrow & \alpha & {\scriptstyle v'}\downarrow & \alpha' & \downarrow{\scriptstyle v''} \\ B & \xrightarrow[g]{} & B' & \xrightarrow[g']{} & B'', \end{array}$$



(OL3; unit laws) For every $f\colon A \to A'$, we require



(OL4; associativity) For every composable triple $A \xrightarrow{f} A' \xrightarrow{f'} A'' \xrightarrow{f''} A'''$, we require that



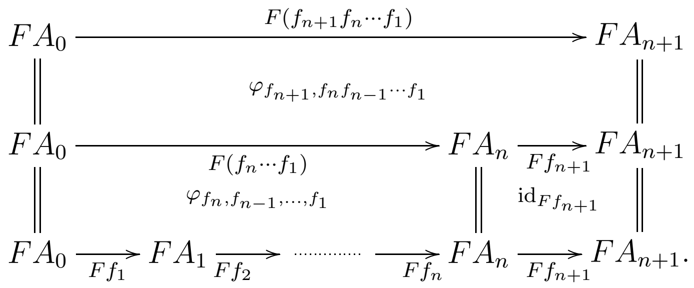
For $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} A_n$, we define the *n-fold comparison morphism*

$$\varphi_{f_n,f_{n-1},\ldots,f_1}\colon F(f_nf_{n-1}\cdots f_1) \to F(f_n)F(f_{n-1})\cdots F(f_1)$$

recursively by

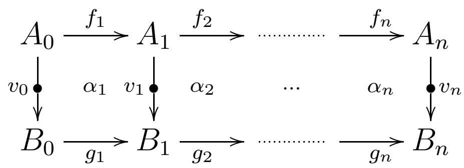
$$\begin{array}{lcll}\varphi & = & \varphi_{A_0} & (n=0)\\ \varphi_f & = & \mathrm{id}_{Ff} & (n=1)\\ \varphi_{f_{n+1},f_n,\ldots,f_1} & = & \left(\mathrm{id}_{Ff_{n+1}}\varphi_{f_n,f_{n-1},\ldots,f_1}\right)\cdot\varphi_{f_{n+1},f_n\cdots f_1} & \end{array}$$

as in the following diagram



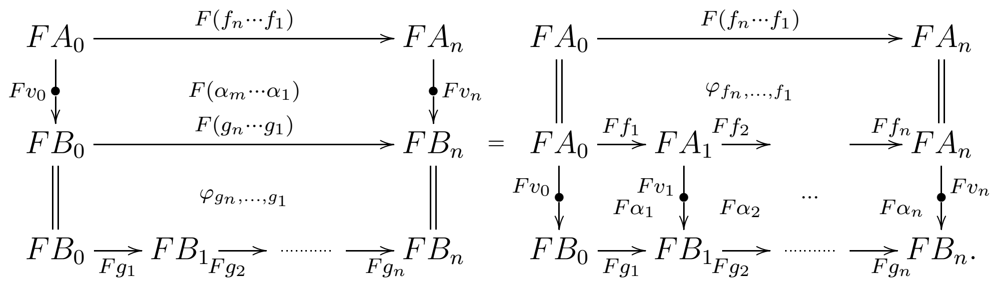
1.10. Proposition.

1. *For any path of cells*

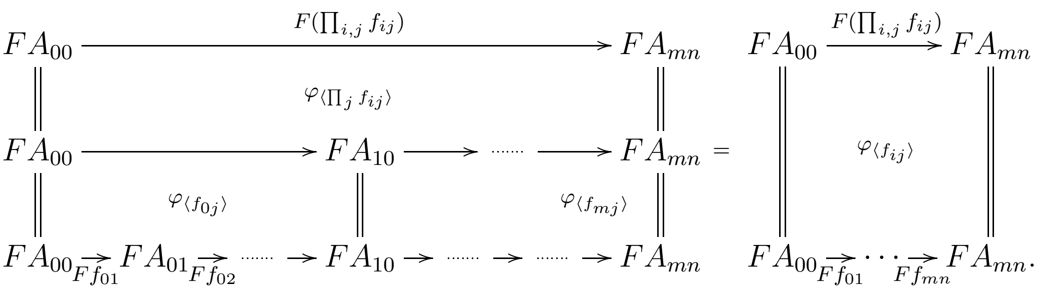


$(n \geqslant 0)$

we have general naturality in the sense that

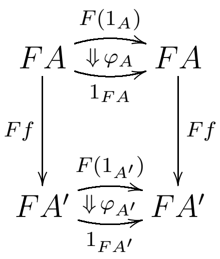


2. *For any path of paths $\langle f_{ij} \rangle$, we have general associativity in the sense that*



Proof. Straightforward induction. ■

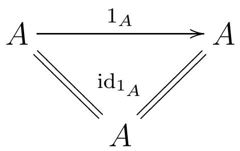
1.11. Remark. Conditions (OL2) - (OL4) are the same as the coherence conditions for oplax morphisms of bicategories [Bé]. In the case of bicategories, condition (OL1) is vacuous as all vertical arrows are identities and $1_{\mathrm{id}_A} = \mathrm{id}_{1_A}$. However, when specifying the cells φ_A for bicategories one might wonder whether further naturality conditions might be needed. It would not make sense to require commutativity of the diagram



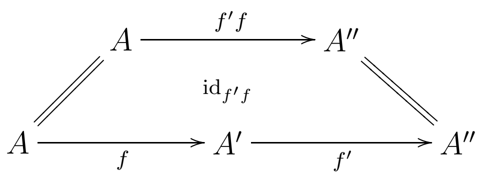
since $Ff\,F1_A \neq F1_{A'}\,Ff$. One might precede these cells by $\varphi_{f,1_A}$ and $\varphi_{1_{A'},f}$ and get an equality, but this is just the condition (OL3). It turns out that the double category formulation of oplaxity very nicely expresses the naturality of φ_A in A.

General naturality for $n = 0$ is exactly (OL1) as the proof shows. Similarly, general associativity has as a special case the unit conditions when the path of paths is made up of two paths where one is of length one and the other one is of length two.

1.12. Example. Let $\mathbb{A}$ be any double category and $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ its double category of paths. Define $\Xi\colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\,\mathbb{A}$ as follows. Ξ is the identity on objects and vertical arrows; also, Ξ takes a cell, or a horizontal arrow, to the corresponding singleton path. For an object A in $\mathbb{A}$, let ξ_A be the cell



and for a path $A \xrightarrow{f} A' \xrightarrow{f'} A''$ of horizontal arrows, let $\xi_{f',f}$ be the cell



.

All conditions for Ξ hold trivially, because all cells are identities. (Note however that although ξ_A and $\xi_{f',f}$ are given by identities, neither is invertible as the indexing functions are not. They are only cartesian morphisms.) So Ξ (equipped with the cells ξ) is an oplax morphism of double categories. We shall show that it is in fact the universal one. In order to express properly what this means, we need the notion of *vertical transformation* of oplax morphisms.

1.13. Definition. *Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be oplax morphisms of double categories. A* vertical transformation *$t\colon F \to G$ assigns to each object A in $\mathbb{A}$ a vertical arrow $tA\colon FA \multimap GA$ and to each horizontal arrow $f\colon A \longrightarrow A'$ a cell*

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ {\scriptstyle tA}\big\downarrow & tf & \big\downarrow{\scriptstyle tA'} \\ GA & \xrightarrow[Gf]{} & GA' \end{array} ,$$

satisfying the following four conditions.

(VT1) *For every vertical arrow $v\colon A \multimap C$ in $\mathbb{A}$, the square*

$$\begin{array}{ccc} FA & \xrightarrow{Fv} & FC \\ \big\downarrow{\scriptstyle tA} & & \big\downarrow{\scriptstyle tB} \\ GA & \xrightarrow[Gv]{} & GC \end{array}$$

commutes.

(VT2) *For every cell*

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle v}\big\downarrow & \alpha & \big\downarrow{\scriptstyle v'} \\ C & \xrightarrow[h]{} & C' \end{array}$$

in $\mathbb{A}$,

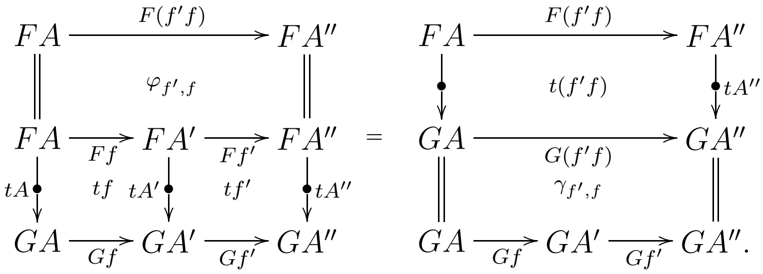
$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ {\scriptstyle tA}\big\downarrow & tf & \big\downarrow{\scriptstyle tA'} \\ GA & \xrightarrow[Gf]{} & GA' \\ {\scriptstyle Gv}\big\downarrow & G\alpha & \big\downarrow{\scriptstyle Gv'} \\ GC & \xrightarrow[Gh]{} & GC' \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ {\scriptstyle Fv}\big\downarrow & F\alpha & \big\downarrow{\scriptstyle Fv'} \\ FA & \xrightarrow[Fh]{} & FC' \\ {\scriptstyle tC}\big\downarrow & th & \big\downarrow{\scriptstyle tC'} \\ GC & \xrightarrow[Gh]{} & GC'. \end{array}$$

(VT3) *For every A in $\mathbb{A}$,*

$$\begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ {\scriptstyle tA}\big\downarrow & t1_A & \big\downarrow{\scriptstyle tA} \\ GA & \xrightarrow[G1_A]{} & GA \\ \| & \gamma_A & \| \\ GA & \xrightarrow[1_{GA}]{} & GA \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ \| & \varphi_A & \| \\ FA & \xrightarrow[1_{FA}]{} & FA \\ {\scriptstyle tA}\big\downarrow & 1_{tA} & \big\downarrow{\scriptstyle tA} \\ GA & \xrightarrow[1_{GA}]{} & GA. \end{array}$$

(VT4) *For every pair of composable horizontal arrows*

$$A \xrightarrow{f} A' \xrightarrow{f'} A'',$$

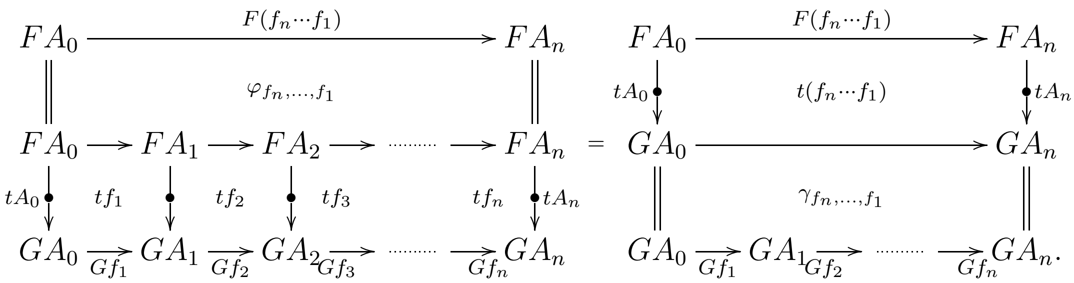


1.14. REMARK. While condition (VT1) is, strictly speaking, implicit in condition (VT2), it is stated separately for greater clarity. These conditions describe what we will call *vertical naturality.* Conditions (VT3) and (VT4) describe *horizontal functoriality.*

1.15. PROPOSITION. *(General Horizontal Functoriality) For every path*

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \longrightarrow \cdots \xrightarrow{f_n} A_n,$$

we have

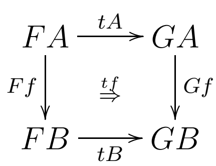


PROOF. Easy induction. ∎

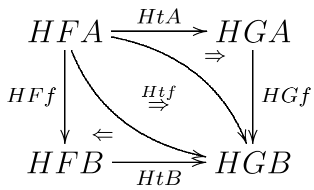
1.16. REMARKS.

1. The special case when $n = 0$ is part of (VT3). The case $n = 1$ is vacuous.

2. Double categories, oplax morphisms and vertical transformations form a 2-category. In fact, as shown in [GP], they are part of a strict double category $\mathbb{D}$oub whose horizontal arrows are oplax morphisms and whose vertical arrows are lax morphisms. This is remarkable, and another instance where the double category point of view is illuminating. Indeed, it is well-known that we cannot form a 2-category consisting of bicategories, oplax morphisms and any of the obvious choices of 2-cells (lax, oplax, or pseudo). (See for example [Lac].) Indeed, although oplax morphisms do compose in a strictly associative way (so we would obtain a 2-category rather than a bicategory if anything) whiskering does not work for any of these choices. Suppose that we have a transformation $t\colon F \to G$ for oplax morphisms $F, G\colon \mathcal{A} \to \mathcal{B}$ with components



where tf must be a cell, an isomorphism or an equality. If $H\colon \mathcal{B} \to \mathcal{C}$ is an oplax morphism then applying H to tf gives us



so this would not compose.

The correct 2-cells are the specialisations of those we introduced above, the vertical cells. In the case of bicategories, these look overly special, since their one-dimensional components are all identities (so there are only 2-cells between morphisms which agree on objects). However, they are not that special, and indeed occur throughout bicategory theory (see for example [CR]).

Let $\mathbf{Doub}(\mathbb{A},\mathbb{B})$ denote the category of double functors from $\mathbb{A}$ to $\mathbb{B}$ with vertical natural transformations, and $\mathbf{Doub}_{\mathrm{Opl}}(\mathbb{A},\mathbb{B})$ the category of oplax morphisms from $\mathbb{A}$ to $\mathbb{B}$ with vertical transformations.

1.17. Theorem. *For any double category $\mathbb{A}$, $\Xi\colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\,\mathbb{A}$ is the universal oplax morphism in the sense that composition with Ξ,*

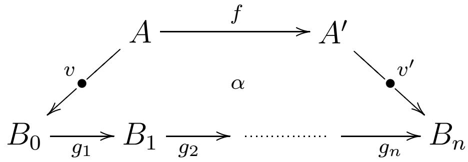
$$\Xi^*\colon \mathbf{Doub}(\mathbb{P}\mathrm{ath}\,\mathbb{A},\mathbb{B}) \to \mathbf{Doub}_{\mathrm{Opl}}(\mathbb{A},\mathbb{B})$$

is an isomorphism of categories.

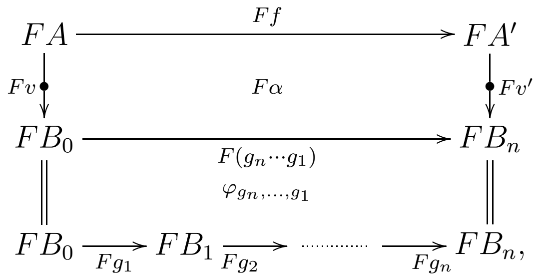
Proof. Given an oplax morphism $F\colon \mathbb{A} \to \mathbb{B}$, define $\Phi(F)\colon \mathbb{P}\mathrm{ath}\,\mathbb{A} \to \mathbb{B}$ to be the same as F on objects and vertical arrows. For a path $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$, define $\Phi(F)(\langle f_i \rangle)$ to be the composite

$$FA_0 \xrightarrow{Ff_1} FA_1 \xrightarrow{Ff_2} \cdots \xrightarrow{Ff_n} FA_n$$

(the empty path goes to 1_{FA_0}, of course). For a cell of the form

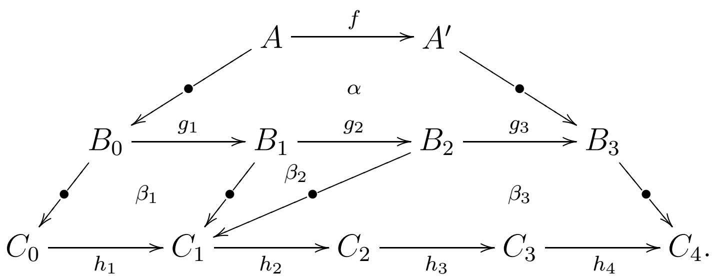


we assign the cell

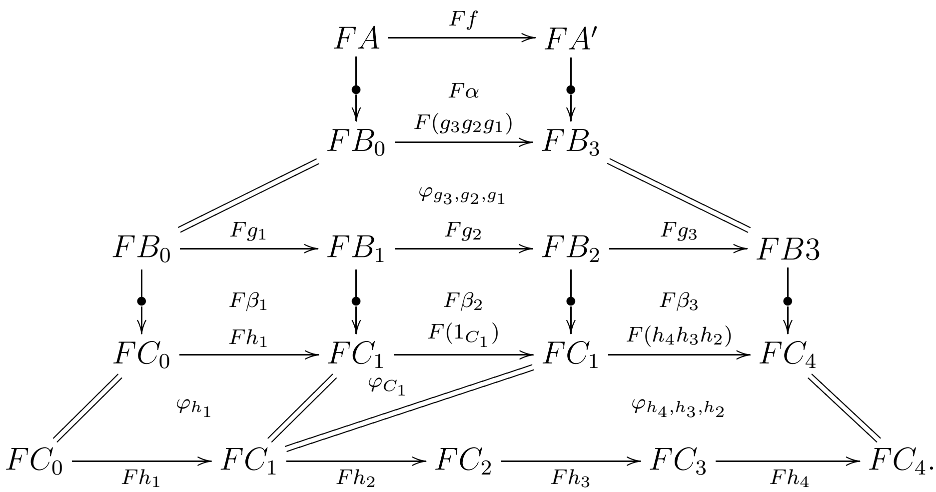


and extend $\Phi(F)$ to cells with vertical domains of arbitrary length, by horizontal composition.

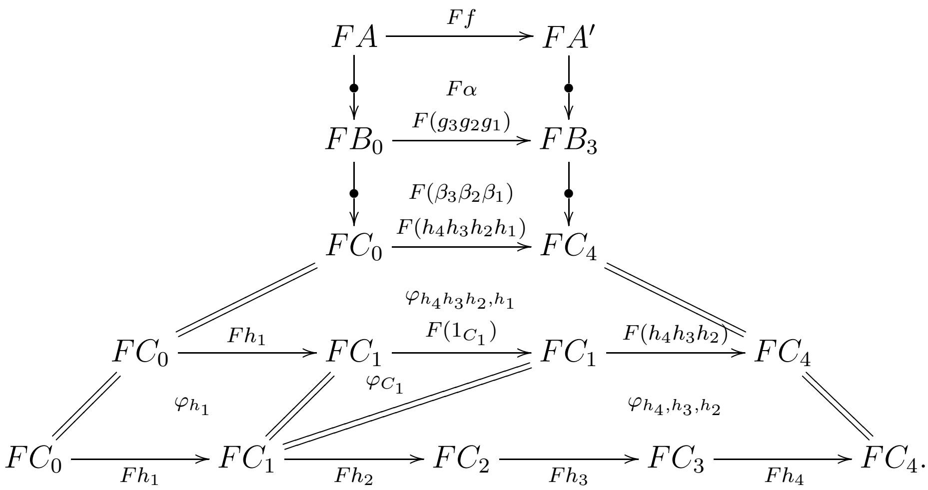
Now $\Phi(F)$ is functorial on vertical arrows because F is, and on horizontal arrows because $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ is free on these. $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ is also free for horizontal composition of cells, so $\Phi(F)$ is also functorial on these. We must verify that $\Phi(F)$ preserves vertical composition of cells (and vertical identities). A typical example will make this clear. Consider the vertical composite



The composite of the images in $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ is the pasting of

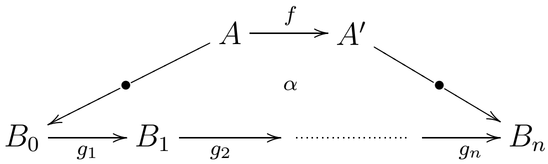


Now apply general naturality to the two middle rows to get

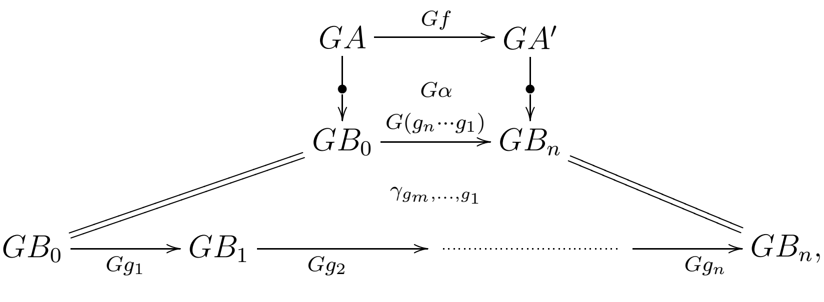


The top two squares compose to $F(\beta_3\beta_2\beta_1 \cdot \alpha)$, and the bottom row composes to $\varphi_{h_4,h_3,h_2,h_1}$ by general associativity, so the whole diagram pastes to $\Phi(F)(\beta_3\beta_2\beta_1 \cdot \alpha)$ as required. (Notwithstanding what we tell our students, sometimes an example is better than a proof.)

Now suppose that we start with a double functor $G\colon \mathbb{P}\mathrm{ath}\,\mathbb{A} \to \mathbb{B}$ and consider $\Phi\Xi^*(G) = \Phi(G \circ \Xi)$. On objects, vertical arrows, and horizontal paths of length one, $\Phi(G \circ \Xi)$ agrees with G, and by freeness, they agree on paths of arbitrary length as well. Also by freeness, we only have to check that they agree on cells of the form

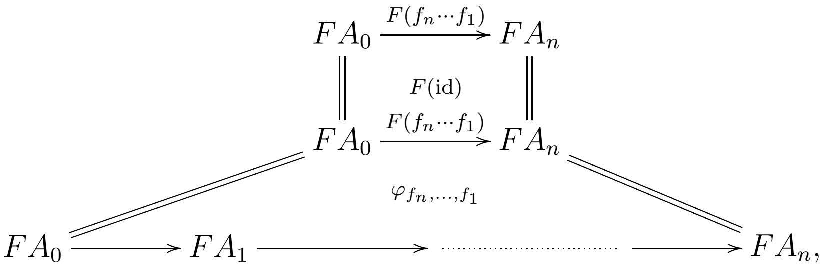


which $\Phi(G \circ \Xi)$ takes to



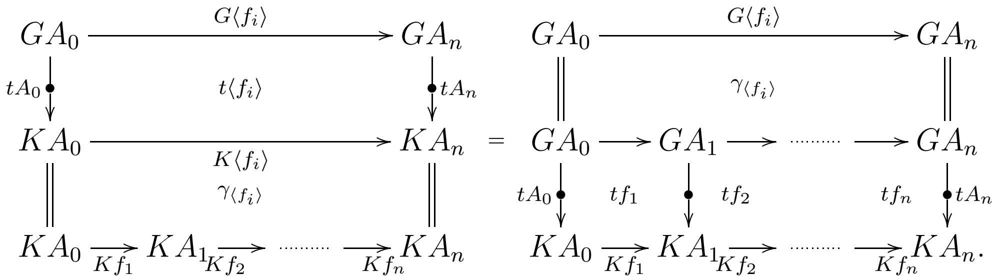
where γ is the generalised "codiagonal" expressing oplaxity of $G \circ \Xi$. The codiagonal for a composite is the composite of the codiagonals in the only sense possible, $(G, \gamma) \circ (F, \varphi) = (GF, G\varphi \cdot \gamma)$, which in this case gives $\gamma_{g_m,\ldots,g_1} = G(\mathrm{id}_{g_n \cdots g_1})$, so that $\Phi(G \circ \Xi)(\alpha) = G(\mathrm{id})G(\alpha) = g(\alpha)$. So $\Phi\Xi^*(G) = G$.

Starting with $F\colon \mathbb{A} \to \mathbb{B}$ an oplax morphism, we have that $\Xi^*\Phi(F) = \Phi(F) \circ \Xi$. Moreover, since $\Phi(F)$ agrees with F on objects, vertical arrows, vertical composition, horizontal arrows of length one, and cells whose codomain has length one, we only need to check that $\Xi^*\Phi(F)$ and F have the same oplaxity cells. For a path $\langle f_i \rangle$ in $\mathbb{A}$, the generalised oplaxity cell of $\Phi(F) \circ \Xi$ is $\Phi(F)(\xi_{f_n,\dots,f_1})$, which by definition of $\Phi(F)$ is



i.e., $\varphi_{f_n,\dots,f_1}$. So $\Phi(F) \circ \Xi = F$. This establishes that Ξ^* is bijective on objects with inverse Φ.

To finish the proof we will show that Ξ^* is full and faithful. Choose double functors $G, K\colon \mathbb{P}\mathrm{ath}\,(\mathbb{A}) \to \mathbb{B}$, and a vertical transformation $t\colon G \circ \Xi \longrightarrow K \circ \Xi$. Then t is specified on objects and horizontal morphisms of $\mathbb{A}$, *i.e.*, paths of length one. We wish to show that it extends uniquely to paths of arbitrary length. For any path $\langle f_i \rangle$, an extension of t would have to satisfy general functoriality, *i.e.*,



As G and K are double functors on $\mathbb{P}\mathrm{ath}\,(\mathbb{A})$, the cells $\gamma_{\langle f_i \rangle}$ and $\kappa_{\langle f_i \rangle}$ are actually vertical identities, so this completely determines $t\langle f_i \rangle$. It is straightforward to check that, thus extended, t is a vertical transformation $G \longrightarrow K$, thus completing the proof. ∎

1.18. The Comonad. Not surprisingly, $\mathbb{P}\mathrm{ath}$ is the functor part of a comonad on the category of double categories. The one-dimensional version, Path, studied in the prologue arose from the *underlying graph - free category* adjoint pair. Here we work in the reverse direction. First we get the comonad structure and then determine, in Section 3, what the two-dimensional analog of 'graph' is.

The universality of $\mathbb{P}\mathrm{ath}$ in the previous section expresses an adjointness which is the two dimensional version of the Kleisli category construction of the prologue. The Kleisli morphisms were "pretty poor morphisms", *i.e.*, 'functors' that don't preserve composition nor identities. In the current context these correspond to oplax morphisms, a much richer notion of morphism which occurs frequently in practice. Once again we see that the two-dimensional structure adds so much more.

Let $\mathcal{A}$ be a 2-category and $\mathbb{G} = (G, E, D)$ a 2-comonad on $\mathcal{A}$. The *Kleisli 2-category* $\mathcal{A}\mathbb{G}$ of $\mathbb{G}$ has the same objects as $\mathcal{A}$ and hom categories defined by

$$\mathcal{A}^{\mathbb{G}}(A, B) = \mathcal{A}(GA, B).$$

Vertical composition of cells is as in $\mathcal{A}$ and horizontal composition of arrows and cells is the usual Kleisli composition: $g \circ f = g(Gf)D$. There is an adjoint pair of 2-functors

$$\mathcal{A} \underset{U^{\mathbb{G}}}{\overset{R^{\mathbb{G}}}{\rightleftarrows}} \mathcal{A}^{\mathbb{G}}$$

$$U^{\mathbb{G}} \dashv R^{\mathbb{G}}$$

inducing the comonad $\mathbb{G}$; among those $\mathcal{A}^{\mathbb{G}}$ is characterised up to isomorphism by the mere fact that the right adjoint is bijective on objects.

Theorem 1.17 has as an immediate corollary the following result.

1.19. Proposition. *The inclusion,* **Doub** $\hookrightarrow$ $\mathbf{Doub}_{\mathrm{Opl}}$ *of the 2-category of double categories, double functors, and vertical transformations into the 2-category of double categories, oplax morphisms, and vertical transformations, has a 2-left adjoint* $\mathbb{P}\mathrm{ath}$ *with unit* Ξ. $\mathbf{Doub}_{\mathrm{Opl}}$ *is the Kleisli 2-category for the 2-comonad* $\mathbb{P}\mathrm{ath}$ *induced on* **Doub**.

1.20. Remark. The fact that $\mathbb{P}\mathrm{ath}$ is a 2-functor is a double category phenomenon. Of course the $\mathbb{P}\mathrm{ath}$ construction would work for 2-categories because 2-categories are special double categories, but vertical transformations look overly special in that setting. In any case, it is the vertical transformations that make things work, so even if we start with 2-categories we must consider them as double categories to get the two dimensional structure of **Doub** and $\mathbb{P}\mathrm{ath}$. In fact, $\mathbb{P}\mathrm{ath}$ is even better than a 2-comonad, it is one in which the structure morphisms are left adjoint to the counits, of the sort studied by Kock [K] and Zöberlein [Z]. Following Kelly and Lack [KL] we call them *oplax idempotent comonads.* (See also [M].)

Let us recall the definition. A 2-comonad $\mathbb{G} = (G, E, D)$ on a 2-category $\mathcal{A}$ is *oplax idempotent* if for every object A, there are 2-cells $\alpha\colon DA \cdot GEA \to 1_{G^2A}$ and $\beta\colon 1_{G^2A} \to DA \cdot EGA$ which together with the triangle identities $1_{GA} = GEA \cdot DA$ and $EGA \cdot DA = 1_{GA}$ give an adjoint triple

$$GEA \dashv DA \dashv EGA.$$

Wood [W] has shown that if $\varepsilon\colon UR \to 1_{\mathcal{B}}$ and $\eta\colon 1_{\mathcal{A}} \to RU$ are the adjunctions for an adjoint pair $\mathcal{A} \underset{R}{\overset{U}{\leftrightarrows}} \mathcal{B}$ of 2-functors, then there is a transformation $\varphi\colon \eta R \cdot R\varepsilon \to 1_{RUR}$ which together with the triangle identity $R\varepsilon \cdot \eta R = 1_R$ makes $R\varepsilon \dashv \eta R$, if and only if there is a transformation $\psi\colon U\eta \cdot \varepsilon U \to 1_{URU}$ which makes $U\eta \dashv \varepsilon U$ with $1_U \to \varepsilon U \cdot U\eta$ as unit, and in this case the comonad $(UR, \varepsilon, U\eta R)$ is an oplax idempotent. This is a useful criterion as it avoids having to deal with G^2 which can be quite complicated.

1.21. THEOREM. *The comonad* $\mathbb{P}\mathrm{ath}$ *on* **Doub** *is an oplax idempotent.*

PROOF. The comonad structure on $\mathbb{P}\mathrm{ath}$ is obtained from the adjoint pair of 2-functors, $\mathbb{P}\mathrm{ath}\colon \mathbf{Doub}_{\mathrm{Opl}} \to \mathbf{Doub}$, left adjoint to the inclusion $\mathbf{Doub} \hookrightarrow \mathbf{Doub}_{\mathrm{Opl}}$. The unit is $\Xi\colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\,\mathbb{A}$ and the counit is $\Gamma\colon \mathbb{P}\mathrm{ath}\,\mathbb{A} \to \mathbb{A}$ is composition. So it is sufficient to show that Γ is left adjoint to Ξ in $\mathbf{Doub}_{\mathrm{Opl}}$. The composite $\Gamma\Xi$ is the identity on $\mathbb{A}$, so our unit $\eta\colon 1_{\mathbb{A}} \longrightarrow\!\!\!\!\!\bullet\;\; \Gamma\Xi$ will be the identity vertical transformation. The counit $\varepsilon\colon \Xi\Gamma \longrightarrow\!\!\!\!\!\bullet\;\; 1_{\mathbb{P}\mathrm{ath}\,\mathbb{A}}$ is the identity on objects, and for every 1-cell in $\mathbb{P}\mathrm{ath}\,\mathbb{A}$, *i.e.*, a path $\langle f_i \rangle$, $\varepsilon(\langle f_i \rangle)$ is the cell

$$\begin{array}{ccccccccc}
A_0 & & & \xrightarrow{\quad f_n\cdots f_1 \quad} & & & & & A_n \\
{\scriptstyle \mathrm{id}_{A_0}}\big\downarrow\bullet & & & \mathrm{id}_{f_n,\ldots,f_1} & & & & & \bullet\big\downarrow{\scriptstyle \mathrm{id}_{A_n}} \\
A_0 & \xrightarrow[f_1]{} & A_1 & \xrightarrow[f_2]{} & A_2 & \longrightarrow & \cdots\cdots & \xrightarrow[f_n]{} & A_n.
\end{array}$$

Note that for paths of length 1, this is actually an identity so that $\varepsilon\Xi$ is the identity. This is one of the triangle equalities. Also notice that $\Gamma\varepsilon$ is the identity, and this is the other triangle equality. ∎

We saw in the prologue that the comonad Path on **Cat** has as its coalgebras free categories on graphs (the category of coalgebras is equivalent to the category of graphs), and that a category has at most one coalgebra structure, despite the fact that Path is not an idempotent comonad. This surprising phenomenon can now be understood as follows. If a category $\mathbf{A}$ is considered as a vertically discrete double category, then a coalgebra structure $F\colon \mathbf{A} \to \mathbb{P}\mathrm{ath}\,\mathbb{A}$ is the same as a coalgebra structure $\mathbf{A} \to \mathrm{Path}\mathbf{A}$, because $\mathbb{P}\mathrm{ath}\,\mathbf{A}$ and $\mathrm{Path}\,\mathbf{A}$ are the same at the level of horizontal arrows. As $\mathbb{P}\mathrm{ath}$ is an oplax idempotent, there can be at most one of these, a (vertical) left adjoint to $E\colon \mathbb{P}\mathrm{ath}\,\mathbb{A} \to \mathbb{A}$.

1.22. REMARK. $\mathbb{P}\mathrm{ath}\,\mathbf{A}$ has a nontrivial double structure even when $\mathbf{A}$ is just a category. Indeed, this is already seen for $\mathbb{P}\mathrm{ath}\,\mathbb{1}$. The vertical arrows are identities (so it is a 2-category), but a cell from a path $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ to a path $B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} B_m$ is an endpoint and order preserving function $\varphi\colon [n] \to [m]$ such that for every $i = 1, \ldots, n$, $f_i = g_{\varphi(i)}^{\varphi(i-1)}$ (so in particular, $B_{\varphi(i)} = A_i$). Although the composites that $\mathbf{A}$ has are forgotten in $\mathrm{Path}\,(\mathbf{A})$, they are remembered in $\mathbb{P}\mathrm{ath}\,(\mathbf{A})$ and, in fact, applying Π_0 locally to $\mathbb{P}\mathrm{ath}\,(\mathbf{A})$ gives us back $\mathbf{A}$ itself.

As the coalgebras for Path are graphs, and graphs are the building blocks for categories, it will be instructive to determine the Eilenberg-Moore coalgebras for the oplax idempotent comonad $\mathbb{P}\mathrm{ath}$ on **Doub**. They should be to oplax morphisms what graphs are to functors, *i.e.*, the true objects on which oplax morphisms are defined. After all, if morphisms are meant to be structure preserving, what can we make of oplax morphisms, which don't preserve horizontal composition? We shall see in Section 3 that the coalgebras are precisely Leinster's **fc**-multicategories, whose theory we review in the next section.

2. Oplax Double Categories

Oplax morphisms don't preserve composition in spite of the fact that morphisms should be structure preserving functions. In a situation such as this, it is often the case that what we do not sufficiently understand is the structure. If composition is not preserved, then it shouldn't be part of the structure. But we do need enough structure to yield oplaxness.

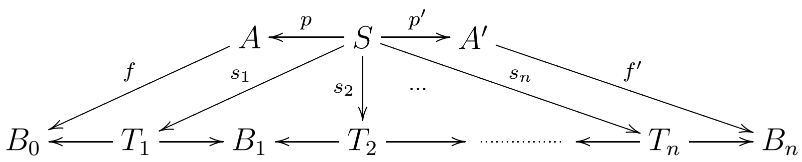
Our main example is the double category $\mathbb{S}\mathrm{pan}\,(\mathbf{A})$ which will be explored in detail in the sequel [DPP4] to the present work. $\mathbb{S}\mathrm{pan}\,(\mathbf{A})$ has the same objects as $\mathbf{A}$, its vertical arrows are the morphisms of $\mathbf{A}$, its horizontal arrows are spans in $\mathbf{A}$, and a typical cell looks like

$$\begin{array}{ccccc} A & \xleftarrow{p} & S & \xrightarrow{p'} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle f'} \\ B & \xleftarrow[q]{} & T & \xrightarrow[q']{} & B'. \end{array}$$

If $\mathbf{A}$ has pullbacks, then $\mathbb{S}\mathrm{pan}(\mathbf{A})$ is a (weak) double category. If $\mathbf{A}$ fails to have pullbacks, we don't have horizontal composition; however, we have not only all the rest of the double category structure (vertical composition of arrows and cells), but also the extra structure that would determine the horizontal composition if it existed. Since composition of spans is performed using pullbacks we know what a morphism into it should be; in fact we know this not just for composites of two spans, but for n-fold composites as well. Indeed, if T is the composite of the spans

$$B = B_0 \xleftarrow{q_1} T_1 \xrightarrow{q_1'} B_1 \xleftarrow{q_2} T_2 \xrightarrow{q_2'} B_2 \xleftarrow{q_3} \cdots \xleftarrow{q_n} T_n \xrightarrow{q_n'} B_n = B',$$

then cells as above are in bijective correspondence with diagrams



and it is clear from representability theory that a knowledge of such 'generalised cells' will determine the composite up to isomorphism, if it exists. We have vertical identities and a generalised composition: given n other generalised cells with domains T_i, they can be vertically composed with the above one, and this composition is associative. This is the kind of structure we want. It is simply the vertical dual of Leinster's **fc**-multicategories [L1, L2, L3], whose theory we now review. They are the double category version of what Hermida [H1] calls 'multicategories with several objects' and what Hermida, Makkai, and Power [HMP] call 'multitopic categories'. We propose to call them lax double categories for reasons hinted at above and which should become clear shortly. (Actually, we shall study the dual oplax double categories, because of the applications we have in mind.)

2.1. Multicategories. All of these concepts are an outgrowth of the notion of multicategory [Lam] which was inspired by the relationship between tensor products and multilinear maps. A (small) *multicategory* consists of a set of objects and set of *multiarrows* with domains finite strings of objects and codomains single objects. Given multiarrows $f\colon \langle B_1, \ldots, B_n\rangle \to C$ and $g_i\colon \langle A_1, \ldots, A_{m_i}\rangle \to B_i$, $i = 1, \ldots, n$, there is a composite $f(g_1, \ldots, g_n)\colon \langle A_i\rangle \to C$. This composition is associative and there are identities satisfying $1_C f = f = f(1_{B_1}, \ldots, 1_{B_n})$.

We can reformulate this in terms of the free monoid monad (T, η, μ). A multicategory consists of two sets A_0, A_1 (objects and arrows) with functions $\partial_0\colon A_1 \to TA_0$, $\partial_1\colon A_1 \to A_0$, $\iota\colon A_0 \to A_1$, and $\gamma\colon A_2 \to A_1$ (domain, codomain, identity, and composition). A_2 is defined by the pullback

$$\begin{array}{ccc} A_2 & \longrightarrow & TA_1 \\ \downarrow & \text{Pb} & \downarrow{\scriptstyle T\partial_1} \\ A_1 & \xrightarrow[\partial_0]{} & TA_0. \end{array}$$

The associativity and unit laws can easily be expressed diagrammatically. In fact, all of this can be carried out in any category with pullbacks and for any *cartesian monad* $\mathbb{T}$, *i.e.*, a monad $\mathbb{T} = (T, \eta, \mu)$ for which T preserves pullbacks and the naturality squares

$$\begin{array}{ccc} A & \xrightarrow{\eta A} & TA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ B & \xrightarrow[\eta B]{} & TB \end{array} \quad \text{and} \quad \begin{array}{ccc} T^2A & \xrightarrow{\mu A} & TA \\ {\scriptstyle T^2 f}\downarrow & & \downarrow{\scriptstyle Tf} \\ T^2B & \xrightarrow[\mu B]{} & TB \end{array}$$

are all pullbacks. This gives Burroni's notion of $\mathbb{T}$-multicategory [Bu].

In keeping with the theme of spans, let us introduce the double category $\mathbb{T}$-$\mathbb{S}$pan$(\mathbf{A})$ for any cartesian monad $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathbf{A}$ with pullbacks. The objects of $\mathbb{T}$-$\mathbb{S}$pan$(\mathbf{A})$ are those of $\mathbf{A}$. A horizontal arrow $A \longrightarrow\!\!\!\!\!\!+\;\; B$ is a span

$$\begin{array}{ccccc} & & S & & \\ & \swarrow & & \searrow & \\ TA & & & & B. \end{array}$$

A vertical arrow is just an arrow of $\mathbf{A}$, and a cell is a commutative diagram

$$\begin{array}{ccccc} TA & \longleftarrow & S & \longrightarrow & B \\ {\scriptstyle Ta}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle b} \\ TA' & \longleftarrow & S' & \longrightarrow & B'. \end{array}$$

Horizontal composition is given by pullback and composition with μ:

$$\begin{array}{ccccccccc} & & & & R\otimes S & & & & \\ & & & \swarrow & & \searrow & & & \\ & & TS & & \text{Pb} & & R & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & \\ & T^2A & & & TB & & & & C \\ {\scriptstyle \mu A}\swarrow & & & & & & & & \\ TA & & & & & & & & \end{array}$$

and identities are

$$\begin{array}{ccc} & A & \\ {\scriptstyle \eta A}\swarrow & & \searrow{\scriptstyle 1_A} \\ TA & & A. \end{array}$$

Horizontal composition of cells is given by the universal property of pullbacks. Proving the unit laws and associativity of horizontal composition is an easy exercise using the cartesianness of $\mathbb{T}$. Now, a $\mathbb{T}$-multicategory is exactly a horizontal monad in $\mathbb{T}$-$\mathbb{S}\text{pan}(\mathbf{A})$ and a $\mathbb{T}$-multifunctor is a vertical morphism of monads in $\mathbb{T}$-$\mathbb{S}\text{pan}(\mathbf{A})$.

Note that multifunctors are defined using vertical arrows, and so their construction uses the double structure of $\mathbb{T}$-$\mathbb{S}\text{pan}(\mathbf{A})$. When category objects are defined as monads in the *bicategory* $\mathbf{Span}(\mathbf{A})$ none of the various notions of morphism give functors. At best we get profunctors and at worst we get things never before observed in nature. Of course we want functors as our morphisms and the usual way of dealing with this is to define morphisms to be *maps* in $\mathbf{Span}(\mathbf{A})$ with left and right actions. While this does work in the present context, it is certainly *ad hoc*. There doesn't seem to be any overriding idea for doing this, except that it gives what we want. Strictly speaking, it doesn't even do that, the representability of maps being intimately related to Cauchy completeness.

2.2. Lax Double Categories. The following discussion can be found in Chapter 5 of Leinster's book [L2]. We include it here so as not to interrupt the logical flow of ideas. Consider the category of graphs, $\mathbf{Gph}$, and on it the free category monad, $\mathbb{T} = (T, \eta, \mu)$, introduced in the Prologue.

2.3. Proposition. $\mathbb{T}$ *is a cartesian monad.*

Proof. Let

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{P_1} & \mathbf{G}_1 \\ {\scriptstyle P_2}\downarrow & & \downarrow{\scriptstyle F_1} \\ \mathbf{G}_2 & \xrightarrow[F_2]{} & \mathbf{G}_0 \end{array}$$

be a pullback of graphs and consider

$$\begin{array}{ccc} T\mathbf{G} & \xrightarrow{TP_1} & T\mathbf{G}_1 \\ {\scriptstyle TP_2}\downarrow & & \downarrow{\scriptstyle TF_1} \\ T\mathbf{G}_2 & \xrightarrow[TF_2]{} & T\mathbf{G}_0\,. \end{array}$$

T is the identity on vertices, so there is no problem there. An edge in $T\mathbf{G}$ is a path of edges in $\mathbf{G}$ and an edge in $\mathbf{G}$ is a pair of edges $(A_1 \xrightarrow{f_1} B_1, A_2 \xrightarrow{f_2} B_2)$ in $\mathbf{G}_1 \times \mathbf{G}_2$ such that $F_1 f_1 = F_2 f_2$. A path of these is a pair of paths of the same length $(\langle f_{1i}\rangle, \langle f_{2i}\rangle)$ such that $F_1 f_{1i} = F_2 f_{2i}$ for all i. Since TF_1 and TF_2 preserve length and two paths in $T\mathbf{G}_0$ are equal if and only if they are identical, such a pair of paths is exactly an edge in the pullback of TF_1 and TF_2. So T preserves pullbacks.

Let $F\colon G \to G'$ be a morphism of graphs and consider the naturality square

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\eta\mathbf{G}} & T\mathbf{G} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle TF} \\ \mathbf{G}' & \xrightarrow[\eta\mathbf{G}']{} & T\mathbf{G}'. \end{array}$$

An edge of the pullback corresponds to an edge g of $\mathbf{G}'$ together with a path $\langle f_i\rangle$ in $\mathbf{G}$ such that $\langle Ff_i\rangle$ is g, *i.e.*, the path $\langle f_i\rangle$ has length one, and $Ff_1 = g$. Thus, an element of the pullback is completely determined by an edge of $\mathbf{G}$, *i.e.*, the above square is a pullback.

Also consider the square

$$\begin{array}{ccc} T^2\mathbf{G} & \xrightarrow{\mu\mathbf{G}} & T\mathbf{G} \\ {\scriptstyle T^2F}\downarrow & & \downarrow{\scriptstyle TF} \\ T^2\mathbf{G}' & \xrightarrow[\mu\mathbf{G}']{} & T\mathbf{G}'. \end{array}$$

An edge of the pullback of $T^2\mathbf{G}'$ and $T\mathbf{G}$ is a path of paths $(m, (n_j, \langle g_{ij}\rangle))$ in $\mathbf{G}'$ and a path $(p, \langle f_k\rangle)$ in $\mathbf{G}$ such that $(\sum_{j\in m} n_j, \langle g_{ij}\rangle) = (p, \langle Ff_k\rangle)$ which is entirely determined by a partition of p into m pieces, and the edges f_k, *i.e.*, a path of paths in $\mathbf{G}$ or an element of $T^2\mathbf{G}$. Thus the μ-naturality square is a pullback. ∎

2.4. Definition. *A* lax double category *is a* $\mathbb{T}$*-multicategory in* **Gph**.

In elementary terms, a lax double category has objects, horizontal and vertical arrows, and multicells. The vertical arrows form a category whereas the horizontal arrows don't. A multicell has domains and codomains as illustrated

$$\begin{array}{ccccccccc} B_0 & \xrightarrow{g_1} & B_1 & \xrightarrow{g_2} & B_2 & \longrightarrow & \cdots & \xrightarrow{g_n} & B_n \\ {\scriptstyle v}\downarrow & & & & \alpha & & & & \downarrow{\scriptstyle v'} \\ A & & & & \xrightarrow[f]{\hspace{6em}} & & & & A'. \end{array}$$

Such an α can be composed with n cells

$$\begin{array}{ccccccc} C_{i0} & \xrightarrow{h_{i1}} & C_{i1} & \xrightarrow{h_{i2}} & \cdots & \xrightarrow{h_{im_i}} & C_{im_i} \\ \downarrow & & & \beta_i & & & \downarrow \\ B_{i-1} & & & \xrightarrow[g_i]{\hspace{6em}} & & & B_i \end{array}$$

to give a cell

$$\begin{array}{ccccccccc} C_{10} & \xrightarrow{h_{11}} & C_{11} & \xrightarrow{h_{12}} & C_{12} & \longrightarrow & \cdots\cdots & \xrightarrow{h_{nm_n}} & C_{nm_n} \\ \big\downarrow & & & & \alpha\circ\langle\beta_i\rangle & & & & \big\downarrow \\ A & & & & \xrightarrow[f]{\hspace{6em}} & & & & A'. \end{array}$$

There are identity multicells

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \Big\| & \mathrm{id}_f & \Big\| \\ A & \xrightarrow[f]{} & A' \end{array}$$

and the composition is associative in the only sense it can be.

A lax double category in which all vertical arrows are identities is what Hermida [H1] calls a multicategory with several objects. A lax double category with a single object is the same as a multicategory as defined in [Lam] (see also [L2]).

2.5. Definition. *A* morphism of lax double categories $F\colon \mathbb{A} \to \mathbb{B}$ *takes arrows to arrows of the same type and cells to cells respecting domains and codomains, and preserving identities and vertical composition of arrows and cells.*

A double category $\mathbb{A}$ gives a lax double category $\mathbb{L}\mathrm{ax}\,\mathbb{A}$ if we take multicells

$$\begin{array}{ccccccc} B_0 & \xrightarrow{g_1} & B_1 & \xrightarrow{g_2} & \cdots\cdots & \xrightarrow{g_n} & B_n \\ \big\downarrow & & & \alpha & & & \big\downarrow v' \\ A & & & \xrightarrow[f]{\hspace{6em}} & & & A' \end{array}$$

to be double cells

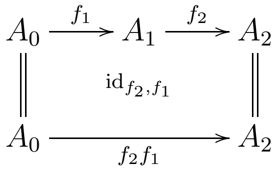
$$\begin{array}{ccc} B_0 & \xrightarrow{(g_n g_{n-1})\cdots g_2)g_1} & B_n \\ v\big\downarrow & \alpha & \big\downarrow v' \\ A & \xrightarrow[f]{\hspace{6em}} & A'. \end{array}$$

That we actually get a lax double category follows from the coherence theorem for (weak) double categories which is a minor variation on the coherence theorem for bicategories [DP2], and in fact is equivalent to it. This is one of Hermida's main points in [H1].

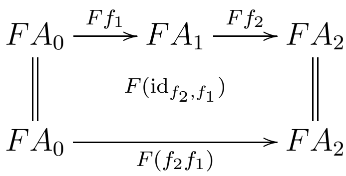
The following theorem is implicitly in [H1] (see 9.4 and 9.5 there).

2.6. Theorem. *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be double categories. A morphism of lax double categories* $\mathbb{L}\mathrm{ax}\,\mathbb{A} \to \mathbb{L}\mathrm{ax}\,\mathbb{B}$ *is the same as a lax morphism of double categories* $\mathbb{A} \to \mathbb{B}$.

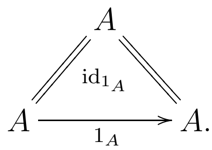
Proof. For $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2$ in $\mathbb{A}$, we have the multicell



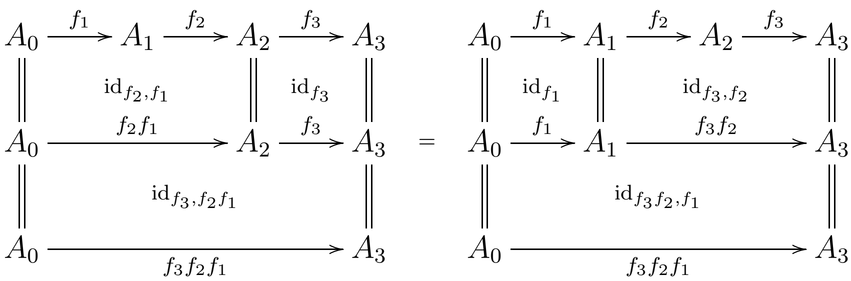
and applying F we get



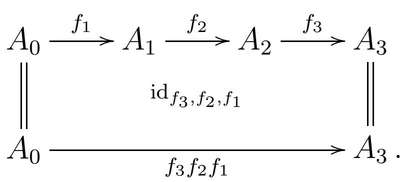
which we take as $\varphi_{f_2,f_1}\colon (Ff_2)(Ff_1) \to F(f_1 f_2)$. For $\varphi_A\colon 1_{FA} \to F(1_A)$ we take the image under F of



The coherence properties that φ must satisfy all follow from the fact that F is a morphism of lax double categories. For example, associativity of φ follows from the identity:



(since both are equal to



)

Naturality follows by applying F to both sides of the identity

$$\begin{array}{ccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 \\ \bullet\!\downarrow & \alpha & \bullet\!\downarrow & \beta & \bullet\!\downarrow \\ B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 \\ \| & & \text{id} & & \| \\ B_0 & & \longrightarrow & & B_2 \end{array} \quad = \quad \begin{array}{ccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 \\ \| & & \text{id} & & \| \\ A_0 & & \longrightarrow & & A_2 \\ \bullet\!\downarrow & & \beta\alpha & & \bullet\!\downarrow \\ B_0 & & \longrightarrow & & B_2. \end{array}$$

■

In order to extend the bijection in the previous theorem to an isomorphism of categories we must define what transformations of morphisms of lax double categories are. Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be morphisms of lax double categories. A *vertical transformation* $t\colon F \to\!\!\!\!\!\bullet\;\, G$ consists of the following data:

1. For each object A of $\mathbb{A}$, a vertical arrow $tA\colon FA \to\!\!\!\!\!\bullet\;\, GA$;
2. For each horizontal arrow $f\colon A \to A'$ in $\mathbb{A}$, a cell

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ {\scriptstyle tA}\,\bullet\!\downarrow & tf & \bullet\!\downarrow\,{\scriptstyle tA'} \\ GA & \xrightarrow[Gf]{} & GA', \end{array}$$

satisfying the following naturality conditions:

1. For each vertical arrow $v\colon A \to\!\!\!\!\!\bullet\;\, \overline{A}$ in $\mathbb{A}$,

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ {\scriptstyle Fv}\,\bullet\!\downarrow & & \bullet\!\downarrow\,{\scriptstyle Gv} \\ F\overline{A} & \xrightarrow[t\overline{A}]{} & G\overline{A} \end{array}$$

   commutes;

2. For each multicell

$$\begin{array}{ccccccccc} A_0 & \xrightarrow{f_1} & A_1 & \xrightarrow{f_2} & A_2 & \longrightarrow & \cdots & \xrightarrow{f_n} & A_n \\ {\scriptstyle v}\,\bullet\!\downarrow & & & & \alpha & & & & \bullet\!\downarrow\,{\scriptstyle w} \\ B & & & & \xrightarrow[g]{} & & & & B' \end{array}$$

we have

$$
\begin{array}{ccccccccc}
FA_0 & \xrightarrow{Ff_1} & FA_1 & \xrightarrow{Ff_2} & FA_2 & \longrightarrow & \cdots & \xrightarrow{Ff_n} & FA_n \\
{\scriptstyle v}\big\downarrow\!\bullet & & & & F\alpha & & & & \bullet\!\big\downarrow{\scriptstyle Fw} \\
FB & & & & \xrightarrow[Fg]{\qquad\qquad\qquad} & & & & FB' \\
{\scriptstyle tB}\big\downarrow\!\bullet & & & & tg & & & & \bullet\!\big\downarrow{\scriptstyle tB'} \\
GB & & & & \xrightarrow[Gg]{\qquad\qquad\qquad} & & & & GB'
\end{array}
$$

$$
= \begin{array}{ccccccccccc}
FA_0 & \xrightarrow{Ff_1} & FA_1 & \xrightarrow{Ff_2} & FA_2 & \longrightarrow & \cdots & \longrightarrow^{Ff_{n-1}} & FA_{n-1} & \xrightarrow{Ff_n} & FA_n \\
{\scriptstyle tA_0}\big\downarrow\!\bullet & tf_1 & {\scriptstyle tA_1}\big\downarrow\!\bullet & tf_2 & {\scriptstyle tA_2}\big\downarrow\!\bullet & & & & {\scriptstyle tA_{n-1}}\big\downarrow\!\bullet & tf_n & \bullet\!\big\downarrow{\scriptstyle tA_n} \\
GA_0 & \xrightarrow[Gf_1]{} & GA_1 & \xrightarrow[Gf_2]{} & GA_2 & \longrightarrow & \cdots & \longrightarrow & GA_{n-1} & \xrightarrow[Gf_n]{} & GA_n \\
{\scriptstyle Gv}\big\downarrow\!\bullet & & & & & G\alpha & & & & & \bullet\!\big\downarrow{\scriptstyle Gw} \\
GB & & & & & \xrightarrow[Gg]{\qquad\qquad\qquad} & & & & & GB'.
\end{array}
$$

For double categories $\mathbb{A}$ and $\mathbb{B}$ and morphisms $F, G\colon \mathbb{L}\mathrm{ax}\,\mathbb{A} \to \mathbb{L}\mathrm{ax}\,\mathbb{B}$, vertical transformations $t\colon F \multimap G$ correspond to vertical transformations of oplax morphisms of double categories.

We have now defined the correct structure so that the natural morphisms are lax morphisms. In the process we have taken care of coherence as well.

In the notion of lax double category we have enough structure to recover horizontal composition if it is there. Indeed,

$$
\begin{array}{ccccc}
A_0 & \xrightarrow{f_1} & A_1 & \xrightarrow{f_2} & A_2 \\
\Big\| & & \mathrm{id}_{f_2,f_1} & & \Big\| \\
A_0 & & \xrightarrow[f_2 f_1]{\qquad\qquad} & & A_2
\end{array}
$$

is a universal cell in the sense that vertical composition with it gives a natural bijection between cells of the form

$$
\begin{array}{ccc}
A_0 & \xrightarrow{f_2 f_1} & A_2 \\
\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow \\
B_0 & \xrightarrow[g]{} & B'
\end{array}
$$

and cells of the form

$$
\begin{array}{ccccc}
A_0 & \xrightarrow{f_1} & A_1 & \xrightarrow{f_2} & A_2 \\
\big\downarrow\!\bullet & & \alpha & & \bullet\!\big\downarrow \\
B & & \xrightarrow[g]{\qquad\qquad} & & B'.
\end{array}
$$

This universal property determines $f_2 f_1$ uniquely up to isomorphism. However, as Hermida makes clear, this condition is not enough in general. If we wish to show that composition defined this way is associative (up to coherent isomorphism) we need what he calls strong representability. Our definition follows the lines in Hermida [H1].

2.7. DEFINITION. *Let* $\mathbb{A}$ *be a lax double category and*

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots\cdots \xrightarrow{f_n} A_n$$

a path of arrows of $\mathbb{A}$. *We say that* the composite of the path is strongly representable *if there is an arrow* $f\colon A_0 \to A_n$ *and a multicell*

$$\begin{array}{ccc} A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} & \cdots\cdots & \xrightarrow{f_n} A_n \\ \Big\| & \eta_{\langle f_n,\ldots,f_1\rangle} & \Big\| \\ A_0 & \xrightarrow[\quad f \quad]{} & A_n \end{array}$$

such that for any paths $\langle x_1,\ldots,x_m\rangle$, *and* $\langle y_p,\ldots,y_1\rangle$, *and cell* α *as below*

$$\begin{array}{ccc} X_m \xrightarrow{x_m} \cdots \longrightarrow X_1 \xrightarrow{x_1} A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n \xrightarrow{y_1} Y_1 \xrightarrow{y_2} \cdots \longrightarrow & & Y_p \\ v \big\downarrow & \alpha & \big\downarrow w \\ B & \xrightarrow[\qquad g \qquad]{} & C \end{array}$$

there is a unique cell $\overline{\alpha}$

$$\begin{array}{ccc} X_m \xrightarrow{x_m} \cdots \longrightarrow X_1 \xrightarrow{x_1} A_0 \xrightarrow{f} A_n \xrightarrow{y_1} Y_1 \xrightarrow{y_2} \cdots \longrightarrow & & Y_p \\ v \big\downarrow & \overline{\alpha} & \big\downarrow w \\ B & \xrightarrow[\qquad g \qquad]{} & C \end{array}$$

such that

$$\begin{array}{ccccccccccc} X_m \xrightarrow{x_m} X_{m-1} \xrightarrow{x_{m-1}} & \cdots & X_1 \xrightarrow{x_1} A_0 & \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} & A_n \xrightarrow{y_1} Y_1 \xrightarrow{y_2} & \cdots & \xrightarrow{y_p} Y_p \\ \Big\| \;\; id \;\; \Big\| & & \Big\| \;\; id \;\; \Big\| & \eta_{\langle f_m,\ldots,f_1\rangle} & \Big\| \;\; id \;\; \Big\| & & \Big\| \\ X_m \xrightarrow{x_m} X_{m-1} \xrightarrow{x_{m-1}} & \cdots & X_1 \xrightarrow{x_1} A_0 & \xrightarrow[\qquad f \qquad]{} & A_n \xrightarrow{y_1} Y_1 \xrightarrow{y_2} & \cdots & \xrightarrow{y_p} Y_p \\ v \big\downarrow & & & \overline{\alpha} & & & \big\downarrow w \\ B & & & \xrightarrow[\qquad g \qquad]{} & & & C \end{array}$$

is equal to α.

We say that the composite of the path $\langle f_n,\ldots,f_1\rangle$ is representable *if we require the above condition only in the case where* $m = p = 0$, *i.e., when there are no* x*'s or* y*'s.*

We will usually choose one representing f and call it $f_n f_{n-1} \cdots f_1$. When $n = 1$, the composite is represented by f_1. When $n = 0$, a case we will be particularly interested in, we say that the identity 1_{A_0} is (strongly) representable. Aside from the case $n = 0$, we will be mostly interested in the case $n = 2$. The following proposition is essentially due to Hermida ([H1], 11.4).

2.8. Proposition. *A lax double category is of the form* $\mathbb{L}\mathrm{ax}\,\mathbb{A}$ *if and only if all composites are strongly representable.*

2.9. Representability of composition. It may appear from the literature so far that strong representability is the only useful condition, and in fact it is the only one we will use in this paper. However, plain representability is also interesting. In fact there is something mysterious about strong representability; after all, representability on its own does determine the composites up to isomorphism. Without strong representability we do not get associativity of composition up to isomorphism. We do get, however, comparison cells $fgh \to (fg)h$ and $fgh \to f(gh)$. More generally, given a path of paths of morphisms $f_{1,1} \ldots f_{1,n_1} \ldots f_{m,1} \ldots f_{m,n_m}$ we have comparison morphisms

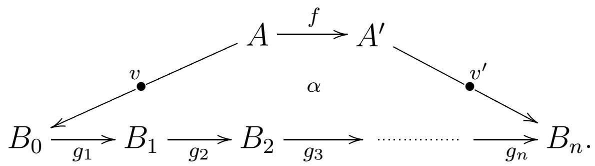
$$f_{m,n_m} \cdots f_{1,1} \to (f_{m,n_m} \cdots f_{m,1}) \cdots (f_{1,n_1} \cdots f_{1,1}) \tag{5}$$

satisfying the obvious associativity conditions. Thus we obtain the double category version of (the dual of) Leinster's lax bicategories (see [L2], p. 121). Let us call these *slack double categories* so as to avoid possible confusion with what we call lax double categories and he calls **fc**-multicategories. The relationship between these two concepts is this. Given a slack double category $\mathbb{A}$ we can construct a lax one, $\mathbb{L}\mathrm{ax}\,\mathbb{A}$, as in the paragraph following Definition 2.5. The composition of multicells there only requires the comparison morphisms (5). Then Proposition 2.8 can be modified to say that a lax double category is of the form $\mathbb{L}\mathrm{ax}\,\mathbb{A}$ for a slack double category $\mathbb{A}$ if and only if all composites are (weakly) representable.

There are also Grandis' related notions of lax bicategories which have comparison cells between certain pairs of associations of composites in a preferred direction (which is chosen to model problems in concurrency theory) [G]. These form a special case of Leinster's lax bicategories where some of the comparison cells are identities.

We believe that our notion of lax double category is the most basic one from which the others can best be understood and so we suggest using this name rather than that of **fc**-multicategory, since the latter does not reflect the central role of this concept.

2.10. Oplax double categories. The example that motivated this section is that of $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ where $\mathbf{A}$ is a category which does not necessarily have pullbacks and this yields an *oplax double category* rather than a lax one. This is simply the vertical dual of the notion of lax double category in which multicells look like



The $\mathbb{S}$pan construction has been central to our work and has guided us in our choice of orientation for the construction of $\Pi_2\mathbf{A}$ ([DPP1]) and consequently $\mathbb{P}\mathrm{ath}\,\mathbb{A}$. Thus we now switch to oplax double categories.

Given a double category $\mathbb{A}$ we now have an oplax double category $\mathbb{O}\mathrm{plax}\,\mathbb{A}$ associated to it where a multicell α as above is a cell

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ v \big\downarrow & \alpha & \big\downarrow v' \\ B_0 & \xrightarrow[g_n^0]{} & B_n \end{array}$$

in $\mathbb{A}$. Morphisms of oplax double categories are the obvious ones (dual to the lax case) as are vertical transformations. It is easy to see that $\mathbb{O}\mathrm{plax}$ extends to double functors and vertical transformations of such.

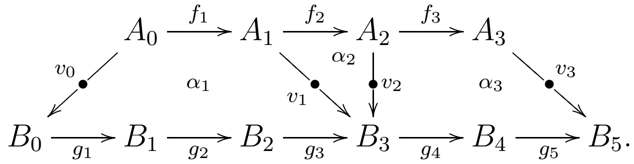
We introduced the 2-categories **Doub** and $\mathbf{Doub}_{\mathrm{Opl}}$ in Section 1. We also have the larger 2-category **Oplax** of oplax double categories, whose objects are oplax double categories, arrows are morphisms of such, and 2-cells are vertical transformations.

2.11. PROPOSITION. $\mathbb{O}\mathrm{plax}$ *is a locally full and faithful 2-functor* **Doub** $\to$ **Oplax**.

PROOF. The only thing that is not completely obvious is the locally full and faithful part, and that is an easy calculation. ∎

Proposition 1.19 of Section 1 showed that $\mathbb{P}\mathrm{ath}$ is left adjoint to the inclusion **Doub** $\hookrightarrow$ $\mathbf{Doub}_{\mathrm{Opl}}$. We now show that $\mathbb{P}\mathrm{ath}$ extends to **Oplax** and is 2-left adjoint to $\mathbb{O}\mathrm{plax}$. This is the natural setting for $\mathbb{P}\mathrm{ath}$. An oplax double category has all the ingredients for a double category except for the horizontal composition. Generating the free double category from one would necessarily involve paths, ordinary paths of 1-cells and paths of double cells, and perhaps some bookkeeping complications for double cells. Our $\mathbb{P}\mathrm{ath}$ construction is exactly the right thing. In fact, it doesn't use horizontal composition of arrows or cells.

Explicitly, let $\mathbb{A}$ be our oplax double category. We construct a (strict) double category $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ with the same objects and vertical arrows as $\mathbb{A}$. A horizontal arrow is a path of horizontal arrows in $\mathbb{A}$ and a double cell is a horizontal path of multicells in $\mathbb{A}$. The vertical domain and codomain of such a path are the path of domains and the concatenation of the paths of the codomains (respectively). For example, a typical cell might look like



The domains and codomains are clear. Vertical composition of cells uses the multicell composition of $\mathbb{A}$, in the obvious way. Both horizontal structures are free categories; vertical composition of arrows also forms a category because it is the same as in $\mathbb{A}$. It is easy to show that vertical composition of cells also forms a category and to check the middle four interchange law.

A loftier point of view will make this all transparent. For a cartesian monad $\mathbb{T}$ in a category $\mathbf{A}$, we have not only the double category $\mathbb{T}$-$\mathbb{S}$pan$(\mathbf{A})$ but also the the double category $\mathbb{S}\mathrm{pan}(\mathbf{A}^{\mathbb{T}})$ of spans in the Eilenberg-Moore category $\mathbf{A}^{\mathbb{T}}$ of $\mathbf{A}$, and a double functor $\tilde{T}\colon \mathbb{T}\text{-}\mathbb{S}\mathrm{pan}(\mathbf{A}) \to \mathbb{S}\mathrm{pan}(\mathbf{A}^{\mathbb{T}})$. $\tilde{T}$ is defined by $\tilde{T}(A) = (TA, \mu A)$ and $\tilde{T}(f) = T(f)$ for an object A and a vertical arrow f of $\mathbb{T}$-$\mathbb{S}$pan. For a $\mathbb{T}$-Span, $TA \xleftarrow{p} S \xrightarrow{g} B$, $\tilde{T}$ is defined as

$$(TA, \mu A) \xleftarrow{\mu A} (T^2A, \mu TA) \xleftarrow{Tp} (TS, \mu S) \xrightarrow{Tg} (TB, \mu B)$$

and for a cell

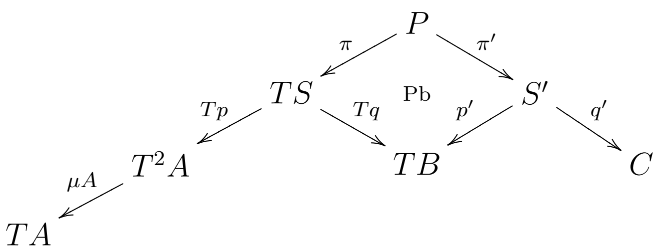
$$\begin{array}{ccccccc} A & \quad & TA & \xleftarrow{p} & S & \xrightarrow{q} & B \\ {\scriptstyle f}\downarrow & & {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle g} \\ A' & & TA' & \xleftarrow[p']{} & S' & \xrightarrow[q']{} & B', \end{array}$$

$\tilde{T}(f, x, g) = (Tf, Tx, Tg)$.

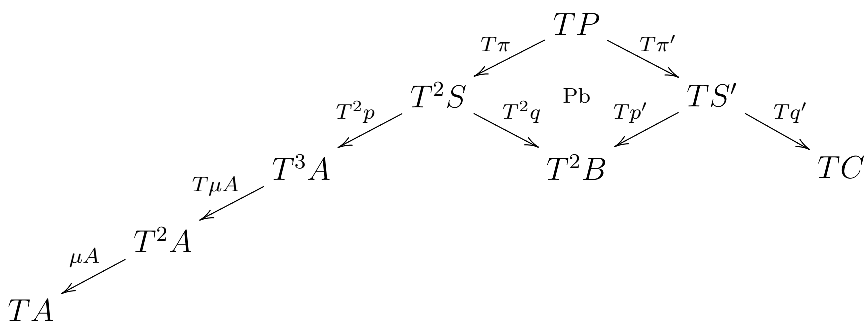
For the identity span $TA \xleftarrow{\eta A} A \xrightarrow{1_A} A$ we get

$$TA \xleftarrow{\mu A} T^2A \xleftarrow{T\eta A} TA \xrightarrow{T1_A} TA$$

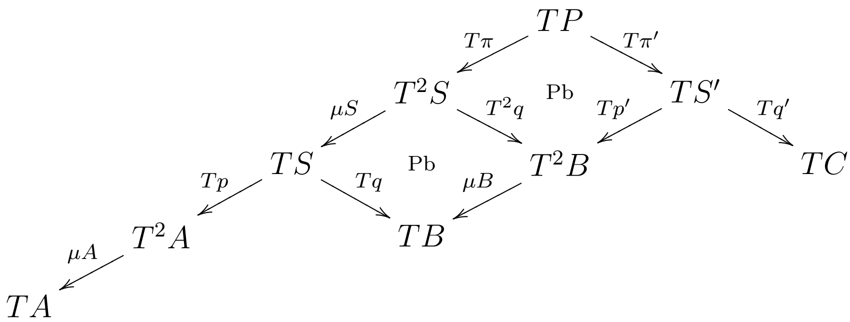
which is the identity in $\mathbb{S}\mathrm{pan}(\mathbf{A}^{\mathbb{T}})$. A composite



is transformed into



whereas the composite $\tilde{T}(S')\tilde{T}(S)$ is given by



where the two squares are pullbacks because $\mathbb{T}$ is cartesian. Finally, the left legs are equal because

$$\begin{array}{ccccc} T^2S & \xrightarrow{T^2p} & T^3A & \xrightarrow{T\mu A} & T^2A \\ {\scriptstyle \mu S}\downarrow & & {\scriptstyle \mu TA}\downarrow & & \downarrow{\scriptstyle \mu A} \\ TS & \xrightarrow[Tp]{} & T^2A & \xrightarrow[\mu A]{} & TA \end{array}$$

commutes.

So $\tilde{T}\colon \mathbb{T}\text{-}\mathbb{S}\mathrm{pan}\,(\mathbf{A}) \to \mathbb{S}\mathrm{pan}(\mathbf{A}^{\mathbb{T}})$ is indeed a double functor and as such preserves monads and their morphisms. If $\mathbf{A}$ is the category of graphs and $\mathbb{T}$ is the free category monad on it, then $\mathbb{T}\text{-}\mathbb{S}\mathrm{pan}\,(\mathbf{A})$ is the double category of lax double categories, $\mathbf{A}^{\mathbb{T}}$ is the category of categories so a monad in $\mathbb{S}\mathrm{pan}(\mathbf{A}^{\mathbb{T}})$ is a double category. This is the higher reason why we can apply the $\mathbb{P}\mathrm{ath}$ construction to a lax double category, and by duality, to an oplax double category to get a genuine double category, $\mathbb{P}\mathrm{ath}\,\mathbb{A}$.

2.12. Theorem. *$\mathbb{P}\mathrm{ath}$ is the object part of a 2-functor* **Oplax** $\to$ **Doub** *which is a 2-left adjoint to the inclusion* $\mathbb{O}\mathrm{plax}$: **Doub** $\to$ **Oplax**.

Proof. That $\mathbb{P}\mathrm{ath}$ is a 2-functor with the obvious extension to morphisms of oplax double categories and vertical transformations is a straightforward calculation.

Let $\mathbb{A}$ be an arbitrary oplax double category. Define $\Xi\colon \mathbb{A} \to \mathbb{O}\mathrm{plax}\,\mathbb{P}\mathrm{ath}\,\mathbb{A}$ to be the identity on objects and vertical arrows and to take horizontal arrows and cells to singleton paths of the same. This is clearly a morphism of oplax double categories.

Given a double category $\mathbb{B}$ we will show that

$$\tilde{\Xi}\colon \mathbf{Doub}\,(\mathbb{P}\mathrm{ath}\,\mathbb{A}, \mathbb{B}) \to \mathbf{Oplax}\,(\mathbb{A}, \mathbb{O}\mathrm{plax}\,\mathbb{B})$$

defined by $\tilde{\Xi}(F) = \mathbb{O}\mathrm{plax}(F) \cdot \Xi$ is an isomorphism of categories. ∎

The referee has pointed out that the above adjunction can also be seen as part of our previously described “loftier point of view”. Indeed, one can extend the usual forgetful functor $\mathbf{A}^{\mathbb{T}} \to \mathbf{A}$ to a lax double functor $\tilde{U}$, which is right adjoint to $\tilde{T}$ in an appropriate double categorical sense. Then taking monads it will induce the inclusion of categories, and yield the adjunction of Theorem 2.12. See [L2], Section 6.6 for further details.

2.13. Theorem. $\mathbb{P}\mathrm{ath} : \mathbf{Oplax} \to \mathbf{Doub}$ *is 2-comonadic.*

Proof. The 2-comonad on **Doub** induced by the adjunction

$$\mathbb{P}\mathrm{ath} \dashv \mathbb{O}\mathrm{plax}$$

is what we called $\mathbb{P}\mathrm{ath}$ in Section 1.18 where we showed that it was oplax idempotent.

A coalgebra for the $\mathbb{P}\mathrm{ath}$ comonad is a double category $\mathbb{A}$ with a double functor $F\colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\,\mathbb{A}$ satisfying the counit law

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{F} & \mathbb{P}\mathrm{ath}\,\mathbb{A} \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle E} \\ & & \mathbb{A} \end{array}$$

and coassociativity

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{F} & \mathbb{P}\mathrm{ath}\,\mathbb{A} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathbb{P}\mathrm{ath}\,F} \\ \mathbb{P}\mathrm{ath}\,\mathbb{A} & \xrightarrow[D]{} & \mathbb{P}\mathrm{ath}^2\,\mathbb{A}. \end{array}$$

The first condition forces F to be the identity on objects and vertical arrows. It also says that F gives a functorial factorisation of horizontal arrows and double cells into n-fold composites. In particular, this assigns a grading to horizontal arrows and double cells. As in the proof of the one-dimensional case from the prologue, this makes the horizontal arrows into a free category on the graph of degree 1 arrows, and similarly for the cells under horizontal composition.

In fact, the grading is given by $\mathbb{A} \xrightarrow{F} \mathbb{P}\mathrm{ath}\,\mathbb{A} \xrightarrow{\mathbb{P}\mathrm{ath}\,(!)} \mathbb{P}\mathrm{ath}\,\mathbb{1}$. It will be recalled from Section 1 that a cell of degree 1 in $\mathbb{P}\mathrm{ath}\,\mathbb{1}$ has the form

$$\begin{array}{ccc} * & \xrightarrow{1} & * \\ \| & & \| \\ * & \xrightarrow[n]{} & * \end{array}$$

which tells us that a cell of degree 1 in $\mathbb{A}$ has boundary

$$\begin{array}{ccccccc} & A & & \xrightarrow{f} & & A' & \\ {\scriptstyle v}\swarrow & & & \alpha & & & \searrow{\scriptstyle v'} \\ B_0 & \xrightarrow[g_1]{} & B_1 & \xrightarrow[g_2]{} & \cdots\cdots & \xrightarrow[g_n]{} & B_n \end{array}$$

where f and each g_i has itself degree 1. It follows that the degree 1 cells form an oplax double category $\mathbb{A}(1)$ and that $\mathbb{A} \cong \mathbb{P}\mathrm{ath}\,\mathbb{A}(1)$. ■

2.14. Remark. We interpret this theorem as validating our claim that oplax double categories are precisely the right structure on which to define oplax morphisms. In particular, **Oplax** is as cocomplete as **Doub** which has all 2-colimits, *i.e.*, **Oplax** is 2-cocomplete. It is also 2-complete because **Doub** is and the comonad $\mathbb{P}$ath preserves pullbacks as is easily seen. Note that $\mathbf{Doub}_{\mathrm{Opl}}$ is neither complete nor cocomplete, a problem that surfaces immediately when working with oplax morphisms. However, we know from monad theory that every coalgebra is an equaliser of two cofree coalgebras, in fact that equaliser is a pullback, even an intersection, so that **Oplax** is a completion of $\mathbf{Doub}_{\mathrm{Opl}}$ under binary intersections. Simply adding intersections makes it complete and cocomplete.

It should also be remarked that since the comonad $\mathbb{P}$ath is oplax idempotent, it is a property of a double category to be $\mathbb{P}$ath of an oplax one, not extra structure.

3. $\mathbb{P}\mathrm{ath}_*$

The pointed case is more complicated but also more interesting and is at the heart of our Π_2 construction. Now we are dealing with the case where identities are preserved. Thus we are in the situation analogous to free categories generated by reflexive graphs, the Path_* construction from the Prologue. Already, in the one dimensional case, there were some complications involving the equivalence relation on paths, but the situation was quite simple and equivalence classes had canonical representatives. In the two dimensional case we do not have to worry about equivalent paths of arrows, but the equivalence relation on cells is much more complicated.

3.1. Definition. *An oplax morphism of double categories* $F\colon \mathbb{A} \to \mathbb{B}$ *is called* normal *if for every* A *the given cell*

$$\begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ \| & \varphi_A & \| \\ FA & \xrightarrow[1_{FA}]{} & FA \end{array}$$

is vertically invertible.

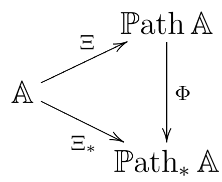
The problem which we shall address is the construction of the universal oplax normal morphism

$$\mathbb{A} \xrightarrow{\Xi_*} \mathbb{P}\mathrm{ath}_*\, \mathbb{A}$$

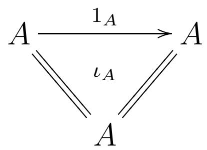
for any double category $\mathbb{A}$.

Normality is important. Without it we can do almost nothing; and we note in particular that our motivating example, $\mathbf{A} \to \mathbb{S}\mathrm{pan}\mathbf{A}$, is normal. This is more or less equivalent to preservation of adjoints. More precisely, a morphism is normal if and only if it preserves adjoints and companions [GP], as will be shown in [DPP4].

3.2. The Necessity of the Equivalence Relation. It is clear that the universal oplax *normal* morphism, $\Xi_*\colon \mathbb{A} \to \mathbb{P}\mathrm{ath}_* \mathbb{A}$ exists as it is described by operations and equations. Our problem is to get a concrete workable description of it. We have already constructed the universal oplax morphism, $\Xi\colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\, \mathbb{A}$ and this will be our starting point. Universality of Ξ gives a unique morphism of double categories, Φ, such that

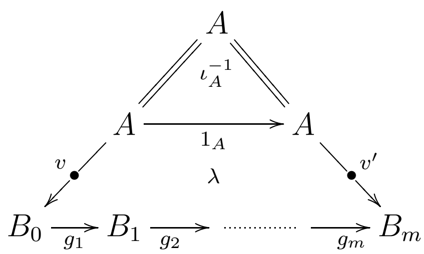


commutes. So $\mathbb{P}\mathrm{ath}_* \mathbb{A}$ will require some inverses that are not present in $\mathbb{P}\mathrm{ath}\, \mathbb{A}$, and these in turn will require certain elements to be identified. Specifically, we must add an inverse for each canonical morphism

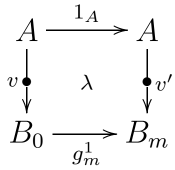


in $\mathbb{P}\mathrm{ath}\, \mathbb{A}$, and then all double categorical consequences of this. Thus the objects and vertical arrows on $\mathbb{P}\mathrm{ath}_* \mathbb{A}$ are those of $\mathbb{A}$, and the horizontal arrows are paths of arrows of $\mathbb{A}$, just like the arrows of $\mathbb{P}\mathrm{ath}\, \mathbb{A}$. However, the double cells are equivalence classes of words in the cells of $\mathbb{P}\mathrm{ath}\, \mathbb{A}$ and the ι_A^{-1}, constructed using horizontal and vertical composition. This, of course, is potentially very complicated. The following proposition hints that the problem may be tractable after all.

There are two simple kinds of words, those that come from single cells in $\mathbb{P}\mathrm{ath}\, \mathbb{A}$, *i.e.*, fences $(\varphi, \langle v_i\rangle, \langle \alpha_i\rangle)\colon \langle f_i\rangle \to \langle g_j\rangle$, which we call *words of type 1*, and those of the form



where λ is a double cell

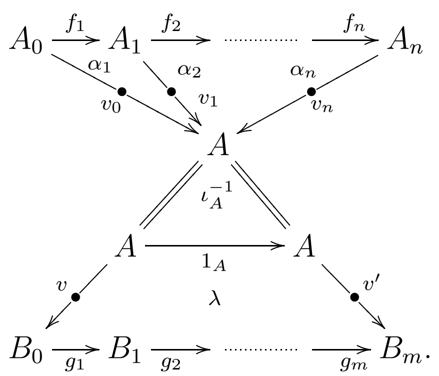


in $\mathbb{A}$, which we call *words of type 2*.

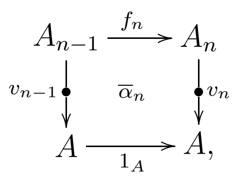
3.3. Proposition. *Each cell of* $\mathbb{P}\mathrm{ath}_* \mathbb{A}$ *with vertical domain of length greater than 0 has a representative of type 1; a cell with vertical domain of length 0 has a representative of type 2.*

Proof. It is clear that the cells of $\mathbb{P}\mathrm{ath}_* \mathbb{A}$ are generated by equivalence classes of words of type 1 and 2, so it will be sufficient to show that they are closed under horizontal and vertical composition.

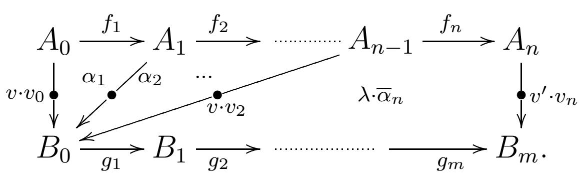
Both horizontal and vertical composition of two words of type 1 yield words of type 1 as they come from $\mathbb{P}\mathrm{ath}\, \mathbb{A}$. Also, a word of type 2 composed vertically with a word of type 1 (where the word of type 2 is above the word of type 1) gives another word of type 2. For a word of type 1 composed vertically with a word of type 2, consider the diagram



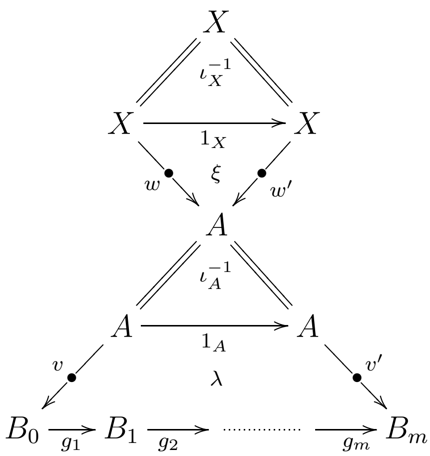
If we write $\alpha_n = \iota_A \cdot \overline{\alpha}_n$ for



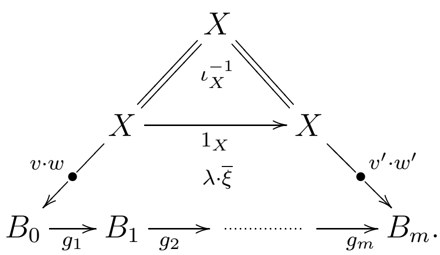
the above composition can be rewritten as



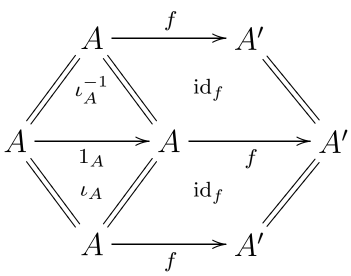
In a similar way, composing two type 2 words vertically



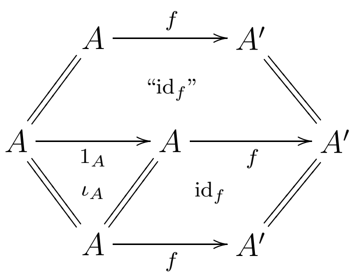
gives



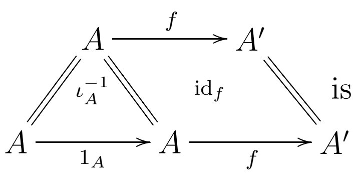
Next, consider the horizontal composite of a word of type 1 with a word of type 2 as above. First note that for any $f\colon A \to A'$, the vertical composites



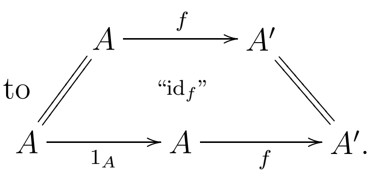
and



are both equal to id_f and the bottom cell is invertible, so

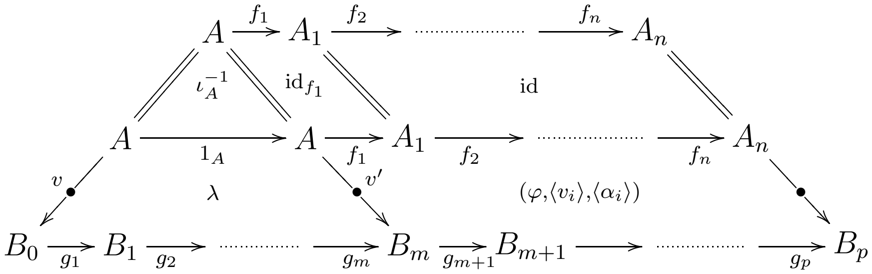


is equal to



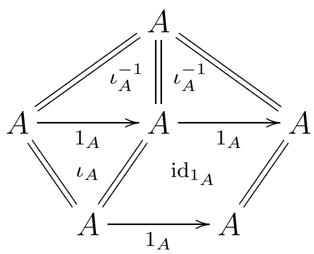
Thus, the horizontal composition of a word of type 2 as above with a word of type 1,

$(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$, can be written as

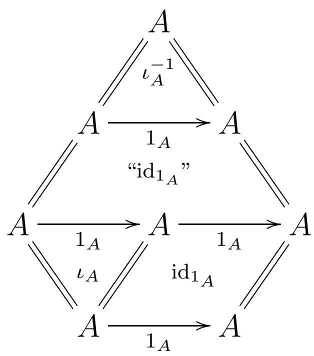


so that the ι_A^{-1} gets absorbed into the id_{f_1} and the whole composition is a word of type 1. Dually, a word of type 1 followed by a word of type 2 is another word of type 1.

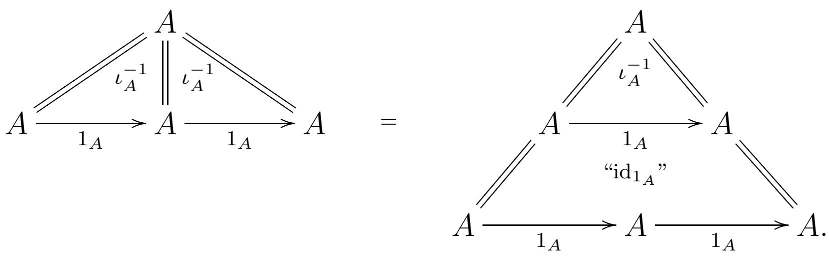
Finally, the horizontal composite of two type 2 words is another type 2 word. Indeed, the two vertical composites



and



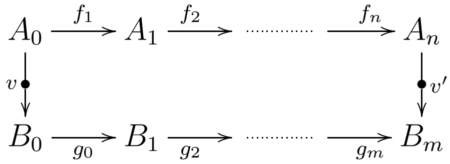
both equal ι_A^{-1} and as $(\mathrm{id}_{1_A})(\iota_A)$ is invertible, we see that



It is now easy to see that the horizontal composite of two type 2 words is again one. ∎

Now that we have an idea of what the equivalence classes are, our next step is to understand the equivalence relation as such. The next result is crucial in this endeavour.

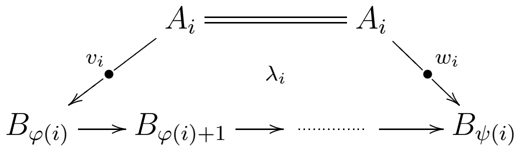
3.4. Proposition. *Let $(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$ and $(\psi, \langle w_i \rangle, \langle \beta_i \rangle)$ be two fences (i.e., words of type 1) with common boundary*



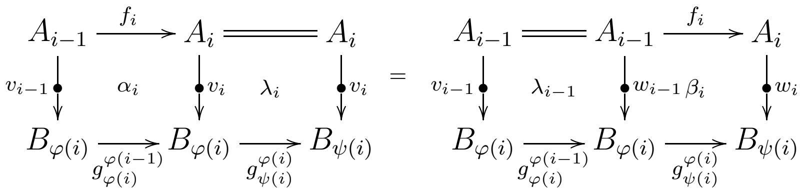
such that:

1. $\varphi \leqslant \psi$;

2. *for each* $i = 0, 1, \ldots, n$, *there are cells* λ_i, *as in*



   such that



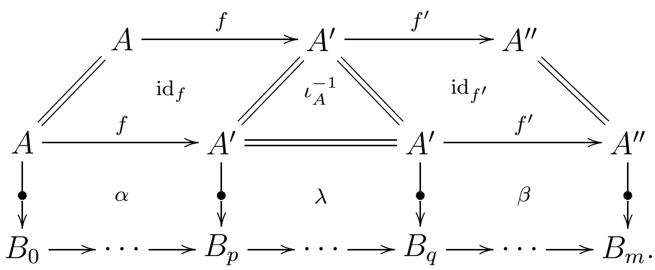
(6)

3. $\lambda_0 = 1_{v_0} = 1_{w_0}$ *and* $\lambda_n = 1_{v_n} = 1_{w_n}$.

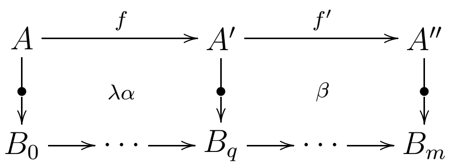
Then $(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$ *and* $(\psi, \langle w_i \rangle, \langle \beta_i \rangle)$ *represent the same cell in* $\mathbb{P}\mathrm{ath}_* \mathbb{A}$.

3.5. Remark. The intertwining of α, β and λ in (6) is nicely expressed in the double index notation, *viz.* $\lambda_i \alpha_i^j = \beta_i^j \lambda_j$.

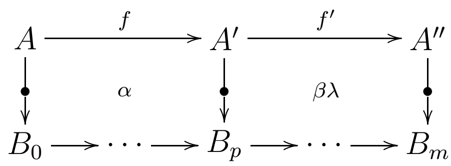
Proof. Consider α, β, and λ as illustrated in



First composing the four cells on the left and the two on the right and then composing the two on the left and the four on the right, we see that the fences



and



represent the same cell in $\mathbb{P}\mathrm{ath}_* \mathbb{A}$. Thus we get the following:

$$\begin{aligned}
\langle \alpha_n, \alpha_{n-1}, \ldots, \alpha_1 \rangle &= \langle \lambda_n \alpha_n, \alpha_{n-1}, \ldots, \alpha_1 \rangle \\
&= \langle \beta_n \lambda_{n-1}, \alpha_{n-1}, \ldots, \alpha_1 \rangle \\
&= \langle \beta_n, \lambda_{n-1} \alpha_{n-1}, \ldots, \alpha_1 \rangle \\
&= \langle \beta_n, \beta_{n-1} \lambda_{n-2}, \ldots, \alpha_1 \rangle \\
&= \langle \beta_n, \beta_{n-1}, \lambda_{n-2} \alpha_{n-2}, \ldots, \alpha_1 \rangle \\
&\quad \ldots \\
&= \langle \beta_n, \beta_{n-1}, \ldots, \lambda_1 \alpha_1 \rangle \\
&= \langle \beta_n, \beta_{n-1}, \ldots, \beta_1 \lambda_0 \rangle \\
&= \langle \beta_n, \beta_{n-1}, \ldots, \beta_1 \rangle
\end{aligned}$$

■

3.6. Remark. For fences whose vertical domain has length 1, there are no non-trivial such systems of λ-cells. Moreover, since type 2 representatives are vertically isomorphic to type 1 representatives with vertical domains of length 1, there are none for that case either. This implies that the equivalence relation for double cells introduced in the next Proposition is trivial for all representatives of type 2, and for those of type 1 with vertical domains of length 1.

3.7. Proposition. *Let $\sim$ be the equivalence relation on fences generated by*

$$(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle) \sim (\psi, \langle w_i \rangle, \langle \beta_i \rangle)$$

if there exist $\langle \lambda_i \rangle$ satisfying the conditions of the previous proposition. Then $\sim$ is a congruence of double categories on $\mathbb{P}\mathrm{ath}\, \mathbb{A}$.

Proof. It is clear that horizontal composition of fences respects $\sim$: simply concatenate the λs.

To show that vertical composition respects $\sim$ it will be sufficient to show that if there are $\langle \lambda_i \rangle$ directly relating $(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$ to $(\psi, \langle w_i \rangle, \langle \beta_i \rangle)$, then

$$(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle) \cdot (\theta, \langle u_i \rangle, \langle \gamma_i \rangle) \text{ is related to } (\psi, \langle w_i \rangle, \langle \beta_i \rangle) \cdot (\theta, \langle u_i \rangle, \langle \gamma_i \rangle)$$

and

$$(\xi, \langle x_k \rangle, \langle \delta_k \rangle) \cdot (\varphi, \langle v_i \rangle, \langle \alpha_i \rangle) \text{ is related to } (\xi, \langle x_k \rangle, \langle \delta_k \rangle) \cdot (\psi, \langle w_i \rangle, \langle \beta_i \rangle).$$

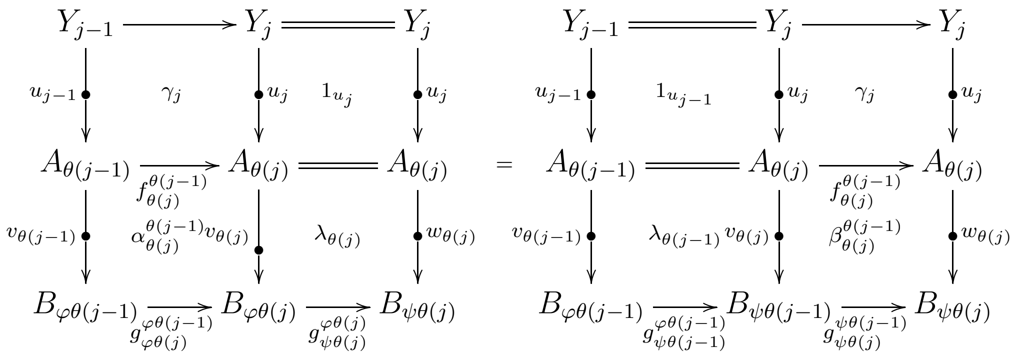
Now

$$(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle) \cdot (\theta, \langle u_i \rangle, \langle \gamma_i \rangle) = (\varphi\theta, \langle v_{\theta(j)} \cdot u_j \rangle, \langle \alpha_{\theta(j)}^{\theta(j-1)} \cdot \gamma_j \rangle)$$

and

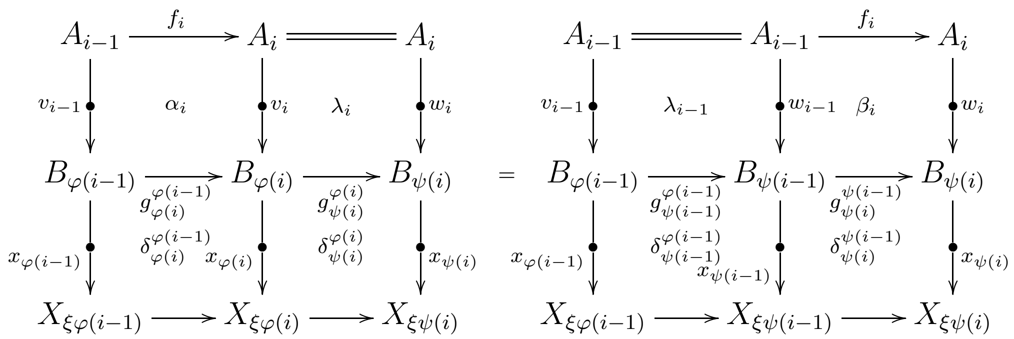
$$(\psi, \langle w_i \rangle, \langle \beta_i \rangle) \cdot (\theta, \langle u_i \rangle, \langle \gamma_i \rangle) = (\psi\theta, \langle w_{\theta(j)} \cdot u_j \rangle, \langle \beta_{\theta(j)}^{\theta(j-1)} \cdot \gamma_j \rangle).$$

We have



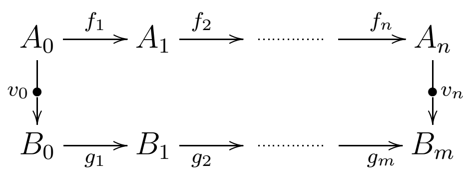
so if we take $\overline{\lambda} = \lambda_{\theta(j)} \cdot 1_{u_j}$ we get our relation.

For the equivalence involving $(\xi, \langle x_k \rangle, \langle \delta_k \rangle)$ take $\overline{\overline{\lambda}}_i = \delta^{\varphi(i)}_{\psi(i)} \cdot \lambda_i$. Then

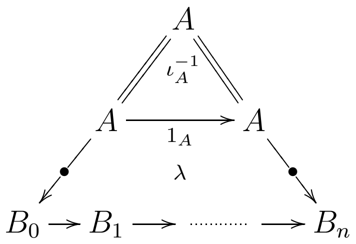


shows that $(\xi, \langle x_k \rangle, \langle \delta_k \rangle) \cdot (\varphi, \langle v_i \rangle, \langle \alpha_i \rangle) \sim (\xi, \langle x_k \rangle, \langle \delta_k \rangle) \cdot (\psi, \langle w_i \rangle, \langle \beta_i \rangle)$. ■

3.8. The Construction of $\mathbb{P}\mathrm{ath}_*$. Let $\mathbb{A}$ be a double category. We construct a new double category $\mathbb{P}\mathrm{ath}_* \mathbb{A}$ which has the same objects and vertical arrows as $\mathbb{A}$. A horizontal arrow of $\mathbb{P}\mathrm{ath}_* \mathbb{A}$ is a path of horizontal arrows of $\mathbb{A}$. For $n \geqslant 1$, a cell with boundary



is an equivalence class of fences $(\varphi, \langle v_i \rangle, \langle \alpha_i \rangle)$ where the equivalence relation is the $\sim$ of Proposition 3.7. Such an equivalence class will be denoted $[\varphi, \langle v_i \rangle, \langle \alpha_i \rangle]$ or $[\alpha_n, \alpha_{n-1}, \ldots, \alpha_1]$. (A more suggestive notation might be $\alpha_n \otimes \alpha_{n-1} \otimes \cdots \otimes \alpha_1$ or $\alpha_n \otimes_{v_{n-1}} \alpha_{n-1} \otimes_{v_{n-2}} \cdots \otimes_{v_1} \alpha_1$.) The number n is called the *degree* of the cell. The degree zero cells or *scalars* are cells of type 2,



which we will denote simply by λ. They are in bijection with cells of degree 1 with vertical domain an identity, but it is important to distinguish between the two. The first notation does: it gives us $[\lambda]$ as opposed to λ, even though the equivalence relation is trivial in degree 1. The $\otimes$ notation does not make the distinction and with this notation we will rely on alphabetical distinction, writing α, β as opposed to λ, μ etc. When confusion is possible, we will revert to the first notation.

We are now ready to define horizontal and vertical composition of cells. First the horizontal:

$$\begin{aligned} [\beta_m, \ldots, \beta_1][\alpha_n, \ldots, \alpha_1] &= [\beta_m, \ldots, \beta_1, \alpha_n, \ldots, \alpha_1] \\ \lambda[\alpha_n, \ldots, \alpha_1] &= [\lambda\alpha_n, \ldots, \alpha_1] \\ [\beta_m, \ldots, \beta_1]\mu &= [\beta_m, \ldots, \beta_1\mu] \\ \lambda\mu &= \lambda\mu. \end{aligned}$$

The vertical structure comes from that of $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ in degree $\geqslant 1$.

$$\begin{aligned} [\psi, \langle w_j \rangle, \langle \beta_j \rangle][\varphi, \langle v_i \rangle, \langle \alpha_i \rangle] &= [\psi\varphi, \langle w_{\varphi(i)} \cdot v_i \rangle, \langle \beta_{\varphi(i)}^{\varphi(i-1)} \cdot \alpha_i \rangle] \\ \lambda \cdot [\alpha_n, \ldots, \alpha_1] &= [\lambda \cdot \alpha_n, 1_v \cdot \alpha_{n-1}, \ldots, 1_v \cdot \alpha_1] \\ [\beta_m, \ldots, \beta_1] \cdot \mu &= (\beta_m \cdots \beta_1) \cdot \mu = \beta_m^0 \cdot \mu \\ \lambda \cdot \mu &= \lambda \cdot \mu. \end{aligned}$$

Also, define $\Xi_* \colon \mathbb{A} \to \mathbb{P}\mathrm{ath}_*\,\mathbb{A}$ to be the identity on objects and vertical arrows and inclusion as paths of length 1 on arrows, and $\xi_*(\alpha) = [\alpha]$ on cells.

3.9. Proposition.

1. $\mathbb{P}\mathrm{ath}_*\,\mathbb{A}$ *is a (strict) double category.*

2. $\Xi_* \colon \mathbb{A} \to \mathbb{P}\mathrm{ath}_*\,\mathbb{A}$ *is an oplax morphism.*

3. Ξ_* *is vertically full and faithful.*

Proof.

1. That composition of horizontal or vertical arrows is associative and unitary is obvious.

There are eight instances of horizontal associativity for cells that must be checked but only one is not entirely obvious:

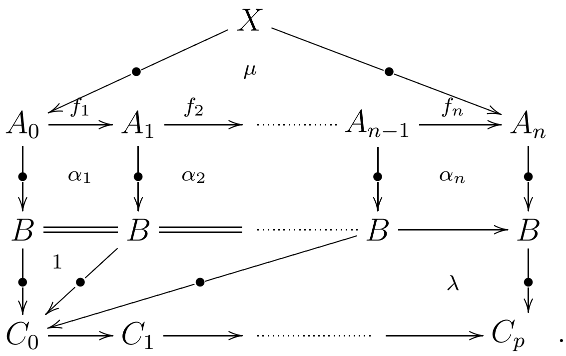
$$\begin{aligned} ([\beta_m, \ldots, \beta_1]\lambda)[\alpha_n, \ldots, \alpha_1] &= [\beta_m, \ldots, \beta_1\lambda, \alpha_n, \ldots, \alpha_1] \\ [\beta_m, \ldots, \beta_1](\lambda[\alpha_n, \ldots, \alpha_1]) &= [\beta_m, \ldots, \beta_1, \lambda\alpha_n, \ldots, \alpha_1] \end{aligned}$$

which are equal as a consequence of the equivalence relation. The horizontal identity on v is given by the scalar 1_v.

There are also eight instances of vertical associativity depending on the type of factors. All of these can be proved by fairly straightforward calculations. We do a representative example, the case of the multiplication of a type 1 cell with a scalar on each side. We have that $\lambda \cdot ([\alpha_n, \ldots, \alpha_1] \cdot \mu) = \lambda \cdot (\alpha_n^0 \cdot \mu) = \lambda \cdot \alpha_n^0 \cdot \mu$, whereas

$$\begin{aligned} &(\lambda \cdot [\alpha_n, \ldots, \alpha_1]) \cdot \mu \\ &\quad = [\lambda \cdot \alpha_n, 1 \cdot \alpha_{n-1}, \ldots, 1 \cdot \alpha_1] \cdot \mu \\ &\quad = ((\lambda \cdot \alpha_n)(1 \cdot \alpha_{n-1}) \cdots (1 \cdot \alpha_1)) \cdot \mu \end{aligned}$$

which is represented by

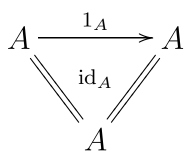


By general associativity [DP1], this is the same as $\lambda \cdot \alpha_n^0 \cdot \mu$.

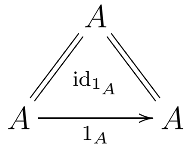
The vertical identity on $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ is $[\mathrm{id}_{f_n}, \ldots, \mathrm{id}_{f_1}]$ and the vertical identity on the empty path A is the scalar $\mathrm{id}_{1_A} = 1_{\mathrm{id}_A}$. The four cases to be checked to verify that these actually work are trivial.

Finally, the middle four interchange law involves checking sixteen cases, which all work either for trivial reasons or because of the equivalence relation.

2. $\Xi_* \colon \mathbb{A} \to \mathbb{P}\mathrm{ath}_* \mathbb{A}$ clearly preserves vertical composition of arrows and cells as well as vertical identities. The cells expressing oplaxity come from those for $\Xi \colon \mathbb{A} \to \mathbb{P}\mathrm{ath}\, \mathbb{A}$, so they satisfy the required coherence conditions. Finally, the cell $(\xi_*)_A \colon \Xi_*(1_A) \to 1_{\Xi_* A}$ comparing units is

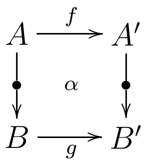


which has the scalar



as its inverse. Indeed, $\mathrm{id}_{1_A} \cdot [\mathrm{id}_{1_A}] = [\mathrm{id}_{1_A} \cdot \mathrm{id}_{1_A}] = [\mathrm{id}_{1_A}]$ which is the identity on 1_A in $\mathbb{P}\mathrm{ath}_* \mathbb{A}$, and $[\mathrm{id}_{1_A}] \cdot \mathrm{id}_{1_A} = \mathrm{id}_{1_A} \cdot \mathrm{id}_{1_A} = \mathrm{id}_{1_A}$.

3. Ξ_* is the identity on vertical arrows and as the equivalence relation is trivial on cells of degree 1, there is a bijection between cells of the form



in $\mathbb{P}\mathrm{ath}_* \mathbb{A}$ and similar ones in $\mathbb{A}$.

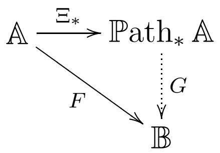
■

We can now state the main result of this section:

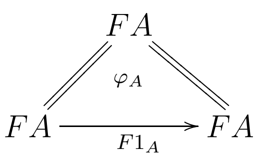
3.10. THEOREM. *For any double category* $\mathbb{A}$, $\Xi_* \colon \mathbb{A} \to \mathbb{P}\mathrm{ath}_* \mathbb{A}$ *is the universal oplax normal morphism, i.e., for any (strict) double category* $\mathbb{B}$ *composing with* Ξ_* *induces an isomorphism of categories*

$$\mathbf{Doub}(\mathbb{P}\mathrm{ath}_*\mathbb{A}, \mathbb{B}) \xrightarrow{\cong} \mathbf{Doub}_{\mathrm{OplN}}(\mathbb{A}, \mathbb{B}).$$

PROOF. Given an oplax normal morphism F as in the diagram below, we wish to show that it extends uniquely to a double functor G



As Ξ_* is the identity on objects and vertical arrows, G is uniquely determined on these, and as $\mathbb{P}\mathrm{ath}_* \mathbb{A}$ is free on horizontal arrows G is also uniquely determined on these. Since F is oplax it extends uniquely to a double functor on $\mathbb{P}\mathrm{ath}\, \mathbb{A}$, so there is only one possible choice for G on cells of type 1. It is easily checked that G respects the instances of $\sim$ described in Proposition 3.4, so G is well-defined on equivalence classes. Finally, any cell of type 2 (scalar) λ can be written as $[\lambda] \cdot 1_A$ and $G(1_A)$ has to be

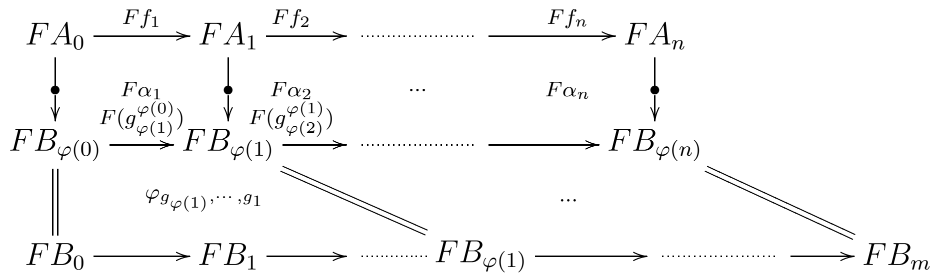


This shows that there is a unique well-defined G extending F and we only need to show that it preserves horizontal and vertical composition. On cells of type 1 there is no problem as the compositions are already preserved at the $\mathbb{P}\mathrm{ath}\,\mathbb{A}$ stage.

To check the other cases we need explicit formulas for $G[\alpha_n,\ldots,\alpha_1]$ and $G(\lambda)$. They are

$G[\alpha_n,\ldots,\alpha_1]$

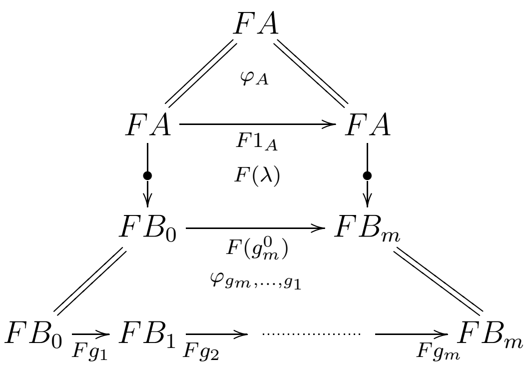
$=$



$$= (\varphi_{g_m,\cdots,g_{\varphi(n-1)+1}}\cdot F\alpha_n)(\varphi_{g_{\varphi(n-1)}\cdots g_{\varphi(n-2)+1}}\cdot F\alpha_{n-1})\cdots(\varphi_{g_{\varphi(1)}\cdots g_1}\cdot F\alpha_1),$$

and

$G(\lambda) =$



.

We need to check that

$$G(\lambda\cdot[\alpha_n,\ldots,\alpha_1]) = G(\lambda)\cdot G([\alpha_n,\ldots,\alpha_1]), \tag{7}$$

$$G([\alpha_n,\ldots,\alpha_1]\cdot\lambda) = G([\alpha_n,\ldots,\alpha_1])\cdot G(\lambda), \tag{8}$$

and

$$G(\lambda\cdot\mu) = G(\lambda)\cdot G(\mu). \tag{9}$$

To prove (7), note that in order for λ to be composable with $[\alpha_n,\ldots,\alpha_1]$, the vertical codomains of all the α_i are identity arrows, so

$$\begin{aligned}
G(\lambda\cdot[\alpha_n,\ldots,\alpha_1]) &= G([\lambda\cdot\alpha_n, 1\cdot\alpha_{n-1},\ldots,1\cdot\alpha_1]) \\
&= (\varphi_{g_m,\ldots,g_1}\cdot F(\lambda\cdot\alpha_n))F\alpha_{n-1}\cdots F\alpha_1 \\
&= (\varphi_{g_m,\ldots,g_1}\cdot F(\lambda)\cdot F(\alpha_n))F(\alpha_{n-1})\cdots F(\alpha_1) \\
&= (\varphi_{g_m,\ldots,g_1}\cdot F(\lambda))\cdot(F(\alpha_n)F(\alpha_{n-1})\cdots F(\alpha_1)) \\
&= G(\lambda)\cdot G([\alpha_n,\ldots,\alpha_1]).
\end{aligned}$$

To prove (8),

$$
\begin{aligned}
&G([\alpha_n, \ldots, \alpha_1] \cdot \lambda) \\
&\quad = \varphi_{g_m,\ldots,g_1} \cdot F(\alpha_n^0 \cdot \lambda) \\
&\quad = \varphi_{g_m,\ldots,g_1} \cdot F(\alpha_n^0) \cdot F(\lambda) \\
&\quad = ((\varphi_{g_m,\cdots,g_{\varphi(n-1)+1}} \cdot F\alpha_n) \cdots (\varphi_{g_{\varphi(1)}\cdots g_1} \cdot F\alpha_1)) \cdot (\varphi_{f_n,\cdots,f_1} \cdot F(\lambda)) \\
&\quad = G([\alpha_n, \ldots, \alpha_1]) \cdot G(\lambda).
\end{aligned}
$$

Finally, (9) is straightforward, and left as an exercise for the reader.

This shows that composing with Ξ_* is bijective on objects. To show that it is also bijective on morphisms, replace $\mathbb{B}$ with $\mathbb{B}^{\dot\downarrow}$, as defined in the next paragraph, and then invoke Proposition 3.11 below. ∎

Let $\dot\downarrow$ represent the double category with two objects and only one non-identity arrow, a vertical arrow. So for any double category $\mathbb{D}$, there is a bijection between double functors $\dot\downarrow \to \mathbb{D}$ and vertical arrows of $\mathbb{D}$. Now **Doub**, being categories in **Cat**, is cartesian closed and we have the vertical arrow double category $\mathbb{B}^{\dot\downarrow}$ of $\mathbb{B}$, for which double functors

$$\mathbb{A} \to \mathbb{B}^{\dot\downarrow}$$

are in bijective correspondence with vertical transformations

$$\mathbb{A} \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \mathbb{B}\,.$$

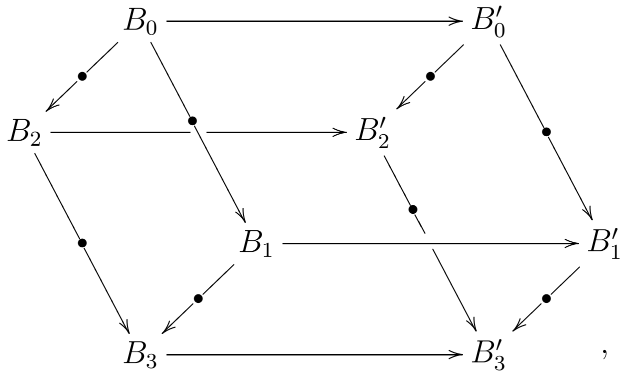
Explicitly, the objects of $\mathbb{B}^{\dot\downarrow}$ are vertical arrows of $\mathbb{B}$, its horizontal arrows are cells

$$
\begin{array}{ccc}
B_0 & \xrightarrow{b_0} & B_0' \\
{\scriptstyle w}\,\dot\downarrow & \beta & \dot\downarrow\,{\scriptstyle w'} \\
B_1 & \xrightarrow[b_1]{} & B_1'
\end{array}
\quad ,
$$

its vertical arrows are commutative squares

$$
\begin{array}{ccc}
B_0 & \overset{\bullet}{\longrightarrow} & B_2 \\
\dot\downarrow & & \dot\downarrow \\
B_1 & \overset{\bullet}{\longrightarrow} & B_3
\end{array}
\quad ,
$$

and its cells are commutative cubes



,

or in plane language,

$$\begin{array}{ccc} B_0 & \longrightarrow & B_0' \\ {\scriptstyle w_1}\,\downarrow\!\bullet & \beta_1 & \bullet\!\downarrow\,{\scriptstyle w_1'} \\ B_1 & \longrightarrow & B_1' \\ {\scriptstyle \overline{w}_1}\,\downarrow\!\bullet & \overline{\beta}_1 & \bullet\!\downarrow\,{\scriptstyle \overline{w}_1'} \\ B_3 & \longrightarrow & B_3' \end{array} \quad = \quad \begin{array}{ccc} B_0 & \longrightarrow & B_0' \\ {\scriptstyle w_2}\,\downarrow\!\bullet & \beta_2 & \bullet\!\downarrow\,{\scriptstyle w_2'} \\ B_2 & \longrightarrow & B_2' \\ {\scriptstyle \overline{w}_2}\,\downarrow\!\bullet & \overline{\beta}_2 & \bullet\!\downarrow\,{\scriptstyle \overline{w}_2'} \\ B_3 & \longrightarrow & B_3' \end{array}.$$

3.11. Proposition. *There are double functors* $D_0, D_1 \colon \mathbb{B}^{\updownarrow} \to \mathbb{B}$ *and a vertical transformation* $v \colon D_0 \multimap D_1$ *which establish a bijection between oplax morphisms* $\mathbb{A} \to \mathbb{B}^{\updownarrow}$ *and pairs of oplax morphisms* $\mathbb{A} \to \mathbb{B}$ *with a vertical transformation between them. This bijection respects normality.*

Proof. Straightforward calculation. ■

3.12. The Comonad $\mathbb{P}\mathrm{ath}_*$. The isomorphism of categories given in Theorem 3.10 shows that $\mathbb{P}\mathrm{ath}_*$ is a left 2-adjoint to the inclusion **Doub** $\hookrightarrow$ $\mathbf{Doub}_{\mathrm{OplN}}$ so in fact $\mathbb{P}\mathrm{ath}_*$ is a 2-comonad in **Doub**. The following proposition is obvious, but nevertheless illuminates the nature of oplax normal morphisms.

3.13. Proposition. $\mathbf{Doub}_{\mathrm{OplN}}$ *is the Kleisli 2-category for the comonad* $\mathbb{P}\mathrm{ath}_*$ *on* **Doub**.

We have the following analog of Theorem 1.21 for the normal case.

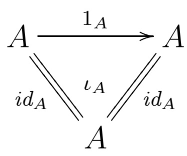
3.14. Theorem. *The comonad* $\mathbb{P}\mathrm{ath}_*$ *on* **Doub** *is oplax idempotent.*

Proof. The proof is the same as that of Theorem 1.21 except for the addition of a few $*$s as subscripts. ■

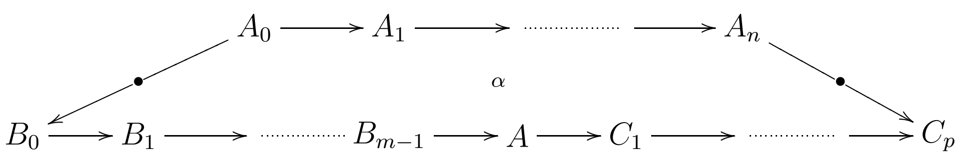
In view of this theorem we should figure out what the 2-category of Eilenberg-Moore coalgebras is. These will be the oplax normal double categories, *i.e.*, the natural objects on which oplax normal morphisms are defined. Because $\mathbb{P}\mathrm{ath}_*$ is oplax idempotent, each double category will have at most one such structure, so being an Eilenberg-Moore coalgebra is again a property rather than extra structure: $\mathbb{A}$ is a coalgebra if and only if $\Xi_* \colon \mathbb{A} \to \mathbb{P}\mathrm{ath}_* \mathbb{A}$ has a left adjoint, which should probably be composition in some form. This description, although accurate, is somewhat misleading. The example of $\mathbb{P}\mathrm{ath}$ suggests identifying $\mathbb{A}$ with an oplax double category of some sort, and this does indeed work.

The vertical dual of Definition 2.7 gives us the notion of an n-fold composite being strongly representable in an oplax double category. For $n = 0$ we talk about an identity being strongly representable.

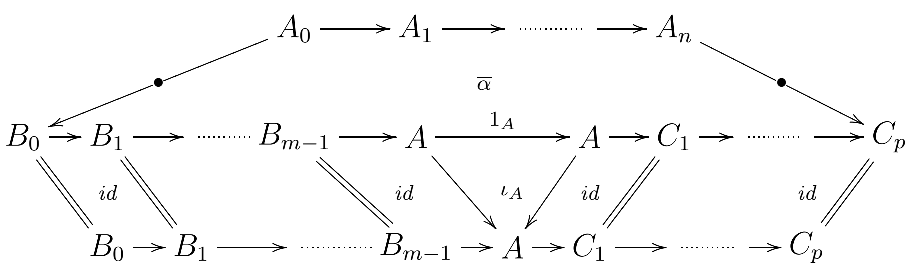
3.15. Definition. *Let $\mathbb{A}$ be an oplax double category and A an object of $\mathbb{A}$. The* identity on A is strongly representable *if there is an arrow $1_A \colon A \to A$ and a cell*



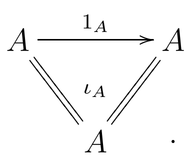
with the following universal property. For any cell



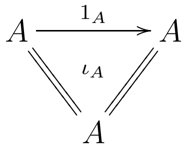
there exists a unique cell $\overline{\alpha}$, as indicated below, such that



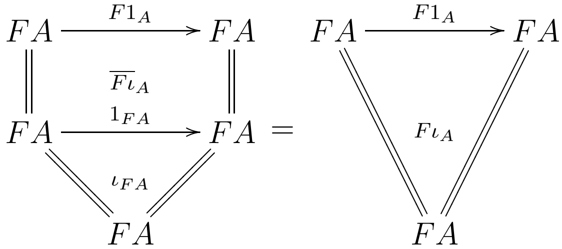
is equal to α. We say that $\mathbb{A}$ is oplax normal *if all identities are strongly representable. A* morphism of oplax normal double categories *is a morphism of oplax double categories which preserves the universal cells*



Actually, as with all definitions involving representability, the arrows of the form 1_A are only determined up to vertical isomorphism. If $\mathbb{A}$ is oplax normal we pick a universal cell



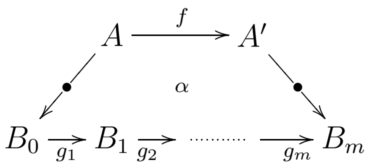
for each object A in $\mathbb{A}$. Then preservation of universal cells says that the unique cell $\overline{F\iota_A}$ such that



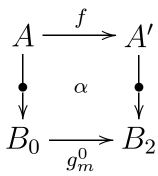
is a vertical isomorphism. (This is analogous to the preservation of limits in ordinary category theory.)

Let **OplaxN** be the locally full and faithful sub-2-category of **Oplax**, determined by the oplax normal double categories and morphisms of such (with arbitrary vertical transformations as 2-cells).

Recall from Section 2.10 the construction $\mathbb{O}$plax which takes a double category $\mathbb{A}$ and considers it as an oplax double category with the same objects, vertical and horizontal arrows, with multicells



given by double cells



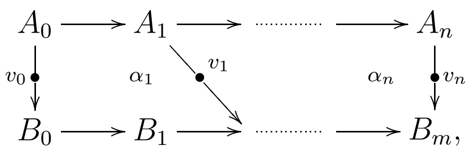
in $\mathbb{A}$.

3.16. Proposition. *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories, then $\mathbb{O}\mathrm{plax}\,\mathbb{A}$ and $\mathbb{O}\mathrm{plax}\,\mathbb{B}$ are normal and a morphism of oplax normal double categories $\mathbb{O}\mathrm{plax}\,\mathbb{A} \to \mathbb{O}\mathrm{plax}\,\mathbb{B}$ is the same as a normal oplax morphism $\mathbb{A} \to \mathbb{B}$. In fact, $\mathbb{O}$plax is a fully faithful and locally fully faithful 2-functor* $\mathbf{Doub}_{\mathrm{OplN}} \to \mathbf{OplaxN}$.

We denote by $\mathbb{O}\mathrm{plax}_*\colon \mathbf{Doub} \to \mathbf{OplaxN}$ the 2-functor $\mathbb{O}$plax whose codomain has been restricted to **OplaxN**. We shall show that $\mathbb{P}\mathrm{ath}_*$ gives a left 2-adjoint to this inclusion. That is, $\mathbb{P}\mathrm{ath}_*\mathbb{A}$ is the free double category generated by an oplax normal $\mathbb{A}$. We first must extend $\mathbb{P}\mathrm{ath}_*$ to **OplaxN**.

3.17. The Extension of $\mathbb{P}\mathrm{ath}_*$. The construction follows closely that of $\mathbb{P}\mathrm{ath}_*$ for actual double categories. First we need some definitions. Let $\mathbb{A}$ be an arbitrary oplax normal double category.

3.18. Definition. *Given a path* $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ *in* $\mathbb{A}$ *and* $0 \leqslant i \leqslant j \leqslant n$*, we let* f^i_j *denote the* path $A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_j} A_j$ *(if* $i = j$ *this is of course the empty path). We also extend this notation to cells: given a fence in* $\mathbb{P}\mathrm{ath}\,\mathbb{A}$*,*

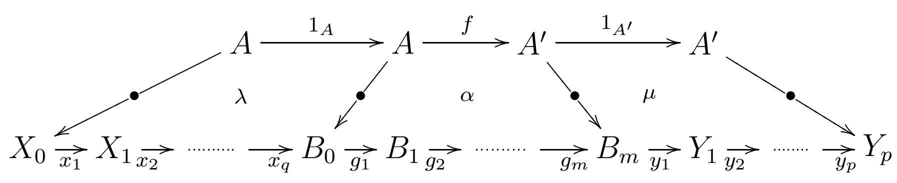


α^i_j *denotes the formal composite* $\alpha_j \alpha_{j-1} \cdots \alpha_{i+1}$*.*

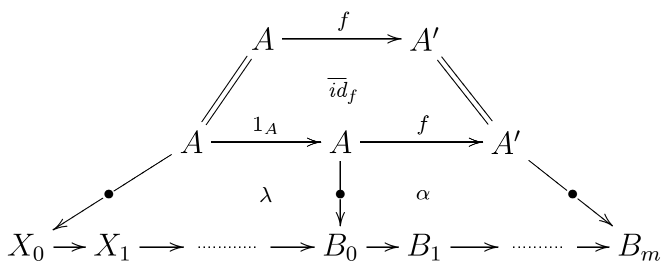
If $\mathbb{A}$ is an actual double category, then the definitions of f^i_j and α^i_j are inconsistent in the sense that $\mathrm{id}_{f^i_j}$ is not the same thing as $\langle \mathrm{id}_f \rangle^j_i$. However, there will be no confusion if we keep the distinction between $\mathbb{O}\mathrm{plax}\,\mathbb{A}$ and $\mathbb{A}$ clear.

The definition of $\mathbb{P}\mathrm{ath}_*$ uses composition of general cells with scalars (cells whose vertical domain is an identity), both in the definition of the equivalence relation on fences and that of horizontal composition.

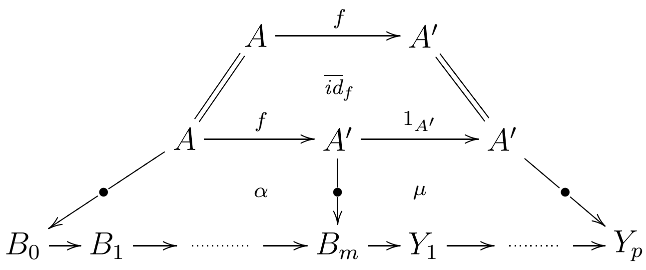
3.19. Definition. *Consider cells*



in $\mathbb{A}$*. The cells* $\alpha\lambda$ *and* $\mu\alpha$ *are defined to be the composites*



and



respectively. (Note that the cells $\overline{id}_f$ are different in these two composites, but this will cause no problem.)

These horizontal composites obey all the associativity and unitary laws that make sense. They are best understood in the following more general context which will be of use later. (Recall the definition of strong representability for composable paths of arrows from Definition 2.7.)

3.20. Proposition. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *be arrows of* $\mathbb{A}$.

1. *If* gf *is strongly representable, then* $h(gf)$ *is strongly representable if and only if* hgf *is. Moreover, in this case there is a canonical isomorphism* $h(gf) \cong hgf$.

2. *If* 1_A *is strongly representable, then so is* $f1_A$ *and* $f1_A \cong f$.

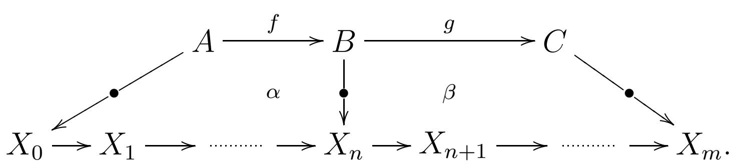
3. *If* 1_B *is strongly representable then so is* $1_B f$ *and* $1_B f \cong f$.

Proof. Straightforward. ∎

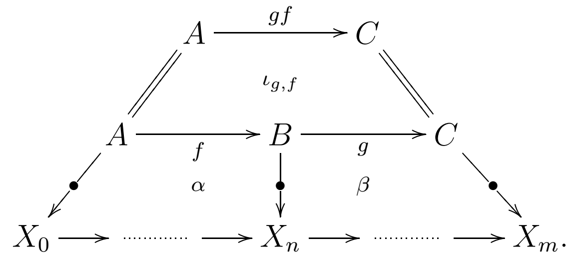
3.21. Corollary. *If* gf *and* hg *are strongly representable, then* $h(gf)$ *is strongly representable if and only if* $(hg)f$ *is. Moreover, in this case we have* $h(gf) \cong (hg)f$.

Once the composite of certain horizontal arrows is strongly representable, the horizontal composite of cells with these as vertical domains is automatically defined and indeed satisfies all reasonable associativity and unitary properties.

3.22. Definition. *Suppose* gf *is strongly representable and consider cells*



The horizontal composite $\beta\alpha$ *is defined to be* $\langle\beta,\alpha\rangle \cdot \iota_{g,f}$



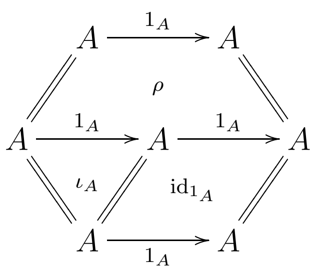
3.23. Proposition. *Let* α, β, γ *be cells with vertical domains* f, g, h*, respectively and assume that* gf*,* hg*,* $h(gf)$ *and* $(hg)f$ *are strongly representable. Then* $\gamma(\beta\alpha) = (\gamma\beta)\alpha$ *(by which we mean that* $(\gamma\beta)\alpha$ *composed with the canonical isomorphism* $h(gf) \cong (hg)f$ *is* $\gamma(\beta\alpha)$*.)*

Proof. This follows immediately from the definition of horizontal composition and the canonical isomorphism. ■

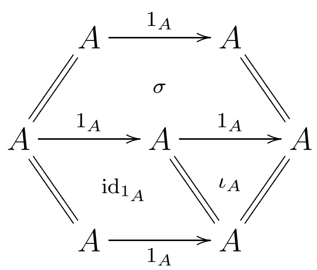
3.24. Corollary. *Let* α *be an arbitrary cell and* λ *and* μ *scalars with the appropriate identities as vertical domains. Then:*

1. *The three possible meanings of* $\mu\lambda$ *agree;*
2. $(\mu\alpha)\lambda = \mu(\alpha\lambda)$*;*
3. $(\alpha\mu)\lambda = \alpha(\mu\lambda)$*;*
4. $\mu(\lambda\alpha) = (\mu\lambda)\alpha$*.*

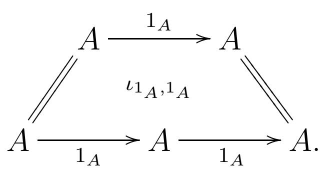
Proof. The only thing to check is part 1. The other results follow from the previous proposition. The three meanings referred to are the following. The universal property of 1_A gives cells ρ and σ such that



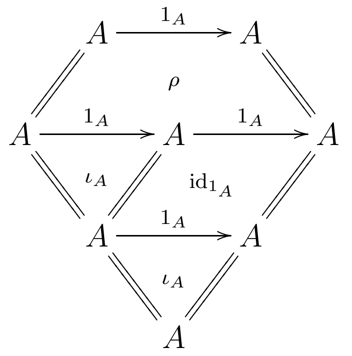
and



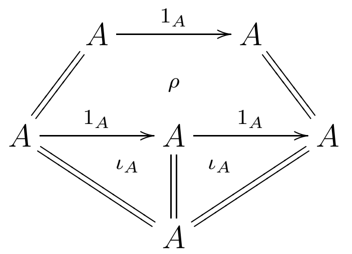
are both equal to id_{1_A}, and representability of the composite gives



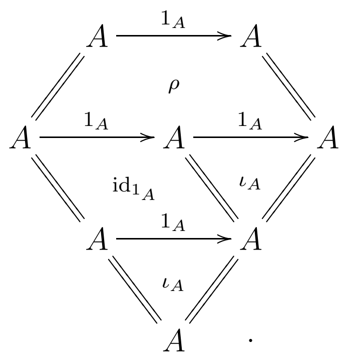
So $\mu\lambda$ could alternately be defined as $\langle\mu,\lambda\rangle\cdot\rho$, $\langle\mu,\lambda\rangle\cdot\sigma$ or $\langle\mu,\lambda\rangle\cdot\iota_{1_A,1_A}$. In fact, showing that $1_A 1_A$ is representable requires us to construct $\iota_{1_A,1_A}$ and depending on whether we invoke property 2 or 3 of Proposition 3.20, it is taken to be $\langle\mu,\lambda\rangle\cdot\rho$ or $\langle\mu,\lambda\rangle\cdot\sigma$ respectively. So we must show that these last two cells are the same and indeed they are. In fact, the composite



is equal, by associativity and the unit law, to



which is by the same token equal to



We can cancel ι off the bottom and thus see that ρ satisfies the same property as σ. ∎

With these technicalities out of the way, we can now define the equivalence relation $\sim$ on the fences of the oplax normal double category $\mathbb{A}$ as in Proposition 3.7 and note that it is still a congruence of double categories on $\mathbb{P}\text{ath}\,\mathbb{A}$. The construction of $\mathbb{P}\text{ath}_*\mathbb{A}$ is exactly the same as in Section 3.8 and it again forms a strict double category (Proposition 3.9). Ξ_* is defined as before which is now understood as a morphism of oplax normal double categories

$$\Xi_*\colon \mathbb{A} \to \mathbb{O}\text{plax}_*\mathbb{P}\text{ath}_*\mathbb{A}.$$

It is still vertically full and faithful in the obvious sense for oplax double categories. We now have the extension (to its natural domain) of Theorem 3.10.

3.25. Theorem. *Let* $\mathbb{A}$ *be an oplax normal double category. Then*

$$\Xi_*\colon \mathbb{A} \to \mathbb{O}\mathrm{plax}_*\mathbb{P}\mathrm{ath}_*\mathbb{A}$$

is the universal oplax normal morphism, i.e., for any strict double category $\mathbb{B}$*, composition with* Ξ_* *induces an* isomorphism *of categories*

$$\mathbf{Doub}(\mathbb{P}\mathrm{ath}_*\mathbb{A}, \mathbb{B}) \xrightarrow{\cong} \mathbf{OplaxN}(\mathbb{A}, \mathbb{O}\mathrm{plax}_*\mathbb{B}).$$

Proof. The proof of Theorem 3.10 was written in such a way as to go through, *mutatis mutandis.* ∎

This theorem shows that $\mathbb{P}\mathrm{ath}_*$ is a left 2-adjoint to the 'inclusion'

$$\mathbb{O}\mathrm{plax}_*\colon \mathbf{Doub} \to \mathbf{OplaxN}$$

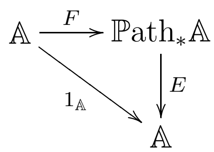
and consequently extends in a unique way to a 2-functor. The 2-comonad $\mathbb{P}\mathrm{ath}_*$ induced on **Doub** is of course the same $\mathbb{P}\mathrm{ath}_*$ introduced at the beginning of Section 3.12.

3.26. Eilenberg-Moore Algebras for $\mathbb{P}\mathrm{ath}_*$. We are now in the position to describe the Eilenberg-Moore algebras for the comonad $\mathbb{P}\mathrm{ath}_*$ on **Doub**.

3.27. Theorem. $\mathbb{P}\mathrm{ath}_*\colon$ **OplaxN** $\to$ **Doub** *is comonadic.*

Proof. $\mathbb{P}\mathrm{ath}_*$ has a right 2-adjoint $\mathbb{O}\mathrm{plax}_*$ and the induced comonad is what we simply called $\mathbb{P}\mathrm{ath}_*$ in Section 3.8.

Let $\mathbb{A}$ be a double category and $F\colon \mathbb{A} \to \mathbb{P}\mathrm{ath}_*\mathbb{A}$ a coalgebra structure. The unit law



tells us that F is the identity on objects and vertical arrows and that it assigns to each horizontal arrow f a factorisation of it, $f = f_n f_{n-1} \cdots f_1$. As in the case of $\mathbb{P}\mathrm{ath}$ (and Path) this makes the horizontal arrows of $\mathbb{A}$ into a free category generated by the arrows of degree 1 (*i.e.*, whose assigned factorisation has only one factor). F also assigns to each cell α a factorisation of it, $\alpha = \alpha_n \cdots \alpha_1$, although this time not a unique one, but in fact an equivalence class of factorisations. Also, the length of this factorisation is not uniquely determined, because it may contain scalars (*i.e.*, cells whose vertical domain has degree 0). However, the equivalence relation is such that scalars can be absorbed into any adjacent nonscalar, so each equivalence class contains a representative factorisation containing no scalars, though not a canonical one (provided of course that the cell itself is not a pure scalar). Call a factorisation of α *reduced* if it is either scalar or contains no scalars. Then F provides a reduced factorisation $\alpha = \alpha_n \cdots \alpha_1$ for each α and the length n is uniquely determined, namely the length of the factorisation of its vertical domain.

We construct an oplax normal double category $\mathbb{A}_0$ as follows. It is the suboplax double category of $\mathbb{O}\text{plax}_*\mathbb{A}$ with the same objects and vertical arrows and whose horizontal arrows and cells are those of degree $\leqslant 1$. It is clear that $\mathbb{A}_0$ is a suboplax double category, since the degree of the cells is the degree of the top cell. It is also clearly normal, since the degree of the canonical cell ι_A has degree 1 and the cells whose existence is required for the universal property of ι_A are the same as the test cells.

The inclusion $\mathbb{A}_0 \subseteq \mathbb{O}\text{plax}_*\mathbb{A}$ corresponds to a double functor $\mathbb{P}\text{ath}_*\mathbb{A}_0 \to \mathbb{A}$ which is nothing but horizontal composition of arrows and cells. We claim that it is an isomorphism of double categories. We have already argued that it is one-to-one and onto on objects and both kinds of arrows, and that it is onto on cells. Suppose that we have two reduced paths of cells $\langle \alpha_n, \ldots, \alpha_1 \rangle$ and $\langle \beta_n, \ldots, \beta_1 \rangle$ whose composites are equal in $\mathbb{A}$, $\alpha_n \cdots \alpha_1 = \beta_n \cdots \beta_1$. Then $F(\alpha_n \cdots \alpha_1) = F(\alpha_n) \cdots F(\alpha_1) = [\alpha_n, \ldots, \alpha_1]$ as F is a double functor and $F(\alpha_i) = [\alpha_i]$. Similarly, $F(\beta_n \cdots \beta_1) = [\beta_n, \ldots, \beta_1]$, so $[\alpha_n, \ldots, \alpha_1] = [\beta_n, \ldots, \beta_1]$ and we see that $\mathbb{P}\text{ath}_*\mathbb{A}_0 \to \mathbb{A}$ is one-to-one on cells. This shows that every coalgebra is of the form $\mathbb{P}\text{ath}_*\mathbb{A}_0$ for a unique oplax normal double category $\mathbb{A}_0$.

If $G \colon \mathbb{B} \to \mathbb{P}\text{ath}_*\mathbb{B}$ is another coalgebra structure, a morphism of coalgebras is a double functor H such that

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{F} & \mathbb{P}\text{ath}_*\mathbb{A} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \mathbb{P}\text{ath}_* H} \\ \mathbb{B} & \xrightarrow[G]{} & \mathbb{P}\text{ath}_*\mathbb{A} \end{array}$$

commutes. As such it preserves the degree of horizontal arrows and cells so it restricts to a morphism of oplax double categories, $H_0 \colon \mathbb{A}_0 \to \mathbb{B}_0$, which is normal as H preserves all identities.

Finally, vertical transformations between coalgebra homomorphisms

$$H \longrightarrow\!\!\!\!\!\!\bullet\;\; K$$

are exactly the same as vertical transformations

$$H_0 \longrightarrow\!\!\!\!\!\!\bullet\;\; K_0 \,.$$

This shows that the 2-category $\mathbf{Doub}_{\mathbb{P}\text{ath}_*}$ of coalgebras on the comonad $\mathbb{P}\text{ath}_*$ is equivalent to the 2-category **OplaxN**, thus concluding the proof. ∎

The preceding theorem shows that oplax normal double categories are the correct objects for studying oplax normal double functors. In the sequel to this paper [DPP4] we will introduce a more restricted class of oplax double categories, the *paranormal* ones and make an argument for their study.

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