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FROBENIUS ALGEBRAS AND AMBIDEXTROUS ADJUNCTIONS

AARON D. LAUDA

Abstract. In this paper we explain the relationship between Frobenius objects in monoidal categories and adjunctions in 2-categories. Specifically, we show that every Frobenius object in a monoidal category M arises from an ambijunction (simultaneous left and right adjoints) in some 2-category $\mathcal{D}$ into which M fully and faithfully embeds. Since a 2D topological quantum field theory is equivalent to a commutative Frobenius algebra, this result also shows that every 2D TQFT is obtained from an ambijunction in some 2-category. Our theorem is proved by extending the theory of adjoint monads to the context of an arbitrary 2-category and utilizing the free completion under Eilenberg-Moore objects. We then categorify this theorem by replacing the monoidal category M with a semistrict monoidal 2-category M, and replacing the 2-category $\mathcal{D}$ into which it embeds by a semistrict 3-category. To state this more powerful result, we must first define the notion of a 'Frobenius pseudomonoid', which categorifies that of a Frobenius object. We then define the notion of a 'pseudo ambijunction', categorifying that of an ambijunction. In each case, the idea is that all the usual axioms now hold only up to coherent isomorphism. Finally, we show that every Frobenius pseudomonoid in a semistrict monoidal 2-category arises from a pseudo ambijunction in some semistrict 3-category.

1. Introduction

In this paper we aim to illuminate the relationship between Frobenius objects in monoidal categories and adjunctions in 2-categories. One of the results we prove is that:

> *Every Frobenius object in* any *monoidal category M arises from simultaneous left and right adjoints in some 2-category into which M fully and faithfully embeds.*

To indicate the *two-handedness* of these simultaneous left and right adjoints we refer to them as ambidextrous adjunctions following Baez [3]. We sometimes refer to an ambidextrous adjunction as an *ambijunction* for short.

Intuitively, the relationship between Frobenius objects and adjunctions can best be understood geometrically. This geometry arises naturally from the language of 2-categorical string diagrams [14, 35]. In string diagram notation, objects A and B of the 2-category $\mathcal{D}$ are depicted as 2-dimensional regions which we sometimes shade to differentiate between

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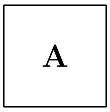
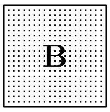
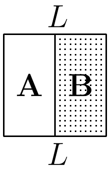
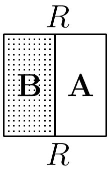
different objects:



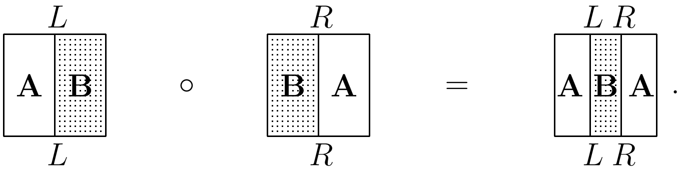


The morphisms of $\mathcal{D}$ are depicted as one dimensional edges. Thus, if $L\colon A \to B$ and $R\colon B \to A$ are morphisms in $\mathcal{D}$, then they are depicted as follows:

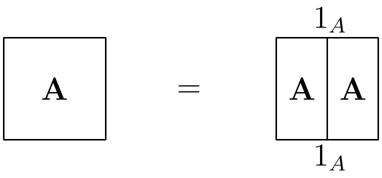




and their composite $RL\colon A \to A$ as:



As a convenient convention, the identity morphisms of objects in $\mathcal{D}$ are not drawn. This convention allows for the identification:

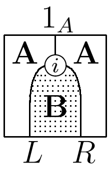


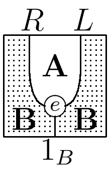
of string diagrams.

The 2-morphisms of $\mathcal{D}$ are drawn as 0-dimensional vertices or as small discs if we want to label them. Hence, the unit $i\colon 1 \Rightarrow RL$ and counit $e\colon LR \Rightarrow 1$ of an adjunction

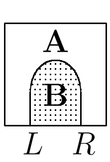
$$\begin{tikzcd} A \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left, "R"] \end{tikzcd}$$

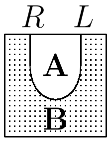
are depicted as:



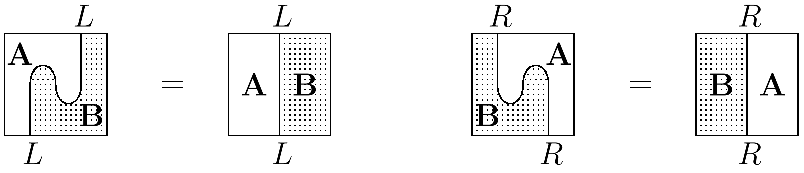


However, using the convention for the identity morphisms mentioned above and omitting the labels we can simplify these string diagrams as follows:

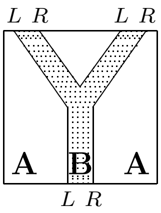




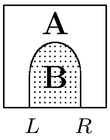
We can also express the axioms for an adjunction, often referred to as the triangle identities or zig-zag identities, by the following equations of string diagrams:



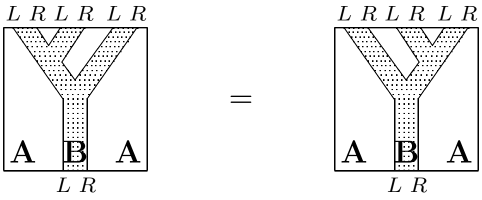
Early work on homological algebra [11, 28] and monad theory [10, 21] showed that an adjunction $A \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} B$ endows the composite morphism RL with a monoid structure in the monoidal category $\mathrm{Hom}(A, A)$. This monoid in $\mathrm{Hom}(A, A)$ can be vividly seen using the language of 2-categorical string diagrams. The multiplication on RL is defined using the unit for the adjunction as seen below:



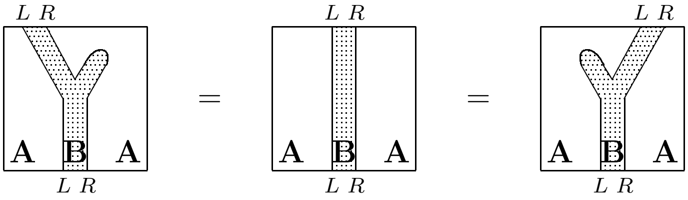
and the unit for multiplication is



the unit of the adjunction. The associativity axiom:



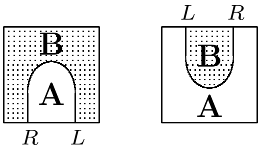
follows from the interchange law in the 2-category $\mathcal{D}$, and the unit laws:



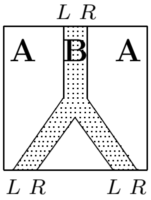
follow from the triangle identities in the definition of an adjunction.

Starting with an adjunction $A \underset{R}{\overset{L}{\underset{\top}{\rightleftarrows}}} B$ where L is the right adjoint produces the color inverted versions of the diagrams above. The unit $j\colon 1_B \Rightarrow LR$ and counit $k\colon RL \Rightarrow 1_A$

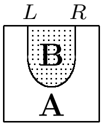
would appear as:



In this case RL becomes a comonoid in $\mathrm{Hom}(A, A)$ whose comultiplication is given by the diagram:

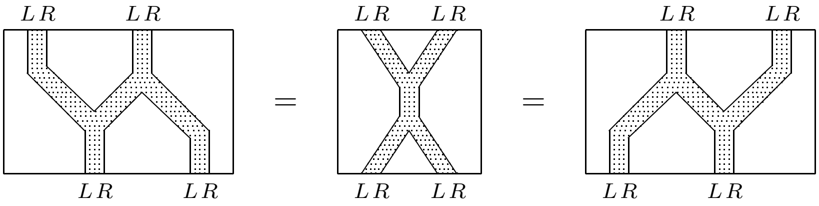


and whose counit is:



the counit of the right adjunction. By similar diagrams as those above, the coassociativity and counit axioms follow from the axioms of a 2-category and the axioms of an adjunction.

When the morphisms L is both left and right adjoint to R the object RL of $\mathrm{Hom}(A, A)$ is both a monoid and a comonoid. These structures satisfy compatibility conditions, known as the Frobenius identities, making RL into a Frobenius object. Indeed, the Frobenius identities:



follow from the interchange axiom of the 2-category $\mathcal{D}$. Thus we have shown that the axioms of an ambijunction in a 2-category beautifully imply all the axioms of a Frobenius object; by drawing string diagrams their relationship becomes much more transparent.

The converse, that every Frobenius object in the monoidal category M actually arises in this way from an ambidextrous adjunction in some 2-category $\mathcal{D}$ into which M fully and faithfully embeds, has thus far not been proven in a completely general context. An attempt to prove this result was made by Müger [31] who showed that, with certain extra assumptions about the monoidal category M, a 2-category $\mathcal{E}$ into which M fully and faithfully embeds can be constructed, and Frobenius objects in M correspond precisely to ambijunctions in this 2-category $\mathcal{E}$. However, we will see that the converse can be proven quite naturally using the language of monad theory where questions like this have already been resolved. Extending the work of Lawvere [27], Street [36] has made substantial progress by proving this result in the context of the 2-category **Cat**. Street's approach suggests a natural framework for proving the completely general result and experts in

2-categorical monad theory and enriched category theory will find that our proof is a straight forward extension of Street's work. Using a wealth of results from category theory, especially the formal theory of monads [32], we extend Street's work and prove this result for Frobenius objects in a generic monoidal category M.

To understand the relevance of monad theory a bit of background is in order. A monad on a category $\mathcal{A}$ can be defined as a monoid in the functor category $[\mathcal{A},\mathcal{A}]$. The theory of monads has been well developed and, in particular, it is well known that every monad $\mathbb{T}$ on a category $\mathcal{A}$ arises from a pair of adjoint functors $\mathcal{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{B}$. The problem of constructing an adjunction from a monad has two well known solutions — the Kleisli construction $\mathcal{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{A}_{\mathbb{T}}$ [21], and the Eilenberg-Moore construction $\mathcal{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{A}^{\mathbb{T}}$ [10]. These two solutions are the initial and terminal solution to the problem of constructing such an adjunction, in the general sense explained in [32]. Similarly, a comonoid in $[\mathcal{A},\mathcal{A}]$ is known as a comonad, and these constructions work equally well to create a pair of adjoint functors where the functor $\mathcal{A}^{\mathbb{T}} \to \mathcal{A}$ is now the left adjoint.

An interesting situation arises when the endofunctor $T\colon \mathcal{A} \to \mathcal{A}$ defining the monad has a specified right adjoint G. In this case, Eilenberg and Moore showed that G can be equipped with the structure of a comonad $\mathbb{G}$, and that the Eilenberg-Moore category of coalgebras $\mathcal{A}^{\mathbb{G}}$ for the comonad $\mathbb{G}$ is isomorphic to the Eilenberg-Moore category of algebras $\mathcal{A}^{\mathbb{T}}$ for the monad $\mathbb{T}$ [10]. The isomorphism $\mathcal{A}^{\mathbb{T}} \cong \mathcal{A}^{\mathbb{G}}$ also has the property that it commutes with the forgetful functors into $\mathcal{A}$.

If the functor T is equipped with a natural transformation from T to the identity of $\mathcal{A}$ such that precomposition with the monad multiplication is the counit for a specified self adjunction, then we call the resulting structure a Frobenius monad. This turns out to be the same as a Frobenius object in $[\mathcal{A},\mathcal{A}]$. Street uses these results to show that a Frobenius monad always arises from a pair of adjoint functors that are both left and right adjoints — an ambijunction in **Cat**.

The approach outlined above is essentially the one taken in this paper. Monads and adjunctions can be defined in any 2-category, and many of the properties of monads and adjunctions in **Cat** carry over to this abstract context. For instance, every adjunction in a 2-category $\mathcal{K}$ gives rise to a monad on an object of $\mathcal{K}$. However, it is not always true that one can find an adjunction generating a given monad. This can be attributed to the lack of an object in $\mathcal{K}$ to play the role of the Eilenberg-Moore category of algebras (or the lack of a Kleisli object, but we will focus on Eilenberg-Moore objects in this paper). When such an object does exist it is called an Eilenberg-Moore object for the monad $\mathbb{T}$. The existence of Eilenberg-Moore objects in a 2-category $\mathcal{K}$ is a completeness property of the 2-category in question. In particular, $\mathcal{K}$ has Eilenberg-Moore objects if it is finitely complete as a 2-category [32, 34].

Recall that every bicategory is biequivalent to a strict 2-category, and hence every monoidal category is biequivalent to a strict monoidal category. Let $\mathcal{M}$ be a monoidal category and denote as $\Sigma(M)$ the suspension of a strictification of M. Then since the 2-category $\Sigma(M)$ has only one object, say $\bullet$, a Frobenius object in M is just a Frobenius monad on the object $\bullet$ in the 2-category $\Sigma(M)$. It is tempting to use Eilenberg and

Moore's theorem on adjoint monads to conclude that this Frobenius monad arose from an ambijunction, but their construction used the fact the 2-category $\mathcal{K}$ was **Cat**. Since **Cat** is finitely complete as a 2-category, this allows the construction of Eilenberg-Moore objects which are a crucial ingredient in Eilenberg and Moore's result. Considering Frobenius monads in $\Sigma(M)$, the strictification of the suspension of the monoidal category M, it is apparent that the required Eilenberg-Moore object is unlikely to exist: the 2-category $\Sigma(M)$ has only one object! Fortunately, there is a categorical construction that enlarges a 2-category into one that has Eilenberg-Moore objects. This is known as the Eilenberg-Moore completion and it will be discussed in greater detail later. The important aspect to bear in mind is that this construction produces a 2-category together with an ambijunction generating our Frobenius object.

Frobenius objects have found tremendous use in topology, particularly in the area of topological quantum field theory. A well known result going back to Dijkgraaf [9] states that 2-dimensional topological quantum field theories are equivalent to commutative Frobenius algebras, see also [1, 22]. Our result then indicates that:

> *Every 2D topological quantum field theory arises from an ambijunction in some 2-category.*

More recently, higher-dimensional analogs of Frobenius algebras have begun to appear in higher-dimensional topology. For example, instances of categorified Frobenius structure have appeared in 3D topological quantum field theory [37], Khovanov homology — the homology theory for tangle cobordisms generalizing the Jones polynomial [20], and the theory of thick tangles [26].

In all of the cases mentioned above, the higher-dimensional Frobenius structures can be understood as instances of a single unifying notion — a 'Frobenius pseudomonoid'. A Frobenius pseudomonoid is a categorified Frobenius algebra — a monoidal category satisfying the axioms of a Frobenius algebra up to coherent isomorphism. Being inherently categorical, our approach to solving the problem of constructing adjunctions from Frobenius objects suggests a quite natural procedure for not only defining a Frobenius pseudomonoid[1], but more importantly, for showing that:

> *Every Frobenius pseudomonoid in a semistrict monoidal 2-category arises from a pseudo ambijunction in a semistrict 3-category.*

The categorified theorem as stated above takes place in the context of a semistrict 3-category, also referred to as a **Gray**-category. We take this as a sufficient context for the generalization since every tricategory or weak 3-category is triequivalent to a **Gray**-category [12]. A **Gray**-category can be defined quite simply using enriched category theory [19]. Specifically, a **Gray**-category is a category enriched in **Gray**. Although a more explicit definition of a **Gray**-category can be given, see for instance Marmolejo [29], we will not be needing it for this paper. Adjunctions as well as monads generalize to this

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[1] Note that our definition of Frobenius pseudomonoid nearly coincides with the notion given by Street [36]; the slight difference is that certain isomorphisms in our definition are made explicit.

context and are called pseudoadjunctions and pseudomonads, respectively. They consist of the usual data, where the axioms now hold up to coherent isomorphism. In the context of an arbitrary **Gray**-category we extend the notion of mateship under adjunction to the notion of mateship under pseudoadjunction. Eilenberg-Moore objects and the Eilenberg-Moore completion also make sense in this context, so we are able to categorify Eilenberg and Moore's theorem on adjoint monads, as well as our theorem relating Frobenius objects to ambijunctions, to the context of an arbitrary **Gray**-category.

We remark that Street has demonstrated that the condition for a monoidal category to be a Frobenius pseudomonoid is identical to the condition of $*$-autonomy [36]. These $*$-autonomous monoidal categories are known to have an interesting relationship with quantum groups and quantum groupoids [8]. Combined with our result relating Frobenius pseudomonoids to pseudo ambijunctions, the relationship with $*$-autonomous categories may have implications to quantum groups, as well as the field of linear logic where $*$-autonomous categories are used extensively.

2. Adjoint monads and Frobenius objects

This section can be viewed as a decategorification of the results in Section 3. The expert is encouraged to skip this section and proceed directly to Section 3. For the non-expert, an extended version of this paper is available where additional details are included [25].

2.1. Preliminaries. In this section we review the concepts of adjunctions and monads in an arbitrary 2-category along with some of the general theory needed later on. A good reference for much of the material presented in this section is [17].

2.2. Definition. *An* adjunction $i, e\colon F \dashv U\colon A \to B$ *in a 2-category* $\mathcal{K}$ *consists of morphisms* $U\colon A \to B$ *and* $F\colon B \to A$*, and 2-morphisms* $i\colon 1 \Rightarrow UF$ *and* $e\colon FU \Rightarrow 1$*, such that*

$$\begin{tikzcd} & UFU \arrow[dr, Rightarrow, "Ue"] & \\ U \arrow[ur, Rightarrow, "iU"] \arrow[rr, Rightarrow] & & U \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & FUF \arrow[dr, Rightarrow, "eF"] & \\ F \arrow[ur, Rightarrow, "Fi"] \arrow[rr, Rightarrow] & & F \end{tikzcd}$$

commute.

2.3. Proposition. *If* $i, e\colon F \dashv U\colon A \to B$ *and* $i', e'\colon F' \dashv U'\colon B \to C$ *are adjunctions in the 2-category* $\mathcal{K}$*, then* $FF' \dashv U'U$ *with unit and counit:*

$$\begin{aligned} \bar{i} &:= 1 \xRightarrow{i'} U'F' \xRightarrow{U'iF'} U'UFF' \\ \bar{e} &:= FF'U'U \xRightarrow{Fe'U} FU \xRightarrow{e} 1 \end{aligned}$$

2.4. Definition. *Let* $i, e\colon F \dashv U\colon A \to B$ *and* $i', e'\colon F' \dashv U'\colon A' \to B'$ *in the 2-category* $\mathcal{K}$*. It was shown by Kelly and Street [17] that if* $a\colon A \to A'$ *and* $b\colon B \to B'$*, then there is* *a bijection between 2-morphisms* $\xi\colon bU \Rightarrow U'a$ *and 2-morphisms* $\zeta\colon F'b \Rightarrow aF$*, where* ζ *is given in terms of* ξ *by the composite:*

$$\zeta = \; F'b \xRightarrow{F'bi} F'bUF \xRightarrow{F'\xi F} F'U'aF \xRightarrow{e'aF} aF$$

and ξ *is given in terms of* ζ *by the composite:*

$$\xi = \; bU \xRightarrow{i'bU} U'F'bU \xRightarrow{U'\zeta U} U'aFU \xRightarrow{U'ae} U'a\,.$$

Under these circumstances we say that ξ *and* ζ *are* mates under adjunction.

The naturality of this bijection can be expressed as an isomorphism of certain double categories, see Proposition 2.2 [17]. In both cases, the objects of the double categories are those of $\mathcal{K}$. The horizontal arrows are the morphisms of $\mathcal{K}$ with the usual composition and the vertical arrows are the adjunctions in $\mathcal{K}$ with the composition given in Proposition 2.3. In the first double category, a square with sides $a\colon A \to A'$, $b\colon B \to B'$,$i, e\colon F \dashv U\colon A \to B$, and $i', e'\colon F' \dashv U'\colon A' \to B'$ is a 2-cell $\xi\colon bU \Rightarrow U'a$. In the second double category a square with the same sides is a 2-cell $\zeta\colon F'b \Rightarrow aF$. The isomorphism between these two double categories makes precise the idea that the association of mateship under adjunction respects composites and identities both of adjunctions and of morphisms in $\mathcal{K}$.

2.5. Definition. *A* monad $\mathbb{T} = (T, \mu, \eta)$ *in a 2-category* $\mathcal{K}$ *on the object* B *of* $\mathcal{K}$ *consists of an endomorphism* $T\colon B \to B$ *together with 2-morphisms called the* multiplication *for the monad* $\mu\colon T^2 \Rightarrow T$*, and the* unit *for the monad* $\eta\colon 1 \Rightarrow T$*, such that*

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, equal] & TT \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "\eta T"'] \arrow[dl, equal] \\
 & T &
\end{tikzcd}$$

and

$$\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "\mu T"] \arrow[d, Rightarrow, "T\mu"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

commute.

A comonad is defined by reversing the directions of the 2-cells. A complete treatment of monads in this generality is presented in [24, 32]. It is clear that if $i, e\colon F \dashv U\colon A \to B$ is an adjunction in $\mathcal{K}$, then (UF, UeF, i) is a monad on B. We now recall a result due to Eilenberg-Moore [10], proven in the context $\mathcal{K} = \mathbf{Cat}$, that easily generalizes to arbitrary $\mathcal{K}$.

2.6. Proposition. *Let* $\mathbb{T} = (T, \mu, \eta)$ *be a monad on an object* B *in a 2-category* $\mathcal{K}$ *such that the endomorphism* $T\colon B \to B$ *has a specified right adjoint* G *with counit* $\sigma\colon TG \to 1$ *and unit* $\iota\colon 1 \to GT$*. Then* $\mathbb{G} = (G, \varepsilon, \delta)$ *is a comonad where* ε *and* δ *are the mates under adjunction of* η *and* μ *with the explicit formulas* $\varepsilon = \sigma.\eta G$ *and* $\delta = G^2\sigma.G^2\mu G.G\iota TG.\iota G$ *and* $\mathbb{G}$ *is said to be a comonad* right adjoint *to the monad* $\mathbb{T}$*, denoted* $\mathbb{T} \dashv \mathbb{G}$*.*

Proof. This statement immediately follows from the composition preserving property of the bijection of mates under adjunction. ■

2.7. Definition. *A monad* $\mathbb{T}$ *in the 2-category* $\mathcal{K}$ *is called a* Frobenius monad *if it is equipped with a morphism* $\varepsilon: T \to 1$ *such that* $\varepsilon.\mu$ *is the counit for an adjunction* $T \dashv T$.

The notion of a Frobenius monad (or Frobenius standard construction as it was originally called) was first defined by Lawvere [27]. In Street [36] several definitions of Frobenius monad are given and proven equivalent. If one regards the monoidal category **Vect** as a one object 2-category $\Sigma(\mathbf{Vect})$, then a Frobenius monad in $\Sigma(\mathbf{Vect})$ is just the usual notion of a Frobenius algebra.

2.8. Definition. *An* action *of the monad* $\mathbb{T}$ *on a morphism* $s: A \to B$ *in the 2-category* $\mathcal{K}$ *is a 2-morphism* $\nu: Ts \Rightarrow s$ *such that*

$$\begin{tikzcd} s \arrow[r, Rightarrow, "\eta s"] \arrow[dr, equal] & Ts \arrow[d, Rightarrow, "\nu"] \\ & s \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T^2 s \arrow[r, Rightarrow, "\mu s"] \arrow[d, Rightarrow, "T\nu"'] & Ts \arrow[d, Rightarrow, "\nu"] \\ Ts \arrow[r, Rightarrow, "\nu"'] & s \end{tikzcd}$$

commute. A morphism s *together with an action is called a* $\mathbb{T}$-algebra *(with domain* A*).*

For any morphism $s: A \to B$ in $\mathcal{K}$, Ts with action $\mu s: T^2 s \Rightarrow Ts$ is a $\mathbb{T}$-algebra. For reasons that will soon become apparent we call the $\mathbb{T}$-algebra $(Ts, \mu s)$ a *free* $\mathbb{T}$*-algebra.* The traditional notion of $\mathbb{T}$-algebra corresponds to the notion presented above when $\mathcal{K} = \mathbf{Cat}$ and A is the one object category. In this case we identify the map $s: 1 \to B$ with its image.

2.9. Definition. *Let* $\mathbb{T}$ *be a monad in* $\mathcal{K}$. *For each* A *in* $\mathcal{K}$ *define the category* $\mathbb{T}\text{-}\mathbf{Alg_A}$ *whose objects are* $\mathbb{T}$*-algebras, and whose morphisms between* $\mathbb{T}$*-algebras* (s, ν) *and* (s', ν') *are those 2-morphisms* $h: s \Rightarrow s'$ *of* $\mathcal{K}$ *making*

$$\begin{tikzcd} Ts \arrow[r, Rightarrow, "Th"] \arrow[d, Rightarrow, "\nu"'] & Ts' \arrow[d, Rightarrow, "\nu'"] \\ s \arrow[r, Rightarrow, "h"'] & s' \end{tikzcd}$$

commute. We call the morphisms in $\mathbb{T}\text{-}\mathbf{Alg_A}$ morphisms of $\mathbb{T}$-algebras.

Given a morphism $K: A' \to A$ in $\mathcal{K}$, one can define a change of base functor $\hat{K}$: $\mathbb{T}\text{-}\mathbf{Alg_A} \to \mathbb{T}\text{-}\mathbf{Alg_{A'}}$. If $h: (s, \nu) \to (s', \nu')$ is in $\mathbb{T}\text{-}\mathbf{Alg_A}$, then its image under $\hat{K}$ is $hK: (sK, \nu K) \to (s'K, \nu' K)$. If $k: K \Rightarrow K'$ in $\mathcal{K}$ then we get a natural transformation $\hat{k}: \hat{K} \Rightarrow \hat{K'}$ such that $\hat{k}_{(s,\nu)} = sk$. In fact, this shows that the construction of $\mathbb{T}$-algebras defines a 2-functor $\mathbb{T}\text{-}\mathbf{Alg}: \mathcal{K}^{\mathrm{op}} \to \mathbf{Cat}$.

As with the case when $\mathcal{K} = \mathbf{Cat}$ we have a forgetful functor $U_A^{\mathbb{T}}: \mathbb{T}\text{-}\mathbf{Alg_A} \to \mathcal{K}(A, B)$ with left adjoint: $F_A^{\mathbb{T}}: \mathcal{K}(A, B) \to \mathbb{T}\text{-}\mathbf{Alg_A}$. This adjunction exists for every A in $\mathcal{K}$. In fact, we have the following:

2.10. Proposition. *The collection of adjunctions* $i^{\mathbb{T}}_A, e^{\mathbb{T}}_A\colon F^{\mathbb{T}}_A \Rightarrow U^{\mathbb{T}}_A\colon \mathbb{T}\textbf{-Alg}_{\mathbf{A}} \to \mathcal{K}(A,B)$ *defined for each* A *in* $\mathcal{K}$ *defines an adjunction*

$$i^{\mathbb{T}}, e^{\mathbb{T}}\colon F^{\mathbb{T}} \Rightarrow U^{\mathbb{T}}\colon \mathbb{T}\textbf{-Alg} \to \mathcal{K}(-,B)$$

in the 2-category $[\mathcal{K}^{\mathrm{op}}, \mathbf{Cat}]$ *consisting of 2-functors* $\mathcal{K}^{\mathrm{op}} \to \mathbf{Cat}$*, 2-natural transformations between them, and modifications.*

2.11. Definition. *We say that an* Eilenberg-Moore object exists for a monad $\mathbb{T}$ *if the 2-functor* $\mathbb{T}\textbf{-Alg}\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{Cat}$ *is representable. An Eilenberg-Moore object for the monad* T *is then just a choice of representation for the 2-functor* $\mathbb{T}\textbf{-Alg}$*, that is an object* $B^{\mathbb{T}}$ *of* $\mathcal{K}$ *together with a specified 2-natural isomorphism from* $\mathbb{T}\textbf{-Alg}$ *to the 2-functor* $\mathcal{K}(-,B^{\mathbb{T}})$*.*

If an Eilenberg-Moore object exists for a monad $\mathbb{T}$ then by the enriched Yoneda lemma, or 2-categorical Yoneda lemma as it is sometimes referred to in this context, the adjunction of Proposition 2.10 arises from an adjunction $i^{\mathbb{T}}, e^{\mathbb{T}}\colon F^{\mathbb{T}} \dashv U^{\mathbb{T}}\colon B \to B^{\mathbb{T}}$ in $\mathcal{K}$.

Given a comonad $\mathbb{G}$ in $\mathcal{K}$ we can define the category $\mathbb{G}\textbf{-CoAlg}_A$ of $\mathbb{G}$-coalgebras $(s,\bar{\nu})$ and maps of $\mathbb{G}$-coalgebras by reversing the directions of the 2-cells in the definition of $\mathbb{T}\textbf{-Alg}_A$ and substituting the appropriate data for $\mathbb{G}$. We also have a 2-functor $\mathbb{G}\textbf{-CoAlg}\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{Cat}$ and the forgetful 2-natural transformation $U^{\mathbb{G}}\colon \mathbb{G}\textbf{-CoAlg} \to \mathcal{K}(-B)$. However, in this case, $U^{\mathbb{G}}$ has a right adjoint $F^{\mathbb{G}}$. An Eilenberg-Moore object for a comonad is just a choice of representation for the 2-functor $\mathbb{G}\textbf{-CoAlg}$. If an Eilenberg-Moore object for $\mathbb{G}$ does exist then, again by the 2-categorical Yoneda lemma, the adjunction $i^{\mathbb{G}}, e^{\mathbb{G}}\colon F^{\mathbb{G}} \vdash U^{\mathbb{G}}\colon \mathbb{G}\textbf{-CoAlg} \to \mathcal{K}(-,B)$ arises from an adjunction $i^{\mathbb{G}}, e^{\mathbb{G}}\colon F^{\mathbb{G}} \vdash U^{\mathbb{G}}\colon B \to B^{\mathbb{G}}$ in $\mathcal{K}$.

2.12. Adjoint monads. Given $\mathbb{T} \dashv \mathbb{G}$ in $\mathbf{Cat}$, Eilenberg and Moore [10] showed that mateship under adjunction of action and coaction defines an isomorphism $B^{\mathbb{T}} \cong B^{\mathbb{G}}$ of categories between the Eilenberg-Moore category of $\mathbb{T}$-algebras for the monad $\mathbb{T}$ and the Eilenberg-Moore category of $\mathbb{G}$-coalgebras for the comonad $\mathbb{G}$. In this section we continue the program for the formal theory of monads begun by Street [32]. In particular, we extend the classical theory of adjoint monads developed by Eilenberg and Moore to the context of an arbitrary 2-category.

We can utilize the results of Eilenberg and Moore by observing that a monad $\mathbb{T}$ on B defines a traditional monad $\mathcal{K}(A,\mathbb{T})$ on the category $\mathcal{K}(A,B)$ for every other object A in the 2-category $\mathcal{K}$. Furthermore, $\mathbb{T}\textbf{-Alg}_A$ is just the category of algebras for this monad.

2.13. Lemma. *Let* $\mathbb{T}$ *be a monad on* $B \in \mathcal{K}$*. If* $\iota, \sigma\colon T \dashv G$ *and* G *is equipped with the comonad structure* $\mathbb{G}$ *from Proposition 2.6, then the category* $\mathbb{T}\textbf{-Alg}_A$ *is isomorphic to the category* $\mathbb{G}\textbf{-CoAlg}_A$ *and this isomorphism commutes with the forgetful functors to* $\mathcal{K}(A,B)$*.*

Proof. By the remarks prior to the statement of the lemma this result follows from the work of Eilenberg-Moore [10]. We denote the isomorphism as

$$\mathcal{M}_A\colon \mathbb{T}\textbf{-Alg}_A \to \mathbb{G}\textbf{-CoAlg}_A\colon h\colon (s,\nu) \to (s',\nu') \mapsto h\colon (s,\bar{\nu}) \to (s',\bar{\nu}')$$

where ν is the mate of $\bar{\nu}$ and ν' is the mate of $\bar{\nu}'$ under the adjunction $T \dashv G$. We denote the inverse as $\overline{\mathcal{M}_A}$. ∎

2.14. Theorem. [The adjoint monad theorem] *Let $\mathbb{T}$ be a monad in $\mathcal{K}$ with $T \dashv G$ and denote the induced comonad of Proposition 2.6 as $\mathbb{G}$. Then there is a 2-natural isomorphism $\mathcal{M}\colon \mathbb{T}\textbf{-Alg} \to \mathbb{G}\textbf{-CoAlg}$ making the following diagram*

$$\begin{tikzcd}
\mathbb{T}\textbf{-Alg} \arrow[rr, "\mathcal{M}"] \arrow[dr, "U^{\mathbb{T}}"'] & & \mathbb{G}\textbf{-CoAlg} \arrow[dl, "U^{\mathbb{G}}"] \\
& \mathcal{K}(-,B) &
\end{tikzcd}$$

commute. Furthermore, if one exists, an Eilenberg-Moore object $B^{\mathbb{T}}$ for the monad $\mathbb{T}$ serves as an Eilenberg-Moore object $B^{\mathbb{G}}$ for the comonad $\mathbb{G}$. So that the above diagram arises via the 2-categorical Yoneda lemma from the commutative diagram:

$$\begin{tikzcd}
B^{\mathbb{T}} \arrow[rr, "\mathcal{M}"] \arrow[dr, "U^{\mathbb{T}}"'] & & B^{\mathbb{G}} \arrow[dl, "U^{\mathbb{G}}"] \\
& B &
\end{tikzcd}$$

in $\mathcal{K}$.

Proof. We show that the collection of natural isomorphisms $\mathcal{M}_A$ defined in Lemma 2.13 define a 2-natural isomorphism $\mathcal{M}\colon \mathbb{T}\textbf{-Alg} \to \mathbb{G}\textbf{-CoAlg}$. The 1-naturality of $\mathcal{M}$ follows from the fact that if $K\colon A' \to A$, then

$$\hat{K}\mathcal{M}_A(s,\nu) = \big(sK, (g\nu.\iota s)K\big) = \big(sK, g\nu K.\iota sK\big) = \mathcal{M}_{A'}\hat{K}(s,\nu).$$

The 2-naturality of $\mathcal{M}$ follows from the fact that $\mathcal{M}_A$ is the identity on morphisms. Hence, $\mathcal{M}$ is a 2-natural transformation. From Lemma 2.13 it is clear that $\mathcal{M}$ commutes with the forgetful functors since this is verified pointwise.

If $B^{\mathbb{T}}$ is an Eilenberg-Moore object for the monad T, then we have a choice of 2-natural isomorphism $\mathcal{K}(-,B^{\mathbb{T}}) \cong \mathbb{T}\textbf{-Alg}$. Composing this 2-natural isomorphism with the 2-natural isomorphism $\mathcal{M}$ equips $B^{\mathbb{T}}$ with the structure of an Eilenberg-Moore object for the comonad $\mathbb{G}$. Since the 2-natural isomorphism $\mathcal{M}$ commutes with the forgetful 2-natural isomorphisms $U^{\mathbb{T}}$ and $U^{\mathbb{G}}$, it is clear that their images under the 2-categorical Yoneda lemma will make the required diagram commute. ∎

This theorem shows that if the monad $\mathbb{T}$ has an adjoint comonad $\mathbb{G}$, and if the Eilenberg-Moore objects exists, then the 'forgetful morphism' $U^{\mathbb{T}}\colon B^{\mathbb{T}} \to B$ has not only a left adjoint $F^{\mathbb{T}}$, but also a right adjoint $\overline{\mathcal{M}}F^{\mathbb{G}}$. We can also extend the classical converse of this theorem to show that if a morphism has both a left and right adjoint, then the induced monad and comonad are adjoint.

2.15. Theorem. *Let* $\begin{tikzcd} B \arrow[r, shift left, "L_1"] \arrow[r, phantom, "\perp"] & C \arrow[l, shift left, "R"] \end{tikzcd}$ and $\begin{tikzcd} B \arrow[r, shift left, "L_2"] \arrow[r, phantom, "\top"] & C \arrow[l, shift left, "R"] \end{tikzcd}$ *be specified adjunctions in the 2-category* $\mathcal{K}$. *Also, let* $\mathbb{T}_1$ *be the monad on* B *induced by the composite* RL_1, *and* $\mathbb{T}_2$ *be the comonad on* B *induced by the composite* RL_2. *Then* $\mathbb{T}_1 \dashv \mathbb{T}_2$ *via a specified adjunction determined from the data defining the adjunctions* $L_1 \dashv R \dashv L_2$.

Proof. Let $i_1, e_1 \colon L_1 \dashv R \colon C \to B$ and $i_2, e_2 \colon R \dashv L_2 \colon B \to C$, then it follows from the composition of adjunctions that $RL_1 \dashv RL_2$ with $\iota = Ri_2L_1.i_1 \colon 1 \Rightarrow RL_2RL_1$ and $\sigma = e_2.Re_1L_2 \colon RL_1RL_2 \Rightarrow 1$. The triangle identities follow from the triangle identities for the pairs (i_1, e_1) and (i_2, e_2). It remains to be shown that $\mu = L_1e_1R$ is mates under adjunction with $\delta = Ri_2L_2$ and $\eta = i_1$ is mates with $\varepsilon = e_2$.

The mate to e_2 is given by the composite:

$$1 \xRightarrow{i_1} RL_1 \xRightarrow{Ri_2L_1} RL_2RL_1 \xRightarrow{e_2RL_1} RL_1$$

but, by one of the triangle identities, this is just the map i_1. The mate to i_1 is given by the composite:

$$RL_2 \xRightarrow{i_1RL_2} RL_1RL_2 \xRightarrow{Re_1L_2} RL_2 \xRightarrow{e_2} 1$$

and by the other triangle identity is equal to e_2. Hence η is the mate of ε. In a similar manner it can be shown that μ is the mate of δ using multiple applications of the triangle identities. ■

2.16. Eilenberg-Moore completions. As we saw in the introduction, one of the aims of this paper is to show that every Frobenius object in *any* monoidal category arises from an ambijunction in some 2-category. To prove this, one is tempted to apply Theorem 2.14. However, when regarding a Frobenius object in a monoidal category as a Frobenius monad on the suspension of the monoidal category caution must be exercised. The 2-category $\Sigma(M)$ has only one object. Thus, it is unlikely that the Eilenberg-Moore objects, supposed to exist in Theorem 2.14, actually exists in $\Sigma(M)$.

Street [34] has shown that an Eilenberg-Moore object can be considered as a certain kind of weighted limit. He has also shown that the weight is finite in the sense of [16]. In *The Formal Theory of Monads. II.* [24], Lack and Street use this result to show that one can define $\mathbf{EM}(\mathcal{K})$; the free completion under Eilenberg-Moore objects of the 2-category $\mathcal{K}$. Since the free completion under a class of colimits is more accessible than completions under the corresponding limits, Lack and Street first construct $\mathrm{Kl}(\mathcal{K})$ — the free completion under Kleisli objects. They then take $\mathbf{EM}(\mathcal{K})$ to be $\mathrm{Kl}(\mathcal{K}^{\mathrm{op}})^{\mathrm{op}}$. Since a Kleisli object is a colimit, to construct $\mathrm{Kl}(\mathcal{K})$ one must complete $\mathcal{K}$ embedded in $[\mathcal{K}^{\mathrm{op}}, \mathbf{Cat}]$, by Yoneda, under the class of Φ-colimits, where Φ consists of the weights for Kleisli objects. This amounts to taking the closure of the representables under Φ-colimits [19]. By the theory of such completions, we obtain a 2-functor $Z \colon \mathcal{K} \to \mathbf{EM}(\mathcal{K})$ with the property that for any 2-category $\mathcal{L}$ with Eilenberg-Moore objects, composition with Z induces an equivalence of categories between the 2-functor category $[\mathcal{K}, \mathcal{L}]$ and the full subcategory of the 2-functor category $[\mathbf{EM}(\mathcal{K}), \mathcal{L}]$ consisting of those 2-functors which preserve Eilenberg-Moore objects [24]. Furthermore, the theory of completions under a class of colimits also tells us that Z will be fully faithful.

The Eilenberg-Moore completion can also be given a concrete description. The object of $\mathbf{EM}(\mathcal{K})$ are the monads in $\mathcal{K}$ and the morphisms are the usual morphisms of monads. Hence, a morphism from $\mathbb{T} = (T\colon B \to B, \mu, \eta)$ to $\mathbb{T}' = (T'\colon C \to C, \mu', \eta')$ in $\mathbf{EM}(\mathcal{K})$ is a morphism $F\colon B \to C$ and a 2-morphisms $\phi\colon T'F \Rightarrow FT$ of $\mathcal{K}$ satisfying two equations:

$$\begin{tikzcd}
T'T'F \arrow[r, Rightarrow, "T'\phi"] \arrow[d, Rightarrow, "\mu' F"'] & T'FT \arrow[r, Rightarrow, "\phi T"] & FTT \arrow[d, Rightarrow, "F\mu"] \\
TF \arrow[rr, Rightarrow, "\phi"'] & & FT
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F \arrow[dl, Rightarrow, "\eta' F"'] \arrow[dr, Rightarrow, "F\eta"] & \\
T'F \arrow[rr, Rightarrow, "\phi"'] & & FT
\end{tikzcd}$$

A crucial observation made by Lack and Street is that the 2-morphisms in $\mathbf{EM}(\mathcal{K})$ are *not* the 2-morphisms of the 2-category of monads. Rather, a 2-morphism from (F, ϕ) to (F', ψ) in $\mathbf{EM}(\mathcal{K})$ consists of a 2-morphism $f\colon F \to F'T$ satisfying

$$\begin{tikzcd}
T'F \arrow[r, Rightarrow, "\phi"] \arrow[d, Rightarrow, "T'f"'] & FT \arrow[r, Rightarrow, "fT"] & F'TT \arrow[d, Rightarrow, "F'\mu"] \\
T'F'T \arrow[r, Rightarrow, "\psi T"'] & F'TT \arrow[r, Rightarrow, "F'\mu"'] & F'T
\end{tikzcd}$$

2.17. **EM-Completions in Vect.** The Eilenberg-Moore completion may seem rather substantial, so in order to gain some insight into this procedure we briefly discuss the implications of this completion for **Vect**. The objects of $\mathbf{EM}(\Sigma(\mathbf{Vect}))$ will be the monads in $\Sigma(\mathbf{Vect})$. In this case, a monad in $\Sigma(\mathbf{Vect})$ is an algebra in the traditional sense of linear algebra — a vector space equipped with an associative, unital multiplication. For the duration of this example 'algebra' is to be interpreted in this sense; not in the sense of an algebra for a monad. A morphism in $\mathbf{EM}(\Sigma(\mathbf{Vect}))$ from an algebra A_1 to an algebra A_2 amounts to a vector space V together with a linear map $V \otimes A_2 \xrightarrow{\phi} A_1 \otimes V$ such that

$$\begin{tikzcd}
V{\otimes}A_2{\otimes}A_2 \arrow[r, "\phi\otimes A_2"] \arrow[d, "V\otimes m_2"'] & A_1{\otimes}V{\otimes}A_2 \arrow[r, "A_1\otimes\phi"] & A_1{\otimes}A_1{\otimes}V \arrow[d, "m_1\otimes V"] \\
V{\otimes}A_2 \arrow[rr, "\phi"'] & & A_1{\otimes}V
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V \arrow[dl, "V\otimes\iota_2"'] \arrow[dr, "\iota_1\otimes V"] & \\
V{\otimes}A_2 \arrow[rr, "\phi"'] & & A_1{\otimes}V
\end{tikzcd}$$

commute, where (m_1, ι_1) and (m_2, ι_2) are the multiplication and unit for the algebras A_1 and A_2 respectively. This might be described as a left-free bimodule: a vector space V with a right A_2 action on $A_1 \otimes V$ given by $A_1 \otimes V \otimes A_2 \xrightarrow{A_1\otimes\phi} A_1 \otimes A_1 \otimes V \xrightarrow{m_1} A_1 \otimes V$. This action makes $A_1 \otimes V$ into a (A_1, A_2)-bimodule. The composite of morphisms $(V, \phi)\colon A_1 \to A_2$ and $(V', \phi')\colon A_2 \to A_3$ is given by $(V \otimes V', \phi \otimes V' \circ V \otimes \phi')\colon A_1 \to A_3$ – the left-free bimodule $A_1 \otimes V \otimes V'$.

A 2-morphism in $\mathbf{EM}(\Sigma(\mathbf{Vect}))$ from $(V,\phi) \Rightarrow (V',\psi)$ is a linear map $\rho\colon V \to A_1 \otimes V'$ making

$$\begin{tikzcd}
V{\otimes}A_2 \arrow[r, "\phi"] \arrow[d, "\rho{\otimes}A_2"'] & A_1{\otimes}V \arrow[r, "A_1{\otimes}\rho"] & A_1{\otimes}A_1{\otimes}V' \arrow[d, "m_1{\otimes}V'"] \\
A_1{\otimes}V'{\otimes}A_2 \arrow[r, "A_1{\otimes}\psi"'] & A_1{\otimes}A_1{\otimes}V' \arrow[r, "m_1{\otimes}V'"'] & A_1{\otimes}V'
\end{tikzcd}$$

commute. This amounts to saying that a 2-morphism is just a bimodule homomorphism of left-free bimodules. To summarize:

> *Every Frobenius algebra in* **Vect** *will be shown to arise from an ambijunction in the 2-category* $\mathbf{EM}(\Sigma(\mathbf{Vect}))$ *consisting of: algebras, left-free bimodules, and bimodule homomorphisms.*

Recall that the Eilenberg-Moore completion was obtained from the Kleisli completion as $\mathrm{Kl}(\mathcal{K}^{\mathrm{op}})^{\mathrm{op}}$. Hence, a similar description of the Kleisli completion of $\Sigma(\mathbf{Vect})$ can be given in terms of right-free bimodules[2].

Ambijunctions in the Eilenberg-Moore completion of $\Sigma(\mathbf{Vect})$ correspond to the notion of a *Frobenius extension* familiar to algebraists, see for example [15]. For an algebra A over the field k we have the inclusion map $\iota\colon k \to A$. The category of A-modules corresponds to the category of algebras for the monad A in $\Sigma(\mathbf{Vect})$. The restriction functor $\mathrm{Res}\colon A{-}\mathrm{mod} \to k{-}\mathrm{mod}$ has left and right adjoint functors: the induction functor $\mathrm{Ind}(M) = A \otimes_k M$ and coinduction $\mathrm{CoInd}(M) = \mathrm{Hom}_k(A,M)$. When A is a Frobenius algebra in **Vect** these functors are isomorphic defining an ambijunction generating A.

2.18. Frobenius monads and ambijunctions.

2.19. Lemma. *Let* $\mathbb{T} = (T,\mu,\eta,\varepsilon)$ *be a Frobenius monad on* $\mathcal{K}$ *with* $\iota, \varepsilon.\mu\colon T \dashv T$. *For notational convenience, denote the induced comonad of Proposition 2.6 on* T *as* $\mathbb{G}$. *Then the 2-natural isomorphism* $\mathcal{M}$ *of Theorem 2.14 satisfies the commuting diagram*

$$\begin{tikzcd}
\mathbb{T}\textbf{-Alg} \arrow[rr, "\mathcal{M}"] \arrow[dr, shift right, "U^{\mathbb{T}}"'] & & \mathbb{G}\textbf{-CoAlg} \arrow[dl, shift left, "U^{\mathbb{G}}"] \\
& \mathcal{K}(-,B) \arrow[ul, shift right, "F^{\mathbb{T}}"'] \arrow[ur, shift left, "F^{\mathbb{G}}"] &
\end{tikzcd}$$

Proof. By Theorem 2.14 all we must show is that $\mathcal{M}F^{\mathbb{T}} = F^{\mathbb{G}}$. Hence, by the remarks prior to Lemma 2.13 this follows from [10]. ∎

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[2] This description of the Eilenberg-Moore completion and Kleisli completion was explained to the author by Steve Lack.

Compare the following two Theorems to Proposition 1.4 and Proposition 1.5 of [36].

2.20. Theorem. *Given a Frobenius monad* $(\mathbb{T}, \mu, \eta, \varepsilon)$ *on an object* B *in* $\mathcal{K}$*, then in* $\mathbf{EM}(\mathcal{K})$ *the left adjoint* $F^{\mathbb{T}}\colon B \to B^{\mathbb{T}}$ *to the forgetful functor* $U^{\mathbb{T}}\colon B^{\mathbb{T}} \to B$ *is also right adjoint to* $U^{\mathbb{T}}$ *with counit* ε*. Hence, the Frobenius monad* $\mathbb{T}$ *is generated by an ambidextrous adjunction in* $\mathbf{EM}(\mathcal{K})$*.*

Proof. Identify $\mathbb{T}$ with its fully faithful image via the 2-functor $Z\colon \mathcal{K} \to \mathbf{EM}(\mathcal{K})$; then an Eilenberg-Moore object $B^{\mathbb{T}}$ for the monad $\mathbb{T}$ exists in $\mathbf{EM}(\mathcal{K})$. Let $\mathbb{G}$ denote the induced comonad structure on T given in Proposition 2.6. Then by the adjoint monad theorem, the object $B^{\mathbb{T}}$ serves as an Eilenberg-Moore object $B^{\mathbb{G}}$ for the comonad $\mathbb{G}$ and we have that $U^{\mathbb{T}} = U^{\mathbb{G}}\mathcal{M}$.

By the remarks following Definition 2.11 we have $i^{\mathbb{T}}, e^{\mathbb{T}}\colon F^{\mathbb{T}} \dashv U^{\mathbb{T}}\colon B \to B^{\mathbb{T}}$ and $i^{\mathbb{G}}, e^{\mathbb{G}}\colon F^{\mathbb{G}} \vdash U^{\mathbb{G}}\colon B \to B^{\mathbb{G}}$. Since $U^{\mathbb{G}}F^{\mathbb{G}}$ generates the comonad $\mathbb{G}$ and ε is the counit for the comonad $\mathbb{G}$, it is clear that $e^{\mathbb{G}} = \varepsilon$ above. All that remains to be shown is that $\mathcal{M}F^{\mathbb{T}} = F^{\mathbb{G}}$. This follows by the 2-categorical Yoneda lemma applied to Lemma 2.19. Hence, the Frobenius monad $\mathbb{T} = U^{\mathbb{T}}F^{\mathbb{T}}$ is generated by an ambijunction $F^{\mathbb{T}} \dashv U^{\mathbb{T}} \dashv F^{\mathbb{T}}$ in $\mathbf{EM}(\mathcal{K})$. ■

2.21. Theorem. *Let* $i, e, j, k\colon F \dashv U \dashv F\colon B \to C$ *be an ambidextrous adjunction in the 2-category* $\mathcal{K}$*. Then the monad* (UF, UiF, e) *generated by the adjunction is a Frobenius monad with* $\varepsilon = k$*.*

Proof. All we must show is that $UF \dashv UF$ with counit $k.UiF$. Define the unit of the adjunction to be $UjF.i$. The zig-zag identities follow from the zig-zag identities for (i, e) and (j, k). ■

2.22. Corollary. *If* $\begin{tikzcd} B \arrow[r, shift left, "F"] \arrow[r, phantom, "\top\perp"] & C \arrow[l, shift left, "U"] \end{tikzcd}$ *is a specified ambijunction in the 2-category* $\mathcal{K}$*, then* UF *is a Frobenius object in the strict monoidal category* $\mathcal{K}(B, B)$*.*

Proof. By Theorem 2.21, UF defines a Frobenius monad on the object B in $\mathcal{K}$. As explained above, this is simply a Frobenius object in the monoidal category $\mathcal{K}(B, B)$. ■

2.23. Corollary. *A Frobenius object in a monoidal category* M *yields an ambijunction in* $\mathbf{EM}(\Sigma(M))$*, where* $\Sigma(M)$ *is the 2-category obtained by the strictification of the suspension of* M*.*

Proof. Recall that a monad on an object B in a 2-category $\mathcal{K}$ can be thought of as a monoid object in the monoidal category $\mathcal{K}(B, B)$. Similarly, a comonad on B is just a comonoid object in $\mathcal{K}(B, B)$. Regarding M as a one object 2-category $\Sigma(M)$, a Frobenius object in M is simply a Frobenius monad in $\Sigma(M)$. Applying Theorem 2.20 completes the proof. ■

2.24. Corollary. *Every Frobenius algebra in the category* **Vect** *arises from an ambidextrous adjunction in the 2-category whose objects are algebras, morphisms are bimodules of algebras, and whose 2-morphisms are bimodule homomorphisms.*

Proof. This follows immediately from Corollary 2.23 and the discussion in subsection 2.17. ∎

2.25. Corollary. *Every 2D topological quantum field theory, in the sense of Atiyah [2], arises from an ambidextrous adjunction in the 2-category whose objects are algebras, morphisms are bimodules of algebras, and whose 2-morphisms are bimodule homomorphisms.*

Proof. Since a 2D topological quantum field theory is equivalent to a commutative Frobenius algebra [1, 22], the proof follows from Corollary 2.24. ∎

3. Categorification

In this section we extend the theory of the previous section to the context of **Gray**-categories. **Gray** is the symmetric monoidal closed category whose underlying category is **2-Cat**; the category whose objects are 2-categories, and whose morphisms are 2-functors. **Gray** differs from **2-Cat** in that **Gray** has a more interesting monoidal structure than the usual cartesian monoidal structure on **2-Cat**. A **Gray**-category, also known as a semistrict 3-category, is defined using enriched category theory [19] as a category enriched in **Gray**. The unusual tensor product in **Gray**, or 'Gray tensor product', has the effect of equipping a **Gray**-category $\mathcal{K}$ with a cubical functor $M\colon \mathcal{K}(B,C) \times \mathcal{K}(A,B) \to \mathcal{K}(A,C)$ for all objects A,B,C in $\mathcal{K}$. This means that if $f\colon F \Rightarrow F'$ in $\mathcal{K}(A,B)$, and $g\colon G \Rightarrow G'$ in $\mathcal{K}(B,C)$, then, rather than commuting on the nose, we have an invertible 3-cell $M_{g,f}$ — denoted g_f following Marmolejo [29] — in the following square:

$$\begin{tikzcd}
GF \arrow[r, Rightarrow, "gF"] \arrow[d, Rightarrow, "Gf"'] \arrow[dr, phantom, "\Downarrow g_f"] & G'F \arrow[d, Rightarrow, "G'f"] \\
GF' \arrow[r, Rightarrow, "gF'"'] & G'F'.
\end{tikzcd}$$

We take this notion to be a sufficiently general extension since every tricategory or weak 3-category is triequivalent to a **Gray**-category [12].

The proof of the adjoint monad theorem relied heavily on the notion of mates under adjunction and the fact that this relationship respected composites of morphisms and adjunctions. In order to categorify this theorem we will first have to categorify the notion of mates under adjunction to the notion of mates under pseudoadjunction. In this case, rather than a bijection between certain morphisms, we will have an equivalence of Hom categories. The naturality of this equivalence will also be discussed.

In Section 3.8 we define the notion of a pseudomonad in a **Gray**-category and review some of the basic theory. Using the notion of mateship under pseudoadjunction it is shown that if a pseudomonad has a specified pseudoadjoint G, then G is a pseudocomonad. All of the theorems from the previous section are then extended into this context and the notion of a Frobenius pseudomonad and Frobenius pseudomonoid are given. The main result that every Frobenius pseudomonoid arises from a pseudo ambijunction is then proven as a corollary of the categorified version of the Eilenberg-Moore adjoint monad theorem in Section 3.17.

3.1. Pseudoadjunctions. We begin with the definition of a pseudoadjunction given by Verity in [38] where they were called *locally-adjoint biadjoint pairs.* For more details see also the discussion by Lack where the 'free living' or 'walking' pseudoadjunction is defined [23].

3.2. Definition. *A* pseudoadjunction $I, E, i, e\colon F \dashv_p U\colon A \to B$ *in a* **Gray**-*category* $\mathcal{K}$ *consists of morphisms* $U\colon A \to B$ *and* $F\colon B \to A$, *2-morphisms* $i\colon 1 \Rightarrow UF$ *and* $e\colon FU \Rightarrow 1$, *and coherence 3-isomorphisms*

$$\begin{tikzcd} & UFU \arrow[dr, Rightarrow, "Ue"] & \\ U \arrow[ur, Rightarrow, "iU"] \arrow[rr, Rightarrow, "1"'] \arrow[rr, phantom, bend left=35, "I\Uparrow"] & & U \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & FUF \arrow[dr, Rightarrow, "eF"] & \\ F \arrow[ur, Rightarrow, "Fi"] \arrow[rr, Rightarrow, "1"'] \arrow[rr, phantom, bend left=35, "E\Downarrow"] & & F \end{tikzcd}$$

such that the following two diagrams are both identities:

$$\begin{tikzcd} & & FU \arrow[ddr, Rightarrow, "e"] & \\ & FI\Downarrow & & \\ FU \arrow[uurr, Rightarrow, bend left, "1"] \arrow[r, Rightarrow, "FiU"] \arrow[ddrr, Rightarrow, bend right, "1"'] & FUFU \arrow[uur, Rightarrow, "FUe"] \arrow[ddr, Rightarrow, "eFU"'] & \Downarrow e_e^{-1} & 1 \\ & EU\Downarrow & & \\ & & FU \arrow[uur, Rightarrow, "e"'] & \end{tikzcd} \qquad \begin{tikzcd} & UF \arrow[ddr, Rightarrow, "iUF"'] \arrow[ddrr, Rightarrow, bend left, "1"] & & \\ & & \Downarrow IF & \\ 1 \arrow[uur, Rightarrow, "i"] \arrow[ddr, Rightarrow, "i"'] & i_i^{-1}\Downarrow & UFUF \arrow[r, Rightarrow, "UeF"] & UF \\ & & \Downarrow UE & \\ & UF \arrow[uur, Rightarrow, "UFi"] \arrow[uurr, Rightarrow, bend right, "1"'] & & \end{tikzcd}$$

We will sometimes denote a pseudoadjunction as $F \dashv_p U$ and say that the morphism U is the *right pseudoadjoint* of F. Likewise, F is said to be the *left pseudoadjoint* of U.

3.3. Proposition. *If* $I, E, i, e\colon F \dashv_p U\colon A \to B$ *and* $I', E', i', e'\colon F' \dashv_p U'\colon B \to C$, *then* $FF' \dashv_p U'U$ *with*

$$\begin{aligned} \bar{i} &:= 1 \xRightarrow{i'} U'F' \xRightarrow{U'iF'} U'UFF' \\ \bar{e} &:= FF'U'U \xRightarrow{Fe'U} FU \xRightarrow{e} 1 \end{aligned}$$

and

$$\bar{I} := \begin{tikzcd}[column sep=small]
 & & U'UFF'U'U \arrow[rd, Rightarrow, "U'UFe'U"] \arrow[d, phantom, "\Uparrow U'i_{e'}U"] & & \\
 & U'F'U'U \arrow[ru, Rightarrow, "U'iF'U'U"] \arrow[rd, Rightarrow, "U'e'U"] \arrow[d, phantom, "\Uparrow I'U"] & & U'UFU \arrow[rd, Rightarrow, "U'eU"] \arrow[d, phantom, "\Uparrow U'I"] & \\
U'U \arrow[ru, Rightarrow, "i'U'U"] \arrow[rr, Rightarrow, "1"'] & & U'U \arrow[ru, Rightarrow, "U'Ui"] \arrow[rr, Rightarrow, "1"'] & & U'U
\end{tikzcd}$$

$$\bar{E} := \begin{tikzcd}[column sep=small]
 & & FF'U'UFF' \arrow[rd, Rightarrow, "Fe'UFF'"] \arrow[d, phantom, "\Downarrow Fi_{e'}^{-1}F'"] & & \\
 & FF'U'F' \arrow[ru, Rightarrow, "FF'U'iF'"] \arrow[rd, Rightarrow, "Fe'F'"] \arrow[d, phantom, "\Downarrow FE'"] & & FUFF' \arrow[rd, Rightarrow, "FeF'"] \arrow[d, phantom, "\Downarrow EF'"] & \\
FF' \arrow[ru, Rightarrow, "FF'i'"] \arrow[rr, Rightarrow, "1"'] & & FF' \arrow[ru, Rightarrow, "iFF'"] \arrow[rr, Rightarrow, "1"'] & & FF'.
\end{tikzcd}$$

Proof. The proof is given in [13] although it is a routine verification and can be checked directly. ■

3.4. Proposition. *Let*

- $I, E, i, e\colon F \dashv_p U\colon A \to B$, *and*
- $I', E', i', e'\colon F' \dashv_p U'\colon A' \to B'$

in the **Gray**-*category* $\mathcal{K}$. *If* $a\colon A \to A'$ *and* $b\colon B \to B'$, *then there is an equivalence of categories* $\mathcal{K}(bU, U'a) \simeq \mathcal{K}(F'b, aF)$ *given by:*

$$\begin{array}{rcl}
\Theta\colon \mathcal{K}(bU, U'a) & \to & \mathcal{K}(F'b, aF) \\
\xi & \mapsto & \zeta = \; F'b \xRightarrow{F'bi} F'bUF \xRightarrow{F'\xi F} F'U'aF \xRightarrow{e'aF} aF
\end{array}$$

$$\omega\colon \xi_1 \Rrightarrow \xi_2 \quad \mapsto \quad \begin{tikzcd}
F'b \arrow[r, Rightarrow, "F'bi"] & F'bUF \arrow[r, Rightarrow, bend left, "F'\xi_1 F"] \arrow[r, Rightarrow, bend right, "F'\xi_2 F"'] \arrow[r, phantom, "\Downarrow F'\omega F"] & F'U'aF \arrow[r, Rightarrow, "e'aF"] & aF
\end{tikzcd}$$

and

$$\begin{array}{rcl}
\Phi\colon \mathcal{K}(F'b, aF) & \to & \mathcal{K}(bU, U'a) \\
\zeta & \mapsto & \xi = \; bU \xRightarrow{i'bU} U'F'bU \xRightarrow{U'\zeta U} U'aFU \xRightarrow{U'ae} U'a
\end{array}$$

$$\varrho\colon \zeta_1 \Rrightarrow \zeta_2 \quad \mapsto \quad \begin{tikzcd}
bU \arrow[r, Rightarrow, "i'bU"] & U'F'bU \arrow[r, Rightarrow, bend left, "U'\zeta_1 U"] \arrow[r, Rightarrow, bend right, "U'\zeta_2 U"'] \arrow[r, phantom, "\Downarrow U'\varrho U"] & U'aFU \arrow[r, Rightarrow, "U'ae"] & U'a\,.
\end{tikzcd}$$

Proof. It is clear that Θ is a functor from its definition above. That is, Θ preserves composites of 3-morphisms along 2-morphisms in $\mathcal{K}$. Let ξ be an object of $\mathcal{K}(bU, U'a)$ so that $\Phi\Theta(\xi)$ is given by the composite

$$bU \xRightarrow{i'bU} U'F'bU \xRightarrow{U'F'biU} U'F'bUFU \xRightarrow{U'F'\xi FU} U'F'U'aFU \xRightarrow{U'e'aFU} U'aFU \xRightarrow{U'ae} U'a\ .$$

Define an isomorphism $\gamma_\xi\colon \xi \Rrightarrow \Phi\Theta(\xi)$ by the diagram

$$\begin{tikzcd}
 & & & {\scriptstyle i'^{-1}_{\xi}\Downarrow} & U'a \arrow[rrd, Rightarrow, "i'U'a"'] \arrow[rrrrd, Rightarrow, "1"] & & {\scriptstyle \Downarrow I'a} & & \\
bU \arrow[rrrru, Rightarrow, "\xi"] \arrow[r, Rightarrow, "i'bU"'] & U'F'bU \arrow[rr, Rightarrow, "1"] \arrow[rd, Rightarrow, "U'F'biU"'] & {\scriptstyle \Downarrow U'F'bE^{-1}} & U'F'bU \arrow[rrr, Rightarrow, "U'F'\xi"] & & {\scriptstyle \Downarrow U'F'\xi_e^{-1}} & U'F'U'a \arrow[rr, Rightarrow, "U'e'a"'] & {\scriptstyle \Downarrow U'e'^{-1}_e} & U'a \\
 & & U'F'bUFU \arrow[ru, Rightarrow, "U'F'bUe"'] \arrow[rrr, Rightarrow, "U'F'\xi FU"'] & & & U'F'U'aFU \arrow[ru, Rightarrow, "U'F'U'ae"'] \arrow[rr, Rightarrow, "U'e'aFU"'] & & U'aFU \arrow[ru, Rightarrow, "U'ae"'] &
\end{tikzcd}$$

which is invertible because I', E and the structural maps in the **Gray**-category are invertible. It is straight forward to check the naturality of this isomorphism. Let $\omega\colon \xi_1 \Rrightarrow \xi_2$; then $\gamma_{\xi_2} \circ \omega = \Phi\Theta(\omega) \circ \gamma_{\xi_1}$ by the invertibility of I', E and the axioms of a **Gray**-category. The isomorphism $\bar{\gamma}_\zeta\colon \zeta \Rrightarrow \Theta\Phi(\zeta)$, for ζ in $\mathcal{K}(F'b, aF)$, is given by:

$$\begin{tikzcd}
 & & & {\scriptstyle \zeta_i\Downarrow} & aF \arrow[rrd, Rightarrow, "aiF"'] \arrow[rrrrd, Rightarrow, "1"] & & {\scriptstyle \Downarrow aI} & & \\
F'b \arrow[rrrru, Rightarrow, "\zeta"] \arrow[r, Rightarrow, "F'bi"'] & F'bUF \arrow[rr, Rightarrow, "1"] \arrow[rd, Rightarrow, "F'i'bUF"'] & {\scriptstyle \Downarrow E'^{-1}bUF} & F'bUF \arrow[rrr, Rightarrow, "\zeta UF"] & & {\scriptstyle \Downarrow e'_{\zeta}UF} & aFUF \arrow[rr, Rightarrow, "aeF"'] & {\scriptstyle \Downarrow e'_{ae}F} & aF \\
 & & F'U'F'bUF \arrow[ru, Rightarrow, "e'F'bUF"'] \arrow[rrr, Rightarrow, "F'U'\zeta UF"'] & & & F'U'aFUF \arrow[ru, Rightarrow, "e'aFUF"'] \arrow[rr, Rightarrow, "F'U'aeF"'] & & F'U'aF \arrow[ru, Rightarrow, "e'aF"'] &
\end{tikzcd}$$

By similar arguments as above this isomorphism is natural. ■

Using this equivalence of categories we extend the notion of mateship under adjunction to the notion of *mateship under pseudoadjunction.* We now express the naturality conditions this equivalence satisfies:

3.5. PROPOSITION. *Consider the collection of pseudoadjunctions and morphisms:*

- $I, E, i, e\colon F \dashv_p U\colon A \to B$,
- $I', E', i', e'\colon F' \dashv_p U'\colon A' \to B'$,

- *$I'', E'', i'', e''\colon F'' \dashv_p U''\colon A'' \to B''$, and*

- *$a\colon A \to A', a'\colon A' \to A'', b\colon B \to B', b'\colon B' \to B''$,*

in the **Gray***-category $\mathcal{K}$. Let*

$$\begin{aligned}
\Theta\colon \mathcal{K}(bU, U'a) &\to \mathcal{K}(F'b, aF) & \Phi\colon \mathcal{K}(F'b, aF) &\to \mathcal{K}(bU, U'a)\\
\Theta'\colon \mathcal{K}(b'U', U''a') &\to \mathcal{K}(F''b', a'F') & \Phi'\colon \mathcal{K}(F''b', a'F') &\to \mathcal{K}(b'U', U''a')\\
\bar{\Theta}\colon \mathcal{K}(b'bU, U''a'a) &\to \mathcal{K}(F''b'b, a'aF) & \bar{\Phi}\colon \mathcal{K}(F''b'b, a'aF) &\to \mathcal{K}(b'bU, U''a'a)
\end{aligned}$$

be the functors from Proposition 3.4 defining the relevant equivalences of categories. Then there exists a natural isomorphism W between the following pasting composites of functors:

$$\begin{aligned}
a'\Theta(-).\Theta'(-)b \quad &: \quad \mathcal{K}(bU, U'a) \times \mathcal{K}(b'U', U''a') \to \mathcal{K}(F''b'b, a'aF)\\
\bar{\Theta}(-a.b'-) \quad &: \quad \mathcal{K}(bU, U'a) \times \mathcal{K}(b'U', U''a') \to \mathcal{K}(F''b'b, a'aF),
\end{aligned}$$

and a natural isomorphism Y between the pasting composites:

$$\begin{aligned}
\Phi'(-)a.b'\Phi(-) \quad &: \quad \mathcal{K}(F'b, aF) \times \mathcal{K}(F''b', a'F') \to \mathcal{K}(b'bU, U''a'a)\\
\bar{\Phi}(a'-.-b) \quad &: \quad \mathcal{K}(F'b, aF) \times \mathcal{K}(F''b', a'F') \to \mathcal{K}(b'bU, U''a'a).
\end{aligned}$$

Proof. Let $\xi \in \mathcal{K}(bU, U'a)$ and $\xi' \in \mathcal{K}(b'U', U''a')$, then $W(\xi \times \xi')$ is given by the following pasting composite of invertible 3-morphisms:

$$\begin{tikzcd}
 & & & F''U''a'F'b \arrow[r, Rightarrow, "e''a'F'b"] \arrow[dr, Rightarrow, "F''U''a'F'bi"] & a'F'b \arrow[rr, Rightarrow, "a'F'bi"] \arrow[d, phantom, "\Downarrow e''_{a'F'bi}"] & & a'F'bUF \arrow[drr, Rightarrow, "a'F'\xi F"] & & & \\
 & F''b'U'F'b \arrow[urr, Rightarrow, "F''\xi'F'b"] \arrow[dr, Rightarrow, "F''b'U'F'bi"] & & \Downarrow F''\xi'_{F'bi} & F''U''a'F'bUF \arrow[urr, Rightarrow, "e''a'F'bUF"'] \arrow[drrr, Rightarrow, "F''U''a'F'\xi F"] & & \Downarrow e''_{a'F'\xi}F & & a'F'U'aF \arrow[dr, Rightarrow, "a'e'aF"] & \\
F''b'b \arrow[ur, Rightarrow, "F''b'i'b"] \arrow[dddr, Rightarrow, "F''b'bi"'] & & F''b'U'F'bUF \arrow[urr, Rightarrow, "F''\xi'F'bUF"] \arrow[drr, Rightarrow, "F''b'U'F'\xi F"] & & \Downarrow F''\xi'_{F'\xi}F & & & F''U''a'F'U'aF \arrow[ur, Rightarrow, "e''a'F'U'aF"] \arrow[dddr, Rightarrow, "F''U''a'e'aF"'] & & a'aF \\
 & \Downarrow F''b'i'_{bi} & & \Downarrow F''b'i'_{\xi}F & F''b'U'F'U'aF \arrow[urrr, Rightarrow, "F''\xi'F'U'aF"] \arrow[dr, Rightarrow, "F''bU'e'aF"] & & & \Downarrow F''\xi'_{e'}aF & \Downarrow e''_{ae'}aF & \\
 & & & & \Downarrow F''b'I'aF & F''b'U'aF \arrow[drrr, Rightarrow, "F''\xi'aF"] & & & & \\
 & F''b'bUF \arrow[uuur, Rightarrow, "F''b'i'bUF"'] \arrow[rr, Rightarrow, "F''b'\xi F"'] & & F''b'U'aF \arrow[uur, Rightarrow, "F''b'i'U'aF"] \arrow[urr, Rightarrow, "1"] \arrow[rrrrr, Rightarrow, "F''\xi'aF"'] & & \| & & & F''U''a'aF \arrow[uuur, Rightarrow, "e''a'aF"'] &
\end{tikzcd}$$

If $\omega\colon \xi_1 \Rrightarrow \xi_2$ and $\omega'\colon \xi_1' \Rrightarrow \xi_2'$ then the naturality of W follows from the axioms of the cubical functor defining the Gray tensor product. Given $\zeta \in \mathcal{K}(F'b, aF)$ and $\zeta' \in \mathcal{K}(F''b', a'F')$

the natural isomorphism Y can be defined similarly:

$$\begin{tikzcd}
 & & & b'U'aFU \arrow[r, Rightarrow, "b'U'ae"] \arrow[dr, Rightarrow, "i''b'U'aFU"'] & b'U'a \arrow[rr, Rightarrow, "i''b'U'a"] & & U''F''b'U'a \arrow[drr, Rightarrow, "U''\zeta'U'a"] & & & \\
 & b'U'F'bU \arrow[urr, Rightarrow, "b'U'\zeta U"] \arrow[dr, Rightarrow, "i''b'U'F'bU"] & & (i''_{b'U'\zeta})^{-1}U \Downarrow & U''F''b'U'aFU \arrow[urr, Rightarrow, "U''F''b'U'ae"'] \arrow[drrr, Rightarrow, "U''\zeta'F'U'aFU"] & (i''_{b'U'aFUe})^{-1} \Downarrow & \Downarrow U''(\zeta'_{U'ae})^{-1} & & U''a'F'U'a \arrow[dr, Rightarrow, "U''a'e'a"] & \\
b'bU \arrow[ur, Rightarrow, "b'i'bU"] \arrow[dddr, Rightarrow, "i''b'bU"'] & & U''F''b'U'F'bU \arrow[urr, Rightarrow, "U''F''b'U'\zeta U"] \arrow[drr, Rightarrow, "U''\zeta'U'F'bU"] & & U''(\zeta'_{U'\zeta})^{-1}U \Downarrow & & & U''a'F'U'aFU \arrow[ur, Rightarrow, "U'a'F'U'ae"] \arrow[dddr, Rightarrow, "U''a'e'aFU"'] & & U''a'a \\
 & (i''b'_{i'})^{-1}bU \Downarrow & & & U''a'F'U'F'bU \arrow[urrr, Rightarrow, "U''a'F'U'\zeta'U"] \arrow[dr, Rightarrow, "U''a'e'F'bU"'] & & \Downarrow U''a'(e'_{F'U'\zeta'})^{-1}U & & \Downarrow U''a'(e'_{ae})^{-1} & \\
 & & \Downarrow U''(\zeta'_{i'})^{-1} & & U''a'E'^{-1}bU \Downarrow & U''a'F'bU \arrow[drrr, Rightarrow, "U''a'\zeta U"] \arrow[d, phantom, "\|"] & & & & \\
 & U''F''b'bU \arrow[uuur, Rightarrow, "U''F''i'bU"'] \arrow[rr, Rightarrow, "U''\zeta'bU"'] & & U''a'F'bU \arrow[uur, Rightarrow, "U''a'F'i'bU"] \arrow[ur, Rightarrow, "1"] \arrow[rrrrr, Rightarrow, "U''a'\zeta U"'] & & & & & U''a'aFU \arrow[uuur, Rightarrow, "U''a'ae"'] &
\end{tikzcd}$$

and by similar arguments, Y can be shown to be natural. ■

We will find it necessary later in this paper to refer to the natural isomorphisms W and Y within the context of some specified choices of composable pseudoadjunctions and morphisms a, a', b, b'. We will refer to these isomorphisms generically as W and Y, even though we will consider many different choices of pseudoadjunctions and morphisms. This is possible since these natural isomorphisms exist for every possible choice of this data. Furthermore, the specific pseudoadjoints and morphisms should be clear from the context. When no confusion is likely to arise we will also denote $\Theta', \bar{\Theta}, \Phi'$, and $\bar{\Phi}$ simply as Θ and Φ, respectively. Note in particular that when $a = 1_A, b = 1_B, a' = 1_A, b' = 1_B$, then we have $\Theta(\xi).\Theta(\xi') \cong \Theta(\xi'.\xi)$, and similarly for Φ.

3.6. Proposition. *Consider the collection of pseudoadjunctions and morphisms:*

- $I, E, i, e\colon F \dashv_p U\colon A \to B$,
- $I', E', i', e'\colon F' \dashv_p U'\colon A' \to B'$,
- $I_1, E_1, i_1, e_1\colon F_1 \dashv_p U_1\colon B \to C$,
- $I_1', E_1', i_1', e_1'\colon F_1' \dashv_p U_1'\colon B' \to C'$, *and*
- $a\colon A \to A', b\colon B \to B', c\colon C \to C'$

in the **Gray***-category* $\mathcal{K}$*. Let*

$$\begin{aligned}
\Theta\colon \mathcal{K}(bU, U'a) &\to \mathcal{K}(F'b, aF) & \qquad \Phi\colon \mathcal{K}(F'b, aF) &\to \mathcal{K}(bU, U'a)\\
\Theta_1\colon \mathcal{K}(cU_1, U_1'b) &\to \mathcal{K}(F_1'c, bF_1) & \qquad \Phi_1\colon \mathcal{K}(F_1'c, bF_1) &\to \mathcal{K}(cU_1, U_1'b)\\
\bar{\Theta}\colon \mathcal{K}(cU_1U, U_1'U'a) &\to \mathcal{K}(F'F_1'c, aFF_1) & \qquad \bar{\Phi}\colon \mathcal{K}(F'F_1'c, aFF_1) &\to \mathcal{K}(cU_1U, U_1'U'a)
\end{aligned}$$

be the functors from Proposition 3.4 defining the relevant equivalences of categories. Here $\bar{\Theta}$ and $\bar{\Phi}$ are the equivalence corresponding to the composite pseudoadjunction defined in

Proposition 3.3. Then there exists a natural isomorphism V between the following pasting composites of functors:

$$\begin{aligned}\Theta(-)F_1.F'\Theta_1(-) &\ : \ \mathcal{K}(bU, U'a) \times \mathcal{K}(cU_1, U_1'b) \to \mathcal{K}(F'F_1'c, aFF_1)\\ \bar{\Theta}(U_1' - . - U) &\ : \ \mathcal{K}(bU, U'a) \times \mathcal{K}(cU_1, U_1'b) \to \mathcal{K}(F'F_1'c, aFF_1),\end{aligned}$$

and a natural isomorphism X between the pasting composites:

$$\begin{aligned}U_1'\Phi(-).\Phi_1(-)U &\ : \ \mathcal{K}(F'b, aF) \times \mathcal{K}(F_1'c, bF_1) \to \mathcal{K}(cU_1U, U_1'Ua)\\ \bar{\Phi}(-F_1.F'-) &\ : \ \mathcal{K}(F'b, aF) \times \mathcal{K}(F_1'c, bF_1) \to \mathcal{K}(cU_1U, U_1'Ua).\end{aligned}$$

Proof. Let $\xi \in \mathcal{K}(bU, U'a)$ and $\xi_1 \in \mathcal{K}(cU_1, U_1'b)$, then $V(\xi \times \xi_1)$ is given by the following pasting composite of invertible 3-morphisms:

$$\begin{tikzcd}
 & & F'F_1'U_1'bF_1 \arrow[rr, Rightarrow, "F'e_1'bF_1"] \arrow[ddddrr, Rightarrow, "FF_1U_1biF_1"'] & & F'bF_1 \arrow[rr, Rightarrow, "F'biF_1"] & & F'bUFF_1 \arrow[dr, Rightarrow, "F'\xi FF_1"] & & \\
 & F'F_1'cU_1F_1 \arrow[ur, Rightarrow, "F'F_1'\xi_1F_1"] \arrow[dd, equal] & & & & & & F'U'aFF_1 \arrow[dr, Rightarrow, "e'aFF_1"] \arrow[dd, equal] & \\
F'F_1'c \arrow[ur, Rightarrow, "FF_1'ci_1"] \arrow[dr, Rightarrow, "F'F_1'ci_1"'] & & F'F_1'\xi_{1_i}F_1 \Downarrow & & F'e'_{1_{bi}}F_1 \Downarrow & & \Downarrow F'e'_{1_\xi}FF_1 & & aFF_1 \\
 & F'F_1'cU_1F_1 \arrow[dr, Rightarrow, "F'F_1'cU_1iF_1"'] & & & & & & F'U'aFF_1 \arrow[ur, Rightarrow, "e'aFF_1"'] & \\
 & & F'F_1'cU_1UFF_1 \arrow[rr, Rightarrow, "F'F_1'\xi_1UFF_1"'] & & F'F_1'U_1'bUFF_1 \arrow[uurr, Rightarrow, "F'e_1'bUFF_1"] \arrow[rr, Rightarrow, "F'F_1'U_1\xi FF_1"'] & & F'F_1U_1UaFF_1 \arrow[ur, Rightarrow, "F'e_1'U'aFF_1"'] & &
\end{tikzcd}$$

Since this 3-isomorphism is composed entirely of **Gray**-naturality isomorphisms, it is clear that V is a natural isomorphisms. Given $\zeta \in \mathcal{K}(F'b, aF)$ and $\zeta_1 \in \mathcal{K}(F_1'c, bF_1)$ the natural isomorphism X can be defined similarly:

$$\begin{tikzcd}
 & & U_1'bF_1U_1U \arrow[rr, Rightarrow, "U_1'be_1U"] \arrow[ddddrr, Rightarrow, "U_1'i'bF_1U_1U"'] & & U_1'bU \arrow[rr, Rightarrow, "U_1'i'bU"] & & U_1'U'F'bU \arrow[dr, Rightarrow, "U_1'U'\zeta'U"] & & \\
 & U_1'F_1'cU_1U \arrow[ur, Rightarrow, "U_1'\zeta_1U_1U"] \arrow[dd, equal] & & & & & & U_1'U'aFU \arrow[dr, Rightarrow, "U_1'U'ae"] \arrow[dd, equal] & \\
cU_1U \arrow[ur, Rightarrow, "i_1'cU_1U"] \arrow[dr, Rightarrow, "i_1'cU_1U"'] & & U_1'(i'_{\zeta_1})^{-1}U_1U \Downarrow & & U_1'(i'b_{e_1U})^{-1} \Downarrow & & \Downarrow U_1'U'(\zeta_{e_1})^{-1}U & & U_1'U'a \\
 & U_1'F_1'cU_1U \arrow[dr, Rightarrow, "U_1'i'F_1'cU_1U"'] & & & & & & U_1'U'aFU \arrow[ur, Rightarrow, "U_1'U'ae"'] & \\
 & & U_1'U'F'F_1'cU_1U \arrow[rr, Rightarrow, "U_1'UF'\zeta_1U_1U"'] & & U_1'U'F'bF_1U_1U \arrow[uurr, Rightarrow, "U_1'U'F'be_1U"] \arrow[rr, Rightarrow, "U_1'U'\zeta F_1U_1U"'] & & U_1'U'aFF_1U_1U \arrow[ur, Rightarrow, "U_1'U'aFe_1U"'] & &
\end{tikzcd}$$

■

As with the isomorphisms W and Y in Proposition 3.5, we will find it necessary to generically refer to the natural isomorphisms V and X even though we may consider many different choices of pseudoadjunctions and composable morphisms a, b, c. Again, this is allowed because these natural isomorphisms exist for every choice of this data.

Before moving on to the theory of pseudomonads we first collect a result about the functors Θ and Φ.

3.7. Proposition. *Let Θ and Φ be as in Proposition 3.4 with $I, E, i, e \colon F \dashv_p U = I', E', i', e' \colon F' \dashv_p U'$ and $a = 1_A$ and $b = 1_B$. Then in the category $\mathcal{K}(U,U)$ the object $\Phi(1_F)$ is isomorphic to the object 1_U, and in the category $\mathcal{K}(F,F)$ the object $\Theta(1_U)$ is isomorphic to the object 1_F.*

Proof. The isomorphisms are I^{-1} and E respectively. ∎

3.8. Pseudomonads. Here we present the theory of pseudomonads. For more details see [23, 29, 30].

3.9. Definition. *A* pseudomonad $\mathbb{T} = (T, \mu, \eta, \lambda, \rho, \alpha)$ *on an object B of the* **Gray***-category $\mathcal{K}$ consists of an endomorphism $T \colon B \to B$ together with* multiplication *for the pseudomonad $\mu \colon T^2 \Rightarrow T$,* unit *for the pseudomonad $\eta \colon 1 \Rightarrow T$, and coherence 3-isomorphisms*

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow] \arrow[dr, phantom, shift left=4, "\lambda\,\Lleftarrow"] & TT \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow] \arrow[dl, phantom, shift right=4, "\Lleftarrow\,\rho"'] \\
& T &
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
& T^3 \arrow[dl, Rightarrow, "T\mu"'] \arrow[dr, Rightarrow, "\mu T"] & \\
T^2 \arrow[dr, Rightarrow, "\mu"'] \arrow[rr, phantom, "\overset{\alpha}{\Rrightarrow}"] & & T^2 \arrow[dl, Rightarrow, "\mu"] \\
& T &
\end{tikzcd}$$

such that the following two equations are satisfied:

$$\begin{tikzcd}
T^4 \arrow[r, Rightarrow, "T^2\mu"] \arrow[dd, Rightarrow, "\mu T^2"'] \arrow[dr, Rightarrow, "T\mu T"] & T^3 \arrow[dr, Rightarrow, "T\mu"] \arrow[d, phantom, "\Lleftarrow T\alpha"] & \\
& T^3 \arrow[r, Rightarrow, "T\mu"'] \arrow[dd, Rightarrow, "\mu T"'] \arrow[dl, phantom, "\alpha T \Lleftarrow"] \arrow[ddr, phantom, "\overset{\alpha}{\Lleftarrow}"] & T^2 \arrow[dd, Rightarrow, "\mu"] \\
T^3 \arrow[dr, Rightarrow, "\mu T"'] & & \\
& T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^4 \arrow[r, Rightarrow, "T^2\mu"] \arrow[dd, Rightarrow, "\mu T^2"'] \arrow[ddr, phantom, "\overset{\mu_\mu^{-1}}{\Lleftarrow}"] & T^3 \arrow[dr, Rightarrow, "T\mu"] \arrow[dd, Rightarrow, "\mu T"] & \\
& & T^2 \arrow[dd, Rightarrow, "\mu"] \arrow[dl, phantom, "\Lleftarrow \alpha"] \\
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "\mu T"'] & T^2 \arrow[d, phantom, "\overset{\alpha}{\Lleftarrow}"] \arrow[dr, Rightarrow, "\mu"'] & \\
& T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

$$\begin{tikzcd}
& & T^2 \arrow[dr, Rightarrow, "\mu"] \arrow[dd, phantom, "\alpha \Downarrow"] & \\
T^2 \arrow[r, Rightarrow, "T\eta T"] & T^3 \arrow[ur, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "\mu T"'] & & T \\
& & T^2 \arrow[ur, Rightarrow, "\mu"'] &
\end{tikzcd}
=
\begin{tikzcd}
& T^3 \arrow[dr, Rightarrow, "T\mu"] \arrow[d, phantom, "T\lambda \Downarrow"] & & \\
T^2 \arrow[ur, Rightarrow, "T\eta T"] \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "T\eta T"'] & {} \arrow[d, phantom, "\rho T \Downarrow"] & T^2 \arrow[r, Rightarrow, "\mu"] & T \\
& T^3 \arrow[ur, Rightarrow, "\mu T"'] & &
\end{tikzcd}$$

This definition was given by F. Marmolejo in [29] and can be understood as a pseudomonoid (in the sense of [7]) in $\mathcal{K}(B,B)$. An elegant treatment of pseudomonads is presented in [23] where the 'free living' or 'walking' pseudomonad is defined. A *pseudocomonad* $\mathbb{G} = (G, \delta, \varepsilon, \bar{\lambda}, \bar{\rho}, \bar{\alpha})$ is defined by reversing the directions of the 2-cells in the definition of a pseudomonad. A pseudocomonad can also be understood as a pseudocomonoid in $\mathcal{K}(B,B)$.

3.10. Proposition. [Lack [23]] *A pseudoadjunction* $F \dashv_p U\colon B \to C$ *in the* **Gray**-*category* $\mathcal{K}$ *induces a pseudomonad* $(UF, i, UeF, IF, UE, Ue_e^{-1})$ *on the object* B *in* $\mathcal{K}$.

3.11. Proposition. *Let* $\mathbb{T} = (T, \mu, \eta, \lambda, \rho, \alpha)$ *be a pseudomonad on an object* B *in a* **Gray**-*category* $\mathcal{K}$ *such that the endomorphism* $T\colon B \to B$ *has a specified right pseudoadjoint* G *with counit* $\sigma\colon TG \to 1$, *unit* $\iota\colon 1 \to GT$, *and coherences* $\Upsilon\colon \sigma T.T\iota \to 1$ *and* $\Sigma\colon 1 \to G\sigma.\iota G$. *Then mateship under pseudoadjunction, together with the natural isomorphisms in Propositions 3.5 and 3.6, define a pseudocomonad* $\mathbb{G} = (G, \varepsilon, \delta, \bar{\lambda}, \bar{\rho}, \bar{\alpha})$ *on* G *with explicit formulas:*

$$\begin{aligned}
\varepsilon &:= \Phi(\eta) = \sigma.\eta G \\
\delta &:= \Phi(\mu) = G^2\sigma.G^2\mu G.G\iota TG.\iota G \\
\bar{\lambda} &:= G\Phi(\eta).\Phi(\mu) \overset{G\Phi(\eta).\Sigma G.\Phi(\mu)}{\Rrightarrow} G\Phi(\eta).\Phi(1_T)G.\Phi(\mu) \overset{X.\Phi(\mu)}{\Rrightarrow} \Phi(\eta T).\Phi(\mu) \overset{Y}{\Rrightarrow} \Phi(\mu.\eta T) \overset{\Phi(\lambda)}{\Rrightarrow} \Phi(1_T) \overset{\Sigma^{-1}}{\Rrightarrow} G \\
\bar{\rho} &:= G \overset{\Sigma}{\Rrightarrow} \Phi(1_T) \overset{\Phi(\rho)}{\Rrightarrow} \Phi(\mu.T\eta) \overset{Y^{-1}}{\Rrightarrow} \Phi(T\eta).\Phi(\mu) \overset{X^{-1}}{\Rrightarrow} \Phi(1_T).\Phi(\eta)G.\Phi(\mu) \overset{\Sigma^{-1}.\Phi(\eta)G.\Phi(\mu)}{\Rrightarrow} \Phi(\eta)G.\Phi(\mu) \\
\bar{\alpha} &:= \Phi(\mu)G.\Phi(\mu) \overset{GG\Sigma.\Phi(\mu)G.\Phi(\mu)}{\Rrightarrow} GG\Phi(1_T).\Phi(\mu)G.\Phi(\mu) \overset{X.\Phi(\mu)}{\Rrightarrow} \Phi(T\mu).\Phi(\mu) \overset{Y}{\Rrightarrow} \Phi(\mu.T\mu) \\
&\quad \overset{\Phi(\alpha)}{\Rrightarrow} \Phi(\mu.\mu T) \overset{Y^{-1}}{\Rrightarrow} \Phi(\mu T).\Phi(\mu) \overset{X^{-1}.\Phi(\mu)}{\Rrightarrow} G\Phi(\mu).\Phi(1_T)G.\Phi(\mu) \overset{G\Phi(\mu).\Sigma^{-1}G.\Phi(\mu)}{\Rrightarrow} G\Phi(\mu).\Phi(\mu)
\end{aligned}$$

Under these circumstances $\mathbb{G}$ *is said to be a pseudocomonad right pseudoadjoint to the pseudomonad* $\mathbb{T}$, *denoted* $\mathbb{T} \dashv_p \mathbb{G}$.

Proof. Mateship under pseudoadjunction preserves composites along morphisms and pseudoadjoints up to natural isomorphism by Propositions 3.5 and 3.6. Therefore because $\mathbb{T} = (T, \mu, \eta, \lambda, \rho, \alpha)$ is a pseudomonad, $\mathbb{G} = (G, \varepsilon, \delta, \bar{\lambda}, \bar{\rho}, \bar{\alpha})$ defines a pseudocomonad on B. ∎

3.12. Definition. *A pseudomonad* $\mathbb{T}$ *in the* **Gray**-*category* $\mathcal{K}$ *is called a* Frobenius pseudomonad *if it is equipped with a map* $\varepsilon\colon T \to 1$ *such that* $\varepsilon.\mu$ *is the counit for a specified pseudoadjunction* $T \dashv_p T$.

We use this notion of Frobenius pseudomonad to define a Frobenius pseudomonoid in a **Gray**-monoid or semistrict monoidal 2-category. A **Gray**-monoid is just a one object **Gray**-category. In particular, if $\mathcal{K}$ is a **Gray**-category and B is an object of $\mathcal{K}$, then $\mathcal{K}(B, B)$ is a **Gray**-monoid. A Frobenius pseudomonad on B is then just a Frobenius pseudomonoid in the **Gray**-monoid $\mathcal{K}(B, B)$. This definition of Frobenius pseudomonoid takes the minimalist approach, a pseudomonoid equipped with the specified pseudoadjoint structure that enables one to construct a pseudocomonoid structure. For a more explicit description of this definition see Street's work [36]. One may prefer the definition of a Frobenius pseudomonoid to be symmetrical: a pseudomonoid, and a pseudocomonoid subject to compatibility conditions. In the sequel to this paper we explain the relationship between these two perspectives which turn out to be equivalent in a precise sense [26].

We now describe the generalization of algebras for a monad and construct the 2-category of pseudoalgebras based at A for a pseudomonad $\mathbb{T}$. Pseudoalgebras for a

2-monad were first explicitly defined by Street [33] and were well known to the Australian category theory community at that time [18]. For a treatment using the powerful machinery of Blackwell-Kelly-Power [5], see [4]. The treatment we give here follows Marmolejo [29].

3.13. DEFINITION. *Let* $\mathbb{T}$ *be a pseudomonad in the* **Gray**-*category* $\mathcal{K}$ *and let* A *be an object of* $\mathcal{K}$. *We define a* pseudoalgebra based at A *for the pseudomonad* $\mathbb{T}$ *to consist of a morphism* $s\colon A \to B$, *a 2-morphisms* $\nu\colon Ts \Rightarrow s$, *and 3-isomorphisms*

$$\begin{tikzcd}
s \arrow[r, Rightarrow, "\eta s"] \arrow[dr, Rightarrow] & Ts \arrow[d, Rightarrow, "\nu"] \arrow[dl, phantom, "\psi\,\swarrow"] \\
& s
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& T^2s \arrow[dl, Rightarrow, "T\nu"'] \arrow[dr, Rightarrow, "\mu s"] & \\
Ts \arrow[rr, phantom, "\overset{\chi}{\Rrightarrow}"] \arrow[dr, Rightarrow, "\nu"'] & & Ts \arrow[dl, Rightarrow, "\nu"] \\
& s &
\end{tikzcd}$$

such that the following two equations are satisfied:

$$\begin{tikzcd}
T^3s \arrow[rr, Rightarrow, "T^2\nu"] \arrow[dr, Rightarrow, "T\mu T"] \arrow[dd, Rightarrow, "\mu T^2"'] & & T^2s \arrow[dr, Rightarrow, "T\nu"] \arrow[d, phantom, "T\chi\,\swarrow"] & \\
& T^2s \arrow[rr, Rightarrow, "T\mu"'] \arrow[dd, Rightarrow, "\mu\nu"'] \arrow[dl, phantom, "\alpha s\,\swarrow"] \arrow[ddrr, phantom, "\chi\,\swarrow"] & {} & Ts \arrow[dd, Rightarrow, "\nu"] \\
T^2s \arrow[dr, Rightarrow, "\mu\nu"'] & & & \\
& Ts \arrow[rr, Rightarrow, "\nu"'] & & s
\end{tikzcd}
=
\begin{tikzcd}
T^3s \arrow[rr, Rightarrow, "T^2\nu"] \arrow[dd, Rightarrow, "\mu Ts"'] \arrow[ddrr, phantom, "\mu_\nu^{-1}\,\swarrow"] & & T^2s \arrow[dr, Rightarrow, "T\nu"] \arrow[dd, Rightarrow, "\mu s"] & \\
& & & Ts \arrow[dd, Rightarrow, "\nu"] \arrow[dl, phantom, "\swarrow\,\chi"] \\
T^2s \arrow[rr, Rightarrow, "T\nu"] \arrow[dr, Rightarrow, "\mu s"'] & & Ts \arrow[dr, Rightarrow, "\nu"'] \arrow[d, phantom, "\chi\,\swarrow"] & \\
& Ts \arrow[rr, Rightarrow, "\nu"'] & {} & s
\end{tikzcd}$$

$$\begin{tikzcd}
& & Ts \arrow[dr, Rightarrow, "\nu"] \arrow[dd, phantom, "\chi\,\Downarrow"] & \\
Ts \arrow[r, Rightarrow, "T\eta s"] & T^2s \arrow[ur, Rightarrow, "Ts"] \arrow[dr, Rightarrow, "\mu s"'] & & s \\
& & Ts \arrow[ur, Rightarrow, "\nu"'] &
\end{tikzcd}
=
\begin{tikzcd}
& T^2s \arrow[dr, Rightarrow, "T\nu"] \arrow[d, phantom, "T\psi\,\Downarrow"] & & \\
Ts \arrow[ur, Rightarrow, "T\eta s"] \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "T\eta s"'] & {} \arrow[d, phantom, "\rho s\,\Downarrow"] & Ts \arrow[r, Rightarrow, "\nu"] & s. \\
& T^2s \arrow[ur, Rightarrow, "\mu s"'] & &
\end{tikzcd}$$

It is clear that for any morphism $r\colon A \to B$ in $\mathcal{K}$, Tr with action $\mu r\colon T^2r \Rightarrow Tr$ and coherence $\lambda r\colon \mu r.\eta Tr \Rrightarrow Tr$ and $\alpha r\colon \mu r.T\mu t \Rrightarrow \mu r.\mu Tr$ is a pseudoalgebra based at A. We call the pseudoalgebra Tr a *free pseudoalgebra.*

3.14. DEFINITION. *Let* $\mathbb{T}$-$\mathbf{Alg_A}$ *be the 2-category whose objects are pseudoalgebras based at* A *for the pseudomonad* $\mathbb{T}$. *A morphism* $(h, \varrho)\colon (s, \nu, \psi, \chi) \to (s', \nu', \psi', \chi')$ *in* $\mathbb{T}$-$\mathbf{Alg_A}$ *consists of a 2-morphism* $h\colon s \Rightarrow s'$ *in* $\mathcal{K}$ *(a morphism in* $\mathcal{K}(A, B)$*), together with an invertible 3-morphism*

$$\begin{tikzcd}
& Ts \arrow[dl, Rightarrow, "Th"'] \arrow[dr, Rightarrow, "\nu"] & \\
Ts' \arrow[rr, phantom, "\overset{\varrho}{\Rrightarrow}"] \arrow[dr, Rightarrow, "\nu'"'] & & s \arrow[dl, Rightarrow, "h"] \\
& s' &
\end{tikzcd}$$

satisfying the following two equations:

$$\begin{tikzcd}
s \arrow[r, Rightarrow, "\eta s"] \arrow[dr, Rightarrow, "1"'] & Ts \arrow[r, Rightarrow, "Th"] \arrow[d, Rightarrow, "\nu"] \arrow[dl, phantom, "\overset{\psi}{\Swarrow}"] & Ts' \arrow[d, Rightarrow, "\nu'"] \arrow[dl, phantom, "\overset{\varrho}{\Swarrow}"] \\
& s \arrow[r, Rightarrow, "h"'] & s'
\end{tikzcd}
=
\begin{tikzcd}
s \arrow[r, Rightarrow, "\eta s"] \arrow[d, Rightarrow, "h"'] \arrow[dr, phantom, "\overset{\eta_h}{\Swarrow}"] & Ts \arrow[d, Rightarrow, "Th"] \\
s' \arrow[r, Rightarrow, "\eta s'"] \arrow[dr, Rightarrow, "1"'] & Ts' \arrow[d, Rightarrow, "\nu'"] \arrow[dl, phantom, "\overset{\psi'}{\Swarrow}"] \\
& s'
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
T^2s \arrow[rr, Rightarrow, "T^2h"] \arrow[dd, Rightarrow, "\mu s"'] \arrow[dr, Rightarrow, "T\nu"] & & T^2s' \arrow[dr, Rightarrow, "T\nu'"] \arrow[dl, phantom, "\overset{T\varrho}{\Swarrow}"] & \\
& Ts \arrow[rr, Rightarrow, "Th"'] \arrow[dd, Rightarrow, "\nu"'] \arrow[dl, phantom, "\overset{\chi}{\Swarrow}"] & & Ts' \arrow[dd, Rightarrow, "\nu'"] \\
Ts \arrow[dr, Rightarrow, "\nu"'] & & \overset{\varrho}{\Swarrow} & \\
& s \arrow[rr, Rightarrow, "h"'] & & s'
\end{tikzcd}
=
\begin{tikzcd}
T^2s \arrow[rr, Rightarrow, "T^2h"] \arrow[dd, Rightarrow, "\mu s"'] \arrow[ddrr, phantom, "\overset{\mu_h^{-1}}{\Swarrow}"] & & T^2s' \arrow[dd, Rightarrow, "\mu s'"] \arrow[dr, Rightarrow, "T\nu'"] & \\
& & & Ts' \arrow[dd, Rightarrow, "\nu'"] \arrow[dl, phantom, "\overset{\chi'}{\Swarrow}"] \\
Ts \arrow[rr, Rightarrow, "Th"] \arrow[dr, Rightarrow, "\nu"'] & & Ts' \arrow[dr, Rightarrow, "\nu'"'] \arrow[dl, phantom, "\overset{\varrho}{\Swarrow}"] & \\
& s \arrow[rr, Rightarrow, "h"'] & & s'
\end{tikzcd} \tag{2}$$

A 2-morphism $\xi\colon (h, \varrho) \Rightarrow (h', \varrho')\colon (\psi, \chi) \to (\psi', \chi')$ *in* $\mathbb{T}$-$\mathbf{Alg_A}$ *is a 3-morphisms* $\xi\colon h \Rrightarrow h'$ *such that the following condition is satisfied:*

$$\begin{tikzcd}
Ts \arrow[rr, Rightarrow, bend left, "Th"] \arrow[rr, Rightarrow, bend right, "Th'"'] \arrow[rr, phantom, "T\xi \Downarrow"] \arrow[dd, Rightarrow, "\nu"'] & & Ts' \arrow[dd, Rightarrow, "\nu'"] \\
& \varrho' \Downarrow & \\
s \arrow[rr, Rightarrow, "h'"'] & & s'
\end{tikzcd}
=
\begin{tikzcd}
Ts \arrow[rr, Rightarrow, "Th"] \arrow[dd, Rightarrow, "\nu"'] & & Ts' \arrow[dd, Rightarrow, "\nu'"] \\
& \varrho \Downarrow & \\
s \arrow[rr, Rightarrow, bend left, "h"] \arrow[rr, Rightarrow, bend right, "h'"'] \arrow[rr, phantom, "\xi \Downarrow"] & & s'
\end{tikzcd} \tag{3}$$

Marmolejo has shown that given a morphism $K\colon A' \to A$ in $\mathcal{K}$, one can define a change of base 2-functor $\hat{K}\colon \mathbb{T}\text{-}\mathbf{Alg_A} \to \mathbb{T}\text{-}\mathbf{Alg_{A'}}$. If $\xi\colon (h, \varrho) \Rightarrow (h', \varrho')\colon (s, \nu, \psi, \chi) \to (s', \nu', \psi', \chi')$ is in $\mathbb{T}$-$\mathbf{Alg_A}$, then its image under $\hat{K}$ is $\xi K\colon (hK, \varrho K) \Rightarrow (h'K, \varrho' K)\colon (sK, \nu K, \psi K, \chi K) \to (s'K, \nu' K, \psi' K, \chi' K)$. If $k\colon K \Rightarrow K'$ in $\mathcal{K}$ then we get a pseudo natural transformation $\hat{k}\colon \hat{K} \Rightarrow \hat{K}'$ such that $\hat{k}_{(s,\nu,\psi,\chi)} = (sk, \nu_k^{-1})$ and $\hat{k}_{(h,\varrho)} = h_k^{-1}$. If $\kappa\colon k \Rrightarrow k'\colon K \Rightarrow K'$, then $\kappa_{(s,\nu,\psi,\chi)} = s\kappa$ defines a modification $\hat{\kappa}\colon \hat{k} \Rrightarrow \hat{k}$. In fact, this shows that the construction of $\mathbb{T}$-pseudoalgebras defines a **Gray**-functor $\mathbb{T}$-$\mathbf{Alg}\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{Gray}$.

For every object A in $\mathcal{K}$ there is a forgetful 2-functor $U_A^{\mathbb{T}}\colon \mathbb{T}\text{-}\mathbf{Alg_A} \to \mathcal{K}(A, B)$

$$\begin{array}{rcl}
U_A^{\mathbb{T}}\colon \mathbb{T}\text{-}\mathbf{Alg_A} & \to & \mathcal{K}(A, B) \\
(s, \nu, \psi, \chi) & \mapsto & s \\
(h, \varrho) & \mapsto & h \\
\xi\colon h \Rrightarrow h' & \mapsto & \xi.
\end{array}$$

This assignment extends to a **Gray**-natural transformation $U^{\mathbb{T}}\colon \mathbb{T}\textbf{-Alg} \to \mathcal{K}(-,B)$. In Proposition 3.15 we will define a left pseudoadjoint $F_A^{\mathbb{T}}$ to the 2-functor $U_A^{\mathbb{T}}$, see also [29]. In Theorem 3.16 we will show that this left pseudoadjoint $F_A^{\mathbb{T}}$ extends to **Gray**-natural transformation $F^{\mathbb{T}}\colon \mathcal{K}(-,B) \to \mathbb{T}\textbf{-Alg}$ left pseudoadjoint to $U^{\mathbb{T}}$ in the **Gray**-category $[\mathcal{K}^{\mathrm{op}}, \mathbf{Gray}]$ described below.

Recall that **Gray** is the symmetric monoidal closed category whose closed structure is given by the internal hom in **Gray**. Hence, for **Gray**-functors $F, G\colon \mathcal{K} \to \mathcal{L}$ the internal hom $\mathbf{Gray}(F,G)$ in **Gray** is the 2-category consisting of 2-functors, pseudo natural transformations, and modifications. It is a standard result from enriched category theory that **Gray**-categories, **Gray**-functors, and **Gray**-natural transformations form a 2-category written **Gray-Cat** [6, 19]. Furthermore, since **Gray** is a complete symmetric monoidal closed category, if $\mathcal{K}$ is small, then the category of **Gray**-functors and **Gray**-natural transformations can be provided with the structure of a **Gray**-category, written $[\mathcal{K}, \mathcal{L}]$.

The objects of $[\mathcal{K}, \mathcal{L}]$ are the **Gray**-functors $F, G\colon \mathcal{K} \to \mathcal{L}$, and the morphisms are the **Gray**-natural transformations between them. The 2-category $\textbf{Gray-Nat}(F,G)$ of **Gray**-natural transformations is given by the following equalizer:

$$\begin{tikzcd}
\textbf{Gray-Nat}(F,G) \arrow[r, tail] & \prod_{A\in\mathcal{K}} \mathcal{L}(FA,GA) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \prod_{A',A''\in\mathcal{K}} [\mathcal{K}(A',A''), \mathcal{L}(FA',GA'')]
\end{tikzcd}$$

where u and v are the morphisms corresponding via adjunction and symmetry to the morphisms[3]:

$$\begin{tikzcd}
\left(\prod_A \mathcal{L}(FA,GA)\right) \otimes \mathcal{K}(A',A'') \arrow[d, "p_{A'}\otimes G_{A'A''}"] & \mathcal{K}(A',A'') \otimes \left(\prod_A \mathcal{L}(FA,GA)\right) \arrow[d, "F_{A'A''}\otimes p_{A''}"] \\
\mathcal{L}(FA',GA') \otimes \mathcal{L}(GA',GA'') \arrow[d, "c_{FA',GA',GA''}"] & \mathcal{L}(FA',FA'') \otimes \mathcal{L}(FA'',GA'') \arrow[d, "c_{FA',FA'',GA''}"] \\
\mathcal{L}(FA',GA'') & \mathcal{L}(FA',GA'')
\end{tikzcd}$$

We will refer to the morphisms and 2-morphisms of the 2-category $\textbf{Gray-Nat}(F,G)$ as **Gray**-modifications and **Gray**-perturbations respectively. This terminology should not be interpreted to mean some sort of '**Gray** enriched modification' or '**Gray** enriched perturbation' since there is no such notion as a $\mathcal{V}$-modification or $\mathcal{V}$-perturbation for arbitrary enriching category $\mathcal{V}$.

Let $\alpha, \beta\colon F \Rightarrow G\colon \mathcal{K} \to \mathbf{Gray}$ be **Gray**-natural transformations with $\mathcal{K}$ a small **Gray**-category. A **Gray**-modification $\theta\colon \alpha \Rrightarrow \beta$ assigns to each object A of $\mathcal{K}$ a pseudo natural transformation $\theta_A\colon \alpha_A \to \beta_A$ such that if $k\colon K \Rightarrow K'\colon A' \to A''$ in $\mathcal{K}$, then the following

---

[3] Here we are using the notation for $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations given in [6].

equality holds:

$$\begin{tikzcd}
\alpha_{A''}FK \arrow[r, Rightarrow, "\alpha_{A''}Fk"] \arrow[d, Rightarrow, "\theta_{A''}FK"'] \arrow[dr, phantom, "\overset{\sim}{\Swarrow}"] & \alpha_{A''}FK' \arrow[d, Rightarrow, "\theta_{A''}FK'"] \\
\beta_{A''}FK \arrow[r, Rightarrow, "\beta_{A''}Fk"'] & \beta_{A''}FK'
\end{tikzcd}
=
\begin{tikzcd}
GK\alpha_{A'} \arrow[r, Rightarrow, "Gk\alpha_{A'}"] \arrow[d, Rightarrow, "GK\theta_{A'}"'] \arrow[dr, phantom, "\overset{\sim}{\Swarrow}"] & GK'\alpha_{A'} \arrow[d, Rightarrow, "GK'\theta_{A'}"] \\
GK\beta_{A'} \arrow[r, Rightarrow, "Gk\beta_{A'}"'] & GK'\beta_{A'}
\end{tikzcd}$$

If $\Omega\colon \theta, \varphi\colon \alpha \Rightarrow \beta\colon F \to G\colon \mathcal{K} \to \mathbf{Gray}$ are **Gray**-modifications, a **Gray**-perturbation assigns to each object $A \in \mathcal{K}$ a modification $\Omega_A\colon \theta_A \to \varphi_A$ such that if $\kappa\colon k \Rrightarrow k'\colon K \Rightarrow K'\colon A' \to A''$ in $\mathcal{K}$, then the following equality holds:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\alpha_{A''}FK \arrow[r, Rightarrow, bend left, "\alpha_{A''}Fk"] \arrow[r, Rightarrow, bend right, "\alpha_{A''}Fk'"'] \arrow[r, phantom, "\alpha_{A''}F\kappa\Downarrow"] \arrow[d, Rightarrow, bend right=50, "\varphi_{A''}FK"'] \arrow[d, Rightarrow, bend left=50, "\theta_{A''}FK"] \arrow[d, phantom, "\overset{\Omega_{A''}FK}{\Leftarrow}"] \arrow[dr, phantom, "\overset{\sim}{\Swarrow}"] & \alpha_{A''}FK' \arrow[d, Rightarrow, "\theta_{A''}FK'"] \\
\beta_{A''}FK \arrow[r, Rightarrow, "\beta_{A''}Fk'"'] & \beta_{A''}FK'
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=huge]
GK\alpha_{A'} \arrow[r, Rightarrow, "Gk\alpha_{A'}"] \arrow[d, Rightarrow, "GK\varphi_{A'}"'] \arrow[dr, phantom, "\overset{\sim}{\Swarrow}"] & GK'\alpha_{A'} \arrow[d, Rightarrow, bend right=50, "GK'\varphi_{A'}"'] \arrow[d, Rightarrow, bend left=50, "GK'\theta_{A'}"] \arrow[d, phantom, "\overset{GK'\Omega_{A'}}{\Leftarrow}"] \\
GK\beta_{A'} \arrow[r, Rightarrow, bend left, "Gk\beta_{A'}"] \arrow[r, Rightarrow, bend right, "Gk'\beta_{A'}"'] \arrow[r, phantom, "G\kappa\beta A'\Downarrow"] & GK'\beta_{A'}
\end{tikzcd}$$

3.15. Proposition. *(see [23]) Let* $\mathbb{T}$ *be a pseudomonad in the* **Gray***-category* $\mathcal{K}$*. Then the forgetful 2-functor* $U_A^{\mathbb{T}}\colon \mathbb{T}\text{-}\mathbf{Alg_A} \to \mathcal{K}(A,B)$ *has a left pseudoadjoint* $F_A^{\mathbb{T}}\colon \mathcal{K}(A,B) \to \mathbb{T}\text{-}\mathbf{Alg_A}$ *in the* **Gray***-category* **Gray** *given by sending each object* r *of* $\mathcal{K}(A,B)$ *to the corresponding free pseudoalgebra* $(Tr, \mu r, \lambda r, \alpha r)$*, each morphism* $h\colon r \to r'$ *to* (Th, μ_h^{-1})*, and each 2-morphism* $\xi\colon h \Rrightarrow h'$ *to* $T\xi\colon Th \Rrightarrow Th'$*.*

3.16. Theorem. *(see [23]) The collection of pseudoadjunctions:*

$$I_A^T, E_A^{\mathbb{T}}, i_A^T, e_A^T\colon F_A^T \dashv_p U_A^T\colon \mathbb{T}\text{-}\mathbf{Alg}_A \to \mathcal{K}(A,B)$$

defined for each A *in Proposition 3.15 extend to a pseudoadjunction*

$$I^{\mathbb{T}}, E^{\mathbb{T}}, i^{\mathbb{T}}, e^{\mathbb{T}}\colon F^{\mathbb{T}} \dashv_p U^{\mathbb{T}}\colon \mathbb{T}\text{-}\mathbf{Alg} \to \mathcal{K}(-,B)$$

in the **Gray***-category* $[\mathcal{K}^{\mathrm{op}}, \mathbf{Gray}]$*. In particular,* $F^{\mathbb{T}}$ *is a* **Gray***-natural transformation,* $i^{\mathbb{T}}, e^{\mathbb{T}}$ *are* **Gray***-modifications, and* $I^{\mathbb{T}}, E^{\mathbb{T}}$ *are* **Gray***-perturbations.*

Note that the previous theorem can also be adapted to the context of a pseudocomonad $\mathbb{G}$ on B. In this case, one obtains a **Gray**-functor $\mathbb{G}\text{-}\mathbf{CoAlg}\colon \mathcal{K}^{\mathrm{op}} \to \mathbf{Gray}$. As before there exists a forgetful **Gray**-natural transformation $U^{\mathbb{G}}\colon \mathbb{G}\text{-}\mathbf{CoAlg} \to \mathcal{K}(-,B)$. However, in this case, $U^{\mathbb{G}}$ has a right pseudoadjoint $F^{\mathbb{G}}$.

3.17. PSEUDOADJOINT PSEUDOMONADS. In Section 2.16 it was explained how thinking of an Eilenberg-Moore object as a weighted limit can be used to construct the free completion of a 2-category under Eilenberg-Moore objects. In this section we will need to generalize the notion of an Eilenberg-Moore object for a monad to an Eilenberg-Moore object for a pseudomonad. It turns out that thinking of an Eilenberg-Moore object as weighted limit will prove useful for this task as well.

Denote the 'free living monad' or 'walking monad' as **mnd**, meaning that a monad in a 2-category $\mathcal{K}$ is a 2-functor $\mathbf{mnd} \to \mathcal{K}$. Street has constructed a 2-functor $J\colon \mathbf{mnd} \to \mathbf{Cat}$ with the property that the Eilenberg-Moore object of a monad $\mathbb{T}\colon \mathbf{mnd} \to \mathcal{K}$ is the J-weighted limit of the 2-functor $\mathbb{T}$ [34]. This idea was used by Lack to construct a **Gray**-category **psm** — the 'free living pseudomonad' — such that a pseudomonad $\mathbb{T}$ in the **Gray**-category $\mathcal{K}$ is a **Gray**-functor $\mathbb{T}\colon \mathbf{psm} \to \mathcal{K}$ [23]. Lack also constructs a **Gray**-functor $P\colon \mathbf{psm} \to \mathbf{Gray}$ with the property that the Eilenberg-Moore object of the pseudomonad $\mathbb{T}$ is the P-weighted limit of the **Gray**-functor $\mathbb{T}\colon \mathbf{psm} \to \mathcal{K}$, denoted $\{P, \mathbb{T}\}$.

This limit does not always exist in $\mathcal{K}$, but $\mathcal{K}$ can always be embedded via the Yoneda embedding $Y\colon \mathcal{K} \to [\mathcal{K}^{\mathrm{op}}, \mathbf{Gray}]$ where the P-weighted limit of $Y\mathbb{T}$ can be formed. Then the Eilenberg-Moore object of $\mathbb{T}$ will exist in $\mathcal{K}$ if and only if $\{P, Y\mathbb{T}\}$ is representable since the Yoneda embedding must preserve any limits which exist. The **Gray**-functor $\{P, Y\mathbb{T}\}$ is just the **Gray**-functor $\mathbb{T}$-**Alg** constructed in the previous section. Thus, an Eilenberg-Moore object for the pseudomonad $\mathbb{T}$ is just a choice of representation for $\mathbb{T}$-**Alg**.

If $\mathbb{T}$-**Alg** is representable, then in our previous notation, it will correspond to the **Gray**-functor $\mathcal{K}(-, B^{\mathbb{T}})$ where $B^{\mathbb{T}}$ is an Eilenberg-Moore object for the pseudomonad $\mathbb{T}$. If an Eilenberg-Moore object for $\mathbb{T}$ does exist then the pseudoadjunction $I^{\mathbb{T}}, E^{\mathbb{T}}, i^{\mathbb{T}}, e^{\mathbb{T}}\colon F^{\mathbb{T}} \dashv_p U^{\mathbb{T}}\colon \mathbb{T}\text{-}\mathbf{Alg} \to \mathcal{K}(-, B)$ of Theorem 3.16 corresponds via the enriched Yoneda lemma to a pseudoadjunction $I, E, i, e\colon F \dashv_p U\colon B \to B^{\mathbb{T}}$ in $\mathcal{K}$.

The limit description of an Eilenberg-Moore object in a **Gray**-category also facilitates the free completion of an arbitrary **Gray**-category to one that has Eilenberg-Moore objects. Indeed, because a **Gray**-category is just a **Gray**-enriched category, the free-completion is achieved using the theory of enriched category theory [19]. With all of the abstract theory in place, we begin by proving the pointwise version of the categorified adjoint monad theorem. In Theorem 3.19 we will prove the full result.

3.18. LEMMA. *If $\Sigma, \Upsilon, \iota, \sigma\colon T \dashv_p G$, then the 2-category $\mathbb{T}$-$\mathbf{Alg_A}$ of pseudoalgebras based at A is 2-equivalent to the 2-category $\mathbb{G}$-$\mathbf{CoAlg_A}$ of pseudocoalgebras based at A for the pseudocomonad $\mathbb{G}$. Furthermore, this 2-equivalence commutes with the forgetful 2-functors $U_A^{\mathbb{T}}\colon \mathbb{T}\text{-}\mathbf{Alg_A} \to \mathcal{K}(A, B)$ and $U_A^{\mathbb{G}}\colon \mathbb{G}\text{-}\mathbf{CoAlg_A} \to \mathcal{K}(A, B)$.*

Proof. This lemma is essentially due to the properties of pseudomates under pseudoadjunction and the fact that this association preserves composites up to natural isomorphism. With Θ and Φ as in Proposition 3.4, define the 2-functor:

$$\mathcal{M}_A\colon \mathbb{T}\text{-}\mathbf{Alg_A} \quad \to \quad \mathbb{G}\text{-}\mathbf{CoAlg_A}$$

$$\begin{array}{rcl}
(s,\nu,\psi,\chi) & \mapsto & \big(s,\Phi(\nu),\ \Phi(\eta)s.\Phi(\nu) \xRightarrow{Y} \Phi(\nu.\eta s) \xRightarrow{\Phi(\psi)} \Phi(s) = s\ ,\\[1ex]
 & & G\Phi(\nu).\Phi(\nu) \xRightarrow{X} \Phi(\nu.T\nu) \xRightarrow{\Phi(\chi)} \Phi(\nu.\mu s) \xRightarrow{Y^{-1}} \Phi(\nu)s.\Phi(\nu)\ \big)\\[1ex]
(h,\varrho) & \mapsto & (h,\ \Phi(\nu').\Phi(h) \xRightarrow{X} \Phi(\nu'.Th) \xRightarrow{\Phi(\varrho)} \Phi(h.\nu) \xRightarrow{X^{-1}} Gh.\Phi(\nu)\ )\\
\xi\colon (h,\varrho) \Rightarrow (h',\varrho') & \mapsto & \xi\colon (h, X^{-1}\circ\Phi(\varrho)\circ X) \Rightarrow (h', X^{-1}\circ\Phi(\varrho')\circ X)
\end{array}$$

This data defines a pseudocoalgebra, morphism of pseudocoalgebras, and 2-morphism of pseudocoalgebras because $\xi\colon (h,\varrho) \Rightarrow (h',\varrho')\colon (s,\nu,\psi,\chi) \to (s',\nu',\psi',\chi')$ is a 2-morphism of pseudoalgebras, and mateship under pseudoadjunction preserves all composites up to natural isomorphism.

Since $\mathcal{M}_A\colon h\colon s \Rightarrow s' \mapsto h\colon s \Rightarrow s'$ it is clear that $\mathcal{M}_A$ preserves 1-morphism identities and to see that $\mathcal{M}_A$ preserves composites of 1-morphisms all we must check is its behavior on ϱ. For this purpose it will be helpful to have the specific form of $\mathcal{M}_A(h,\varrho)$. By plugging in the relevant pseudoadjunctions, one can check that $\mathcal{M}_A(h,\varrho) = \big(h, \Phi(\varrho)\circ G\nu'.(\iota_h)\big)$. Hence, using the definition of $\Phi(\varrho)$ and the **Gray** axioms it is easy to verify the following chain of equalities:

$$\begin{array}{rcl}
\mathcal{M}_A(h',\varrho').\mathcal{M}_A(h,\varrho) & = & \big(h'.h, Gh'.G\varrho.\iota s \circ Gh'.G\nu'.\iota_h \circ G\varrho'.\iota s'.h \circ G\nu''.\iota_{h'}.h\big)\\
 & = & \big(h'.h, Gh'.G\varrho.\iota s \circ G\varrho'.GTh.\iota s \circ G\nu''.\iota_{h'.h}\big)\\
 & = & \mathcal{M}_A(h'.h, \varrho'.Th \circ h'.\varrho).
\end{array}$$

Since $\mathcal{M}_A$ maps 2-morphisms to themselves, it is clear that $\mathcal{M}_A$ preserves composition of 2-morphisms on the nose as well. Hence, $\mathcal{M}_A\colon \mathbb{T}\text{-}\mathbf{Alg}_A \to \mathbb{G}\text{-}\mathbf{CoAlg}_A$ is a 2-functor. Now we define the other 2-functor taking part in the equivalence:

$$\begin{array}{rcl}
\overline{\mathcal{M}}_A\colon \mathbb{G}\text{-}\mathbf{CoAlg_A} & \to & \mathbb{T}\text{-}\mathbf{Alg_A}\\
(s,\bar\nu,\bar\psi,\bar\chi) & \mapsto & \big(s,\Theta(\bar\nu),\ \Theta(\bar\nu).\Theta(\varepsilon)s \xRightarrow{W} \Theta(\varepsilon s.\bar\nu) \xRightarrow{\Theta(\bar\psi)} \Theta(s) = s\ ,\\[1ex]
 & & \Theta(\bar\nu).T\Theta(\bar\nu) \xRightarrow{V} \Theta(G\bar\nu.\bar\nu) \xRightarrow{\Theta(\bar\chi)} \Theta(\delta s.\bar\nu) \xRightarrow{W^{-1}} \Theta(\bar\nu).\Theta(\delta)s\ \big)\\[1ex]
(h,\bar\varrho) & \mapsto & (h,\ \Theta(\bar\nu').Th \xRightarrow{V} \Theta(\bar\nu.h) \xRightarrow{\Theta(\bar\varrho)} \Theta(Gh.\bar\nu) \xRightarrow{V^{-1}} \Theta(h).\Theta(\bar\nu) = h.\Theta(\bar\nu)\ )\\
\xi\colon (h,\bar\varrho) \Rightarrow (h',\bar\varrho') & \mapsto & \xi\colon (h, V^{-1}\circ\Theta(\bar\varrho)\circ V) \Rightarrow (h', V^{-1}\circ\Theta(\bar{\varrho}')\circ V)
\end{array}$$

This will define a 2-functor, again by the functoriality of mateship under pseudoadjunction and the axioms of **Gray**-category. It will be helpful to have the explicit formula for $\overline{\mathcal{M}}_A(h,\bar\varrho)$. By plugging in the relevant pseudoadjunctions one can check that

$$\overline{\mathcal{M}}_A(h,\bar\varrho) = (h, \sigma_h.T\bar\nu \circ \Theta(\bar\varrho)).$$

We now show that $\mathcal{M}_A$ and $\overline{\mathcal{M}}_A$ define a 2-equivalence of 2-categories. Define the 2-natural isomorphism $\Gamma_A\colon 1_{\mathbb{T}\text{-}\mathbf{Alg_A}} \Rightarrow \overline{\mathcal{M}}_A\mathcal{M}_A$ as follows: Denote $\overline{\mathcal{M}}_A\mathcal{M}_A\big((s,\nu,\psi,\chi)\big)$

as $(s, \tilde{\nu}, \tilde{\psi}, \tilde{\chi})$. Define the morphism of pseudoalgebras $\Gamma_{(s,\nu,\psi,\chi)} \colon (s, \nu, \psi, \chi) \to (s, \tilde{\nu}, \tilde{\psi}, \tilde{\chi})$ by letting $h \colon s \Rightarrow s$ be the identity, so that ϱ is just a map $\tilde{\nu} \Rightarrow \nu$. From the definition of $\mathcal{M}_A$ and $\overline{\mathcal{M}}_A$ we know that $\tilde{\nu} = \Theta\Phi(\nu)$. Hence we can choose ϱ to be the isomorphism $\bar{\gamma}_\nu^{-1} \colon \Theta\Phi(\nu) \Rightarrow \nu$ defined in Proposition 3.4. The pair $(1_s, \bar{\gamma}_\nu^{-1})$ is a morphism of pseudoalgebras by the naturality of the isomorphism $\bar{\gamma}$ of Proposition 3.4 applied to the 3-morphisms ψ and χ. The explicit form of the isomorphism $\bar{\gamma}_\nu^{-1}$ is $\nu.\Upsilon s \circ \sigma_\nu^{-1}.T\iota s$.

To see that Γ_A is natural in the one dimensional sense, suppose that $(h, \varrho) \colon (\psi, \chi) \to (\psi', \chi')$ is an arbitrary 1-cell in $\mathbb{T}$-$\mathbf{Alg_A}$. Consider the following diagram:

$$\begin{tikzcd}
(s, \nu, \psi, \chi) \arrow[r, "{(h,\varrho)}"] \arrow[d, "{(1,\bar{\gamma}_\nu^{-1})}"'] & (s', \nu', \psi', \chi') \arrow[d, "{(1,\bar{\gamma}_{\nu'}^{-1})}"] \\
(s, \tilde{\nu}, \tilde{\psi}, \tilde{\chi}) \arrow[r, "{\overline{\mathcal{M}}_A\mathcal{M}_A(h,\varrho)}"'] & (s', \tilde{\nu}', \tilde{\psi}', \tilde{\chi}')
\end{tikzcd}$$

Note that since $h \colon s \Rightarrow s'$ for some $s' \colon A \to B$, the pseudoadjunction determining the mate of h is the identity adjunction so that h is its own mate under pseudoadjunction. Thus, this diagram of pseudoalgebra maps commutes if and only if

$$h.\gamma_\nu \circ \tilde{\varrho} = \varrho \circ \gamma_{\nu'}.Th. \tag{4}$$

Using the explicit formulas given above we have that

$$\overline{\mathcal{M}}_A\mathcal{M}_A(h, \varrho) = (h, \sigma_h^{-1}.TG\nu.T\iota s \circ \sigma s'.TG\varrho.T\iota s \circ \sigma s'.TG\nu'.T\iota_h^{-1}).$$

In order to prove the naturality of Γ_A we will need the following equalities that all follow directly from the axioms of a **Gray**-category:

$$\begin{aligned}
h.\sigma_\nu^{-1} \circ \sigma_h^{-1}.TG\nu &= \sigma_{h.\nu}^{-1} \\
\sigma_{h.\nu}^{-1} \circ \sigma s'.TG\varrho &= \varrho.\sigma Ts \circ \sigma_{\nu'.Th}^{-1} \\
\sigma_{\nu'.Th}^{-1} &= \nu'.\sigma_{Th}^{-1} \circ \sigma_{\nu'}^{-1}.TGTh \\
\sigma_{Th}^{-1}.T\iota s \circ \sigma Ts'.T\iota_h^{-1} &= (\sigma.T\iota)_h^{-1} \\
Th.\Upsilon s \circ (\sigma.T\iota)_h^{-1} &= Th_s \circ \Upsilon s'.Th = \Upsilon s'.Th
\end{aligned}$$

The proof of equation 4 above is as follows:

$$\begin{aligned}
h.\gamma_\nu \circ \tilde{\varrho} &= h.\nu.\Upsilon s \circ h.\sigma_\nu^{-1}.T\iota s \circ \sigma_h^{-1}.TG\nu.T\iota s \circ \sigma s'.TG\varrho.T\iota s \circ \sigma s'.TG\nu'.T\iota_h^{-1} \\
&= h.\nu.\Upsilon s \circ \sigma_{h.\nu}^{-1}.T\iota s \circ \sigma s'.TG\varrho.T\iota s \circ \sigma s'.TG\nu'.T\iota_h^{-1} \\
&= h.\nu.\Upsilon s \circ \varrho.\sigma Ts.T\iota s \circ \sigma_{\nu'.Th}^{-1}.T\iota s \circ \sigma s'.TG\nu'.T\iota_h^{1} \\
&= h.\nu.\Upsilon s \circ \varrho.\sigma Ts.T\iota s \circ \nu'.\sigma Ts'.T\iota_h^{-1} \circ \sigma_{\nu'}^{-1}.T\iota s'.Th \quad (Interchange) \\
&= h.\nu.\Upsilon s \circ \varrho.\sigma Ts.T\iota s \circ \nu'.(\sigma.T\iota)_h^{-1} \circ \sigma_{\nu'}^{-1}.T\iota s'.Th \\
&= \varrho \circ \nu'.Th.\Upsilon s \circ \nu'.(\sigma.T\iota)_h^{-1} \circ \sigma_{\nu'}^{-1}.T\iota s'.Th \quad (Interchange) \\
&= \varrho \circ \nu'.\Upsilon s'.Th \circ \sigma_{\nu'}^{-1}.T\iota s'.Th \\
&= \varrho \circ \gamma_{\nu'}.Th
\end{aligned}$$

To see the 2-naturality of Γ_A let $\xi\colon (h,\rho) \Rightarrow (h',\varrho')$, then the equality:

$$\begin{tikzcd}[column sep=large]
(\psi,\chi) \arrow[r, bend left, "{(h,\varrho)}"] \arrow[r, bend right, "{(h',\varrho')}"'] \arrow[r, phantom, "\Downarrow\xi"] & (\psi',\chi') \arrow[r, "{(1,\bar{\gamma}_{\nu'}^{-1})}"] & (\tilde{\psi},\tilde{\chi}) & =
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
(\psi,\chi) \arrow[r, "{(1,\bar{\gamma}_{\nu}^{-1})}"] & (\psi',\chi') \arrow[r, bend left, "{\overline{\mathcal{M}}_A\mathcal{M}_A(h,\varrho)}"] \arrow[r, bend right, "{\overline{\mathcal{M}}_A\mathcal{M}_A(h',\varrho')}"'] \arrow[r, phantom, "\Downarrow\overline{\mathcal{M}}_A\mathcal{M}_A\xi"] & (\tilde{\psi},\tilde{\chi}).
\end{tikzcd}$$

follows from the fact that $\overline{\mathcal{M}}_A\mathcal{M}_A(\xi) = \xi$ and the naturality of $\bar{\gamma}$ applied to the 3-morphism ξ in $\mathcal{K}$. A 2-natural isomorphism $\overline{\Gamma}_A\colon \mathcal{M}_A\overline{\mathcal{M}}_A \Rightarrow 1_{\mathbb{G}\text{-}\mathbf{CoAlg}_{\mathbf{A}}}$ can be defined in a similar way.

To see that this 2-equivalence of 2-categories commutes with the forgetful 2-functors, note that in the above proof we have shown that the 2-equivalence is the identity on the base map s of the pseudoalgebra. Furthermore, in the discussion of naturality we have shown that for any map (h,ϱ) of pseudoalgebras $\mathcal{M}_A$ is the identity on h and $\mathcal{M}_A$ also acts as the identity on every 3-cell defining a 2-morphism of pseudoalgebras. Thus, by the definition of the forgetful 2-functors $U_A^{\mathbb{T}}$ and $U_A^{\mathbb{G}}$ it is clear that the equivalence $\mathcal{M}_A$ commutes with the forgetful 2-functors. ■

3.19. Theorem. [The categorified adjoint monad theorem] *If* $\Sigma, \Upsilon, \iota, \sigma\colon T \dashv_p G$ *in* $\mathcal{K}$, *then the* **Gray**-*functors* $\mathbb{T}\text{-}\mathbf{Alg}$ *and* $\mathbb{G}\text{-}\mathbf{Alg}$ *are* **Gray**-*equivalent in the* **Gray**-*category* $[\mathcal{K}^{\mathrm{op}}, \mathbf{Gray}]$. *This means that there exists* **Gray**-*natural transformations* $\mathcal{M}\colon \mathbb{T}\text{-}\mathbf{Alg} \to \mathbb{G}\text{-}\mathbf{CoAlg}$, $\overline{\mathcal{M}}\colon \mathbb{G}\text{-}\mathbf{CoAlg} \to \mathbb{T}\text{-}\mathbf{Alg}$ *and invertible* **Gray**-*modifications* $\Gamma\colon 1_{\mathbb{T}\text{-}\mathbf{Alg}} \Rightarrow \overline{\mathcal{M}}\mathcal{M}$, $\overline{\Gamma}\colon \mathcal{M}\overline{\mathcal{M}} \Rightarrow 1_{\mathbb{G}\text{-}\mathbf{CoAlg}}$. *Furthermore, this* **Gray**-*equivalence commutes with the forgetful* **Gray**-*natural transformations* $U^{\mathbb{T}}$ *and* $U^{\mathbb{G}}$.

Proof. Define a **Gray**-natural transformation $\mathcal{M}\colon \mathbb{T}\text{-}\mathbf{Alg} \to \mathbb{G}\text{-}\mathbf{CoAlg}$ which assigns to each object in $\mathcal{K}^{\mathrm{op}}$ the 2-functor $\mathcal{M}_A$ defined in the preceding lemma. To see the naturality of $\mathcal{M}$ let $K\colon A' \to A$ in $\mathcal{K}$ and note that $\Phi(fK) = \Phi(f)K$ for $f = \nu, \psi, \chi$ so that:

$$\begin{aligned}
\hat{K}\mathcal{M}_A(s,\nu,\psi,\chi) &= \big(sK, \Phi(\nu)K, \Phi(\psi)K \circ YK, Y^{-1}K \circ \Phi(\chi)K \circ XK\big)\\
&= \big(sK, \Phi(\nu K), \Phi(\psi K) \circ Y, Y^{-1} \circ \Phi(\chi K) \circ X\big)\\
&= \mathcal{M}_{A'}\hat{K}(s,\nu,\psi,\chi).
\end{aligned}$$

A similar check shows that the collection of 2-functors $\overline{\mathcal{M}}_A\colon \mathbb{G}\text{-}\mathbf{CoAlg}_A \to \mathbb{T}\text{-}\mathbf{Alg}_A$ defines a **Gray**-natural transformation $\overline{\mathcal{M}}\colon \mathbb{G}\text{-}\mathbf{CoAlg} \to \mathbb{T}\text{-}\mathbf{Alg}$.

Define an invertible **Gray**-modification $\Gamma\colon 1_{\mathbb{T}\text{-}\mathbf{Alg}} \to \overline{\mathcal{M}}\mathcal{M}$ which assigns to each object in $\mathcal{K}^{\mathrm{op}}$ the 2-natural isomorphism Γ_A defined in the preceding lemma. To see that this data defines a **Gray**-modification first note that the following diagrams commute:

$$\begin{tikzcd}
\hat{K} \arrow[r, Rightarrow, "\hat{k}"] \arrow[d, Rightarrow, "\Gamma_{A'}\hat{K}"'] & \hat{K}' \arrow[d, Rightarrow, "\Gamma_{A'}\hat{K}'"] \\
\overline{\mathcal{M}}_{A'}\mathcal{M}_{A'}\hat{K} \arrow[r, Rightarrow, "\overline{\mathcal{M}}_{A'}\mathcal{M}_{A'}\hat{k}"'] & \overline{\mathcal{M}}_{A'}\mathcal{M}_{A'}\hat{K}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{K} \arrow[r, Rightarrow, "\hat{k}"] \arrow[d, Rightarrow, "\hat{K}\Gamma_{A}"'] & \hat{K}' \arrow[d, Rightarrow, "\hat{K}'\Gamma_{A}"] \\
\hat{K}\overline{\mathcal{M}}_{A}\mathcal{M}_{A} \arrow[r, Rightarrow, "\hat{k}\overline{\mathcal{M}}_{A}\mathcal{M}_{A}"'] & \hat{K}'\overline{\mathcal{M}}_{A}\mathcal{M}_{A}
\end{tikzcd}$$

In the first case we must show that the composites of pseudoalgebra homomorphisms $(sK', \bar{\gamma}^{1}_{\nu K'}).(sk, \nu_k^{-1})$ and $(sk, \Theta\Phi(\nu)_k^{-1}).(sK, \bar{\gamma}^{-1}_{\nu K})$ are equal. In the second diagram we must show that $(sK', \bar{\gamma}_{\nu}^{-1}K').(sK, \nu_k^{-1}) = (sK', \Theta\Phi(\nu)_k^{-1}).(sK, \bar{\gamma}_{\nu}^{-1}K)$. These equalities follow from the fact that $\Theta\Phi(\nu K') = \Theta\Phi(\nu)K'$, $\Theta\Phi(\nu K) = \Theta\Phi(\nu)K$ and the **Gray** axioms. Finally, Γ is a **Gray**-modification since $\bar{\gamma}_{\nu}^{-1}K' = \bar{\gamma}^{-1}_{\nu K'}$ and $\bar{\gamma}_{\nu}^{-1}K = \bar{\gamma}^{-1}_{\nu K}$.

In a similar fashion one can define an invertible **Gray**-modification $\overline{\Gamma}\colon \mathcal{M}\overline{\mathcal{M}} \Rightarrow 1_{\mathbb{G}\textbf{-CoAlg}}$. Hence, we have shown that the **Gray**-functors $\mathbb{T}$**-Alg** and $\mathbb{G}$**-CoAlg** are **Gray**-equivalent in the **Gray**-category $[\mathcal{K}^{\mathrm{op}}, \textbf{Gray}]$. Lemma 3.18 shows that this **Gray**-equivalence commutes with the forgetful functors. ∎

3.20. THEOREM. *Let $\mathbb{T} = (T, \mu, \eta, \lambda, \rho, \alpha, \varepsilon)$ be a Frobenius pseudomonad on the object B of the* **Gray***-category $\mathcal{K}$ with $\Sigma, \Upsilon, \iota, \varepsilon.\mu\colon T \dashv_p T$. Denote the induced pseudocomonad structure on T by $\mathbb{G}$. Then there exists an invertible* **Gray***-modification $\Xi\colon \mathcal{M}F^{\mathbb{T}} \to F^{\mathbb{G}}$.*

Proof. We begin by defining a 2-natural isomorphism $\Xi_A\colon \mathcal{M}_A F_A^{\mathbb{T}} \Rightarrow F_A^{\mathbb{G}}\colon \mathcal{K}(A, B) \to \mathbb{G}\textbf{-CoAlg}_A$. Recall that $F_A^{\mathbb{T}}(s) = (Ts, \mu s, \lambda s, \alpha s)$ and that $F_A^{\mathbb{G}}(s) = (Ts, \delta s, \bar{\rho}^{-1}s, \bar{\alpha}s)$ with $\delta, \bar{\rho}, \bar{\alpha}$ of the form given in Proposition 3.11. Hence,

$$\begin{aligned}
\mathcal{M}_A F_A^{\mathbb{T}}(s) &= \big(Ts, \Phi\left((\mu s)\right), \Phi\left((\lambda s)\right), \Phi\left((\alpha s)\right)\big) \\
&= \big(Ts, T\mu s.\iota s, \Phi\left((\lambda s)\right), \Phi\left((\alpha s)\right)\big) \\
F_A^{\mathbb{G}}(s) &= \big(Gs, \delta s, \bar{\rho}^{-1}s, \bar{\alpha}s\big) = \big(Ts, T^2(\varepsilon.\mu)s.T^2\mu Ts.T\iota T^2 s.\iota Ts, \bar{\rho}^{-1}s, \bar{\alpha}s\big).
\end{aligned}$$

The double parenthesis are used in order to distinguish which pseudoadjunction is intended. For example, $\Phi(\mu s)$ is determined by the pseudoadjunctions with $F = T, U = T, F' = T^2, U' = G^2$ and morphisms $a = b = s$. While $\Phi\left((\mu s)\right)$ is given by the pseudoadjunctions with $F = U = 1_B, F' = T, U' = T$ and morphisms $a = b = Ts$.

We define the isomorphism of pseudocoalgebras $(h, \bar{\varrho}s)\colon \mathcal{M}_A F_A^{\mathbb{T}}(s) \to F_A^{\mathbb{G}}(s)$ by taking $h = 1_{Ts}$, and $\bar{\varrho}s$ given by the following diagram:

$$\begin{tikzcd}
 & T^3s \arrow[r, Rightarrow, "T\mu s"] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow T\iota_{\mu}^{-1}s"] & T^2s \arrow[rrr, Rightarrow, "T^2s"] \arrow[dr, Rightarrow, "T\iota Ts"] & & & T^2s \\
Ts \arrow[ur, Rightarrow, "\iota Ts"] \arrow[dr, Rightarrow, "\iota Ts"'] & & & T^4s \arrow[ur, phantom, "\Downarrow T\Sigma s"] \arrow[dr, Rightarrow, "T^2\mu s"] \arrow[dd, phantom, "\Downarrow T^2\alpha s"] & & \\
 & T^3s \arrow[r, Rightarrow, "T\iota T^2s"'] & T^5s \arrow[ur, Rightarrow, "T^3\mu Ts"] \arrow[dr, Rightarrow, "T^2\mu Ts"'] & & T^3s \arrow[uur, Rightarrow, "T^2\varepsilon s"'] & \\
 & & & T^4s \arrow[ur, Rightarrow, "T^2\mu s"'] & &
\end{tikzcd}$$

One can verify using the definitions of $\mathcal{M}_A$, $\bar{\psi}$, and $\bar{\chi}$ that this map is indeed a morphism of pseudocoalgebras.

Let $h\colon s \Rightarrow s'$ in $\mathcal{K}(A, B)$. To establish the 1-naturality of Ξ_A we must show that:

$$\begin{tikzcd}
\mathcal{M}_A(Ts, \mu s, \lambda s, \alpha s) \arrow[r, "\mathcal{M}_A(Th,\mu_h^{-1})"] \arrow[d, "(Ts,\bar{\varrho}s)"'] & \mathcal{M}_A(Ts', \mu s', \lambda s', \alpha s') \arrow[d, "(Ts',\bar{\varrho}s')"] \\
(Ts, \delta s, \bar{\rho}^{-1}s, \bar{\alpha}s) \arrow[r, "(Th,\delta_h^{-1})"'] & (Ts', \delta s', \bar{\rho}^{-1}s', \bar{\alpha}s')
\end{tikzcd}$$

commutes where

$$\begin{aligned}
\mathcal{M}_A(Th, \mu_h^{-1}) &= \Phi((\mu_h^{-1})) \circ T\mu s'.(\iota_{Th}^{-1}) \\
&= T\mu_{Th}^{-1}.\iota Ts \circ T\mu s'.(\iota_{Th}^{-1}).
\end{aligned}$$

This amounts to the equality of the following diagrams:

$$\begin{tikzcd}
Ts \arrow[r, Rightarrow, "\iota Ts"] \arrow[d, Rightarrow, "Th"'] & T^3s \arrow[rrrr, Rightarrow, "T\mu s"] \arrow[d, Rightarrow, "T^3h"] & & & & T^2s \arrow[d, Rightarrow, "T^2h"] \\
Ts' \arrow[r, Rightarrow, "\iota Ts'"] \arrow[ddr, Rightarrow, "\iota Ts'"'] & T^3s' \arrow[r, Rightarrow, "T\mu s'"] \arrow[dd, equal] & T^2s' \arrow[rrr, equal] \arrow[dr, Rightarrow, "T\iota Ts'"] & & & T^2s' \\
& & & T^4s' \arrow[dr, Rightarrow, "T^2\mu s'"] & & \\
& T^3s' \arrow[r, Rightarrow, "T\iota T^2s'"'] & T^5s' \arrow[ur, Rightarrow, "T^3\mu Ts'"] \arrow[dr, Rightarrow, "T^2\mu Ts'"'] & & T^3s' \arrow[uur, Rightarrow, "T^2\varepsilon s'"'] & \\
& & & T^4s' \arrow[ur, Rightarrow, "T^2\mu s'"'] & &
\arrow[from=1-1, to=2-2, phantom, "\Downarrow\iota_{Th}"]
\arrow[from=1-2, to=2-6, phantom, "\Downarrow T\mu_{Th}^{-1}"]
\arrow[from=2-2, to=4-3, phantom, "\Downarrow T\iota_\mu^{-1}s'"]
\arrow[from=2-4, to=3-6, phantom, "\Downarrow T\Sigma s'"]
\arrow[from=3-4, to=5-4, phantom, "\Downarrow T^2\alpha s'"]
\end{tikzcd}$$

$$\begin{tikzcd}
& T^3s \arrow[r, Rightarrow, "T\mu s"] \arrow[dd, equal] & T^2s \arrow[dr, Rightarrow, "T\iota Ts"] \arrow[ddrrr, Rightarrow, bend left, "T^2s"] & & & \\
& & & T^4s \arrow[dr, Rightarrow, "T^2\mu s"] & & \\
Ts \arrow[uur, Rightarrow, "\iota Ts"] \arrow[r, Rightarrow, "\iota Ts"'] \arrow[dd, Rightarrow, "Th"'] & T^3s \arrow[r, Rightarrow, "T\iota T^2s"'] & T^5s \arrow[ur, Rightarrow, "T^3\mu Ts"] \arrow[dr, Rightarrow, "T^2\mu Ts"'] & & T^3s \arrow[r, Rightarrow, "T^2\varepsilon s"'] & T^2s \arrow[dd, Rightarrow, "T^2h"] \\
& & & T^4s \arrow[ur, Rightarrow, "T^2\mu s"'] & & \\
Ts' \arrow[r, Rightarrow, "\iota Ts'"'] & T^3s' \arrow[r, Rightarrow, "T\iota T^2s'"'] & T^5s' \arrow[dr, Rightarrow, "T^2\mu Ts'"'] & & T^3s' \arrow[r, Rightarrow, "T^2\varepsilon s'"'] & T^2s' \\
& & & T^4s' \arrow[ur, Rightarrow, "T^2\mu s'"'] & &
\arrow[from=1-2, to=2-4, phantom, "\Downarrow T\iota_\mu^{-1}s"]
\arrow[from=1-4, to=2-6, phantom, "\Downarrow T\Sigma s"]
\arrow[from=2-4, to=4-4, phantom, "\Downarrow T^2\alpha s"]
\arrow[from=4-4, to=6-4, phantom, "\Downarrow \delta_h^{-1}"]
\end{tikzcd}$$

which are equal by a routine verification using the **Gray**-category axioms. The 2-naturality of Ξ_A follows from the fact that both $F_A^{\mathbb{T}}$ and $F_A^{\mathbb{G}}$ map the 2-morphism $\xi\colon h \Rightarrow$

$h'\colon s \to s'$ to $T\xi$, and the fact that the 2-functor $\mathcal{M}_A$ is the identity on 2-morphisms of pseudocoalgebras.

The collection of Ξ_A define a **Gray**-modification by the commutativity of the following diagrams:

$$\begin{tikzcd}
\mathcal{M}_{A'}F^{\mathbb{T}}_{A'}\mathcal{K}(K,B) \arrow[r, Rightarrow, "\mathcal{M}_{A'}F^{\mathbb{T}}_{A'}\mathcal{K}(k,B)"] \arrow[d, Rightarrow, "\Xi_{A'}\mathcal{K}(K,B)"'] & \mathcal{M}_{A'}F^{\mathbb{T}}_{A'}\mathcal{K}(K',B) \arrow[d, Rightarrow, "\Xi_{A'}\mathcal{K}(K',B)"] \\
F^{\mathbb{G}}_{A'}\mathcal{K}(K,B) \arrow[r, Rightarrow, "F^{\mathbb{G}}_{A'}\mathcal{K}(k,B)"'] & F^{\mathbb{G}}_{A'}\mathcal{K}(K',B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{K}\mathcal{M}_A F^{\mathbb{T}}_A \arrow[r, Rightarrow, "\hat{k}\mathcal{M}_A F^{\mathbb{T}}_A"] \arrow[d, Rightarrow, "\hat{K}\Xi_A"'] & \hat{K}'\mathcal{M}_A F^{\mathbb{T}}_A \arrow[d, Rightarrow, "\hat{K}'\Xi_A"] \\
\hat{K}F^{\mathbb{G}}_A \arrow[r, Rightarrow, "\hat{k}F^{\mathbb{G}}_A"'] & \hat{K}'F^{\mathbb{G}}_A
\end{tikzcd}$$

which are both equal to

$$\begin{tikzcd}
(TsK, \mu sK) \arrow[r, "\Big(Tsk{,}\Phi\big((\mu s_k^{-1})\big)\Big)"] \arrow[d, "(TsK{,}\bar{\varrho}sK)"'] & (TsK', \mu sK') \arrow[d, "(TsK'{,}\bar{\varrho}sK')"] \\
\big(TsK, \Phi\big((\mu sK)\big)\big) \arrow[r, "\big(Tsk{,}\Phi(\mu)s_k\big)"'] & (TsK', \Theta\Phi(\mu)sK')
\end{tikzcd}$$

■

3.21. PROPOSITION. *Let* $B \underset{R}{\overset{L_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\perp_p}}}} C$ and $B \underset{R}{\overset{L_2}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\top_p}}}} C$ *be pseudoadjunctions in the* **Gray**-*category* $\mathcal{K}$. *Also, let* $\mathbb{T}_1$ *be the pseudomonad on* B *induced by the composite* RL_1, *and* T_2 *be the endomorphism on* B *induced by the composite* RL_2. *Then* $T_1 \dashv_p T_2$ *are pseudoadjoint morphisms, hence* T_2 *is with the pseudocomonad structure induced via mateship is a right pseudoadjoint pseudocomonad for the pseudomonad* $\mathbb{T}$.

Proof . The composites RL_1 and RL_2 of pseudoadjoints are pseudoadjoint by Proposition 3.3. Thus, if we let $\mathbb{T}_2$ be the pseudocomonad on B determined via mateship from the pseudomonad $\mathbb{T}_1$ then it is clear that $\mathbb{T}_1 \dashv_p \mathbb{T}_2$. ■

3.22. THEOREM. *If* $I, E, J, K, i, e, j, k\colon F \dashv_p U \dashv_p F\colon A \to B$ *is a pseudo ambijunction in the* **Gray**-*category* $\mathcal{K}$, *then the induced pseudomonad* UF *on* B *is Frobenius with* $\varepsilon = k$.

PROOF. All we must show is that $UF \dashv_p UF$ with counit $k.UiF$. Define the unit of the pseudo adjunction to be $UjF.i$. Then $UF \dashv_p UF$ follows by Proposition 3.3. ■

We now make use of the fact that every **Gray**-category $\mathcal{K}$ can be freely completed to a **Gray**-category $\mathbf{EM}(\mathcal{K})$ where an Eilenberg-Moore object exists for every pseudomonad in $\mathcal{K}$.

3.23. Theorem. *Given a Frobenius pseudomonad* $(\mathbb{T}, \varepsilon)$ *on an object* B *in the* **Gray**-*category* $\mathcal{K}$, *then in* $\mathbf{EM}(\mathcal{K})$ *the left pseudoadjoint* $F^{\mathbb{T}}\colon B \to B^{\mathbb{T}}$ *to the forgetful* **Gray**-*functor* $U^{\mathbb{T}}\colon B^{\mathbb{T}} \to B$ *is also right pseudoadjoint to* $U^{\mathbb{T}}$ *with counit* ε. *Hence, the Frobenius pseudomonad* $\mathbb{T}$ *is generated by an ambidextrous pseudo adjunction in* $\mathbf{EM}(\mathcal{K})$.

Proof. In $\mathbf{EM}(\mathcal{K})$ an Eilenberg-Moore object exists for the pseudomonad $\mathbb{T}$. In particular, this means that the **Gray**-functor $\mathbb{T}$-**Alg** is represented by $\mathcal{K}(-, B^{\mathbb{T}})$ for some $B^{\mathbb{T}}$ in $\mathbf{EM}(\mathcal{K})$. Hence, the pseudoadjunction

$$I^{\mathbb{T}}, E^{\mathbb{T}}, i^{\mathbb{T}}, e^{\mathbb{T}}\colon F^{\mathbb{T}} \dashv_p U^{\mathbb{T}}\colon \mathbb{T}\text{-}\mathbf{Alg} \to \mathcal{K}(-, B)$$

of Theorem 3.16 arises via the enriched Yoneda lemma from a pseudoadjunction

$$I^{\mathbb{T}}, E^{\mathbb{T}}, i^{\mathbb{T}}, e^{\mathbb{T}}\colon F^{\mathbb{T}} \dashv_p U^{\mathbb{T}}\colon B \to B^{\mathbb{T}}$$

in $\mathbf{EM}(\mathcal{K})$. Furthermore, since $\mathbb{T}$ is a Frobenius pseudomonad we can equip the endomorphism T with the induced pseudocomonad structure of Proposition 3.11. We denote this pseudocomonad as $\mathbb{G}$. Then the pseudoadjunction:

$$I^{\mathbb{G}}, E^{\mathbb{G}}, i^{\mathbb{G}}, e^{\mathbb{G}}\colon \overline{\mathcal{M}}F^{\mathbb{G}} \vdash_p U^{\mathbb{G}}\mathcal{M}\colon \mathbb{T}\text{-}\mathbf{Alg} \to \mathcal{K}(-, B)$$

given by the construction of pseudocoalgebras composed with the **Gray**-equivalence $\mathbb{T}$-**Alg** $\simeq$ $\mathbb{G}$-**CoAlg** must also arise via the enriched Yoneda lemma from a pseudoadjunction:

$$I^{\mathbb{G}}, E^{\mathbb{G}}, i^{\mathbb{G}}, e^{\mathbb{G}}\colon U^{\mathbb{G}}\mathcal{M} \dashv_p \overline{\mathcal{M}}F^{\mathbb{G}}\colon B \to B^{\mathbb{T}}$$

in $\mathbf{EM}(\mathcal{K})$. Since this pseudoadjunction generates the pseudocomonad $\mathbb{G}$, and $\mathbb{G}$ is defined by mateship under the self pseudoadjunction determined by ε, we have that $e^{\mathbb{G}} = \varepsilon$.

By Theorem 3.19 we have that $U^{\mathbb{G}}\mathcal{M} = U^{\mathbb{T}}$. Since $\mathbb{T}$-**Alg** is representable in $\mathbf{EM}(\mathcal{K})$ the isomorphism $\mathcal{M}F^{\mathbb{T}} \cong F^{\mathbb{G}}$ of Proposition 3.20 arises via the enriched Yoneda lemma from an isomorphism between the morphisms $\mathcal{M}F^{\mathbb{T}}$ and $F^{\mathbb{G}}$ in $\mathbf{EM}(\mathcal{K})$. Hence, $F^{\mathbb{T}}\colon B \to B^{\mathbb{T}}$ is both a left and right pseudoadjoint to $U^{\mathbb{T}}$, so that the Frobenius pseudomonad $\mathbb{T}$ is induced from an ambidextrous pseudoadjunction. ■

3.24. Corollary. *A Frobenius pseudomonoid in a semistrict monoidal 2-category* $\mathcal{M}$ *(or* **Gray**-*monoid) yields a pseudo ambijunction in* $\mathbf{EM}(\Sigma(\mathcal{M}))$, *where* $\Sigma(\mathcal{M})$ *is the* **Gray**-*category obtained from the suspension of* $\mathcal{M}$.

Proof. Recall that a Frobenius pseudomonoid in the **Gray**-monoid $\mathcal{M}$ is just a Frobenius pseudomonad in the **Gray**-category $\Sigma(\mathcal{M})$. The result follows by Theorem 3.23. ■

3.25. Corollary. *If* $B \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\top_p\ \perp_p}}}} C$ *is a pseudo ambijunction in the* **Gray**-*category* $\mathcal{K}$, *then* UF *is a Frobenius pseudomonoid in the semistrict monoidal 2-category* $\mathcal{K}(B, B)$.

Proof. By Theorem 3.22, UF is a Frobenius pseudomonad on B in $\mathcal{K}$. By definition this is a Frobenius pseudomonoid in $\mathcal{K}(B, B)$. ■

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Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge,
Cambridge CB3 0WB
UK

Email: `email: a.lauda@dpmms.cam.ac.uk`

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