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CARTESIAN BICATEGORIES II

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Abstract. The notion of *cartesian bicategory*, introduced in [C&W] for locally ordered bicategories, is extended to general bicategories. It is shown that a cartesian bicategory is a symmetric monoidal bicategory.

1. Introduction

1.1 We recall that in [C&W] a locally ordered bicategory $\mathbf{B}$ was said to be *cartesian* if the subbicategory of left adjoints, Map$\mathbf{B}$, had finite products; each hom-category $\mathbf{B}(B, C)$ had finite products; and a certain derived tensor product on $\mathbf{B}$, extending the product structure of Map$\mathbf{B}$, was functorial. It was shown that cartesian structure provides an elegant base for sets of axioms characterizing bicategories of

i) relations in a regular category

ii) ordered objects and order ideals in an exact category

iii) additive relations in an abelian category

iv) relations in a Grothendieck topos.

Notable was an axiom, *groupoidalness*, that captures the *discrete* objects in a cartesian locally ordered bicategory and gives rise to a very satisfactory approach to duals.

It was predicted in [C&W] that the notion of cartesian bicategory would be developable without the restriction of local orderedness, so as to capture

v) spans in a category with finite limits

vi) profunctors in an elementary topos.

It is this development of the unrestricted notion of cartesian bicategory that is our present concern. In a sequel [CKWW] we shall give such further characterization theorems but this paper is concerned with the basic development.

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A cartesian bicategory is a bicategory with various *properties*. In contrast to many bicategorical studies, *no constraint data subject to coherence conditions are assumed*. Constraint data is constructed from the existence clauses of the universal properties provided by the definition of cartesian bicategory and it is tempting to believe that all coherence conditions for such constraints will automatically follow from the uniqueness clauses of such universal properties. After all, a mere category with finite products has $- \times -$ as a candidate for a monoidal structure $- \otimes -$. In this case it is easy to construct arrows $a_{X,Y,Z}:(X \times Y) \times Z \to X \times (Y \times Z)$ and prove that they are invertible and satisfy Mac Lane's pentagon condition, for example.

A cartesian bicategory need not have all finite products (even in the sense of bilimits). However, it admits a canonical $- \otimes -$ connective that one immediately suspects to be the main ingredient of a symmetric monoidal structure. One of the axioms for a cartesian bicategory requires that the locally full subbicategory determined by the left adjoint arrows admits all finite products in the sense of bilimits. We show that another bicategory associated with a cartesian bicategory also has finite products in this sense. Assuming that a bicategory with finite products underlies a symmetric monoidal bicategory we are then able to show that a cartesian bicategory also underlies a symmetric monoidal bicategory.

When we started this project we assumed it was well known that a bicategory with finite products in the sense of bilimits *is* a symmetric monoidal bicategory. All to whom we spoke about the matter agreed with us but we were unable to find a published account. The universal property for products in the sense of bilimits is given in terms of equivalences (rather than isomorphisms) of hom-categories. At first this seems to be a rather weak beginning when one considers that a monoidal bicategory is a one-object tricategory whose definition appears very complicated to the uninitiated. It seemed that rather a lot had to be constructed, and even more to be proved, starting from very little. To make matters worse we need symmetry in our sequel paper [CKWW] and a symmetric monoidal bicategory is a one-object, one-arrow, one-2-cell, one-3-cell weak 6-category. Of course one has the more informative definitions of a symmetric monoidal bicategory provided collectively by [D&S] and [McC] but we ultimately decided that our assumptions about finite products required proofs.

Accordingly, we begin in Section 2 with the study of a bicategory $\mathbf{A}$ with finite products, concluding it with Theorem 2.15 stating that $(\mathbf{A}, \times, 1, \cdots)$ is a symmetric monoidal bicategeory. In Section 3 we find it convenient to define a bicategory $\mathbf{B}$ to be *precartesian* if the locally full subbicategory determined by the left adjoints, $\mathbf{M} := \mathrm{Map}\mathbf{B}$, has finite products and all the hom categories have finite products (as mere categories). We construct from $\mathbf{B}$ and $\mathbf{M}$ a further bicategory $\mathbf{G}$, whose objects are general arrows of $\mathbf{B}$ and whose arrows between them are given by squares containing a 2-cell in which the arrow components are left adjoints. We show that $\mathbf{G}$ too has finite products, for a precartesian $\mathbf{B}$, and use $\mathbf{G}$ to define arrows

$$\otimes:\mathbf{B} \times \mathbf{B} \to \mathbf{B} \leftarrow \mathbf{1}:I$$

which are proved to be lax functors. Finally, in Section 4 we define $\mathbf{B}$ to be cartesian if it is precartesian and enjoys the further property that the constructed constraints for

$\otimes$ and I are invertible, thus making these pseudofunctors. We prove Theorem 4.6, that $(\mathbf{B}, \otimes, I, \cdots)$ is a symmetric monoidal bicategeory, and further basic facts that will be used in [CKWW].

We should point out that we have deliberately made no use of (proarrow) equipments in this paper. The papers [CKW] and [CKVW] sought to capture the universal properties of $\otimes$ and I in a way quite different from that we have just described. The cartesian bicategories we listed at the beginning of 1.1 are all parametrized by mere categories $\mathcal{E}$ enjoying such properties as having finite limits, being regular, or being an elementary topos. The effect of "change of base" along a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ on the cartesian bicategories Rel$\mathcal{E}$, Span$\mathcal{E}$, and so on, is an important one but it does seem to require arrows between bicategories more general than lax functors and colax functors.

By concentrating on the underlying equipments of cartesian bicategories, such as Rel$\mathcal{E}$ and Span$\mathcal{E}$, one is led to a 2-category of these wherein adjointness reveals $\otimes$ and I as right adjoints to diagonals. This notion first appeared in [CKW], but only for locally ordered equipments. In [CKVW] the locally ordered restriction was removed but application of the formulation given there essentially restricted one to (cartesian) bicategories in which the subbicategory of left adjoints (maps) actually forms a mere category. Verity's thesis [VER] had confronted the change of base problem for equipments in full generality without either of the simplifying assumptions above. Moreover, the anonymous referee informs us that some of the results of this paper can be shown to follow from results in [VER]. We do not doubt this and we acknowledge that the relationship between cartesian bicategories and cartesian equipments warrants considerable study; but this paper is a direct sequel to [C&W].

G.M. "Max" Kelly died during the preparation of this paper. He was actively working on it on the day of his passing. The other authors express their gratitude for his work here and for so much more that he had shared with us as a friend and a colleague over many years. We regret too that he was unable to provide a proof reading of our final draft.

2. Bicategories with Finite Products

2.1 We consider a bicategory $\mathbf{A}$. We write $X, Y, Z, ...$ for objects, $R, S, T, ...$ for arrows, and $\alpha, \beta, \gamma, ...$ for 2-cells of $\mathbf{A}$. We usually omit parentheses in three-fold composites but then RST is to be understood as $(RS)T$. In this vein we define $\mathbf{A}(T, R){:}\mathbf{A}(X, A) \to \mathbf{A}(Y, B)$ as the composite $\mathbf{A}(X, A) \xrightarrow{\mathbf{A}(1,R)} \mathbf{A}(X, B) \xrightarrow{\mathbf{A}(T,1)} \mathbf{A}(Y, B)$, where $R{:}A \to B$ and $T{:}Y \to X$ in $\mathbf{A}$, and use this choice in defining the hom pseudofunctor

$$\mathbf{A}(-,-){:}\mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{CAT} \tag{1}$$

where **CAT** denotes the 2-category of categories. Note that we use *pseudofunctor* for what is also called a *homomorphism of bicategories* and *lax functor* for what is also called a *morphism of bicategories*.

We find it convenient to assume all our bicategories to be *normal*, in the sense that the constraints $1R \cong R$ and $R1 \cong R$ are identities. It is easy to replace any bicategory by a biequivalent normal one: we have only to provide a new identity.

A functor $F : X \to Y$, or more generally an arrow $F : X \to Y$ in a bicategory, is said to be an *equivalence* if there is an arrow $U : Y \to X$ and invertible 2-cells $\epsilon : FU \to 1$ and $\alpha : 1 \to UF$. It is a bicategorical formality that in such a situation one can find an invertible $\eta : 1 \to UF$ giving an adjunction $\eta, \epsilon : F \dashv U$. We may as well suppose each equivalence to come with a specified right adjoint with invertible unit and invertible counit.

An arrow R in a bicategory is called a *map* if it has a right adjoint. We shall usually denote maps by lowercase Roman letters; and if f is a map, we write $\eta_f, \epsilon_f : f \dashv f^*$ for a chosen adjunction that makes it so. We write Map$\mathbf{A}$ for the locally-full subbicategory of $\mathbf{A}$ determined by the maps.

2.2 We begin by supposing the bicategory $\mathbf{A}$ to have binary and nullary *products* (in the *bilimit* sense, which is the only one appropriate to bicategories). For the binary case, this means simply that to each pair (X, Y) of objects is assigned an object $X \times Y$ and arrows $p_{X,Y} : X \times Y \to X$ and $r_{X,Y} : X \times Y \to Y$, called projections, such that the functor $\mathbf{A}(A, X \times Y) \to \mathbf{A}(A, X) \times \mathbf{A}(A, Y)$ they determine is an equivalence for each A. (We use lowercase for projections because in the case of interest they are hypothesized to be maps.) In elementary terms, $\mathbf{A}(A, X \times Y) \to \mathbf{A}(A, X) \times \mathbf{A}(A, Y)$ is essentially surjective on objects and fully faithful. Thus for $R : A \to X$ and $S : A \to Y$ we have $\langle R, S\rangle : A \to X \times Y$ and invertible 2-cells $\mu_{R,S} : p_{X,Y}\langle R, S\rangle \to R$ and $\nu_{R,S} : r_{X,Y}\langle R, S\rangle \to S$. And for $T, U : A \to X \times Y$ together with 2-cells $\alpha : p_{X,Y}T \to p_{X,Y}U$ and $\beta : r_{X,Y}T \to r_{X,Y}U$ there exists a unique $\gamma : T \to U$ having $p_{X,Y}\gamma = \alpha$ and $r_{X,Y}\gamma = \beta$. In particular, for 2-cells $\phi : R \to R' : A \to X$ and $\psi : S \to S' : A \to Y$, there is a unique 2-cell $\langle \phi, \psi\rangle : \langle R, S\rangle \to \langle R', S'\rangle$ making $\mu_{R,S}$ and $\nu_{R,S}$ natural in R and S.

Next, since *each* pair of objects has such a product, we can, as is well known, make $\times$ into a pseudofunctor $\times : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ in such a way that the $p_{X,Y}$ and $r_{X,Y}$ become the components of pseudonatural transformations p and r. In more detail, for $R : X \to A$ and $S : Y \to B$ we have $R \times S : X \times Y \to A \times B$ and invertible 2-cells $p_{R,S} : p_{A,B}(R \times S) \to Rp_{X,Y}$ and $r_{R,S} : r_{A,B}(R \times S) \to Sr_{X,Y}$, making p and r into pseudonatural transformations $p : \times \to P$ and $r : \times \to R$, where $P, R : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ are the projection pseudofunctors. Of course we take $R \times S$ to be $\langle Rp_{X,Y}, Sr_{X,Y}\rangle$ in the language of the last paragraph, while $p_{R,S}$ is a suitable μ and $r_{R,S}$ is a suitable ν; but we don't have to remember their origins.

As a matter of fact we have a problem here with notation, because later we use $p_{R,S}$ in a different sense in connection with the product projection $R \times S \to R$ in another bicategory $\mathbf{G}$ whose objects are the arrows of a bicategory $\mathbf{B}$. To avoid confusion we henceforth denote the 2-cells of the last paragraph, namely the pseudonaturality isomorphisms of p and r, by $p'_{R,S}$ and $r'_{R,S}$ instead of $p_{R,S}$ and $r_{R,S}$. More generally, if t is a pseudonatural transformation with components t_X, we will write t'_R for the pseudonaturality isomorphism corresponding to an arrow $R : X \to Y$.

Of course $\times : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ is the right pseudoadjoint of the diagonal pseudofunctor $\Delta : \mathbf{A} \to \mathbf{A} \times \mathbf{A}$, with counit constituted by the pseudonatural p and r. For the unit

d we can take the components d_X to be $\langle 1,1\rangle : X \nrightarrow X \times X$, with the pseudonaturality isomorphism $d'_R : d_Y R \nrightarrow (R \times R) d_X$ coming from the evident isomorphisms of its domain and its codomain with $\langle R, R\rangle$.

There remains the nullary product, namely the *terminal object* of $\mathbf{A}$. Here we are given an object 1 of $\mathbf{A}$ for which the functor $\mathbf{A}(A,1) \nrightarrow \mathbf{1}$ is an equivalence for each A, so that $1 : \mathbf{1} \nrightarrow \mathbf{A}$ is the right pseudoadjoint of $! : \mathbf{A} \nrightarrow \mathbf{1}$. The unit is given by choosing an arrow $t_X : X \nrightarrow 1$ for each X; then for $R : X \nrightarrow A$ there is a unique 2-cell $t'_R : t_A R \nrightarrow t_X$, and it is invertible.

2.3 The bicategory $\mathbf{A}$ is said to have *finite products* if, for every finite set I and every family $X = (X_i)_{i\in I}$ of objects of $\mathbf{A}$, there is an object P and a family of arrows $p = (p_i : P \nrightarrow X_i)_{i\in I}$ with domain P so that, for each A, the functor

$$\mathbf{A}(A,P) \nrightarrow \prod_{i\in I} \mathbf{A}(A, X_i) \tag{2}$$

induced by the p_i is an equivalence. Such a family $p = (p_i : P \nrightarrow X_i)_{i\in I}$ is called a *product cone over* X.

2.4. Proposition. *A bicategory with binary products and a terminal object has finite products.*

Proof. The cases of I having cardinality 2 or 0 are covered explicitly by the hypotheses. They are a matter of identifying $\mathbf{A}^I$ with $\mathbf{A} \times \mathbf{A}$ or $\mathbf{1}$. For I having cardinality 1 and an I-indexed family X, any equivalence $P \nrightarrow X$, in particular the identity $1_X : X \nrightarrow X$, is a product cone. Suppose that we have a product cone for each I-indexed family X with cardinality of I less than $n+1$ and consider a family Z indexed by a set J of cardinality $n+1$. We can see J as a sum of sets $I + \mathbf{1}$ so that the J-family Z is an I-family X together with a single object Y. Let $(p_i : P \nrightarrow X_i)_{i\in I}$ be a product cone for X and consider the J-family of arrows $q = (q_j : P \times Y \nrightarrow Z_j)_{j\in J}$ given by $q_j = p_i . p_{P,Y} : P \times Y \nrightarrow X_i$, for $j = i \in I$ and $q_j = r_{P,Y} : P \times Y \nrightarrow Y)$ for $j = * \in \mathbf{1}$. Since a composite of equivalences is an equivalence, this family is a product cone over Z. ∎

Because a finite set I with cardinality n admits a bijection to the set $\{1, 2, \cdots, n\}$, the bicategory $\mathbf{A}$ with binary and nullary products, equivalently all finite products, admits, for each natural number n, a pseudofunctor

$$\Pi_n : \mathbf{A} \times \cdots \times \mathbf{A} = \mathbf{A}^n \nrightarrow \mathbf{A}$$

with domain the n-fold product bicategory, right pseudoadjoint to the diagonal pseudofunctor. We can write, for example, $\Pi_3(X,Y,Z) = X \times Y \times Z$ *without parentheses.*

2.5 We want to exhibit $\mathbf{A}$ as underlying a monoidal bicategory with $\times = \Pi_2$ as its tensor product. To this end, for a family $X = (X_i)_{i\in I}$ we write $\mathbf{A}(X) = \mathbf{A}((X_i)_{i\in I})$ for the bicategory whose objects are the product cones over X, an arrow from $(b_i : B \nrightarrow X_i)$ to $(c_i : C \nrightarrow X_i)$ being an arrow $R : B \nrightarrow C$ in $\mathbf{A}$ along with isomorphisms $\mu_i : c_i R \nrightarrow b_i$, and

a 2-cell $(R, \mu_i) \to (S, \nu_i)$ being a 2-cell $\alpha : R \to S$ in $\mathbf{A}$ having $\nu_i.(c_i\alpha) = \mu_i$. There is an evident forgetful pseudofunctor from $\mathbf{A}(X)$ to $\mathbf{A}$.

But since we know that finite products exist, we can abbreviate in the above by writing an object $(b_i : B \to X_i)$ of $\mathbf{A}(X)$ as $b : B \to X$, treating X here as a name of a product $\prod_{i \in I} X_i$, so that $b : B \to X$ is an equivalence. Now an arrow $(R, \mu) : (B, b) \to (C, c)$ of $\mathbf{A}(X)$ is an arrow $R : B \to C$ of $\mathbf{A}$ along with a single isomorphism $\mu : cR \to b$, and a 2-cell $(R, \mu) \to (S, \nu)$ is an $\alpha : R \to S$ satisfying $\nu.(c\alpha) = \mu$.

Recall that a bicategory is biequivalent to the bicategory **1** precisely when

i) the set of objects is not empty,

ii) for any objects B and C there is an arrow $R : B \to C$, and

iii) for any two arrows $R, S : B \to C$ there is a unique 2-cell $R \to S$.

2.6. Proposition. *Each bicategory* $\mathbf{A}(X)$ *is biequivalent to* **1**.

Proof. First, the existence of finite products in $\mathbf{A}$ ensures that each $\mathbf{A}(X)$ is not empty. Next, for any objects (B, b) and (C, c) of $\mathbf{A}(X)$, there is an arrow (R, μ) from (B, b) to (C, c) because, (C, c) being a product cone, (2) (with P replaced by C) is essentially surjective. Finally, if (S, ν) is a second arrow from (B, b) to (C, c), then to give a 2-cell $\alpha : (R, \mu) \to (S, \nu)$ is to give an $\alpha : R \to S$ having $c_i\alpha = (\nu_i)^{-1}(\mu_i)$; and there is a unique such 2-cell because, (C, c) being a product cone, (2) (with P replaced by C) is fully faithful. ∎

Note that, as a consequence, every arrow in $\mathbf{A}(X)$ is an equivalence; so that its underlying arrow in $\mathbf{A}$ is also an equivalence. Again, every 2-cell in $\mathbf{A}(X)$ is an isomorphism, so that its underlying 2-cell in $\mathbf{A}$ is also an isomorphism.

2.7 In the definition of a monoidal bicategory the tensor product (here $\times = \Pi_2$) is to be a pseudofunctor. We have remarked in 2.2 that $\times = \Pi_2$ is a pseudofunctor $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$, and that p, r, and t are pseudonatural. Then there is a pseudofunctor $\Pi_2 \times 1 : \mathbf{A} \times \mathbf{A} \times \mathbf{A} \to \mathbf{A} \times \mathbf{A}$, and hence a pseudofunctor $\Pi_2(\Pi_2 \times 1) : \mathbf{A} \times \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ sending (X, Y, Z) to $(X \times Y) \times Z$. Similarly there is a pseudofunctor $\Pi_2(1 \times \Pi_2) : \mathbf{A} \times \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ sending (X, Y, Z) to $X \times (Y \times Z)$. Each of the pseudofunctors $\Pi_2(\Pi_2 \times 1)$ and $\Pi_2(1 \times \Pi_2)$ is an object of the bicategory $[\mathbf{A}^3, \mathbf{A}]$ of pseudofunctors, pseudonatural transformations, and modifications.

Starting with the pseudonatural p and r we construct the projections

$$p_{X,Y}.p_{(X\times Y),Z}, \quad r_{X,Y}.p_{(X\times Y),Z}, \quad \text{and} \quad r_{(X\times Y),Z} \tag{3}$$

from $(X \times Y) \times Z$ to X, Y, and Z respectively, and the projections

$$p_{X,(Y\times Z)}, \quad p_{Y,Z}.r_{X,(Y\times Z)}, \quad \text{and} \quad r_{Y,Z}.r_{X,(Y\times Z)} \tag{4}$$

from $X \times (Y \times Z)$ to X, Y, and Z respectively. Like p and r, the projections (3) and (4) are pseudonatural when we treat $(X \times Y) \times Z$, $X \times (Y \times Z)$, X, Y, and Z as objects of

$[\mathbf{A}^3, \mathbf{A}]$, regarding them as alternative names for $\Pi_2(\Pi_2 \times 1)$, for $\Pi_2(1 \times \Pi_2)$, and for the first, second, and third projections of $\mathbf{A}^3$ onto $\mathbf{A}$. Accordingly, we can see the projections (3) and (4) as arrows of $[\mathbf{A}^3, \mathbf{A}]$.

The projections (3) and (4) constitute product cones, exhibiting $(X \times Y) \times Z$ and $X \times (Y \times Z)$ as products in $\mathbf{A}$ of X, Y, and Z. However products in $[\mathbf{A}^3, \mathbf{A}]$ are formed pointwise from those in $\mathbf{A}$. Accordingly, when we see the projections (3) and (4) as arrows of $[\mathbf{A}^3, \mathbf{A}]$, they form product cones there, exhibiting $(X \times Y) \times Z$ and $X \times (Y \times Z)$ as products of X, Y, and Z not only in $\mathbf{A}$ but also in $[\mathbf{A}^3, \mathbf{A}]$.

2.8 We now apply Proposition 2.6 to the bicategory $[\mathbf{A}^3, \mathbf{A}](X, Y, Z)$. We have $\Pi_3 : \mathbf{A}^3 \to \mathbf{A}$. The product cones (3) and (4) correspond to equivalences $h : \Pi_2(\Pi_2 \times 1) \to \Pi_3$ and $k : \Pi_2(1 \times \Pi_2) \to \Pi_3$, with components $h_{X,Y,Z} : (X \times Y) \times Z \to X \times Y \times Z$ and $k_{X,Y,Z} : X \times (Y \times Z) \to X \times Y \times Z$.

It follows from Proposition 2.6 that there is a pseudonatural transformation

$$a : \Pi_2(\Pi_2 \times 1) \to \Pi_2(1 \times \Pi_2) \tag{5}$$

and an invertible modification

$$\mu : ka \to h \tag{6}$$

Moreover a is an equivalence, and (a, μ) is unique to within a unique isomorphism. The components of the pseudonatural a are equivalences

$$a_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z)$$

in $\mathbf{A}$, and the components of μ are isomorphisms

$$\mu_{X,Y,Z} : k_{X,Y,Z}.a_{X,Y,Z} \to h_{X,Y,Z}$$

In a similar way we produce the pseudonatural equivalences $l : \Pi_2(\Pi_0 \times 1) \to \Pi_1$ and $r : \Pi_1 \to \Pi_2(1 \times \Pi_0)$.

2.9 In $[\mathbf{A}^4, \mathbf{A}]$ we have (writing XY for $X \times Y$ and so on) the composite

$$((XY)Z)W \to (X(YZ))W \to X((YZ)W) \to X(Y(ZW)) \tag{7}$$

of $a_{X,Y,Z}W$, $a_{X,YZ,W}$, and $Xa_{Y,Z,W}$; on the face of it these are arrows in $\mathbf{A}$, but we can also see them (as in the second paragraph of 2.7) as naming pseudonatural equivalences which are arrows of $[\mathbf{A}^4, \mathbf{A}]$. We also have the composite

$$((XY)Z)W \to (XY)(ZW) \to X(Y(ZW)) \tag{8}$$

of $a_{XY,Z,W}$ and $a_{X,Y,ZW}$. Let us write (7) and (8) as $m, n : U \to V$, where $U = \Pi_2(\Pi_2(\Pi_2 \times 1) \times 1)$ and $V = \Pi_2(1 \times \Pi_2(1 \times \Pi_2))$; and denote by $u : U \to \Pi_4$ and $v : V \to \Pi_4$ the equivalences formed from projections like (3) and (4).

Each of these arrows in (7) and (8) comes with an invertible modification, namely $\mu_{X,Y,Z}W$, $\mu_{X,YZ,W}$, $X\mu_{Y,Z,W}$, $\mu_{XY,Z,W}$, and $\mu_{X,Y,ZW}$ respectively, where μ is the modification in (6). These modifications fit together and compose to give invertible modifications $\alpha : vm \nrightarrow u$ and $\beta : vn \nrightarrow u$. Now (m, α) and (n, β) are arrows from U to V in $[\mathbf{A}^4, \mathbf{A}](X, Y, Z, W)$, so that there is by Proposition 2.6 a unique 2-cell $\pi{:}(m, \alpha) \nrightarrow (n, \beta)$, which is invertible. That is to say, $\pi : m \nrightarrow n$ is an invertible modification satisfying $\beta.(v\pi) = \alpha$.

This is the modification π of (TD7) in [GPS]. There are three similar modifications called μ, λ, and ρ in [GPS] involving respectively (a, l, r), (a, l, l) and (a, r, r), and these appear similarly as 2-cells in $[\mathbf{A}^3, \mathbf{A}](X, Y, Z)$.

2.10 On page 10 of [GPS] there is a diagram (TA1) to be satisfied in a monoidal bicategory. It demands the equality in $[\mathbf{A}^5, \mathbf{A}]$ of two modifications

$$X(Ya).Xa.a.(Xa)V.aV.(aU)V \Longrightarrow a.a.a$$

They are in fact equal by Proposition 2.6, as they are 2-cells in $[\mathbf{A}^5, \mathbf{A}](X, Y, Z, U, V)$.

The remaining two axioms follow in a similar way.

2.11 We next show that the monoidal bicategory we have constructed is symmetric; or more correctly that we can endow it with a symmetry. According to [D&S], a *symmetry* for a monoidal bicategory consists of a braiding and a syllepsis, with the syllepsis satisfying a certain symmetry condition: see Definition 18 on page 131 of [D&S]. Although [D&S] consider only the special case of a Gray monoid, where the associativity is an identity, their definition above of a symmetry is surely meant to apply generally. The meanings of braiding and syllepsis (but not of symmetry) are given in [McC], as follows (see pages 133 to 145).

The basic datum for a braiding is a pseudonatural equivalence

$$s{:} \otimes \nrightarrow \otimes S \tag{9}$$

where $S{:}\mathbf{A}^2 \nrightarrow \mathbf{A}^2$ sends (X, Y) to (Y, X) and so on, and where $\otimes{:}\mathbf{A}^2 \nrightarrow \mathbf{A}$ is the tensor product, which for us is $\times$, here more conveniently called Π. There are two further data, consisting of invertible modifications sitting in hexagonal diagrams, and the data are to satisfy four axioms which are equations between invertible modifications.

A syllepsis for a braided monoidal bicategory is an invertible modification σ from the identity of $\otimes$ to $sS.s{:} \otimes \nrightarrow \otimes SS = \otimes$, with components

$$\sigma_{X,Y}{:}1_\otimes \nrightarrow s_{Y,X}s_{X,Y} \tag{10}$$

which is to satisfy two axioms consisting of equations between invertible modifications. Note that $1_\otimes S = 1_{\otimes S}$ and we have also $\sigma S{:}1_{\otimes S} \nrightarrow s.sS$, with components

$$\sigma_{Y,X}{:}1_{\otimes S} \nrightarrow s_{X,Y}s_{Y,X}$$

The braiding and the syllepsis constitute a symmetry if (see [D&S] p.131]) the syllepsis satisfies the further condition $s\sigma = (\sigma S)s$, which in terms of components is

$$s_{X,Y}\sigma_{X,Y} = \sigma_{Y,X}s_{X,Y} \tag{11}$$

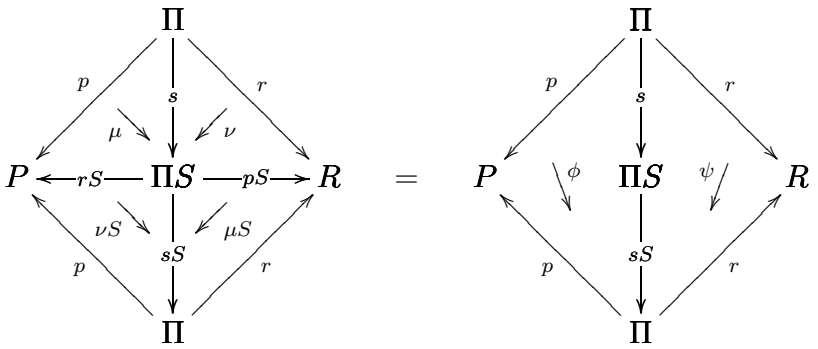
2.12 For the monoidal bicategory arising as above from a bicategory $\mathbf{A}$ with finite products, we construct a symmetry as follows. In $[\mathbf{A}^2, \mathbf{A}]$ we have the product cone

$$P \xleftarrow{p} \Pi \xrightarrow{r} R$$

of 2.2, where $P, R{:}\mathbf{A}^2 \to \mathbf{A}$ are the projection pseudofunctors, but also

$$P = RS \xleftarrow{rS} \Pi S \xrightarrow{pS} PS = R$$

is a product cone. By the essential surjectivity aspect of the universal property of the latter, we have in the top two triangles on the left below an arrow s (necessarily a pseudonatural equivalence), and invertible 2-cells μ and ν as shown.



(12)

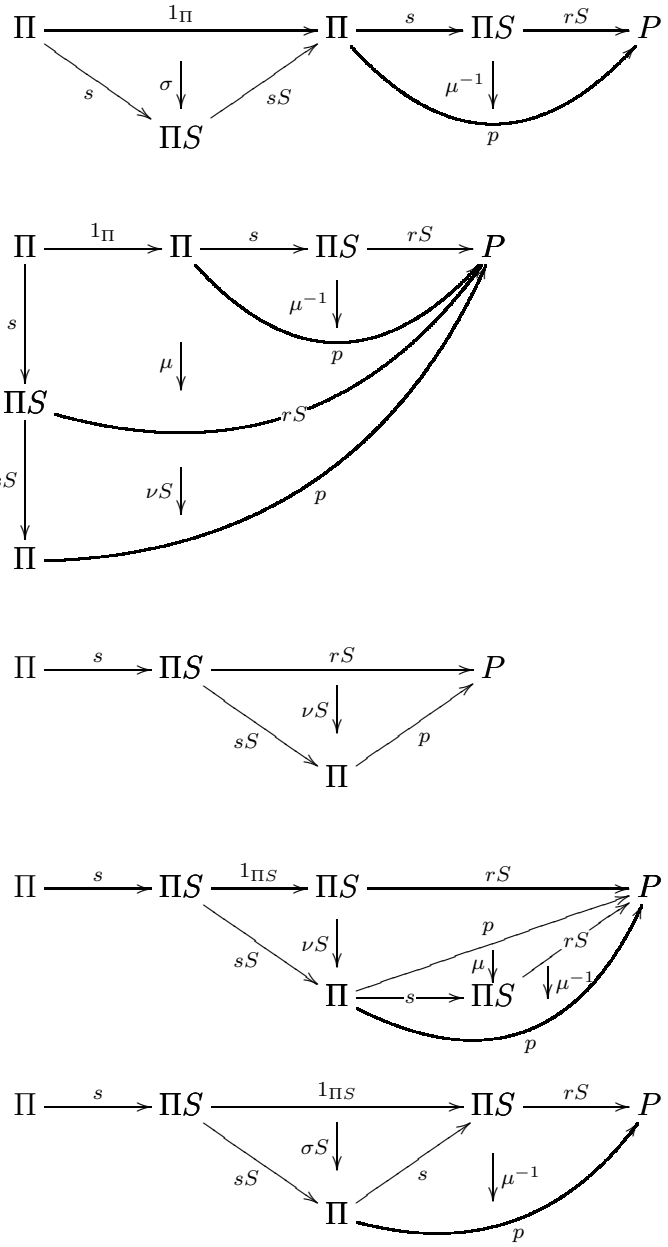
Note that precomposing the top two triangles on the left above with S produces the bottom two triangles on the left above. These triangles give rise to pasting composites that we have displayed, and named, on the right above. By normality of $\mathbf{A}$, and hence of $[\mathbf{A}^2, \mathbf{A}]$, we have $\phi{:}p1_\Pi \to p(sS)s$ and $\psi{:}r1_\Pi \to r(sS)s$. It follows, by the fully faithful aspect of the universal property of the product cone of 2.2 that there is a unique 2-cell

$$\sigma{:}1_\Pi \to (sS)s \tag{13}$$

with $p\sigma = \phi$ and $r\sigma = \psi$ in $[\mathbf{A}^2, \mathbf{A}]$. So σ is by construction a modification. Moreover, σ is invertible since ϕ and ψ are so. If we precompose the entire left side of (12) with S then we get a unique 2-cell $\sigma S{:}1_{\Pi S} \to s(sS)$ with $(pS)(\sigma S) = \phi S$ and $(rS)(\sigma S) = \psi S$. It follows that sS is an inverse equivalence of s. We will take s and the σ as constructed here as the basic datum for a braiding (9) and the datum for a syllepsis (10).

2.13 Before dealing with the further data for a braiding and the braiding and syllepsis equations we will establish the symmetry equation (11) directly from the descriptions of σ and σS. To show that $s\sigma = (\sigma S)s$ it suffices to show that $(rS)s\sigma = (rS)(\sigma S)s$ and $(pS)s\sigma = (pS)(\sigma S)s$. We show the first equality by showing equality of the pasting

composites of each side with the invertible $\mu^{-1}\colon (rS)s \to p$. We have:



The second equality, $(pS)s\sigma = (pS)(\sigma S)s$, can be shown, similarly, by showing equality of the pasting composites of each side with the invertible $\nu^{-1}\colon (pS)s \to r$.

2.14 It is now convenient to again treat $X \times Y$, $Y \times X$, X, and Y as objects in $[\mathbf{A}^2, \mathbf{A}]$, regarding them as alternative names for $\Pi = \times$, for ΠS, and for the first and second projections of $\mathbf{A}^2$ onto $\mathbf{A}$.

In $[\mathbf{A}^3, \mathbf{A}]$ we have (again writing XY for $X \times Y$ and so on) the composite

$$(XY)Z \xrightarrow{a} X(YZ) \xrightarrow{s} (YZ)X \xrightarrow{a} Y(ZX) \tag{14}$$

and the composite

$$(XY)Z \xrightarrow{sZ} (YX)Z \xrightarrow{a} Y(XZ) \xrightarrow{Ys} Y(ZX) \tag{15}$$

We now write the composites in (14) and (15) as $m, n : U \nrightarrow V$, where $U = \Pi(\Pi \times 1)$ and $V = \Pi(1 \times \Pi)(1 \times S)(S \times 1)$, and we write $u : U \nrightarrow \Pi_3$ and $v : V \nrightarrow \Pi_3$ for the equivalences constructed using projections. Each of the arrows in (14) and (15) comes with modifications constructed from either the μ in (6) or the μ and ν in (12) and these fit together and compose to give modifications $\alpha : vm \nrightarrow u$ and $\beta : vn \nrightarrow u$. Now (m, α) and (n, β) are arrows from U to V in $[\mathbf{A}^3, \mathbf{A}](X, Y, Z)$, so by Proposition 2.6 there is a unique invertible 2-cell $(m, \alpha) \nrightarrow (n, \beta)$. This is the modification 'R' of [McC]. The modification 'S' of [McC] (which relates s and a^*, the adjoint inverse equivalence of a) is constructed in a similar way.

In [McC] there are four diagrams, (BA1)–(BA4), to be satisfied in a braided monoidal category. Three of these demand equality of two invertible 2-cells in $[\mathbf{A}^4, \mathbf{A}]$ (modifications) and the other is an equality of two invertible 2-cells in $[\mathbf{A}^3, \mathbf{A}]$. In each case we use Proposition 2.6 to demonstrate equality. For the first three, the modifications in question are shown to be 2-cells in $[\mathbf{A}^4, \mathbf{A}](X, Y, Z, W)$. For the fourth, the modifications are shown to be 2-cells in $[\mathbf{A}^3, \mathbf{A}](X, Y, Z)$.

In [McC] there are two diagrams, (SA1) and (SA2), to be satisfied in a sylleptic (braided) monoidal category. Both demand equality of two invertible 2-cells in $[\mathbf{A}^3, \mathbf{A}]$. In each case, Proposition 2.6 can be used after showing that the modifications in question provide 2-cells in $[\mathbf{A}^3, \mathbf{A}](X, Y, Z)$. As with the braiding equations, we forego an explicit demonstration. To summarize:

2.15. THEOREM. *A bicategory* $\mathbf{A}$ *with binary product* $\times$ *and terminal object* 1 *underlies a symmetric monoidal bicategory with* $\times$ *as its tensor product and* 1 *as its unit object.* ∎

3. Precartesian Bicategories

3.1 A bicategory $\mathbf{B}$ is said to be *precartesian* if

i) the bicategory $\mathbf{M} = \mathrm{Map}\mathbf{B}$ has finite products;

ii) each category $\mathbf{B}(X, Y)$ has finite products.

We henceforth assume $\mathbf{B}$ to be a *precartesian* bicategory.

We have discussed the notations $\times : \mathbf{M} \times \mathbf{M} \nrightarrow \mathbf{M}$ and $p : \times \nrightarrow P$ and $r : \times \nrightarrow R$ for the binary product in a bicategory. Here we find it convenient to write I (rather than 1) for the terminal object of $\mathbf{M}$. For the binary product in each $\mathbf{B}(X, Y)$ we write $R \wedge S$, with $\pi : R \wedge S \nrightarrow R$ and $\rho : R \wedge S \nrightarrow S$ for the projections, and we use $\top$, or $\top_{X,Y}$, for the terminal object of $\mathbf{B}(X, Y)$. For the adjunction units we write $\delta : R \nrightarrow R \wedge R$ and $\tau : R \nrightarrow \top$.

3.2. Example.

i) All the examples of cartesian bicategories provided in [C&W];

ii) $\mathbf{B} = \mathrm{Span}\mathcal{E}$, the bicategory of spans in $\mathcal{E}$, for $\mathcal{E}$ a category with finite limits;

iii) $\mathbf{B} = \mathrm{Prof}\mathcal{E}$, the bicategory of categories and profunctors in $\mathcal{E}$, for $\mathcal{E}$ an elementary topos;

iv) $\mathbf{B} = \mathrm{Cart}\mathcal{E}$, the 2-category of categories with finite products, finite product preserving functors, and natural transformations in $\mathcal{E}$, for $\mathcal{E}$ a category with finite limits.

3.3 We now describe a certain bicategory $\mathbf{G}$ associated to $\mathbf{B}$. It is in fact the bicategory formed by applying Street's [STR] two-sided Grothendieck construction to the pseudofunctor

$$\mathbf{M}^{\mathrm{op}} \times \mathbf{M} \xrightarrow{i^{\mathrm{op}} \times i} \mathbf{B}^{\mathrm{op}} \times \mathbf{B} \xrightarrow{\mathbf{B}(-,-)} \mathbf{CAT} \tag{16}$$

wherein i is the inclusion of $\mathbf{M}$ in $\mathbf{B}$ and $\mathbf{B}(-,-)$ is the hom pseudofunctor of (1). An object of $\mathbf{G}$ is a triple (X, R, A) where R is an object of the category $\mathbf{B}(X, A)$ — that is, a general arrow $R{:}X \nrightarrow A$ in $\mathbf{B}$. We shall often write $R{:}X \nrightarrow A$, or just R, for the object (X, R, A). An arrow in $\mathbf{G}$ from (X, R, A) to (Y, S, B) consists of a triple (f, α, u) where $f{:}X \nrightarrow Y$ and $u{:}A \nrightarrow B$ are *maps* in $\mathbf{B}$ and $\alpha{:}uR \nrightarrow Sf$ is a 2-cell in $\mathbf{B}$ as in

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle R}\downarrow & \xrightarrow{\ \alpha\ } & \downarrow{\scriptstyle S} \\ A & \xrightarrow[u]{} & B \end{array} \tag{17}$$

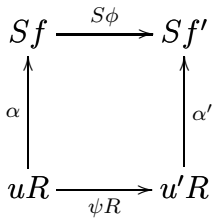
and such arrows are composed by bicategorical pasting as discussed in Verity's thesis [VER].

There is another way of describing an arrow of $\mathbf{G}$. Since the map u has a right adjoint u^*, to give the 2-cell $\alpha{:}uR \nrightarrow Sf$ is equally to give a 2-cell $\beta{:}R \nrightarrow u^*Sf$ as in

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle R}\downarrow & \xrightarrow{\ \beta\ } & \downarrow{\scriptstyle S} \\ A & \xleftarrow[u^*]{} & B \end{array} \tag{18}$$

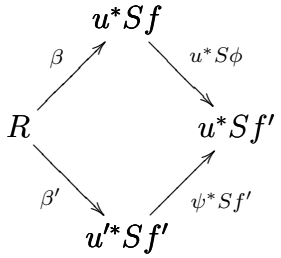
namely the *mate* of α in the sense of Kelly and Street [K&S]. We call the description of an arrow of $\mathbf{G}$ by a triple (f, α, u) as in (17) its *primary form* and that by the triple (f, β, u) as in (18) its *secondary form*. It is easy to see that composition of arrows expressed in their secondary forms is again given by bicategorical pasting.

Given arrows (f, α, u) and (f', α', u') of $\mathbf{G}$, expressed in primary form, a 2-cell in $\mathbf{G}$ from (f, α, u) to (f', α', u') consists of 2-cells $\phi: f \to f'$ and $\psi: u \to u'$ in $\mathbf{M}$ for which the diagram



(19)

commutes. If the secondary forms of the same arrows are (f, β, u) and (f', β', u') then the condition (19) for 2-cells $\phi: f \to f'$ and $\psi: u \to u'$ to constitute a 2-cell in $\mathbf{G}$ becomes the commutativity of

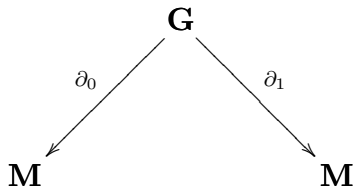


(20)

where $\psi^* : u'^* \to u^*$ is the mate with respect to the adjunctions $u \dashv u^*$ and $u' \dashv u'^*$ of $\psi: u \to u'$.

3.4. Proposition. *The typical arrow (f, α, u) of G, expressed in primary form, is an equivalence if and only if f and u are equivalences in $\mathbf{M}$ and α is invertible in $\mathbf{B}(X, B)$.* ∎

There are pseudofunctors ∂_0 and ∂_1 as shown in (21) below



(21)

given (using either description for arrows) by

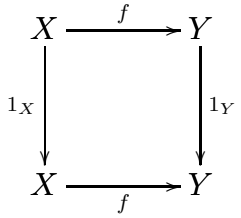
$$\partial_0[(\phi, \psi): (f, \alpha, u) \to (f', \alpha', u'): (X, R, A) \to (Y, S, B)] = \phi: f \to f': X \to Y$$

and

$$\partial_1[(\phi, \psi): (f, \alpha, u) \to (f', \alpha', u'): (X, R, A) \to (Y, S, B)] = \psi: u \to u': A \to B$$

We note that ∂_0 and ∂_1 have the common section $\iota: \mathbf{M} \to \mathbf{G}$ which sends a map $f: X \to Y$

to the square



with the identity 2-cell understood (and meaningful by the normality of $\mathbf{B}$).

3.5 We are going to show that the bicategory $\mathbf{G}$ has finite products. We shall use $R \otimes S$ for the binary product in $\mathbf{G}$, to distinguish $f \otimes g$ from $f \times g$ when f and g are maps.

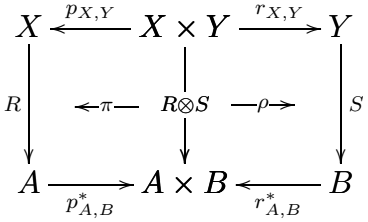
Given objects $R{:}X \nrightarrow A$ and $S{:}Y \nrightarrow B$ of $\mathbf{G}$, we define $R \otimes S{:}X \times Y \nrightarrow A \times B$ by

$$R \otimes S = p^*_{A,B} R p_{X,Y} \wedge r^*_{A,B} S r_{X,Y}$$

which can be abbreviated to $p^*Rp \wedge r^*Sr$ when X, Y, A, and B are clear; and we define arrows $p_{R,S}{:}R \otimes S \nrightarrow R$ and $r_{R,S}{:}R \otimes S \nrightarrow S$ of $\mathbf{G}$, in their secondary forms, by

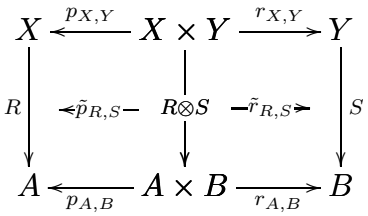
$$p_{R,S} = (p_{X,Y}, \pi, p_{A,B}) \quad \text{and} \quad r_{R,S} = (r_{X,Y}, \rho, r_{A,B})$$

as in the diagram



(22)

wherein π and ρ are the projections of the product $\wedge$ in $\mathbf{B}(X \times Y, A \times B)$. Accordingly, the primary forms of $p_{R,S}$ and of $r_{R,S}$ are $(p_{X,Y}, \tilde{p}_{R,S}, p_{A,B})$ and $(r_{X,Y}, \tilde{r}_{R,S}, r_{A,B})$ as in



(23)

where the 2-cell $\tilde{p}_{R,S}$ is the mate $p(p^*Rp \wedge r^*Sr) \xrightarrow{p\pi} pp^*Rp \xrightarrow{\epsilon_p Rp} Rp$ of π and similarly for $\tilde{r}_{R,S}$.

To prove that $R \otimes S$, with the projections $p_{R,S}$ and $r_{R,S}$, is indeed the binary product in $\mathbf{G}$ is to show that, for any object $T{:}Z \nrightarrow C$ of $\mathbf{G}$, the functor

$$\mathbf{G}(T, R \otimes S) \xrightarrow{(\mathbf{G}(T,p_{R,S}),\mathbf{G}(T,r_{R,S}))} \mathbf{G}(T, R) \times \mathbf{G}(T, S)$$

is essentially surjective on objects and is fully faithful. These properties are established in the following lemmas, in the first of which we use the fact that a functor of the form $\mathbf{B}(f, u^*)$, where f and u are maps, preserves products, being the right adjoint of $\mathbf{B}(f^*, u)$.

3.6. LEMMA. *For each object $T:Z \to C$ in $\mathbf{G}$, the functor*

$$\mathbf{G}(T, R \otimes S) \xrightarrow{(\mathbf{G}(T,p_{R,S}),\mathbf{G}(T,r_{R,S}))} \mathbf{G}(T,R) \times \mathbf{G}(T,S)$$

is essentially surjective on objects. In fact, more is true: for any

$$(f,\alpha,u):(Z,T,C) \to (X,R,A) \quad \textit{and} \quad (g,\beta,v):(Z,T,C) \to (Y,S,B)$$

in $\mathbf{G}$ and any

$$(h,w):(Z,C) \to (X \times Y, A \times B)$$

with invertible 2-cells

$$(\mu_0,\mu_1):(p_{X,Y},p_{A,B})(h,w) \to (f,u) \quad \textit{and} \quad (\nu_0,\nu_1):(r_{X,Y},r_{A,B})(h,w) \to (g,v)$$

provided by the products in $\mathbf{M} \times \mathbf{M}$, there exists a unique γ making (h,γ,w) an arrow $T \to R \otimes S$ in $\mathbf{G}$ with invertible 2-cells

$$(\mu_0,\mu_1):p_{R,S}(h,\gamma,w) \to (f,\alpha,u) \quad \textit{and} \quad (\nu_0,\nu_1):r_{R,S}(h,\gamma,w) \to (g,\beta,v).$$

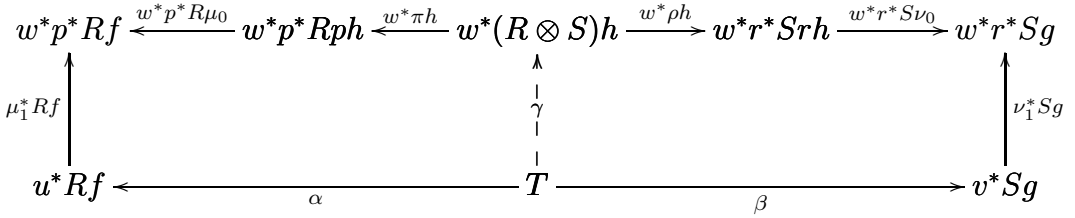
PROOF. Here we use the secondary forms for all arrows in $\mathbf{G}$. The product diagram

$$p^*Rp \xleftarrow{\pi} p^*Rp \wedge r^*Sr \xrightarrow{\rho} r^*Sr$$

in $\mathbf{B}(X \times Y, A \times B)$ gives rise to the further product diagram

$$w^*p^*Rf \xleftarrow{w^*p^*R\mu_0} w^*p^*Rph \xleftarrow{w^*\pi h} w^*(p^*Rp \wedge r^*Sr)h \xrightarrow{w^*\rho h} w^*r^*Srh \xrightarrow{w^*r^*S\nu_0} w^*r^*Sg$$

in $\mathbf{B}(Z,C)$ because $\mathbf{B}(h,w^*)$ preserves products and μ_0 and ν_0 are invertible. Now consider the following diagram:



To give such a 2-cell γ is to give an arrow $(h,\gamma,w):T \to R \otimes S$ in $\mathbf{G}$; and by (20), the condition for $(\mu_0,\mu_1):p_{R,S}(h,\gamma,w) \to (f,\alpha,u)$ to provide a 2-cell in $\mathbf{G}$ is the commutativity of the left square, while the condition for $(\nu_0,\nu_1) : r_{R,S}(h,\gamma,w) \to (g,\beta,v)$ to provide a 2-cell in $\mathbf{G}$ is the commutativity of the right square. Since the top row of the diagram is a product, there is a unique such γ. ∎

3.7. COROLLARY. *If two arrows $(h,\gamma,w),(h,\delta,w):(Z,T,C) \to (X \times Y, R \otimes S, A \times B)$ in $\mathbf{G}$ satisfy $p_{R,S}(h,\gamma,w) = p_{R,S}(h,\delta,w)$ and $r_{R,S}(h,\gamma,w) = r_{R,S}(h,\delta,w)$ then $\gamma = \delta$ so that $(h,\gamma,w) = (h,\delta,w)$.*

PROOF. We can apply Lemma 3.6 with $(f,\alpha,u) = p_{R,S}(h,\gamma,w)$ and with $(g,\beta,v) = r_{R,S}(h,\gamma,w)$, taking μ_0, μ_1, ν_0, and ν_1 to be identities. ∎

3.8. LEMMA. *For each object $T{:}Z \nrightarrow C$ in $\mathbf{G}$, the functor*

$$\mathbf{G}(T, R \otimes S) \xrightarrow{(\mathbf{G}(T,p_{R,S}),\mathbf{G}(T,r_{R,S}))} \mathbf{G}(T, R) \times \mathbf{G}(T, S)$$

is fully faithful.

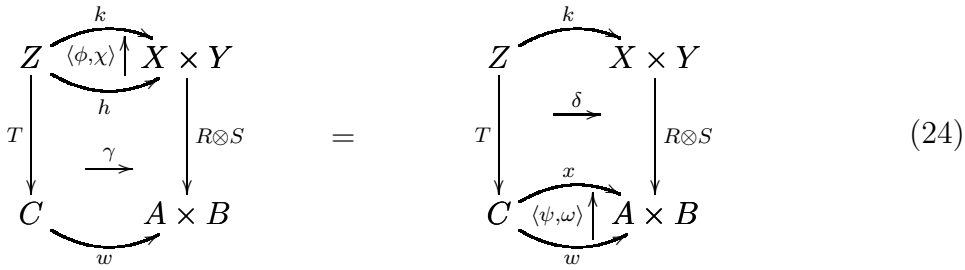
PROOF. Let $(h, \gamma, w), (k, \delta, x){:}(Z, T, C) \nrightarrow (X \times Y, R \otimes S, A \times B)$ be arrows of $\mathbf{G}$ in primary form, and consider 2-cells

$$(\phi, \psi){:}p_{R,S}(h, \gamma, w) \nrightarrow p_{R,S}(k, \delta, x) \quad \text{and} \quad (\chi, \omega){:}r_{R,S}(h, \gamma, w) \nrightarrow r_{R,S}(k, \delta, x)$$

Since these data further provide 2-cells $(\phi, \psi) : (p_{X,Y}, p_{A,B})(h, w) \nrightarrow (p_{X,Y}, p_{A,B})(k, x)$ and $(\chi, \omega) : (r_{X,Y}, r_{A,B})(h, w) \nrightarrow (r_{X,Y}, r_{A,B})(k, x)$ in $\mathbf{M} \times \mathbf{M}$ and since the bicategory $\mathbf{M} \times \mathbf{M}$ has finite products, there are unique 2-cells $\langle \phi, \chi \rangle{:}h \nrightarrow k$ and $\langle \psi, \omega \rangle{:}w \nrightarrow x$ in $\mathbf{M}$ satisfying

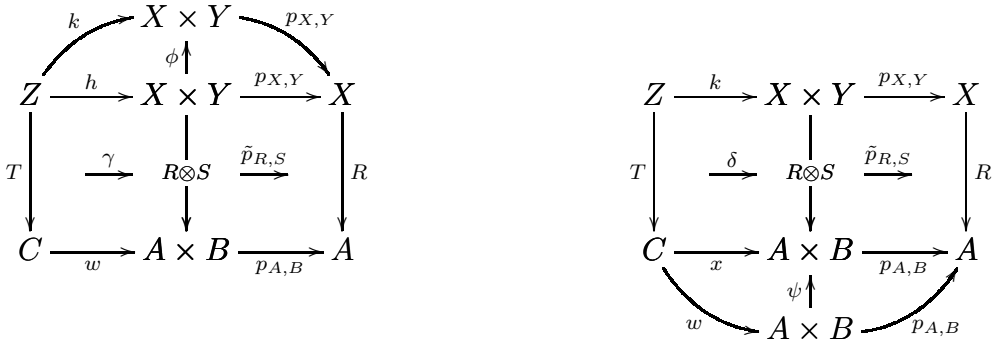
$$(p_{X,Y}, p_{A,B})(\langle \phi, \chi \rangle, \langle \psi, \omega \rangle) = (\phi, \psi) \quad \text{and} \quad (r_{X,Y}, r_{A,B})(\langle \phi, \chi \rangle, \langle \psi, \omega \rangle) = (\chi, \omega).$$

It only remains to show that $(\langle \phi, \chi \rangle, \langle \psi, \omega \rangle)$ constitutes a 2-cell $(h, \gamma, w) \nrightarrow (k, \delta, x)$ in $\mathbf{G}$, for which the requisite condition is



(24)

Because both pasting composites displayed are of the form $(k, ?, w){:}T \nrightarrow R \otimes S$ in $\mathbf{G}$, Corollary 3.7 ensures their equality if they are coequalized by both $p_{R,S}$ and $r_{R,S}$. Composing the left side of (24) with $p_{R,S}$ gives the left pasting composite below, while composing the right side of (24) with $p_{R,S}$ gives the right pasting composite below.



However, the composites above are equal since (ϕ, ψ) was assumed to provide a 2-cell in $\mathbf{G}$. A similar argument applies to composition with $r_{R,S}$ and this completes the proof. ∎

3.9. THEOREM. **G** *has finite products preserved by* ∂_0 *and* ∂_1.

PROOF. The binary products are provided by Lemmas 3.6 and 3.8; we now show that $\top = \top_{I,I}: I \to I$ provides a terminal object for **G**. In fact, for each object (X, R, A) in **G** we have the maps $t_X: X \to I$ and $t_A: A \to I$ of 2.2; and the right adjoint functor $\mathbf{B}(t_X, t_A^*)$ sends $\top_{I,I}$ to $\top_{X,A}$. Accordingly, there is a unique 2-cell $\tau: R \to t_A^* \top t_X$, which is the description in secondary form of an arrow $t_R: R \to \top$ in **G**. For any arrow $(f, \alpha, u): (X, R, A) \to (Y, S, B)$ we have a unique 2-cell $t_S(f, \alpha, u) \to t_R$ given by (t'_f, t'_u) and it is invertible. ∎

3.10. REMARK. For the primary form of the arrow t_R we use $(t_X, \tilde{t}_R, t_A)$. It is the component of a pseudonatural transformation $t: 1_{\mathbf{G}} \to \top!$ which is the unit for a pseudoadjunction $! \dashv \top: \mathbf{1} \to \mathbf{G}$.

3.11 When $Y = X$ and $B = A$ in (23), we have $d_A^*(R \otimes S)d_X = d_A^*(p^*Rp \wedge r^*Sr)d_X \cong d_A^*p^*Rpd_X \wedge d_A^*r^*Srd_X \cong R \wedge S$; thus

$$R \wedge S \cong d_A^*(R \otimes S)d_X$$

We have in particular $R \wedge R \cong d_A^*(R \otimes R)d_X$; and composing this isomorphism with $\delta: R \to R \wedge R$ gives a 2-cell we can still call δ as in

$$\begin{array}{ccc} X & \xrightarrow{d_X} & X \times X \\ {\scriptstyle R}\downarrow & \xrightarrow{\delta} & \downarrow{\scriptstyle R \otimes R} \\ A & \xleftarrow[d_A^*]{} & A \times A \end{array}$$

giving, in secondary form, an arrow $d_R: R \to R \otimes R$, for whose primary form we write $(d_X, \tilde{d}_R, d_A)$. The d_R are the components of a pseudonatural transformation $d: 1_{\mathbf{G}} \to \otimes\Delta$, which is the unit for a pseudoadjunction $\Delta \dashv \otimes : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ (for which the counit has components given by the pairs $(p_{R,S}, r_{R,S}) = ((p_{X,Y}, \tilde{p}_{R,S}, p_{A,B}), (r_{X,Y}, \tilde{r}_{R,S}, r_{A,B}))$ of (23)). We stress that the $\tilde{p}_{R,S}$, $\tilde{r}_{R,S}$, $\tilde{d}_R$, and $\tilde{t}_R$ are not, in general, invertible.

3.12 Each 2-cell $\alpha: R \to R': X \to A$ in **B** gives an arrow $(1_X, \alpha, 1_A)$ of **G**, as in

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle R}\downarrow & \xrightarrow{\alpha} & \downarrow{\scriptstyle R'} \\ A & \xrightarrow[1]{} & A \end{array}$$

If $\gamma: R' \to R'': X \to A$ is another 2-cell, the composite arrow $(1, \gamma, 1)(1, \alpha, 1)$ is $(1, \gamma\alpha, 1)$, since the normality of **B** forces the constraint $(1R')1 \cong 1(R'1)$ to be an identity.

Like $\times$ in 2.2 the product $\otimes$ of **G** provides a pseudofunctor $\otimes: \mathbf{G} \times \mathbf{G} \to \mathbf{G}$. Consider the product $(1_X, \alpha, 1_A) \otimes (1_Y, \beta, 1_B)$, where $\alpha: R \to R': X \to A$ and $\beta: S \to S': Y \to B$. Like

the $f \times g$ of 2.2, it is determined to within isomorphism by the existence of invertible 2-cells

$$\mu : p_{R',S'}(1_X, \alpha, 1_A) \otimes (1_Y, \beta, 1_B) \cong (1_X, \alpha, 1_A)p_{R,S},$$
$$\nu : r_{R',S'}(1_X, \alpha, 1_A) \otimes (1_Y, \beta, 1_B) \cong (1_Y, \beta, 1_B)r_{R,S}.$$

Recall from 3.5 that, in secondary form, $p_{R,S} = (p_{X,Y}, \pi, p_{A,B})$ and $r_{R,S} = (r_{X,Y}, \rho, r_{A,B})$. Since we have equalities

$$(p_{X,Y}, p_{A,B})(1_{X\times Y}, 1_{A\times B}) = (1_X, 1_A)(p_{X,Y}, p_{A,B}),$$
$$(r_{X,Y}, r_{A,B})(1_{X\times Y}, 1_{A\times B}) = (1_Y, 1_B)(r_{X,Y}, r_{A,B}),$$

it follows from Lemma 3.6 that there is a unique $\phi : R \otimes S \to R' \otimes S'$ for which we have equalities

$$p_{R',S'}(1_{X\times Y}, \phi, 1_{A\times B}) = (1_X, \alpha, 1_A)p_{R,S} \tag{25}$$
$$r_{R',S'}(1_{X\times Y}, \phi, 1_{A\times B}) = (1_Y, \beta, 1_B)r_{R,S} \tag{26}$$

We write $\alpha \otimes \beta$ for this value of ϕ. Inserting the values above of $\tilde{p}_{R,S}$ and so on in these last equalities gives

$$\pi(\alpha \otimes \beta) = (p^*\alpha p)\pi$$
$$\rho(\alpha \otimes \beta) = (r^*\beta r)\rho$$

from which we deduce that

$$\alpha \otimes \beta = (p^*\alpha p) \wedge (r^*\beta r)$$

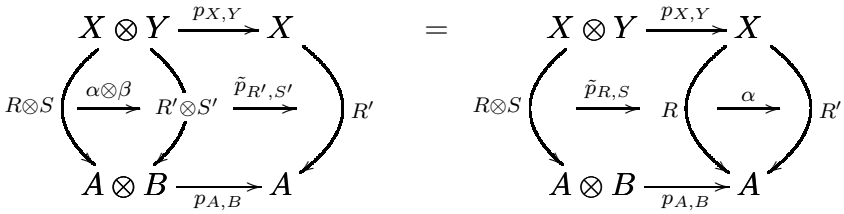
Thus our formula $R \otimes S = (p^*Rp) \wedge (r^*Sr)$ extends to 2-cells to give a functor

$$\otimes_{(X,Y),(A,B)} : \mathbf{B} \times \mathbf{B}((X,Y),(A,B)) = \mathbf{B}(X,A) \times \mathbf{B}(Y,B) \to \mathbf{B}(X \times Y, A \times B)$$

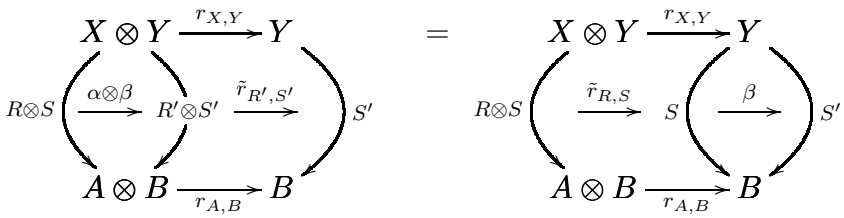
namely the composite

$$\mathbf{B}(X,A) \times \mathbf{B}(Y,B) \xrightarrow{\mathbf{B}(p,p^*)\times\mathbf{B}(r,r^*)} \mathbf{B}(X\times Y, A\times B) \times \mathbf{B}(X\times Y, A\times B) \xrightarrow{\wedge} \mathbf{B}(X\times Y, A\times B)$$

3.13. Remark. We observe for later reference that (25) and (26) can be displayed as



and



saying that the $\tilde{p}_{R,S}$ and $\tilde{r}_{R,S}$ are natural in R and S.

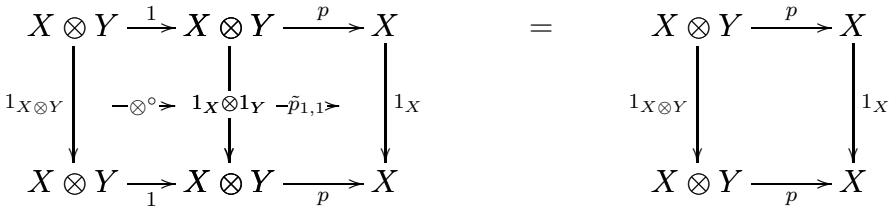
3.14 We extend the definition of $\otimes$ to objects by setting $X \otimes Y = X \times Y$. We shall now show that the functors

$$\otimes_{(X,Y),(A,B)} : \mathbf{B}(X, A) \times \mathbf{B}(Y, B) \to \mathbf{B}(X \otimes Y, A \otimes B)$$

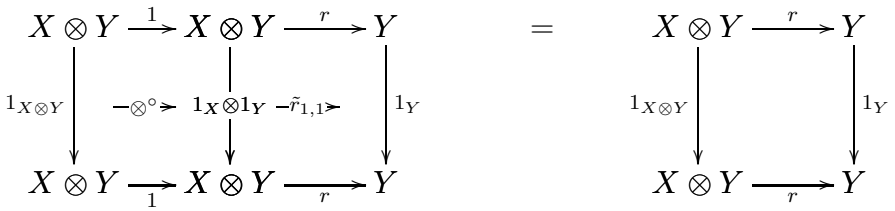
provide the effect on homs for a *lax* functor

$$(\otimes, \widetilde{\otimes}, \otimes^\circ) : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$$

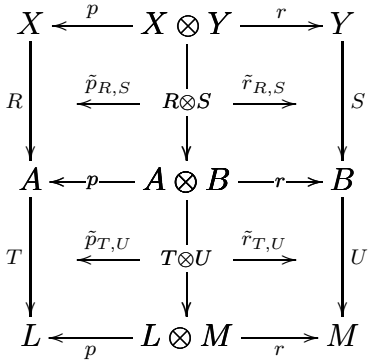
First, for objects X and Y of $\mathbf{B}$, there is by Lemma 3.6 a unique $\otimes^\circ : 1_{X\otimes Y} \to 1_X \otimes 1_Y$ satisfying



and



Given $R : X \to A$, $S : Y \to B$, $T : A \to L$ and $U : B \to M$, vertical pasting as in the diagram



(27)

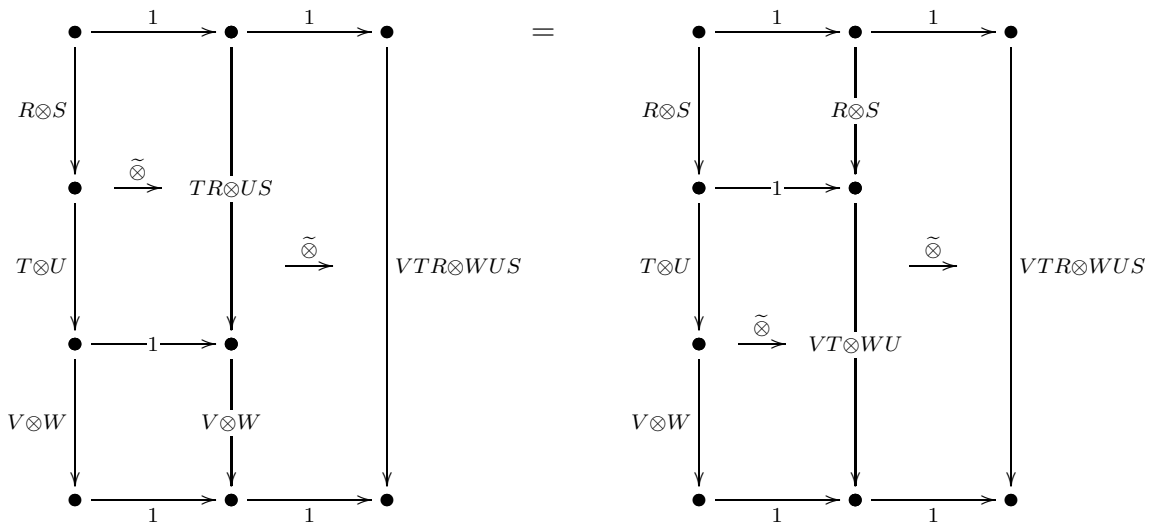
gives arrows in $\mathbf{G}$ from $(T \otimes U)(R \otimes S)$ to TR and US. Accordingly, there is by Lemma 3.6 a unique

$$\widetilde{\otimes} : (T \otimes U)(R \otimes S) \to (TR) \otimes (US)$$

whose composites with $p_{TR,US}$ and $r_{TR,US}$ are these vertical pastings.

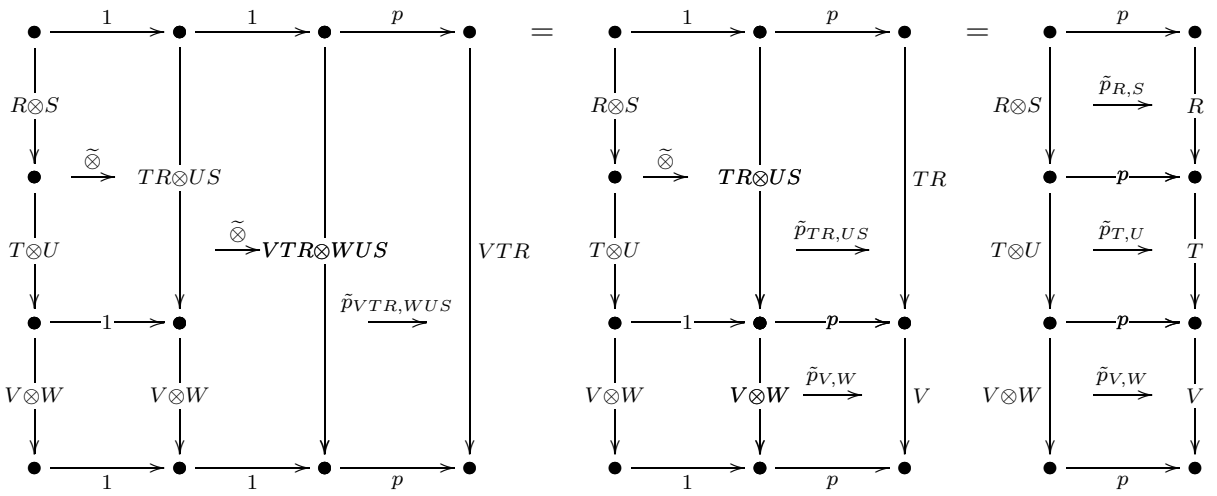
The first requirement for $(\otimes, \widetilde{\otimes}, \otimes^\circ)$ to be a lax functor $\mathbf{B}\times\mathbf{B}\to\mathbf{B}$ is that $\widetilde{\otimes}$ be natural in T, R, U, and S; and it is so because the assignment $\alpha\mapsto(1,\alpha,1)$ respects *both* vertical *and* horizontal composition of 2-cells.

For the associativity coherence condition, consider the data in (27) along with further arrows $V : L \to C$ and $W : M \to D$ in $\mathbf{B}$. We require the following equality of pasting composites:



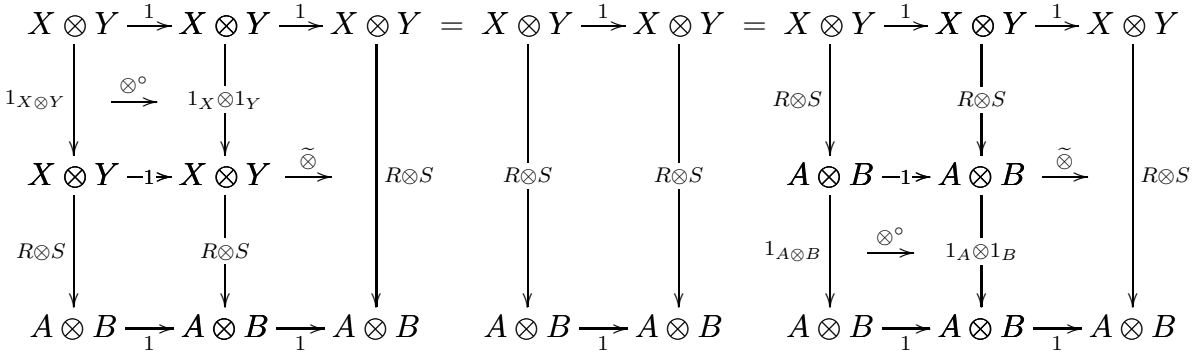
(28)

To establish this equality consider the effect of pasting $p_{VTR,WUS}$ on the right to the left side of Equation (28):



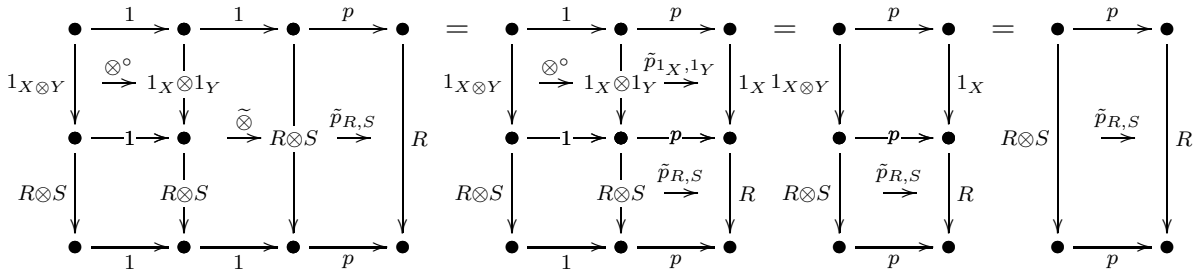
It is clear that when $p_{VTR,WUS}$ is pasted to the right side of Equation (28) the result is the same. Similarly, the left side of (28) pasted to $r_{VTR,WUS}$ is equal to the right side of (28) pasted to $r_{VTR,WUS}$. This suffices by Corollary 3.7 to prove the condition satisfied.

For the unitary coherence conditions we require:



(29)

To prove the first of these consider



If all instances of p in the sequence of diagrams directly above are replaced by r (with accompanying changes of codomains), the equations continue to hold and the two sequences of equations then jointly establish, by Corollary 3.7, the first unitary coherence equation. Derivation of the second unitary coherence equation is similar.

Thus we have proved:

3.15. THEOREM. *For a precartesian bicategory* $\mathbf{B}$*, the data* $(\otimes, \widetilde{\otimes}, \otimes^\circ)$ *constitute a lax functor* $\mathbf{B} \times \mathbf{B} \to \mathbf{B}$. ∎

3.16. REMARK. Our main interest in the $\tilde{p}_{R,S}$ and the $\tilde{r}_{R,S}$ is in their role as components of the $p_{R,S}$ and the $r_{R,S}$, which are the components of the pseudonatural transformations comprising the product projections $p : P \leftarrow \otimes \to R : r$, for the product pseudofunctor $\otimes : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$. However we can say more. The naturality of the $\tilde{p}_{R,S}$ and the $\tilde{r}_{R,S}$ in R and S noted in Remark 3.13, together with the equations *defining* $\otimes^\circ$ and $\tilde{\otimes}$, show that the $\tilde{p}_{R,S}$ and the $\tilde{r}_{R,S}$ provide lax naturality 2-cells making the $p_{X,Y}$ and the $r_{X,Y}$ components of *lax natural* transformations $p : P \leftarrow \otimes \to R : r$, whose domain is the lax functor $(\otimes, \widetilde{\otimes}, \otimes^\circ) : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$. It then follows that the $\tilde{d}_R$ provide lax naturality 2-cells making the d_X components of a *lax natural* transformation $d : 1_{\mathbf{B}} \to \otimes D : \mathbf{B} \to \mathbf{B}$, where $D : \mathbf{B} \to \mathbf{B} \times \mathbf{B}$ is the diagonal pseudofunctor.

3.17 Next, for a precartesian bicategory $\mathbf{B}$, we describe a lax functor $\mathbf{1} \to \mathbf{B}$, which amounts to giving an object of $\mathbf{B}$ and a monad on this. The object we take is I, the terminal object of Map($\mathbf{B}$). The monad we take on the object I has underlying arrow

$\top = \top_{I,I}$, the terminal object of $\mathbf{B}(I,I)$. The multiplication $\widetilde{I}:\top\top \to \top$ and the unit $I^\circ:1_I \to \top$ are the unique 2-cells into the terminal object, which trivially satisfy the three monad equations.

3.18. Proposition. *For a precartesian bicategory* $\mathbf{B}$*, the object* I *of* $\mathbf{B}$*, the arrow* $\top: I \to I$*, and the 2-cells* $\widetilde{I}:\top\top \to \top$ *and* $I^\circ:1_I \to \top$ *constitute a lax functor* $I:\mathbf{1} \to \mathbf{B}$*.* ∎

3.19. Remark. Further to Remark 3.16 we note that the $\tilde{t}_R$ provide lax naturality 2-cells making the t_X components of a *lax natural* transformation $t:1_{\mathbf{B}} \to I!:\mathbf{B} \to \mathbf{B}$, where $!:\mathbf{B} \to \mathbf{1}$ is the unique such pseudofunctor.

3.20 We have the inclusion pseudofunctor $i:\mathbf{M} \to \mathbf{B}$. Given maps $f:X \to A$ and $g:Y \to B$ (in $\mathbf{M}$ of course) we have, as in 2.2, (in $\mathbf{M}$ and hence in $\mathbf{B}$) the map $f \times g:X \times Y \to A \times B$ with invertible 2-cells $p'_{f,g} : p_{A,B}(f \times g) \to f p_{X,Y}$ and $r'_{f,g} : r_{A,B}(f \times g) \to g r_{X,Y}$. Since $f \otimes g:X \otimes Y \to A \otimes B$ is the product of f and g in $\mathbf{G}$, there is by Lemma 3.6 a unique 2-cell $m'_{f,g}:f \times g \to f \otimes g$ for which the arrow $(1, m'_{f,g}, 1):(X \times Y, f \times g, A \times B) \to (X \otimes Y, f \otimes g, A \otimes B)$ of $\mathbf{G}$ satisfies the following two equations:

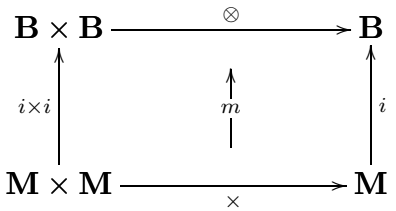
$$p_{f,g}(1, m'_{f,g}, 1) = (p_{X,Y}, p'_{f,g}, p_{A,B}),$$

$$r_{f,g}(1, m'_{f,g}, 1) = (r_{X,Y}, r'_{f,g}, r_{A,B}).$$

For the pseudofunctor $\times : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ we can take $1_X \times 1_Y$ to be $1_{X \times Y}$ with $p'_{1,1}$ and $r'_{1,1}$ identities, and then take $\times^\circ : 1_{X \times Y} \to 1_X \times 1_Y$ to be again an identity; while $\tilde{\times}:(u \times v)(f \times g) \to uf \times vg$ is the evident 2-cell.

For objects X and Y of $\mathbf{M}$ we define $m_{X,Y}:X \times Y \to X \otimes Y$ to be the identity.

3.21. Proposition. *For a precartesian bicategory* $\mathbf{B}$*, the* $m_{X,Y}$ *and the* $m'_{f,g}$ *above constitute a lax natural transformation* m *as in*



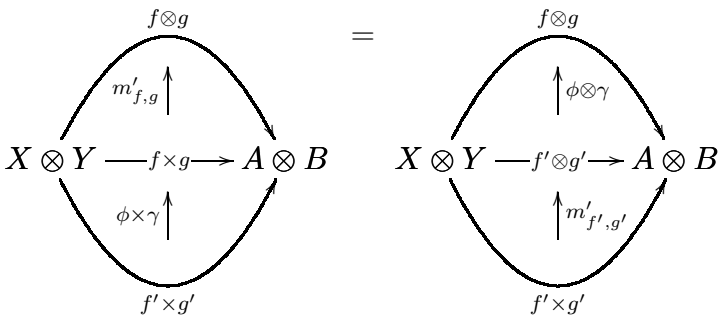
Proof. For each lax-naturality equation we appeal to Corollary 3.7 For naturality in f and g, we have

$$p_{f,g}(1, m'_{f,g}, 1)(1, \phi \times \gamma, 1) = (p_{X,Y}, p'_{f,g}, p_{A,B})(1, \phi \times \gamma, 1) = (1, \phi, 1)(p_{X,Y}, p'_{f',g'}, p_{A,B}),$$

where the first equality is by 3.20 and the second is by the pseudonaturality of p; while

$$p_{f,g}(1, \phi \otimes \gamma, 1)(1, m'_{f',g'}, 1) = (1, \phi, 1)\tilde{p}_{f',g}(1, m'_{f',g'}, 1) = (1, \phi, 1)(p_{X,Y}, p'_{f',g'}, p_{A,B}),$$

where the first equality is by 3.12 and the second is by the equations defining the $m'_{f,g}$ in 3.20. Similarly, $(1, m'_{f,g}, 1)(1, \phi \times \gamma, 1)$ and $(1, \phi \otimes \gamma, 1)(1, m'_{f',g'}, 1)$ have the same composite with $\tilde{r}_{f,g}$, so that Corollary 3.7 gives



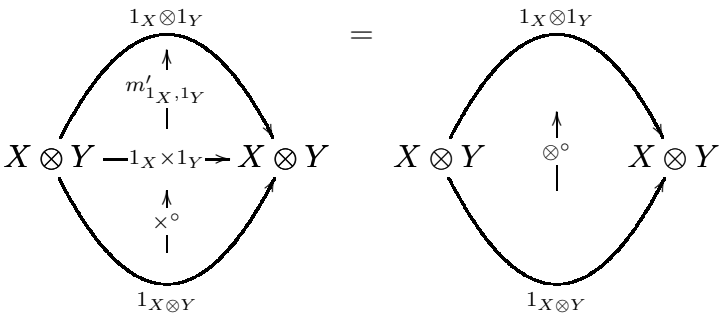
For the nullary coherence condition, we have seen that $\times^\circ$ and $p'_{1,1}$ can be taken to be identities; and then

$$p_{1,1}(1, m'_{1,1}, 1)(1, \times^\circ, 1) = (p_{X,Y}, p'_{1,1}, p_{X,Y}) = (p_{X,Y}, 1, p_{X,Y}),$$

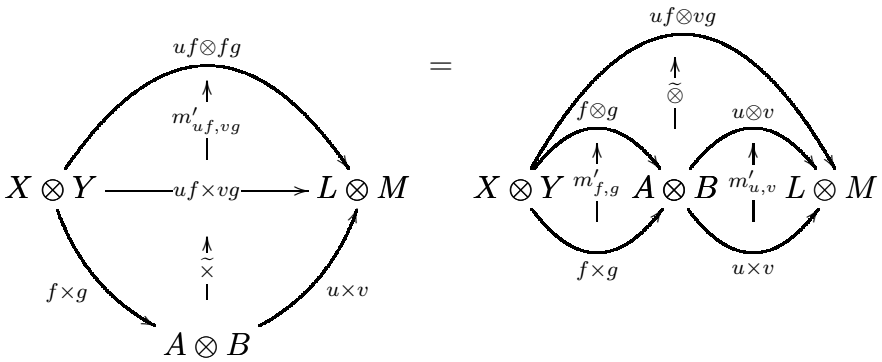
where the first equality is by 3.20, while

$$p_{1,1}(1, \otimes^\circ, 1) = (p_{X,Y}, 1, p_{X,Y})$$

by 3.14. Similarly with $r_{1,1}$ in place of $p_{1,1}$, whereupon Corollary 3.7 gives



Finally, for the binary coherence equation we require, for $f{:}X \to A$, $g{:}Y \to B$, $u{:}A \to L$, and $v{:}B \to M$ in $\mathbf{M}$,



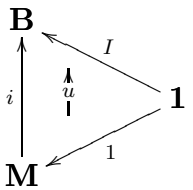
Now the composite of the right side with $p_{uf,vg}$ equals by 3.14 the vertical pasting of $p_{f,g}(1, m'_{f,g}, 1)$ and $p_{u,v}(1, m'_{u,v}, 1)$ along $p_{A,B}$, and thus by 3.20 equals the vertical pasting along $p_{A,B}$ of $(p_{X,Y}, p'_{f,g}, p_{A,B})$ and $(p_{A,B}, p'_{u,v}, p_{L,M})$—which, because p is pseudonatural, equals the composite of $(p_{X,Y}, p'_{uf,vg}, p_{L,M})$ with $\tilde{\times}$. But by 3.20 this is also equal to the composite of the left side with $p_{uf,vg}$. In the same way the two sides have equal composites with $r_{uf,vg}$, whence they are equal by Corollary 3.7. ∎

3.22 Consider objects $R{:}X \nrightarrow A$ and $S{:}Y \nrightarrow B$ of $\mathbf{G}$, along with their product $R \otimes S$ and its projections (in their primary forms) $(p_{X,Y}, \tilde{p}_{R,S}, p_{A,B})$ and $(r_{X,Y}, \tilde{r}_{R,S}, r_{A,B})$. Consider also maps $f{:}L \to X$ and $g{:}M \to Y$, along with the invertible 2-cells $p'_{f,g} : p_{X,Y}(f \times g) \nrightarrow f p_{L,M}$ and $r'_{f,g} : r_{X,Y}(f \times g) \nrightarrow g r_{L,M}$ of 2.2. Now form the vertical pasting of $(p_{X,Y}, \tilde{p}_{R,S}, p_{A,B})$ and $(p_{L,M}, p'_{f,g}, p_{X,Y})$ along $p_{X,Y}$ and similarly the vertical pasting of $(r_{X,Y}, \tilde{r}_{R,S}, r_{A,B})$ and $(r_{L,M}, r'_{f,g}, r_{X,Y})$ along $r_{X,Y}$. These pastings constitute arrows in $\mathbf{G}$ from $(R \otimes S)(f \times g)$ to Rf and to Sg, and thus determine an arrow in $\mathbf{G}$ from $(R \otimes S)(f \times g)$ to $Rf \otimes Sg$.

In fact this arrow is an isomorphism. To see this, observe that the secondary forms of these pastings are made by pasting $(p_{L,M}, p'_{f,g}, p_{X,Y})$ and $(r_{L,M}, r'_{f,g}, r_{X,Y})$ to the secondary forms of $(p_{X,Y}, \tilde{p}_{R,S}, p_{A,B})$ and $(r_{X,Y}, \tilde{r}_{R,S}, r_{A,B})$, which are $(p_{X,Y}, \pi, p_{A,B})$ and $(r_{X,Y}, \rho, r_{A,B})$ where π and ρ are the projections from $R \otimes S = p^*Rp \wedge r^*Sr$. Since *precomposition* with the map $f \times g$, being a right adjoint, preserves products, the projections $\pi(f \times g)$ and $\rho(f \times g)$ express $(R \otimes S)(f \times g)$ as a product $p^*Rp(f \times g) \wedge r^*Sr(f \times g)$. Then, because the 2-cells $p'_{f,g}{:}p_{X,Y}(f \times g) \nrightarrow f p_{L,M}$ and $r'_{f,g}{:}r_{X,Y}(f \times g) \nrightarrow g r_{L,M}$ are invertible, $(R \otimes S)(f \times g)$ is also a product $p^*Rfp \wedge r^*Sgr$ with projections the vertical pastings of $(p_{X,Y}, \pi, p_{A,B})$ to $(p_{L,M}, p'_{f,g}, p_{X,Y})$ and of $(r_{X,Y}, \rho, r_{A,B})$ to $(r_{L,M}, r'_{f,g}, r_{X,Y})$. This completes the proof. Similarly, for maps $u{:}Z \nrightarrow A$ and $v{:}W \nrightarrow B$, we can construct an isomorphism $(u \times v)^*(R \otimes S) \xrightarrow{\simeq} u^*R \otimes v^*S$. (These isomorphisms should be seen in the light of the **L**-homomorphisms of (2.15) of [CKVW].)

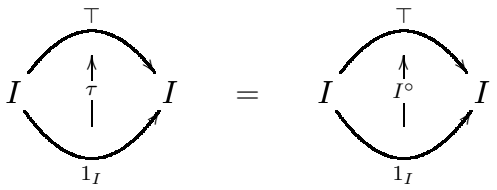
If we apply the first result above with $A = X$, $B = Y$, $R = 1_X$ and $S = 1_Y$, we have an isomorphism $\xi{:}(1_X \otimes 1_Y)(f \times g) \nrightarrow f \otimes g$, whose projections are the pasting of $(p_{L,M}, p'_{f,g}, p_{X,Y})$ to $(p_{X,Y}, p_{1,1}, p_{X,Y})$ and the same with r in place of p. Composing ξ with $\otimes^\circ(f \times g)$ gives by 2.13 a 2-cell $f \times g \nrightarrow f \otimes g$ whose projections are the pasting of $(p_{L,M}, p'_{f,g}, p_{X,Y})$ to an identity, and the same with r in place of p. By 3.20, therefore, this arrow $\xi.\otimes^\circ(f \times g){:}f \times g \nrightarrow f \otimes g$ is $m'_{f,g}$.

3.23. Proposition. *For a precartesian bicategory* $\mathbf{B}$*, there is a lax natural transformation* u *as in*

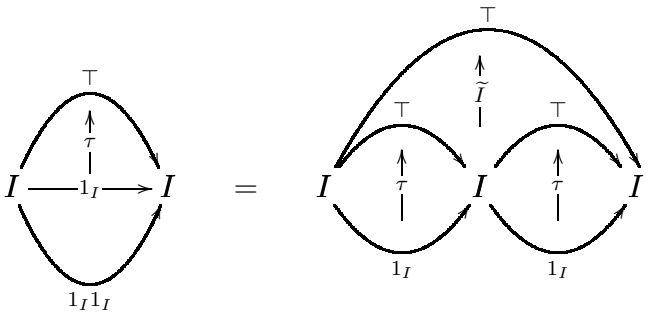


whose component u *at the unique object of* $\mathbf{1}$ *is* $1_I{:}I \nrightarrow I$ *and whose lax naturality 2-cell at the unique arrow of* $\mathbf{1}$ *is* $\tau{:}1 \nrightarrow \top$*, the lax naturality here being equivalent to the two*

equations:



and



which hold automatically since $\top$ *is the terminal object of* $\mathbf{B}(I, I)$. ∎

3.24. Proposition. *For a precartesian bicategory* $\mathbf{B}$*, the lax natural transformation*

$$m : i. \times \nrightarrow \otimes .(i \times i) : \mathbf{M} \times \mathbf{M} \nrightarrow \mathbf{B}$$

is a pseudonatural transformation if and only if the constraints $\otimes^\circ$ *are invertible, and then* m *itself is invertible. Again, the lax transformation*

$$u : i.1 \nrightarrow I : \mathbf{1} \nrightarrow \mathbf{B}$$

is a pseudonatural transformation if and only if the constraint $I^\circ : 1_I \nrightarrow \top$ *is invertible, and then* $\tilde{I}$ *and* u *itself are invertible.*

Proof. The lax natural m is pseudonatural when the 2-cells $m'_{f,g}$ are invertible. By the last paragraph of 3.22, each $m'_{f,g}$ is invertible when $\otimes^\circ$ is invertible; on the other hand $\otimes^\circ$ is invertible if $m'_{1_X,1_Y}$ is invertible by the second equation of Proposition 3.21. Then m is invertible, because each $m_{X,Y}$ is an identity. The truth of the assertions about u is immediate from Proposition 3.23. ∎

4. Cartesian Bicategories

4.1. Definition. *A precartesian bicategory* $\mathbf{B}$ *is said to be* cartesian *when the*

$$\mathbf{B} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{B} \xleftarrow{I} \mathbf{1}$$

are pseudofunctors, meaning that $\widetilde{\otimes}$*,* $\otimes^\circ$*,* I° *(and hence* $\widetilde{I}$*) are invertible.*

Since pseudofunctors carry adjunctions to adjunctions, it should be noted that in a cartesian bicategory, the map $f \times g$ arising from maps f and g has adjunction data

$$\eta_f \otimes \eta_g, \epsilon_f \otimes \epsilon_g : f \times g \dashv f^* \otimes g^*$$

4.2. Proposition. *For a cartesian bicategory* $\mathbf{B}$*, the pseudofunctors*

$$\mathbf{B} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{B} \xleftarrow{I} \mathbf{1}$$

1) restrict to $\mathbf{M}$ *giving a right adjoint to*

$$\mathbf{M} \times \mathbf{M} \xleftarrow{\Delta} \mathbf{M}$$

with unit d *and a right adjoint to*

$$\mathbf{M} \xrightarrow{!} \mathbf{1}$$

with unit t*;*

2) the composites

$$\mathbf{B}(X,Y) \times \mathbf{B}(X,Y) \xrightarrow{\otimes} \mathbf{B}(X \otimes X, Y \otimes Y) \xrightarrow{\mathbf{B}(d_X, d_Y^*)} \mathbf{B}(X,Y)$$

and

$$\mathbf{B}(X,Y) \xleftarrow{\mathbf{B}(t_X, t_Y^*)} \mathbf{B}(I,I) \xleftarrow{\top} \mathbf{1}$$

provide right adjoints to

$$\mathbf{B}(X,Y) \times \mathbf{B}(X,Y) \xleftarrow{\Delta} \mathbf{B}(X,Y) \quad \textit{and} \quad \mathbf{B}(X,Y) \xrightarrow{!} \mathbf{1}$$

Moreover, for any other pair of pseudofunctors $\mathbf{B} \times \mathbf{B} \xrightarrow{\otimes'} \mathbf{B} \xleftarrow{I'} \mathbf{1}$*, if they satisfy 1) and 2) then* $\otimes' \cong \otimes$ *and* $I' \cong I$ *(as pseudofunctors). Thus, a cartesian bicategory can be described alternatively as a bicategory* $\mathbf{B}$ *together with pseudofunctors* $\otimes$ *and* I*, as under consideration, which satisfy 1) and 2).*

Proof. We have seen that $\otimes$ and I satisfy the conditions of the proposition. Only their essential uniqueness remains to be shown. If $\otimes'$ and I' satisfy *1)* then their satisfaction of *2)* is equivalent to their providing right adjoints to

$$\mathbf{G} \times \mathbf{G} \xleftarrow{\Delta} \mathbf{G} \xrightarrow{!} \mathbf{1}$$

commuting via ∂_0 and ∂_1 with the corresponding adjoints for $\mathbf{M}$. By essential uniqueness of products we have diagrams

$$\begin{array}{ccc} X \otimes' Y & \xrightarrow{k_{X,Y}} & X \otimes Y \\ {\scriptstyle R \otimes' S}\downarrow & \xrightarrow{\tilde{k}_{R,S}} & \downarrow{\scriptstyle R \otimes S} \\ A \otimes' B & \xrightarrow[k_{A,B}]{} & A \otimes B \end{array} \qquad \begin{array}{ccc} I' & \xrightarrow{l_*} & I \\ {\scriptstyle \top}\downarrow & \xrightarrow{\tilde{l}_{1*}} & \downarrow{\scriptstyle \top'} \\ I' & \xrightarrow[l_*]{} & I \end{array}$$

which are equivalences in $\mathbf{G}$. By Proposition 3.4 the $k_{X,Y}$ and l_* are equivalences in $\mathbf{M}$ and the $\tilde{k}_{R,S}$ and $\tilde{l}_{1*}$ are isomorphisms. These provide the components for pseudonatural equivalences $k\colon \otimes' \to \otimes$ and $l\colon I' \to I$. ∎

4.3. Remark. Since a terminal object in a (mere) category is unique to within unique isomorphism and we have $I^\circ : 1_I \to \top$ invertible in a cartesian bicategory we may as well as assume, in a cartesian bicategory, that we have chosen $\top = 1_I$.

4.4 We turn now to further analysis of the pseudofunctors

$$\mathbf{B} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{B} \xleftarrow{I} \mathbf{1}$$

Since $\otimes$ is binary product in $\mathbf{M}$ we have the pseudonatural (adjoint) equivalences

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z) \qquad l : I \otimes X \longrightarrow X \qquad r : X \otimes I \longrightarrow X$$

$$\text{and} \qquad s : X \otimes Y \longrightarrow Y \otimes X$$

in $\mathbf{M}$ as constructed and studied in Section 2. However, we can say more:

4.5. Proposition. *The equivalence maps $a : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l : I \otimes X \to X$, $r : X \otimes I \to X$, and $s : X \otimes Y \to Y \otimes X$ extend to pseudonatural equivalences between the relevant $\mathbf{B}$-valued functors.*

Proof. Because $\mathbf{G}$ has finite products which commute via ∂_0 and ∂_1 with those of $\mathbf{M}$ we have equivalences

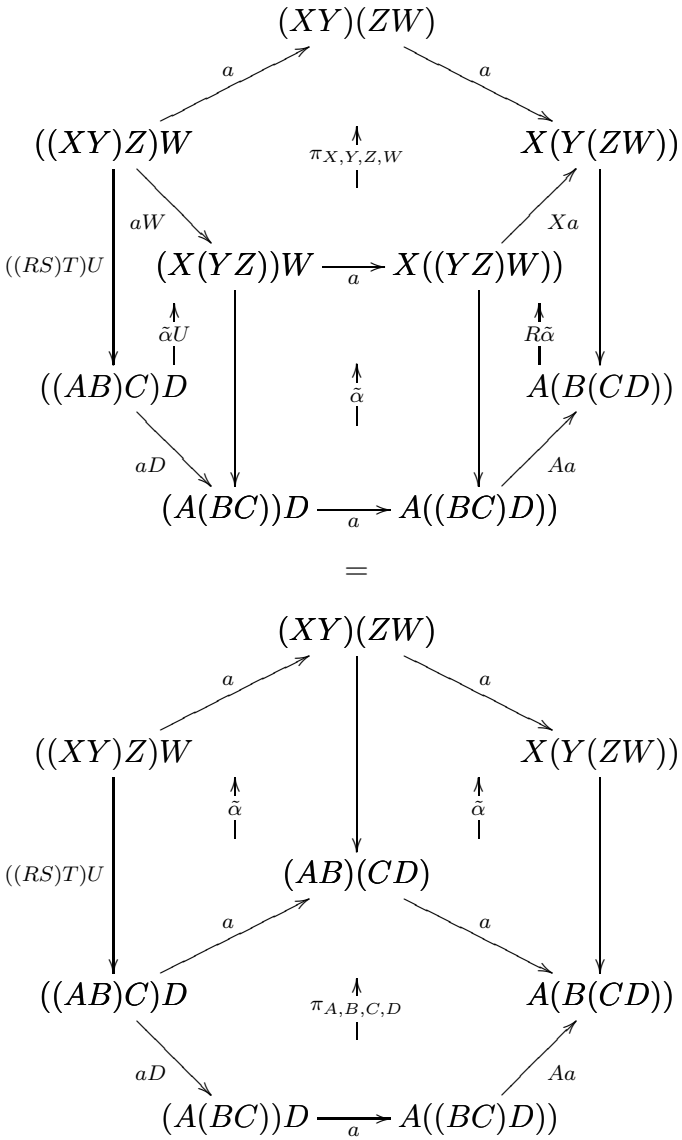
$$\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{a} & X \otimes (Y \otimes Z) \\
{\scriptstyle (R \otimes S) \otimes T}\downarrow & \xrightarrow{\tilde{a}_{R,S,T}} & \downarrow{\scriptstyle R \otimes (S \otimes T)} \\
(A \otimes B) \otimes C & \xrightarrow[a]{} & A \otimes (B \otimes C)
\end{array}
\qquad
\begin{array}{ccc}
I \otimes X & \xrightarrow{l} & X \\
{\scriptstyle I \otimes R}\downarrow & \xrightarrow{\tilde{l}_R} & \downarrow{\scriptstyle R} \\
I \otimes A & \xrightarrow[l]{} & A
\end{array}
\qquad
\begin{array}{ccc}
X \otimes I & \xrightarrow{r} & X \\
{\scriptstyle R \otimes I}\downarrow & \xrightarrow{\tilde{r}_R} & \downarrow{\scriptstyle R} \\
A \otimes I & \xrightarrow[r]{} & A
\end{array}
\qquad
\begin{array}{ccc}
X \otimes Y & \xrightarrow{s} & Y \otimes X \\
{\scriptstyle R \otimes S}\downarrow & \xrightarrow{\tilde{s}_{R,S}} & \downarrow{\scriptstyle S \otimes R} \\
A \otimes B & \xrightarrow[s]{} & B \otimes A
\end{array}$$

in $\mathbf{G}$. From Proposition 3.4 we have that the $\tilde{a}_{R,S,T}$, $\tilde{l}_R$, $\tilde{r}_R$, and $\tilde{s}_{R,S}$ are invertible so that the squares above can be seen as providing the data for pseudonatural transformations between $\mathbf{B}$-valued functors. We omit verification of the pseudonaturality equations. ∎

4.6. Theorem. *The data $(\mathbf{B}, \otimes, I, a, l, r, s)$ extend to provide a symmetric monoidal bicategory.*

Proof. By Theorem 2.15 the bicategories $\mathbf{M}$ and $\mathbf{G}$ are symmetric monoidal bicategories. To construct the modifications π, μ, λ, and ρ of [GPS] observe that we have invertible 2-cells $\pi_{X,Y,Z,W}$ as in the first pentagonal region below (in which we have abbreviated $\otimes$ by juxtaposition) because $\mathbf{M}$ is monoidal. Since $\mathbf{M}$ is a subbicategory of $\mathbf{B}$ the π are also 2-cells in $\mathbf{B}$. For these to constitute a $\mathbf{B}$-valued modification we require the following

equality of pastings.



However, this equation follows simply because the pair $(\pi_{W,X,Y,Z}, \pi_{A,B,C,D})$ is an invertible 2-cell in the monoidal bicategory $\mathbf{G}$. Finally, to see for example that the π satisfy the non-abelian cocycle condition in [GPS] observe that the pentagons and squares in that diagram have boundaries composed entirely of maps so that the condition is satisfied because $\mathbf{M}$ is a monoidal bicategory. The other symmetric monoidal bicategory data is dealt with in a similar manner. ∎

As we stressed earlier the 2-cells in the product projections displayed in (23) are, in general, by no means invertible in $\mathbf{B}$. However, for those of the special form $\tilde{p}_{R,1_Y}$ or $\tilde{r}_{1_X,S}$ we have:

4.7. Proposition. *In a cartesian bicategory, for any arrow $R{:}X \to A$ and any object Y, the 2-cells*

$$\begin{tikzcd} X \otimes Y \arrow[r, "p_{X,Y}"] \arrow[d, "R\otimes 1_Y"'] \arrow[dr, phantom, "\xrightarrow{\tilde{p}_{R,1_Y}}"] & X \arrow[d, "R"] \\ A \otimes Y \arrow[r, "p_{A,Y}"'] & A \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X \otimes Y \arrow[r, "1_X\otimes t_Y"] \arrow[d, "R\otimes 1_Y"'] \arrow[dr, phantom, "\xrightarrow{1_R\otimes \tilde{t}_{1_Y}}"] & X \otimes I \arrow[d, "R\otimes \top"] \\ A \otimes Y \arrow[r, "1_A\otimes t_Y"'] & A \otimes I \end{tikzcd}$$

are invertible. Similarly, for any object X and any arrow $S{:}Y \to B$, the 2-cell $r_{1_X,S}$ is invertible.

Proof. In any bicategory with finite products, the equivalence $X \to X \times I$ identifies $p_{X,Y}{:}X \times Y \to X$ and $1_X \times t_Y{:}X \times Y \to X \times I$ to within isomorphism. In particular, this applies to $\mathbf{G}$ where the inverse of the relevant equivalence is

$$\begin{tikzcd} X \otimes I \arrow[r, "p_{X,I}"] \arrow[d, "R\otimes 1_I"'] \arrow[dr, phantom, "\xrightarrow{\tilde{p}_{R,1_I}}"] & X \arrow[d, "R"] \\ A \otimes I \arrow[r, "p_{A,I}"'] & A \end{tikzcd}$$

in which, by Proposition 3.4, the 2-cell $\tilde{p}_{R,1_I}$ is invertible. When the equivalence square above is pasted to the square on the right in the statement the result is the square on the left in the statement. In the square on the right of the statement observe first that since $\top = 1_I$ we have $\tilde{t}_{1_Y} = 1_{t_Y}$ by uniqueness of such arrows to $\top{:}I \to I$ in $\mathbf{G}$. Now $1_R \otimes \tilde{t}_{1_Y} = 1_R \otimes 1_{t_Y}$ is invertible being a $\otimes$-product of invertible 2-cells. (We note for clarity though that our notation here has $1_R \otimes 1_{t_Y}$ being the $\otimes$-product of the identity $1_A R \to R 1_X$ and the identity $t_Y 1_Y \to 1_I t_Y$ by the assumed normality of $\mathbf{B}$. As a paste composite of invertibles, $\tilde{p}_{R,1_Y}$ is also invertible. Similarly, since $1_{t_X} \otimes 1_S$ is invertible, $\tilde{r}_{1_X,S}$ is invertible. ∎

A similar result holds for the mates of the $\tilde{p}_{R,1_Y}$ and the $\tilde{r}_{1_X,S}$.

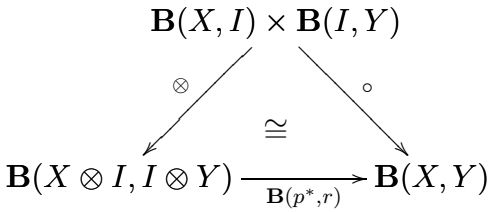
4.8. Proposition. *In a cartesian bicategory, for any arrow $R{:}X \to A$ and any object Y, the mate of $\tilde{p}_{R,1_Y}$ as in*

$$\begin{tikzcd} X \otimes Y \arrow[d, "R\otimes 1_Y"'] \arrow[dr, phantom, "\xrightarrow{\tilde{p}^*_{R,1_Y}}"] & X \arrow[l, "p^*_{X,Y}"'] \arrow[d, "R"] \\ A \otimes Y & A \arrow[l, "p^*_{A,Y}"] \end{tikzcd}$$

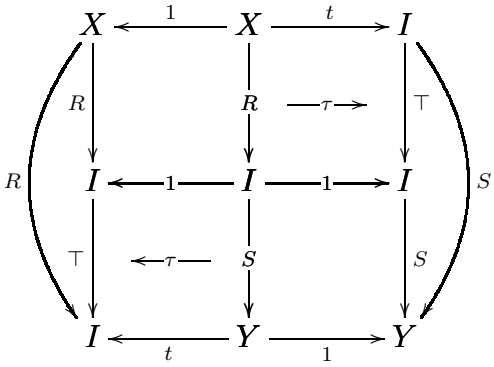
is invertible. Similarly, for any arrow $S{:}Y \to B$ and any object X, the mate $\tilde{r}^*_{1_X,S}$ of $\tilde{r}_{1_X,S}$ is invertible.

PROOF. As in Proposition 4.7 the task reduces to showing that the mate of $1_R \otimes \tilde{t}_{1_Y} = 1_R \otimes 1_{t_Y}$ with respect to the adjunctions $1_X \otimes t_Y \dashv (1_X \otimes t_Y)^*$ and $1_A \otimes t_Y \dashv (1_A \otimes t_Y)^*$ is invertible. Since $\otimes$ is a psedofunctor it preserves adjunctions and the mate operation. The mate at issue is easily seen to be $1_R \otimes 1_{t_Y^*}$ which being a $\otimes$-product of invertibles is invertible. ∎

4.9. PROPOSITION. *For X and Y in a cartesian bicategory there is a natural isomorphism*

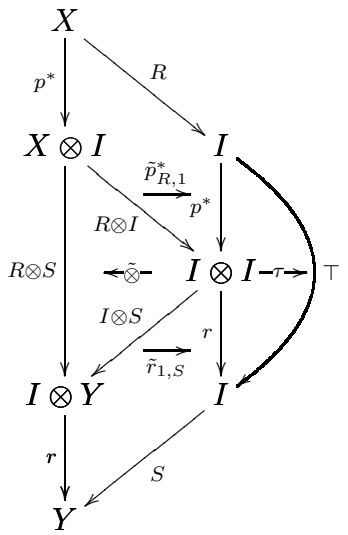


where we have used $\circ$ to denote composition in $\mathbf{B}$. Moreover, the 2-cells



provide product projections for $SR{:}X \nrightarrow Y$ seen as a product of $R{:}X \nrightarrow I$ and $S{:}I \nrightarrow Y$ in $\mathbf{G}$.

PROOF. We have



in which all the 2-cells are invertible, those in the parallelograms being so by Propositions 4.8 and 4.7. From this explicit description the second clause of the statement follows easily. ∎

References

[C&W] A. Carboni and R.F.C. Walters. *Cartesian bicategories I.* J. Pure Appl. Algebra **49** (1987), 11–32.

[CKW] A. Carboni, G.M. Kelly, and R.J. Wood. *A 2-categorical approach to change of base and geometric morphisms I*, Cahiers Topol. Géom. Diff. **32** (1991), No. 1, 47–95.

[CKVW] A. Carboni, G.M. Kelly, D. Verity and R.J. Wood. *A 2-categorical approach to change of base and geometric morphisms II*, Theory Applic. Categ. **4** (1998), No. 5, 73–136.

[CKWW] A. Carboni, G.M. Kelly, R.F.C. Walters and R.J. Wood. *Cartesian bicategories III.* To appear.

[D&S] B.J. Day and R. Street. *Monoidal bicategories and Hopf algebroids.* Adv. Math. 129 (1997), 99–157.

[GPS] R. Gordon, J. Power, and R. Street. *Coherence for tricategories.* Mem. Amer. Math. Soc. 117 (1995).

[K&S] G.M. Kelly and R. Street. *Review of the elements of 2-categories.* Lecture Notes in Math. 420 (1974), 75–103.

[McC] P. McCrudden. *Balanced coalgebroids.* Theory Applic. Categ. **7** (2000), 71–147.

[STR] R. Street. *Fibrations in bicategories*, Cahiers Topol. Géom. Diff. **21** (1980), 111–160.

[VER] D. Verity. *Enriched categories, Internal Categories, and Change of Base.* PhD Thesis, Cambridge, 1992.

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