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AN ALGEBRAIC DESCRIPTION OF LOCALLY MULTIPRESENTABLE CATEGORIES

JIŘÍ ADÁMEK, JIŘÍ ROSICKÝ

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Abstract. Locally finitely presentable categories are known to be precisely the categories of models of essentially algebraic theories, i.e., categories of partial algebras whose domains of definition are determined by equations in total operations. Here we show an analogous description of locally finitely multipresentable categories. We also prove that locally finitely multipresentable categories are precisely categories of models of sketches with finite limit and countable coproduct specifications, and we present an example of a locally finitely multipresentable category not sketchable by a sketch with finite limit and finite colimit specifications.

Introduction.

We have shown in [AR_1] how each locally finitely presentable category $\mathcal{K}$ in the sense of [GU] can be described by an essentially algebraic theory. This means that there exists a (finitary, many-sorted) signature $\Sigma = \Sigma_t \cup \Sigma_p$ such that $\mathcal{K}$ is equivalent to the category of partial Σ-algebras A such that

(i) each operation σ_A with $\sigma \in \Sigma_t$ is total (i.e., everywhere defined)

(ii) for each $\sigma \in \Sigma_p$ a finite set Def_σ of equations in signature Σ_t is given and $\sigma_A(a_1 \dots a_n)$ is defined iff all equations of Def_σ are fulfilled in $(a_1, \dots, a_n)$.

and

(iii) a set of equations is given which all partial algebras of the given category satisfy.

In the present paper we discuss locally finitely multipresentable categories, as introduced by Y. Diers [D]. Recall that a category is locally finitely multipresentable iff it has

(a) connected limits (or, equivalently, multicolimits)

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and

(b) a set of finitely presentable objects $\mathcal{A}$ such that each object is a directed colimit of $\mathcal{A}$-objects.

Examples: fields and homomorphisms, linearly ordered sets and strictly increasing maps. We describe a natural generalization of the concept of essentially algebraic theory, called essentially multialgebraic theory, and we show that a category is locally finitely multipresentable iff it is equivalent to the category of models of such a theory. Here we again work with partial algebras of signature $\Sigma = \Sigma_t \cup \Sigma_p$ (where Σ_t are the total operations). But rather than having a finite set of equations for each element of Σ_p, we assume that Σ_p is decomposed into a doubly indexed union $\Sigma_p = \bigcup_{\gamma\in\Gamma} \bigcup_{i\in I_\gamma} \Sigma_{\gamma i}$ and for each $\gamma \in \Gamma$ a finite set Def_γ of equations in the signature Σ_t is given. In each model A the equations of Def_γ determine the domain of definition of all operations of $\bigcup_{i\in I_\gamma} \Sigma_{\gamma i}$ as follows: if all equations of Def_γ are fulfilled in $(a_1, \ldots, a_n)$ then there exists precisely one $i \in I_\gamma$ such that $\sigma_A(a_1, \ldots, a_n)$ are defined for all $\sigma \in \Sigma_{\gamma i}$, whereas all $\tau(a_1, \ldots, a_n)$ for $\tau \in \Sigma_\gamma - \Sigma_{\gamma i}$ are undefined.

We consider next sketches describing locally finitely multipresentable categories: we have shown in [AR$_1$] that a category is locally finitely multipresentable iff it can be sketched by a (finite limit, coproduct)-sketch. We now prove that, assuming the non-existence of measurable cardinals, we can restrict ourselves to (finite limit, countable coproduct)-sketches. On the other hand, finite coproducts are not sufficient: a locally finitely multipresentable category $\mathcal{K}$ is exhibited which cannot be sketched by a (finite limit, finite coproduct)-sketch. Curiously, $\mathcal{K}$ can be sketched by a (finite limit, finite colimit)-sketch.

I. Essentially Multialgebraic Theory

Throughout the paper we work with partial Σ-algebras, where Σ is a many-sorted (finitary) signature. That is, if S is the set of sorts under consideration, then Σ is a set of operation symbols σ with prescribed arities

$$\sigma : s_1 \times \cdots \times s_k \to s\,.$$

This means that a partial Σ-algebra A consists of an S-sorted set, i.e., a collection $(A_s)_{s\in S}$ of sets, together with partial maps

$$\sigma_A : A_{s_1} \times \cdots \times A_{s_k} \to A_s$$

for each $\sigma : s_1 \times \cdots \times s_k \to s$. A *homomorphism* from a partial Σ-algebra A into a partial Σ-algebra B is an S-sorted map $h = (h_s)_{s\in S} : A \to B$ (i.e., $h_s : A_s \to B_s$ are maps) preserving the operations in the strong sense: whenever $\sigma_A(a_1, \ldots, a_k)$ is defined in A, it follows that $\sigma_B(h_{s_1}(a_1), \ldots, h_{s_k}(a_k))$ is defined in B and has the value $h(\sigma_A(a_1, \ldots, a_k))$. This results in the category $\mathrm{PAlg}\,\Sigma$ of partial algebras and homomorphisms.

A partial algebra is said to *satisfy* an equation provided that whenever a tuple is such that both sides of the equation are defined in it, then the results are equal.

DEFINITION. By an *essentially multialgebraic theory* is understood a quadruple

$$\mathbf{T} = (\Sigma, E, \Gamma, \mathrm{Def})$$

where

- Σ is a signature;
- E is a set of equations (in the signature Σ);
- Γ is a set of collections $\gamma = (\Sigma_{\gamma_i})_{i\in I_\gamma}$ of finite subsets $\Sigma_{\gamma_i} \subseteq \Sigma$ such that Σ_{γ_i} are pairwise disjoint for $\gamma \in \Gamma$, $i \in I_\gamma$;
- Def is a map assigning to each $\gamma \in \Gamma$ a finite set Def_γ of equations in the signature

$$\Sigma_t \stackrel{\text{def}}{=} \Sigma - \bigcup_{\gamma\in\Gamma} \bigcup_{i\in I_\gamma} \Sigma_{\gamma_i}$$

  such that variables appearing in the equations of Def_γ are the same as those used by any operation of $\bigcup\limits_{i\in I_\gamma} \Sigma_{\gamma_i}$.

REMARK. (1) The collection γ can also be empty, i.e., the case $I_\gamma = \emptyset$ is not excluded.

(2) The last clause concerning variables is only needed because the set $\bigcup_{i\in I_\gamma} \Sigma_{\gamma_i}$ can be infinite.

DEFINITION. By a *model* of an essentially multialgebraic theory $\mathbf{T}$ we understand a partial Σ-algebra A such that

(1) A satisfies all equations in E;
(2) σ_A is a total operation for each $\sigma \in \Sigma_t$;
(3) Given $\gamma \in \Gamma$, then whenever $\sigma_A(a_1, \ldots, a_n)$ is defined for some $\sigma \in \bigcup\limits_{i\in I_\gamma} \Sigma_{\gamma_i}$ then all equations of Def_γ hold in the n-tuple $(a_1, \ldots, a_n)$;
(4) Given $\gamma \in \Gamma$, then when all equations of Def_γ hold in some n-tuple $(a_1, \ldots, a_n)$, then there exists a unique $i_0 \in I_\gamma$ such that $\sigma_A(a_1, \ldots, a_n)$ is defined for each $\sigma \in \Sigma_{\gamma_{i_0}}$.

The category of all models of $\mathbf{T}$ and homomorphisms is denoted by $\mathrm{Mod}\,\mathbf{T}$.

REMARK. In case $I_\gamma = \emptyset$, a model A of $\mathbf{T}$ has the property that for no n-tuple of A all equations of Def_γ are satisfied.

EXAMPLES. (1) **Fields**. We start with the algebraic theory (Σ_t, E_t) of rings with a unit element. We introduce two unary operation symbols o and i (where $o(x)$ will be defined precisely when $x = 0$ and $i(x)$ is the inverse of x for all $x \neq 0$). We put

$$\Sigma = \Sigma_t \cup \{o, i\}$$
$$\Gamma = \{\gamma, \gamma'\}$$

where

$$\gamma = \{\{o\}, \{i\}\}$$
$$\gamma' = \emptyset$$
$$\mathrm{Def}_\gamma = \emptyset$$
$$\mathrm{Def}_{\gamma'} = \{0 = 1\}$$
$$E = E_t \cup E'$$

where E' are the following four equations:

$$i(x) \cdot x = 1$$
$$x \cdot i(x) = 1$$
$$o(x) = 1$$
$$x \cdot o(x) = 0\,.$$

Thus, a model of this theory is a ring with a unit and with two partial unary operations such that for each element x exactly one of $i(x)$ and $o(x)$ is defined, and 0 nonequal 1. Moreover, if $i(x)$ is defined then $i(x) = x^{-1}$ and if $o(x)$ is defined, then $x = 0$ (and $o(0) = 1$). This precisely describes all fields.

(2) **Graphs**. The category of graphs, i.e., sets with a binary relation, and homomorphisms is locally finitely presentable. Thus, it can be described by an essentially algebraic theory. The following theory was presented in [AR_1]: we have sorts edge and vertex. We put $\Sigma = \{\sigma, \tau, \rho\}$ where

$$\sigma, \tau : \text{edge} \to \text{vertex}$$

are the operations of source and target, respectively, and

$$\rho : \text{edge} \times \text{edge} \to \text{edge}$$

is an auxiliary operation to express the implication

$$\sigma(x) = \sigma(y) \wedge \tau(y) = \tau(y) \to x = y\,.$$

Thus, $\Sigma_t = \{\sigma, \tau\}$ and

$$\mathrm{Def}_{\{\rho\}} = \{\sigma(x) = \sigma(y), \quad \tau(x) = \tau(y)\}$$

whereas E consists of two equations

$$\rho(x, y) = x \tag{1}$$
$$\rho(x, y) = y\,. \tag{2}$$

A model of this theory is a pair of sets A_{edge}, A_{vertex} together with two operations

$$\sigma_A, \tau_A : A_{\text{edge}} \to A_{\text{vertex}}$$

satisfying the above implications. This can be identified with the graph on the set A_{vertex} with vertices p, q connected by an edge iff $p = \sigma(x)$ and $q = \tau(x)$ for some $x \in A_{\text{edge}}$.

(3) **Linearly ordered sets.** We will now model the full subcategory of the category of graphs consisting of all strictly linearly ordered sets. We extend the above essentially algebraic theory as follows:

(i) Linearity. We introduce two binary operations α_1, α_2 of arity

$$\text{vertex} \times \text{vertex} \to \text{edge}$$

and one binary operation α_3 of arity

$$\text{vertex} \times \text{vertex} \to \text{vertex}.$$

We put

$$\gamma = \{\{\alpha_i\}\}_{i=1,2,3}$$
$$\text{Def}_\gamma = \emptyset$$

and add the following equations to E:

$$\sigma\alpha_1(u,v) = u \quad \text{and} \quad \tau\alpha_1(u,v) = v \tag{3}$$

$$\sigma\alpha_2(u,v) = v \quad \text{and} \quad \tau\alpha_2(u,v) = u \tag{4}$$

$$\alpha_3(u,v) = u \quad \text{and} \quad \alpha_3(u,v) = v\,. \tag{5}$$

Thus, in a model A, given vertices u and v precisely one of $\alpha_i(u,v)$ is defined for $i = 1, 2, 3$. In case $i = 1$ we have $u < v$ (since $\alpha_1(u,v)$ is the edge $u \to v$), if $i = 2$ we have $v < u$, and in case $i = 3$ we have $u = v$.

(ii) Irreflexivity. We introduce $\gamma' = \emptyset$ with

$$\text{Def}_{\gamma'} = \{\sigma(x) = \tau(x)\}\,.$$

(iii) Transitivity. Choose an operation $\beta : \text{edge} \times \text{edge} \to \text{edge}$ and put

$$\gamma'' = \{\{\beta\}\}$$
$$\text{Def}_{\gamma''} = \{\tau(x) = \sigma(y)\}$$

and add to E the equations

$$\sigma\beta(x,y) = \sigma(x) \quad \text{and} \quad \tau\beta(x,y) = \tau(y)\,. \tag{6}$$

The theory of signature

$$\Sigma^* = \{\sigma, \tau, \rho, \alpha_1, \alpha_2, \alpha_3, \beta\}$$

with the equation (1)-(6) forming E, with

$$\Gamma = \{\gamma, \gamma', \gamma''\}$$

and with the above Def axiomatizes linearly ordered sets.

Remark. We recall from [AR_1] 4.30 a characterization of locally finitely multipresentable categories as precisely the ω-cone-orthogonality classes in locally finitely presentable categories:

Let $\mathscr{L}$ be a locally finitely presentable category. For each set $\mathcal{M}$ of cones $m = (D_m \overset{d_{mj}}{\to} D_{mj})_{j\in J_m}$ in $\mathscr{L}$ we denote by $\mathcal{M}^\perp$ the full subcategory of $\mathscr{L}$ consisting of all objects L such that given $m \in \mathcal{M}$ then L is orthogonal to m, i.e., given a morphism $f : D_m \to L$, there exists a unique $j \in J_m$ such that f factorizes through d_{mj}, and also the factorization is unique (i.e., there exists a unique morphism $f' : D_{mj} \to L$ with $f = f' \cdot d_{mj}$). We call $\mathcal{M}^\perp$ a cone-orthogonality class, and in case each D_m and D_{mj} is finitely presentable in $\mathscr{L}$, $\mathcal{M}^\perp$ is called an ω-cone-orthogonality class. The latter subcategories of $\mathscr{L}$ are locally finitely multipresentable. Conversely, every locally finitely multipresentable category is equivalent to an ω-cone-orthogonality class of some locally finitely presentable category.

Proposition 1. *For every essentially multialgebraic theory* $\boldsymbol{T}$ *the category* $Mod\,\mathbf{T}$ *is locally finitely multipresentable.*

Proof. Denote by $\mathscr{L}$ the full subcategory of $\mathrm{PAlg}\,\Sigma$ which consists of all partial Σ-algebras which satisfy all equations in E and in which all Σ_t-operations are everywhere defined. It is easy to see that $\mathscr{L}$ is a locally finitely presentable category. We are now going to present a set $\mathcal{M}$ of cones such that $\mathcal{M}^\perp$ is equal to $\mathrm{Mod}\,\mathbf{T}$.

For each element $\gamma \in \Gamma$ we construct a cone $(F_\gamma \overset{f_{\gamma_i}}{\to} F_{\gamma_i})_{i\in I_\gamma}$ in $\mathscr{L}$ as follows. Let $x_0, \ldots, x_{n-1}$ be all variables which appear in Def_γ. (Since the same variables appear in operations of $\bigcup\limits_{i\in I_\gamma} \Sigma_{\gamma_i}$, we can consider each of these operations σ as n-ary, and write $\sigma(x_0, \ldots, x_{n-1})$. This is done for technical convenience only since we can simply forget those variables on which σ does not depend.) Denote by F'_γ the free algebra generated by $\{x_0, \ldots, x_{n-1}\}$ in the equational class of all (total) Σ_t-algebras satisfying the equations of Def_γ. We consider F'_γ as a partial Σ-algebra by leaving all operations of $\Sigma - \Sigma_t$ nowhere defined. Then F'_γ has a reflection in $\mathscr{L}$, say, $r_\gamma : F'_\gamma \to F_\gamma$. Since both the set of generators and Def_γ are finite sets, it follows that F_γ is finitely presentable in $\mathscr{L}$. Next, for each $i \in I_\gamma$ we form a free algebra F'_{γ_i} generated by $\{x_0, \dots, x_{n-1}\} \cup \Sigma_{\gamma_i}$ in the above equational class, and we consider $F'_{\gamma i}$ as a partial Σ-algebra as follows: for any $\sigma \in \Sigma_{\gamma_i}$ we define

$$\sigma_{F'_{\gamma_i}}([x_0], \dots, [x_{n-1}]) = [\sigma]$$

(where [ ] denotes the inclusion of generators to F'_{γ_i}) and leave $\sigma_{F_{\gamma_i}}$ undefined in all other n-tuples; for any $\sigma \in \Sigma - (\Sigma_t \cup \Sigma_{\gamma_i})$ the operation $\sigma_{F_{\gamma_i}}$ is nowhere defined. Let $r_{\gamma_i} : F'_{\gamma_i} \to F_{\gamma_i}$ be a reflection of F'_{γ_i} in $\mathscr{L}$; again, F_{γ_i} is finitely presentable in $\mathscr{L}$. The inclusion of the sets of generators extends to a Σ-homomorphism $f'_{\gamma_i} : F'_\gamma \to F'_{\gamma_i}$ whose reflections is denoted by $f_{\gamma_i} : F_\gamma \to F_{\gamma_i}$. Put

$$m_\gamma = (F_\gamma \xrightarrow{f_{\gamma_i}} F_{\gamma_i})_{i \in I_\gamma} .$$

If an algebra $A \in \mathscr{L}$ is orthogonal to the cone m_γ then it has the property that

(∗) for each n-tuple $(a_0, \dots, a_{n-1})$ in which all Def_γ-equations are satisfied there exists precisely one $i \in I_\gamma$ such that $\sigma_A(a_0, \dots, a_{n-1})$ is defined for each $\sigma \in \Sigma_{\gamma_i}$.

In fact, since the n-tuple is satisfied by all equations of Def_γ, we have a unique Σ-homomorphism $h' : F'_\gamma \to A$ with $h'([x_k]) = a_k$ for $k = 0, \dots, n-1$; let $h : F_\gamma \to A$ be the Σ-homomorphism with $h' = h \cdot r_\gamma$. We have a unique $i \in I_\gamma$ such that h factorizes through f_{γ_i}. Let us show that, then, $\sigma_A(a_0, \dots, a_{n-1})$ is defined for each $\sigma \in \Sigma_{\gamma_i}$. Given a Σ-homomorphism $\bar{h} : F_{\gamma_i} \to A$ with $h = \bar{h} \cdot f_{\gamma_i}$, then since $\sigma_{F'_{\gamma_i}}([x_0], \dots, [x_{n-1}]) = [\sigma]$, and since $f'_{\gamma_i}([x_k]) = [x_k]$ for $k = 0, \dots, n-1$, we obtain

$$\sigma_A(\bar{h} \cdot r_{\gamma_i}([x_0]), \dots, \bar{h} \cdot r_{\gamma_i}([x_{n-1}])) = \bar{h} \cdot r_{\gamma_i}([\sigma]),$$

thus

$$\sigma_A(a_0, \dots, a_{n-1}) = \bar{h} \cdot r_{\gamma_i}([\sigma]) .$$

Conversely, if $j \in I_\gamma$ has the property that all $\sigma_A(a_0, \dots, a_{n-1})$ with $\sigma \in \Sigma_{\gamma_i}$ are defined, then $i = j$ because the unique Σ_t-homomorphism $\bar{h}' : F'_{\gamma_i} \to A$ with $\bar{h}'([x_k]) = a_k$ and $\bar{h}'([\sigma]) = \sigma_A(a_0, \dots, a_n)$ for all $\sigma \in \Sigma_{\gamma_j}$ then is, obviously, a homomorphism $\bar{h}' : F'_{\gamma_j} \to A$ of partial Σ-algebras. Let $\tilde{h} : F_{\gamma_j} \to A$ be the Σ-homomorphism with $\bar{h}' = \tilde{h} \cdot r_{\gamma_j}$, then we have

$$h \cdot r_\gamma([x_k]) = \tilde{h} \cdot r_{\gamma_j} \cdot f'_{\gamma_j}([x_k]) \ \text{ for } \ k = 0, \dots, n-1,$$

thus $h \cdot r_\gamma = \tilde{h} \cdot r_{\gamma_j} \cdot f'_{\gamma_j} = \tilde{h} \cdot f_{\gamma_j} \cdot r_\gamma$, from which it follows that $h = \tilde{h} \cdot f_{\gamma_j}$. Therefore, h factorizes through f_{γ_j}, which proves $i = j$. Thus, (∗) is established.

Conversely, let $A \in \mathscr{L}$ have the property $(*)$, then A is orthogonal to m_γ. In fact, for each homomorphism $h : F_\gamma \to A$ the n-tuple $a_k = h \cdot r_\gamma([x_k])$, $k = 0, \ldots, n-1$ satisfies all the equations of Def_γ. Thus, there is a unique $i \in I_\gamma$ with all $\sigma_A(a_0, \ldots, a_{n-1})$, $\sigma \in \Sigma_{\gamma_i}$, defined. It follows that h factorizes through f_{γ_i}: let $\bar{h}' : F'_{\gamma_i} \to A$ be the unique Σ_t-homomorphism with $\bar{h}'([x_k]) = a_k$ and $\bar{h}'([\sigma]) = \sigma_A(a_0, \ldots, a_{n-1})$ for all $\sigma \in \Sigma_{\gamma_i}$, then $\bar{h}' = F'_{\gamma_i} \to A$ is a homomorphism of $\mathrm{P\,Alg}\,\Sigma$, thus, $\bar{h}' = \bar{h} \cdot r_{\gamma_i}$ for a homomorphism $\bar{h} : F_{\gamma_i} \to A$; it follows that $h = \bar{h} \cdot f_{\gamma_i}$. Moreover, this factorization is unique: suppose $h = h^* \cdot f_{\gamma_i}$, then we prove $h^* \cdot r_{\gamma_i} = \bar{h} \cdot r_{\gamma_i}$ (which implies $h^* = \bar{h}$) by showing that these homomorphisms agree on all the generators of F'_{γ_i}. In fact,

$$\begin{aligned} h^* \cdot r_{\gamma_i}([x_k]) &= h^* \cdot r_{\gamma_i} \cdot f'_{\gamma_i}([x_k]) \\ &= h^* \cdot f_{\gamma_i} \cdot r_\gamma([x_k]) \\ &= h \cdot r_\gamma([x_k]) \\ &= a_k \end{aligned}$$

and for $[\sigma]$, where $\sigma \in \Sigma_{\gamma_i}$, the equality $\sigma_{F'_{\gamma_i}}([x_0], \ldots, [x_{k-1}]) = [\sigma]$ implies

$$\begin{aligned} h^* \cdot r_{\gamma_i}([\sigma]) &= \sigma_A(h^* \cdot r_{\gamma_i}([x_0]), \ldots h^* \cdot r_{\gamma_i}([x_{n-1}])) \\ &= \sigma_A(a_0, \ldots, a_{n-1}) \,. \end{aligned}$$

This shows that h factorizes uniquely through f_{γ_i}. Conversely, if h factorizes through f_{γ_i}, we show that $j = i$ by verifying that $\sigma_A(a_0, \ldots, a_{n-1})$ is defined for all $\sigma \in \Sigma_{\gamma_j}$ – this is performed precisely as in the above proof that $h^* \cdot r_{\gamma_i}([\sigma]) = \sigma_A(a_0, \ldots, a_{n-1})$.

We conclude that

$$\mathrm{Mod}\,\mathbf{T} = \{m_\gamma; \gamma \in \Gamma\}^\perp \,.$$

Since all domains and codomains appearing in m_γ are finitely presentable (for all $\gamma \in \Gamma$), we conclude by the preceding Remark that $\mathrm{Mod}\,\mathbf{T}$ is locally finitely multipresentable. ∎

Remark. Locally finitely multipresentable categories were also characterized in [AR$_1$] 4.30 as multireflective subcategories of $\mathrm{Alg}\,\Sigma$ (categories of total Σ-algebras) closed under directed colimits. More precisely, a category is locally finitely multipresentable iff there exists a finitary signature Σ and a full subcategory $\mathcal{K}$ of $\mathrm{Alg}\,\Sigma$ equivalent to the given category, which has the following properties:

(i) Every Σ-algebra A has a multireflection in $\mathcal{K}$, i.e., a cone $(A \xrightarrow{r_i} A_i)_{i \in I}$ such that all A_i lie in $\mathcal{K}$ and any homomorphism $h : A \to B$ with B in $\mathcal{K}$ factorizes through a unique r_i, $i \in I$, and the factorization h' with $h = h' r_i$ is also unique;

(ii) $\mathcal{K}$ is closed under directed colimits, or, equivalently, a Σ-algebra K lies in $\mathcal{K}$ whenever it is orthogonal to the multireflection of each finitely presentable Σ-algebra A. If $(A \xrightarrow{r_i} A_i)_{i \in I}$ is such a multireflection, then it is easy to verify that A_i are also finitely presentable for all i.

Theorem 1. *A category is locally finitely multipresentable iff it is equivalent to the category of models of some essentially multialgebraic theory.*

Proof. Due to the preceding Proposition and Remark, it is sufficient to show that any full subcategory $\mathcal{K}$ of Alg Σ satisfying (i) and (ii) above is equivalent to Mod $\mathbf{T}$ for some essentially multialgebraic theory $\mathbf{T}$. The signature Σ^* of the theory $\mathbf{T}$ we define now will be an extension of the given signature Σ (where the operations of Σ will be precisely the total operations, $\Sigma = \Sigma^*_t$). Let $(A_i)_{i\in I}$ be a set of representatives of all finitely presentable Σ-algebras w.r.t. isomorphism. For each $i \in I$ we have, since A_i is finitely presentable, a finite set of variables $x_1, \ldots, x_k$ (of sorts $t_1, \ldots, t_k$, respectively) and a finite set called Def_i of equations in those variables such that A_i is presented by those variables and equations. That is, A_i can be identified with the quotient algebra of the free Σ-algebra $F(x_1, \ldots, x_k)$ generated by $\{x_1, \ldots, x_k\}$ modulo the congruence $\sim_i$ generated by Def_i. We also form a multireflection $(A_i \stackrel{m_{ij}}{\to} A_{ij})_{j\in J_i}$ and, since A_{ij} is finitely presentable, we can choose a finite set G_{ij} of generators of A_{ij} for each $j \in J_i$ (containing $m_{ij}([x_n])$, $n = 1, \ldots, k$ where $[x_n]$ is the congruence class of x_n in $A_i = F(x_1, \ldots, x_k)/\sim_i$).

We now extend Σ to Σ^* by adding, for each $i \in I$, $j \in J_i$ and each element a of sort $s(a)$ in A_{ij}, an operation

$$\sigma_{ija} : t_1 \times \cdots \times t_k \to s(a)\,.$$

(The informal idea of σ_{ija} in models K is the following: the defining equations Def_i guarantee that if $\sigma_{ija}\ (x_1, \ldots, x_k)$ is defined, then the interpretation of the variables $x_1, \ldots, x_k$ yields a homomorphism $h : A_i \to K$. Then h factors as $h = h'\, m_{ij}$ and the result of σ_{ija} will be the image of a under h'.) Thus, we put $\Sigma^* = \Sigma \cup \{\sigma_{ija};\ i \in I,\ j \in J_i$ and $a \in G_{ij}\}$ and

$$\Sigma^*_{ij} = \{\sigma_{ija}; a \in G_{ij}\}.$$

We defined Def_i above for each $i \in I$. It remains to define the set E of equations in the signature Σ^*. These will be of two types:

(a) $x_n = \sigma_{ijx^*_n}(x_1, \ldots, x_k)$ for all $i \in I$, $j \in J_i$, $n = 1, \ldots, k$ where $x_1, \ldots, x_k$ are the variables of the given presentation of A_i, and $x^*_n = m_{ij}([x_n])$.

(b) Since A_{ij} is generated by $G_{ij} = \{g_1, \ldots, g_m\}$, we can choose, for each element $a \in A_{ij}$, a Σ-term $\rho'_a(x_1, \ldots, x_m)$ such that $(\rho'_a)_{A_{ij}}(g_1, \ldots, g_m) = a$; we denote by $\rho_a = \rho'_a(\sigma_{ijg_1}, \ldots, \sigma_{ijg_m})$ the corresponding Σ^*-term. For each $\tau \in \Sigma$ and each instance of computation of τ_A, say, $\tau_A(a_1, \ldots, a_n) = a$, we add to E the equation

$$\tau(\rho_{a_1}, \ldots, \rho_{a_n}) = \rho_a.$$

We thus defined an essentially multialgebraic theory

$$\mathbf{T} = (\Sigma^*, E, \Gamma, \mathrm{Def})$$

where $\Gamma = \{\{\Sigma^*_{ij}\}_{j\in J_i}\}_{i\in I}$. We will prove that $\mathcal{K}$ is equivalent to Mod $\mathbf{T}$. For each Σ-algebra K in $\mathcal{K}$ we define a partial Σ^*-algebra $H(K)$ as follows. Given $i \in I$ and $j \in J_i$

then for $a \in G_{ij}$ we define $(\sigma_{ija})_K$ in a tuple $(b_1, \ldots, b_k)$ iff there exists a homomorphism $h : A_{ij} \to K$ with $(h \cdot m_{ij})_{t_n}([x_n]) = b_n$, and then

$$(\sigma_{ija})_K(b_1, \ldots, b_k) = h_{s(a)}(a)\,.$$

Let us verify $H(K)$ is a model of $\mathbf{T}$. It satisfies the equations of E:

(a) If $(\sigma_{ijx_n^*})_K$ is defined in $(b_1, \ldots, b_k)$, then we have $h : A_{ij} \to K$ with $(h \cdot m_{ij})_{t_n}([x_n]) = b_n$ and

$$(\sigma_{ijx_n^*})_K(b_1, \ldots, b_n) = h_{t_n}(x_n^*) = h_{t_n}(m_{ij})_{t_n}([x_n]) = b_n\,.$$

(b) The equations $\tau_A(\rho_{a_1}, \ldots, \rho_{a_n}) = \rho_a$ are satisfied since h is a Σ-homomorphism.

Furthermore, if all equations of Def_i are satisfied in $(b_1, \ldots, b_k)$, then the homomorphism $f : F(x_1, \ldots, x_k) \to K$ determined by $f_{t_n}(x_n) = b_n$, $n = 1, \ldots, k$, factorizes through $A_i = F(x_1, \ldots, x_k)/\sim_i$, i.e., we have a homomorphism $g : A_i \to K$ with $g_{t_n}([x_n]) = b_n$. Since $K \in \mathcal{K}$, there exists a unique j such that $g = h\,m_{ij}$ for a unique Σ-homomorphism $h : A_{ij} \to K$. Since $(h \cdot m_{ij})_{t_n}([x_n]) = b_n$, we conclude that $(\sigma_{ija})_K(b_1, \ldots, b_k)$ is defined for each $a \in G_{ij}$. Conversely, given j' such that $(\sigma_{ij'a'})_K(b_1, \ldots, b_k)$ is defined for each $a' \in G_{ij'}$, it follows that a homomorphism $h' : A_{ij'} \to K$ exists with $h' \cdot m_{ij'} = h \cdot m_{ij}$ – consequently, $j = j'$ and $h = h'$. (In fact, define h' by $h'_{s(a)}(a) = \sigma_{ij'a'}(b_1, \ldots, b_k)$; the equations in (b) guarantee that h' is a Σ-homomorphism and those of (a) guarantee $h = h' \cdot m_{ij'}$.)

We conclude that $H(K)$ is a model of $\mathbf{T}$. We obtain a functor

$$H : \mathcal{K} \to \mathrm{Mod}\,\mathbf{T}$$

defined on homomorphisms by $H(h) = h$. To show that H is an equivalence of categories, it is sufficient to verify that whenever a partial Σ^*-algebra B is a model of $\mathbf{T}$ then its Σ-reduct B_0, i.e., the Σ-algebra obtained by forgeting the operations in $\Sigma^* - \Sigma$, lies in $\mathcal{K}$ – it then follows that $B = H(B_0)$. We can assume, without loss of generality, that B_0 is a finitely presentable Σ-algebra. In fact, since B is a model of $\mathbf{T}$, for each Σ-subalgebra C of B_0 the restriction of the partial operations of B to C defines a partial Σ^*-algebra C' which is also a model of $\mathcal{K}$. Suppose that we know already that this implies that C (the reduct of C') lies in $\mathcal{K}$, then B_0 also lies in $\mathcal{K}$ since $\mathcal{K}$ is closed under directed colimits (and B_0 is a directed colimit of all such C.) Thus, B_0 can be assumed to be A_i for some $i \in I$. The equations of Def_i are all satisfied in the tuple $([x_1], \ldots, [x_k])$ of B, thus, there exists $j \in J_i$ such that $(\sigma_{ija})_B$ are defined in that tuple for each element a of G_{ij} (because B is a model of $\mathbf{T}$) and thus, also $(\rho_a)_B([x_1], \ldots, [x_n])$ are defined. This allows us to define

$$k(a) = (\sigma_{ija})_B([x_1], \ldots, [x_k]) \text{ for all } a \in G_{ij}\,.$$

The equations of E guarantee that k is a Σ-homomorphism $k : A_{ij} \to B$ satisfying

$$k \cdot m_{ij} = id_{A_i}\,.$$

In fact, let $\tau_{A_{ij}}(a_1,\ldots,a_n) = a$ for some $\tau : s_1 \times \cdots \times s_n \to s$ in Σ, then from the equation in (b) above we conclude that

$$\begin{aligned}
k(a) &= k((\rho'_a)_{A_{ij}}(g_1,\ldots,g_m)) \\
&= (\rho'_a)_B(k(g_1),\ldots,k(g_m)) \\
&= (\rho'_a)_B((\sigma_{ijg_1})_B([x_1],\ldots,[x_k]),\ldots,(\sigma_{ijg_m})_B([x_1],\ldots,[x_k]) \\
&= (\rho_a)_B([x_1],\ldots,[x_k]) \\
&= \tau_B((\rho_{a_1})_B([x_1],\ldots,[x_k]),\ldots,(\rho_{a_n})_B([x_1],\ldots,[x_k])) \\
&= \tau_B((\rho'_{a_1})_B((\sigma_{ijg_1})_B([x_1],\ldots,[x_k]),\ldots,(\sigma_{ijg_n})_B([x_1],\ldots,[x_k])),\ldots) \\
&= \tau_B((\rho'_{a_1})_B(k(g_1),\ldots,k(g_m)),\ldots,(\rho_{a_n})_B(k(g_1),\ldots,k(g_m)) \\
&= \tau_B(k(a_1),\ldots,k(a_n)),
\end{aligned}$$

thus, k is a homomorphisms. To verify $k \cdot m_{ij} = id_{A_i}$, it is thus sufficient to prove that $k \cdot m_{ij}([x_n]) = [x_n]$ for all $n = 1,\ldots,k$, and this follows from the equation (a). This shows that m_{ij} is an isomorphism: from

$$(m_{ij} \cdot k) \cdot m_{ij} = m_{ij} = id_{A_{ij}} \cdot m_{ij}$$

and from $A_{ij} \in \mathcal{K}$ we get, by the definition of multireflection, that $m_{ij} \cdot k = id_{A_{ij}}$, thus $k = m_{ij}^{-1}$. This proves that $B_0 = A_i \cong A_{ij}$ lies in $\mathcal{K}$. Consequently, $H : \mathcal{K} \to \mathrm{Mod}\,\mathbf{T}$ is an equivalence of categories. ∎

Remark. The above theorem has an immediate generalization to infinitary algebras; given a regular cardinal λ, we can introduce essentially multialgebraic theories of λ-ary partial algebras – here Def_γ is a set of less than λ equations and Γ is a collection of subsets of Σ of cardinality less than λ, otherwise the definition is quite analogous. A category is locally λ-multipresentable iff it is equivalent to the category of models of some essentially multialgebraic λ-ary theory.

II. Finitary Sketches for Locally Multipresentable categories

Recall that a *sketch* is a quadruple $\mathscr{S} = (\mathcal{A}, \mathbf{L}, \mathbf{C}, \sigma)$ where $\mathcal{A}$ is a small category, $\mathbf{L}$ and $\mathbf{C}$ are sets of diagrams in $\mathcal{A}$ and σ assigns to each diagram $D \in \mathbf{L}$ a cone $\sigma(D)$ and to each diagram $D \in \mathbf{C}$ a cocone $\sigma(D)$. A *model* of $\mathscr{S}$ is a functor $F : \mathcal{A} \to \mathrm{Set}$ such that for each diagram $D \in \mathbf{L}$ the image of $\sigma(D)$ under F is a limit of FD and for each diagram $D \in \mathbf{C}$ the image of $\sigma(D)$ is a colimit of FD.

A category is said to be *sketchable* by $\mathscr{S}$ if it is equivalent to the full subcategory of $\mathrm{Set}^{\mathcal{A}}$ consisting of all models of $\mathscr{S}$. It is proved in [AR$_1$] 4.32 that a category is locally finitely multipresentable iff it is sketchable by a (finite limit, coproduct)-sketch (i.e. a sketch with all diagrams in $\mathbf{L}$ finite and all diagrams in $\mathbf{C}$ discrete). We will now present an example showing that it is, in general, not sufficient to work with finite limits and finite coproducts:

EXAMPLE. (4) The following locally finitely multipresentable category is not sketchable by a (finite limit, finite coproduct) - sketch.

Objects: 0 and all primes $p > 2$.

Morphisms: (a) a unique morphism $0 \to p$ for each object p
(b) $\hom(p, p)$ is the cyclic group of order p
(c) no other morphisms except those in (a), (b).

This category has only trivial directed colimits, thus, each object is finitely presentable. It has connected limits: an equalizer of any pair of distinct morphisms in $\hom(p, p)$ is $0 \to p$, analogously with pullbacks. Thus, the category is locally finitely multipresentable.

Any (finite limit, finite coproduct)-sketchable category $\mathcal{K}$ an be axiomatized by a first-order theory. That is, there exists a many-sorted, finitary signature Σ and a theory T of the first-order logic of Σ-structures such that $\mathcal{K}$ is equivalent to the category of all T-models and Σ-homomorphisms – see [MP]. Thus, it is sufficient to prove that our category $\mathcal{K}$ cannot be axiomatized. In fact, suppose that E is an equivalence between $\mathcal{K}$ and the category $\mathrm{Mod}\, T$ of all models of T. It is well-known that $\mathrm{Mod}\, T$ is closed under ultraproducts in the category $\mathrm{Str}\, \Sigma$ of all Σ-structures. Let $\mathcal{U}$ be a free ultrafilter on the set of all objects of $\mathcal{K}$. Since the ultraproduct $\prod_{\mathcal{U}} E(p)$ lies in $\mathrm{Mod}\, T$, there exists an object q with $E(q)$ isomorphic to $\prod_{\mathcal{U}} E(p)$. To obtain the desired contradiction, we will show that the above ultraproduct has infinitely many endomorphisms, although $\hom(E(q), E(q)) \cong \hom(q, q)$ is finite. In fact, for each element f of the product $\prod_{p \in \mathcal{K}} \hom(p, p)$ we obtain an endomorphism f' of the ultraproduct, defined by $f'([x_p]) = [Ef_p(x_p)]$, and for two such elements f, g we have $f' = g'$ iff $\mathcal{U}$ contains the set $\{p;\ f_p = g_p\}$. It is sufficient to choose an infinite set $M \subseteq \prod \hom(p, p)$ such that for $f, q \in M$ with $f \neq q$ the set of all p's with $f_p = g_p$ is always finite, then $\{f';\ f \in M\}$ is an infinite set of endomorphisms of $\prod_{\mathcal{U}} E(p)$. This proves that $\mathcal{K}$ is not axiomatizable in first-order logic.

THEOREM 2. *Assuming the non-existence of measurable cardinals, each locally finitely multipresentable category can be sketched by a (finite limit, countable coproduct)-sketch.*

PROOF. Let $\mathcal{K}$ be a locally finitely multipresentable category. By [AR_1] 4.32 there exists a (finite limit, coproduct)-sketch $\mathscr{S}$ with $\mathcal{K}$ equivalent to $\mathrm{Mod}\, \mathscr{S}$. Let α be a cardinal such that each coproduct-specification of $\mathscr{S}$ has less than α objects. We will construct a (finite limit, countable coproduct)-sketch $\mathscr{S}^*$ with $\mathrm{Mod}\, \mathscr{S} \cong \mathrm{Mod}\, \mathscr{S}^*$.

Since no cardinal is measurable, for each cardinal β there exists a basic theory T_β which is one-sorted, has constants c_i ($i \in \beta$) in the signature, and has, up to isomorphism, a unique model B such that $(c_i)_B \neq (c_j)_B$ for $i \neq j$ and $\{(c_i)_B\}_{i \in \beta}$ is the underlying set of B. In fact, T_β is explicitely described in [AJMR]. By inspecting that description, it can be easily seen that all disjunctions used in all sentences of T_β are actually disjoint. This means that whenever $p \vee q$ was used we can write, instead, $(p \vee q) \wedge (p \wedge q \to \bot)$ and where $\bigvee_{n \in \omega} p_n$ was used, we can write $\bigvee_{n \in \omega} p_n \wedge \bigwedge_{n \neq m} (p_n \wedge p_m \to \bot)$.

Here $\perp$ denotes the formula *false*. It means that T_β uses only

(a) finite conjunctions
(b) finite quantifications

and

(c) finite or countable disjunctions which are disjoint.

We now apply the procedure of [MP] 3.2.5 associating with each basic theory a sketch $\mathscr{S}$ with Mod $\mathscr{S}$ equivalent to the category of all models of that theory. Let $\mathscr{S}_\beta$ be the sketch associated to T_β. Due to (a)–(c), $\mathscr{S}_\beta$ is a (finite limit, countable coproduct)-sketch. $\mathscr{S}_\beta$ contains the object A_β describing the unique sort, and each of the constants c_i gives us a morphism $c_i^\beta : 1_\beta \to A_\beta$ such that the (essentially unique) model of $\mathscr{S}_\beta$ has the above properties.

II. The rest of the proof is analogous to that of Theorem 12 of [AJMR]. Let $\mathcal{K}$ be a locally finitely multipresentable category. There exists a (finite limit, coproduct)-sketch $\mathscr{S}$ sketching $\mathcal{K}$, see [AR$_1$] 4.32. For each uncountable coproduct-specification

$$(B_i \xrightarrow{b_i} B)_{i\in\beta} \qquad (\beta \text{ a uncountable cardinal})$$

of $\mathscr{S}$ we take a copy of the above sketch $\mathscr{S}_\beta$ and we denote by $\bar{\mathscr{S}}$ the disjoint union of the sketch $\mathscr{S}$ and all these sketches $\mathscr{S}_\beta$. Let $\mathscr{S}^*$ be the sketch we obtain from $\bar{\mathscr{S}}$ by deleting all of the uncountable coproduct specifications and, instead, by adding to $\bar{\mathscr{S}}$

(a) a formal new morphism $f : B \to A_\beta$
(b) the following pullback-specifications for all $i \in \beta$

$$\begin{array}{ccc} B_i & \xrightarrow{\;b_i\;} & B \\ {\scriptstyle t_i}\downarrow & & \downarrow{\scriptstyle f} \\ 1_\alpha & \xrightarrow[\;c_i^\alpha\;]{} & A_\beta \end{array}$$

where $t_i = t_\beta \cdot f \cdot b_i$.

It is easy to see that the sketches $\mathscr{S}$,$\bar{\mathscr{S}}$, and $\mathscr{S}^*$ have equivalent categories of models. ∎

Remark. The statement of the Theorem 2 is actually logically equivalent to the non-existence of measurable cardinals. That is, assuming that a measurable cardinal λ exists, there are locally finitely multipresentable categories not sketchable by (finite limit, countable coproduct)-sketch. In fact, not sketchable by λ-*ary sketches*, i.e., sketches in which all diagrams in $\mathbf{L} \cup \mathbf{C}$ have sizes smaller than λ:

Example. (5) If λ is a measurable cardinal, the following category is locally finitely multipresentable, but it is not sketchable by a λ-ary sketch:

Objects: all ordinals $i < \lambda$.

Morphisms: (a) a unique morphism $0 \to i$ for each object i

(b) $\hom(i,i)$ is a group of cardinality $\aleph_i$ for all $i > 0$

(c) no other morphisms except those in (a), (b).

The proof of local finite multipresentability is analogous to that in Example (4). Also the proof that the category is not sketchable by a λ-ary sketch is quite analogous: by [MP], any λ-ary sketchable category can be axiomatized by infinitary first-order theory of the λ-ary logic $L_{\lambda,\lambda}$. Thus, if our category $\mathcal{K}$ would be sketchable by a λ-ary sketch, there would exist an equivalence E between $\mathcal{K}$ and $\mathrm{Mod}\,T$ for some first-order theory in a λ-ary signature Σ. Then $E(\mathcal{K})$ would be closed in $\mathrm{Mod}\,\Sigma$ under ultraproducts over all λ-complete ultrafilters. Since λ is measurable, there exists a nontrivial λ-complete ultrafilter $\mathcal{U}$ on the set of all objects of $\mathcal{K}$. The proof is then concluded by showing that the ultraproduct $\prod_{\mathcal{U}} E(i)$ has $\aleph_\lambda$ pairwise distinct endomorphisms (whereas $\hom(i,i)$ has cardinality $\aleph_i < \aleph_\lambda$ for each object i of $\mathcal{K}$).

Remark. The category $\mathcal{K}$ of Example (4) has the following remarkable property: we have seen above that

(a) $\mathcal{K}$ is sketchable by a (finite limit, coproduct)-sketch

and

(b) $\mathcal{K}$ is not sketchable by a (finite limit, finite coproduct)-sketch.

We now show that, nevertheless,

(c) $\mathcal{K}$ is sketchable by a (finite limit, finite colimit)-sketch.

In fact, as proved in Theorem 3 of [AJMR], for (c) it is sufficient to find an axiomatization of $\mathcal{K}$ by a *σ-coherent theory* in the logic $L_{\omega_1,\omega}$. A σ-coherent theory consists of sentences

$$(\forall x_1, \dots, x_n)(\varphi \to \psi)$$

where φ and ψ are formulas built from atomic formulas by finite conjunctions, countable disjunctions, and finite existential quantification. Now observe that $\mathcal{K}$ is equivalent to the category of all graphs (i.e. Σ-structures where Σ consists of one binary relation symbol R) isomorphic either to G_0, the empty graph, or G_p, the circuit of length p (with vertices $0, 1, \dots, p-1$ and edges $i \to i+1$ for all $i < p-1$ and $p-1 \to 0$) for all primes $p > 2$. The latter category has the following σ-coherent axiomatization in which $R^2(x,y)$ is a shorthand for $(\exists z)(R(x,z) \wedge R(z,y))$, analogously with $R^3, R^4, \dots$:

(1) $(\forall x)(\exists\,!y)R(x,y)$

(2) $(\forall y)(\exists\,!x)R(x,y)$

(3) $(\forall x)(R^{nm}(x,x) \to R^n(x) \vee R^m(x))$
where n, m are arbitrary natural numbers

(4) $(\forall x)(\exists y)((x = y) \to \bot)$

(5) $(\forall x, y) \bigvee_{n\in\omega} R^{2m}(x, y)$ $\qquad$ ($R^0(x, y)$ means $x = y$).

In fact, each G_n is a model of (1)–(5). Let $G \neq G_0$ be a model of (1)–(5). By (5), each pair of vertices can be connected by a path of even length. By (1) and (2), G is a circuit or an infinite path, but (5) excludes the latter and forces the length P of the circuit to be larger than 2. Finally, by (3) we know that p is a prime. Thus, $G \cong G_p$, and G_p is a model of (1)–(5).

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J. Adámek
Technical University Braunschweig
Postfach 3329
38023 Braunschweig, Germany

J. Rosický
Masaryk University
Janáčkovo nám. 2a, 662 95 Brno
Czech Republic

E-mail: adamek@iti.cs.tu-bs.de and rosicky@math.muni.cz

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