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HIGHER COSPANS AND WEAK CUBICAL CATEGORIES (COSPANS IN ALGEBRAIC TOPOLOGY, I)

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Abstract.

We define a notion of *weak cubical category*, abstracted from the structure of n-cubical cospans $x\colon \wedge^n \to \mathbf{X}$ in a category $\mathbf{X}$, where $\wedge$ is the 'formal cospan' category. These diagrams form a cubical set with compositions $x +_i y$ in all directions, which are computed using pushouts and behave 'categorically' in a weak sense, up to suitable comparisons. Actually, we work with a 'symmetric cubical structure', which includes the transposition symmetries, because this allows for a strong simplification of the coherence conditions. These notions will be used in subsequent papers to study topological cospans and their use in Algebraic Topology, from tangles to cobordisms of manifolds.

We also introduce the more general notion of a *multiple category*, where - to start with - arrows belong to different sorts, varying in a countable family, and symmetries must be dropped. The present examples seem to show that the symmetric cubical case is better suited for topological applications.

Introduction

A cospan in a category is a diagram of shape

$$u = (u^-\colon X^- \to X^0 \leftarrow X^+ : u^+), \tag{1}$$

viewed as a morphism $u\colon X^- \nrightarrow X^+$; they are composed with pushouts, forming a bicategory; or, also, the weak arrows of a larger structure, the pseudo double category $\mathbb{C}\mathrm{osp}(\mathbf{X})$, as in [11]. Typically, the bicategories of cobordisms between manifolds used in Topological Quantum Field Theories and the bicategories of tangles are of this type.

This is the first paper in a series devoted to topological cospans in Algebraic Topology (i.e., cospans of continuous mappings), together with their higher dimensional versions. We begin by preparing the cubical structure of higher cospans $\mathbb{C}\mathrm{osp}_*(\mathbf{X})$ on a category $\mathbf{X}$ with pushouts, and abstract from the construction the general notion of a 'weak cubical category'.

An n-cubical cospan in $\mathbf{X}$ is defined as a functor $x\colon \wedge^n \to \mathbf{X}$, where $\wedge$ is the category

$$\wedge : \qquad -1 \to 0 \leftarrow 1 \qquad \text{(the } \textit{formal cospan}\text{)}. \tag{2}$$

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Plainly, these diagrams form a cubical set (cf. Kan [15, 16]), equipped with compositions $x +_i y$ of i-consecutive n-cubes, for $i = 1, ..., n$; such compositions are computed by pushouts, and behave 'categorically' in a weak sense, up to suitable comparisons.

To make room for the latter, the n-th component of $\mathbb{C}\mathrm{osp}_*(\mathbf{X})$

$$\mathbb{C}\mathrm{osp}_n(\mathbf{X}) = \mathbf{Cat}(\wedge^n, \mathbf{X}), \tag{3}$$

will not be just the *set* of functors $x\colon \wedge^n \to \mathbf{X}$ (the *n-cubes*, or *n-dimensional objects*, of the structure) but the *category* of such functors and their natural transformations $f\colon x \to x'\colon \wedge^n \to \mathbf{X}$ (the *n-maps* of the structure, which should actually be viewed as *$(n+1)$-cells*). The comparisons will be *invertible* n-maps; but general n-maps are also important, e.g. to define limits and colimits (cf. 4.6).

Thus, a *weak cubical category* will have countably many *weak* (or *cubical*) directions $i = 1, 2, ..., n, ...$ all of the same sort, and one *strict* (or *transversal*) direction 0, which can be of a different sort. The compositions $x +_i y$ along the cubical directions behave weakly and, typically, are obtained as (co)limits; the composition gf in the transversal direction behaves strictly and, typically, arises from composition in a ordinary category; the comparisons for the weak compositions are isomorphisms of the strict one. It is tempting to view the transversal direction as 'temporal' and the cubical ones as 'spatial', but this interpretation might clash with applications in physics and we will not follow it.

Truncating the cubical structure in degree 1 (see 4.5), we get a *weak 2-cubical category*, with one strict direction and one weak direction. This coincides with a pseudo (or weak) double category, as defined and studied in [11, 12]. The theory of weak cubical categories will likely be an extension of the theory of weak double categories developed in those papers.

In a *strict* cubical category, i.e. a weak cubical category where all comparisons are identities, there are no weak laws and we prefer to speak of (countably many) cubical directions and (one) transversal direction; the former are of the same sort, generally different from the transversal one.

In Section 1 we begin by the strict case, defining cubical categories and treating a simple example: the cubical category $\mathbb{C}\mathrm{ub}_*(\mathbf{X})$ of commutative cubical diagrams $x\colon \mathbf{i}^n \to \mathbf{X}$ on a category $\mathbf{X}$ (with their natural transformations). The construction is based on the structure of the ordinal category $\mathbf{i}$

$$\mathbf{i} = \mathbf{2} = \{0 \to 1\} \qquad\qquad (\text{the } \textit{formal arrow}), \tag{4}$$

as a *formal interval* (see 1.3), with faces, degeneracy and a concatenation pushout (16).

The substructure $\mathbb{C}\mathrm{ub}_*(\mathbf{X}; \mathbf{X}_0, \mathbf{X}')$ defined in 1.1 shows an example of a cubical category where the transversal and cubical sorts are distinct. Then, in Section 2, we introduce the transposition symmetries, for cubical sets and cubical categories; in the case of $\mathbb{C}\mathrm{ub}_*(\mathbf{X})$, such symmetries are generated by the basic transposition $s\colon \mathbf{i}^2 \to \mathbf{i}^2$.

In Sections 3 and 4, we construct the symmetric weak cubical category $\mathbb{C}\mathrm{osp}_*(\mathbf{X})$ mentioned above, based on a similar structure of (symmetric) formal interval for $\wedge$; and abstract from this construction the general notion of a symmetric weak cubical category.

Notice that the interchange of *weak* compositions only works *in a weak sense*, even in this relatively simple construction; symmetries allow us to reduce the interchange comparisons to one (in each dimension), and this fact strongly simplifies the coherence problems. Other examples, like spans and diamonds (or bispans), are sketched (see 4.7); higher cobordisms will be dealt with in a sequel. A strict cubical category $\mathbb{R}\mathrm{el}_*(\mathbf{Ab})$, of higher relations for abelian groups, can be obtained as a quotient of the weak cubical structures for spans or cospans (4.8).

We end in Section 5, dealing with (strict) multiple categories, where the cubical directions can be of different sorts. One might think that this should be the natural extension of double categories in higher dimension; yet, various examples of topological or homotopical kind fall into the cubical case, where all the weak directions are of the same sort and - moreover - transpositions permute them.

Size problems can be easily dealt with, fixing a hierarchy of two universes, $\mathcal{U}_0 \in \mathcal{U}$, and assuming that 'small' category means $\mathcal{U}$-small. Then, for instance, we can apply $\mathbb{C}\mathrm{osp}_*(-)$ to the small categories **Set** and **Top** of $\mathcal{U}_0$-small sets or spaces. **Cat** will be the category (or 2-category) of $\mathcal{U}$-small categories, to which **Set** and **Top** belong.

Cubical categories *with connections* have been studied by Al-Agl, Brown and Steiner [1], and proved to be equivalent to globular categories. Monoidal n-categories of higher spans can be found in Batanin [3]. A structure for *cobordisms with corners*, using 2-cubical cospans, has been recently proposed by J. Morton [17] and J. Baez [2], in the form of a 'Verity double bicategory' [18]; its relations with the present 2-truncated version $2\mathbb{C}\mathrm{osp}_*(\mathbf{X})$ (a *weak 3-cubical category*, according to our terminology) are briefly examined in 4.5. See also Cheng-Gurski [6].

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1. Cubical categories

We begin by the notion of a (strict) cubical category. Symmetries will be introduced in the next section. The index α takes the values 0 and 1, but is written $-, +$ in superscripts.

1.1. Commutative cubes and their transformations. For a small category $\mathbf{X}$, we will construct in this section a cubical category $\mathbb{C}\mathrm{ub}_*(\mathbf{X})$ of commutative cubes, of any dimension; the present subsection is an overview of this construction.

First, we start with a *reduced* cubical category $\mathrm{Cub}_*(\mathbf{X})$ (note the different notation), which - loosely speaking - is a cubical set with categorical compositions in any direction, satisfying the interchange property. In the present case, 0-cubes are *points* $x \in \mathbf{X}$ (i.e., objects of $\mathbf{X}$), 1-cubes are *arrows* $x\colon x_0 \to x_1$ in $\mathbf{X}$, 2-cubes are *commutative squares* of $\mathbf{X}$, and so on. Faces and degeneracies are obvious, as well as the i-directed composition of i-consecutive n-cubes, for $1 \leqslant i \leqslant n$.

All the structure will be obtained from a cocubical object, based on the ordinal

$$\mathbf{i} = \mathbf{2} = \{0 \to 1\} \qquad \text{(the } \textit{formal arrow}\text{)}, \tag{5}$$

an order category on two objects (identities being understood, as we will generally do). In fact, an n-cube is the same as a functor $x\colon \mathbf{i}^n \to \mathbf{X}$ and the n-th component of $\mathrm{Cub}_*(\mathbf{X})$ is the set

$$\mathrm{Cub}_n(\mathbf{X}) = \mathbf{Cat}(\mathbf{i}^n, \mathbf{X}) \qquad (n \geqslant 0). \tag{6}$$

But this component is naturally a category $\mathbb{C}\mathrm{ub}_n(\mathbf{X})$, whose morphisms are the natural transformations $f\colon x \to x'\colon \mathbf{i}^n \to \mathbf{X}$ of commutative n-cubes. The (non-reduced) *cubical category* $\mathbb{C}\mathrm{ub}_*(\mathbf{X})$ will also contain such *n-maps* between n-cubes, forming a category object within reduced cubical categories, or equivalently a reduced cubical category within categories.

In the present case, a natural transformation of n-dimensional commutative cubes is just an $(n+1)$-commutative cube of $\mathbf{X}$, and $\mathbb{C}\mathrm{ub}_*(\mathbf{X})$ can be viewed as a *re-indexing* of $\mathrm{Cub}_*(\mathbf{X})$. But it is easy to construct examples where this is not the case: for instance, the sub-structure $\mathbb{C}\mathrm{ub}_*(\mathbf{X}; \mathbf{X}_0, \mathbf{X}')$ defined by two subcategories $\mathbf{X}_0, \mathbf{X}'$ of $\mathbf{X}$, with cubes belonging to $\mathrm{Cub}_*(\mathbf{X}')$ and natural transformations $f\colon x \to x'\colon \mathbf{i}^n \to \mathbf{X}$ restrained to have components in $\mathbf{X}_0$, so that a map $f\colon x \to x'$ of 0-cubes is not the same as a 1-cube with the same faces. (More generally, fixing a subcategory $\mathbf{X}_i$ for any direction $i \geqslant 0$, one would obtain a *multiple category,* see Section 5.) A cubical category of higher relations will be constructed in 4.8.

Introducing the new maps will be crucial for two goals:

(a) defining *weak* cubical categories, where the 'cubical' composition laws only behave well up to invertible n-maps,

(b) defining limits and colimits in (weak or strict) cubical categories.

The notions studied here should not be confused with a category enriched in the *cartesian* or *monoidal* category of cubical sets. (The latter is important in homotopy theory, since any cylinder functor automatically produces such a structure.)

1.2. The reduced case. Let us begin defining a *reduced cubical category* $\mathbf{A}$ as a cubical set equipped with compositions in all directions, which are strictly categorical (i.e., strictly associative and unital) and satisfy the interchange property.

(cub.1) A *reduced cubical category* $\mathbf{A}$ is, first of all, a cubical set $((A_n), (\partial_i^\alpha), (e_i))$ in the usual sense [15, 16, 5]: a sequence of sets A_n, for $n \geqslant 0$ (whose elements are called *n-cubes*, or *n-dimensional objects*), with *faces* ∂_i^α and *degeneracies* e_i which satisfy the usual cubical relations

$$\begin{array}{lll}
\partial_i^\alpha\colon A_n \rightleftarrows A_{n-1} : e_i & & (i = 1, ..., n;\ \alpha = \pm), \\
\partial_i^\alpha . \partial_j^\beta = \partial_j^\beta . \partial_{i+1}^\alpha & & (j \leqslant i), \\
e_j . e_i = e_{i+1} . e_j & & (j \leqslant i), \\
\partial_i^\alpha . e_j = e_j . \partial_{i-1}^\alpha\ (j < i), & \text{or id}\ (j = i), & \text{or}\ e_{j-1} . \partial_i^\alpha\ (j > i).
\end{array} \tag{7}$$

(Cubical sets form a category of presheaves, as we will recall in 1.5.)

(cub.2) Moreover, for $1 \leqslant i \leqslant n$, the *i-concatenation* $x +_i y$ (or i-composition) of two n-cubes x, y is defined when x, y are *i-consecutive*, i.e. $\partial_i^+(x) = \partial_i^-(y)$, and satisfies the following 'geometrical' interactions with faces and degeneracies

$$\begin{array}{lll} \partial_i^-(a +_i b) & = \partial_i^-(a), & \partial_i^+(a +_i b) = \partial_i^+(b), \\ \partial_j^\alpha(a +_i b) & = \partial_j^\alpha(a) +_{i-1} \partial_j^\alpha(b) & (j < i), \\ & = \partial_j^\alpha(a) +_i \partial_j^\alpha(b) & (j > i), \end{array} \tag{8}$$

$$\begin{array}{lll} e_j(a +_i b) & = e_j(a) +_{i+1} e_j(b) & (j \leqslant i \leqslant n), \\ & = e_j(a) +_i e_j(b) & (i < j \leqslant n+1). \end{array} \tag{9}$$

(cub.3) For $1 \leqslant i \leqslant n$, we have a category $A_i^n = (A_{n-1}, A_n, \partial_i^-, \partial_i^+, e_i, +_i)$, where faces give domains and codomains, and degeneracy yields the identities.

(cub.4) For $1 \leqslant i < j \leqslant n$, and n-cubes x, y, z, u, we have

$$(x +_i y) +_j (z +_i u) = (x +_j z) +_i (y +_j u) \qquad (\textit{middle-four interchange}), \tag{10}$$

whenever these compositions make sense:

$$\begin{array}{l} \partial_i^+(x) = \partial_i^-(y), \\ \partial_i^+(z) = \partial_i^-(u), \\ \partial_j^+(x) = \partial_j^-(z), \\ \partial_j^+(y) = \partial_j^-(u), \end{array} \qquad \begin{array}{|c|c|} \hline x & y \\ \hline z & u \\ \hline \end{array} \qquad \begin{array}{l} \overset{i}{\rightarrow} \\ \downarrow j \end{array} \tag{11}$$

Notice that the *nullary* interchange is already expressed above, in (9).

A *cubical functor* $F\colon \mathbf{A} \to \mathbf{B}$ between reduced cubical categories is a morphism of cubical sets which preserves all composition laws.

1.3. Commutative cubes. We formalise now the construction of the reduced cubical category $\mathrm{Cub}_*(\mathbf{X})$, where an n-cube is a commutative n-cubical diagram of the given category $\mathbf{X}$.

Recall that an n-cube is a functor $x\colon \mathbf{i}^n \to \mathbf{X}$ (1.1). The category $\mathbf{i} = \mathbf{2}$ has the structure of a *formal interval* (or reflexive cograph), with respect to the cartesian product in **Cat**: in other words, it comes equipped with two (obvious) faces ∂^α, defined on the singleton category $\mathbf{1} = \{*\} = \mathbf{i}^0$ and a (uniquely determined) degeneracy e

$$\{*\} \underset{e}{\overset{\partial^\alpha}{\rightrightarrows}} \mathbf{i} \qquad \partial^\alpha(*) = \alpha \qquad (\alpha = 0, 1). \tag{12}$$

These maps (trivially) satisfy the equations $e\partial^\alpha = \mathrm{id}$.

Thus, a 1-cube $x\colon \mathbf{i} \to \mathbf{X}$ amounts to an arrow $x\colon x_0 \to x_1$ and has faces $\partial^\alpha(x) = x.\partial^\alpha = x_\alpha$, while the degeneracy, or identity, of an object x is $e(x) = x.e\colon \mathbf{i} \to \mathbf{X}$.

Then, as usual in formal homotopy theory (based on a formal interval), the functors

$$(-)_i^n = \mathbf{i}^{i-1} \times - \times \mathbf{i}^{n-i} \colon \mathbf{Cat} \to \mathbf{Cat} \qquad (1 \leqslant i \leqslant n), \tag{13}$$

produce the higher faces and degeneracies of the interval

$$\begin{array}{ll} \partial_i^\alpha \colon \mathbf{i}^{n-1} \to \mathbf{i}^n, & \partial_i^\alpha(t_1, ..., t_{n-1}) = (t_1, ..., \alpha, ..., t_{n-1}), \\ e_i \colon \mathbf{i}^n \to \mathbf{i}^{n-1}, & e_i(t_1, ..., t_n) = (t_1, ..., \hat{t}_i, ..., t_n) \qquad (t_j = 0, 1). \end{array} \tag{14}$$

(Note that these functors between order-categories are determined by their action on objects; the dimension n is often omitted in notation.)

By a contravariant action, we get the faces and degeneracies of the cubical set $\mathrm{Cub}_*(\mathbf{X})$, denoted in the same way:

$$\partial_i^\alpha(x) = x.\partial_i^\alpha, \qquad e_i(x) = x.e_i \qquad (i = 1, ..., n;\ \alpha = \pm). \tag{15}$$

Composition of 1-cubes comes, formally, from the *concatenation pushout*, in **Cat**, which gives the category $\mathbf{i}_2 = \mathbf{3} = \{0 \to 1 \to 2\}$, equipped with a *concatenation map* k

$$\begin{array}{ccc} \{*\} & \xrightarrow{\partial^+} & \mathbf{i} \\ {\scriptstyle \partial^-}\downarrow & & \downarrow{\scriptstyle k^-} \\ \mathbf{i} & \xrightarrow[k^+]{} & \mathbf{i}_2 \end{array} \qquad \begin{array}{l} k \colon \mathbf{i} \to \mathbf{i}_2, \\ \\ k(0) = 0,\ k(1) = 2. \end{array} \tag{16}$$

And indeed, given two consecutive 1-cubes $x, y \colon \mathbf{i} \to \mathbf{X}$ (with $\partial_1^+ x = \partial_1^- y$), their composite $z = yx$ is computed using the functor $[x, y] \colon \mathbf{i}_2 \to \mathbf{X}$ determined by the pushout in $\mathbf{X}$

$$z = [x, y].k \colon \mathbf{i} \to \mathbf{i}_2 \to \mathbf{X}. \tag{17}$$

Moreover, acting on the concatenation pushout and the concatenation map k, the functors $(-)_i^n$ produce the *n-dimensional i-concatenation pushout* $\mathbf{i}_2^{ni}$ and the *n-dimensional i-concatenation map* $k_i \colon \mathbf{i}^n \to \mathbf{i}_2^{ni}$

$$\begin{array}{ccc} \mathbf{i}^{n-1} & \xrightarrow{\partial^+} & \mathbf{i}^n \\ {\scriptstyle \partial^-}\downarrow & & \downarrow{\scriptstyle k^-} \\ \mathbf{i}^n & \xrightarrow[k^+]{} & \mathbf{i}_2^{ni} \end{array} \qquad \begin{array}{l} \mathbf{i}_2^{ni} = \mathbf{i}^{i-1} \times \mathbf{i}_2 \times \mathbf{i}^{n-i}, \\ \\ k_i = \mathbf{i}^{i-1} \times k \times \mathbf{i}^{n-i} \colon \mathbf{i}^n \to \mathbf{i}_2^{ni}. \end{array} \tag{18}$$

Now, given two *i-consecutive* n-cubes $x, y \colon \mathbf{i}^n \to \mathbf{X}$ (with $\partial_i^+ x = \partial_i^- y$), their *$i$-concatenation* $z = x +_i y$ is computed using the functor $[x, y] \colon \mathbf{i}_2^{ni} \to \mathbf{X}$ determined by the pushout in $\mathbf{X}$

$$z = [x, y].k_i \colon \mathbf{i}^n \to \mathbf{i}_2^{ni} \to \mathbf{X}. \tag{19}$$

Plainly, a functor $F \colon \mathbf{X} \to Y$ can be extended to a cubical functor F_*, which coincides with F in degree 0 (up to identifying $\mathbf{X}$ with $\mathrm{Cub}_0(\mathbf{X})$)

$$F_* \colon \mathrm{Cub}_*(\mathbf{X}) \to \mathrm{Cub}_*(\mathbf{Y}), \qquad F_*(x \colon \mathbf{i}^n \to \mathbf{X}) = F \circ x \colon \mathbf{i}^n \to Y. \tag{20}$$

In the next section we will add to $\mathrm{Cub}_*(\mathbf{X})$ further structure, produced by the transposition of coordinates.

1.4. Cubical categories. As envisioned above, the reduced cubical category $\mathrm{Cub}_*(\mathbf{X})$ has a natural extension $\mathbb{C}\mathrm{ub}_*(\mathbf{X})$, introducing *transversal maps* $f\colon x \to x'$ of n-cubes (also called *n-maps*, or *$(n+1)$-cells*) as natural transformations $f\colon x \to x'\colon \mathbf{i}^n \to \mathbf{X}$, so that the n-th component $\mathbb{C}\mathrm{ub}_n(\mathbf{X}) = \mathbf{Cat}(\mathbf{i}^n, \mathbf{X})$ is now a category. The new faces, degeneracy and composition are written

$$\partial_0^- f = x, \qquad \partial_0^+ f = x', \qquad e_0 x = \mathrm{id}(x), \qquad c_0(f,g) = gf\colon x \to x'', \tag{21}$$

where gf is the ordinary (vertical) composition of natural transformations.

The new structure we are interested in, a *cubical category* $\mathbb{A}$, is a category object within reduced cubical categories (and their cubical functors)

$$\mathbb{A}^0 \underset{e_0}{\overset{\partial_0^\alpha}{\rightleftarrows}} \mathbb{A}^1 \xleftarrow{c_0} \mathbb{A}^2 \qquad (\alpha = \pm) \tag{22}$$

or equivalently a reduced cubical category within categories

$$\mathbb{A} = ((\mathbb{A}_n), (\partial_i^\alpha), (e_i), (+_i)), \qquad \mathbb{A}_n = (A_n, M_n, \partial_0^\alpha, e_0, c_0). \tag{23}$$

Explicitly, the latter statement means that $\mathbb{A}$ is a reduced cubical category where each component $\mathbb{A}_n$ is a category, while the cubical faces, degeneracies and compositions are functors

$$\partial_i^\alpha\colon \mathbb{A}_n \rightleftarrows \mathbb{A}_{n-1} : e_i, \qquad +_i : \mathbb{A}_n \times_i \mathbb{A}_n \to \mathbb{A}_n. \tag{24}$$

(The pullback $\mathbb{A}_n \times_i \mathbb{A}_n$ is the category of pairs of i-consecutive n-cubes.)

A *cubical functor* $F\colon \mathbb{A} \to \mathbb{B}$ between cubical categories strictly preserves the whole structure. A reduced cubical category amounts to a cubical category all of whose n-maps are identities.

1.5. Truncation. An *n-cubical set* $A = ((A_k), (\partial_i^\alpha), (e_i))$ has components indexed on $k = 0, ..., n$. Of course, also its faces $\partial_i^\alpha\colon A_k \to A_{k-1}$ and degeneracies $e_i\colon A_{k-1} \to A_k$ undergo the restriction $k \leqslant n$, and satisfy the cubical relations as far as applicable.

Cubical sets are presheaves $A\colon \mathbb{I}^{\mathrm{op}} \to \mathbf{Set}$, the *cubical site* $\mathbb{I}$ being the subcategory of $\mathbf{Set}$ with objects 2^n, for $2 = \{0,1\}$ and $n \geqslant 0$, together with those mappings $2^m \to 2^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates. (A study of this site and its extensions, including connections and symmetries, can be found in [10]). And, of course, an n-cubical set is a presheaf on its truncation $\mathbb{I}_n$, with objects $k \leqslant n$.

The truncation functor

$$tr_n\colon \mathbf{Cub} \to n\mathbf{Cub}, \qquad sk_n \dashv tr_n \dashv cosk_n, \tag{25}$$

has left and right adjoint, called *n-skeleton* and *n-coskeleton*, which can be constructed by means of left or right Kan extensions along the embedding $\mathbb{I}_n \subset \mathbb{I}$ of the n-cubical site into the cubical one

(26)

Recalling that a k-map between k-cubes is viewed as a $(k+1)$-*dimensional cell* (1.4), an n-truncated cubical category will be called an $(n+1)$-*cubical category*. For instance

$$2\mathbb{C}\mathrm{ub}_*(\mathbf{X}) = tr_2\mathbb{C}\mathrm{ub}_*(\mathbf{X}), \qquad 2\mathbb{C}\mathrm{ub}_*(\mathbf{X}; \mathbf{X}_0, \mathbf{X}') = tr_2\mathbb{C}\mathrm{ub}_*(\mathbf{X}; \mathbf{X}_0, \mathbf{X}'),$$

are *3-cubical categories*; and, indeed, their 2-maps are commutative 3-dimensional cubes.

Thus, a 1-cubical category is a category, a 2-cubical category amounts to a (strict) double category, and a 3-cubical category amounts to a (strict) triple category of a particular kind, with:

- objects (of one type);
- arrows in directions 0, 1 and 2, *where the last two types coincide*;
- 2-dimensional cells in directions 01, 02, 12, *where the first two types coincide*;
- and 3-dimensional cells (of one type).

2. Symmetric cubical categories

We develop here a notion of *symmetric* cubical category, where the symmetric group S_n operates on the n-dimensional component, i.e. on n-cubes and n-maps. The presence of these symmetries will grant a relatively simple description of the weak case, in the next section.

2.1. Symmetries of the interval. The standard interval $\mathbf{I} = [0, 1]$ of topological spaces has two basic symmetries

$$\begin{array}{lll} r\colon \mathbf{I} \to \mathbf{I}, & r(t) = -t, & (\textit{reversion}), \\ s\colon \mathbf{I}^2 \to \mathbf{I}^2, & s(t_1, t_2) = (t_2, t_1) & (\textit{transposition}), \end{array} \tag{27}$$

They produce the n-dimensional symmetries, applying $(-)^n_i$ and $(-)^{n-1}_i$, respectively:

$$r_i\colon \mathbf{I}^n \to \mathbf{I}^n \quad (i = 1, ..., n), \qquad s_i\colon \mathbf{I}^n \to \mathbf{I}^n \quad (i = 1, ..., n-1). \tag{28}$$

Together, they generate the group of symmetries of the euclidean n-cube $\mathbf{I}^n$, also called the hyperoctahedral group, which is isomorphic to the semidirect product $(\mathbf{Z}/2)^n \rtimes S_n$. The transpositions s_i generate the subgroup of permutations of coordinates, isomorphic to the symmetric group S_n, under the Moore relations

$$s_i.s_i = 1, \qquad s_i.s_j.s_i = s_j.s_i.s_j \ \ (i = j-1), \qquad s_i.s_j = s_j.s_i \ \ (i < j-1), \tag{29}$$

(see Coxeter-Moser [7], 6.2; or Johnson [14], Section 5, Thm. 3). Of course, in this isomorphism, the transposition $s_i\colon \mathbf{I}^n \to \mathbf{I}^n$ corresponds to the permutation $s_i = (i, i+1)\colon \{1, ..., n\} \to \{1, ..., n\}$.

Now, as is generally the case in *directed* algebraic topology (see [8, 9] and references therein), our formal interval $\mathbf{i}$, in $\mathbf{Cat}$, has no reversion. But it has transpositions

$$s\colon \mathbf{i}^2 \to \mathbf{i}^2, \qquad s(t_1,t_2)=(t_2,t_1), \qquad s_i = \mathbf{i}^{i-1}\times s\times \mathbf{i}^{n-1-i}\colon \mathbf{i}^n \to \mathbf{i}^n \ \ (i=1,...,n-1). \tag{30}$$

They operate, contravariantly, on every category $\mathbb{C}\mathrm{ub}_n(\mathbf{X}) = \mathbf{Cat}(\mathbf{i}^n, \mathbf{X})$

$$s_i(x) = x.s_i\colon \mathbf{i}^n \to \mathbf{i}^n \to \mathbf{X}, \tag{31}$$

together with the whole symmetric group S_n.

With faces and degeneracies, transpositions generate the *symmetric cubical site* $\mathbb{I}_s$, a subcategory of the *extended cubical site* $\mathbb{K}$ of [10] (which also contains the connections). $\mathbb{I}_s$ is generated by faces, degeneracies and transpositions under the ordinary cocubical relations (for faces and degeneracies), the Moore relations (29) and other equations which link transpositions with faces and degeneracies; we write down, below, their duals.

2.2. Symmetric cubical sets. A *symmetric cubical set* will be a cubical set (1.2)

$$A = ((A_n), (\partial_i^\alpha), (e_i)),$$

which is further equipped with *transpositions*

$$s_i\colon A_n \to A_n \qquad\qquad (i=1,...,n-1). \tag{32}$$

The latter must satisfy the Moore relations (29) and the following equations:

$$\begin{array}{lcccc} & j<i & j=i & j=i+1 & j>i+1 \\ \partial_j^\alpha.s_i = & s_{i-1}.\partial_j^\alpha & \partial_{i+1}^\alpha & \partial_i^\alpha & s_i.\partial_j^\alpha \\ s_i.e_j = & e_j.s_{i-1} & e_{i+1} & e_i & e_j.s_i. \end{array} \tag{33}$$

The *symmetric cubical relations* consist thus: of the cubical relations (7), of the (self-dual) Moore relations (29) and of the above equations (33).

2.3. Symmetric cubical categories. A *symmetric cubical category*

$$\mathbb{A} = ((\mathbb{A}_n), (\partial_i^\alpha), (e_i), (+_i), (s_i)),$$

will be a cubical category (1.4) equipped with transposition *functors* $s_i\colon \mathbb{A}_n \to \mathbb{A}_n$ which make it a symmetric cubical set. Furthermore, cubical compositions and transpositions must be consistent, in the following sense

$$s_{i-1}(x +_i y) = s_{i-1}(x) +_{i-1} s_{i-1}(y), \qquad s_j(x +_i y) = s_j(x) +_i s_j(y) \qquad (j \neq i-1, i). \tag{34}$$

$\mathrm{Cub}_*(\mathbf{X})$ is a symmetric cubical category, with transpositions defined as above, in (31). The *involutive* case, further equipped with reversions under axioms which can be easily deduced from [10], is also of interest - not for commutative cubes, but certainly for higher cospans; however, we will not go here into such details. A *symmetric cubical functor* is a cubical functor which also preserves transpositions.

3. A formal interval for cubical cospans

We construct, here and in the next section, a cubical structure for higher cospans. The index α takes now the values $-1, 1$, also written $-, +$.

3.1. The setting. We shall follow a formal procedure, similar to the previous one for commutative cubes, in order to describe cospans in a category $\mathbf{X}$ and their higher dimensional versions.

The model of the construction will be the *formal cospan* $\wedge$, together with its cartesian powers

$$-1 \to 0 \leftarrow 1 \qquad \wedge,$$

$$\begin{array}{ccccc} (-1,-1) & \longrightarrow & (0,-1) & \longleftarrow & (1,-1) \\ \downarrow & & \downarrow & & \downarrow \\ (-1,0) & \longrightarrow & (0,0) & \longleftarrow & (1,0) \\ \uparrow & & \uparrow & & \uparrow \\ (-1,1) & \longrightarrow & (0,1) & \longleftarrow & (1,1) \end{array} \qquad \wedge^2.$$

$$\begin{array}{ccc} \bullet & \xrightarrow{1} & \\ \downarrow{\scriptstyle 2} & & \\ & & \end{array}$$

(35)

(In such diagrams, displaying finite categories, identities and composed arrows are always understood.) On the other hand, a *pt-category*, or *category with distinguished pushouts*, will be a ($\mathcal{U}$-small) category where *some* spans (f, g) have *one distinguished* pushout

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \bullet & \xrightarrow[f']{} & \bullet \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{f} & x' \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ x & \xrightarrow[f]{} & x' \end{array} \tag{36}$$

and we assume the following *unitarity constraints*:

(i) each square of identities is a distinguished pushout,

(ii) *if* the span $(f, 1)$ has a distinguished pushout, this is $(1, f)$; and symmetrically (see the right diagram above).

A *pt-functor* $F\colon \mathbf{X} \to \mathbf{Y}$ is a functor between pt-categories which *strictly* preserves the distinguished pushouts. We speak of a *full* (resp. *trivial*) *choice*, or of a *category* $\mathbf{X}$ *with full* (resp. *trivial*) *distinguished pushouts*, when all spans in $\mathbf{X}$ (resp. only the pairs of identities) have a distinguished pushout.

We will work in the category pt$\mathbf{Cat}$ of pt-categories and pt-functors, which is $\mathcal{U}$-complete and $\mathcal{U}$-cocomplete. For instance, the product of a family $(\mathbf{X}_i)_{i\in I}$ of pt-categories indexed on a $\mathcal{U}$-small set is the cartesian product $\mathbf{X}$ (in $\mathbf{Cat}$), equipped with those (pushout) squares in $\mathbf{X}$ whose projection in each factor $\mathbf{X}_i$ is a distinguished pushout. In particular, the terminal object of pt$\mathbf{Cat}$ is the terminal category $\mathbf{1}$ with the trivial (and unique) choice: its only square is a distinguished pushout.

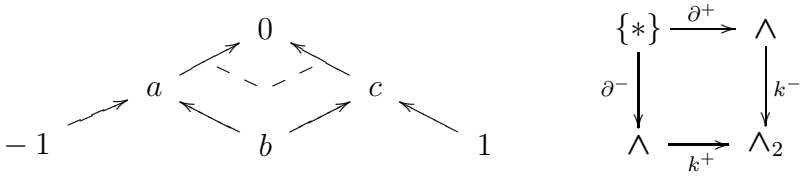
Cat embeds in pt**Cat**, equipping a small category with the trivial choice of pullbacks (left adjoint to the forgetful functor). Limits and colimits are preserved by the embedding. Our construction will require this sort of double setting $\mathbf{Cat} \subset \mathrm{pt}\mathbf{Cat}$, with 'models' $\wedge^n$ having a trivial choice and cubical cospans $\wedge^n \to \mathbf{X}$ living in categories with a full choice (which is necessary to compose them).

Notice that $\wedge, \wedge^2$ (and all powers) *have* all pushouts; however, should we use the full choice suggested by diagram (35), the pt-functors $\wedge^2 \to \mathbf{X}$ would only reach very particular 2-cubical cospans. Notice also that, in the absence of the unitarity constraint (i) on the choice of pushouts, the terminal object of pt**Cat** would still be the same, but a functor $x\colon \mathbf{1} \to \mathbf{X}$ could only reach an object whose square of identities is distinguished. On the other hand, condition (ii) will just simplify things, making our units work strictly; one might prefer to omit it, to get a 'more general' behaviour.

3.2. The interval structure. As **i** in the previous section, the category $\wedge$ has a basic structure of formal symmetric interval, with respect to the cartesian product in **Cat** (and pt**Cat**)

$$\begin{array}{llll} \partial^\alpha\colon \mathbf{1} \rightrightarrows \wedge, & e\colon \wedge \to \mathbf{1}, & s\colon \wedge^2 \to \wedge^2 & (\alpha = \pm 1), \\ \partial^\alpha(*) = \alpha, & & s(t_1, t_2) = (t_2, t_1). & \end{array} \tag{37}$$

Composition is - formally - more complicated. The *model of binary composition* will be the pt-category $\wedge_2$ displayed below, with one non-trivial distinguished pushout



(38)

Now, the commutative square at the right hand above is *not* a pushout; in fact, in **Cat** or pt**Cat**, the corresponding pushout is the subcategory $\wedge_{(2)}$ lying at the basis of $\wedge_2$:

$$-1 \to a \leftarrow b \to c \leftarrow 1 \qquad \wedge_{(2)}. \tag{39}$$

But the relevant fact is that a category **X** with *full* distinguished pushouts 'believes' that the square above is (also) a pushout. Explicitly, we have the following *para-universal* property of $\wedge_2$ (which, in itself, does not determine it, since it is also satisfied by $\wedge_{(2)}$)

(a) given two cospans $x, y\colon \wedge \to \mathbf{X}$, with values in a category **X** with full distinguished pushouts and $\partial_1^+ x = \partial_1^- y$, there is precisely one pt-functor $[x, y]\colon \wedge_2 \to \mathbf{X}$ such that $[x, y].k^- = x$, $[x, y].k^+ = y$.

The *concatenation map*

$$k\colon \wedge \to \wedge_2, \tag{40}$$

is an embedding, already displayed above by the labelling of objects in $\wedge_2$.

Also here, the functors $(-)^n_i = \wedge_{i-1} \times - \times \wedge_{n-i} \colon \mathrm{pt}\mathbf{Cat} \to \mathrm{pt}\mathbf{Cat}$ produce the higher structure of the interval, for $1 \leqslant i \leqslant n$ and $\alpha = \pm 1$

$$\begin{array}{ll} \partial^\alpha_i \colon \wedge^{n-1} \to \wedge^n, & \partial^\alpha_i(t_1, ..., t_{n-1}) = (t_1, ..., \alpha, ..., t_{n-1}), \\ e_i \colon \wedge^n \to \wedge^{n-1}, & e_i(t_1, ..., t_n) = (t_1, ..., \hat{t}_i, ..., t_n), \\ s_i \colon \wedge^{n+1} \to \wedge^{n+1}, & s_i(t_1, ..., t_{n+1}) = (t_1, ..., t_{i+1}, t_i, , ..., t_n). \end{array} \tag{41}$$

Moreover, acting on $\wedge_{(2)}$ (in (39)) and k, these functors yield the *n-dimensional i-concatenation model* $\wedge^{ni}_2$ and the *n-dimensional i-concatenation map* $k_i \colon \wedge^n \to \wedge^{ni}_2$

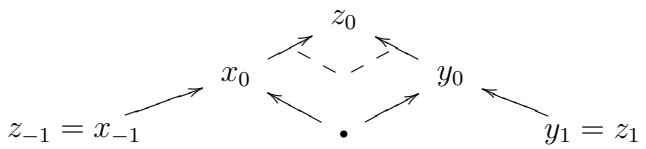
$$\begin{array}{ccc} \wedge^{n-1} & \xrightarrow{\partial^+_i} & \wedge^n \\ {\scriptstyle \partial^-_i}\downarrow & & \downarrow{\scriptstyle k^-_i} \\ \wedge^n & \xrightarrow[k^+_i]{} & \wedge^{ni}_2 \end{array} \qquad \begin{array}{l} \wedge^{ni}_2 = \wedge^{i-1} \times \wedge_2 \times \wedge^{n-i}, \\ \\ k_i = \wedge^{i-1} \times k \times \wedge^{n-i} \colon \wedge^n \to \wedge^{ni}_2. \end{array} \tag{42}$$

Again, the square above is not a pushout, but $\mathbf{X}$ (having a full choice of pushouts) believes it is. The formal interval $\wedge$ has much further structure, which is certainly of interest but will not be used here: for instance, the reversion symmetry $r \colon \wedge \to \wedge$, $r(t) = -t$, and the connections (cf. [10]).

3.3. Ordinary cospans. Let $\mathbf{X}$ be a *category with full distinguished pushouts*. A pt-functor $x \colon \wedge \to \mathbf{X}$ is just a functor, and amounts to a cospan $x_{-1} \to x_0 \leftarrow x_1$ in $\mathbf{X}$, i.e. a 1-cube with faces $\partial^\alpha_1(x) = x.\partial^\alpha = x_\alpha$. The degenerate 1-cube at the vertex x is the constant functor $e_1(x) = x.e = (x \to x \leftarrow x)$, with id$x$ at both arrows. The concatenation $z = x +_1 y$ of two consecutive cospans $x, y \colon \wedge \to \mathbf{X}$ (with $\partial^+_1 x = \partial^-_1 y$) is computed using the pt-functor $[x, y] \colon \wedge_2 \to \mathbf{X}$ determined by the para-universal property of $\wedge_2$ (3.2)

$$z = [x, y].k \colon \wedge \to \wedge_2 \to \mathbf{X}. \tag{43}$$

This amounts to the usual composition of cospans, by a *distinguished* pushout in $\mathbf{X}$ (because the pt-functor $[x, y]$ preserves the choice)



(44)

The bicategory of cospans in $\mathbf{X}$ [Be] will not be used directly, even though it lies within $\mathbb{C}\mathrm{osp}_*(\mathbf{X})$.

3.4. The symmetric pre-cubical category of cospans. A *symmetric pre-cubical category*

$$\mathbf{A} = ((A_n), (\partial^\alpha_i), (e_i), (s_i), (+_i)), \tag{45}$$

will be a symmetric cubical set with compositions, satisfying the consistency axioms (cub.1-2) of 1.2, where transpositions and compositions agree (in the sense of (34)). Thus,

we are not assuming that the i-compositions behave in a categorical way or satisfy interchange, in any sense, even weak.

For a category $\mathbf{X}$ with full distinguished pushouts, there is such a structure $\mathbf{A} = \mathrm{Cosp}_*(\mathbf{X})$. An n-cube, or *n-dimensional object*, or *n-cubical cospan*, is a functor $x\colon \wedge^n \to \mathbf{X}$; faces, degeneracies and transpositions are computed according to the formulas (41) for the interval $\wedge$

$$\mathrm{Cosp}_n(\mathbf{X}) = \mathbf{Cat}(\wedge^n, \mathbf{X}) = \mathrm{pt}\mathbf{Cat}(\wedge^n, \mathbf{X}),$$

$$\begin{array}{ll} \partial_i^\alpha(x) = x.\partial_i^\alpha\colon \wedge^{n-1} \to \wedge^n \to \mathbf{X}, & \partial_i^\alpha(x)(t_1,...,t_{n-1}) = x(t_1,...,\alpha,...,t_{n-1}), \\ e_i(x) = x.ei\colon \wedge^n \to \wedge^{n-1} \to \mathbf{X}, & e_i(x)(t_1,...,t_n) = x(t_1,...,\hat{t}_i,...,t_n), \\ s_i(x) = x.s_i\colon \wedge^{n+1} \to \wedge^{n+1} \to \mathbf{X}, & s_i(x)(t_1,...,t_{n+1}) = x(t_1,...,t_{i+1},t_i,,...,t_{n+1}). \end{array} \tag{46}$$

The i-composition $x +_i y$ is computed on the i-concatenation model $\wedge_2^{ni}$ (42), as

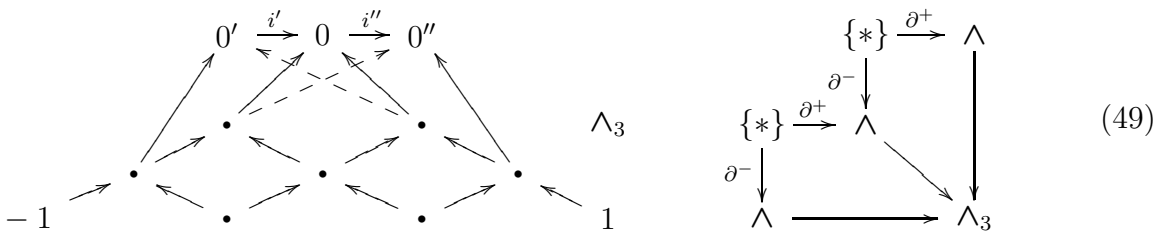
$$x +_i y = [x,y].k_i\colon \wedge^n \to \wedge_2^{ni} \to \mathbf{X} \qquad (\partial_i^+(x) = \partial_i^-(y)). \tag{47}$$

A *symmetric cubical functor* $F\colon \mathbf{A} \to \mathbf{B}$ between symmetric pre-cubical categories will be a morphism of symmetric cubical sets which preserves all composition laws. Plainly, a pt-functor $F\colon \mathbf{X} \to \mathbf{Y}$ between categories with full distinguished pushouts can be extended to a symmetric cubical functor F_* (which coincides with F in degree 0)

$$F_*\colon \mathrm{Cosp}_*(\mathbf{X}) \to \mathrm{Cosp}_*(\mathbf{Y}), \qquad F_*(x\colon \wedge^n \to \mathbf{X}) = F\circ x\colon \wedge^n \to \mathbf{Y}. \tag{48}$$

3.5. Formal associativity comparison. To prepare the next section, we extend now the basic structure of the directed interval with formal comparisons for associativity (and then for middle-four interchange).

The *model of ternary compositions* $\wedge_3$ is the order-category displayed below, at the left, with five non-trivial distinguished pushouts (as made explicit below)



(49)

The role of $\wedge_3$ will come forth from the right-hand diagram, in pt**Cat**, which *behaves as a colimit* for a category $\mathbf{X}$ with full distinguished pushouts. The true colimit $\wedge_{(3)}$, in **Cat** and pt**Cat**, consists of the six arrows along the bottom of the diagram of $\wedge_3$. Furthermore, $\wedge_3$ also contains:

- a symmetric construction of three *distinguished* pushouts, ending up at the vertex 0,
- two more *distinguished* pushouts, which attain the vertices 0' and 0",
- three coherent isomorphisms $i'\colon 0' \to 0$, $i''\colon 0 \to 0''$ and $i = i''i'\colon 0' \to 0''$ (so that each of these three objects is a colimit of the inclusion $\wedge_{(3)} \to \wedge_3$).

The functors $\kappa', \kappa'' \colon \wedge \to \wedge_3$ with the following images

$$\kappa' : \quad -1 \to 0' \leftarrow 1 \qquad\qquad \kappa'' : \quad -1 \to 0'' \leftarrow 1 \tag{50}$$

correspond to two iterated concatenations of the obvious three consecutive cospans $\wedge \to \wedge_3$ which 'cover' $\wedge_{(3)}$. They are linked by a functorial isomorphism

$$\begin{array}{lll} \kappa \colon \kappa' \to \kappa'' \colon \wedge \to \wedge_3 & & (\textit{formal associativity comparison}), \\ \kappa(\alpha) = \mathrm{id}\alpha, & \kappa(0) = i \colon 0' \to 0'' & (\alpha = \pm 1). \end{array} \tag{51}$$

Now, given three consecutive cospans x, y, z in $\mathbf{X}$ (having a full choice), the pt-functor $w = [x, y, z] \colon \wedge_3 \to \mathbf{X}$ resulting from the para-universal property contains both iterated concatenations $x +_1 (y +_1 z)$ and $(x +_1 y) +_1 z$. Thus, these consecutive cospans produce a natural isomorphism

$$\kappa(x, y, z) = [x, y, z].\kappa \colon x +_1 (y +_1 z) \to (x +_1 y) +_1 z \ \ (\textit{associativity comparison}). \tag{52}$$

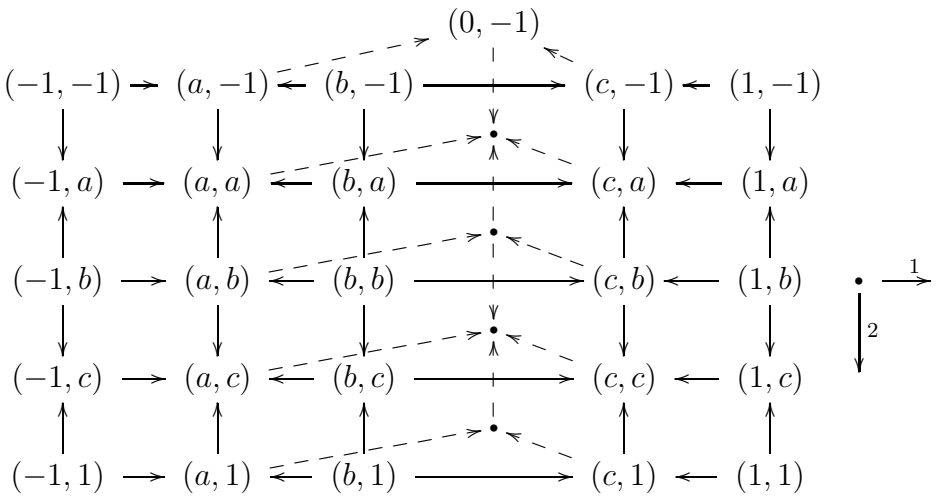
(One can note that the functor $[x, y, z]$ also 'contains' an intermediate *regular ternary concatenation* $x +_1 y +_1 z$, through the object $w(0)$).

3.6. Formal interchange comparisons. The pt-category $\wedge^2 = \wedge \times \wedge$ (a product in **Cat** and pt**Cat**, with trivial choice) is represented in (35). We have already remarked that $\wedge^2$ *has pushouts, but none of them is distinguished* (except the trivial ones, the squares of identities); which is what we need to represent all double cospans in $\mathbf{X}$ as pt-functors $\wedge^2 \to \mathbf{X}$. Double cospans can be concatenated in two directions (as will be formalised below, in any dimension $\geqslant 2$). The model $\wedge_{2\times 2}$ for the 2-dimensional interchange of concatenations is constructed below, starting with the colimit in **Cat** and pt**Cat** of the following diagram (again, any category $\mathbf{X}$ with a full choice will believe that also $\wedge_{2\times 2}$ is a colimit of the same diagram)

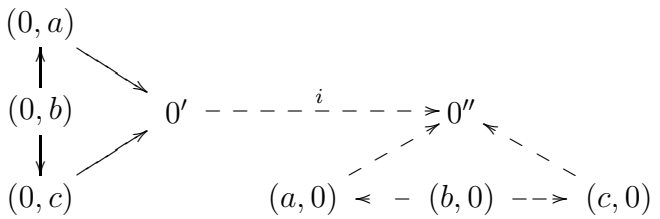
$$\begin{array}{ccccc} \wedge^2 & \xleftarrow{\partial_1^+} & \wedge & \xrightarrow{\partial_1^-} & \wedge^2 \\ {\scriptstyle \partial_2^+}\uparrow & & & & \uparrow{\scriptstyle \partial_2^+} \\ \wedge & & & & \wedge \\ {\scriptstyle \partial_2^-}\downarrow & & & & \downarrow{\scriptstyle \partial_2^-} \\ \wedge^2 & \xleftarrow[\partial_1^+]{} & \wedge & \xrightarrow[\partial_1^-]{} & \wedge^2 \end{array} \tag{53}$$

The (true) colimit is the pasting of four copies of $\wedge^2$, displayed in the solid diagram

below, and amounts to the product $\wedge_{(2)} \times \wedge_{(2)}$ (cf. (39))



(54)



(55)

By definition, the category $\wedge_{2\times 2}$ also contains two constructions, which correspond to two symmetric procedures: first composing in direction 1 and then in direction 2, or vice versa; namely:

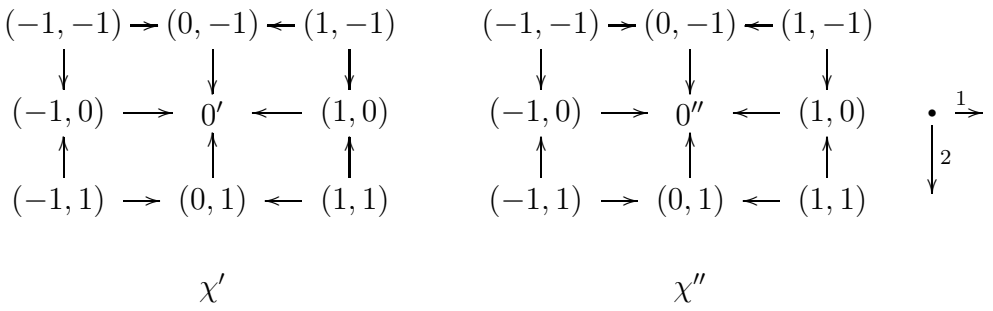
(a) a copy of $\wedge_2 \times \wedge_{(2)}$ (adding in the dashed arrows pertaining to the five distinguished pushouts $(0, j)$, with $j = -1, a, b, c, 1$), together with the (solid) distinguished pushout $0'$ displayed in (55);

(b) a symmetric construction, not displayed above: a copy of $\wedge_{(2)} \times \wedge_2$ (with five distinguished pushouts $(j, 0)$), together with the (dashed) distinguished pushout $0''$ displayed in (55);

(c) a coherent isomorphism $i\colon 0' \to 0''$ which links these two objects, so that each of them becomes a colimit of the inclusion $\wedge_{(2)} \times \wedge_{(2)} \to \wedge_{2\times 2}$.

The two symmetric procedures correspond to the functors $\chi', \chi''\colon \wedge^2 \to \wedge_{2\times 2}$ displayed

below



(56)

and are linked by a natural isomorphism, all of whose components are identities except the central one:

$$\chi\colon \chi' \to \chi''\colon \wedge^2 \to \wedge_{2\times 2}, \quad \chi(0,0) = i\colon 0' \to 0'' \qquad (\textit{middle-four interchange}). \tag{57}$$

3.7. Higher interchanges. Applying the functor $(-)_i^{n-1}$, for $1 \leqslant i < n$, gives a natural transformation *which concerns the interchange of the cubical compositions in directions* i *and* $i+1$

$$\begin{array}{l}\chi_i = \wedge^{i-1}\times\chi\times\wedge^{n-i-1}\colon \wedge^{i-1}\times\chi'\times\wedge^{n-i-1} \to \wedge^{i-1}\times\chi''\times\wedge^{n-i-1}\colon \\ \wedge^n \to \wedge^{i-1}\times\wedge_{2\times 2}\times\wedge^{n-i-1} \qquad (\textit{formal } i\textit{-interchange comparison}),\end{array} \tag{58}$$

Plainly, this is sufficient, since transpositions allow us to permute any two directions; actually, the single comparison $\chi_1 = \chi\times\wedge^{n-2}$ will suffice.

Without transpositions, in order to define 'general' weak cubical categories, we should construct formal interchanges

$$\chi_{ij}\colon \chi'_{ij} \to \chi''_{ij}\colon \wedge^n \to \wedge^{ni}_j \qquad (1 \leqslant i < j \leqslant n), \tag{59}$$

generalising the previous procedure, on the basis of a new diagram (53) containing the faces $\partial_i^\alpha, \partial_j^\beta\colon \wedge^{n-1} \to \wedge^n$. This is not complicated in itself, but would make much more complicated the coherence axioms of the next section.

4. Symmetric weak cubical categories

This section contains our main definitions and examples. Relations with weak double categories [11, 12] and Morton's structure of 2-cubical cospans [17] are examined in 4.5. Again, the index α takes the values ± 1, also written $-, +$.

4.1. Introducing transversal maps. As in Section 1, we introduce now a richer structure, having maps between n-objects *in a new direction* 0, which can be viewed as strict or 'transversal' in opposition with the previous weak or 'cubical' directions. The comparisons for units, associativity and interchange will be *invertible* maps of this kind. Being invertible, their orientation is inessential; but, for a possible extension to the lax

case, we will choose *the* orientation which is consistent with directed homotopy, see [9]. The new maps will also be used, below, to introduce limits (and then, one cannot restrict to the invertible ones).

Let us start with considering a general *category object* $\mathbb{A}$ within the category of symmetric pre-cubical categories and their functors

$$\mathbb{A}^0 \underset{e_0}{\overset{\partial_0^\alpha}{\leftleftarrows}} \mathbb{A}^1 \xleftarrow{c_0} \mathbb{A}^2 \qquad (\alpha = \pm). \tag{60}$$

We have thus:

(wcub.1) A symmetric pre-cubical category $\mathbb{A}^0 = ((A_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-cubes*, or *n-dimensional objects* of $\mathbb{A}$.

(wcub.2) A symmetric pre-cubical category $\mathbb{A}^1 = ((M_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-maps*, or *$(n+1)$-cells*, of $\mathbb{A}$.

(wcub.3) Symmetric cubical functors ∂_0^α and e_0, called 0-*faces* and 0-*degeneracy*, with $\partial_0^\alpha . e_0 = \mathrm{id}$.

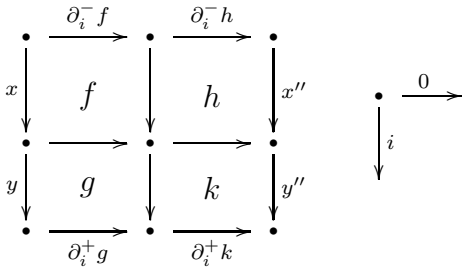
Typically, an n-map will be written as $f\colon x \to x'$, where $\partial_0^- f = x$, $\partial_0^+ f = x'$ are n-cubes. Every n-dimensional object x has an *identity* $e_0(x)\colon x \to x$. Note that ∂_0^α and e_0 preserve cubical faces (∂_i^α, with $i > 0$), cubical degeneracies (e_i), transpositions (s_i) and cubical concatenations ($+_i$). In particular, given two i-consecutive n-maps f, g, their 0-faces are also i-consecutive and we have:

$$f +_i g\colon x +_i y \to x' +_i y' \qquad (\text{for } f\colon x \to x',\ g\colon y \to y';\ \partial_i^+ f = \partial_i^- g). \tag{61}$$

(wcub.4) A composition law c_0 which assigns to two 0-consecutive n-maps $f\colon x \to x'$ and $h\colon x' \to x''$ (of the same dimension), an n-map $hf\colon x \to x''$ (also written $h.f$). This composition law is (strictly) categorical, and forms a category $\mathbb{A}_n = (A_n, M_n, \partial_0^\alpha, e_0, c_0)$. It is also consistent with the symmetric pre-cubical structure, in the following sense

$$\partial_i^\alpha(hf) = (\partial_i^\alpha h).(\partial_i^\alpha f), \qquad e_i(hf) = (e_i h)(e_i f), \qquad s_i(hf) = (s_i h)(s_i f), \tag{62}$$

$$(h +_i k).(f +_i g) = hf +_i kg, \tag{63}$$



The last condition is the (strict) middle-four interchange between the strict composition c_0 and any weak one. An n-map $f\colon x \to x'$ is said to be *special* if its 2^n vertices are identities

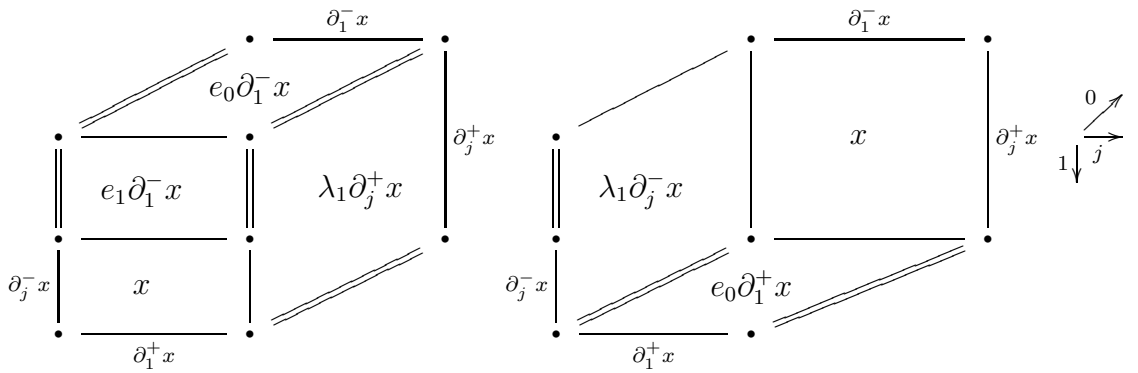
$$\partial^{\boldsymbol{\alpha}} f\colon \partial^{\boldsymbol{\alpha}} x \to \partial^{\boldsymbol{\alpha}} x', \qquad \partial^{\boldsymbol{\alpha}} = \partial_1^\alpha \circ \partial_2^\alpha \circ \ldots \partial_n^\alpha \qquad (\alpha_i = \pm). \tag{64}$$

In degree 0, this just means an identity.

4.2. Comparisons. We can now define a *symmetric weak cubical category* $\mathbb{A}$ as a category object within the category of symmetric pre-cubical categories and symmetric cubical functors, as made explicit in the preceding section, which is further equipped with invertible special transversal maps, playing the role of comparisons for units, associativity and cubical interchange, as follows. (We only assign the comparisons in direction 1 or 1, 2; all the others can be obtained with transpositions.)

(wcub.5) For every n-cube x, we have an invertible special n-map $\lambda_1 x$, which is natural on n-maps and has the following faces (for $n > 0$)

$$\begin{array}{ll} \lambda_1 x\colon (e_1\partial_1^- x) +_1 x \to x & (\textit{left-unit 1-comparison}), \\ \partial_1^\alpha \lambda_1 x = e_0\partial_1^\alpha x, \qquad \partial_j^\alpha \lambda_1 x = \lambda_1 \partial_j^\alpha x & (1 < j \leqslant n) \end{array} \tag{65}$$

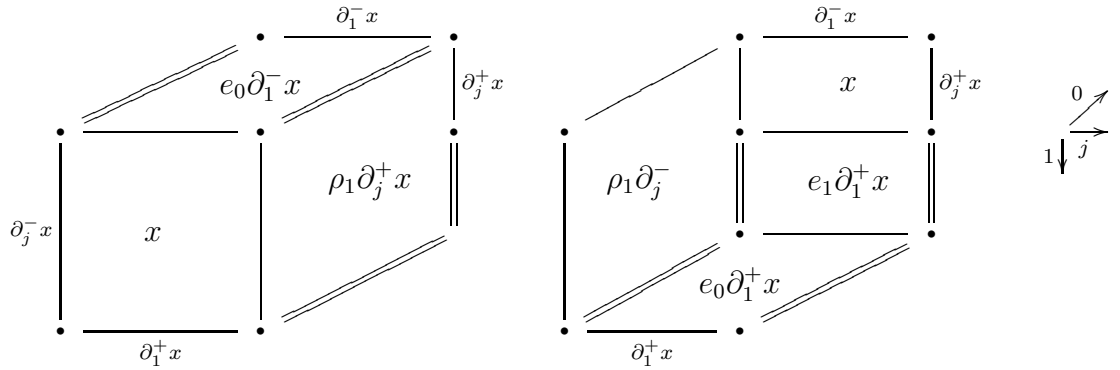


The naturality condition means that, for every n-map $f\colon x \to x'$, the following square of n-maps commutes

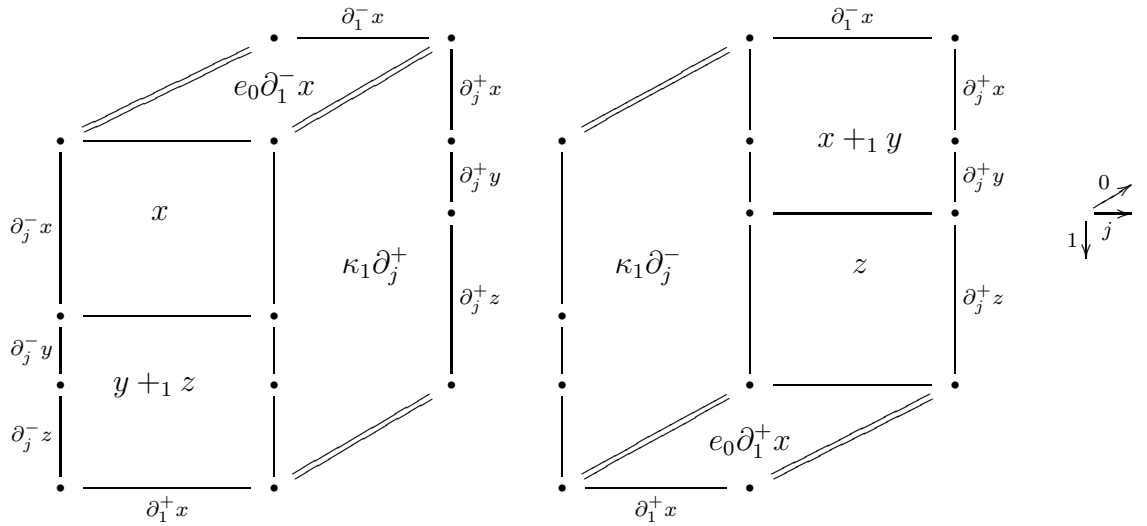
$$\begin{array}{ccc} (e_1\partial_1^- x) +_1 x & \xrightarrow{\lambda_1 x} & x \\ {\scriptstyle (e_1\partial_1^- f)+_1 f}\downarrow & & \downarrow{\scriptstyle f} \\ (e_1\partial_1^- x') +_1 x' & \xrightarrow[\lambda_1 x']{} & x' \end{array} \tag{66}$$

(wcub.6) For every n-cube x, we have an invertible special n-map $\rho_1 x$, which is natural on n-maps and has the following faces

$$\begin{array}{ll} \rho_1 x\colon x \to x +_1 (e_1\partial_1^+ x), & (\textit{right-unit 1-comparison}), \\ \partial_1^\alpha \rho_1 x = e_0\partial_1^\alpha x, \qquad \partial_j^\alpha \rho_1 x = \rho_1 \partial_j^\alpha x & (1 < j \leqslant n) \end{array} \tag{67}$$



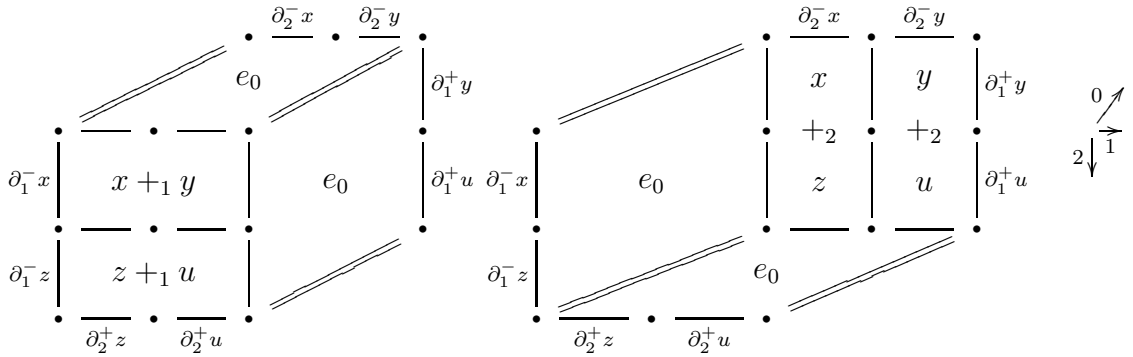
(wcub.7) For three 1-consecutive n-cubes x, y, z, we have an invertible special n-map $\kappa_1(x, y, z)$, which is natural on n-maps and has the following faces



(wcub.8) Given four n-cubes x, y, z, u which satisfy the boundary conditions (11) for $i = 1$ and $j = 2 \leqslant n$, we have an invertible n-map $\chi_1(x, y, z, u)$, which is natural on n-maps and has the following faces (partially displayed below)

$$\chi_1(x, y, z, u)\colon (x +_1 y) +_2 (z +_1 u) \to (x +_2 z) +_1 (y +_2 u) \qquad (\textit{interchange 1-comparison}),$$

$$\begin{aligned}
&\partial_1^- \chi_1(x, y, z, u) = e_0(\partial_1^- x +_2 \partial_1^- z), && \partial_1^+ \chi_1(x, y, z, u) = e_0(\partial_1^+ y +_2 \partial_1^+ u),\\
&\partial_2^- \chi_1(x, y, z, u) = e_0(\partial_2^- x +_1 \partial_2^- y), && \partial_2^- \chi_1(x, y, z, u) = e_0(\partial_2^- x +_1 \partial_2^- y), \qquad (68)\\
&\partial_j^\alpha \chi_1(x, y, z, u) = \chi_1(\partial_j^\alpha x, \partial_j^\alpha y, \partial_j^\alpha z, \partial_j^\alpha u) && (2 < j \leqslant n),
\end{aligned}$$



(wcub.9) Finally, these comparisons must satisfy some conditions of coherence, listed below (4.3). We say that $\mathbb{A}$ is unitary if the comparisons λ, ρ are identities.

4.3. Coherence. The coherence axiom (wcub.9) means that the following diagrams of transversal maps commute (assuming that all the cubical compositions make sense):

(i) *coherence pentagon for* $\kappa = \kappa_1$:

$$\begin{array}{ccccc} & & x +_1 (y +_1 (z +_1 u)) & & \\ & {\scriptstyle 1+\kappa}\swarrow & & \searrow{\scriptstyle \kappa} & \\ x +_1 ((y +_1 z) +_1 u) & & & & (x +_1 y) +_1 (z +_1 u) \\ {\scriptstyle \kappa}\downarrow & & & & \\ (x +_1 (y +_1 z)) +_1 u & & & \searrow{\scriptstyle \kappa} & \\ & {\scriptstyle \kappa+1}\searrow & & & ((x +_1 y) +_1 z) +_1 u \end{array} \tag{69}$$

(ii) *coherence hexagon for* $\chi = \chi_1$ and $\kappa = \kappa_1$:

$$\begin{array}{ccc} (x +_1 (y +_1 z)) +_2 (x' +_1 (y' +_1 z')) & \xrightarrow{\kappa+\kappa} & ((x +_1 y) +_1 z) +_2 ((x' +_1 y') \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \chi} \\ (x +_2 x') +_1 ((y +_1 z) +_2 (y' +_1 z')) & & ((x +_1 y) +_2 (x' +_1 y')) +_1 (z +_2 z') \\ {\scriptstyle 1+\chi}\downarrow & & \downarrow{\scriptstyle \chi+1} \\ (x +_2 x') +_1 ((y +_2 y') +_1 (z +_2 z')) & \xrightarrow[\kappa]{} & ((x +_2 x') +_1 (y +_2 y')) +_1 (z +_2 z') \end{array} \tag{70}$$

(iii) *coherence triangle for* $\lambda_1, \rho_1, \kappa_1$:

$$\begin{array}{ccc} x +_1 (e_1\partial_1^- y +_1 y) & \xrightarrow{\kappa} & (x +_1 e_1\partial_1^+ x) +_1 y \\ & {\scriptstyle 1+\lambda}\searrow \quad \nearrow{\scriptstyle \rho+1} & \\ & x +_1 y & \end{array} \tag{71}$$

In the unitary case, one replaces the last condition requiring this occurrence of κ to be an identity. Now, the question arises whether we have indeed listed a *sufficient system* of

coherence relations, which would allow us to prove a Coherence Theorem: 'all diagrams naturally constructed with comparisons commute'. This problem will not be addressed here.

4.4. The weak cubical category of cospans. Now, starting from a category $\mathbf{X}$ with full distinguished pushouts, we have a symmetric weak cubical category $\mathbb{C}\mathrm{osp}_*(\mathbf{X})$, which is unitary (under the unitarity constraint for the choice of pushouts in $\mathbf{X}$).

(a) The symmetric pre-cubical category of n-dimensional objects is our previous $\mathrm{Cosp}_*(\mathbf{X})$ of 3.4.

(b) An *n-map* $f\colon x \to x'$ (also called an *$(n+1)$-cell*) is a natural transformation of n-cubes $f\colon x \to x'\colon \wedge^n \to \mathbf{X}$, or equivalently an n-cube in the pt-category $\mathbf{X^2}$ of morphisms of $\mathbf{X}$ (with the coherent choice of distinguished pushouts). They form thus the symmetric pre-cubical category $\mathrm{Cosp}_*(\mathbf{X^2})$, with:

$$\begin{array}{ll} \partial_i^\alpha f = f.\partial_i^\alpha \colon x\partial_i^\alpha \to x'\partial_i^\alpha \colon \wedge^{n-1} \to \mathbf{X} & (i \leqslant n, \alpha = \pm 1), \\ e_i f = f.e_i \colon x.e_i \to x'.e_i \colon \wedge^n \to \mathbf{X} & (i \leqslant n), \\ s_i f = f.s_i \colon x.s_i \to x'.s_i \colon \wedge^n \to \mathbf{X} & (i \leqslant n-1), \\ f +_i g = [f,g].k_i \colon \wedge^n \to \mathbf{X^2} & (\partial_i^+ f = \partial_i^- g). \end{array} \tag{72}$$

(c) The symmetric pre-cubical functors of 0-faces and 0-degeneracies come forth, contravariantly, from the obvious functors linking the categories $\mathbf{1}$ and $\mathbf{2}$

$$e_0 \colon \mathrm{Cosp}_*(\mathbf{X}) \rightleftarrows \mathrm{Cosp}_*(\mathbf{X^2}) : \partial_0^\alpha \qquad (\partial_0^\alpha \colon \mathbf{1} \rightrightarrows \mathbf{2} : e_0, \ \ \alpha = \pm). \tag{73}$$

(d) The composite $hf\colon x \to x''$ of 0-consecutive n-maps is the composition of natural transformations. It is categorical and preserves the symmetric cubical structure.

(e) The cubical composition laws behave categorically up to suitable comparisons for associativity and interchange, which are invertible maps. These are defined as follows, for $n \geqslant 1$ and $i = 1, ..., n$ (even if we only need the case $i = 1$ to build up the required structure).

(i) Given three i-consecutive n-cospans $x, y, z\colon \wedge^n \to \mathbf{X}$, the formal associativity comparison $\kappa\colon \kappa' \to \kappa''\colon \wedge \to \wedge_3$ (51) gives the *associativity i-comparison*, a natural isomorphism $\kappa_i = \kappa_i(x,y,z)$

$$\kappa_i = [x,y,z].(\wedge^{i-1} \times \kappa \times \wedge^{n-i})\colon x +_i (y +_i z) \to (x +_i y) +_i z \colon \wedge^n \to \mathbf{X}. \tag{74}$$

(ii) Given four n-cospans $x, y, z, u\colon \wedge^n \to \mathbf{X}$, which satisfy the boundary conditions (11) for $i < i+1 \leqslant n$, the formal interchange comparison $\chi\colon \chi' \to \chi''\colon \wedge^2 \to \wedge_{2\times 2}$ (57) gives the following natural isomorphism, the *interchange i-comparison* $\chi_i = \chi_i(x,y,z,u)$ (for the directions $i, i+1$)

$$\chi_i = \begin{bmatrix} x & y \\ z & u \end{bmatrix}.\chi_i \colon (x +_i y) +_j (z +_i u) \to (x +_j z) +_i (y +_j u) \colon \wedge^n \to \mathbf{X}. \tag{75}$$

The coherence axioms hold, as the terms of *each* diagram in 4.3 are computed by different systems of distinguished pushouts, which end up with various constructions of the same colimit in $\mathbf{X}$, and therefore are linked by coherent isomorphisms.

4.5. Truncation. As in 1.5, n-truncation yields the structure of a *symmetric weak* $(n+1)$*-cubical category*. Thus $n\mathbb{C}\mathrm{osp}_*(\mathbf{X})$, which contains the k-cubes and k-maps of $\mathbb{C}\mathrm{osp}_*(\mathbf{X})$ for $k \leqslant n$, is the symmetric weak $(n+1)$-*cubical* category of n-cubical cospans.

In the 1-truncated case there is only one cubical direction and no transposition, so that we drop the term 'symmetric'. A *weak 2-cubical category*, like $1\mathbb{C}\mathrm{osp}_*(\mathbf{X}) = \mathbb{C}\mathrm{osp}(\mathbf{X})$, amounts, precisely, to a weak (or pseudo) double category, as defined in [11], Section 7: a structure with a strict 'horizontal' composition and a weak 'vertical' composition, under strict interchange. According to the terminology of [11, 12], a 0-cube is an *object*, a 0-map is a *horizontal arrow*, a 1-cube is a *vertical arrow*, and finally a 1-map is a *double cell*. (Notice that in [11], 7.1, the axioms are written in the *one-sorted* approach, where everything is a double cell, so that degeneracies do not appear.)

The 2-truncated structure $2\mathbb{C}\mathrm{osp}_*(\mathbf{X})$, a symmetric weak 3-cubical category, is related to Morton's construction [Mo], which consists of a 'Verity double bicategory' [18]. Loosely speaking, and starting from $2\mathbb{C}\mathrm{osp}_*(\mathbf{X})$, one should omit the transpositions and restrict transversal maps to the special ones.

4.6. Cubical limits. Limits in weak double categories (i.e. weak 2-cubical categories) have been dealt with in [11]; this study can likely be extended to the general cubical case. Here, we only deal with products, in order to show the importance of having *general* n-maps, as opposed to only using the invertible ones - which would be sufficient for comparisons.

Let $\mathbb{A}$ be a weak cubical category, or even a pre-cubical one. We say that $\mathbb{A}$ *has cubical products* if:

(i) for every $n \geqslant 0$, the ordinary category $\mathbb{A}_n$ of n-cubes and n-maps of $\mathbb{A}$ has products: i.e., every family $(x_i)_{i\in I}$ of n-cubes (indexed on a small set) has an n-cube $x = \prod x_i$ equipped with a family $p_i \colon x \to x_i$ of n-maps ($i \in I$) satisfying the usual universal property;

(ii) such products are preserved by faces and degeneracies.

If the pt-category $\mathbf{X}$ has products, then the weak cubical category $\mathbb{C}\mathrm{osp}_*(\mathbf{X})$ has cubical products. Indeed, the category of functors $\mathbb{C}\mathrm{osp}_n(\mathbf{X}) = \mathbf{Cat}(\wedge^n, \mathbf{X})$ has products, which are computed pointwise; and these are preserved by faces and degeneracies, which are computed via maps $\wedge^{n-1} \rightleftarrows \wedge^n$.

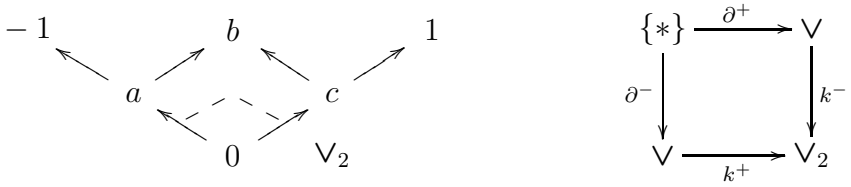
In the 1-truncated case, a weak double category $\mathbb{A}$ has cubical products if and only if, according to the definition of [11], $\mathbb{A}$ has *a lax functorial choice of products*; this follows straightforwardly from the characterisation of this property given in Lemma 4.4 of [11]. (Note that products in the pt-category $\mathbf{X}$ need not preserve pushouts, and - generally - will only give 'lax functors' with respect to weak compositions.)

4.7. Spans and diamonds. Dualising the construction of cospans, we have the symmetric weak cubical category $\mathbb{S}\mathrm{p}_*(\mathbf{X})$ of cubical spans, for a small category $\mathbf{X}$ with full distinguished pullbacks. The construction is based on the formal span $\vee = \wedge^{\mathrm{op}}$ (with a trivial choice of pullbacks)

$$\begin{array}{ll} \vee : & -1 \leftarrow 0 \rightarrow 1 \\ \mathbb{S}\mathrm{p}_*(\mathbf{X}) = \mathbb{C}\mathrm{osp}_*(\mathbf{X}^{\mathrm{op}}), & \mathbb{S}\mathrm{p}_n(\mathbf{X}) = \mathbf{Cat}(\vee^n, \mathbf{X}), \end{array} \tag{76}$$

and its structure as a formal symmetric interval, dual to the one of $\wedge$.

Notice that, formally, weak compositions are still based on *pushouts* (and their generalisations). The concatenation model is the pb-category $\vee_2 = (\wedge_2)^{\mathrm{op}}$, displayed at the left



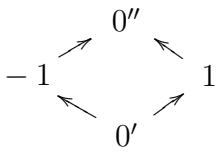
(77)

The category $\vee_2$ can be inserted in the commutative diagram above, at the right. It is obtained from the corresponding pushout $\vee_{(2)}$ in **Cat**, adding a distinguished *pullback* with vertex 0; but any category with a full choice of pullbacks believes that also $\vee_2$ is a pushout.

Similarly, for a category $\mathbf{X}$ *with full distinguished pullbacks and pushouts*, we have a weak cubical category of *cubical bispans*, or *cubical diamonds*

$$\mathbb{B}\mathrm{isp}_*(\mathbf{X}), \qquad \mathbb{B}\mathrm{isp}_n(\mathbf{X}) = \mathbf{Cat}(\Diamond^n, \mathbf{X}). \tag{78}$$

The construction is based on the category $\Diamond$



(the *formal bispan*), (79)

$\Diamond$

which is just a 'formal commutative square', but becomes a *formal bispan* when equipped with the obvious structure of formal symmetric interval, combining the structures of $\wedge$ and $\vee$: faces end up at ± 1, and so on.

4.8. Cubical relations. We end this section sketching the construction of a strict cubical category $\mathbb{R}\mathrm{el}_*(\mathbf{Ab})$, as a quotient of $\mathbb{C}\mathrm{osp}_*(\mathbf{Ab})$. One could equivalently use $\mathbb{S}\mathrm{p}_*(\mathbf{Ab})$; and the same can be done with any abelian category. (For relations of sets, one *should* work with spans.)

In dimension 1, let us say that two cospans $x, x' \colon \wedge \to \mathbf{Ab}$ with the same cubical faces (at ± 1) are equivalent when they have the same pullback-span (which is jointly monic); they define thus one relation $[x] = [x'] \colon x^- \!\!\rightarrow x^+$, represented by this jointly monic span.

Moreover, any transversal map $f\colon x \to y$ provides a transversal map $[f]\colon [x] \to [y]$, defined as a transversal map in $\mathbb{S}\mathrm{p}_*(\mathbf{Ab})$ between jointly monic spans. Composition is induced by the composition of cospans, which is consistent with our equivalence relation. (Equivalently, one can take the jointly monic span associated to a concatenation of jointly monic spans.)

In higher dimension, two n-cospans $x, x'\colon \wedge^n \to \mathbf{Ab}$ with the same *outer vertices*

$$x(\mathbf{t}) = x'(\mathbf{t}), \qquad \mathbf{t} \in \{-1, 1\}^n, \tag{80}$$

are equivalent when all the 'homologous' pairs of 1-cospans which x, x' 'contain'

$$\{t_1, ..., t_{i-1}\} \times \wedge \times \{t_{i+1}, ..., t_n\} \to \wedge^n \rightrightarrows \mathbf{Ab}, \tag{81}$$

are equivalent, as above.

The 1-truncated cubical category $\mathbb{R}\mathrm{el}(\mathbf{Ab})$ coincides with the double category of abelian groups, homomorphisms and relations considered in [11]. Notice that a double cell, i.e. a transversal map $f\colon u \to v$ of 1-cubes in $\mathbb{R}\mathrm{el}_*(\mathbf{Ab})$, amounts to a *lax*-commutative square of relations

$$\begin{array}{ccc} X^- & \xrightarrow{f^-} & Y^- \\ {\scriptstyle u}\downarrow & \leqslant & \downarrow{\scriptstyle v} \\ X^+ & \xrightarrow[f^+]{} & Y^+ \end{array} \qquad\qquad f^+u \leqslant vf^-. \tag{82}$$

which need *not* commute: generally, f is *not* a transversal map in the cubical category of cubes of relations.

5. Strict multiple categories

We end with a generalisation of the cubical structure, having a countable family of cubical directions, of *different sorts*.

5.1. The geometry. Loosely speaking, a strict multiple category $\mathbb{A}$ will be a generalised strict cubical category where the cubical directions can be of different sorts, or *colours*. The index $i \geqslant 0$ will represent such sorts, *including the transversal one*, which - in the strict case - has no reason to be treated differently. Thus, we will have

- a set $A_\emptyset$ of objects,
- a set A_i of i-arrows, or *i-coloured* arrows, for every index $i \geqslant 0$ (with faces in $A_\emptyset$),
- a set $A_{i_1 i_2}$ of 2-dimensional (i_1, i_2)-cells, for indices $i_1 < i_2$ (with faces in A_{i_1} and A_{i_2}),
- and generally a set $A_{\mathbf{i}} = A_{i_1 i_2 ... i_n}$ of n-dimensional $\mathbf{i}$-cells, for every *multi-index* $\mathbf{i}$

$$\mathbf{i} = (i_1, ..., i_n), \qquad 0 \leqslant i_1 < i_2 < ... < i_n \qquad (n \geqslant 0), \tag{83}$$

(with faces in the various $A_{i_1 ... \hat{i}_j ... i_n}$).

5.2. Multiple sets. We begin by extending the presheaf category $\mathbf{Set}^{\mathbb{I}^{\mathrm{op}}}$ of cubical sets. While the site $\mathbb{I}$ (1.5) has objects 2^n, i.e. the powers of the cardinal $2 = \{0, 1\}$, the new site $\mathbb{M}$ is based on a sequence of disjoint copies $2_i = \{0_i, 1_i\}$ of the cardinal 2, for $i \geqslant 0$, representing the various 1-dimensional directions. Then, each multi-index $\mathbf{i}$ (as in (83) gives an object of $\mathbb{M}$, namely the cartesian product of the corresponding copies of 2

$$2_{\mathbf{i}} = 2_{i_1} \times 2_{i_2} \times ... \times 2_{i_n} \qquad (n \geqslant 0;\ \ 0 \leqslant i_1 < i_2 < ... < i_n), \tag{84}$$

Starting from the basic faces and degeneracy in direction $i \geqslant 0$

$$\partial_i^\alpha \colon 1 \rightleftarrows 2_i : e_i, \tag{85}$$

the arrows of $\mathbb{M}$ are generated, under composition in $\mathbf{Set}$, by the following higher faces and degeneracies

$$\begin{array}{l} \partial_{i_j}^\alpha = 2_{i_1} \times ... \times \partial_{i_j}^\alpha \times ... \times 2_{i_n} \colon 2_{i_1} \times ... \times 1 \times ... \times 2_{i_1} \to 2_{\mathbf{i}}, \\ e_{i_j} = 2_{i_1} \times ... \times e_{i_j} \times ... \times 2_{i_n} \colon 2_{\mathbf{i}} \to 2_{i_1} \times ... \times 1 \times ... \times 2_{i_n} \quad (i_1 < ... < i_j < ... < i_n). \end{array} \tag{86}$$

Note that the indexing we are using is (also here) incomplete: domains and codomains must be written down; for a complete indexing, one can write $\partial^\alpha_{\mathbf{i}i_j}$ and $e_{\mathbf{i}i_j}$.

Thus, a *multiple set* is a functor $A \colon \mathbb{M}^{\mathrm{op}} \to \mathbf{Set}$, which means a system of sets and mappings

$$\begin{array}{ll} A_{\mathbf{i}} = A_{i_1 i_2 ... i_n} & (n \geqslant 0;\ \ 0 \leqslant i_1 < i_2 < ... < i_n), \\ \partial_{i_j}^\alpha \colon A_{\mathbf{i}} \to A_{i_1 ... \hat{i}_j ... i_n}, \quad e_{i_j} \colon A_{i_1 ... \hat{i}_j ... i_n} \to A_{\mathbf{i}} & (i_1 < ... < i_j < ... < i_n), \end{array} \tag{87}$$

satisfying the *multiple relations*

$$\begin{array}{ll} \partial_i^\alpha . \partial_j^\beta = \partial_j^\beta . \partial_i^\alpha \ \ (i \neq j), & e_i e_j = e_j e_i \ \ (i \neq j), \\ \partial_i^\alpha . e_j = e_j . \partial_i^\alpha \ \ (i \neq j) \quad or \quad \mathrm{id} \ \ (i = j). & \end{array} \tag{88}$$

These relations look simpler then the cubical ones because here an index i stands for a particular sort, instead of a mere position, and is never 'renamed'. A *reduced* multiple set works with indices $i \geqslant 1$.

Alternatively, as suggested by the referee, one can view $\mathbb{M}$ as the direct limit of its truncations

$$\mathbb{M}_0 \to \mathbb{M}_1 \to \mathbb{M}_2 \to .... \tag{89}$$

where $\mathbb{M}_1 = \mathbb{I}_1 = \{0 \rightleftarrows 1\}$ is the site of 1-cubical sets (or 1-simplicial sets, or reflexive graphs) and $\mathbb{M}_n = (\mathbb{M}_1)^n$ is a cartesian power. The functor $\mathbb{M}_0 = \mathbf{1} \to \mathbb{M}_1$ takes value at the object 1, and the following ones are obtained by repeatedly applying $- \times \mathbb{M}_1$.

5.3. Multiple categories. We extend now the definition of a *strict cubical category*, given in 1.4.

(mlt.1) A *multiple category* $\mathbb{A}$ is, first of all, a multiple set of components $A_{\mathbf{i}}$, whose elements will be called $\mathbf{i}-$cells; as above, $\mathbf{i}$ is any multi-index $(0 \leqslant i_1 < i_2 < ... < i_n)$.

(mlt.2) Moreover, given two $\mathbf{i}$-cells x, y which are i_j-*consecutive* $(\partial^+_{i_j}(x) = \partial^-_{i_j}(y))$, the i_j - *concatenation* $x +_{i_j} y$ (or i_j-composition) is defined and satisfies the following 'geometrical' interactions with faces and degeneracies

$$\begin{aligned} &\partial^-_{i_j}(a +_{i_j} b) = \partial^-_{i_j}(a), && \partial^+_{i_j}(a +_{i_j} b) = \partial^+_{i_j}(b), \\ &\partial^\alpha_{i_h}(a +_{i_j} b) = \partial^\alpha_{i_h}(a) +_{i_j} \partial^\alpha_{i_h}(b) \ \ (h \neq j), && e_{i_h}(a +_{i_j} b) = e_{i_h}(a) +_{i_j} e_{i_h}(b). \end{aligned} \tag{90}$$

(mlt.4) For $i < j$ we have

$$(a +_i b) +_j (c +_i d) = (a +_j c) +_i (b +_j d) \qquad (\textit{middle-four interchange}), \tag{91}$$

whenever these composites make sense. (Again, the *nullary* interchange is already expressed above.)

A *multiple functor* $F\colon \mathbb{A} \to \mathbb{B}$ between multiple categories is a morphism of multiple sets which preserves all composition laws.

Within the cubical category $\mathbb{C}\mathrm{ub}_*(\mathbf{X})$, every sequence $(\mathbf{X}_i)_{i\geqslant 0}$ of subcategories of $\mathbf{X}$ brings forth a multiple category $\mathbb{C}\mathrm{ub}_*(\mathbf{X}, (\mathbf{X}_i))$, where the i-directed arrows belong to $\mathbf{X}_i$ (and, for higher cells, form commutative diagrams of $\mathbf{X}$). For instance, take as $\mathbf{X}$ the category of smooth manifolds and continuous mappings, with $\mathbf{X}_i$ the subcategory of mappings of class $\mathcal{C}^i$.

Truncation works in the obvious way, as in 1.5. In particular, a *strict triple category* coincides with the notion introduced by Gray [13], which amounts to a category object in the category of double categories and double functors.

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