DOCTRINES WHOSE STRUCTURE FORMS A FULLY FAITHFUL ADJONT STRING

F. MARMOLEJO
Transmitted by Ross Street

ABSTRACT. We pursue the definition of a KZ-doctrine in terms of a fully faithful adjoint string $Dd \dashv m \dashv dD$. We give the definition in any Gray-category. The concept of algebra is given as an adjunction with invertible counit. We show that these doctrines are instances of more general pseudomonads. The algebras for a pseudomonad are defined in more familiar terms and shown to be the same as the ones defined as adjunctions when we start with a KZ-doctrine.

1. Introduction

Free co-completions of categories under suitable classes of colimits were the motivating examples for the definition of KZ-doctrines. We introduce in this paper a not-strict version of such doctrines defined via a fully faithful adjoint string. Thus, a non-strict KZ-doctrine on a 2-category $\mathcal{K}$ consists of a normal endo homomorphism $D : \mathcal{K} \to \mathcal{K}$, and strong transformations $d : 1_\mathcal{K} \to D$, and $m : DD \to D$ in such a way that $Dd \dashv m \dashv dD$ forms a fully faithful adjoint string, satisfying one equation involving the unit of $Dd \dashv m$ and the counit of $m \dashv dD$. Given an object $C$ in $\mathcal{K}$, we think of $DC$ as the co-completion of $C$, consisting of suitable diagrams over $C$, $dC : C \to DC$ as the functor that assigns to every object of $C$ the diagram on that object with identities for every arrow in the diagram, and $mC : DDC \to DC$ as a colimit functor. The idea of pursuing the adjoint string as definition follows in the steps of [3] and was suggested by R. J. Wood.

Now, $Dd \dashv m \dashv dD$ being a fully faithful adjoint string means that the counit $\beta : m \circ dD \to Id$ of $m \dashv dD$ is invertible (equivalently, the unit $\eta : Id \to m \circ Dd$ is invertible [7]).

Recall that A. Kock’s algebraic presentation of KZ-doctrines [9] asks for equalities $m \circ dD = Id$ and $Id = m \circ Dd$, and for a 2-cell $\delta : Dd \to dD$ satisfying four equations.

We can produce from the adjoint string a 2-cell $\delta : Dd \to dD$, namely, the pasting of $\beta^{-1}$ and the unit for the adjunction $Dd \dashv m$. This $\delta$ satisfies similar (‘non-strict’ versions of) the conditions required for a KZ-doctrine in [9]. Thus, the KZ-doctrines of [9] are particular instances of our KZ-doctrines.

Since the algebras for a KZ-doctrine are given in terms of adjunctions it seems reasonable to define the doctrine in terms of adjunctions. Instead of having equality as in [9] we have the invertible 2-cells $\beta$ and $\eta$. This laxification is justified if only because associativity...
in [9] is deduced up to isomorphism, but that paper also mentions some shortcomings of insisting on normalized algebras. We believe also that the approach via the adjoint string gives us a better insight into the nature of $\delta : Dd \to dD$.

We work in the framework of enriched category theory [2], where the category $\mathbf{V}$ is equal to the category $\text{Gray}$ with strict tensor product [5] (see [4] as well). By working in the context of $\text{Gray}$-categories we are developing the ‘formal theory of KZ-doctrines’ in the way that, by working in a 2-category, [13] develops the ‘formal theory of monads’. Notice that this is a very general setting since every tricategory is equivalent to a $\text{Gray}$-category [5]. The idea of defining KZ-doctrines in an enriched setting is also suggested in [9].

We adopt the definition of a pseudomonoid given in [1]. We show that every KZ-doctrine is a pseudomonad (pseudomonoid in the Gray monoid determined by an object of the $\text{Gray}$-category), and that the 2-categories of algebras defined as adjunctions coincide with the classical algebras for a pseudomonad (Theorem 10.7). We follow [13] in defining the algebras for a pseudomonad and the algebras for a KZ-doctrine with arbitrary objects of the $\text{Gray}$-category as domains.

R. Street [13] gives a conceptual global account of KZ-doctrines in terms of the simplicial category $\Delta$. Recall that in that context a doctrine on a bicategory $\mathcal{K}$ is a homomorphism of bicategories $\Delta \to \text{Hom}(\mathcal{K}, \mathcal{K})$ that preserves the monoid structure (ordinal addition on the domain and composition on the codomain), with $\Delta$ considered as a locally discrete 2-category. A KZ-doctrine is a doctrine that agrees in the common domain with a homomorphism of bicategories $\Delta^+ \to \text{Hom}(\mathcal{K}, \mathcal{K})$ where $\Delta^+$ is the 2-category of non-empty finite ordinals, order and last element preserving functions and inequalities. As pointed out in [9] this definition explicitly excludes the left most adjoint $Dd \dashv m$, without any indication as to whether it can be put back on. We show that, for a pseudomonad to be a KZ-doctrine either one of the adjunctions $Dd \dashv m$ or $m \dashv dD$ is enough.

For examples of free cocompletions of categories under different kinds of colimits we refer the reader to the bibliography of [9].

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2. Background

We work in the context of $\text{Gray}$-categories, where $\text{Gray}$ is the symmetric monoidal closed category whose underlying category is $2\text{-Cat}$ with the tensor product as in [5]. A $\text{Gray}$-category is then a category enriched in the category $\text{Gray}$ as in [2]. If $\mathcal{A}$ is a $\text{Gray}$-category and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are objects of $\mathcal{A}$, then the multiplication

$$\mathcal{A}(\mathcal{A}, \mathcal{B}) \otimes \mathcal{A}(\mathcal{B}, \mathcal{C}) \to \mathcal{A}(\mathcal{A}, \mathcal{C})$$

corresponds to a cubical functor of two variables

$$M : \mathcal{A}(\mathcal{A}, \mathcal{B}) \times \mathcal{A}(\mathcal{B}, \mathcal{C}) \to \mathcal{A}(\mathcal{A}, \mathcal{C}).$$
We denote the action of $M$ by juxtaposition $M(F, G) = GF$. Given $f : F \to F'$ in $A(A, B)$ and $g : G \to G'$ in $A(B, C)$ we denote the invertible 2-cell $M_{f,g}$ by

\[
\begin{array}{c}
GF 
\xrightarrow{gF} G'F
\\
\downarrow{Gf} \\
GF' 
\xleftarrow{gF'} G'F'.
\end{array}
\]

$M$ being a cubical functor implies that $(\_): A(B, C) \to A(A, C)$ and $G(\_): A(A, B) \to A(A, C)$ are 2-functors. It also implies that $(\_)f : (\_)F \to (\_)F'$ and $g(\_): G(\_) \to G'(\_)$ are strong transformations. Furthermore, if $\varphi : f \to f'$ and $\gamma : g \to g'$ then $(\_)\varphi : (\_)f \to (\_)f'$ and $\gamma(\_): g(\_) \to g'(\_)$ are modifications. Given $f'' : F'' \to F$ and $g'' : G' \to G''$ we also have that $g(f'' \circ f) = (G' f'' \circ g_f) \cdot (g_f'' \circ G_f)$ and $(g' \circ g)_f = (g'' \circ g F') \cdot (g'' F' \circ g_f)$. If $h : H \to H'$ is a 1-cell in $A(C, D)$, then properties like $h_g F = h_F g$ follow from the pentagon, and properties like $1_A F = F$ follow from the triangle that define a Gray-category. We will use these properties in the sequel without explicit mention.

3. KZ-Doctrines

3.1. Let $A$ be a Gray-category and $K$ be an object in $A$.

3.2. Definition. A KZ-doctrine $D$ on $K$ consists of an object $D$, 1-cells $d : 1_K \to D$, and $m : DD \to D$ in $A(K, K)$ and a fully-faithful adjoint string $\eta, \epsilon : Dd \dashv m$; and $\alpha, \beta : m \dashv dD : D \to DD$ such that

\[
\begin{array}{ccc}
1_K & d & DD \\
\downarrow{dD} & \downarrow{\beta} & \downarrow{m} \\
Dd & \downarrow{\eta} & DD \\
\downarrow{Dd} & \downarrow{\alpha} & \downarrow{m} \\
D & \downarrow{dD} & D.
\end{array}
\]

The adjoint string being fully-faithful means that the counit $\beta$ is invertible. It follows from a folklore result, whose statement and proof can be found in [7], that this is the case if and only if the unit $\eta$ is also invertible.

Compare this condition with condition T0 of [9], in which strict equality $m \circ Dd = m \circ dD = Id$ is asked for. As a matter of fact that paper points out some limitations that arise by requiring commutativity on the nose. Furthermore, associativity of $m$ is deduced there only up to isomorphism.

The other piece of information given in [9] is a 2-cell from $Dd$ to $dD$. In our case, this 2-cell comes from the adjoint string.
Define $\delta : Dd \rightarrow dD$ to be the pasting

$$
\begin{array}{c}
\xymatrix{
D \ar[r]^{Id_D} & D \\
DD \ar[r]_{Id_{DD}} & DD \\
& Dd \\
Dd \ar@/_1pc/[rr]_{\delta} \ar@/^1pc/[rr]^{Dd} & & DD. \\
\end{array}
$$

(2)

We know from [12] that $\delta$ is equal to the pasting $(dD \circ \eta^{-1}) \cdot (\alpha \circ Dd)$ and that it is unique with the property $m \circ \delta = \beta^{-1} \cdot \eta^{-1}$.

Condition T1 from [9] now takes the form:

**Proposition 3.1**

$$
1_K \xymatrix{ d \ar[r] & D \ar@/_1pc/[rr]_{\delta} \ar@/^1pc/[rr]^{Dd} & DD \\
dD \ar@/_1pc/[rr]_{dD} \ar@/^1pc/[rr]^{dD} & & DD. \\
}\quad = \quad 1_K \xymatrix{ d \ar[r] & D \ar@/_1pc/[rr]_{\delta} \ar@/^1pc/[rr]^{Dd} & DD. \\
dD \ar@/_1pc/[rr]_{dD} \ar@/^1pc/[rr]^{dD} & & DD. \\
}$$

**Proof.** Observe that as a consequence of (1), $d_d$ is equal to the pasting

$$
\begin{array}{c}
\xymatrix{
1_K \ar[r]^{(d_d)^{-1}} & DD \ar@/_1pc/[rr]_{m} \ar@/^1pc/[rr]^{Dd} & D \ar[r]^{d} & DD. \\
D \ar@/_1pc/[rr]_{d_d} \ar@/^1pc/[rr]^{dD} & & Dd \\
& D \ar[u]^{\beta^{-1}} \ar[u]_{\beta^{-1}} \\
D \ar[u]_{dD} \ar[u]^{dD} & & dD. \\
D \ar[u]_{dD} \ar[u]^{dD} & & dD. \\
& D. \\
\end{array}
$$

Notice that $\epsilon \circ Dd = Dd \circ \eta^{-1}$ (consequence of one of the triangular identities). Cancel $d_d$ with its inverse. Finally observe that $\delta \circ d = (\epsilon \circ Dd \circ d) \cdot (Dd \circ \beta^{-1} \circ d)$. ■

The condition T2 of [9] takes the form of the uniqueness property for $\delta$ mentioned above. We write it as a lemma.

**Lemma 3.2** $m \circ \delta = \beta^{-1} \cdot \eta^{-1}$ ■

We define the algebras for a KZ-doctrine with an arbitrary object of the Gray-category $A$ as domain. This is in agreement with [8], where the algebras for a monad on a 2-category are defined over arbitrary objects of the 2-category.

3.3. **Definition.** Let $\mathcal{X}$ be an object of $A$. A $D$-algebra with domain $\mathcal{X}$ is an adjunction

$$
\varphi, \psi : x \dashv dX : X \rightarrow DX
$$

in $A(\mathcal{X}, \mathcal{K})$, with the counit $\psi$ invertible.
A D-algebra as above, produces a co-fully-faithful adjoint string $Dx \dashv DdX \dashv mX$. As in the definition of $\delta$, we obtain

\[
\begin{array}{c}
\xymatrix{DX \ar[r]^{Id_{DX}} & DX \\
DDX \ar[r]^{Id_{DDX}} & DDX }
\end{array} = \begin{array}{c}
\xymatrix{DDX \ar[r]^{Id_{DDX}} & DDX \\
DX \ar[r]^{Id_{DX}} & DX }
\end{array}
\]

The following proposition tells us that for a D-algebra, the unit is uniquely determined by the counit

**Proposition 3.3** If $\varphi, \psi : x \dashv dX : X \to DX$ is a D-algebra, then $\varphi$ is equal to the pasting

\[
\begin{array}{c}
\xymatrix{DX \ar[r]^{Id_{DX}} \ar@/^/[r]^{DdX} & DDX \ar[r]^{D\psi^{-1}} \ar@/_/[r]_{ddX} & DX }
\end{array}
\]

**Proof.** Start with the above pasting. Replace $\delta X$ by $(\epsilon X \circ dDX) \cdot (DdX \circ \beta X^{-1})$. Use (4). Since the pasting of $d_x$ and $d_dX$ is equal to $d(d_{X \circ x})$, we have that

\[
\begin{array}{c}
\xymatrix{DDX \ar[r]^{DdX} \ar@/^/[r]^{dDX} & DX \ar[r]^{D\psi} \ar@/_/[r]_{ddX} & DDX }
\end{array} = \begin{array}{c}
\xymatrix{DX \ar[r]^{dDX} \ar@/^/[r]^{d\psi} & DX \ar[r]^{Id_{DX}} & DDX }
\end{array}
\]

Therefore we have that $(D\varphi \circ dDX) \cdot (DdX \circ d_x) = (dDX \circ \varphi) \cdot (d_{dX}^{-1} \circ x)$. Make this last substitution. As a consequence of (1) we have that the pasting of $d_{dX}^{-1}, \beta X^{-1}$ and $\eta X^{-1}$ is the identity.

Observe that, for any invertible 2-cell $\psi : x \circ dX \to Id_X$ the pasting (5) is always defined. Denote this pasting by $\hat{\psi}$. Now we show that one of the triangular identities is always satisfied.

**Lemma 3.4** If $\psi : x \circ dX \to Id_X$ is an invertible 2-cell in $A(\mathcal{X}, A)$, then the pasting

\[
\xymatrix{DX \ar[r]^{Id_{DX}} \ar@/_/[r]_{\psi} & DX }
\]

is the identity on $dX$. 
Proof. We know from Proposition 3.1 that \( \delta X \circ dX = d_{dX} \). The pasting of \( d_{dX} \) with \( d_x \) is \( d_{(xodX)} \). The pasting of this last 2-cell with \( D\psi^{-1} \) is equal to \( dX \circ \psi^{-1} \).

So, in order to see if an invertible \( \psi : x \circ dX \to Id_X \) determines a (necessarily unique, in view of Proposition 3.3) \( D \)-algebra, all we have to do is to check the other triangular identity.

**Proposition 3.5** An invertible 2-cell \( \psi : x \circ dX \to Id_X \) in \( A(X, A) \) is the counit of an adjunction \( x \dashv dX \) if and only if the pasting

\[
\begin{align*}
DX & \xrightarrow{Id_{DX}} DX \\
x & \xrightarrow{\psi} x \\
X & \xrightarrow{id_X} X
\end{align*}
\]

is the identity on \( x \).

Since we have \( m \dashv dD \) with invertible counit \( \beta \), we have as a corollary the condition corresponding to condition T3 in [9]

**Corollary 3.6** The pasting

\[
\begin{align*}
DD & \xrightarrow{Id_{DD}} DD \\
D & \xrightarrow{\eta} D
\end{align*}
\]

is the identity on \( m \).

Observe that a KZ-doctrine in \( A \), gives with the same data a KZ-doctrine in \( A^{trop} \) but with the roles of \( \alpha, \epsilon \) and \( \beta, \eta \) interchanged. Here \( A^{op} \) is the dual in the enriched sense, whereas \( A^{tr} \) is such that for every \( A \) and \( B \) in \( A \), we have \( A^{tr}(A, B) = A(A, B)^{co} \). We thus obtain the condition corresponding to T3* of [9]

**Corollary 3.7** The pasting

\[
\begin{align*}
DD & \xrightarrow{Id_{DD}} DD \\
D & \xrightarrow{\eta} D
\end{align*}
\]

is the identity on \( m \).
4. Normalized KZ-doctrines vs. KZ-doctrines

In this section we make explicit the comparison between the definition of KZ-doctrines in [9] and the definition given in this paper. Notice first that our definition is given in a general Gray-category, whereas the definition in [9] is given in 2-Cat. Notice furthermore, that we have replaced invertible 2-cells where the definition in [9] asked for strict equalities.

The definition given in [9] makes sense in a general Gray-category provided that the 2-cell \( d_d \) is an identity. So what we do is to compare the definitions in this more general setting.

Let’s assume first then, that we have a KZ-doctrine \( D \) in our sense, such that \( \beta, \eta \) and \( d_d \) are identities. Define \( \delta = \epsilon \circ D (\text{pasting (2)}) \). In this case the conditions corresponding to T1, T2 and T3 above are identical to the conditions T1, T2 and T3 of [9].

Conversely, assume we have \( (D, d, m, \delta) \) a KZ-doctrine in the sense of [9] (where we are assuming that \( d_d \) is an identity). It follows from the work done in [9] that \( Dd \dashv m \) with identity unit and \( m \dashv dD \) with identity counit. We have therefore a KZ-doctrine in our sense. All we have to show now is that \( \delta = \epsilon \circ D \), where \( \epsilon \) is the counit of \( Dd \dashv m \). But this is clear since \( \delta \) is unique with the property \( m \circ \delta = \beta^{-1} \cdot \eta^{-1} \).

5. Associativity up to isomorphism for KZ-doctrines

We deduce associativity up to isomorphism for a KZ-doctrine \( D \) as a corollary to the following technical proposition. Recall that \( D \)-algebras have objects of \( A \) as domains.

**Proposition 5.1** Let \( \psi : x \circ dX \to Id_X \) and \( \zeta : z \circ dZ \to Id_Z \) be \( D \)-algebras with the same object \( X \) of \( A \) as domain. Let \( h : X \to Z \) be a 1-cell in \( A(X, K) \). If \( h \) has a right adjoint then the pasting

\[
\begin{align*}
DX &\xrightarrow{Id_{DX}} DX &\xrightarrow{Dh} &\xrightarrow{\zeta \downarrow} DZ \\
X &\xrightarrow{\psi \downarrow} dX &\xrightarrow{h} &\xrightarrow{dZ} Z
\end{align*}
\]

is invertible.

**Proof.** Assume \( \pi, \chi : h \vdash k \). The inverse of the above pasting is

\[
\begin{align*}
DX &\xrightarrow{D\pi \downarrow} DX &\xrightarrow{d_k \downarrow} &\xrightarrow{\psi \downarrow} X \\
DZ &\xrightarrow{\zeta \downarrow} Z &\xrightarrow{X \downarrow} &\xrightarrow{\chi \downarrow} Z
\end{align*}
\]
As a corollary we have,

**Proposition 5.2** The pasting

\[
\begin{array}{c}
\text{DDD} \xrightarrow{\text{Id}_{\text{DDD}}} \text{DDD} \\
\downarrow \alpha D \\
\text{DD} \xrightarrow{m} \text{D}
\end{array}
\quad
\begin{array}{c}
\text{DDD} \xrightarrow{Dm} \text{DD} \\
\downarrow d_m \\
\text{DD} \xrightarrow{d} \text{D}
\end{array}
\quad
\begin{array}{c}
\text{DD} \xrightarrow{D} \text{D} \\
\downarrow m \\
\text{D}
\end{array}
\]

is invertible.

**Proof.** Apply 5.1 with \( \psi = \beta D \), \( \zeta = \beta \) and \( h = m \).

As a corollary of the following lemma, we are able to write (6) in terms of \( D\epsilon, m_d \) and \( \eta \).

**Lemma 5.3** Denoting pasting (6) by \( \mu \), we have

\[
\begin{array}{c}
\text{DDD} \xrightarrow{\text{Id}_{\text{DDD}}} \text{DDD} \\
\downarrow D\eta \\
\text{DD} \xrightarrow{\text{DD}\mu} \text{DD} \xrightarrow{\text{DD}m} \text{D}
\end{array}
\quad
\begin{array}{c}
\text{DD} \xrightarrow{m} \text{D} \\
\downarrow m \\
\text{D}
\end{array}
\quad
\begin{array}{c}
\text{DDD} \xrightarrow{\text{DD}\mu} \text{DD} \xrightarrow{\text{DD}m} \text{D} \\
\downarrow m \\
\text{D} \\
\end{array}
\]

**Proof.** Start on the left hand side. Substitute (6) for \( \mu \). Make the substitution

\[
\begin{array}{c}
\text{DD} \xrightarrow{\text{DD}\mu} \text{DD} \xrightarrow{\text{DD}m} \text{DD} \xrightarrow{\text{DD}\mu} \text{DD} \\
\downarrow m \\
\text{D} \\
\end{array}
\]

Then the substitution

\[
\begin{array}{c}
\text{DD} \xrightarrow{\text{DD}\mu} \text{DD} \xrightarrow{\text{DD}m} \text{DD} \\
\downarrow m \\
\text{D} \\
\end{array}
\]

recalling that \( D\epsilon_d = dD_d \). Finally, use the fact that \( \alpha \) and \( \beta \) define an adjunction.
Corollary 5.4 Pasting (6) equals

Another corollary to Proposition 5.1 is

Proposition 5.5 For any $D$-algebra $(X, x, \psi)$, the pasting

is invertible.

Proof. Apply 5.1 with $\psi = \beta X$, $\zeta = \psi$ and $h = x$.

Denote pasting (7) by $\chi_\psi$.

Proposition 5.6 For any $D$-algebra $(X, x, \psi)$, we have that

Proof. Replace $\chi_\psi$ by (7). Notice then that the pasting of $\eta X$ with $\alpha X$ produces $\delta X$. Now paste with $D\psi$ and its inverse and use 3.5.

6. 2-categories of algebras for a KZ-doctrine

Fix an object $\mathcal{X}$ in $A$. Define the 2-category $D\text{-Alg}_X$ of $D$-algebras with domain $\mathcal{X}$ as follows: The objects of $D\text{-Alg}_X$ are $D$-algebras $\psi : x \circ dX \to \text{Id}_X$ with domain $\mathcal{X}$. Given another $D$-algebra $\zeta : z \circ dZ \to \text{Id}_Z$ with domain $\mathcal{X}$, define $D\text{-Alg}_X(\psi, \zeta)$ to be the full
subcategory of \( A(\mathcal{X}, \mathcal{K})(X, Z) \) determined by those 1-cells \( h : X \to Z \) with the property that

\[
\begin{array}{cccc}
D_X & \xrightarrow{Id_{D_X}} & D_X & \xrightarrow{Dh} \to D_Z \\
\downarrow x & & \downarrow d_X & \downarrow d_Z \\
X & \to Z & \xrightarrow{h} & \to Z \\
\end{array}
\]

(8)

is invertible. The horizontal composite of \( h : \psi \to \zeta \) and \( k : \zeta \to \tau \) is \( k \circ h \).

There is a forgetful 2-functor \( U_\mathcal{X} : \text{D-Alg}_\mathcal{X} \to A(\mathcal{X}, \mathcal{K}) \) with \( U_\mathcal{X}(\psi) = X \). The left biadjoint \( F_\mathcal{X} : A(\mathcal{X}, \mathcal{K}) \to \text{D-Alg}_\mathcal{X} \) is defined as follows: For every \( X \) in \( A(\mathcal{X}, \mathcal{K}) \) define \( F_\mathcal{X}(X) = \beta X \). If \( \gamma : h \to h' : X \to Z \), define \( F_\mathcal{X}(h) = Dh \) and \( F_\mathcal{X}(\gamma) = D\gamma \). It is straightforward to show that \( F_\mathcal{X} \) is a 2-functor provided we know that \( Dh : \beta X \to \beta Z \) is a 1-cell in \( \text{D-Alg}_\mathcal{X} \). To see this we need a lemma.

**Lemma 6.1** For every 1-cell \( h : X \to Z \) in \( A(\mathcal{X}, \mathcal{K}) \) we have that the pasting

\[
\begin{array}{cccc}
DDX & \xrightarrow{Id_{DDX}} & DDX & \xrightarrow{DDh} \to DDZ \\
\downarrow m_X & & \downarrow dDX & \downarrow dZ \\
DX & \to DZ & \xrightarrow{h} & \to DZ \\
\end{array}
\]

is equal to \( m_h^{-1} \).

**Proof.** Since

\[
\begin{array}{cccc}
DX & \xrightarrow{dDX} & DDX & \xrightarrow{m_X} \to DX \\
\downarrow Dh & & \downarrow DDX & \downarrow Dh \\
DX & \to DDX & \xrightarrow{m_h} \to DX \\
\end{array}
\]

we have that \((\beta Z \circ Dh) \cdot (mZ \circ dDh) = (Dh \circ \beta X) \cdot (m_h^{-1} \circ dDX)\). Make this last substitution on the pasting of the lemma, and use the fact that \( \alpha \) and \( \beta \) define an adjunction.

Notice that \( F_\mathcal{X} \circ U_\mathcal{X} = D(\_ : A(\mathcal{X}, \mathcal{K}) \to A(\mathcal{X}, \mathcal{K}) \). The unit for the biadjunction \( F_\mathcal{X} + U_\mathcal{X} \) is \( d(\_ : 1_{A(\mathcal{X}, \mathcal{K})} \to D(\_ \mathcal{X}). \) The counit \( s : F_\mathcal{X} \circ U_\mathcal{X} \to 1_{\text{D-Alg}_\mathcal{X}} \) is given by the structure maps, that is to say, for \( \psi : x \circ dX \to Id_X \) we put \( s_\psi = x : \beta X \to \psi \). Notice that Proposition 5.5 says that \( x \) is a 1-cell in \( \text{D-Alg}_\mathcal{X} \). Given \( h : \psi \to \zeta \) in \( \text{D-Alg}_\mathcal{X} \), we define the transition 2-cell \( s_h \) as the inverse of (8).

The invertible modification \( Id_{F_\mathcal{X}} \to (sF_\mathcal{X}) \circ (F_\mathcal{X}d(\_ \mathcal{X}) \) is defined to be \( \eta X \) at every \( X \) in \( A(\mathcal{X}, \mathcal{K}) \). The invertible modification \( (U_\mathcal{X}s) \circ (d(\_ \mathcal{X})U_\mathcal{X} \to Id_{U_\mathcal{X}} \) is defined to be \( \psi \) at every \( \psi \) in \( \text{D-Alg}_\mathcal{X} \). To see that this defines a modification we have to show:
Lemma 6.2 \( h \circ \psi \) is equal to the pasting

\[
\begin{array}{c}
X \xrightarrow{dX} DX \xrightarrow{x} X \\
\downarrow \quad \downarrow h \\
Z \xrightarrow{dZ} DZ \xrightarrow{\zeta} Z.
\end{array}
\]

Proof. Consider the inverse of the above pasting composite and use the definition of \( s_h \).

\( \triangleright \)

Change of base. Assume that we have two objects \( \mathcal{X} \) and \( \mathcal{Z} \) of \( A \), and \( H \) an object in \( A(\mathcal{X}, \mathcal{Z}) \). Then the 2-functor \( (\_\_)H : A(\mathcal{Z}, \mathcal{K}) \rightarrow A(\mathcal{X}, \mathcal{K}) \) induces a change of base 2-functor \( \hat{H} : D-\text{Alg}_\mathcal{X} \rightarrow D-\text{Alg}_\mathcal{Z} \) such that

\[
D-\text{Alg}_\mathcal{Z} \xrightarrow{\hat{H}} D-\text{Alg}_\mathcal{X} \\
U_Z \downarrow \quad \downarrow U_X \\
A(\mathcal{Z}, \mathcal{K}) \xrightarrow{(\_\_)H} A(\mathcal{X}, \mathcal{K})
\]

commutes.

7. The Gray-category of D-algebras

We can, by allowing the domain to change, define the Gray-category \( D-\text{Alg} \) made up of D-algebras for a KZ-doctrine \( D \).

The objects of \( D-\text{Alg} \) are D-algebras with any object of \( A \) as domain. Given D-algebras \( \psi : x \circ dX \rightarrow Id_X \) with domain \( \mathcal{X} \) and \( \zeta : z \circ dZ \rightarrow Id_Z \) with domain \( \mathcal{Z} \), the 2-category \( D-\text{Alg}(\psi, \zeta) \) is defined as follows:

The objects of \( D-\text{Alg}(\psi, \zeta) \) are pairs \((N, h)\), where \( N \) is an object in \( A(\mathcal{X}, \mathcal{Z}) \) and \( h : X \rightarrow ZN \) is a 1-cell in \( A(\mathcal{X}, \mathcal{K}) \), such that the pasting

\[
\begin{array}{c}
DX \xrightarrow{Id_{DX}} DX \xrightarrow{Dh} DZN \\
\downarrow \quad \downarrow dX \quad \downarrow dZ_N \\
X \xrightarrow{h} ZN \xrightarrow{Id_{ZN}} ZN
\end{array}
\]

is invertible.

A 1-cell \((n, \bar{n}) : (N, h) \rightarrow (N', h')\) in \( D-\text{Alg}(\psi, \zeta) \) consists of a 1-cell \( n : N \rightarrow N' \) in \( A(\mathcal{X}, \mathcal{Z}) \) and a 2-cell \( \bar{n} : Zn \circ h \rightarrow h' \) in \( A(\mathcal{X}, \mathcal{K}) \).

A 2-cell \( \nu : (n, \bar{n}) \rightarrow (n', \bar{n}') \) is a 2-cell \( \nu : n \rightarrow n' \) in \( A(\mathcal{X}, \mathcal{Z}) \) such that \( \bar{n} = \bar{n}' \cdot (Y \nu \circ h) \). Vertical composition is the obvious one.
Define $Id_{(N,h)} = (Id_N, id_h)$.

Given $(n, \bar{n}) : (N, h) \to (N', h')$, and $(\ell, \bar{\ell}) : (N', h') \to (N'', h'')$ define $(\ell, \bar{\ell}) \circ (n, \bar{n}) = (\ell \circ n, \bar{\ell} \cdot (Z\ell \cdot \bar{n}))$. If $\lambda : (\ell, \bar{\ell}) \to (\ell', \bar{\ell}')$ and $\nu : (n, \bar{n}) \to (n', \bar{n}')$ define $\lambda \circ (n, \bar{n}) = \lambda \circ n$ and $(\ell, \bar{\ell}) \circ \nu = \ell \circ \nu$. This completes the definition of the 2-category $\text{DAlg}(\psi, \zeta)$.

Define $1_\psi = (1_X, Id_X)$.

For another D-algebra $\tau : y \circ dY \to Id_Y$ with domain $Y$, we define the cubical functor

$$M : \text{DAlg}(\psi, \zeta) \times \text{DAlg}(\zeta, \tau) \to \text{DAlg}(\psi, \tau)$$

denoted by juxtaposition as for $A$, as follows:

Given $(N, h)$ in $\text{DAlg}(\psi, \zeta)$ and $\omega : (o, \bar{o}) \to (o', \bar{o}') : (O, g) \to (O', g')$ in $\text{DAlg}(\zeta, \tau)$, define $(O, g)(N, h) = (O N, g N \circ h)$, and $(o, \bar{o})(N, h) = (o N, \bar{o} N \circ h)$, and $\omega(N, h) = \omega N$.

On the other hand, given $\nu : (n, \bar{n}) \to (n', \bar{n}') : (N, h) \to (N', h')$ in $\text{DAlg}(\psi, \zeta)$ and $(O, g)$ in $\text{DAlg}(\zeta, \tau)$ we define $(O, g)(N, h) = (O N, g N \circ h)$, and $(O, g)(n, \bar{n}) = (O n, (g N' \circ \bar{n}) \cdot (g_n \circ h))$, and $(O, g)\nu = O \nu$. The proof that we obtain 2-functors with these definitions is fairly straightforward.

For $(n, \bar{n}) : (N, h) \to (N', h')$ and $\omega : (o, \bar{o}) : (O, g) \to (O', g')$ we define the invertible 2-cell $(o, \bar{o})(n, \bar{n}) = a_n : (O', g')(n, \bar{n}) \circ (o, \bar{o})(N, h) \to (o, \bar{o})(N', h') \circ (O, g)(n, \bar{n})$.

These definitions give us a cubical functor since we have a cubical functor $A(X, Z) \times A(Z, Y) \to A(X, Y)$.

We have to show now that the diagrams required for a Gray-category are satisfied. We only do the pentagon. Given another D-algebra $\theta : w \circ dW \to Id_W$ with domain $W$, we have that the pentagon commutes if and only if the diagram of cubical functors

$$
\begin{array}{c}
\text{DAlg}(\psi, \zeta) \times \text{DAlg}(\zeta, \tau) \times \text{DAlg}(\tau, \theta) \\
\downarrow \text{DAlg}(\psi, \zeta) \times \text{M} \\
\text{DAlg}(\psi, \zeta) \times \text{DAlg}(\zeta, \theta) \times \text{DAlg}(\tau, \theta) \\
\downarrow M \\
\text{DAlg}(\psi, \theta)
\end{array}
$$

commutes. This is equivalent to the following six conditions for $(n, \bar{n}) : (N, h) \to (N', h')$ in $\text{DAlg}(\psi, \zeta)$, $(o, \bar{o}) : (O, g) \to (O', g')$ in $\text{DAlg}(\zeta, \tau)$ and $(p, \bar{p}) : (P, k) \to (P', k')$ in $\text{DAlg}(\tau, \theta)$:

1. $((\_)(N, h)) \circ ((\_)(O, g)) = ((\_)((O, g)(N, h))) : \text{DAlg}(\tau, \theta) \to \text{DAlg}(\psi, \theta)$.
2. $((p, k)(\_)) \circ ((\_)(N, h)) = ((\_)(N, h)) \circ ((p, k)(\_)) : \text{DAlg}(\zeta, \tau) \to \text{DAlg}(\psi, \theta)$.
3. $((P, k)(\_)) \circ (O, g)(\_)) = ((P, k)(O, g)(\_)) : \text{DAlg}(\psi, \zeta) \to \text{DAlg}(\psi, \theta)$.
4. $(p, \bar{p})(o, \bar{o})(N, h) = ((p, \bar{p})(o, \bar{o}))(N, h)$.
5. $(p, \bar{p})(O, g)(n, \bar{n}) = ((p, \bar{p})(O, g))(n, \bar{n})$.
6. $(P, k)(o, \bar{o})(n, \bar{n}) = ((P, k)(o, \bar{o}))(n, \bar{n})$.

All the above conditions follow from the definitions and the corresponding facts for the Gray-category $A$. 

8. Pseudomonads

We adopt the definition of pseudomonoid given in [1]. That is, given a Gray-category $A$, and an object $K$ in $A$, we define a pseudomonad $D$ on $K$ to be a pseudomonoid in the Gray monoid $A(K, K)$. Explicitly, $D$ consists of an object $D$ in $A(K, K)$ together with 1-cells $d : 1_K \to D$ and $m : DD \to D$ and invertible 2-cells satisfying the following two conditions:

\[
\begin{array}{c}
\xymatrix{DD & DD \\
& D \\
\ar@{=>}[r]^{\beta} & \ar@{=>}[r]_{\eta} & D \\
\ar@{=>}[rr]_{\alpha} & & \ar@{=>}[r]_{\mu} & \ar@{=>}[rr]^{m} & & D }
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{DDD & DD \\
& D \\
\ar@{=>}[r]^{\mu} & \ar@{=>}[r]_{m} & D \\
\ar@{=>}[rr]_{m^{-1}} & & \ar@{=>}[r]_{m} & \ar@{=>}[rr]_{m} & & D }
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{DD & DD \\
& D \\
\ar@{=>}[r]^{\mu} & \ar@{=>}[r]_{m} & D \\
\ar@{=>}[rr]_{m} & & \ar@{=>}[r]_{m} & \ar@{=>}[rr]_{m} & & D }
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{DD & DD \\
& D \\
\ar@{=>}[r]^{\mu} & \ar@{=>}[r]_{m} & D \\
\ar@{=>}[rr]_{m} & & \ar@{=>}[r]_{m} & \ar@{=>}[rr]_{m} & & D }
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{DD & DD \\
& D \\
\ar@{=>}[r]^{\mu} & \ar@{=>}[r]_{m} & D \\
\ar@{=>}[rr]_{m} & & \ar@{=>}[r]_{m} & \ar@{=>}[rr]_{m} & & D }
\end{array}
\]

**Warning:** The direction of the arrows $\eta$ and $\mu$ is the opposite to that given in [1]. Since they are invertible this represents no problem.

As pointed out in [1], a pseudomonoid in the cartesian closed 2-category $\text{Cat}$ of categories, functors and natural transformations is precisely a monoidal category, where condition (9) corresponds to the pentagon and condition (10) corresponds to the triangle that has the distinguished object $I$ in the middle. It is well known that in this case the commutativity of these diagrams implies the commutativity of the two triangles that have $I$ on one extreme or the other, and that the ‘right’ and ‘left’ arrows $I \otimes I \to I$ coincide [6]. (This in turn implies the commutativity of all the diagrams [11]). Results like those of [6] can be shown in the present context.
Proposition 8.1 If \( D = (D, d, m, \beta, \eta, \mu) \) is a pseudomonad on an object \( K \), then we have the following equalities:

1. \[ 1_K \xrightarrow{d} D \xleftarrow{dD} D \xrightarrow{m} D \] \[ 1_K \xrightarrow{(dd)^{-1}} DD \xleftarrow{m} D. \]

2. \[ DD \xrightarrow{dDD} DDD \xrightarrow{Dm} DD \] \[ D \xrightarrow{d} D \xleftarrow{m} \]

3. \[ DD \xrightarrow{m} D \]

Proof. To show 2 start with the following pasting
Make the substitution

\[
\begin{array}{ccc}
DD & \xrightarrow{d_{DD}} & D \\
\downarrow{d_{DD}} & & \downarrow{Dm} \\
DDD & \xrightarrow{D} & DD \\
\end{array}
\]

(\text{using the fact that } d^{-1}_{m\circ Dm\circ DDD} \text{ is equal to the pasting of } d^{-1}_{DD}, d^{-1}_{Dm} \text{ and } d^{-1}_{m}). \text{ Make the substitution (10) multiplied on the right by } D. \text{ Now make the substitution (9). Make the substitution }

\[
\begin{array}{ccc}
DD & \xrightarrow{D} & DD \\
\downarrow{d_{DD}} & & \downarrow{Dm} \\
DDD & \xrightarrow{DD} & DDD \\
\end{array}
\]

Then the substitution

\[
\begin{array}{ccc}
D & \xrightarrow{d} & DD \\
\downarrow{d} & & \downarrow{Dm} \\
DD & \xrightarrow{DD} & DDD \\
\end{array}
\]

Notice that, as consequence of (10), the pasting of \(D\beta^{-1}\) and \(\mu\) is equal to \(m \circ \eta D\). The pasting of \(\eta D, Dd^{-1}_m\) and \(m^{-1}_m\) is equal to \(Dm \circ \eta DD\). Observe that the bottom part of the resulting diagram is equal to the bottom part of the pasting we started from. Since all the 2-cells are invertible, we conclude 2.

3 can be proved similarly or by duality.
To show 1, we show first that the pasting

\[
\begin{array}{c}
1 \xrightarrow{d} D \\
\xrightarrow{dD} DD \xrightarrow{\eta D} D
\end{array}
\xrightarrow{dD} DDD \xrightarrow{DdD} D \xrightarrow{m} D
\]

is the identity. To do this, replace \( m \circ \eta D \) by a pasting of \( D\beta^{-1} \) and \( \mu \), using (10). Use condition 2 of the proposition proved above. The pasting of \( D\beta^{-1} \), \( d_{dD} \) and \( d_m \) is \( dD \circ \beta^{-1} \). We thus obtain an identity.

Start again with (11). Paste \( d_d \) and its inverse on top of it. Now, \( \eta D \circ Dd \) is equal to the pasting of \( DD_d, m_d \) and \( \eta \). The pasting of \( DD_d, d_d \) and \( d_dD \) is equal to the pasting of \( d_d, DD_d \) and \( d_d \). The pasting of \( dD_d, m_d \) and \( \beta D \) is \( Dd \circ \beta \). Since (11) is an identity, the resulting pasting is an identity. We thus obtain another identity if we remove \( d_d \) and its inverse. Now paste with \( \eta \) and \( \eta^{-1} \).

9. 2-categories of algebras for a Pseudomonad

As in the case of algebras for a KZ-doctrine we define the algebras for a pseudomonad with an object of \( A \) for domain.

Let \( D \) be a pseudomonad on an object \( K \) of the \( \text{Gray} \)-category \( A \). Let \( X \) be an object of \( A \). We define the 2-category \( D\text{-Alg}_X \) of \( D \)-algebras with domain \( X \) as follows.

An object of \( D\text{-Alg}_X \) consists of an object \( X \) in \( A(X, K) \), together with a 1-cell \( x : DX \rightarrow X \), and invertible 2-cells

\[
X \xrightarrow{dX} DX \\
\xrightarrow{\psi} \xrightarrow{x} X
\]

\[
DDX \xrightarrow{Dx} DX \\
\xrightarrow{mX} \xrightarrow{\chi} X
\]

This data must satisfy the following two conditions

\[
\begin{array}{c}
DDDX \xrightarrow{DDx} DDX
\end{array}
\]

\[
\begin{array}{c}
DDDX \xrightarrow{DDx} DDX
\end{array}
\]
We denote an object in $D$-$\text{Alg}_X$ by the pair $(\psi, \chi)$.

Given another $D$-algebra $(\zeta, \theta)$ with $\zeta : z \circ dZ \longrightarrow Id_Z$, a 1-cell in $D$-$\text{Alg}_X$ is a pair $(h, \rho) : (\psi, \chi) \longrightarrow (\zeta, \theta)$, where $h : X \longrightarrow Z$ is a 1-cell in $A(X, K)$ and

$$
\begin{array}{c}
\begin{array}{c}
D\chi \downarrow \\
\chi
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
DX \longrightarrow DZ \\
Dh
\end{array}
\end{array}
\end{array}.

(13)

$$

is an invertible 2-cell in $A(X, K)$, such that the following two conditions are satisfied.

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
D\chi \downarrow \\
\chi
\end{array}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
DX \longrightarrow DZ \\
Dh
\end{array}
\end{array}
\end{array}

(14)

$$

Given $(h, \rho), (h', \rho') : (\psi, \chi) \longrightarrow (\zeta, \theta)$, a 2-cell $\xi : (h, \rho) \longrightarrow (h', \rho')$ is a 2-cell $\xi : h \longrightarrow h'$ in $A(X, K)$ such that $(\xi \circ x) \cdot \rho = \rho' \cdot (z \circ D\xi)$. Vertical composition is the obvious one.

Horizontal composition: for $(h, \rho) : (\psi, \chi) \longrightarrow (\zeta, \theta)$ and $(k, \pi) : (\zeta, \theta) \longrightarrow (\tau, \sigma)$ we define $(k, \pi) \circ (h, \rho) = (k \circ h, (k \circ \rho) \cdot (\pi \circ Dh))$.

This completes the definition of $D$-$\text{Alg}_X$.

A proof very similar to that of condition 2 of Proposition 8.1 produces:
Lemma 9.1 For every $D$-algebra $(\psi, \chi)$ we have

$$
\begin{array}{c}
DX \xrightarrow{dDX} DDX \xrightarrow{Dx} DX \\
\downarrow \beta \downarrow \downarrow mX \downarrow \xleftarrow{} \chi \\
DX \xrightarrow{\downarrow x} X
\end{array}
\quad
\begin{array}{c}
DX \xrightarrow{dDX} DDX \\
\downarrow x \downarrow \downarrow Dx \\
X \xrightarrow{\downarrow x} X
\end{array}
(16)
$$

As a matter of fact, condition 2 of Proposition 8.1 is the above lemma applied to the $D$-algebra $(\beta, \mu)$.

The Gray-category $D$-$\text{Alg}$ of algebras for a pseudomonad $D$ can be defined along the same lines as the Gray-category $D$-$\text{Alg}$ of algebras for a KZ-doctrine.

10. Every KZ-doctrine is a pseudomonad

Assume we have a KZ-doctrine $D = (D, d, m, \alpha, \beta, \eta, \epsilon)$ as in Section 3. Define $\mu$ as pasting (6). We already know that $\mu$ is invertible.

Proposition 10.1 $D = (D, d, m, \beta, \eta, \mu)$ is a pseudomonad.

Proof. Condition (10) is Proposition 5.6 applied to the $D$-algebra $\beta$. As for the other condition, start on the left hand side of (9). Substitute (6) and (6) multiplied by $D$ on the right for $\mu$ and $\mu D$ respectively. The pasting of $\beta D$ and $\alpha D$ is the identity. The pasting of $d_m D$, $d_m$ and $D \mu$ equals the pasting of $d_D m$, $d_m$ and $\mu$. Paste with $(dDD \circ \beta) \cdot (\alpha D \circ dDD)$ in the middle. Use Lemma 6.1.

To be able to say anything meaningful on this connection between KZ-doctrines and pseudomonads, we must show first that the categories of algebras $D$-$\text{Alg}_X$ and $D$-$\text{Alg}_X$ for any $\mathcal{X}$ are essentially the same. We devote the rest of this section to show that they are 2-isomorphic. So we fix an object $\mathcal{X}$ of $A$, and a KZ-doctrine $D$ on $\mathcal{K}$. We take $D$ as the pseudomonad induced by $D$ as in the above proposition.

We start by stating the recognition lemma [13] in the form we will use it

Lemma 10.2 Given $\psi : x \circ dX \rightarrow Id_X$ and $\zeta : z \circ dZ \rightarrow Id_Z$ in $D$-$\text{Alg}_X$, $h : X \rightarrow Z$ a 1-cell in $A(\mathcal{X}, \mathcal{K})$ and $\rho : z \circ Dh \rightarrow h \circ x$ a 2-cell, we have that

$$
\begin{array}{c}
 DX \xrightarrow{Id DX} DX \xrightarrow{dh} DZ \\
\downarrow x \downarrow \downarrow dX \downarrow \xleftarrow{} dZ \downarrow \zeta \downarrow \downarrow IdZ \\
 X \xrightarrow{h} Z \xrightarrow{IdZ} Z
\end{array}
$$
if and only if

\[
\begin{array}{c}
X \overset{dX}{\longrightarrow} DX \overset{Dh}{\longrightarrow} DZ
\end{array}
\]

\[
\begin{array}{c}
X \overset{\psi}{\longleftarrow} \overset{\rho}{\Longleftarrow} \overset{\zeta}{\downarrow} Z
\end{array}
\]

\[
\begin{array}{c}
X \overset{dX}{\longrightarrow} DX \overset{Dh}{\longrightarrow} DZ
\end{array}
\]

\[
\begin{array}{c}
X \overset{\psi}{\longleftarrow} \overset{\rho}{\Longleftarrow} \overset{\zeta}{\downarrow} Z
\end{array}
\]

Let \( \psi : x \circ dX \rightarrow Id_X \) be an object in \( D\text{-Alg}_X \). Let \( \chi_\psi \) be equal to pasting (7).

**Lemma 10.3** \((\psi, \chi_\psi)\) is a \( D \)-algebra.

**Proof.** Condition (12) is shown as condition (9) in Proposition 10.1. Condition (13) is Proposition 5.6.

Conversely

**Lemma 10.4** If \((\psi, \chi)\) is a \( D \)-algebra with \( \psi : x \circ dX \rightarrow Id_X \), then \( \psi \) is a \( D \)-algebra and \( \chi = \chi_\psi \) (pasting 7).

**Proof.** To show that \( \psi \) is a \( D \)-algebra it suffices to show that the pasting in Proposition 3.5 is the identity on \( x \). Substitute pasting (5) for \( \hat{\psi} \). Paste with \( \chi \) and its inverse. Use (13) on the pasting of \( D\psi^{-1} \) and \( \chi \). By Lemma 3.2 the pasting of \( \eta X \) and \( \delta X \) is \( \beta X^{-1} \). Now use (16). The condition for \( \chi \) follows from Lemma 10.2 and (16).

**Lemma 10.5** Let \( \psi : x \circ dX \rightarrow Id_X \) and \( \zeta : z \circ dZ \rightarrow Id_Z \) be objects and \( h : \psi \rightarrow \zeta \) be a 1-cell in \( D\text{-Alg}_X \). Define \( \rho_h \) as pasting (8). Then we have that \( (h, \rho_h) : (\psi, \chi_\psi) \rightarrow (\zeta, \chi_\zeta) \) is a 1-cell in \( D\text{-Alg}_X \).

**Proof.** Condition (14) follows immediately from the definition of \( \rho_h \). The proof of (15) is very similar to the proof of condition (9) in Proposition 10.1.

Conversely

**Lemma 10.6** If \((h, \rho) : (\psi, \chi) \rightarrow (\zeta, \theta)\) is a 1-cell in \( D\text{-Alg}_X \), then \( h : \psi \rightarrow \zeta \) is a 1-cell in \( D\text{-Alg}_X \) and \( \rho = \rho_h \) (pasting (8)).

The situation for 2-cells is similar. We thus have

**Theorem 10.7** If we define \( \Phi : D\text{-Alg}_X \rightarrow D\text{-Alg}_X \) such that for every \( \xi : h \rightarrow h' : \psi \rightarrow \zeta \) in \( D\text{-Alg}_X \) we have \( \Phi(\psi) = (\psi, \chi_\psi) \), \( \Phi(h) = (h, \rho_h) \) and \( \Phi(\xi) = \xi \), we obtain a 2-isomorphism.

It can also be shown that the Gray-categories \( D\text{-Alg} \) and \( D\text{-Alg} \) are isomorphic.
11. Pseudomonads vs. KZ-doctrines

In [13], the leftmost adjoint in the definition of KZ-doctrine is explicitly excluded. A question raised in [9] asks whether it can be put back on. The answer given here is in the affirmative.

**Theorem 11.1** If \( D = (D,d,m,\beta,\eta,\mu) \) is a pseudomonad on an object \( K \) of a Gray-category \( A \), then, the following statements are equivalent

1. \( m \dashv dD \) with counit \( \beta \).
2. \( Dd \dashv m \) with unit \( \eta \).

**Proof.** Assume \( \alpha, \beta : m \dashv dD \). Notice that we can still define \( \delta \) as

\[
\begin{array}{ccc}
\eta^{-1} \downarrow & & \downarrow \alpha \\
\downarrow & & \\
\end{array}
\]

Define \( \epsilon \) as the pasting

\[
\begin{array}{ccc}
DD & \xymatrix{D \ar[r]^{m} & Dd \ar@{.>}[r] & DD} & \ar[l]_{Dd} DDD \\
\ar[r]_{DDd} & DD \ar[r]_{mD} & DD \\
\ar[u]_{Dd} & & \ar[u]_{mD} \end{array}
\]

Then \( \epsilon \) is the counit for an adjunction \( \eta, \epsilon : Dd \dashv m \). The converse follows similarly or by duality.

**References**


Instituto de Matemáticas
Universidad Nacional Autónoma de México
Area de la Investigación Científica
México D. F. 04510
México

Email: quico@matem.unam.mx
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Jiri Rosicky, Masaryk University: rosicky@math.muni.cz
James Stasheff, University of North Carolina: jds@charlie.math.unc.edu
Ross Street, Macquarie University: street@macadam.mpce.mq.edu.au
Walter Tholen, York University: tholen@mathstat.yorku.ca
Myles Tierney, Rutgers University: tierney@math.rutgers.edu
Robert F. C. Walters, University of Sydney: Walters_b@maths.su.oz.au
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