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# GLUEING ANALYSIS FOR COMPLEMENTED SUBTOPOSES

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Transmitted by S. Niefield

ABSTRACT. We prove how any (elementary) topos may be reconstructed from the data of two complemented subtoposes together with a pair of left exact "glueing functors". This generalizes the classical glueing theorem for toposes, which deals with the special case of an open subtopos and its closed complement.

Our glueing analysis applies in a particularly simple form to a locally closed subtopos and its complement, and one of the important properties (prolongation by zero for abelian groups) can be succinctly described in terms of it.

## 1. Double Glueing

Recall that a subtopos $\underline{H}$ of a topos $\underline{M}$ is given by a geometric morphism $h : \underline{H} \to \underline{M}$ such that the direct image functor $h_* : \underline{H} \to \underline{M}$ is full and faithful; or it may be given by a Lawvere-Tierney topology (nucleus) on $\Omega_{\underline{M}}$. (See e.g. [1] 4.IV.9 or [3] 4.1.) Subtoposes of $\underline{M}$ form a complete lattice, in fact a coframe. It therefore makes sense to ask whether $\underline{H} \hookrightarrow \underline{M}$ has a complement, meaning a subtopos $k : \underline{K} \hookrightarrow \underline{M}$ such that $\underline{H} \cap \underline{K} = 0$, $\underline{H} \cup \underline{K} = 1 (= \underline{M})$ in this lattice; as in any distributive lattice, such a complement is uniquely determined (up to equivalence of subtoposes) and one says then that $\underline{H}$ is a *complemented* subtopos.

If $h : \underline{H} \to \underline{M}$ is an open subtopos, i.e. of the form $\Pi_i : \underline{M}/U \to \underline{M}$ where $i : U \to 1_{\underline{M}}$ is a subobject of the terminal object of $\underline{M}$, then it has a complement $k : \underline{K} \to \underline{M}$, and such subtoposes are called *closed*; this is classical, see [1] IV.9.2-9.3. Furthermore, in this case $\underline{M}$ can be reconstructed from $\underline{H}$, $\underline{K}$, and the "glueing" or "fringe" functor $k^* \circ h_* : \underline{H} \to \underline{K}$, see [1] IV.9.5 (for the case of Grothendieck toposes) and [3] Theorem 4.25 for the case of elementary toposes ("Artin-Wraith glueing"). The latter case may be derived from the theory of left exact comonads on toposes (cf. e.g. [3] Theorem 2.32); this will also be our main tool.

We shall consider two arbitrary complementary subtoposes $\underline{H}$ and $\underline{K}$ of an elementary topos $\underline{M}$, with inclusions $h$ and $k$, respectively ($h$ given by its direct image functor $h_*$ and its left adjoint $h^*$, and similarly for $\underline{K}, k$).

Unlike the classical (Artin-Wraith) case where $\underline{H}$ and $\underline{K}$ are an open and a closed

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subtopos, respectively, and where only one fringe functor is used, we shall use *two* fringe functors to reconstruct $\underline{M}$ from $\underline{H}$ and $\underline{K}$; pictorially, we need to "put glue on *both* the items".

1.1. THEOREM. *Let $f : \underline{H} \to \underline{K}$ and $g : \underline{K} \to \underline{H}$ be left exact functors between (elementary) toposes, together with natural transformations $\eta : id_{\underline{H}} \to g \circ f$ and $\kappa : id_{\underline{K}} \to f \circ g$, satisfying the compatibility conditions that for all $H \in \underline{H}$ and $K \in \underline{K}$*

$$\eta_{gK} = g(\kappa_K) \text{ and } \kappa_{fH} = f(\eta_H), \tag{1}$$

*such that*

$$\eta_H \text{ an isomorphism implies } H = 1 \tag{2}$$

*and*

$$\kappa_K \text{ an isomorphism implies } K = 1.$$

*Then there exists a topos $\underline{M}$ in which $\underline{H}$ and $\underline{K}$ appear as complementary subtoposes, via geometric inclusion morphisms $h : \underline{H} \to \underline{M}$ and $k : \underline{K} \to \underline{M}$, and such that $f = k^* h_*$, $g = h^* k_*$, and such that $\eta$ and $\kappa$ arise out of the front adjunction for the adjoint pairs $k^* \dashv k_*$ and $h^* \dashv h_*$ respectively, in the evident way.*

*Furthermore, every topos $\underline{M}$ with a pair of complementary subtoposes arises this way.*

PROOF. Out of the data, we shall construct a left exact comonad $(G, \epsilon, \psi)$ on the product category $\underline{H} \times \underline{K}$. The functor $G$ takes $(H, K) \in \underline{H} \times \underline{K}$ to $(H \times gK, K \times fH)$ (and is similarly defined on maps). The transformation $\epsilon$ associates to the object $(H, K)$ the pair of projections $H \times gK \to H, K \times fH \to K$, and $\psi : G \to G \circ G$ associates similarly to $(H, K)$ the map

$$(H \times gK, K \times fH) \to (H \times gK \times g(K \times fH), K \times fH \times f(H \times gK))$$

described as follows. It is a pair of maps; we only describe the first of them, the second can be deduced by symmetry. So we want to describe a map

$$H \times gK \to H \times gK \times gK \times gfH$$

(utilizing for the codomain that $g$ preserves binary products), and this map, we describe in "elementwise" terms, as if we were dealing with sets, as follows: $(h, k) \mapsto (h, k, k, \eta_H(h))$ for $h \in H, k \in gK$. Keeping track of the identifications (in the style of $\eta_{H \times gK} = \eta_H \times \eta_{gK}$ arising from the fact that the functors $g$ and $f$ preserve products), one sees with some straightforward labour, using (1), that the counit- and coassociative law holds; and the left exactness of $G$ follows from that of $f$ and $g$. From the general theory of left exact monads on elementary toposes, cf. e.g. [3] Theorem 2.32, we get a topos of coalgebras $(\underline{H} \times \underline{K})^G$ which we call $\underline{M}$. Seen from the viewpoint of the category of geometric morphisms, $\underline{H} \times \underline{K}$ is the *co*product $\underline{H} + \underline{K}$, and the geometric surjection from this coproduct to the topos $\underline{M}$ of coalgebras already proves that $\underline{H}$ and $\underline{K}$ together cover $\underline{M}$, via the composites $\underline{H} \to \underline{H} + \underline{K} \to \underline{M}$, $\underline{K} \to \underline{H} + \underline{K} \to \underline{M}$. We describe these functors explicitly below. We first have a more transparent way of seeing a coalgebra as "two items with two sorts of glue":

A $G$-costructure $\xi : (H, K) \to (H \times gK, K \times fH)$ on $(H, K)$ is a pair of maps $\xi_1 : H \to H \times gK$ and $\xi_2 : K \to K \times fH$, with certain properties: By the counit law for $\xi$, one immediately sees that $\xi_1$ must be of form $< id_H, x >$, where $x : H \to gK$, and similarly $\xi_2 =< id_K, y >$, where $y : K \to fH$. We think of $x$ and $y$ as the two kinds of glue. The coassociative law for the costructure $\xi$ will hold exactly when $x$ and $y$ are compatible in the sense that

$$f(x) \circ y = \eta_H \text{ and } g(y) \circ x = \kappa_K.$$

If $g = 1$, the data $x$ is vacuous, so the only glueing data is the $y : K \to fH$ which is the usual data for glueing an object $K$ of the closed subtopos to an object $H$ of the open subtopos, to get an object of the glued topos, by Artin-Wraith Glueing, cf. [3] p 112.

We can now make explicit the "inclusion" functor $\underline{H} \to \underline{M}$ ($\underline{M}$ = the topos of coalgebras), and similarly for $\underline{K}$. An object $H$ of $\underline{H}$ is sent to $(H, fH) \in \underline{H} \times \underline{K}$, equipped with the costructure defined by the "glue" $x : H \to gfH$ equals $\eta_H$, and $y : fH \to fH$ equals the identity map of $fH$; the left adjoint to the inclusion functor takes $(H, K), \xi$ to $H$ (from which also the fact that the described $\underline{H} \to \underline{M}$ is in fact an inclusion follows). Similarly for the other inclusion.

The last assumption in the Theorem is only used to ensure that the two subtoposes $\underline{H}$ and $\underline{K}$ of $\underline{M}$ are really disjoint; expressed in terms of categories, this is to say that the intersection of the two subcategories consists of the terminal object only (it is known that meet of two subtoposes corresponds to category theoretic intersection, cf. e.g. [3], Exercise 3.9). So assume $(H, fH)$, with glueing data $(\eta_H, id_{fH})$, is isomorphic to an object of the form $(gK, K)$, with glueing data $(id_{gK}, \kappa_K)$. This means that there are isomorphisms $a : H \to gK$ in $\underline{H}$ and $b : fH \to K$ in $\underline{K}$, making two squares commute, one of which is

$$\begin{tikzcd} H \arrow[r, "\eta_H"] \arrow[d, "a"'] & gfH \arrow[d, "g(b)"] \\ gK \arrow[r, "id_{gK}"'] & gK. \end{tikzcd}$$

Since the vertical maps are isomorphisms, it follows that $\eta_H$ is. From (2) then follows that $H \cong 1$.

Finally let us prove that any two complementary subtoposes of a topos arise this way. First note that for any pair of geometric morphisms with common codomain $h : \underline{H} \to \underline{M}$ and $k : \underline{K} \to \underline{M}$, we get, using the universal property of $\underline{H} \times \underline{K}$ as a topos theoretic coproduct, a geometric morphism $p : \underline{H} \times \underline{K} \to \underline{M}$ (with inverse image $M \mapsto (h^*M, k^*M)$). The adjoint pair $p^* \dashv p_*$ gives rise to a left exact comonad $G$ on $\underline{H} \times \underline{K}$, and a comparison functor $\underline{M} \to (\underline{H} \times \underline{K})^G$ (which itself is the inverse image of a geometric morphism; in fact, it is the inclusion part of the surjection/inclusion factorization of $p$, cf. [3] Remark 4.16.). The functor part of the comonad $G$ is given by

$$(H, K) \mapsto (h^*(h_*H \times k_*K), k^*(h_*H \times k_*K)),$$

and this object is isomorphic to $(H \times h^*k_*K, K \times k^*h_*H)$ if $h$ and $k$ are inclusions. So in this case, we have a comonad of the form considered in the Theorem, with $g = h^*k_*$, $f = k^*h_*$; $\eta$ and $\kappa$ are now derived as components of the comultiplication $\psi$ for the comonad, and the compatibility laws (1) are derived from the coassociativity for $\psi$. (Explicitly, $\eta_H$ is the composite

$$H \xrightarrow{\cong} h^*h_*H \xrightarrow{h^*(\text{front})} h^*k_*k^*h_*H$$

where front denotes the unit (front adjunction) for $k^* \dashv k_*$. And $\kappa$ is derived similarly from the front adjunction for the pair $h^* \dashv h_*$.) Because inverse image functors of surjective geometric morphisms are precisely the (left exact) comonadic functors, it follows that the comparison $\underline{M} \to (\underline{H} \times \underline{K})^G$ is an equivalence if and only if the join of $\underline{H} \to \underline{M}$ and $\underline{K} \to \underline{M}$ is all of $\underline{M}$.

Finally, if $\underline{H}$ and $\underline{K}$ are disjoint subtoposes, and $\eta_H : H \to h^*k_*k^*h_*H = gfH$ is an isomorphism, we want to deduce that $H$ is the terminal object. It suffices to prove $h_*H \cong k_*fH$, since the only object in the intersection of the two subcategories is the terminal object. Since $h$ and $k$ are jointly surjective, $h^*$ and $k^*$ are jointly conservative, so it suffices to see that

$$h^*h_*H \cong h^*k_*fH \text{ and } k^*h_*H \cong k^*k_*fH.$$

Since $h$ is an inclusion, the first half of the statement is equivalent to $H \cong h^*k_*fH = gfH$, which follows from the assumption on $\eta_H$. Since $k$ is an inclusion, the second half is equivalent to $k^*h_*H \cong fH$ which is true by the definition of $f$. This proves the Theorem.

■

We finish this section with a lemma, to be used in the next section, but which is of general topos theoretic character.

Let $\underline{H}$ and $\underline{K}$ be complementary subtoposes of an elementary topos $\underline{M}$, with inclusions $h : \underline{H} \hookrightarrow \underline{M}$ and $k : \underline{K} \hookrightarrow \underline{M}$, respectively. We further assume that $\underline{H}$ is dense in $\underline{M}$, meaning that if $Z \in \underline{M}$ has $h^*(Z) = \emptyset$ in $\underline{H}$, then $Z = \emptyset$.

1.2. Lemma. *With the above assumptions, if there exists a $Y \in \underline{M}$ with $h^*Y = 1 + 1$ and $k^*Y = 1$, then $H$ is an open subtopos.*

Proof. It is standard that there exist subobjects $U'$ and $U''$ of $Y$ which by $h^*$ go to $1'$ and $1''$ (these denoting the two copies of 1 in $1 + 1$). Since the meet of these two copies is $\emptyset$ and $h^*$ commutes with meets, it follows that $h^*(U' \cap U'') = \emptyset$, and therefore, by density of $h$, that $U' \cap U'' = \emptyset$. Recall that the *support* $supp(X)$ of an object $X$ in a topos is the image of $X$ under the unique map $X \to 1$. We consider the subobject $U$ of $1 = 1_{\underline{M}}$ given by $U = supp(U') \cap supp(U'')$. We have $h^*(supp(U')) = supp(h^*(U')) = supp(1') = 1_{\underline{H}}$ and similarly $h^*(supp(U'')) = 1_{\underline{H}}$, hence $h^*(supp(U') \cap supp(U'')) = 1_{\underline{H}}$ so that the subtopos $\underline{H}$ is contained in $\underline{M}/U$. We prove the converse inclusion $\underline{H} \supseteq \underline{M}/U$ by showing that $\underline{K} \cap (\underline{M}/U) = \emptyset$, i.e. $k^*U = \emptyset$. Now $k^*U = k^*supp(U') \cap k^*supp(U'')$. Of course in general a construction like $k^* \circ supp$ does not preserve finite meets of subobjects, but it is easy to see that it *does* so for subobjects of any object $Y$ with $k^*Y = 1_{\underline{K}}$. In fact, $k^* \circ supp = supp \circ k^*$, where the map *supp* on the right hand side is the support map for subobjects of $k^*Y$, and this support map is an isomorphism if $k^*Y = 1_{\underline{K}}$.

We conclude that $k^*U = \emptyset$, so $\underline{H} \supseteq \underline{M}/U$, hence $\underline{K}$ is equivalent to the open subtopos $\underline{M}/U$, proving the lemma. ∎

## 2. Locally closed subtoposes

Among subtoposes which always have complements are the locally closed ones. This notion was defined in SGA4, [1] IV.9.4.9: a subtopos of $\underline{M}$ is locally closed if it is the meet of an open and a closed one. There are several equivalent formulations of the notion, and they are patterned over the similar formulations for the case of topological spaces, or for the case of locales. In fact, the localic case is a special case of the topos theoretic one, by taking $\underline{M}$ to be the topos of sheaves on a locale. (If one does internal locale theory in a topos, then the topos theoretic case is identical to the localic one; for, a subtopos of a topos $\underline{M}$ is given by a sublocale of the terminal internal locale $1_{\underline{M}}$ in $\underline{M}$.)

Common to all three cases is the fact that any subtopos (-locale, -space) contains a maximal open one, and is contained in a minimal closed one. The ensuing operations of taking interior and closure are relative to the ambient topos/locale/space $\underline{M}$, but are preserved by intersection with any subtopos (sublocale,...) $\underline{M}'$; any open subtopos (sublocale ...) of $\underline{M}'$ comes about in this way from an open one in $\underline{M}$, and similarly for closed subtoposes. We write $cl(\underline{H})$ for the closure of the subtopos (sublocale, ..) $\underline{H}$ inside $\underline{M}$; this will always mean: closure with respect to the *maximal* topos (locale, ..) under consideration - which is usually denoted $\underline{M}$ or $M$. Note that if $M$ is a locale and $\underline{M} = sh(M)$ the topos of sheaves on it, any subtopos of $\underline{M}$ comes from a sublocale of $M$, and the notions of open, closed, closure, etc. are preserved by this bijective correspondence between subtoposes and sublocales.

For the topos theoretic case, which is our main concern, all these notions appear in [1] IV 9.4.8; we shall elaborate on their description of "locally closed". Further properties of locally closed sublocales and subtoposes were studied by Niefield [6]. In particular, she characterized locally closed inclusions as being the exponentiable inclusions in the appropriate slice category.

Although the following is well known, it is included for the sake of completeness.

2.1. Proposition. *Let $\underline{H} \subseteq \underline{M}$ be a subtopos. Then t.f.a.e.*

*1) $\underline{H} = U \cap F$, where $U \subseteq \underline{M}$ is an open subtopos, and $F \subseteq \underline{M}$ is a closed one;*

*2) $\underline{H}$ is open in its closure $cl(\underline{H})$;*

*3) The inclusion $\underline{H} \hookrightarrow \underline{M}$ may be factored into an open followed by a closed inclusion;*

*4) The inclusion $\underline{H} \hookrightarrow \underline{M}$ may be factored into a closed followed by an open inclusion.*

**Note.** We note that if 1) holds, then we may always replace $F$ by a closed subtopos $F'$ with the property that $\neg F' \subseteq U$ in the lattice of subtoposes of $\underline{M}$ (in which any open, or closed, subtopos is indeed complemented). For, we may simply take $F' = F \cup \neg U$.

PROOF. Assume 1). Since $U$ is open, it is open in $cl(U)$, hence $U \cap F$ is open in $cl(U) \cap F$. But since $cl(U)$ and $F$ are closed, then so is $cl(U) \cap F$, and hence $cl(U) \cap F = cl(U \cap F)$. Therefore $U \cap F$ is open in $cl(U \cap F)$, so 2) holds. Assume 2). Then the factorization $\underline{H} \subseteq cl(\underline{H}) \subseteq \underline{M}$ is a factorization into an open followed by a closed inclusion, so 3) holds. Assume 3), so we have a factorization $\underline{H} \subseteq F \subseteq \underline{M}$ with $F$ closed in $\underline{M}$, and $\underline{H} \subseteq F$ open. The latter means that there is an open $U$ in $\underline{M}$ with $U \cap F = \underline{H}$. So 1) follows. Finally, if 1) holds, $\underline{H} = U \cap F \subseteq U \subseteq \underline{M}$ is a factorization by a closed followed by an open inclusion. And conversely, if $\underline{H} \subseteq U \subseteq \underline{M}$ is such a factorization, then $\underline{H}$, being closed in $U$, is of form $U \cap F$ for some closed $F \subseteq \underline{M}$. ∎

It is clear that a finite meet of locally closed subtoposes is again locally closed. Since closed, as well as open, subtoposes are complemented in the lattice $Sub(\underline{M})$ of subtoposes of $\underline{M}$, it follows that each locally closed subtopos is complemented.

The complement of a locally closed subtopos need not be locally closed. Also, the join of an open and a closed subtopos need not be locally closed. In fact, since open, or closed, sublocales of spatial locales are spatial, then so are locally closed sublocales; and for spaces, it is easy to construct examples in $\mathbf{R}^2$, say (the strictly positive $x$-axis is locally closed in $\mathbf{R}^2$, but its complement is dense, but not open in $\mathbf{R}^2$).

Using the characterizations 3) and 4) of the Proposition, we also immediately conclude that the composite of two locally closed inclusions is locally closed.

For sublocales of a locale, it is possible to get some more explicit algebraic formulations, namely by passing to the frame viewpoint. We use the notation that $O(M)$ is the frame corresponding to the locale $M$. Recall that a nucleus $j$ on a frame $O(M)$ is *open* if it is of the form $o(z) = z \to -$ for some $z \in O(M)$, and that it is *closed* if it is of the form $c(x) = x \vee -$ for some $x \in O(M)$. The open sublocale given by $z$, i.e. by the nucleus $z \to -$, we shall denote by $Z$, or even by $z$; the closed one given by $z$, we denote $\neg z$ or $\neg Z$, since $z$ and $\neg z$ are complements in the lattice $Sub(M)$ of sublocales of $M$.

If $x \leq z$, an elementary calculation with Heyting algebras (cf. [6] or [5]) shows that the composite $o(z) \circ c(x)$, i.e. the operator

$$y \mapsto z \to (x \vee y) \tag{3}$$

is idempotent, hence a nucleus, and hence the join of the nuclei $o(z)$ and $c(x)$ in the lattice of nuclei on $O(M)$. Equivalently, they represent the meet of the open sublocale $Z \subseteq M$ corresponding to $z$ and the closed sublocale $\neg x$ given by the nucleus $x \vee -$ (and which is the complement of the open sublocale given by $x$).

Since by the note after Proposition 1, every locally closed sublocale is of the form $z \cap \neg x$ with $x, z$ open and $x \subseteq z$ (i.e. $x \leq z$), it follows that the nuclei corresponding to locally closed sublocales are those which are given by a nucleus of the form (3) (with $x \leq z$). - We may also denote $z \cap \neg x$ by $z - x$.

The fact that any locally closed sublocale is given by a nucleus of the form (3) (with $x \leq z$) leads to the following "interval" characterization of frame quotient maps that come about from locally closed $j$; recall that if $x \leq z$ in a frame $O(M)$, then the *interval* $[x, z] \subseteq O(M)$, given as $\{y \mid x \leq y \leq z\}$, is also a frame.

2.2. Proposition. *Assume $j$ is a locally closed nucleus on $O(M)$, witnessed by $x, z$ with $x \leq z$. Then the frame quotient $O(M) \to O(M)_j$ is isomorphic to the map*

$$h^* : O(M) \to [x, z]$$

*given by*

$$y \mapsto z \wedge (x \vee y)(= x \vee (z \wedge y)).$$

*Conversely, a map $O(M) \to [x, z]$ of form $y \mapsto z \wedge (x \vee y)$ (where $x \leq z$) is quotient map for a locally closed nucleus.*

Proof. The exhibited map has a right adjoint $h_*$ given by $u \mapsto (z \to u)$, as an elementary calculation shows, and the composite endomap on $O(M)$ is

$$y \mapsto z \to (z \wedge (x \vee y)) = (z \to z) \wedge (z \to (x \vee y)) = z \to (x \vee y),$$

thus equals the given locally closed nucleus. The other composite endomap: $[x, z] \to [x, z]$ is seen to be the identity. The last statement is now clear: the locally closed nucleus is $z \to (x \vee -)$. ∎

We note that the inclusion $[x, z] \subseteq O(M)$ is not in general equal to $h_*$ (unless $z = 1$, i.e. in the case of a closed nucleus $j$), nor is it left adjoint to $h^*$ in general (unless $x = 0$, i.e. in the case of an open nucleus). However, the inclusion $[x, z] \subseteq O(M)$ does preserve all inhabited suprema, and all inhabited infima. It is also easy to see by elementary calculation that it satisfies certain Frobenius conditions for $h_\sharp$, (4), (5), and (6) below, as stated in the following result (essentially announced in [7]).

2.3. Proposition. *In order that a nucleus $j = h_* \circ h^*$ corresponds to a locally closed sublocale $H \subseteq M$, it is necessary and sufficient that there exists a map $h_\sharp : O(H) \to O(M)$ satisfying*

$$h^* h_\sharp 0 = 0, h^* h_\sharp 1 = 1, \tag{4}$$

*as well as one of the following two conditions*

$$h_\sharp(u \wedge h^* v) = h_\sharp u \wedge v \text{ for } v \geq h_\sharp 0 \tag{5}$$

$$h_\sharp(u \vee h^* v) = h_\sharp u \vee v \text{ for } v \leq h_\sharp 1. \tag{6}$$

*If such $h_\sharp$ exists, one may always find an $h_\sharp$ which furthermore preserves all inhabited (in particular filtered) suprema and infima, and also satisfies* both *of the conditions (5) and (6).*

PROOF. Assume such $h_\sharp$ exists. Take $x = h_\sharp 0$, $z = h_\sharp 1$, then $x \leq z$ since $h_\sharp$ is order preserving. Consider the sublocale corresponding to the nucleus $y \mapsto z \to (x \vee y)$, as in (3), i.e. defining the locally closed sublocale $z - x = z \cap \neg x$. To see that $H \subseteq z - x$, it suffices to see that $h^*z = 1$ and $h^*x = 0$ which is what the conditions (4) say. Also, since $h_\sharp$ is order preserving, it follows that $h_\sharp : O(H) \to O(M)$ factors through $[x, z] \subseteq O(M)$. Assume now that (5) holds. Then for $v \in [x, z]$, we have the second equality sign in

$$h_\sharp h^* v = h_\sharp(0_H \vee h^* v) = h_\sharp 0_H \vee v = x \vee v = v.$$

It follows that $h^*$ restricted to $[x, z]$ is injective; but it is also surjective, since $h : H \to M$ factors through the sublocale $z - x$. This proves that $h : H \to M$ as a sublocale is isomorphic to $z - x$. If $h_\sharp$ now does not already satisfy (6) and preservation of inhabited sup and inf, we just replace it by the inclusion $[x, z] \subseteq O(M)$. - If it is (6) rather than (5) that is assumed to hold, the proof is similar, by taking $u$ to be 1.

We shall include also some calculations concerning the complement of a locally closed sublocale. If the locally closed sublocale of $M$ is given by $x \leq z \in O(M)$ as above, the nucleus corresponding to it is the join of the nuclei $o(z)$ and $c(x)$. Hence the complement is the meet of the nuclei $c(z)$ and $o(x)$. But meets of nuclei are computed pointwise, so that this nucleus is given as the operator

$$y \mapsto (z \vee y) \wedge (x \to y). \tag{7}$$

(The fixpoints of this nucleus may, by easy calculation, be seen to be those $y$ which satisfy $x \to y = z \to y$.)

We now return to considerations of subtoposes of topos $\underline{H}$ or $\underline{M}$, but keep some of the frame/locale theoretic notation. Thus, if $w \hookrightarrow 1_{\underline{H}}$ is a subobject of the terminal object of $\underline{H}$, we denote the open subtopos it defines, by $w$ rather than $\underline{H}/w$; its closed complement is denoted $\neg w \hookrightarrow \underline{H}$. If $h : \underline{H} \hookrightarrow \underline{M}$ is a geometric morphism, $h_*(w) \hookrightarrow 1_{\underline{M}}$, and thus $h_*w$ denotes a certain open suptopos of $\underline{M}$. Now consider the composite

$$\neg w \hookrightarrow \underline{H} \overset{h}{\hookrightarrow} \underline{M};$$

$\neg w$ is closed in $\underline{H}$, but not necessarily in $\underline{M}$; however, its closure may be described explicitly as $\neg(h_*w)$. This may be summarized in the following principle:
*if $w \hookrightarrow 1_{HH}$ and $F$ are complements inside $\underline{H}$, then $h_*w$ and $cl(F)$ are complements inside $\underline{M}$.*

Let $\underline{H}$ and $\underline{K}$ be complementary subtoposes of a topos $\underline{M}$, with inclusions $h : \underline{H} \hookrightarrow \underline{M}$ and $k : \underline{K} \hookrightarrow \underline{M}$, respectively.

Then we have left exact "fringe" functors

$$\underline{H} \xrightarrow{h_*} \underline{M} \xrightarrow{k^*} \underline{K}$$

and

$$\underline{K} \xrightarrow{k_*} \underline{M} \xrightarrow{h^*} \underline{H}.$$

By their left exactness, these functors extend to left exact functors on the categories of abelian-group objects, $Ab(\underline{H}) \to Ab(\underline{K})$ and $Ab(\underline{K}) \to Ab(\underline{H})$.

Local closedness of $\underline{H}$ can be expressed in terms of these left exact functors:

2.4. PROPOSITION. *Let $\underline{H}$ be a subtopos of a topos $\underline{M}$. Then the following two conditions are equivalent*
*1) $\underline{H}$ is locally closed in $\underline{M}$.*
*2) $\underline{H}$ has a complement $\underline{K}$, and the "double fringe functor"*

$$\underline{H} \xrightarrow{h_*} \underline{M} \xrightarrow{k^*} \underline{K} \xrightarrow{k_*} \underline{M} \xrightarrow{h^*} \underline{H}$$

*is constant 1.*

PROOF. Of course, condition 2) is equivalent to saying that $h^*k_*k^*h_*(0_{\underline{H}}) = 1_{\underline{H}}$. Let us analyze the significance of $h^*k_*k^*h_*(0_{\underline{H}})$ for general complementary subtoposes $\underline{H}$ and $\underline{K}$. This is achieved by applying the above principle a couple of times. We have

- $0_{\underline{H}}$ and $1_{\underline{H}} = \underline{H}$ are complements inside $\underline{H}$, hence
- $h_*0_{\underline{H}}$ and $cl(\underline{H})$ are complements inside $\underline{M}$, hence
- $\underline{K} \cap h_*0_{\underline{H}} = k^*h_*0_{\underline{H}}$ and $\underline{K} \cap cl(\underline{H})$ are complements inside $\underline{K}$, hence
- $k_*k^*h_*0_{\underline{H}}$ and $cl(\underline{K} \cap cl(\underline{H}))$ are complements inside $\underline{M}$, hence
- $\underline{H} \cap k_*k^*h_*0_{\underline{H}} = h^*k_*k^*h_*0_{\underline{H}}$ and $\underline{H} \cap cl(\underline{K} \cap cl(\underline{H}))$ are complements inside $\underline{H}$.

So to say $h^*k_*k^*h_*0_{\underline{H}} = \underline{H}$ is equivalent to saying $\underline{H} \cap cl(\underline{K} \cap cl(\underline{H})) = 0$, or, since $\underline{H}$ and $\underline{K}$ are complements, to saying that $cl(\underline{K} \cap cl(\underline{H})) \subseteq \underline{K}$. Writing $\underline{K} \cap cl(\underline{H})$ as $cl(\underline{H}) - \underline{H}$ (by $\underline{H}$ and $\underline{K}$ being complements), the condition in turn is equivalent to $cl(cl(\underline{H}) - \underline{H}) \subseteq \underline{K}$, and hence to

$$cl(cl(\underline{H}) - \underline{H}) \subseteq \underline{K} \cap cl(\underline{H}),$$

since $cl(cl\underline{H}) - \underline{H}) \subseteq cl(\underline{H})$ is automatic by idempotency of $cl$. But finally, the displayed inclusion relation may be written $cl(cl(\underline{H}) - \underline{H}) \subseteq cl(\underline{H}) - \underline{H}$ (by $\underline{H}$ and $\underline{K}$ being complements); and this is equivalent to saying that $cl(\underline{H}) - \underline{H}$ is closed in $\underline{M}$, or equivalently closed in $cl(\underline{H})$, which is to say that $\underline{H}$ is open in $cl(\underline{H})$, which is one way of expressing that $\underline{H}$ is locally closed. This proves the Proposition. ∎

From the above analysis (applied to locales rather than to toposes), we also see that $h^*k_*k^*h_*0_H$ is the open complement of what Isbell in [2] p.355 denotes by $H'$, and which is described in loc. cit. (in a slightly different notation) as "the meet of $H$ with the closure of the join of all sublocales of $cl(H)$ disjoint from $H$ (telegraphically $H' = H \cap (cl(cl(H) \setminus H)))$", here $K \setminus H$ denotes the sublocale $\bigvee\{A \leq K \mid A \wedge H = 0\}$ for arbitrary sublocales $K$ and $H$ of $M$; the operation $(-)'$ is in loc. cit. iterated transfinitely to form a decreasing sequence of closed sublocales (which eventually reaches $\emptyset$) of the complemented sublocale $H$; locally closed sublocales are precisely those sublocales for which this sequence already terminates after the first step.

A general structure theory for complemented sublocales is given in [2] and [8]. We haven't investigated to what extent this theory is constructively valid, respectively, to what extent it carries over to elementary toposes. The first question which would have to be answered is whether for any inclusion $\underline{E} \hookrightarrow \underline{F}$, each complemented subtopos $\underline{H}$ of $\underline{E}$ is the restriction of a complemented subtopos of $\underline{F}$ to $\underline{E}$. For localic toposes over **Set** the answer is positive [2, 1.10]. This result can also be extended to Grothendieck toposes over **Set**, but we don't know how to prove the general case. For the present paper, this does not matter, since we only consider the case where the subtopos $H$ is complemented by virtue of being locally closed.

## 3. Prolongation by 0

Let $\underline{H}$ and $\underline{K}$ be complementary subtoposes of a topos $\underline{M}$, with inclusions $h : \underline{H} \hookrightarrow \underline{M}$ and $k : \underline{K} \hookrightarrow \underline{M}$, respectively. Then to say that an abelian group object $A$ in $\underline{H}$ admits prolongation by zero means that there exists an abelian group $Y$ in $\underline{M}$ with $h^*Y \cong A$ and $k^*Y \cong 0$; and we say that $\underline{H}$ admits prolongation by zero if every abelian group object in $\underline{H}$ does; cf. [9] for the classical theory (for topological spaces).

It is clear that if $\underline{H} \subseteq \underline{H}' \subseteq M$ and if an abelian group $A \in \underline{H}$ admits prolongation by zero when $\underline{H}$ is viewed as a (complemented) subtopos of $\underline{M}$, then it also admits prolongation by zero when $\underline{H}$ is viewed as a subtopos of $\underline{H}'$ (of which it is automatically a complemented subtopos). This in particular applies when we take $\underline{H}'$ to be the closure of $\underline{H}$ in $\underline{M}$; note that $\underline{H}$ is dense in $\underline{H}'$.

In case the (constant) group $\mathbf{Z}_2 = 1 + 1$ in $\underline{H} \subseteq \underline{M}$ admits prolongation by zero, it therefore follows immediately from the Lemma in the end of the previous section that $\underline{H}$ is locally closed in $\underline{M}$.

So we restrict from the outset of this section our considerations to a locally closed subtopos $\underline{H}$ of $\underline{M}$ with complement $\underline{K}$. In terms of the functors $h_*, h^*, k_*.k^*$ this means that $h^*k_*k^*h_*$ is constant zero. We denote $k^*h_*$ by $f : \underline{H} \to \underline{K}$ and $h^*k_*$ by $g : \underline{K} \to \underline{H}$, and the local closedness therefore becomes expressed by the equation $g \circ f = 1$.

By the glueing analysis of Section 1, we represent objects in $\underline{M}$ by objects $(H, K)$ in the product category $\underline{H} \times \underline{K}$ equipped with coalgebra structure $\xi$, where the information of $\xi$ in turn is given by a pair $x : H \to gK$, $y : K \to fH$ of "glueing data".

In the following Proposition, the notation $h_!A$ for a prolongation by zero of $A$ anticipates that this is a *functorial* construction, but the functoriality is not assumed, but will be deduced.

3.1. Proposition. *Let $h_!A$ in $\underline{M}$ be a prolongation of the abelian group $A$ in $\underline{H}$. Let the homomorphism $i : h_!A \to h_*A$ correspond to the assumed isomorphism $h^*h_!A \cong A$ under the adjointness $h^* \dashv h_*$; then $i$ is a monomorphism, and it has the universal property that if $b : B \to h_*A$ is a homomorphism in $Ab(\underline{M})$, and $k^*(b) = 0$, then $b$ factors (uniquely) through $i : h_!A \to h_*A$.*

Proof. Let $p : \underline{H} \times \underline{K} \to \underline{M}$ be the geometric morphism with inverse image $M \mapsto (h^*M, k^*M)$, and let $G$ be the comonad on it with the property that its coalgebra topos is $\underline{M}$, as in the previous section. Let $b : B \to h_*A$ be given with $k^*b = 0$. Applying $p^*$ to $b$ gives the top map in the commutative triangle

$$\begin{tikzcd}
(h^*B, k^*B) \arrow[r, "{(h^*b, 0)}"] \arrow[dr, "{(\hat{b}, 0)}"'] & (h^*h_*A, k^*h_*A) \\
& (h^*h_!A = A, k^*h_!A = 0) \arrow[u, "{(i, 0)}"']
\end{tikzcd}$$

(where $\hat{b} : h^*B \to A = h^*h_!A$ corresponds to $b : B \to h_*A$ under the adjointness $h^* \dashv h_*$). The top map is $p^*b$, thus a $G$-coalgebra homomorphism, and the right hand map is $p^*i$, thus likewise a $G$-coalgebra homomorphism, and it is clearly monic in $\underline{H} \times \underline{K}$, hence in $(\underline{H} \times \underline{K})^G \simeq \underline{M}$. But by the Lemma below, $(\hat{b}, 0)$ is a coalgebra homomorphism, thus of form $p^*(\beta)$ for some unique $\beta : B \to h_!A$, which gives the desired factorization $b = i \circ \beta$.

3.2. Lemma. *If the functor part $G$ of a comonad $(G, \epsilon, \psi)$ preserves monics, and the top and right hand map in a commutative triangle, with $s$ monic,*

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "t"'] & Z \\
& Y \arrow[u, "s"']
\end{tikzcd}$$

*are coalgebra homomorphisms (with respect to given costructures on $X$, $Y$, and $Z$), then so is the third map $t$.*

Proof. This amounts to proving the left hand square in

$$\begin{tikzcd}
GX \arrow[r, "Gt"] & GY \arrow[r, "Gs"] & GZ \\
X \arrow[u] \arrow[r, "t"] & Y \arrow[u] \arrow[r, "s"] & Z \arrow[u]
\end{tikzcd}$$

commutative (where the vertical maps are the given costructures). But since $Gs$ is monic, this commutativity follows from the commutativity of the total square and the right hand square, and we do have these commutativities, since $r$ and $s$ are $G$-homomorphisms. ∎

Assume now that every abelian group in $\underline{H}$ admits a prolongation $h_!A$ by zero. The universal property of $i : h_!A \to h_*A$ then in the usual way implies that $h_!$ carries the structure of a functor, in fact a subfunctor of $h_* : Ab(\underline{H}) \to Ab(\underline{M})$.

3.3. Theorem. *A complemented subtopos $\underline{H} \hookrightarrow \underline{M}$ admits prolongation by zero if and only if it is locally closed. And then the prolongation functor $h_! : Ab(\underline{H}) \to Ab(\underline{M})$ admits a right adjoint.*

Proof. We have already seen that if $\underline{H} \hookrightarrow \underline{M}$ admits prolongation by zero, then it is locally closed. To prove the converse is to perform a construction for any $A \in Ab(\underline{H})$. One could do it by consider separately closed inclusions and open inclusions, but we think that the uniform construction in terms of (double) glueing that we shall give, is more transparent, and it does not involve any choices. So given $A \in Ab(\underline{H})$, we construct $h_!A \in \underline{M}$ by constructing an abelian group in $Ab((\underline{H} \times \underline{K})^G)$. This is simply the abelian group $(A, 0) \in Ab(\underline{H} \times \underline{K})$ with coalgebra structure given in terms of "double glueing data" by the pair of homomorphisms $x : A \to g0$, $y : 0 \to fA$ (where $f$ and $g$, as in the previous section denote $k^*h_*$ and $h^*k_*$, respectively, $k : \underline{K} \hookrightarrow \underline{M}$ being the complementary subtopos to $\underline{H}$). The compatibility condition $f(x) \circ y = \kappa_0$ is trivial, since it compares two homomorphisms out of the zero group (and into the zero group, in fact). The other compatibility condition compares two maps $g(y) \circ x$ and $\eta_A$ from $A$ to $gfA$, but by the assumption of local closedness, $gfA$ is terminal, so this one holds as well. Finally it is clear that the constructed coalgebra has the right restrictions, $A$ and 0, respectively, along $h^*$ and $k^*$.

Finally, the right adjoint $h^!$ is described in coalgebra terms as follows. Consider an abelian group object $C$ in $\underline{M} \simeq (\underline{H} \times \underline{K})^G$. In terms of the latter category, $C$ is given by a pair of abelian groups $H$ and $K$ in $\underline{H}$ and $\underline{K}$, respectively, and a pair of group homomorphisms $x : H \to gK$, $y : K \to fH$. We define $h^!(C)$ to be the kernel of the homomorphism $x$. The front adjunction for the adjointness is then simply the identity; the back adjunction $h_!h^!C \to C$ is the pair $Ker(x) \to H$ (inclusion) and $0 \to K$, which is a morphism of coalgebras, i.e. is compatible with the glueing data, by commutativity of the square

$$\begin{tikzcd}
Ker(x) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H \arrow[r, "x"'] & gK
\end{tikzcd}$$

in $\underline{H}$, and and a square in $\underline{K}$ having the 0 group in its upper left hand corner, and therefore automatically commutative. The triangle equations for front- and back- adjunction for the claimed adjointness in this case (where the front adjunction is an identity) just amounts to the requirement that $h^!$ inverts the back adjunction. But $Ker(x)$ is clearly isomorphic to $Ker(\overline{x})$, where $\overline{x}$ is part of the coalgebra structure for $h_!H$, namely the map $Ker(x) \to 0$. This proves the Theorem. ∎

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