Theory and Applications of Categories, Vol. 2, No. 5, 1996, pp. 55–61.

FINITENESS OF A NON-ABELIAN TENSOR PRODUCT OF GROUPS

NICK INASSARIDZE
Transmitted by Ronald Brown

ABSTRACT. Some sufficient conditions for finiteness of a generalized non-abelian tensor product of groups are established extending Ellis' result for compatible actions.

The non-abelian tensor product of groups was introduced by Brown and Loday [2,3] following works of A.Lue [4] and R.K.Dennis [7]. It was defined for any groups A and B which act on themselves by conjugation (${}^x y = xyx^{-1}$) and each of which acts on the other such that the following *compatibility conditions* hold:

$$^{({}^a b)}a' = {}^a\left({}^b({}^{a^{-1}}a')\right), \qquad ^{({}^b a)}b' = {}^b\left({}^a({}^{b^{-1}}b')\right) \tag{1}$$

for all $a, a' \in A$ and $b, b' \in$B. These compatibility conditions are very important in the subsequent theory of the tensor product. In particular they play a crucial role in Ellis's proof [5] that the tensor product of finite groups is finite.

The definition of the non-abelian tensor product was generalized in [6] so as to deal with the case when the compatibility conditions (1) do not hold. The present paper is concerned solely with this generalized tensor product; we obtain conditions which are sufficient for its finiteness.

Henceforth, let A and B be groups with a chosen action of A on B and a chosen action of B on A. We assume that A and B act on themselves by conjugation. These actions yield, in an obvious way, actions of the free product $A * B$ on A and on B. We recall the following definition from [6].

1. DEFINITION. *The* non-abelian tensor product $A \otimes B$ *is the group generated by the symbols* $a \otimes b$, $(a \in A, b \in B)$ *subject to the relations*

$$\begin{aligned} aa' \otimes b &= ({}^a a' \otimes {}^a b)(a \otimes b) \\ a \otimes bb' &= (a \otimes b)({}^b a \otimes {}^b b') \\ (a \otimes b)(a' \otimes b') &= ({}^{[a,b]}a' \otimes {}^{[a,b]} b')(a \otimes b) \\ (a' \otimes b')(a \otimes b) &= (a \otimes b)({}^{[b,a]}a' \otimes {}^{[b,a]} b') \end{aligned}$$

The research described in this publication was made possible in part by Grant IFS MXH200 and by Grant INTAS 93-436. The author would also like to thank the referee and editor for helpful comments.

Received by the editors 29 January 1996 and, in revised form, 19 August 1996.

Published on 28 August 1996

1991 Mathematics Subject Classification : 18G50.

Key words and phrases: Non-abelian tensor product of groups,Comp-subgroup,Comp-pairs, compatibility resolution,half compatible actions.

for all $a, a' \in A$ *and* $b, b' \in B$*, where* $[a, b] = aba^{-1}b^{-1} \in A * B$.

2. Remark. Calculations in [3] imply that $A \otimes B$ coincides with the tensor product of Brown and Loday if conditions (1) are satisfied.

3. Definition. *The Comp*-subgroup of A with B*, denoted by* $Comp\ A(B)$*, is the normal subgroup of* A *generated by the elements*

$$^{(^a b)}a' \ \left(^{aba^{-1}}a'\right)^{-1}$$

where $a, a' \in A$, $b, b' \in B$.

Note that if A and B act on each other compatibly then $Comp\ A(B)$ and $Comp\ B(A)$ are trivial groups.

4. Definition. *It will be said that* A *acts on* B perfectly *if the action of* A *on* B *induces an action of* A *on* $Comp\ B(A)$*, and that* A*,* B *are* groups with perfect actions *if under these actions* A *acts on* B *perfectly and* B *acts on* A *perfectly.*

5. Definition. *Then the Comp*-pairs *of* (A, B) *are the pairs* $(Comp\ A(B), B)$ *and* $(A, Comp\ B(A))$ *of groups.*

The pair (A, B) gives rise to two Comp-pairs. If (A, B) are groups with perfect actions, then these Comp-pairs in turn give rise to four Comp-pairs. If the two "first stage" Comp-pairs are each groups with perfect actions, then the "second stage" Comp-pairs yield eight "third stage" Comp-pairs. And so on.

6. Definition. *The family of pairs thus obtained will be called the* compatibility resolution *of the pair* (A, B).

7. Definition. *It will be said that* A *and* B *act on each other* half compatibly *if for every pair of groups* (C, D) *of the compatibility resolution of* (A, B) *the actions are perfect and the following conditions hold:*
(i) $^h dd^{-1} \in Comp\ D(C)$, $^g cc^{-1} \in Comp\ C(D)$ *for each* $c \in C$, $d \in D$, $h \in Comp\ C(D), g \in Comp\ D(C)$*, and (ii) on some* n*-th stage,* $n \geq 0$*, of the compatibility resolution of* (A, B)*, every pair* (A_n, B_n) *is a pair of groups with compatible actions.*

Clearly if A and B act on each other compatibly (i.e. conditions (1) hold) then they act on each other half compatibly.

There exists an example of half compatible actions that are not compatible. Suppose we have finite groups A and B such that $[A, A]$ is abelian and $[A, A] \not\subset Z(A)$. We shall show that the actions of $A \times B$ on A by conjugation and of A on $A \times B$ trivially are half compatible. (It is shown in Example 10 below that these actions are not compatible.) In effect, since $CompA \times B(A) = 1$, the "first stage" *Comp*-pair $(CompA \times B(A), A)$ of the compatibility resolution of $(A \times B, A)$ is a pair with compatible actions. We have the following equality:

$$\begin{aligned}
&(a^{'},b^{'})(a_3,1)^{((a_1,1)(a,b))}(a_2,1)({}^{(a_1,1)(a,b)(a_1^{-1},1)}(a_2,1))^{-1}(a_3^{-1},1)(a^{'-1},b^{'-1})\\
&=(a^{'}a_3a^{'-1},1)^{((a^{'}a_1a^{'-1},1)(a^{'}aa^{'-1},b^{'}bb^{'-1}))}(a^{'}a_2a^{'-1},1)\\
&\qquad\qquad({}^{(a^{'}a_1a^{'-1},1)(a^{'}aa^{'-1},b^{'}bb^{'-1})(a^{'}a_1^{-1}a^{'-1},1)}(a^{'}a_2a^{'-1},1))^{-1}(a^{'}a_3^{-1}a^{'-1},1),
\end{aligned}$$

where $(a_3,1)^{((a_1,1)(a,b))}(a_2,1)({}^{(a_1,1)(a,b)(a_1^{-1},1)}(a_2,1))^{-1}(a_3^{-1},1)$ is a generator of $CompA(A\times B)$ as a subgroup of the group A. Hence the action of $A\times B$ on A induces an action of $A\times B$ on $CompA(A\times B)$.
Since

$$\begin{aligned}
&{}^{((a_1,1)(a,b))}(a_2,1)({}^{(a_1,1)(a,b)(a_1^{-1},1)}(a_2,1))^{-1}=(aa_2a^{-1}a_1aa_1^{-1}a_2^{-1}a_1a^{-1}a_1^{-1},1)\\
&=((aa_2a^{-1}a_2^{-1}a^{-1})(aa_2a_1a_2^{-1}a^{-1})(aa_2aa_2^{-1}a^{-1})(aa_2a_1^{-1}a_2^{-1}a^{-1})(aa_1a^{-1}a_1^{-1}),1),
\end{aligned}$$

a generator of $CompA(A\times B)$ as a subgroup of the group A is a product of two commutators of A and from the commutativity of $[A,A]$ we have that $CompA(A\times B)$ is abelian. Therefore the other "first stage" *Comp*-pair $(A\times B, CompA(A\times B))$ of the compatibility resolution of $(A\times B, A)$ is a pair with compatible actions. The final part of the half compatibility is easy to verify.

8. Theorem. *Let A and B be finite groups acting on each other half compatibly. Then $A\otimes B$ is finite.*

Proof. Since A and B act on each other half compatibly there exists $n\geq 1$ such that every pair (A_n,B_n) of the n-th stage of the compatibility resolution of (A,B) is a pair of groups with compatible actions. Consider an arbitrary pair (A_{n-1},B_{n-1}) of groups on the $(n-1)$'th stage of the compatibility resolution of (A,B) and consider the following exact sequences of groups:

$$1\longrightarrow CompA_{n-1}(B_{n-1})\longrightarrow A_{n-1}\longrightarrow A_{n-1}/CompA_{n-1}(B_{n-1})\longrightarrow 1,$$

$$1\longrightarrow CompB_{n-1}(A_{n-1})\longrightarrow B_{n-1}\longrightarrow B_{n-1}/CompB_{n-1}(A_{n-1})\longrightarrow 1,$$

where $A_{n-1}/Comp\ A_{n-1}(B_{n-1})$ and $B_{n-1}/Comp\ B_{n-1}(A_{n-1})$ act on each other by the induced actions. It is clear that these homomorphisms preserve the actions.

From [6, Theorem 1(b)] we have the exact sequence of groups

$$\begin{aligned}
&(A_{n-1}\otimes CompB_{n-1}(A_{n-1}))\times(CompA_{n-1}(B_{n-1})\otimes B_{n-1})\longrightarrow(A_{n-1}\otimes B_{n-1})\longrightarrow\\
&\qquad (A_{n-1}/CompA_{n-1}(B_{n-1}))\otimes(B_{n-1}/CompB_{n-1}(A_{n-1}))\longrightarrow 1, \quad (2)
\end{aligned}$$

where the first map is only a map of sets. Since the following are pairs of groups with compatible actions:

$(A_{n-1}/CompA_{n-1}(B_{n-1}), B_{n-1}/CompB_{n-1}(A_{n-1}))$
$(A_{n-1}, CompB_{n-1}(A_{n-1}))$ and
$(CompA_{n-1}(B_{n-1}), B_{n-1})$

we have from [5] that the associated tensor products

$$\begin{array}{l}(A_{n-1}/CompA_{n-1}(B_{n-1}))\otimes(B_{n-1}/CompB_{n-1}(A_{n-1}))\\ A_{n-1}\otimes CompB_{n-1}(A_{n-1})\\ CompA_{n-1}(B_{n-1})\otimes B_{n-1}\end{array}$$

are finite groups. From the exact sequence (2), $A_{n-1}\otimes B_{n-1}$ is also a finite group.

Since $A_{n-1}\otimes B_{n-1}$ is a finite group for any pair (A_{n-1}, B_{n-1}) on the $(n-1)$-th stage of the compatibility resolution of (A, B), using the descent method from the right exactness of the non-abelian tensor product [6, Theorem 1] one can prove the finiteness of $C\otimes D$ for any pair (C, D) of groups on i-th stage, $0\leq i<n-1$, of the compatibility resolution of (A, B) and therefore $A\otimes B$ is a finite group. The proof is complete. ∎

9. THEOREM. *Suppose that the action of B on A is trivial, that A and B are both finite, and that B is soluble. Then $A\otimes B$ is finite.*

PROOF. Let $B^{(0)}=B$, and $B^{(k)}=[B^{(k-1)}, B^{(k-1)}]$ for $k\geq 1$. Then $B^{(n)}$ is abelian for some $n\geq 1$.

Consider the short exact sequence of groups

$$1\longrightarrow B^{(n)}\longrightarrow B^{(n-1)}\longrightarrow B^{(n-1)}/B^{(n)}\longrightarrow 1.$$

Since A acts on $B^{(i)}$ for any i and these homomorphisms preserve the actions, from [6, Theorem 1(a)] we have the following exact sequence of groups

$$A\otimes B^{(n)}\longrightarrow A\otimes B^{(n-1)}\longrightarrow A\otimes(B^{(n-1)}/B^{(n)})\longrightarrow 1. \tag{3}$$

Since $B^{(n)}$ and $B^{(n-1)}/B^{(n)}$ are abelian, the pairs A, $B^{(n)}$ and A, $B^{(n-1)}/B^{(n)}$ are pairs with compatible actions and so from [5] their tensor products $A\otimes B^{(n)}$ and $A\otimes(B^{(n-1)}/B^{(n)})$ are finite groups. It follows from (3) that $A\otimes B^{(n-1)}$ is also a finite group.

Using the descent method, from the right exactness of the non-abelian tensor product [6, Theorem 1] one can prove the finiteness of $A\otimes B^{(i)}$, $1\leq i<n-1$.

Finally consider the following short exact sequence of groups:

$$1\longrightarrow[B, B]\longrightarrow B\longrightarrow B^{ab}\longrightarrow 1.$$

From [6, Theorem 1(a)] we have the exact sequence of groups

$$A\otimes[B, B]\longrightarrow A\otimes B\longrightarrow A\otimes B^{ab}\longrightarrow 1. \tag{4}$$

Therefore from (4) $A\otimes B$ is finite. The proof is complete. ∎

10. EXAMPLE. Let A and B be finite groups. Let $[A, A]$ be abelian and $[A, A]\not\subset Z(A)$. Then the actions of $A\times B$ on A by conjugation and of A on $A\times B$ trivially do not satisfy the compatibility conditions (1).

In fact, since $[A,A] \not\subset Z(A)$, there exists $a, a', a'' \in A$ such that

$$a'aa'^{-1}a^{-1}a'' \neq a''a'aa'^{-1}a^{-1}.$$

Then we have

$$^{((a',1)(a^{-1},b))}(a'',1) \neq {}^{(a',1)(a^{-1},b)(a'^{-1},1)}(a'',1).$$

So these actions do not satisfy the compatibility conditions (1).

From Theorem 9 $(A \times B) \otimes A$ is finite.

Now we will show the existence of such a finite group A. Suppose $M = \mathbf{Z}_p^+$ (additive group) and $N = \mathbf{Z}_p^\times \backslash \{0\}$ (multiplicative group) for any prime $p > 2$. Assume N acts on M by multiplication of $\mathbf{Z}_p$ i.e. $^{[n]}[m] = [nm]$ for all $[n] \neq 0$, $[m] \in \mathbf{Z}_p$. Let us consider $M \bowtie N$ (semi-direct product of M and N). Then the commutant $[M \bowtie N, M \bowtie N] = M$ and therefore is abelian. In fact

$$\begin{aligned}
&([m],[n])([m'],[n'])(-[n]^{-1}[m],[n]^{-1})(-[n']^{-1}[m'],[n']^{-1})\\
&= ([m+nm'],[nn'])\cdot(-[n]^{-1}[m],[n]^{-1})(-[n']^{-1}[m'],[n']^{-1})\\
&= ([m+nm'-n'm],[n'])(-[n']^{-1}[m'],[n']^{-1})\\
&= ([m+nm'-n'm-m'],1)
\end{aligned}$$

i.e. $[M \bowtie N, M \bowtie N] \subset M$.

Next

$$\begin{aligned}
[([m],[n]),([m],[n-1])] &= ([m+nm-nm+m-m],1)\\
&= ([m],1)
\end{aligned}$$

for $p \nmid n, n-1$ and any $[m] \in M$. So $M \subset [M \bowtie N, M \bowtie N]$.

Now we have to show $[M \bowtie N, M \bowtie N] \not\subset Z(M \bowtie N)$. Let $x \in [M \bowtie N, M \bowtie N]$, i.e. $x = ([m],1)$ and suppose that $p \nmid m$. Then

$$([m],1)([m'],[n']) \neq ([m'],[n'])([m],1)$$

for some $m', n' \in \mathbf{Z}$ where $p \nmid (n'-1)$.

Thus $M \bowtie N$ is an example of the above mentioned finite group A and therefore $\{(M \bowtie N) \times B\} \otimes (M \bowtie N)$ is finite for any finite group B.

11. Definition. *Let A and B be groups and A acts on B. Then $[A,B]$ is a normal subgroup of B generated by the elements $^a b b^{-1}$ for all $a \in A$, $b \in B$, and we can define*

$$[A,B]^n = [A,[A,B]^{n-1}], n > 1,$$

since the action of A on B induces the action of A on $[A,B]$.

12. Theorem. *Let A and B be finite groups. Suppose B acts on A trivially. If $[A,B]^n$ is abelian for some $n \geq 1$ then $A \otimes B$ is finite.*

PROOF. Consider the exact sequence of groups

$$1 \longrightarrow [A,B]^n \longrightarrow [A,B]^{n-1} \longrightarrow [A,B]^{n-1}/[A,B]^n \longrightarrow 1$$

Clearly these homomorphisms preserve the actions and from [6, Theorem 1(a)] we have the exact sequence of groups

$$A \otimes [A,B]^n \longrightarrow A \otimes [A,B]^{n-1} \longrightarrow A \otimes [A,B]^{n-1}/[A,B]^n \longrightarrow 1 \qquad (5)$$

Since $[A,B]^n$ is abelian, the induced actions of A and $[A,B]^n$ satisfy the compatibility conditions (1) and from [5] $A \otimes [A,B]^n$ is a finite group. From the construction of $[A,B]^n$ the induced action of A on $[A,B]^{n-1}/[A,B]^n$ is trivial. Since the action of $[A,B]^{n-1}/[A,B]^n$ on A is also trivial, from [1] we have $A \otimes [A,B]^{n-1}/[A,B]^n \approx A^{ab} \otimes ([A,B]^{n-1}/[A,B]^n)^{ab}$ and therefore is a finite group.

From (5), $A \otimes [A,B]^{n-1}$ is also a finite group.

Using the descent method from the right exactness of the non-abelian tensor product [6, Theorem 1] one can prove the finiteness of $A \otimes [A,B]^i$ for $1 \leq i < n-1$.

Finally consider the following short exact sequence of groups

$$1 \longrightarrow [A,B] \longrightarrow B \longrightarrow B/[A,B] \longrightarrow 1.$$

From [6, Theorem 1(a)] we have the exact sequence of groups

$$A \otimes [A,B] \longrightarrow A \otimes B \longrightarrow A \otimes B/[A,B] \longrightarrow 1.$$

By the same reasons as above $A \otimes [A,B]$ and $A \otimes B/[A,B]$ are finite groups and therefore $A \otimes B$ is also a finite group. The proof is complete. ∎

Note that Example 10 is available as an example for Theorem 12.

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A.Razmadze Mathematical Institute,
Georgian Academy of Sciences,
M.Alexidze St. 1,
Tbilisi 380093. Georgia

Email: `inas@imath.acnet.ge`

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