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THE CHU CONSTRUCTION

Dedicated to the memory of Robert W. Thomason, 1952–1995

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Abstract. We take another look at the Chu construction and show how to simplify it by looking at it as a module category in a trivial Chu category. This simplifies the construction substantially, especially in the case of a non-symmetric biclosed monoidal category. We also show that if the original category is accessible, then for any of a large class of "polynomial-like" functors, the category of coalgebras has cofree objects.

1. Introduction

In a recent paper, I showed how the Chu construction, given originally in [Chu, 1979] for symmetric monoidal closed categories, could be adapted to monoidal biclosed (but not necessarily symmetric) categories. The construction, although well motivated by the necessity of providing a doubly infinite family of duals, was rather complicated with many computations involving indices. Recently I have discovered that the $*$-autonomous structure of Chu categories can be put into the familiar context of bimodules over a not necessarily commutative "algebra" object (really, it is a monoid object, but the dual has always been called a coalgebra). It is a familiar fact that over an ordinary ring or algebra, the category of bimodules is a monoidal biclosed category. It was, in fact, the motivating example that led Eilenberg and Kelly to include the non-symmetric case of closed categories in their original paper in 1966.

We begin with a rational reconstruction of how the Chu construction might have been discovered. This is very far from how it actually happened, of course, which we describe briefly at the end. We then extend this construction to the non-symmetric case, which is rather easy. In fact, this whole idea of relating the Chu construction to bimodules was discovered in the non-symmetric case, where the analogy with bimodules became compelling. This turns out to be no analogy, just an instance. We then show how one instance of this gives the non-symmetric Chu construction described in [Barr, 1995].

One of the interesting properties of the symmetric Chu construction is that if the original category is locally presentable (that is, accessible and complete), then the Chu category has cofree cocommutative coalgebras, which means there is a ! construction. This

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is interesting especially in light of the fact that the Chu category of a complete category cannot be accessible unless the original category is a poset. In the non-symmetric case, the notion of cocommutativity is not even definable, so that a ! cannot be constructed that way. Nonetheless, it is interesting to observe that for an interesting class of functors R, the category of R coalgebras has a cofree cotriple. This class consists of the smallest class of functors closed under product, sum and tensor product. Such functors are polymorphic over the class of locally presentable monoidal categories, a crucial point in the proof, which is by induction on the structural complexity.

2. A rational reconstruction

2.1. THE SYMMETRIC CASE. For simplicity, we deal first with the symmetric case, although this works perfectly well in the non-symmetric case and was, in fact, first discovered in that context.

During the year I spent at the ETH in 1975-76, I was interested in self-dual categories. Now it is very easy to find self-dual categories; $\mathscr{V}\times\mathscr{V}^{\mathrm{op}}$ is self dual regardless of the nature of $\mathscr{V}$. So more specifically, I was interested in finding closed self-dual categories in which the dual was given as the internal hom into some "dualizing object". Suppose $\mathscr{V}$ is a monoidal category. What about $\mathscr{V}\times\mathscr{V}^{\mathrm{op}}$? If you make no particular requirements on the tensor (except that it be a bifunctor), you can simply define $(U,U')\otimes(V,V') = (U\otimes V, U'\otimes V')$. But if $\mathscr{V}$ is closed, $\mathscr{V}\times\mathscr{V}^{\mathrm{op}}$ is unlikely to be closed since the tensor will commute with limits in the second coordinate, instead of colimits.

So let us suppose that the original category $\mathscr{V}$ is symmetric closed monoidal and try to see what the corresponding autonomous structure on $\mathscr{V}\times\mathscr{V}^{\mathrm{op}}$ ought to be. Let us suppose that we have $(U,U')\otimes(V,V') = (U\otimes V, X)$ and we are trying to determine X. Let us also suppose that the first coordinate of $(U,U')\multimap(V,V')$ is the internalization of $\mathrm{Hom}((U,U'),(V,V')) = \mathrm{Hom}(U,V)\times\mathrm{Hom}(V',U')$. This suggests that $(U,U')\multimap(V,V') = ((U\multimap V)\times(V'\multimap U'), Y)$ where Y is to be determined. Finally, we observe that if the duality on the category is determined by the internal hom into a dualizing object $\perp$, then we must have

$$\begin{aligned}((U,U')\otimes(V,V'))^{\perp} &= (U,U')\otimes(V,V')\multimap\perp\\ &= (U,U')\multimap((V,V')\multimap\perp)\\ &= (U,U')\multimap(V',V) = ((U\multimap V')\times(V\multimap U'),Y)\end{aligned}$$

from which we conclude that

$$(U,U')\otimes(V,V') = (U\otimes V,(U\multimap V')\times(V\multimap U'))$$

Similarly,[1]

$$(U,U')\multimap(V,V') = ((U\multimap V)\times(V'\multimap U'),U\otimes V')$$

[1]The $\times$ sign in the next formula was mistakenly a $\otimes$ in the previously published version.

Does this work? It certainly does. A map

$$(U,U') \otimes (V \otimes V') \longrightarrow (W,W')$$

is equivalent to a pair of arrows $U \otimes V \longrightarrow W$ and $W' \longrightarrow (U \multimap V') \times (V \multimap U')$, which is equivalent to three arrows $U \otimes V \longrightarrow W$, $W' \longrightarrow U \multimap V'$ and $W' \longrightarrow V \multimap U'$. Similarly, a map $(U,U') \longrightarrow (V,V') \multimap (W,W')$ is a map $(U,U') \longrightarrow ((V \multimap W) \times (W' \multimap V'), V \otimes W')$, which corresponds to three maps $U \longrightarrow V \multimap W$, $U \longrightarrow W' \multimap V'$ and $V \otimes W' \longrightarrow U'$. These two sets of arrows are clearly in one-one correspondence.

2.2. The non-symmetric case. With some care, this also extends to the case of a biclosed monoidal category. We suppose that $\mathscr{V}$ is such a category with two internal homs $\multimap$ and $\multimapinv$ that satisfy

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(V, U \multimap W) \cong \mathrm{Hom}(V, W \multimapinv U)$$

We note that there is some arbitrariness in which one to denote $\multimap$ and which one is $\multimapinv$. The final choice was suggested by the example of a group considered as a biclosed monoidal category in which the elements of the group are the objects, the only maps are identities and the group multiplication is the tensor. In that case, $u \multimap v = u^{-1}v$ and $v \multimapinv u = vu^{-1}$, while if the other choice were made, we would have to interchange the elements. Incidentally, there is more to this analogy. For example, the internalizations of those hom isomorphisms are

$$U \otimes V \multimap W \cong V \multimap (U \multimap W) \qquad W \multimapinv (U \otimes V) \cong (W \multimapinv V) \multimapinv U$$

However, there is one further isomorphism that turns out to be important.

2.3. Proposition.

$$(U \multimap W) \multimapinv V \cong U \multimap (W \multimapinv V)$$

Proof. We have, for any object T,

$$\begin{aligned}\mathrm{Hom}(T, (U \multimap W) \multimapinv V) &\cong \mathrm{Hom}(T \otimes V, U \multimap W) \cong \mathrm{Hom}(U \otimes T \otimes V, W) \\ &\cong \mathrm{Hom}(U \otimes T, W \multimapinv V) \cong \mathrm{Hom}(T, U \multimap (W \multimapinv V))\end{aligned}$$

■

This proof clearly depends on the associativity of the tensor and the proposition gives a kind of associativity between the left and right internal homs. We will often write $U \multimap W \multimapinv V$ for either one, just as with the tensor.

We will now suppose that $\mathscr{V}$ is a biclosed monoidal category in this sense and also that $\sigma : \mathscr{V} \longrightarrow \mathscr{V}$ is a given isomorphism that preserves all the structure. The role of σ will be explained later; for the time being, you can take it to be the identity.

We now define structures on $\mathscr{V} \times \mathscr{V}^{\mathrm{op}}$:

$$\begin{aligned}
(U,U') \otimes (V,V') &= (U \otimes V, (V \multimap U') \times (V' \circ\!\!- \sigma^{-1}U)) \\
(U,U') \multimap (V,V') &= ((U \multimap V) \times (U' \circ\!\!- V'), V' \otimes \sigma^{-1}U) \\
(V,V') \circ\!\!- (U,U') &= ((V \circ\!\!- U) \times (\sigma V' \multimap \sigma U'), U \otimes V') \\
(U,U')^{\perp} &= (U', \sigma^{-1}U) \\
{}^{\perp}(U,U') &= (\sigma U', U)
\end{aligned} \tag{$*$}$$

2.4. THEOREM. *The structures defined by* $(*)$ *give a* $*$*-autonomous category.*

PROOF. First we show that it is a monoidal biclosed category, which means that for all objects (U,U'), (V,V') and (W,W'), we have

$$\begin{aligned}
\mathrm{Hom}((U,U') \otimes (V,V'), (W,W') &\cong \mathrm{Hom}((V,V'), (U,U') \multimap (W,W')) \\
&\cong \mathrm{Hom}((U,U'), (W,W') \circ\!\!- (V,V'))
\end{aligned}$$

For the first isomorphism, we calculate that an arrow

$$(U,U') \otimes (V,V') = (U \otimes V, (V \multimap U') \times (V' \circ\!\!- \sigma^{-1}U)) \longrightarrow (W,W')$$

is given by three arrows

$$U \otimes V \longrightarrow W, \qquad W' \longrightarrow (V \multimap U'), \qquad W' \longrightarrow (V' \circ\!\!- \sigma^{-1}U)$$

which transpose to

$$U \otimes V \longrightarrow W, \qquad V \otimes W' \longrightarrow U', \qquad W' \otimes \sigma^{-1}U \longrightarrow V'$$

An arrow

$$(V,V') \longrightarrow (U,U') \multimap (W,W') = ((U \multimap W) \times (U' \circ\!\!- W'), W' \otimes \sigma^{-1}U)$$

is given by three arrows

$$V \longrightarrow (U \multimap W), \qquad V \longrightarrow (U' \circ\!\!- W'), \qquad W' \otimes \sigma^{-1}U \longrightarrow V'$$

which transpose to

$$V \otimes U \longrightarrow W), \qquad V \otimes W' \longrightarrow U', \qquad W' \otimes \sigma^{-1}U \longrightarrow V'$$

which are the same data. For the second isomorphism, we find that a map

$$(U,U') \longrightarrow (W,W') \circ\!\!- (V,V') = ((W \circ\!\!- V) \times (\sigma W' \multimap \sigma V'), V \otimes W')$$

is given by three arrows

$$U \longrightarrow W \circ\!\!- V, \qquad U \longrightarrow \sigma W' \multimap \sigma V', \qquad V \otimes W' \longrightarrow U'$$

which transpose to

$$U \otimes V \longrightarrow W, \qquad \sigma W' \otimes U \longrightarrow \sigma V', \qquad V \otimes W' \longrightarrow U'$$

Since σ is an isomorphism that preserves all structure, an arrow $\sigma W' \otimes U \longrightarrow \sigma V'$ is the same as an arrow $W' \otimes \sigma^{-1} U \longrightarrow V'$.

This demonstrates the closed monoidal structure. For the rest of the $*$-autonomous structure, we must show that for any object $\mathbf{U}$ of the Chu category, $\mathbf{U}^\perp = \mathbf{U} \multimap \top^\perp$ and that the second dual map $\mathbf{U} \longrightarrow {}^\perp(\mathbf{U}^\perp)$ is an isomorphism. We see that $\top^\perp = (1, \top)$, which we will denote $\perp$. Then

$$(U, U') \multimap (1, \top) = (U \multimap 1 \times U' \multimapinv \top, \top \otimes \sigma^{-1} U) \cong (U', \sigma^{-1} U)$$

as required. For the second dual, we must show that the map $\mathbf{U} \longrightarrow \perp \multimapinv (\mathbf{U} \multimap \perp)$, gotten by twice transposing the identity $\mathbf{U} \multimap \perp \longrightarrow \mathbf{U} \multimap \perp$, is an isomorphism.

We begin by working out what map $\mathbf{U} \longrightarrow \mathbf{V} \multimapinv \mathbf{W}$ corresponds to the second transpose of a given map $\mathbf{W} \longrightarrow \mathbf{U} \multimap \mathbf{V}$. Assuming $\mathbf{U} = (U, U')$ and so on, the latter comes, as we have seen, from three arrows:

$$W \xrightarrow{f} U \multimap V, \qquad W \xrightarrow{g} U' \multimapinv V', \qquad V' \otimes \sigma^{-1} U \xrightarrow{h} W'$$

This corresponds to the map $\mathbf{U} \longrightarrow \mathbf{V} \multimapinv \mathbf{W}$:

$$U \longrightarrow V \multimapinv W, \qquad U \longrightarrow (\sigma V' \multimap \sigma W'), \qquad W \otimes V' \longrightarrow U'$$

described as follows. The first is the second transpose of f, the second is σ applied to the second transpose of h and the third is the transpose of g. Now we consider the special case that $\mathbf{V} = \perp$, $\mathbf{W} = \mathbf{U} \multimap \perp$ and we begin with the identity map. Then we begin with

$$\begin{aligned} p_1 : (U \multimap 1) \times (U' \multimapinv \top) &\longrightarrow U \multimap 1 \\ p_2 : (U \multimap 1) \times (U' \multimapinv \top) &\longrightarrow U' \multimapinv \top \\ \mathrm{id} : \top \otimes \sigma^{-1} U &\longrightarrow \top \otimes \sigma^{-1} U \end{aligned}$$

The corresponding maps $\mathbf{U} \longrightarrow \perp \multimapinv (\mathbf{U} \multimap \perp)$ are

$$\begin{aligned} U &\longrightarrow 1 \multimapinv [(U \multimap 1) \times (U' \multimapinv \top)] = 1 \\ U &\longrightarrow \top \multimap (\sigma \top \otimes \sigma \sigma^{-1} U) \\ (U' \multimapinv \top) \otimes \top &\longrightarrow U' \end{aligned}$$

The first map is the only map $U \longrightarrow 1$, the second is gotten by applying σ to the second transpose of $\mathrm{id} : \top \otimes \sigma^{-1} U \longrightarrow \top \otimes \sigma^{-1} U$. The coherence identities involving transposition and $\top$ make this an isomorphism. The result is that the combined map

$$U \longrightarrow (1 \multimapinv [(U \multimap 1) \times (U' \multimapinv \top)]) \times \big(\top \multimap (\sigma \top \otimes \sigma \sigma^{-1} U)\big)$$

is an isomorphism. The coherence identities also force the second transpose of $\mathrm{id} : U' \multimapinv \top \longrightarrow U' \multimapinv \top$ to be an isomorphism. It follows from Definition B of Section 2 of [Barr, 1995] that $\mathscr{V} \times \mathscr{V}$ is, with the given structure, a $*$-autonomous category.

2.5. Algebras and modules. The notion of algebra (associative, unitary) makes sense in any tensored category. Let us work out what is required for (U, U') to be an algebra in $\mathscr{V} \times \mathscr{V}^{\mathrm{op}}$. The tensor unit is $(\top, 1)$ so a unit is a map $(\top, 1) \longrightarrow (U, U')$, which is nothing but a map $\top \longrightarrow U$ in $\mathscr{V}$. A multiplication is a map

$$(U, U') \otimes (U, U') = (U \otimes U, U \multimap U' \times U' \circ\!\!- \sigma^{-1}U) \longrightarrow (U, U')$$

which corresponds to maps $U \otimes U \longrightarrow U$, $U' \longrightarrow U \multimap U'$ and $U' \longrightarrow U' \circ\!\!- \sigma^{-1}U$ in $\mathscr{V}$. The first of these, together with the unit, makes U into an algebra in $\mathscr{V}$. The second arrow defines a left U-module structure on U' and the third defines a right $\sigma^{-1}U$-module structure on U'.

Since $\top = \sigma^{-1}(\top)$ is obviously an algebra object in $\mathscr{V}$—the initial algebra object, in fact—and every object of $\mathscr{V}$ is a bimodule over $\top$, it follows immediately that every object of the form $(\top, \perp)$ is an algebra, regardless of the nature of $\perp$. These are the ones that will be of interest to us later.

3. The biclosed monoidal structure

In this section, we describe how the category of bimodules for an algebra object in a monoidal biclosed category is monoidal biclosed and $*$-autonomous when the original category is. Let $\mathscr{V}$ be a monoidal biclosed category and suppose that K is an algebra object, associative and unitary. The development parallels the familiar one for rings. We will talk of right, left and two-sided modules, always meaning with respect to the algebra K. These facts are essentially known, see [Street, 1983] or [Koslowski, forthcoming]. Each of these papers gives the proofs in a more general context. In Street's paper, the algebra has many objects and in Koslowski's the category of left K, right L-bimodules is the hom category in a bicategory whose objects are algebras. We give the constructions and omit the proofs.

3.1. The basic definitions. Suppose K is an algebra object, M is a right K-module and N a left K-module. Let the two actions be denoted $\rho(M, K) : M \otimes K \longrightarrow M$ and $\lambda(K, N) : K \otimes N \longrightarrow N$. Define $M \otimes_K N$ so that

$$M \otimes K \otimes N \underset{M \otimes \lambda(K,N)}{\overset{\rho(M,K)\otimes N}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_K N$$

is a coequalizer.

For the next construction, suppose that M and N are left K-modules. Then define $M \underset{K}{\multimap} N$ so that

$$M \underset{K}{\multimap} N \longrightarrow M \multimap N \rightrightarrows (K \otimes M) \multimap N$$

is an equalizer. One of the two maps $M \multimap N \longrightarrow (K \otimes M) \multimap N$ is just induced by $\lambda(K, M) : K \otimes M \longrightarrow M$ and the other is the transpose of the map

$$K \otimes M \otimes (M \multimap N) \xrightarrow{\text{eval}} K \otimes N \xrightarrow{\lambda(K,N)} N$$

Thus this is the internalization of the set of arrows that preserve the K structure.

The third construction begins with right K-modules M and N and defines $N \multimapinv_K M$ so that

$$N \multimapinv_K M \longrightarrow N \multimapinv M \rightrightarrows N \multimapinv (M \otimes K)$$

is an equalizer. The two arrows are defined similarly to the preceding case.

There appears to be no way of defining $M \multimap_K N$ for two right modules or $N \multimapinv_K M$ for two left modules. This is fortunate, since it means that when M and N are bimodules, the notations $M \multimap_K N$ and $N \multimapinv_K M$ are unambiguous. It is analogous to the unambiguous nature of $M \otimes_K N$, using the right structure on M and the left one on N. Another point to note is that, given that the given tensor is non-symmetric, there appears to be no way of defining an opposite algebra K^{op} in such a way that right K-modules are left K^{op}-modules. Similarly, there is no notion, given two algebras K and L, of K, L-bimodule. What there is is a notion of left K, right L-bimodule, to which we now turn.

3.2. Bimodules. In this section, we are primarily interested in left and right K-bimodules. However we investigate the notion of left K, right L bimodule, not so much for the added generality, but also it is less confusing and certainly no harder.

By a left K, right L bimodule, we mean an object M that is both a left K-module and a right L-module such that the square

$$\begin{tikzcd}
K \otimes M \otimes L \arrow[r, "K\otimes\rho(M{,}L)"] \arrow[d, "\lambda(K{,}M)\otimes L"'] & M \otimes L \arrow[d, "\rho(M{,}L)"] \\
K \otimes M \arrow[r, "\lambda(K{,}M)"'] & M
\end{tikzcd}$$

commutes. For convenience, we will use the notation ${}_KM_L$ to indicate that M is a left K, right L-bimodule. If ${}_KM_L$ and ${}_KN_L$, I will write $\mathrm{Hom}_K(M, N)_L$ to denote the set of left K, right L morphisms. If K or L is the tensor unit, it can be omitted from the notation.

3.3. Proposition. *Suppose that J, K and L are algebras and that ${}_JM_K$ and ${}_KN_L$. Then $M \otimes_K N$ gets the structure of a left J, right L-bimodule.*

As mentioned above, we omit the proof, but give the construction. Since $\otimes$ is a left adjoint, it preserves coequalizers so that the top line of the diagram

$$\begin{tikzcd}
J \otimes M \otimes K \otimes N \otimes I \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\lambda(J{,}M)\otimes K\otimes\rho(N{,}I)"'] & J \otimes M \otimes N \otimes I \arrow[r] \arrow[d, "\lambda(J{,}M)\otimes\rho(N{,}I)"] & J \otimes M \otimes_K N \otimes I \\
M \otimes K \otimes K \otimes N \arrow[r, shift left] \arrow[r, shift right] & M \otimes N \arrow[r] & M \otimes_K N
\end{tikzcd}$$

is a coequalizer. The diagram serially commutes and thus induces an arrow $J \otimes M \otimes_K N \otimes I \longrightarrow M \otimes_K N$, which gives the structure.

3.4. Proposition. *Suppose J, K and L are algebras. Suppose ${}_KM_J$ and ${}_KN_L$. Then $M \underset{K}{\multimap} N$ has the structure of a left J, right L bimodule.*

Here is the construction. The composite

$$M \otimes J \otimes (M \multimap N) \otimes L \longrightarrow M \otimes (M \multimap N) \otimes L \longrightarrow N \otimes L \longrightarrow N$$

has as exponential transpose an arrow $J \otimes (M \multimap N) \otimes L \longrightarrow M \multimap N$. Similarly, the composite

$$K \otimes M \otimes J \otimes ((K \otimes M) \multimap N) \otimes L \longrightarrow K \otimes M \otimes ((K \otimes M) \multimap N) \otimes L \longrightarrow N \otimes L \longrightarrow N$$

has as exponential transpose an arrow $J \otimes ((K \otimes M) \multimap N) \otimes L \longrightarrow (K \otimes M) \multimap N$. These are the vertical arrows in the diagram

$$\begin{tikzcd}
J \otimes (M \underset{K}{\multimap} N) \otimes L \arrow[r] & J \otimes (M \multimap N) \otimes L \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & J \otimes ((K \otimes M \multimap N)) \otimes L \arrow[d] \\
M \underset{K}{\multimap} N \arrow[r] & M \multimap N \arrow[r, shift left] \arrow[r, shift right] & ((K \otimes M \multimap N)
\end{tikzcd}$$

The diagram is serially commutative and the bottom row is an equalizer, so there is induced an arrow $J \otimes (M \underset{K}{\multimap} N) \otimes L \longrightarrow M \underset{K}{\multimap} N$ which is the induced structure giving the left J, right L structure on the internal homset.

By interchanging left and right, we also have,

3.5. Proposition. *Suppose J, K and L are algebras. Suppose ${}_LM_K$ and ${}_JN_K$. Then $N \underset{K}{\multimapinv} M$ has the structure of a left J, right L bimodule.*

These structures are related by the following theorem. Although it is stated in terms of objects, these are actually instances of natural equivalences, natural in M, N and P.

3.6. Theorem. *Suppose J, K and L are algebra objects and ${}_JM_K$, ${}_KN_L$ and ${}_JP_L$. Then*

$$\mathrm{Hom}_J(M \otimes_K N, P)_L \cong \mathrm{Hom}_K(N, M \underset{J}{\multimap} P)_L \cong \mathrm{Hom}_J(M, P \underset{L}{\multimapinv} N)_K$$

There are also isomorphisms of the internal homs that look similar to the above. They are not really internalizations of the above because they concern homs of only left or right structure, not the bimodule structure.

3.7. Theorem.

1. *In the situation ${}_JM_K$, ${}_KN_I$, ${}_JP_L$, we have*

$$M \otimes_K N \underset{J}{\multimap} P \cong N \underset{K}{\multimap} (M \underset{J}{\multimap} P)$$

as left I, right L-modules.

2. In the situation ${}_JM_K$, ${}_KN_I$, ${}_LP_I$, we have

$$P \mathrel{\underset{I}{\multimapinv}} M \otimes_K N \cong (P \mathrel{\underset{I}{\multimapinv}} N) \mathrel{\underset{K}{\multimapinv}} M$$

as left L, right J-bimodules.

Proof. We prove the first, the second is similar. Let Q be any left I, right L-bimodule. Then

$$\begin{aligned}
\mathrm{Hom}_I(Q, M \otimes_K N \mathrel{\underset{J}{\multimap}} P)_L &\cong \mathrm{Hom}_J(M \otimes_K N \otimes_I Q, P)_L \\
&\cong \mathrm{Hom}_K(N \otimes_I Q, M \mathrel{\underset{J}{\multimap}} P)_L \\
&\cong \mathrm{Hom}_I(Q, N \mathrel{\underset{K}{\multimap}} (M \mathrel{\underset{J}{\multimap}} P))_L
\end{aligned}$$

Since these isomorphisms are evidently natural in Q, the conclusion follows from Yoneda. ∎

It follows from the above that when M is a right K-module, $M^* = M \multimap \top^*$ is a left K-module and similarly when M is a left K-module, M^* is a right K-module and when M is a two-sided K-module, so is M^*. In the same way, so is ${}^*M$. Thus when $\mathscr{C}$ is $*$-autonomous, so is the category of two-sided K-modules.

Street proved these results under somewhat stronger hypotheses, but he allowed his "algebras" to have many objects. Our hypotheses of equalizers and coequalizers suffice for this case.

What Koslowski actually proves is that if you begin with a biclosed monoidal category there is a biclosed monoidal bicategory whose objects are the algebra objects in the original category and for which the hom category between the algebra objects K and L has as objects left K, right L bimodules and morphisms thereof. The monoidal structure and the biclosed structure is exactly as described above and Koslowski shows that this all works as advertised.

3.8. The symmetric case. Suppose that the tensor product in $\mathscr{V}$ is symmetric. This means that there are natural isomorphisms $c = c(U,V) : U \otimes V \longrightarrow V \otimes U$ that satisfy the usual coherence isomorphisms, including that $c(V,U) \circ c(U,V) = \mathrm{id}$. In that case, we can consider the case that K is a commutative algebra object and then define a module to be symmetric when the left and right actions coincide in the sense that

$$\begin{tikzcd}
K \otimes V \arrow[rr, "c(K,V)"] \arrow[dr, "\lambda(K,V)"'] & & V \otimes K \arrow[dl, "\rho(V,K)"] \\
& V &
\end{tikzcd}$$

commutes. The subcategory of symmetric modules can be shown to be closed under tensor and right and left internal hom and the last two are also isomorphic.

3.9. Example. In Section 2.5 we saw that no matter what object $\perp$ of $\mathscr{V}$ we take, there is a natural bimodule structure on $K = (\top, \perp)$. Here I want to describe the category of bimodules for this algebra.

A left K-module (V, V') has structure given by

$$(\top, \perp) \otimes (V, V') = (\top \otimes V, \top \multimap V' \times \perp \circ\!\!- \sigma^{-1}V) \longrightarrow (V, V')$$

which corresonds to three arrows, $\top \otimes V \longrightarrow V$, $V \longrightarrow \top \multimap V'$, and $V' \longrightarrow \perp \circ\!\!- \sigma^{-1}V$. The first two of these are forced by the unitary identity to be the standard isomorphisms. The third transpose to an arrow $V' \otimes \sigma^{-1}V \longrightarrow \perp$. This third arrow is equivalent to an arrow $\sigma V' \otimes V \longrightarrow \perp$, which is sometimes more convenient. The triviality of the algebra means that associativity imposes no further conditions. Thus a left K-module is a pair (V, V') equipped with an arrow $\sigma V' \otimes V \longrightarrow \perp$. One easily checks that if (U, U') is another such left K-module, then an arrow $(f, f') : (U, U') \longrightarrow (V, V')$ preserves the module structure if and only if the diagram

$$\begin{tikzcd}
\sigma U' \otimes V \arrow[r, "\sigma U' \otimes f"] \arrow[d, "\sigma f' \otimes V"'] & \sigma U' \otimes U \arrow[d] \\
\sigma V' \otimes V \arrow[r] & \perp
\end{tikzcd}$$

commutes.

In a similar way, we can see that a right K-module is given by a pair (V, V') equipped with an arrow $V \otimes V' \longrightarrow \perp$ and arrow between two such is one for which the diagram

$$\begin{tikzcd}
U \otimes V' \arrow[r, "U \otimes f'"] \arrow[d, "f \otimes V'"'] & U \otimes U' \arrow[d] \\
V \otimes V' \arrow[r] & \perp
\end{tikzcd}$$

commutes. A K-bimodule has both structures and no additional coherence is imposed in this case. This gives non-symmetric Chu construction as the category of K-bimodules and it will be a non-symmetric $*$ -autonomous category.

4. The original non-symmetric Chu construction

In this section, we will show how the original non-symmetric Chu construction of [Barr, 1995] fits in as special case of what we have done. We start with a brief exposition of the construction given in that paper. Let $\mathscr{V}$ be a monoidal biclosed category and $\mathscr{W}$ be the category of $\mathbf{Z}$-graded $\mathscr{V}$ objects. That is, an object of $\mathscr{W}$ is a doubly infinite sequence $\mathbf{V} = (\ldots, V_{-1}, V_0, V_1, \ldots)$ of objects of $\mathscr{V}$, equipped with arrows $V_n \otimes V_{n+1} \longrightarrow \perp$ for all

$n \in \mathbf{Z}$. If $\mathbf{U} = (\ldots,\ U_{-1},\ U_0,\ U_1,\ \ldots)$ is another such object, then an arrow $\mathbf{U} \longrightarrow \mathbf{V}$ is given by a sequence of arrows $\mathbf{f} = (\ldots,\ f_{-1},\ f_0,\ f_1,\ \ldots)$ of arrows such that $f_n : U_n \longrightarrow V_n$ when n is even and $f_n : V_n \longrightarrow U_n$ when n is even, subject to the conditions that the two diagrams following commute for all $n \in \mathbf{Z}$:

$$\begin{tikzcd}
U_{2n} \otimes V_{2n+1} \arrow[r, "U_{2n}\otimes f_{2n+1}"] \arrow[d, "f_{2n}\otimes V_{2n+1}"'] & U_{2n} \otimes U_{2n+1} \arrow[d] \\
V_{2n} \otimes V_{2n+1} \arrow[r] & \perp
\end{tikzcd}$$

$$\begin{tikzcd}
V_{2n-1} \otimes U_{2n} \arrow[r, "f_{2n-1}\otimes U_{2n}"] \arrow[d, "V_{2n-1}\otimes f_{2n}"'] & U_{2n-1} \otimes U_{2n} \arrow[d] \\
V_{2n-1} \otimes V_{2n} \arrow[r] & \perp
\end{tikzcd}$$

For more details, in particular, the full description of the $*$ -autonomous structure, see the cited paper. The formulas are complicated and it is not at all clear where they came from.

The descriptions of this paper allow a completely different construction of an equivalent category, one that is described easily. Let $\mathscr{V}$ be a not necessarily symmetric autonomous category and let $\mathrm{Gr}(\mathscr{V})$ denote the category of $\mathbf{Z}$-graded objects of $\mathscr{V}$. The objects are the same as those of $\mathscr{W}$ above, but it is not the same category. In $\mathrm{Gr}(\mathscr{V})$, all the structure is computed component-wise, so that for objects $\mathbf{U} = (\ldots,\ U_{-1},\ U_0,\ U_1,\ \ldots)$ and $\mathbf{V} = (\ldots,\ V_{-1},\ V_0,\ V_1,\ \ldots)$ of $\mathrm{Gr}(\mathscr{V})$, we have $\mathrm{Hom}(\mathbf{U},\mathbf{V}) = \prod_{i\in\mathbf{Z}} \mathrm{Hom}_{\mathscr{V}}(U_i, V_i)$, $\mathbf{U}\otimes\mathbf{V} = (U_i \otimes V_i)$, $\mathbf{U} \multimap \mathbf{V} = (U_i \multimap V_i)$ and $\mathbf{V} \multimapinv \mathbf{U} = (V_i \multimapinv U_i)$. We let $\sigma : \mathrm{Gr}(\mathscr{V}) \longrightarrow \mathrm{Gr}(\mathscr{V})$ be the map that increments by 1 the index on each object. Thus $(\sigma(\mathbf{V}))_i = V_{i-1}$.

Let $\perp$ be an object of $\mathscr{V}$. We also use it to denote the constant object of $\mathrm{Gr}(\mathscr{V})$. An object of $\mathrm{Chu}(\mathrm{Gr}(\mathscr{V}), \perp)$ then consists of a pair we will, for convenience, denote $(\mathbf{V}', \mathbf{V}'')$ along with arrows $\mathbf{V}' \otimes \mathbf{V}'' \longrightarrow \perp$ and $\sigma(\mathbf{V}'') \otimes \mathbf{V}' \longrightarrow \perp$. The first of these is given by arrows ${V'}_i \otimes {V''}_i \longrightarrow \perp$ and the second by ${V''}_{i-1} \otimes {V'}_i \longrightarrow \perp$, for all $i \in \mathbf{Z}$. If we now define $V_{2i} = {V'}_i$ and $V_{2i+1} = {V''}_i$, then this is summarized by saying that a Chu structure is given by a doubly infinite sequence (V_i), $i \in \mathbf{Z}$ and arrows $V_i \otimes V_{i+1} \longrightarrow \perp$ for all $i \in \mathbf{Z}$. It is clear that the arrows in the category will alternate directions and it is not hard to verify that this construction proves the following.

4.1. THEOREM. *For any autonomous category* $\mathscr{V}$*, the category* $\mathrm{Chu}(\mathrm{Gr}(\mathscr{V}))$ *as just described is equivalent to the Chu category described in [Barr, 1995].*

The proof is not hard, but there are a lot of details to verify.

5. Cofree coalgebras

In this section, we will suppose that $\mathscr{V}$ is a locally presentable, that is accessible and complete, category. See [Makkai and Paré, 1990] as well as [Gabriel and Ulmer, 1971]. In the symmetric case, this implies that there are cofree objects in the category of cocommutative, coassociative, counitary coalgebras, which gives a natural model for the ! construction of linear logic, [Barr, 1991]. In the non-symmetric case, cocommutativity cannot be defined and we cannot model ! in this way. Nonetheless, it could be useful to know that cofree coalgebras exist. We will actually show here that cofree coalgebras of the form $C \longrightarrow RC$ exist for a large class of functors R. Among other things, this implies that final R-coalgebras (the cofree generated by the terminal object) exist.

5.1. Theorem. *Suppose $\mathscr{V}$ is a locally presentable category and $\perp$ is an object of $\mathscr{V}$. Let $\mathscr{R}$ denote the least class of functors that is closed under arbitrary products and sums and under finite (including empty) tensor products. Then for any $R \in \mathscr{R}$, the category of R-coalgebras in* $\mathrm{Chu}(\mathscr{V}, \perp)$ *has cofree coalgebras.*

The proof will occupy the rest of this section. Note that the class $\mathscr{R}$ makes sense in any bicomplete monoidal category. This is why we did not include all the constants, although it includes the initial and terminal objects as well as the tensor unit as constant functors and the identity functor as a unary sum (or product or tensor).

We begin with the fact that the functors in $\mathscr{R}$ are accessible on $\mathscr{V}$. Since $\mathscr{V}$ is an accessible category (which $\mathrm{Chu}(\mathscr{V}, \perp)$ is not), this shows that for any $R \in \mathscr{R}$, there are cofree R-coalgebras in $\mathscr{V}$.

5.2. Proposition. *The functors in $\mathscr{R}$ are accessible on $\mathscr{V}$.*

Proof. It is standard that the sum and product of accessible functors is accessible, so we need worry only about tensor products. But the tensor with a fixed object has a right adjoint, hence commutes with arbitrary colimits. We require the following, whose proof is left to the reader.

5.3. Lemma. *Suppose α is an infinite cardinal and $\mathscr{I}$ is an α-filtered category. Then the diagonal $\Delta : \mathscr{I} \longrightarrow \mathscr{I} \times \mathscr{I}$ is cofinal.* ∎

Now if $D : \mathscr{I} \longrightarrow \mathscr{V}$ is an is an α-filtered diagram and R_1, $R_2 : \mathscr{V} \longrightarrow \mathscr{V}$ are α-accessible, then

$$\begin{aligned}\mathrm{colim}_{i\in\mathscr{I}}(R_1 \otimes R_2)Di &\cong \mathrm{colim}_{i\in\mathscr{I}}(R_1Di \otimes R_2Di) \cong \mathrm{colim}_{i\in\mathscr{I}}\mathrm{colim}_{j\in\mathscr{I}}(R_1Di \otimes R_2Dj)\\ &\cong \mathrm{colim}_{i\in\mathscr{I}}(R_1Di \otimes R_2(\mathrm{colim}_{j\in\mathscr{I}}Dj))\\ &\cong R_1(\mathrm{colim}_{i\in\mathscr{I}}Di) \otimes R_2(\mathrm{colim}_{j\in\mathscr{I}}Dj)\end{aligned}$$

It follows from [Makkai, Paré, 1990] that for any $R \in \mathscr{R}$, the underlying functor from the category of R-coalgebras in $\mathscr{V}$ to $\mathscr{V}$ has a right adjoint. We will denote this functor by G_R. It is the functor part of a cotriple $\mathbf{G}_R = (G_R, \epsilon_R, \delta_R)$ on $\mathscr{V}$ for which the category of coalgebras is just the category of R-coalgebras and homomorphisms.

5.4. PROPOSITION. *The functor R on* $\mathrm{Chu}(\mathscr{V},\perp)$ *has the form*

$$R(V,V') = (RV, \widetilde{R}(V,V'))$$

where $\widetilde{R} : \mathscr{V}^{\mathrm{op}} \times \mathscr{V} \longrightarrow \mathscr{V}$ *is a functor.*

PROOF. Since $\prod(V_i, V_i') = (\prod V_i, \sum V_i')$ and $\sum(V_i, V_i') = (\sum V_i, \prod V_i')$ the assertion follows by structural induction for sums and products of such functions. As for tensor products, the conclusion follows from

$$(U,U') \otimes (V,V') = (U \otimes V, (V \multimap U') \times_{V \multimap \perp \circ\!\!- \sigma^{-1}U} (V' \circ\!\!- \sigma^{-1}U))$$

in which the first coordinate is just the tensor product of the first coordinates and the second is functorial in both, contravariant in the first and covariant in the second. Thus if $R = R_1 \otimes R_2$, then

$$\widetilde{R}(V,V') = R_2(V) \multimap \widetilde{R}_1(V,V') \times_{R_2(V) \multimap \perp \circ\!\!- \sigma^{-1}R_1(V)} \widetilde{R}_2(V,V') \circ\!\!- \sigma^{-1}R_1(V)$$

■

5.5. PROPOSITION. *Let $\mathscr{V}$ be a monoidal biclosed accessible category. For a fixed object U of $\mathscr{V}$, the functors $U \multimap -$ and $- \circ\!\!- U$ are accessible.*

PROOF. It is clearly sufficient to do either one, say $U \multimap -$. Let Γ be a set of generators for $\mathscr{V}$ and let α be the sup of the presentation rank of all objects of the form W and $U \otimes W$, where $W \in \Gamma$. Now let $V = \mathrm{colim} V_i$, the colimit taken over an α filtered diagram. Then for any $W \in \Gamma$, we have

$$\begin{aligned}\mathrm{Hom}(W, U \multimap V) &\cong \mathrm{Hom}(U \otimes W, V) \cong \mathrm{colimHom}(U \otimes W, V_i)\\ &\cong \mathrm{colimHom}(W, U \multimap V_i) \cong \mathrm{Hom}(W, \mathrm{colim}(U \multimap V_i))\end{aligned}$$

Since W ranges over a generating set, it follows that the induced map $\mathrm{colim}(U \multimap V_i) \longrightarrow U \multimap V$ is an isomorphism. ■

5.6. PROPOSITION. *For a fixed object V of $\mathscr{V}$ and $R \in \mathscr{R}$, the corresponding functor $\widetilde{R}$ is accessible as a functor $\mathscr{V} \longrightarrow \mathscr{V}$.*

PROOF. If $R = \prod R_i$ and $\widetilde{R}_i$ is the functor corresponding to R_i, then

$$R(V,V') = \left(\prod R_i(V), \sum \widetilde{R}_i(V,V')\right)$$

so that $\widetilde{R}(V,V') = \sum \widetilde{R}_i(V,V')$ and the sum of accessible functors is accessible. A similar argument, since the product of accessible functors is accessible, allows us to draw the same consequence for sums. Finally, we have to deal with the tensor product. Suppose that $R = R_1 \otimes R_2$, $R_1 = (R_1, \widetilde{R}_1)$ and $R_2 = (R_2, \widetilde{R}_2)$ and we know that $\widetilde{R}_1$ and $\widetilde{R}_2$ are accessible in their second variable. Then

$$R(V,V') = (RV, (R_1 V \multimap \widetilde{R}_2(V,V')) \times_{R_1 V \multimap \perp \circ\!\!- \sigma R_2 V} (\widetilde{R}_1(V,V') \circ\!\!- \sigma R_2(V)))$$

The objects $R_1(V)$ and $R_2(V)$ are fixed. Thus by the preceding proposition, it follows that $\widetilde{R}$ is accessible in the second variable. ■

We introduce some temporary notation in order to reduce the notational complication of the rest of the argument. If V is an object of $\mathscr{V}$, let $V^* = V \multimap \perp \times \perp \mathrel{\circ\!\!-} \sigma^{-1}V$. This is not exactly a dual, but it is the case that there is a one-one correspondence between arrows $W \longrightarrow V^*$ and $\sigma^{-1}V \longrightarrow W^*$, although we will not use that fact. What we will use are the obvious facts that a Chu structure on (V, V') is given by an arrow $V' \longrightarrow V^*$ and that $(f, f') : (V, V') \longrightarrow (W, W')$ is a pair of arrows for which the square

$$\begin{tikzcd}
W' \arrow[r, "f'"] \arrow[d] & V' \arrow[d] \\
W^* \arrow[r, "f^*"'] & V^*
\end{tikzcd}$$

commutes.

We now fix an $R \in \mathscr{R}$. Since R is accessible, there is a cofree R-coalgebra cotriple on $\mathscr{V}$ that we will denote $\mathbf{G} = (G, \epsilon, \delta)$. If (V, V') is an object of $\mathrm{Chu}(\mathscr{V}, \perp)$ there is a natural Chu structure on (GV, V'), namely

$$GV \otimes V' \xrightarrow{\epsilon V \otimes 1} V \otimes V' \longrightarrow \perp$$

$$\sigma V' \otimes GV \xrightarrow{1 \otimes \epsilon V} \sigma V' \otimes V \longrightarrow \perp$$

We denote the R-coalgebra structure on GV by $\rho = \rho V : GV \longrightarrow RGV$. If (C, C') is an R-coalgebra and $(f, f') : (C, C') \longrightarrow (V, V')$ is an arrow in $\mathrm{Chu}(\mathscr{V}, \perp)$, then C is an R-coalgebra in $\mathscr{V}$ and $f : C \longrightarrow V$ is an arrow. Thus f factors as $f = \epsilon V \circ \hat{f}$ where $\hat{f} : C \longrightarrow GV$ is a coalgebra morphism. It is immediate that (f, f') factors as $(f, f') = (\epsilon V, 1) \circ (\hat{f}, f')$. We claim that each of these maps is in $\mathrm{Chu}(\mathscr{V}, \perp)$. These follow from the commutativity of

$$\begin{tikzcd}
V' \arrow[r, "f'"] \arrow[d] & C' \arrow[dd] \\
V^* \arrow[d, "(\epsilon V)^*"'] \arrow[dr, "f^*"] & \\
(GV)^* \arrow[r, "(\hat{f})^*"'] & C^*
\end{tikzcd}
\qquad
\begin{tikzcd}
V' \arrow[r, "\mathrm{id}"] \arrow[d] & V' \arrow[d] \\
V^* \arrow[r, "(\epsilon V)^*"'] & (GV)^*
\end{tikzcd}$$

In the left diagram, the upper trapezoid commutes because $(C, C') \longrightarrow (V, V')$ is an arrow in $\mathrm{Chu}(\mathscr{V}, \perp)$, while the lower triangle commutes from the application of the contravariant functor $()^*$ to the identity $\epsilon V \circ \hat{f} = f$. The right hand square commutes by the definition of the Chu structure on (GV, V').

Now define an object $V^\sharp$ of $\mathscr{V}$ so that

$$\begin{tikzcd} V^\sharp \arrow[r, "\widetilde{f}"] \arrow[d] & C' \arrow[d] \\ (GV)^* \arrow[r, "(\hat{f})^*"'] & C^* \end{tikzcd}$$

is a pullback.

5.7. Proposition. *Let $V^\sharp$ be constructed as above. Then*

1. $(GV, V^\sharp)$ *is an object of* $\mathrm{Chu}(\mathscr{V}, \perp)$*;*
2. $(\hat{f}, f') : (C, C') \longrightarrow (GV, V')$ *factors through* $(GV, V\sharp)$*;*
3. $(GV, V^\sharp$ *is an R- coalgebra;*
4. *the first factor of this factorization is by an R-coalgebra morphism.*

Proof. We give it the structure of a Chu object by using the pullback diagram

$$\begin{tikzcd} V' \arrow[dr, "g"] \arrow[drr, "f'"] \arrow[ddr] & & \\ & V^\sharp \arrow[r] \arrow[d] & C' \arrow[d] \\ & (GV)^* \arrow[r, "(\hat{f})^*"'] & C^* \end{tikzcd}$$

to define g. The other figure commutes because $(\hat{f}, f')$ is a Chu morphism. This gives the structure of a Chu object. The commutativity of the left hand triangle shows that the arrow (id, g) is in $\mathrm{Chu}(\mathscr{V}, \perp)$. The commutativity of the square shows that $(\hat{f}, \widetilde{f})$ is in $\mathrm{Chu}(\mathscr{V}, \perp)$, while the commutativity of the upper triangle gives the factorization $(\hat{f}, f') = (\mathrm{id}, g) \circ (\hat{f}, \widetilde{f})$.

A coalgebra structure on an object (C, C') is determined by arrows $c : C \longrightarrow RC$ and $c' : \widetilde{R}(C, C') \longrightarrow C'$ such that the diagram

$$\begin{tikzcd} \widetilde{R} \arrow[r, "c'"] \arrow[d] & C' \arrow[d] \\ (RC)^* \arrow[r, "c^*"'] & C^* \end{tikzcd}$$

commutes. Consider the diagram

$$\begin{tikzcd}
\widetilde{R}G(V,V^\sharp) \arrow[rrr] \arrow[dddd] & & & \widetilde{R}(C,C') \arrow[dl] \arrow[dddd] \\
& V^\sharp \arrow[r] \arrow[dd] & C' \arrow[dd] & \\
& & & \\
& (GV)^* \arrow[r] & C^* & \\
(RGV)^* \arrow[rrr] \arrow[ur] & & & (RC)^* \arrow[ul]
\end{tikzcd}$$

The outer square commutes because it just expresses the fact that there is a morphism $(C,C') \longrightarrow (V,V^\sharp)$ is a morphism and hence so is $R(C,C') \longrightarrow R(V,V^\sharp)$. The right hand trapezoid commutes because (C,C') is an R-coalgebra and the bottom trapezoid commutes since $C \longrightarrow GV$ is an R-coalgebra morphism in $\mathscr{V}$. It readily follows since the inner square is a pullback that there is induced a map $\widetilde{R}(GV,V^\sharp) \longrightarrow V^\sharp$ that makes the other two trapezoids commute. The commutation of the left one implies that $(GV,V^\sharp)$ is an R-coalgebra and of the top and bottom one together that $(C,C') \longrightarrow (GV,V^\sharp)$ is a morphism of R-coalgebras. ∎

This proposition shows that the class of coalgebras of the form (GV,U) constitute a cofinal class among all those mapping to (V,V'). It is understood, of course, that the R-coalgebra structure restricts to $\rho : GV \longrightarrow RGV$ on the first coordinate. We have not yet cut that class down to a set because the second component is still unrestricted.

A coalgebra of the form (GV,U) can be described as a map $U \longrightarrow (GV)^*$ and a map $\widetilde{R}(GV,U) \longrightarrow U$ such that the square

$$\begin{tikzcd}
\widetilde{R}(GV,U) \arrow[r] \arrow[d] & U \arrow[d] \\
(RGV)^* \arrow[r, "(\rho GV)^*"'] & (GV)^*
\end{tikzcd}$$

commutes. Here the arrow $\widetilde{R}(GV,U) \longrightarrow (RGV)^*$ is the one that results from the Chu structure on $R(GV,U)$. Another way of saying this is to say that U and $(RGV)^*$ are objects of the slice category $\mathscr{V}/(GV)^*$ and that $\widetilde{R}$ induces a functor on that category as indicated.

Therefore, let $\mathscr{U}$ denote the category $\mathscr{V}/(GV)^*$ and let $S : \mathscr{U} \longrightarrow \mathscr{U}$ be the functor defined by

$$S(U \longrightarrow (GV)^*) = \widetilde{R}(GV, U) \longrightarrow (RGV)^* \longrightarrow (GV)^*$$

Then an S-coalgebra in $\mathscr{U}$ is exactly the same thing as an R-coalgebra of the form (GV, U), whose first component is ρ. Since colimits in $\mathscr{U}$ are created by the forgetful functor $\mathscr{U} \longrightarrow \mathscr{V}$, it follows from Proposition 5.6 that S is an accessible category and hence that there is a small solution set for the existence of cofree S-coalgebras. In particular there is a small cofinal set for $V \longrightarrow (GV)^*$, which implies that there is a cofree coalgebra generated by (V, V'). This completes the proof of Theorem 5.1.

6. The true history of Chu categories

Here is, insofar as I am aware, the true history of Chu categories. As I was preparing the preliminary work on $*$-autonomous categories, for example, [Barr, 1976], I read a number of books on topological vector spaces, for example, [Schaeffer, 1970]. One thing these books always mentioned and usually developed in some detail was the theory of pairs of spaces. A pair (E, E') consisted of two topological vector spaces and a bilinear pairing $E \times E' \longrightarrow \mathbf{K}$ ($\mathbf{K}$ is the ground field, either the real or complex numbers). Mostly, but not always, it was assumed that the pairing was separated and extensional. This construction was first given by Mackey, published in [1945], based on his 1942 doctoral dissertation. It is interesting to note that Mackey wrote in 1945 that the use of dual pairs, "makes it possible to regard a normable topological linear space as a linear space together with a distinguished family of linear functionals, rather than as a linear space with a topology", which is exactly the point I have been making in [Barr, 1995] and [Barr, to appear].

Grothendieck [1973], a book based on a notes from a 1954 course, develops the idea further, dropping the extensional hypothesis that Mackey and Schaeffer assumed. The theory was developed for these objects, but, although it was evident what maps had to be, they were not made into a category. Still less, was any closed or monoidal structure considered. Nonetheless, this is an instance of $\mathrm{Chu}(\mathscr{V}, \mathbf{K})$ and appears to have been the first. As far as I have been able to determine (based on not very extensive searching) no author actually discussed the notion of a morphism between pairs except in the separated case, where it is a subset of the set of linear maps between the first components consisting of those that induce maps on the second.

So I wrote down the obvious definition of morphism. It then occurred to me to wonder if the resultant category was closed or monoidal and quickly discovered that it was both, modulo the checking of a morass of details. Since Chu needed a topic for his master's degree, I set him this task and the appendix to [Barr, 1979] was the result. But the story does not end there. Or rather, it would have ended there since neither I nor anyone else found any interest in this rather formal construction. But the year 1987 saw a renewed interest in $*$-autonomous categories as models of Girard's linear logic and then Vaughan Pratt and his student Vineet Gupta rediscovered the Chu construction and began to study it in earnest.

Pratt describes this as follows.

> Vineet and I had developed the category PDLat of partial distributive lattices as an extension of event structures for modeling concurrent processes, discovering along the way that it was concretely equivalent to Chu($\mathbf{Set}, 2$) [Pratt, 1993, Gupta, 1994]. We already knew that PDLat's were in a certain sense universal for Stone duality, and my result that the category of k-ary relational structures embeds concretely in Chu($\mathbf{Set}, 2^k$) led me to advance the thesis that Chu spaces were universal for concrete mathematics [Pratt, 1995]. If we regard pure first order logic as the logic just of sets as relational structures with the empty sort, and interpret linear logic propositions as Chu spaces, the implication of this thesis is that linear logic, thus far understood as a logic of resources, is better understood as the logic of the rest of concrete mathematics.

This interest eventually convinced me to look at the Chu construction again as a suitable vehicle in which to express virtually all duality theories. One of the results of this was the paper [Barr, to appear] in which much of the work of [Barr, 1979] is redone in this new light.

Probably the main conclusion to draw from this history is that, as with most mathematical discoveries, it is a mistake to attribute it any one person. It was born out of need, with a number of midwives, but no real discoverer.

There is one more historical note that may be of interest. The construction described in Section 4 came first. In the process of trying to understand it, it seemed to me that there were many analogies between that construction and bimodules. It occurred to me wonder if it really was a category of bimodules and of course it is. In working out the details of where the automorphism σ was to appear in the formulas, I referred repeatedly to the explicit original construction. Thus, although this rational reconstruction sounds convincing, it may be that the original messy construction had to be done first before the one given here could be properly understood.

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