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THE REFLECTIVENESS OF COVERING MORPHISMS IN ALGEBRA AND GEOMETRY

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ABSTRACT. Each full reflective subcategory $\mathcal{X}$ of a finitely-complete category $\mathcal{C}$ gives rise to a factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$, where $\mathcal{E}$ consists of the morphisms of $\mathcal{C}$ inverted by the reflexion $I : \mathcal{C} \to \mathcal{X}$. Under a simplifying assumption which is satisfied in many practical examples, a morphism $f : A \to B$ lies in $\mathcal{M}$ precisely when it is the pullback along the unit $\eta B : B \to IB$ of its reflexion $If : IA \to IB$; whereupon f is said to be a *trivial covering* of B. Finally, the morphism $f : A \to B$ is said to be a *covering* of B if, for some effective descent morphism $p : E \to B$, the pullback p^*f of f along p is a trivial covering of E. This is the *absolute* notion of covering; there is also a more general *relative* one, where some class Θ of morphisms of $\mathcal{C}$ is given, and the class $\mathrm{Cov}(B)$ of coverings of B is a subclass – or rather a subcategory – of the category $\mathcal{C}{\downarrow}B \subset \mathcal{C}/B$ whose objects are those $f : A \to B$ with $f \in \Theta$. Many questions in mathematics can be reduced to asking whether $\mathrm{Cov}(B)$ is reflective in $\mathcal{C}{\downarrow}B$; and we give a number of disparate conditions, each sufficient for this to be so. In this way we recapture old results and establish new ones on the reflexion of local homeomorphisms into coverings, on the Galois theory of commutative rings, and on generalized central extensions of universal algebras.

1. Introduction

In our joint work [4] with Carboni and Paré, there emerged a surprising connexion between certain notions introduced by Janelidze in his papers [11 - 16] on Galois theory in categories, and notions arising in the study of factorization systems by Cassidy, Hébert, and Kelly [5].

Starting from an adjunction

$$\eta, \epsilon : I \dashv H : \mathcal{X} \longrightarrow \mathcal{C} \tag{1.1}$$

where $\mathcal{C}$ is a category with pullbacks, Janelidze constructs a Galois theory in the category $\mathcal{C}$, while the other authors above construct on $\mathcal{C}$ a prefactorization system $(\mathcal{E}, \mathcal{M})$, which

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is usually a factorization system; we recall more details a little later in this Introduction. Each of these constructions becomes simpler under certain additional conditions on the adjunction; and the first surprising observation is that the *admissibility* condition used by Janelidze is clearly equivalent to the *semi-left-exactness* condition of [5] — to be precise, when H is fully faithful, I is semi-left-exact if and only if each object of $\mathcal{C}$ is admissible. The second surprising observation is that, under the admissibility condition, the class $\mathcal{M}$ above coincides with the class of *trivial covering morphisms* in Janelidze's Galois theory.

In any category $\mathcal{C}$ we have the notion of an *effective descent morphism* $p : E \longrightarrow B$, whose precise definition we recall in Section 5 below; informally, we may think of these morphisms as the "good surjections". Some property of $f : A \longrightarrow B$ may be said to hold *locally*, when there is a pullback diagram

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \quad , \tag{1.2}$$

with p an effective descent morphism, such that the property in question holds for $g : D \longrightarrow E$. In particular, $f : A \longrightarrow B$ is said to be a *covering* when it is locally a trivial covering; that is, when there is a pullback diagram (1.2) with p an effective descent morphism and with g a trivial covering morphism. In brief, we may call the class $\mathcal{M}^*$ of coverings the *localization* of the class $\mathcal{M}$. The purpose of Galois theory is the study of the class $\mathcal{M}^*$ of all coverings, or equivalently of the full subcategories $\mathrm{Cov}(B) = \mathcal{M}^*/B$ of $\mathcal{C}/B$ for each B in $\mathcal{C}$.

As is shown in [4], there are classical cases, such as the (purely-inseparable, separable) factorization in field theory, and the (monotone, light) factorization in topology, where the class $\mathcal{M}^*$ forms part of a new factorization system $(\mathcal{E}^*, \mathcal{M}^*)$, and where moreover $\mathcal{E}^*$ is pullback-stable : the latter requirement forces $\mathcal{E}^*$ to be the *stabilization* of $\mathcal{E}$, consisting of those $f : A \longrightarrow B$ every pullback of which lies in $\mathcal{E}$. However we have this situation only under very strong additional assumptions on the adjunction (1.1); counter-examples in [4] show that in general $\mathcal{M}^*$ is not closed under composition, so that it is certainly not part of a factorization system $(\mathcal{E}^*, \mathcal{M}^*)$; moreover, when it *is* part of such a factorization system, $\mathcal{E}^*$ need not be pullback–stable.

Accordingly it is natural to pose:

Question 1.1 Under what conditions on the adjunction (1.1) is the class $\mathcal{M}^*$ of coverings at least *reflective*, in the sense that every morphism $f : A \longrightarrow B$ in $\mathcal{C}$ has a *universal* factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & A' \arrow[ur, "f'"'] & \end{tikzcd} \tag{1.3}$$

with $f' \in \mathcal{M}^*$?

Since $\mathcal{M}^*$ is pullback-stable and contains the identities, we can express the reflectiveness of $\mathcal{M}^*$ equivalently (see [10, Proposition 5.8]) as the assertion that $\mathcal{M}^*$, seen now as the full subcategory of the arrow category $\mathcal{C}^{\mathbf{2}}$ with the coverings as its objects, is a reflective subcategory of $\mathcal{C}^{\mathbf{2}}$. The case where there is a factorization system $(\mathcal{E}^*, \mathcal{M}^*)$ is that (see [10, Theorem 5.10]) where $\mathcal{M}^*$ is reflective as above *and* is closed under composition.

Investigating this Question 1.1, one soon observes:

1°. It follows from [5, Proposition 3.5] that we may assume, without loss of generality, that H is a full embedding; in other words, that $\mathcal{X}$ is just a reflective full subcategory of $\mathcal{C}$. We may further assume $\mathcal{X}$ to be *replete*, meaning that $C \in \mathcal{X}$ whenever some isomorph of C in $\mathcal{C}$ lies in $\mathcal{X}$.

2°. The category $\mathrm{Cov}(B)$ of coverings of a given object B of $\mathcal{C}$ is in fact the union

$$\mathrm{Cov}(B) \quad = \quad \bigcup_p \ \mathrm{Spl}(E, p) \ , \tag{1.4}$$

where $\mathrm{Spl}(E,p)$ is the category of those coverings of B which are "split by the given (E,p)", meaning those $f : A \longrightarrow B$ whose pullback $g : D \longrightarrow E$ along the given p, as in (1.2), is a trivial covering; in this union, p runs through all the effective descent morphisms $p : E \longrightarrow B$ with codomain B. In order to investigate the reflectiveness of $\mathrm{Cov}(B)$, therefore, we are led to investigate, for a fixed $p : E \longrightarrow B$, the reflectiveness of $\mathrm{Spl}(E,p)$ in the slice category $\mathcal{C}/B$. (Recall that there is often an individual effective descent morphism $p : E \longrightarrow B$ with $\mathrm{Cov}(B) = \mathrm{Spl}(E,p)$, as for instance when there is an effective descent morphism $p : E \longrightarrow B$ with E projective with respect to all effective descent morphisms; see below in this Section, and again in Section 8.)

3°. On the other hand, it is reasonable to consider the more general question of the reflectiveness of $\mathcal{M}^* \cap \Theta$ inside some "good" class Θ of morphisms of $\mathcal{C}$, rather than that of $\mathcal{M}^*$ inside the class of *all* morphisms, and to redefine $\mathrm{Cov}(B)$ accordingly.

These observations suggest that we work in the following context. We fix a *Galois structure* consisting of (i) a category $\mathcal{C}$; (ii) a replete full reflective subcategory $\mathcal{X}$ whose inclusion $H : \mathcal{X} \longrightarrow \mathcal{C}$ has the left adjoint $I : \mathcal{C} \longrightarrow \mathcal{X}$ with reflexion-unit $\eta_A : A \longrightarrow HIA = IA$ and with identity counit $IH = 1$; and (iii) a class Θ of morphisms in $\mathcal{C}$ which contains the isomorphisms, is closed under composition, is pullback-stable (in the sense that the pullback of a morphism in Θ along any morphism in $\mathcal{C}$ *exists* and lies in Θ), and is mapped into itself by $HI : \mathcal{C} \longrightarrow \mathcal{C}$. (Note that $\mathcal{C}$ is no longer required to admit *all* pullbacks, except when Θ is the class of *all* morphisms.)

We have on $\mathcal{C}$ the prefactorization system $(\mathcal{E},\mathcal{M})$ defined, in the notation both of the original treatment [5] and of the more recent and complete exposition [4], by

$$\mathcal{E} = (H(\mathrm{mor}\,\mathcal{X}))^{\uparrow} \ , \ \mathcal{M} = (H(\mathrm{mor}\,\mathcal{X}))^{\uparrow\downarrow} ;$$

this is the smallest prefactorization system $(\mathcal{E},\mathcal{M})$, measuring by the size of $\mathcal{M}$, for which $\mathrm{mor}\mathcal{X} \subset \mathcal{M}$. Since $\mathcal{E}$ clearly consists precisely of those $f : A \longrightarrow B$ in $\mathcal{C}$ with If invertible, this prefactorization system has the property that $g \in \mathcal{E}$ wherever $fg \in \mathcal{E}$ and $f \in \mathcal{E}$. By Theorem 3.3 of [5], repeated as Proposition 3.3 of [4], $(\mathcal{E},\mathcal{M})$ is actually a *factorization system* whenever $\mathcal{C}$ admits finite limits and all intersections of strong subobjects — and thus in particular whenever $\mathcal{C}$ is complete and well-powered; we do not make explicit use of this however, all we need about $(\mathcal{E},\mathcal{M})$ being reviewed in Section 2 below.

For an object B of $\mathcal{C}$, let us write $\mathcal{C}{\downarrow}B$ for the full subcategory of the slice-category $\mathcal{C}/B$ given by those $f : A \longrightarrow B$ lying in Θ; we sometimes denote such an object $f : A \longrightarrow B$ of $\mathcal{C}{\downarrow}B$ by (A, f) or just by f. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ that lies in both Θ and $\mathcal{M}$ will be called a *trivial covering of* B; these constitute a full subcategory of $\mathcal{C}{\downarrow}B$ which we shall denote by $\mathcal{M}{\downarrow}B$. Observe now that, for any morphism $p : E \longrightarrow B$ in $\mathcal{C}$, pulling back along p gives a functor $p^* : \mathcal{C}{\downarrow}B \longrightarrow \mathcal{C}{\downarrow}E$ which, because $\mathcal{M}$ is pullback-stable, maps $\mathcal{M}{\downarrow}B$ into $\mathcal{M}{\downarrow}E$. Of course $p^*(f)$ may be a trivial covering of E when f is not a trivial covering of B; we write $\mathrm{Spl}(E, p)$ for the full subcategory of $\mathcal{C}{\downarrow}B$ given by those $f : A \longrightarrow B$ in Θ with $p^*(f) \in \mathcal{M}$, and we say of such an f that it is *split* by p (or by (E, p)). (We remark that this concept is of interest chiefly when $p : E \longrightarrow B$ is a Θ-*effective-descent morphism* in the sense of Janelidze and Tholen [18], which we recall in Section 5 below.)

We call $f : A \longrightarrow B$ a *covering* of B if it lies in $\mathrm{Spl}(E, p)$ for *some* Θ-effective descent morphism $p : E \longrightarrow B$ which itself belongs to Θ; and we write $\mathrm{Cov}(B)$ for the full subcategory of $\mathcal{C}{\downarrow}B$ given by the coverings of B. Accordingly we have, in the ordered set of full subcategories of $\mathcal{C}{\downarrow}B$, the union (1.4) above, wherein $p : E \longrightarrow B$ now runs through the Θ-effective descent morphisms lying in Θ and having codomain B; and this is a *directed* union, as in [4, Section 5]. As we said, there is often an individual Θ-effective-descent morphism $p : E \longrightarrow B$, lying in Θ, for which $\mathrm{Cov}(B) = \mathrm{Spl}(E, p)$; we then say that B is *locally simply connected* and that $p : E \longrightarrow B$ *splits all coverings*. Now our reflectiveness question has three parts:

Question 1.2. Under what conditions

(a) does the inclusion $\mathrm{Spl}(E, p) \longrightarrow \mathcal{C}{\downarrow}B$ have a left adjoint — so that, for the given $p : E \longrightarrow B$, every morphism $f : A \longrightarrow B$ in Θ has a universal factorization (1.3) with (A', f') in $\mathrm{Spl}(E, p)$?

(b) is every object in $\mathcal{C}$ locally simply connected?

(c) (extending the original Question 1.1) does the inclusion $\mathrm{Cov}(B) \longrightarrow \mathcal{C}{\downarrow}B$ have a left adjoint for each $B \in \mathcal{C}$?

In fact we scarcely expect to have a simple answer to each of these questions, consisting of necessary and sufficient conditions; but must content ourselves instead with having a number of sufficient conditions covering important examples both old and new. This is similar to the situation where one would like to prove the reflectivity, in a functor category $\mathcal{C}^{\mathcal{K}}$, of the full subcategory given by the functors sending certain cones to limit-cones; as was first observed in Freyd's book [6, Ch.5, Exercise F], there is a proof when $\mathcal{C} = \mathbf{Set}$ and another when $\mathcal{C} = \mathbf{Set}^{\mathrm{op}}$, and these proofs have nothing in common; later, the proof for $\mathcal{C} = \mathbf{Set}$ was extended in [9], [22], and [7] to the case of locally-presentable (and even locally-bounded) $\mathcal{C}$, while that for $\mathcal{C} = \mathbf{Set}^{\mathrm{op}}$ was extended in [2] and [25] to the case of $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$ locally presentable.

The first purpose of this paper is to show that these old results give various sufficient conditions for Question 1.2 (a). Then, briefly discussing Question 1.2 (b), we describe various special cases in most of which every object in $\mathcal{C}$ is locally simply connected, so that our sufficient conditions for Question 1.2 (a) give sufficient conditions for Question 1.2 (c).

2. Admissibility

The description of the trivial coverings of E, and hence of $\mathrm{Spl}(E,p)$, simplifies when E has a property called *admissibility*, which is in fact needed for our positive results below on the reflectiveness of $\mathrm{Spl}(E,p)$ in $\mathcal{C}{\downarrow}B$. The discussion of admissibility which follows also provides an opportunity to revise the little we need concerning the prefactorization system $(\mathcal{E},\mathcal{M})$ arising from our Galois structure.

A morphism $g : X \longrightarrow Y$ of $\mathcal{X}$ is said to lie in Θ if it does so when seen as a morphism $Hg : HX \longrightarrow HY$ of $\mathcal{C}$; and we write $\mathcal{X}{\downarrow}Y$ for the full subcategory of the slice-category $\mathcal{X}/Y$ given by those $g : X \longrightarrow Y$ lying in Θ. Since HI maps Θ into itself, the morphism $If : IA \longrightarrow IB$ of $\mathcal{X}$ lies in Θ whenever the morphism $f : A \longrightarrow B$ of $\mathcal{C}$ does so; accordingly we have a functor $I^B : \mathcal{C}{\downarrow}B \longrightarrow \mathcal{X}{\downarrow}IB$ sending f to If. Clearly this functor has the right adjoint $H^B : \mathcal{X}{\downarrow}IB \longrightarrow \mathcal{C}{\downarrow}B$ sending $g : X \longrightarrow IB$ to the $v : C \longrightarrow B$ given by the pullback

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "v"'] & X \arrow[d, "g"] \\ B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \quad . \tag{2.1}$$

Moreover, with the notation of (2.1), the g-component of the counit $\epsilon^B : I^B H^B \longrightarrow 1$ of this adjunction is the morphism

$$\begin{tikzcd} IC \arrow[rr, "Iu"] \arrow[dr, "Iv"'] & & X \arrow[dl, "g"] \\ & IB & \end{tikzcd} \tag{2.2}$$

of $\mathcal{X}{\downarrow}IB$; so the elementary theory of adjunctions and the definition of $\mathcal{E}$ give:

2.1 Proposition. *For $B \in \mathcal{C}$, the following are equivalent:*

(i) $H^B : \mathcal{X}\downarrow IB \longrightarrow \mathcal{C}\downarrow B$ *is fully faithful;*
(ii) ϵ^B *is invertible;*
(iii) *we have $u \in \mathcal{E}$ in (2.1) for all $g \in \mathcal{X}\downarrow IB$.*

This proposition leads us to the following definition : An object B of $\mathcal{C}$ for which these equivalent assertions hold is said to be *admissible*.

Consider now the unit $\eta^B : 1 \longrightarrow H^B I^B$ of the adjunction $I^B \dashv H^B$; the f-component of η^B is the morphism w in the following diagram, wherein the square is a pullback:

$$\begin{tikzcd} A \arrow[dr, "w"] \arrow[drr, "\eta_A"] \arrow[ddr, "f"'] & & \\ & C \arrow[r, "u"'] \arrow[d, "v"] & IA \arrow[d, "If"] \\ & B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \quad . \tag{2.3}$$

Here, since $If \in \text{mor}\,\mathcal{X} \subset \mathcal{M}$ and since $If \in \Theta$ because we are supposing f to lie in Θ, and since moreover both $\mathcal{M}$ and Θ are pullback-stable, we have

$$v \in \mathcal{M} \cap \Theta\,, \tag{2.4}$$

so that v is a trivial covering of B. It suffices for our purposes to consider only the case of an *admissible* B; then $u \in \mathcal{E}$ by Proposition 2.1, whence $w \in \mathcal{E}$ since ($I\eta_A$ being invertible) we have $\eta_A \in \mathcal{E}$. Thus $f = vw$ is the $(\mathcal{E}, \mathcal{M})$-factorization of f, and we may see $w : f \longrightarrow v$ as both the reflexion of $f \in \mathcal{C}/B$ into the full subcategory $\mathcal{M}/B$ of $\mathcal{C}/B$, and the reflexion of $f \in \mathcal{C}\downarrow B$ into the full subcategory $\mathcal{M}\downarrow B$ of $\mathcal{C}\downarrow B$.

2.2 Proposition. *For B admissible and for $f : A \longrightarrow B$ lying in the class Θ, the following properties of f are equivalent:*

(i) *the object f of $\mathcal{C}\downarrow B$ lies in the replete image of the fully-faithful*

$$H^B : \mathcal{X}\downarrow IB \longrightarrow \mathcal{C}\downarrow B;$$

(ii) *w is invertible in (2.3);*
(iii) *the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & IA \arrow[d, "If"] \\ B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \tag{2.5}$$

is a pullback;
(iv) *f lies in $\mathcal{M}$; that is, it is a trivial covering of B.*

PROOF. (i) is equivalent to (ii) because w is the unit of the adjunction; (ii) implies (iii) because the square in (2.3) is a pullback; and (iii) implies (iv) because $If \in \mathrm{mor}\,\mathcal{X} \subseteq \mathcal{M}$ and $\mathcal{M}$ is pullback-stable. Finally, $f \in \mathcal{M}$ gives $w \in \mathcal{M}$ since $v \in \mathcal{M}$; but $w \in \mathcal{E}$ as we have seen, so that (iv) implies (ii). ∎

2.3 COROLLARY. *If $f : A \longrightarrow B$ is a trivial covering of the admissible B, then A is admissible.*

PROOF. In the diagram

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "v"'] & X \arrow[d, "g"] \\ A \arrow[r, "\eta_A"'] \arrow[d, "f"'] & IA \arrow[d, "If"] \\ B \arrow[r, "\eta_B"'] & IB \end{tikzcd} ,$$

let the top square be a pullback with $g \in \mathcal{X}\downarrow IA$; then, since the bottom square is a pullback by Proposition 2.2, the exterior too is a pullback, so that $u \in \mathcal{E}$ because B is admissible. We conclude that A is admissible. ∎

The following technical result plays an important role in the Galois theory that we shall recall in Section 5 below:

2.4 PROPOSITION. *The functor $I : \mathcal{C} \longrightarrow \mathcal{X}$ preserves those pullbacks*

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "h"'] & A \arrow[d, "f"] \\ D \arrow[r, "g"'] & B \end{tikzcd} \tag{2.6}$$

for which f lies in the class $\mathcal{M}\downarrow B$ of trivial coverings of B (whence necessarily h lies in $\mathcal{M}\downarrow D$) and for which B and D are admissible.

PROOF. Since (2.6) is a pullback by hypothesis and (2.5) is a pullback by Proposition 2.2, the exterior of ∎

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "h"'] & A \arrow[r, "\eta_A"] \arrow[d, "f"] & IA \arrow[d, "If"] \\ D \arrow[r, "g"'] & B \arrow[r, "\eta_B"'] & IB \end{tikzcd}$$

is a pullback; but this is equally the exterior of

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "h"'] & IC \arrow[r, "Ik"] \arrow[d, "Ih"] & IA \arrow[d, "If"] \\ D \arrow[r, "\eta_D"'] & ID \arrow[r, "Ig"'] & IB \end{tikzcd} \quad . \tag{2.7}$$

Since $If \in \Theta$, its pullback along Ig exists; suppose that it is

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "v"'] & IA \arrow[d, "If"] \\ ID \arrow[r, "Ig"'] & IB \end{tikzcd} \quad ,$$

and that $w : IC \longrightarrow X$ is the comparison morphism with $vw = Ih$ and $uw = Ik$. Since the exterior of (2.7) *is* a pullback, the functor $\eta_D^* : \mathcal{X}{\downarrow}ID \longrightarrow \mathcal{C}{\downarrow}D$ given by pulling back along η_D sends w to an isomorphism. This η_D^*, however, is just the functor H^D, which is fully faithful because D is admissible; so w is already an isomorphism, as desired.

Our definition of $\mathrm{Spl}(E,p)$ for $p : E \longrightarrow B$ may be re-expressed by saying that we have in **Cat** the pullback

$$\begin{tikzcd} \mathrm{Spl}(E,p) \arrow[r, "\subset"] \arrow[d] & \mathcal{C}{\downarrow}B \arrow[d, "p^*"] \\ \mathcal{M}{\downarrow}E \arrow[r, "\subset"'] & \mathcal{C}{\downarrow}E \end{tikzcd} \quad . \tag{2.8}$$

When E is admissible, however, it follows from Proposition 2.2 that $\mathcal{M}{\downarrow}E$ is the image of the fully-faithful $H^E : \mathcal{X}{\downarrow}IE \longrightarrow \mathcal{C}{\downarrow}E$, so that we have an equivalence of categories

$$\mathcal{X}{\downarrow}IE \simeq \mathcal{M}{\downarrow}E \,, \tag{2.9}$$

allowing the following rephrasal of (2.8):

2.5 Theorem. *For any $p : E \longrightarrow B$ in $\mathcal{C}$ with E admissible, we have in* **Cat** *a pullback*

$$\begin{tikzcd} \mathrm{Spl}(E,p) \arrow[r, "\subset"] \arrow[d] & \mathcal{C}{\downarrow}B \arrow[d, "p^*"] \\ \mathcal{X}{\downarrow}IE \arrow[r, "H^E"'] & \mathcal{C}{\downarrow}E \end{tikzcd} \tag{2.10}$$

with H^E fully faithful.

2.6 Remark. It is often the case in practical examples that every $B \in \mathcal{C}$ is admissible. When this is so and Θ consists of all morphisms, the reflexion of $\mathcal{C}$ onto $\mathcal{X}$ was said in [5] to be *semi-left-exact*; and this property was shown in [5, Theorem 4.3] to be *equivalent* to the preservation by I of the pullback of f and g whenever $g \in \mathcal{M}$: compare this with Proposition 2.4 above, and for a more recent account of this and related properties, see [4, Section 3]. In their study [17] of central extensions, Janelidze and Kelly considered an exact category $\mathcal{C}$ along with a reflective subcategory $\mathcal{X}$ closed under both subobjects and quotient objects, with Θ the class of strong epimorphisms; they called $\mathcal{X}$ *admissible* when each $B \in \mathcal{C}$ was admissible, showing *every* such $\mathcal{X}$ to be admissible when the congruence-lattice of each object of $\mathcal{C}$ is modular, and thus in particular whenever $\mathcal{C}$ is a Maltsev or a Goursat variety. From the point of view of Galois theory – see Section 5 below – to exhibit $\mathrm{Spl}(E,p)$ as equivalent to the category of "Θ-actions on $\mathcal{X}$ of the Galois pregroupoid of (E,p)", we need the admissibility not only of E but also of $E \times_B E$ and $E \times_B E \times_B E$. An important case where not every (E,p) has this property, while Θ consists neither of all morphisms nor of the strong epimorphisms, is that considered by Brown and Janelidze in [3].

3. Some relevant results of Wolff and Kelly

We need the notion of a *well-pointed endofunctor* (S,σ) on a category $\mathcal{A}$, introduced by Kelly in [20]. A *pointed endofunctor* (S,σ) consists of an endofunctor $S : \mathcal{A} \longrightarrow \mathcal{A}$ together with a natural transformation $\sigma : 1 \longrightarrow S$ whose domain is the identity endofunctor. By an *action* of (S,σ) on an object A of $\mathcal{A}$ is meant a morphism $a : SA \longrightarrow A$ in $\mathcal{A}$ for which $a.\sigma A = 1_A$; and when a is such an action, we call the pair (A,a) an (S,σ)-*algebra*, or an S-*algebra* for short. The S-algebras are the objects of a category S-Alg, wherein a morphism $f : (A,a) \longrightarrow (B,b)$ is an $f : A \longrightarrow B$ in $\mathcal{A}$ with $fa = b.Sf$; and there is a faithful and conservative forgetful functor $U : S\text{-Alg} \longrightarrow \mathcal{A}$ sending (A,a) to A.

The pointed endofunctor (S,σ) is said to be *well-pointed* when $S\sigma = \sigma S : S \longrightarrow S^2$. When this is so it follows easily that any action $a : SA \longrightarrow A$ also satisfies $\sigma A.a = 1_{SA}$; so A admits an S-action precisely when σA invertible, and then admits the unique action $a = (\sigma A)^{-1}$. In this well-pointed case, therefore, $U : S\text{-Alg} \longrightarrow \mathcal{A}$ is fully faithful; and we may identify S-Alg with the full replete subcategory of $\mathcal{A}$ given by those A with σA invertible.

This subcategory S-Alg is clearly closed in $\mathcal{A}$ under any limits that $\mathcal{A}$ may admit, and numerous existence theorems in mathematics reduce to showing the *reflectiveness* in $\mathcal{A}$ of such an S-Alg — that is, to showing the existence of the *free S-algebra* on each $A \in \mathcal{A}$. Various sufficient conditions for this reflectiveness are known; many were gathered together in [20] and expressed there as two theorems, which we now recall.

Sometimes the reflexion of $\mathcal{A}$ in S-Alg not only exists but takes on a very simple form. Supposing $\mathcal{A}$ to admit filtered colimits, we define a functor $\hat{S} : \infty \longrightarrow \mathcal{A}^{\mathcal{A}}$, where ∞ is the ordered set of all small ordinals and $\mathcal{A}^{\mathcal{A}}$ is the category of endofunctors of $\mathcal{A}$. We write S^β for the value $\hat{S}\beta$ of $\hat{S}$ at the small ordinal β, and write $S^\alpha_\beta : S^\beta \longrightarrow S^\alpha$ for the connecting morphism when $\beta \le \alpha$, defining these by transfinite induction as follows. Take $S^0 = 1_{\mathcal{A}}$; take $S^{\beta+1} = SS^{\beta}$ with $S^{\beta+1}_{\beta}$ equal to $\sigma S^{\beta} : S^{\beta} \longrightarrow SS^{\beta}$; and, for a limit-ordinal α, take $S^{\alpha} = \mathrm{colim}_{\beta<\alpha} S^{\beta}$, with the S^{α}_{β} being the generators of the colimit-cone. One easily sees that, for $A \in \mathcal{A}$ and $B \in S$-Alg, every $f : A \longrightarrow B$ factorizes uniquely through each $S^{\beta}_0 A : A \longrightarrow S^{\beta} A$. It follows that, if $S^{\alpha} A \in S$-Alg for some α, then $S^{\alpha}_0 A : A \longrightarrow S^{\alpha} A$ is the reflexion of A in S-Alg. This is of course the case precisely when $\sigma S^{\alpha} A : S^{\alpha} A \longrightarrow SS^{\alpha} A$ is invertible; then $S^{\gamma}_{\beta} A$ is invertible whenever $\alpha \le \beta \le \gamma$; and we say that *the transfinite sequence* $(S^{\beta} A)_{\beta<\infty}$ *converges at* α.

When each $(S^{\beta} A)$ does converge — perhaps at some α depending on A — we may say that *free S-algebras exist constructively*, or that S-Alg *is constructively reflective in* $\mathcal{A}$. (That S-Alg may be reflective in a complete and cocomplete $\mathcal{A}$ without being constructively so is shown in [20, Remark 5.4].)

The simplest and best-known of the sufficient conditions for the reflectiveness of S-Alg is:

3.1 Proposition. *Let A admit filtered colimits, and suppose that, for some regular cardinal α, the endofunctor S preserves α-filtered colimits. Then each transfinite sequence $(S^{\beta} A)$ converges at α, so that $S^{\alpha}_0 A : A \longrightarrow S^{\alpha} A$ is the reflexion of A into S-Alg.*

This, although a special case of [20, Theorem 6.2], is easy to prove directly : since $(S^{\alpha}_{\beta} : S^{\beta} \longrightarrow S^{\alpha})_{\beta<\alpha}$ is an α-filtered colimit-cone by construction, so too is $(SS^{\alpha}_{\beta} : S^{\beta+1} \longrightarrow S^{\alpha+1})_{\beta<\alpha}$ a colimit-cone, by hypothesis; but $(S^{\alpha}_{\beta+1} : S^{\beta+1} \longrightarrow S^{\alpha})_{\beta<\alpha}$ is itself a colimit-cone, since the $\beta + 1$ for $\beta < \alpha$ are cofinal in the ordered set of all β with $\beta < \alpha$. Moreover the composite of $\sigma S^{\alpha} = S^{\alpha+1}_{\alpha}$ with $S^{\alpha}_{\beta+1}$, which is $S^{\alpha+1}_{\beta+1}$, is easily shown (using the well-pointedness of S) to coincide with SS^{α}_{β} – see [20, Lemma 5.5]; so σS^{α} is indeed invertible.

An important particular case of this proposition is that where α is the first infinite ordinal ω; of course ω-filtered colimits are just filtered colimits; and a functor that preserves them is said to be *finitary*.

Theorem 6.2 of [20] in fact strengthens Proposition 3.1 by weakening its hypotheses in two ways. First, let $(\mathcal{E}, \mathcal{M})$ be a factorization system on $\mathcal{A}$ for which $\mathcal{E}$ consists of epimorphisms, and consider a functor $X : \mathcal{K} \longrightarrow \mathcal{A}$ where $\mathcal{K}$ is some α-filtered small category; we write X_{β} for the value of X at the object β of $\mathcal{K}$. An inductive cone $(r_{\beta} : X_{\beta} \longrightarrow N)_{\beta\in\mathcal{K}}$ is said to be *$\mathcal{E}$-tight* when the induced morphism colim $X \longrightarrow N$ lies in $\mathcal{E}$. We say that *S preserves the $\mathcal{E}$-tightness of α-filtered cones* if, whenever a cone as above (with $\mathcal{K}$ an α-filtered category) is $\mathcal{E}$-tight, so is the cone $(Sr_{\beta} : SX_{\beta} \longrightarrow SN)$. When the factorization system $(\mathcal{E}, \mathcal{M})$ is (isomorphisms, all morphisms), this is just to say that S preserves α-filtered colimits; so the following does generalize Proposition 3.1:

3.2 Proposition. *Let $\mathcal{A}$ admit filtered colimits, and let $(\mathcal{E}, \mathcal{M})$ be a factorization system, with $\mathcal{E}$ consisting of epimorphisms, such that $\mathcal{A}$ is $\mathcal{E}$-cowellpowered. Suppose that, for some regular cardinal α, the endofunctor S preserves the $\mathcal{E}$-tightness of α-filtered cones. Then each transfinite sequence $(S^{\beta} A)$ converges (at some $\gamma(A)$ depending in* *general on A), so that S-Alg is constructively reflective in $\mathcal{A}$.*

Once again, although this is a special case of [20, Theorem 6.2], it is easy to prove directly : by transfinite induction on the small ordinals $\gamma \geq \alpha$, one sees that each cone $(S^\gamma_\beta A : S^\beta A \longrightarrow S^\gamma A)_{\beta<\alpha}$ with $\gamma \geq \alpha$ is $\mathcal{E}$-tight, so that $S^\gamma_\alpha A$ lies in $\mathcal{E}$; by the $\mathcal{E}$-cowellpoweredness, therefore, the transfinite sequence $(S^\gamma A)$ converges at some $\gamma \geq \alpha$.

Since the $X : \mathcal{K} \longrightarrow \mathcal{A}$ forming the base of an $\mathcal{E}$-tight α-filtered cone could be the functor constant at some object M of $\mathcal{A}$, the hypotheses of Proposition 3.2 cannot be satisfied unless $S\mathcal{E} \subset \mathcal{E}$. This restriction disappears when we take the final step to the full strength of [20, Theorem 6.2], requiring S now to preserve the $\mathcal{E}$-tightness only of *some* α-filtered cones. In more detail, we take on $\mathcal{A}$ a second factorization system $(\mathcal{E}', \mathcal{M}')$, again with $\mathcal{E}'$ consisting of epimorphisms, and call $(r_\beta : X_\beta \longrightarrow N)_{\beta\in\mathcal{K}}$ an *$\mathcal{M}'$-cone* when each r_β lies in $\mathcal{M}'$. We say that S *preserves the $\mathcal{E}$-tightness of α-filtered $\mathcal{M}'$-cones* if, whenever the α-filtered $\mathcal{M}'$-cone above is $\mathcal{E}$-tight, so too is the cone $(Sr_\beta : SX_\beta \longrightarrow SN)$ — which need not, however, be an $\mathcal{M}'$-cone. The full version of [20, Theorem 6.2] is:

3.3 Proposition. *Let $\mathcal{A}$ admit filtered colimits, and let $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ be factorization systems on $\mathcal{A}$ for which $\mathcal{E}$ and $\mathcal{E}'$ consist of epimorphisms, and such that $\mathcal{A}$ is both $\mathcal{E}$-cowellpowered and $\mathcal{E}'$-cowellpowered. Suppose that, for some regular cardinal α, the endofunctor S preserves the $\mathcal{E}$-tightness of α-filtered $\mathcal{M}'$-cones. Then each transfinite sequence $(S^\beta A)$ converges, so that S-Alg is constructively reflective in $\mathcal{A}$.*

The proof, which is given in [20], is now less direct; it depends on a construction, due to Koubek and Reiterman and recalled in Section 4 of [20], that replaces certain of the cones $(S^\gamma_\beta : S^\beta \longrightarrow S^\gamma)_{\beta<\alpha'}$ for $\gamma \geq \alpha'$ by associated $\mathcal{M}'$-cones having the same colimit. One begins by considering the $(\mathcal{E}', \mathcal{M}')$-factorizations (e_γ, m_γ) of the $S^\gamma_0 : A \longrightarrow S^\gamma A$, observing that $e_{\gamma+1} = e' e_\gamma$ for some $e' \in \mathcal{E}'$; by the $\mathcal{E}'$-cowellpowerness, therefore, the e_γ become stationary at some γ, which we rename $1'$; setting also $0' = 0$ for uniformity, we henceforth write

$$A = S^0 A = S^{0'} A \underset{f_0}{\longrightarrow} Y_0 \underset{i_0}{\longrightarrow} S^{1'} A$$

for the $(\mathcal{E}', \mathcal{M}')$-factorization $(e_{1'}, m_{1'})$. Starting again now with $S^{1'}A$ instead of $A = S^{0'}A$, we this time get a stationary factorization

$$S^{1'} A \underset{f_1}{\longrightarrow} Y_1 \underset{i_1}{\longrightarrow} S^{2'} A$$

at some ordinal $2' > 1'$; and we continue transfinitely thus, setting $\delta' = \sup_{\beta<\delta} \beta'$ for a limit-ordinal δ, and so obtaining morphisms of transfinite sequences

$$S^{\beta'} A \underset{f_\beta}{\longrightarrow} Y_\beta \underset{i_\beta}{\longrightarrow} S^{(\beta+1)'} A$$

such that each $(S^{\gamma}_{(\beta+1)'} i_\beta : Y_\beta \longrightarrow S^{\gamma}A)_{(\beta+1)'<\gamma}$ is an $\mathcal{M}'$-cone and that $\mathrm{colim}_{\beta<\delta}\, Si_\beta$ is invertible for each limit-ordinal δ. From this point the proof proceeds fairly straightforwardly to the stated conclusion.

When the factorization system $(\mathcal{E}', \mathcal{M}')$ is (isomorphisms, all morphisms), Proposition 3.3 reduces of course to Proposition 3.2. An important case of Proposition 3.3 in practice, however, is that where $(\mathcal{E}', \mathcal{M}') = (\mathcal{E}, \mathcal{M})$, with this being a *proper* factorization system (so that $\mathcal{M}$ consists entirely of monomorphisms). In this case, to say that the $\mathcal{M}$-cone $(r_\beta : X_\beta \longrightarrow N)$ is $\mathcal{E}$-tight is to say that $N = \bigcup X_\beta$, where this union is the join in the ordered set of $\mathcal{M}$-subobjects. Thus to say that S preserves the $\mathcal{E}$-tightness of this $\mathcal{M}$-cone is to say that $SN = \bigcup \mathrm{im}\,(Sr_\beta)$; and when $S\mathcal{M} \subset \mathcal{M}$ it is to say that $SN = \bigcup SX_\beta$ — for instance, in Proposition 3.3, that S *preserves α-filtered unions of $\mathcal{M}$-subobjects.*

Note that, when $\mathcal{A}$ is the category of topological spaces and S is the endofunctor sending A to the discrete topological space on the set $\mathcal{A}(X, A)$, where X is the two-point space with the chaotic topology, S clearly preserves filtered unions of *subspaces.* Yet there is no α for which S preserves α-filtered colimits, or even those α-filtered colimits $(r_\beta : X_\beta \longrightarrow \mathrm{colim}\, X)$ for which the r_β are injections : see [20, Section 3.2]. Since we can move from any endofunctor to a well-pointed endofunctor on a related category, as in [20, Section 18], such examples show that Proposition 3.3 is strictly stronger than Proposition 3.1 — even though the latter does suffice for many important applications.

The alternative set of sufficient conditions for the reflectiveness of S-Alg given in [20, Theorem 7.5] has a different proof that follows up a suggestion of Barr : one first establishes a reflexion of $\mathcal{A}$ onto the $\mathcal{M}'$-closure $\mathcal{B}$ of S-Alg, and then — exhibiting S-Alg as also being S'-Alg for a well-pointed endofunctor (S', σ') on $\mathcal{B}$ — one looks at the reflectivity of S'-Alg in $\mathcal{B}$. The essential result may be stated thus:

3.4 Proposition. *If one seeks to establish only the reflectivity of S-Alg in $\mathcal{A}$, and not its constructive reflectivity, one can in the case $(\mathcal{E}, \mathcal{M}) = (\mathcal{E}', \mathcal{M}')$ and in the case $(\mathcal{E}, \mathcal{M}) =$ (isomorphisms, all morphisms) discard in Proposition 3.3 the hypothesis that $\mathcal{A}$ is $\mathcal{E}'$-cowellpowered, replacing it by the requirement that $\mathcal{A}$ admit arbitrary cointersections — even large ones if need be — of epimorphisms in $\mathcal{E}'$.*

In practice, of course, the cowellpoweredness is so commonly present that one may as well have the stronger result of Proposition 3.3. An example where Proposition 3.4 applies and Proposition 3.3 does not is that where $\mathcal{A} = \infty + 1$, the ordered set of all small ordinals together with the first non-small ordinal ∞, and where S is given by $S\beta = \beta + 1$ for $\beta < \infty$ and by $S\infty = \infty$, with the unique $\sigma : 1 \longrightarrow S$.

Note that, if $\mathcal{Z}$ is a full replete reflective subcategory of $\mathcal{B}$, with inclusion $K : \mathcal{Z} \longrightarrow \mathcal{B}$ and reflexion $R : \mathcal{B} \longrightarrow \mathcal{Z}$, the unit being $\pi : 1 \longrightarrow P$ where $P = KR$ and the counit being the identity, then (P, π) is an idempotent monad on $\mathcal{B}$, with the identity $P^2 = P$ for its multiplication; in particular, (P, π) is a well-pointed endofunctor of $\mathcal{B}$, since $P\pi = \pi P = 1$; and the corresponding subcategory P-Alg of $\mathcal{B}$ is just $\mathcal{Z}$. An important process giving rise to well-pointed endofunctors was described by Wolff in [29, Theorem 2.1] and repeated as [20, Proposition 9.2] : the following is a somewhat special case, whose fairly straightforward proof we shall not repeat here.

3.5 Proposition. *Let $\mathcal{A}$ admit pushouts, let $F \dashv U : \mathcal{A} \longrightarrow \mathcal{B}$ be an adjunction with counit $\epsilon : FU \longrightarrow 1$, and let $\mathcal{Z}$ be a full replete reflective subcategory of $\mathcal{B}$, whose corresponding idempotent monad is (P, π) as above. In the category of endofunctors of $\mathcal{A}$ form the pushout*

$$\begin{tikzcd} FU \arrow[r, "F\pi U"] \arrow[d, "\epsilon"'] & FPU \arrow[d, "\theta"] \\ 1 \arrow[r, "\sigma"'] & S \end{tikzcd} \quad . \tag{3.1}$$

Then the pointed endofuctor (S, σ) is well pointed, and S-Alg consists of those $A \in \mathcal{A}$ with $UA \in \mathcal{Z}$; in other words, we have in **CAT** *a pullback*

$$\begin{tikzcd} S\text{-Alg} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d, "U"] \\ \mathcal{Z} \arrow[r, "K"'] & \mathcal{B} \end{tikzcd} \quad . \tag{3.2}$$

4. A direct application of the Wolff-Kelly results

The diagram (2.10) above is an instance of the diagram (3.2) of Proposition 3.5, provided only that $p^* : \mathcal{C}\!\downarrow\! B \longrightarrow \mathcal{C}\!\downarrow\! E$ has a left adjoint. It certainly does so if $p \in \Theta$; for then the left adjoint $p_! : \mathcal{C}/E \longrightarrow \mathcal{C}/B$ of $p^* : \mathcal{C}/B \longrightarrow \mathcal{C}/E$, given by composition with p, restricts to a functor $p_! : \mathcal{C}\!\downarrow\! E \longrightarrow \mathcal{C}\!\downarrow\! B$, which provides the desired adjoint. Thus Theorem 2.5 and Proposition 3.5 give :

4.1 Proposition. *For a Galois structure $(\mathcal{C}, \mathcal{X}, H, I, \eta, \Theta)$, let $p : E \longrightarrow B$ be a morphism of $\mathcal{C}$ lying in Θ with E admissible. Suppose that $\mathcal{C}\!\downarrow\! B$ admits pushouts — which is certainly the case if $\mathcal{C}$ admits pushouts and Θ is closed under pushouts in $\mathcal{C}^{\mathbf{2}}$. Then* $\mathrm{Spl}(E, p)$ *is S-Alg for the well-pointed endofunctor (S, σ) of $\mathcal{C}\!\downarrow\! B$ defined by the following pushout in the category of endofunctors of $\mathcal{C}\!\downarrow\! B$, wherein $\epsilon : p_! p^* \longrightarrow 1$ is the counit of the adjunction $p_! \dashv p^*$:*

$$\begin{tikzcd} p_! p^* \arrow[r, "p_! \eta^E p^*"] \arrow[d, "\epsilon"'] & p_! H^E I^E p^* \arrow[d, "\theta"] \\ 1 \arrow[r, "\sigma"'] & S \end{tikzcd} \quad . \tag{4.1}$$

Let us apply to this (S, σ) the simplest of the criteria for reflectiveness in Section 3 above — namely that given by Proposition 3.1. To ensure that $\mathcal{C}/B$ admits filtered colimits, let us suppose that $\mathcal{C}$ admits filtered colimits and that $\mathcal{C}\downarrow B$ is closed in $\mathcal{C}/B$ under filtered colimits; note that the latter is certainly satisfied if Θ is closed in $\mathcal{C}^{\mathbf{2}}$ under filtered colimits. Since colimits commute with colimits, the pushout S in (4.1) preserves whatever colimits are preserved by $p_! p^*$, by $p_! H^E I^E p^*$, and by 1. Because the left adjoints $p_!, I^E$, and 1 preserve all colimits, these are the colimits preserved by p^* and by H^E. Now $p^* : \mathcal{C}\downarrow B \longrightarrow \mathcal{C}\downarrow E$ preserves α-filtered colimits if $p^* : \mathcal{C}/B \longrightarrow \mathcal{C}/E$ does so; while $H^E : \mathcal{X}\downarrow IB \longrightarrow \mathcal{C}\downarrow B$ preserves α-filtered colimits if $H : \mathcal{X} \longrightarrow \mathcal{C}$ does so and pulling back along η_B as in (2.1) does so. Thus we get:

4.2 Theorem. *For a Galois structure $(\mathcal{C}, \mathcal{X}, H, I, \eta, \Theta)$, let $\mathcal{C}$ admit pushouts and filtered colimits, and for each B let $\mathcal{C}\downarrow B$ be closed in $\mathcal{C}/B$ under these colimits; the latter is certainly the case if Θ is closed under these colimits in $\mathcal{C}^{\mathbf{2}}$. Let $p : E \longrightarrow B$ be a morphism of $\mathcal{C}$ lying in Θ with E admissible. Then* $\mathrm{Spl}(E, p)$ *is reflective in $\mathcal{C}\downarrow B$ if, for some regular cardinal α, the functor H preserves α-filtered colimits and so does the pullback-functor $f^* : \mathcal{C}\downarrow A \longrightarrow \mathcal{C}\downarrow C$ for each $f : C \longrightarrow A$ in $\mathcal{C}$. If $\mathcal{C}$ admits all pullbacks, it suffices in place of the last condition to suppose that each $f^* : \mathcal{C}/A \longrightarrow \mathcal{C}/C$ preserves α-filtered colimits; which is a* fortiori *the case if pullbacks commute with α-filtered colimits in $\mathcal{C}$.*

The reader will not find it hard to imagine cases where the stronger conditions of Proposition 3.3 are needed : in the category of topological spaces, for example, pullbacks do not preserve α-filtered colimits for any α, but they preserve filtered unions. Yet Theorem 4.2 above does suffice in numerous practical examples. The sufficient conditions we go on to develop now are not so much directly stronger, as different in kind.

5. Internal discrete opfibrations

It is convenient to recall here some properties of discrete opfibrations that are central to descent theory and to Galois theory; see [15] for a fuller treatment.

Let us write Γ for the category generated by the graph

$$\begin{tikzcd} 2 \arrow[r, shift left=2, "q"] \arrow[r, "m"] \arrow[r, shift right=2, "r"'] & 1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & 0 \arrow[l, "e"'] \end{tikzcd} \tag{5.1}$$

subject to the relations

$$de = ce = 1\ ,\ dr = cq\ ,\ dm = dq\ ,\ cm = cr\ . \tag{5.2}$$

By an *internal precategory* in a category $\mathcal{C}$, or just a *precategory in* $\mathcal{C}$ for short, is meant in object P of the functor category $\mathcal{C}^{\Gamma}$; thus a morphism $\phi : Q \longrightarrow P$ of precategories in $\mathcal{C}$ is just a natural transformation, which we might display as

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
Q_2 \arrow[r, shift left=3, "Q(q)"] \arrow[r, "Q(m)"] \arrow[r, shift right=3, "Q(r)"'] \arrow[d, "\phi_2"'] & Q_1 \arrow[d, "\phi_1"] \arrow[r, shift left=3, "Q(d)"] \arrow[r, shift right=3, "Q(c)"'] & Q_0 \arrow[l, "Q(e)"'] \arrow[d, "\phi_0"] \\
P_2 \arrow[r, shift left=3, "P(q)"] \arrow[r, "P(m)"] \arrow[r, shift right=3, "P(r)"'] & P_1 \arrow[r, shift left=3, "P(d)"] \arrow[r, shift right=3, "P(c)"'] & P_0 \arrow[l, "P(e)"']
\end{tikzcd}
\quad . \tag{5.3}
$$

We say that this morphism ϕ is a *discrete opfibration*, or that (Q, ϕ) is a *discrete opfibration over* P, if each of the diagrams

$$
\begin{tikzcd}
Q_2 \arrow[r, "Q(q)"] \arrow[d, "\phi_2"'] & Q_1 \arrow[d, "\phi_1"] \\
P_2 \arrow[r, "P(q)"'] & P_1
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
Q_1 \arrow[r, "Q(d)"] \arrow[d, "\phi_1"'] & Q_0 \arrow[d, "\phi_0"] \\
P_1 \arrow[r, "P(d)"'] & P_0
\end{tikzcd}
\tag{5.4}
$$

is a pullback. These discrete opfibrations over P form a full subcategory, denoted by $\mathcal{C}^P$, of the slice category $\mathcal{C}^\Gamma/P$; and we shall be particularly concerned with the further full subcategory $\mathcal{C}^{\downarrow P}$ of $\mathcal{C}^P$ given by those discrete opfibrations ϕ over P for which ϕ_0 (and hence ϕ_1 and ϕ_2) lie in Θ. Note that, when $\mathcal{C}$ is the category of sets and P is a small category, $\mathcal{C}^P$ is equivalent to the usual functor category $\mathbf{Set}^P$; more generally (see [19]) those discrete opfibrations $\phi : Q \longrightarrow P$ where P and Q are internal categories have been intensively studied. (The usual definition of discrete opfibration for internal *categories* requires only the second diagram in (5.4) to be a pullback; in fact the first is then automatically a pullback.) One easily sees that the class of all discrete opfibrations contains all isomorphisms, is closed under composition, and is stable by pullbacks.

The diagonal functor $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^\Gamma$ being fully faithful, we may identify an object B of $\mathcal{C}$ with its image ΔB, which is the category

$$
\begin{tikzcd}
B \arrow[r, shift left=3, "1"] \arrow[r, "1"] \arrow[r, shift right=3, "1"'] & B \arrow[r, shift left=3, "1"] \arrow[r, shift right=3, "1"'] & B \arrow[l, "1"']
\end{tikzcd}
\quad .
$$

A precategory Q is isomorphic to such a B precisely when all of its six structural morphisms $Q(q), \cdots, Q(c)$ are invertible; whereupon we call Q a *discrete category*. One easily sees that every $f : A \longrightarrow B$ in $\mathcal{C}$ is a discrete opfibration, and that a general $\phi : Q \longrightarrow B$ in $\mathcal{C}^\Gamma$ with B in $\mathcal{C}$ is a discrete opfibration if and only if Q is a discrete category. Thus we have a full inclusion

$$\mathcal{C}/B \longrightarrow \mathcal{C}^B \tag{5.5}$$

which is an equivalence of categories; and it restricts to an equivalence

$$\mathcal{C}{\downarrow}B \;\simeq\; \mathcal{C}^{\downarrow B}\;. \tag{5.6}$$

An internal category P is said to be an *equivalence relation* when each $\mathcal{C}(A,P)$ is an equivalence relation in **Set** in the usual sense; for each $B \in \mathcal{C}$ the discrete category $B = \Delta B \in \mathcal{C}^\Gamma$ is an example of an equivalence relation. Given any $p : E \longrightarrow B$ in $\mathcal{C}$, its kernel-pair $d, c : E \times_B E \longrightarrow E$ extends in an obvious way to an equivalence relation

$$\begin{tikzcd} E \times_B E \times_B E \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & E \times_B E \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & E \end{tikzcd} \tag{5.7}$$

in $\mathcal{C}$, which we denote by Eq (p); and the map p admits in $\mathcal{C}^\Gamma$ an evident factorization

$$E \underset{p''}{\longrightarrow} \text{Eq}\,(p) \underset{p'}{\longrightarrow} B\;. \tag{5.8}$$

Since every morphism $\phi : Q \longrightarrow P$ of precategories induces a functor $\mathcal{C}^{\downarrow\phi} : \mathcal{C}^{\downarrow P} \longrightarrow \mathcal{C}^{\downarrow Q}$ by pulling back, we have commutativity to within isomorphism in

$$\begin{tikzcd} \mathcal{C}^{\downarrow B} \arrow[rr, "\mathcal{C}^{\downarrow p}"] \arrow[dr, "\mathcal{C}^{\downarrow p'}"'] & & \mathcal{C}^{\downarrow E} \\ & \mathcal{C}^{\downarrow \text{Eq}(p)} \arrow[ur, "\mathcal{C}^{\downarrow p''}"'] & \end{tikzcd} \tag{5.9}$$

,

and hence by (5.6) in

$$\begin{tikzcd} \mathcal{C}{\downarrow}B \arrow[rr, "p^*"] \arrow[d, "\simeq"'] & & \mathcal{C}{\downarrow}E \arrow[d, "\simeq"] \\ \mathcal{C}^{\downarrow B} \arrow[rr, "\mathcal{C}^{\downarrow p}"] \arrow[dr, "\mathcal{C}^{\downarrow p'}"'] & & \mathcal{C}^{\downarrow E} \\ & \mathcal{C}^{\downarrow \text{Eq}(p)} \arrow[ur, "\mathcal{C}^{\downarrow p''}"'] & \end{tikzcd} \tag{5.10}$$

.

One can describe in elementary terms the objects of the category $\mathcal{C}^{\downarrow\text{Eq}(p)}$, which is called the category of *descent data* for $p : E \longrightarrow B$; when $p \in \Theta$, so that composition with p provides a left adjoint $p_!$ to p^*, one easily identifies the category $\mathcal{C}^{\downarrow\text{Eq}(p)}$ with the Eilenberg-Moore category of algebras for the monad $p^*p_!$ on $\mathcal{C}{\downarrow}E$, and identifies

$$\mathcal{C}{\downarrow}B \;\simeq\; \mathcal{C}^{\downarrow B} \underset{\mathcal{C}^{\downarrow p'}}{\longrightarrow} \mathcal{C}^{\downarrow\text{Eq}(p)} \tag{5.11}$$

with the corresponding comparison functor. In any case, $p : E \longrightarrow B$ is said to be a Θ-*effective-descent* morphism when (5.11) is an equivalence: when $p \in \Theta$, therefore, this is so precisely when p_* is monadic. Note that $\mathcal{C}^{\downarrow p'}$ in fact sends the object $f : A \longrightarrow B$ of $\mathcal{C}\!\downarrow\! B$ to the discrete opfibration $\phi : Q \longrightarrow \mathrm{Eq}(p)$, where

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "\phi"'] & A \arrow[d, "f"] \\ \mathrm{Eq}(p) \arrow[r, "p'"'] & B \end{tikzcd} \tag{5.12}$$

is a pullback in $\mathcal{C}^\Gamma$; so that the squares of the diagram

$$\begin{tikzcd} Q_2 \arrow[r, "q"] \arrow[d, "\phi_2"'] & Q_1 \arrow[r, "d"] \arrow[d, "\phi_1"] & Q_0 \arrow[r] \arrow[d, "\phi_0"] & A \arrow[d, "f"] \\ E \times_B E \times_B E \arrow[r, "q"'] & E \times_B E \arrow[r, "d"'] & E \arrow[r, "p"'] & B \end{tikzcd} \tag{5.13}$$

are pullbacks in $\mathcal{C}$, the right square of (5.13) being the 0-component of (5.12).

If we now have a Galois structure $(\mathcal{C}, \mathcal{X}, H, I, \eta, \Theta)$ as before, we have a reflexion $I^\Gamma \dashv H^\Gamma : \mathcal{X}^\Gamma \longrightarrow \mathcal{C}^\Gamma$ and hence an induced Galois structure for the functor categories; membership of the Θ^Γ for $\mathcal{C}^\Gamma$ is just pointwise membership of Θ, and we shall in fact write Θ for Θ^Γ. Again, the prefactorization system $(\mathcal{E}^\Gamma, \mathcal{M}^\Gamma)$ arising from the reflexion of $\mathcal{C}^\Gamma$ into $\mathcal{X}^\Gamma$ has $\phi \in \mathcal{E}^\Gamma$ precisely when each of ϕ_0, ϕ_1, ϕ_2 lies in $\mathcal{E}$, and similarly for $\mathcal{M}^\Gamma$; so here too we shall write $\mathcal{E}$ and $\mathcal{M}$ rather than $\mathcal{E}^\Gamma$ and $\mathcal{M}^\Gamma$. Finally, it is clear that an object P of $\mathcal{C}^\Gamma$ is admissible if and only if each of P_0, P_1, P_2 is so in $\mathcal{C}$.

Consider now a Θ-effective-descent morphism $p : E \longrightarrow B$ in $\mathcal{C}$ with Eq (p) admissible in $\mathcal{C}^\Gamma$ — which is to say that each of the objects E, $E \times_B E$, and $E \times_B E \times_B E$ of (5.7) is admissible in $\mathcal{C}$. Then by Proposition 2.1 the functor $H^{\mathrm{Eq}(p)} : \mathcal{X}^\Gamma\!\downarrow\! I(\mathrm{Eq}(p)) \longrightarrow \mathcal{C}^\Gamma\!\downarrow\!\mathrm{Eq}(p)$ is fully faithful. In fact it restricts to the discrete opfibrations lying in Θ, as in

$$\begin{array}{ccc} \mathcal{X}^\Gamma\!\downarrow\! IP & \xrightarrow{H^P} & \mathcal{C}^\Gamma\!\downarrow\! P \\ \cup & & \cup \\ \mathcal{X}^{\downarrow IP} & \xrightarrow[K^P]{} & \mathcal{C}^{\downarrow P} \end{array} \quad , \tag{5.14}$$

where P is now short for $\mathrm{Eq}(p)$ and K^P denotes the restriction of H^P; for H^P is given by pulling back along η_P as in (2.1), and pulling back preserves discrete opfibrations — of course the discrete opfibrations in $\mathcal{X}^\Gamma$ are just those that are such in $\mathcal{C}^\Gamma$.

Let us now expand diagram (5.14) as follows:

$$
\begin{tikzcd}
\mathcal{X}^{\Gamma}\!\downarrow\! IP \arrow[r, "\simeq"] & \mathcal{M}\!\downarrow\! P \arrow[r, "\subset"] & \mathcal{C}^{\Gamma}\!\downarrow\! P \\
\cup & \cup & \cup \\
\mathcal{X}^{\downarrow IP} \arrow[r, "\simeq"'] & \mathcal{C}^{\downarrow P} \cap \mathcal{M}\!\downarrow\! P \arrow[r, "\subset"] & \mathcal{C}^{\downarrow P} \\
 & & \mathcal{C}^{\downarrow B} \arrow[u, "\simeq"'] \\
 & \mathrm{Spl}(E,p) \arrow[uu, "\simeq"] \arrow[r, "\subset"'] & \mathcal{C}\!\downarrow\! B \arrow[u, "\simeq"']
\end{tikzcd}
\quad . \tag{5.15}
$$

In the top row $\mathcal{M}\downarrow P$ is the image of H^P by Proposition 2.2, so that the first arrow is an equivalence as in (2.9). The inverse of this equivalence is of course given by restricting the functor $I^P : \mathcal{C}^{\Gamma}\downarrow P \longrightarrow \mathcal{X}^{\Gamma}\downarrow IP$, and this restriction $I^P : \mathcal{M}\downarrow P \longrightarrow \mathcal{X}^{\Gamma}\downarrow IP$ sends discrete opfibrations to discrete opfibrations by Proposition 2.4; whence the equivalence $\mathcal{X}^{\downarrow IP} \simeq \mathcal{C}^{\downarrow P} \cap \mathcal{M}\downarrow P$ of the second row. The equivalence $\mathcal{C}\downarrow B \simeq \mathcal{C}^{\downarrow P}$ of the last column is that of (5.11), expressing that p is a Θ-effective-descent morphism and sending $f : A \longrightarrow B$ to the $\phi = (\phi_0, \phi_1, \phi_2)$ of (5.13); since this lies in the pullback-stable $\mathcal{M}$ precisely when ϕ_0 does so, which is just to say that $f \in \mathrm{Spl}(E,p)$, we have the equivalence $\mathrm{Spl}(E,p) \simeq \mathcal{C}^{\downarrow P} \cap \mathcal{M}\downarrow P$ of the second column. This equivalence and that of the second row combine to give the chief theorem of Galois theory:

5.1 Theorem. *When $p : E \longrightarrow B$ is a Θ-effective-descent morphism and the objects $E, E \times_B E, E \times_B E \times_B E$ are admissible, we have an equivalence of categories*

$$
\mathrm{Spl}(E,p) \;\simeq\; \mathcal{X}^{\downarrow I(\mathrm{Eq}(p))} \; . \tag{5.16}
$$

Note that the equivalence (5.11) when p is a Θ-effective-descent morphism may be seen as the special case of (5.16) obtained by taking $I : \mathcal{X} \longrightarrow \mathcal{C}$ to be the identity $1 : \mathcal{C} \longrightarrow \mathcal{C}$.

For our present purposes of seeking a left adjoint to the inclusion $\mathrm{Spl}(E,p) \longrightarrow \mathcal{C}\downarrow B$, the point is that we now have a new description of this inclusion, alternative to that given by Theorem 2.5 : namely, with the hypotheses of Theorem 5.1, this inclusion agrees to within equivalence with the fully-faithful K^P of (5.14). This gives us two clearly-sufficient conditions for the reflectiveness of $\mathrm{Spl}(E,p)$ in $\mathcal{C}\downarrow B$:

5.2 Proposition. *When $p : E \longrightarrow B$ satisfies the conditions of Theorem 5.1,* $\mathrm{Spl}(E,p)$ *is reflective in $\mathcal{C}\downarrow B$ if either (a) or (b) below is the case:*

(a) *for any internal precategory R in $\mathcal{X}$, the inclusion*

$$
\mathcal{X}^{\downarrow R} \longrightarrow \mathcal{X}^{\Gamma}\downarrow R
$$

has a left adjoint;

(b) *the left adjoint* $I^P : \mathcal{C}^\Gamma{\downarrow}P \longrightarrow \mathcal{X}^\Gamma{\downarrow}IP$ *of* H^P*, when restricted to* $\mathcal{C}^{\downarrow P}$*, takes its values in* $\mathcal{X}^{\downarrow IP}$*.*

We devote the next section to the pursuit of (a), and shall make use of (b) in the following one. Since a discrete opfibration may be seen as a diagram wherein two squares are to be pullback-squares, we must recall what is known about diagrams in which certain cones are limit-cones.

6. Functors sending certain cones to limit-cones

Consider a small category $\mathcal{K}$ and a class Ψ of projective cones in $\mathcal{K}$: each such cone ψ consists of a small category $\mathcal{J}_\psi$, a functor $N_\psi : \mathcal{J}_\psi \longrightarrow \mathcal{K}$, an object M_ψ of $\mathcal{K}$, and the cone ψ itself, which is a natural transformation $\psi : \Delta M_\psi \longrightarrow N_\psi$, with components $(\psi(j) : M_\psi \longrightarrow N_\psi(j))_{j\in\mathcal{J}_\psi}$. For a category $\mathcal{C}$ one may consider, within the functor category $\mathcal{C}^\mathcal{K}$ of all functors $T : \mathcal{K} \longrightarrow \mathcal{C}$, the full subcategory $\mathcal{C}^K(\Psi)$ given by those T for which each $T\psi : \Delta TM_\psi \longrightarrow TN_\psi$ is a limit-cone in $\mathcal{C}$. Finding criteria for $\mathcal{C}^\mathcal{K}(\Psi)$ to be reflective in $\mathcal{C}^\mathcal{K}$ has been called the *continuous functor problem.*

The earliest treatment for a $\mathcal{C}$ other than **Set** and for a possibly large Ψ was given by Freyd and Kelly [7] and later refined in [20]. For simplicity, take $\mathcal{C}$ to be complete and cocomplete. Each cone ψ as above gives rise in $\mathbf{Set}^\mathcal{K}$ to a morphism

$$\mathrm{colim}_j\ \mathcal{K}(N_\psi(j), -) \underset{k_\psi}{\longrightarrow} \mathcal{K}(M_\psi, -)\ , \tag{6.1}$$

which we may abbreviate to $k_\psi : U_\psi \longrightarrow V_\psi$. For a set X and for $C \in \mathcal{C}$ write $X \cdot C \in \mathcal{C}$ for the coproduct of X copies of C, and consider the morphisms $k_\psi \cdot C : U_\psi \cdot C \longrightarrow V_\psi \cdot C$ for $C \in \mathcal{C}$; clearly $T \in \mathcal{C}^\mathcal{K}$ lies in $\mathcal{C}^\mathcal{K}(\Psi)$ precisely when it is *orthogonal* to each $k_\psi \cdot C$, in the sense that $k_\psi \cdot C$ is inverted by $\mathcal{C}^\mathcal{K}(-, T)$. If the $k_\psi \cdot C$ formed a small set, one could now by an easy application of Wolff's process in Proposition 3.5 above construct on $\mathcal{C}^\mathcal{K}$ a well-pointed endofunctor with $\mathcal{C}^\mathcal{K}(\Psi)$ as its algebras, and thus find a criterion for the reflectivity of $\mathcal{C}^\mathcal{K}(\Psi)$ in $\mathcal{C}^\mathcal{K}$ by appealing to Proposition 3.3.

However the class Ψ of cones may be large (although in our application it is not), and certainly the class $\{C\}$ of objects of $\mathcal{C}$ is usually large. One gets a first reduction of the size of the class $\{k_\psi \cdot C\}$ by supposing $\mathcal{C}$ to admit a small generating set $\mathcal{G}$ with respect to a suitable factorization system $(\mathcal{F}, \mathcal{N})$ on $\mathcal{C}$, so that it suffices for T to be orthogonal to the $k_\psi \cdot G$ with $G \in \mathcal{G}$. Moreover, even if the class Ψ is large, the number of different codomains V_ψ for the k_ψ is small, since by (6.1) these codomains are *representables* in $\mathbf{Set}^\mathcal{K}$; and this leads to a further reduction when we take the $(\mathcal{F}, \mathcal{N})$-factorizations of the $k_\psi \cdot G$ and observe that orthogonality to a large class of morphisms can be handled with no trouble when these are epimorphisms.

On examining the details, with the intention of applying Proposition 3.3, one is led to make the following definition (whose definitive form was first given in [21, Section 6.1]):

6.1 Definition. *A category $\mathcal{C}$ is said to be* locally bounded *if*

(i) *it is locally small, complete, and cocomplete;*

(ii) *with respect to some proper factorization system $(\mathcal{F},\mathcal{N})$ it has a small generating set $\mathcal{G}$ — that is to say, the canonical morphism*

$$\sum_{G\in\mathcal{G}} \mathcal{C}(G,C)\cdot G \longrightarrow C$$

lies in $\mathcal{F}$ for each $C \in \mathcal{C}$;

(iii) *every family $(C \longrightarrow D_i)_{i\in I}$ of morphisms in $\mathcal{F}$, however large, admits a cointersection in $\mathcal{C}$;*

(iv) *there is some (small, infinite) regular cardinal α such that $\mathcal{C}(G,-) : \mathcal{C} \longrightarrow$ **Set** preserves α-filtered unions of $\mathcal{N}$-subobjects for each $G \in \mathcal{G}$.*

This definition gives the criterion we want : for applying Proposition 3.3 to the considerations above yields [20, Theorem 12.1], part of which reads (see also [21, Theorem 6.11]):

6.2 Proposition. *Whenever the category $\mathcal{C}$ is locally bounded, $\mathcal{C}^{\mathcal{K}}(\Psi)$ is reflective in $\mathcal{C}^{\mathcal{K}}$ for any small $\mathcal{K}$ and any class Ψ of projective cones.*

6.3 Remark. Every locally-presentable category $\mathcal{C}$ (in the sense of Gabriel and Ulmer [9]) is locally bounded, with the monomorphisms for $\mathcal{N}$. The category **Top** of topological spaces is not locally presentable, but is locally bounded with the subspace-inclusions for $\mathcal{N}$; while the category of Banach spaces and norm-decreasing linear maps is locally bounded with the inclusions of closed subspaces (with the induced norm) for $\mathcal{N}$.

There is another set of criteria for the reflectivity of $\mathcal{C}^{\mathcal{K}}(\Psi)$ in $\mathcal{C}^{\mathcal{K}}$, virtually incompatible with that above — for it is never the case that both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are locally presentable, unless $\mathcal{C}$ is a complete lattice. For in fact we have:

6.4 Proposition. *Whenever the category $\mathcal{C}^{\mathrm{op}}$ is locally presentable, $\mathcal{C}^{\mathcal{K}}(\Psi)$ is reflective in $\mathcal{C}^{\mathcal{K}}$ for any small $\mathcal{K}$ and any class Ψ of projective cones. The same is true when $\mathcal{C} =$ **Top**$^{\mathrm{op}}$, the dual of the category of topological spaces.*

6.5 Remark. For the cases $\mathcal{C} = \mathbf{Set}^{\mathrm{op}}$ and $\mathcal{C} = \mathbf{Ab}^{\mathrm{op}}$, the truth of Proposition 6.4 is asserted in Freyd's early book [6, Ch.5, Exercise F], along with the observation that there is a proof using the special adjoint functor theorem; it was in checking that assertion long ago that Kelly verified (unpublished) that it works equally well for **Top**$^{\mathrm{op}}$. The rest of Proposition 6.4 was given in unpublished notes [26], [27], [28] of Ulmer, reported at the Oberwolfach meetings on Category Theory in 1975 and 1977. A simple and attractive proof was given in Bird's thesis ([25, Theorem 4.18]), as part of his study of limits and colimits in the 2-category of locally-presentable categories; but this too is unpublished. There seems to be no *published* account of this precise result; but it follows easily from the result [25, Cor. 6.2.5] of Makkai and Paré, when combined with Freyd's Special Adjoint Functor Theorem.

We can now apply the above to the question raised in (a) of Proposition 5.2, concerning the reflectivity of $\mathcal{X}^{\downarrow R}$ in $\mathcal{X}^{\Gamma}\!\downarrow\! R$. The occurrence of Θ in the definitions of $\mathcal{X}^{\downarrow R}$ and of $\mathcal{X}^{\Gamma}\!\downarrow\! R$ often causes no problem:

6.6 LEMMA. *If the class Θ has the property that $\phi \in \Theta$ whenever some composite ϕg lies in Θ, then to prove $\mathcal{X}^{\downarrow R}$ reflective in $X^{\Gamma}\!\downarrow\! R$ it suffices to prove $\mathcal{X}^{R}$ reflective in $\mathcal{X}^{\Gamma}/R$.*

PROOF. Let $f : Q \longrightarrow R$ lie in Θ and hence in $\mathcal{X}^{\Gamma}\!\downarrow\! R \subset X^{\Gamma}/R$, and let its reflexion in $\mathcal{X}^{R}$ be the discrete opfibration $\phi : S \longrightarrow R$ — the unit being $g : Q \longrightarrow S$, so that $\phi g = f$. Now, since ϕ is in Θ, it is clearly the reflexion of $f \in \mathcal{X}^{\Gamma}\!\downarrow\! R$ into $\mathcal{X}^{\downarrow R}$. ∎

6.7 LEMMA. *To prove $\mathcal{X}^{R}$ reflective in $\mathcal{X}^{\Gamma}/R$, it suffices to prove reflective in the arrow category $(\mathcal{X}^{\Gamma})^{\mathbf{2}}$ the full subcategory $\mathcal{D}$ consisting of the discrete opfibrations.*

PROOF. Since the class $\mathcal{D}$ contains the identities, a simple argument (given by Im and Kelly in [10, Proposition 5.1]) shows that the reflexion of $(\mathcal{X}^{\Gamma})^{\mathbf{2}}$ onto $\mathcal{D}$, sending say $f : Q \longrightarrow R$ to $f' : Q' \longrightarrow R'$ with unit (u, v) as in

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "u"'] & R \arrow[d, "v"] \\ Q' \arrow[r, "f'"'] & R' \end{tikzcd} ,$$

∎

has v invertible. Since, therefore, we may as well take v to be the identity 1_R, our reflexion of $(\mathcal{X}^{\Gamma})^{\mathbf{2}}$ onto $\mathcal{D}$ restricts to a reflexion of $\mathcal{X}^{\Gamma}/R$ onto $\mathcal{X}^{R}$.

Since $(\mathcal{X}^{\Gamma})^{\mathbf{2}} = \mathcal{X}^{\Gamma \times \mathbf{2}}$ and since $\mathcal{D}$ consists of those functors $\Gamma \times \mathbf{2} \longrightarrow \mathcal{X}$ sending certain squares to the pullback squares of (5.4), we can apply Propositions 6.2 and 6.4. Combining the above with Proposition 5.2 gives:

6.8 THEOREM. *Let the Galois structure $(\mathcal{C}, \mathcal{X}, H, I, \eta, \Theta)$ be such that $\phi \in \Theta$ whenever some $\phi g \in \Theta$, and let $p : E \longrightarrow B$ be a Θ-effective-descent morphism for which the objects $E, E \times_B E$, and $E \times_B E \times_B E$ are admissible. Then* $\mathrm{Spl}(E, p)$ *is reflective in $\mathcal{C}\!\downarrow\! B$ if $\mathcal{X}$ is locally bounded, or if $\mathcal{X}^{\mathrm{op}}$ is locally presentable, or if $\mathcal{X}^{\mathrm{op}} =$ **Top**.*

7. Some special situations

By analogy with the classical Galois theory of field extensions, we call $p : E \longrightarrow B$ *normal* in a general Galois structure when (E, p) itself lies in $\mathrm{Spl}(E, p)$ — which of course makes sense only when $p \in \Theta$. In this case the diagram (5.13), connecting $f \in \mathcal{C}\!\downarrow\! B$ with its image ϕ in $\mathcal{C}^{\downarrow P}$ where $P = \mathrm{Eq}(p)$, has for $f = p$ the special case

$$\begin{tikzcd}
E \times_B E \times_B E \times_B E \arrow[r] \arrow[d] & E \times_B E \times_B E \arrow[r, "q"] \arrow[d] & E \times_B E \arrow[r, "d"] \arrow[d, "c"] & E \arrow[d, "p"] \\
E \times_B E \times_B E \arrow[r, "q"'] & E \times_B E \arrow[r, "d"'] & E \arrow[r, "p"'] & B
\end{tikzcd} \quad ; \tag{7.1}$$

from which we see that the pullback d of p along p and the pullback q of d along c lie in $\mathcal{M} \cap \Theta$. Accordingly, for any $\phi : Q \longrightarrow P$ in $\mathcal{C}^{\downarrow P}$, it follows from Proposition 2.4 that I preserves both the pullbacks in (5.4), provided that Q_0 and Q_1 are admissible. When this is so, we have (b) of Proposition 5.2, leading to the reflectivity of $\mathrm{Spl}(E, p)$ in $\mathcal{C}{\downarrow}B$; however we need some hypothesis to ensure the admissibility of Q_0 and Q_1, such as the final condition in the following:

7.1 Theorem. *For a Galois structure as above, let $p : E \longrightarrow B$ be a Θ-effective-descent morphism lying in Θ, and such that (E, p) is normal. Then* $\mathrm{Spl}(E, p)$ *is reflective in $\mathcal{C}{\downarrow}B$, provided that every $C \in \mathcal{C}$ admitting some morphism $C \longrightarrow E$ is admissible.*

After all our talk of criteria for the reflectivity of $\mathrm{Spl}(E, p)$ in $\mathcal{C}{\downarrow}B$, we remind the reader that, as we said in the Introduction, our interest is often rather in the reflectivity of Cov(B) in $\mathcal{C}{\downarrow}B$ — the latter being a consequence of the former when some $p : E \longrightarrow B$ splits all coverings. Suppose for the following remarks that Θ consists of all morphisms, and consequently that $\mathcal{C}$ admits pullbacks. We observed in Section 1 that, since $\mathrm{Cov}(B)$ is pullback-stable, and since identity morphisms are coverings, to say that such $\mathrm{Cov}(B)$ is reflective in $\mathcal{C}/B$ is equally to say that the class $\mathcal{M}^*$ of all coverings is reflective in the arrow-category $\mathcal{C}^{\mathbf{2}}$; see [10], or [4, Section 2.12]. The examples of the present article show this to be so for many Galois structures — but it is considerably less common for the class $\mathcal{M}^*$ of coverings to be closed under composition; see [4, Section 10.3] for a counter-example.

In fact $\mathcal{M}^*$ is closed under composition precisely when it is part of a factorization system $(\mathcal{E}^*, \mathcal{M}^*)$ on $\mathcal{C}$, as is shown in each of the references above. When this *is* the case, the class $\mathcal{E}^*$ need not be pullback-stable, as is shown in [4, Section 10.2]; when it *is* pullback stable, it coincides with the class of those e every pullback of which lies in $\mathcal{E}$.

Thus the Galois structures for which the $\mathrm{Cov}(B) = \mathcal{M}^*/B$ are not only reflective in $\mathcal{C}/B$ but arise from a *pullback-stable* factorization system $(\mathcal{E}^*, \mathcal{M}^*)$ are extremely special. Yet there are several important examples — including that adumbrated in Example 1.3 above, where the $(\mathcal{E}^*, \mathcal{M}^*)$ factorization is Eilenberg's monotone-light factorization for maps of compact Hausdorff spaces. The situation was studied by the present authors along with Carboni and Paré in [4], and the central result, in one of its forms, is:

7.2 Theorem. *In a Galois structure as above, let $\mathcal{C}$ admit pullbacks and let Θ consist of all morphisms. Then the class $\mathcal{M}^*$ of coverings is part of a pullback-stable factorization system $(\mathcal{E}^*, \mathcal{M}^*)$ if and only if, for each $f : A \longrightarrow B$ in $\mathcal{C}$, there is some effective descent* *morphism $p : E \longrightarrow B$ such that, if $p^*(f)$ has me for its $(\mathcal{E}, \mathcal{M})$-factorization, every pullback of e lies in $\mathcal{E}$. In such a case, of course, each Cov(B) $= \mathcal{M}^*/B$ is reflective in $\mathcal{C}/B$.*

Simplifications of the criterion of the theorem adapting it to practical examples are too special and too technical to give here : see [4] for further details.

8. Principal examples

Given a Galois structure $(\mathcal{C}, \mathcal{X}, H, I, \eta, \Theta)$, we say that the object B of $\mathcal{C}$ is *locally simply connected in the geometric sense* if there exists a Θ-effective-descent morphism $p : E \longrightarrow B$, itself lying in Θ, such that every covering of E is trivial; such a B is of course locally simply connected in the sense of Section 1 above, $\mathrm{Cov}(B)$ coinciding with $\mathrm{Spl}(E, p)$ since covering-morphisms are pullback-stable; the point is that the present notion agrees with the geometrical idea of a locally-simply-connected space — see Example 8.2 below. Furthermore, we say that B is *locally projective* if there exists in Θ a Θ-effective-descent morphism $p : E \longrightarrow B$ having E projective with respect to Θ-effective-descent morphisms. The observation that a locally projective B is locally simply connected in the geometric sense, although obvious, will be useful in Example 8.4 below.

We turn now to a variety of examples.

8.1 Barr-Diaconescu covering theory.
As was shown in [16], the covering theory of Barr and Diaconescu [1] can be obtained as a special case of the categorical Galois theory expounded above, by considering the following Galois structure:

(i) $\mathcal{C}$ is a connected and locally-connected cocomplete elementary topos;
(ii) $\mathcal{X} = \mathbf{Set}$, and $H : \mathbf{Set} \longrightarrow \mathcal{C}$ is the fully-faithful functor sending X to $X \cdot 1$, the coproduct of X copies of the terminal object 1;
(iii) $I : \mathcal{C} \longrightarrow \mathbf{Set}$, as the left adjoint of H, sends an object C of $\mathcal{C}$ to the set IC of its *connected components* (called *molecules* by Barr and Diaconescu), with the evident unit $\eta C : C \longrightarrow HIC$;
(iv) Θ is the class of all morphisms in $\mathcal{C}$.

For this Galois structure, every object of $\mathcal{C}$ is admissible, and every epimorphism in $\mathcal{C}$ is an effective descent morphism (and so a Θ-effective-descent morphism). Moreover, for every epimorphism $p : E \longrightarrow B$ in $\mathcal{C}$, all the conditions required in Theorem 4.2 and in Theorem 6.8 are in fact satisfied.

Indeed, for Theorem 4.2 we have:

(a) $\mathcal{C}$ admits pushouts and filtered colimits because it admits all colimits;
(b) $\mathcal{C}{\downarrow}B$ is closed in $\mathcal{C}/B$ under colimits because $\mathcal{C}{\downarrow}B = \mathcal{C}/B$;
(c) E is admissible because every object in C is admissible;
(d) H preserves filtered colimits because, as in [1], it has the right adjoint $\Gamma = \mathcal{C}(1, -)$;

(e) $f^* : \mathcal{C}{\downarrow}A \longrightarrow \mathcal{C}{\downarrow}C$ preserves filtered colimits because it coincides with $f^* : \mathcal{C}/A \longrightarrow \mathcal{C}/C$ which ($\mathcal{C}$ being a topos) has a right adjoint $f_* : \mathcal{C}/C \longrightarrow \mathcal{C}/A$.

Again, for Theorem 6.8 we have:

(f) $\phi \in \Theta$ whenever $\phi g \in \Theta$ because Θ is the class of all morphisms;
(g) E, $E \times_B E$, and $E \times_B E \times_B E$ are admissible because all objects are so;
(h) $\mathcal{X}$, being **Set**, is locally bounded.

Thus each of the theorems 4.2 and 6.8 tells us that the inclusion Spl $(E,p) \longrightarrow \mathcal{C}{\downarrow}B$ has a left adjoint. This was of course proved (for $B = 1$) in [1]; and our proof of Theorem 4.2 may be seen as a generalized (and improved) version of the proof in [1].

In order to conclude that $\mathrm{Cov}(B)$ is reflective in $\mathcal{X}{\downarrow}B = \mathcal{C}/B$, we need to add the requirement that B is locally simply connected — or equivalently that, for each connected component C of B, the *topos* $\mathcal{C}/C$ is locally simply connected in the sense of [1]. In particular, if $\mathcal{C}/C$ is locally simply connected for every connected object C in $\mathcal{C}$, then the class of all coverings in $\mathcal{C}$ is reflective. This is the case when $\mathcal{C}$ is the category of simplicial sets; for then every object in $\mathcal{C}$ is locally projective.

8.2 Reflexion of local homeomorphisms into étale maps.
This is just the special case of the previous example given by taking for $\mathcal{C}$ the topos $\mathrm{Shv}(S)$ of sheaves on a connected and locally-connected topological space S. Here $f : A \longrightarrow B$ is a covering in our sense precisely when, considered as a map between the corresponding étale spaces, it is a covering map in the classical geometrical sense. To say that S is locally simply connected in the geometric sense is to say that there is a surjective local homeomorphism $S' \longrightarrow S$ with S' simply connected (in the sense that every covering of S' is trivial); which is certainly the case if every point of S has a simply-connected open neighbourhood. When this is true of S, it is also true of any space having a local homeomorphism from itself to S; then every object in $\mathrm{Shv}(S)$ is locally simply connected, so that the class of coverings in $\mathrm{Shv}(S)$ is reflective. The geometric meaning of this is very simple : for locally simply connected spaces A and B, every local homeomorphism $f : A \longrightarrow B$ has, among its factorizations $f = f'f''$ with f' a covering map, a universal one.

Note that the reflectiveness of the class of coverings could also have been deduced from Theorem 7.1, taking $p : E \longrightarrow B$ therein to be the *universal covering* of B; however this would be circular as it stands, since we *use* the reflectiveness, proved by Theorems 4.2 or 6.8, to *establish the existence* of the universal covering — namely by taking the reflexion of $0 \longrightarrow B$ into the coverings of B, where 0 is the empty space.

8.3 Galois theory of commutative rings.
As was shown in [11] and [12], we can obtain Magid's Galois theory of commutative rings [24] as a special case of the categorical Galois theory above by taking the following Galois structure:

(i) $\mathcal{C}$ is the dual of the category of commutative rings (with 1);
(ii) $\mathcal{X}$ is the category of Stone spaces — also called profinite spaces, or again compact totally-disconnected spaces;

(iii) $H : \mathcal{X} \longrightarrow \mathcal{C}$ sends a space X to the ring of all locally-constant maps from X to the ring $\mathbf{Z}$ of integers;
(iv) $I : \mathcal{C} \longrightarrow \mathcal{X}$ has $I(A) = \mathrm{Spec}\ B(A)$, the Stone space of the boolean algebra of idempotents in A – this is also the space $\mathrm{Comp\,Spec\,A}$ of connected components of the Zariski spectrum of A;
(v) Θ is the class of all morphisms.

For this Galois structure, every object of $\mathcal{C}$ is admissible — see [12, Theorem 2.1]. An epimorphism $p : E \longrightarrow B$ in $\mathcal{C}$ may be identified with a ring extension $B \subset E$; and it is known (Joyal and Tierney, unpublished) that the effective descent morphisms are those epimorphisms $p : E \longrightarrow B$ for which the inclusion $B \subset E$ is a *pure monomorphism* of B-modules.

For each such p, the inclusion $\mathrm{Spl}(E, p) \longrightarrow \mathcal{C}{\downarrow}B = \mathcal{C}/B$ has a left adjoint by Theorem 6.8, whose conditions are satisfied for simple "general reasons" :

(a) $\phi \in \Theta$ wherever $\phi g \in \Theta$ because Θ is the class of all morphisms;
(b) E, $E \times_B E$, and $E \times_B E \times_B E$ are admissible because all objects of $\mathcal{C}$ are so;
(c) $\mathcal{X}^{\mathrm{op}}$, being the category of Boolean algebras, is locally presentable.

Moreover every object B of $\mathcal{C}$ is locally simply connected : if we take E to be the separable closure $\overline{B}$ of B in the sense of Magid [24] and $p : E \longrightarrow B$ to be the canonical inclusion $B \subset \overline{B}$, then $p : E \longrightarrow B$ splits all coverings — indeed, since $\overline{B}$ has no non-trivial coverings, B is locally simply connected in the geometrical sense. Again, since it is well known that the separable closure is a faithfully flat module, $p : E \longrightarrow B$ is an effective descent morphism by a classical result going back to Grothendieck. Thus the class of all coverings in $\mathcal{C}$ is reflective.

Note that, since $\overline{B}$ is normal, we could also have used Theorem 7.1 to conclude that coverings are reflective.

Recall from [12] that the epimorphism $f : A \longrightarrow B$ corresponding to the extension $B \subset A$ is a covering if and only if this ring-extension is quasi-separable — that is, component-wise locally strongly separable. So the reflectiveness of coverings tells us that, for every ring-extension $R \subset S$, there is a largest R-subalgebra S' of S for which the extension $R \subset S'$ is quasi-separable; we might call the elements of S' the *separable elements* over R. Indeed, this notion coincides with the ordinary one if R and S are fields. Moreover, whether or not R is a field, if $u \in R[x]$ is a separable polynomial over R, then all the roots of u in S are separable elements.

8.4 Generalized central extensions of universal algebras.
Consider the following Galois structure:

(i) $\mathcal{C}$ is a variety of universal algebras;
(ii) $\mathcal{X}$ is a subvariety of $\mathcal{C}$;
(iii) $H : \mathcal{X} \longrightarrow \mathcal{C}$ is the inclusion, I is its left adjoint, and η the unit of the adjunction;
(iv) Θ is the class of surjective homomorphisms.

Since $\mathcal{C}$ is an exact category, the effective descent morphisms are the surjections — and in fact the Θ-effective-descent morphisms are also the surjections, provided that the (one-sorted) variety $\mathcal{C}$ is non-degenerate. For any surjective $p : E \longrightarrow B$ in $\mathcal{C}$ with E admissible, $\mathrm{Spl}(E, p)$ is reflective in $\mathcal{X}\!\downarrow\! B$ since the conditions of Theorem 4.2 are satisfied:

(a) $\mathcal{C}$ admits pushouts and filtered colimits because it admits all colimits;
(b) $\mathcal{C}\!\downarrow\! B$ is closed in $\mathcal{C}/B$ under colimits because the class Θ of surjections is closed under colimits in $\mathcal{C}^{\mathbf{2}}$;
(c) E is admissible by hypothesis;
(d) H preserves filtered colimits since these are formed both in $\mathcal{C}$ and in $\mathcal{X}$ as they are in **Set**;
(e) $f^* : \mathcal{C}\!\downarrow\! A \longrightarrow \mathcal{C}\!\downarrow\! C$ preserves filtered colimits because pullbacks commute with filtered colimits in $\mathcal{C}$ as they do in **Set**.

Moreover, when not only E but also $E \times_B E$ and $E \times_B E \times_B E$ are admissible, the conditions of Theorem 6.8 are also satisfied:

(f) ϕ is surjective whenever ϕg is so;
(g) $E, E \times_B E, E \times_B E \times_B E$ are admissible by hypothesis;
(h) the variety $\mathcal{X}$, being locally presentable, is locally bounded.

We showed in [17, Theorem 3.4] that, when $\mathcal{C}$ is a congruence-modular variety, every object in $\mathcal{C}$ is admissible. In that case, therefore, we conclude either from Theorem 4.2 or Theorem 6.8 that $\mathrm{Spl}(E, p)$ is reflective in $\mathcal{C}\!\downarrow\! B$ for every surjective $p : E \longrightarrow B$. Since for each $B \in \mathcal{C}$ we have a surjective $p : E \longrightarrow B$ with E free, every B is in fact locally projective, so that the class of coverings is reflective.

The covering morphisms in this case — or rather the *covering surjections*, since Θ consists of the surjections — were called in [17] the *central extensions*; the justification for this name arising in part from the fact that, when $\mathcal{C}$ and $\mathcal{X}$ are varieties of Ω-groups, our coverings coincide with the central extensions in the sense of A. Fröhlich — see Fröhlich [8] and Lue [23] — which include the ordinary central extensions of groups. In this last classical case, the reflectiveness of the coverings is obvious : every surjective group homomorphism has the universal factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & A/[A, \mathrm{Ker}\ f] \arrow[ur, "f'"'] & \end{tikzcd}$$

with f' central.

Note that one may take the adjunction provided by the reflectiveness of the coverings, along with a suitable class Θ', as itself giving a new Galois structure, and construct thereby a theory of *double central extensions* — as was done in [14] for the case of ordinary central extensions of groups; whereupon, proving next the reflectiveness of double central extensions, one may define triple central extensions and so on. One may further pursue this line for coverings in general, and so construct a “higher-dimensional Galois theory”.

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