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GEOMETRIC CONSTRUCTION OF THE LEVI-CIVITA PARALLELISM

To Bill Lawvere on the occasion of his 60th birthday

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Transmitted by Michael Barr

ABSTRACT. In terms of synthetic differential geometry, we give a variational characterization of the connection (parallelism) associated to a pseudo-Riemannian metric on a manifold.

Introduction

A basic result in differential geometry is: to a Riemannian metric on a manifold, there exists a unique symmetric affine connection compatible with it. This connection is the *Levi-Civita* or *Riemannian* connection. ('Parallelism' is the original word used for what nowadays is called 'affine connection', cf. [10]. We shall follow modern usage, even though the word 'parallelism' has more geometric connotations.) The (modern) standard proof/construction (cf e.g. [2], Theorem 3.6) is an elegant algebraic manipulation with vector fields, their Lie brackets, their inner products, and their covariant derivatives along each other. This manipulation is purely algebraic, and the geometry is not very explicit. A more geometric, or rather dynamic, construction, is due to J. Radon, cf. the Chapter in [3] entitled: "J. Radons mechanische Herleitung des Parallelismus von T. Levi-Civita". We shall give an alternative construction which is purely geometric (involving a variational principle). We do this by utilizing the method of Synthetic Differential Geometry. This method, quite generally, allows one to interweave more closely geometric language and figures with the notions of differential geometry; for instance, the data of an affine connection can be understood (cf. [7]) as a prescription for forming infinitesimal parallelograms, in a sense we shall recall and accompany by exact figures.

In the present note, I shall present a geometric construction of these parallelograms, starting with a pseudo-Riemannian metric g on the manifold M. The construction stays entirely within the geometry of *points* of M, thus does not deal with tangent vectors; formally, consideration of the tangent bundle, and the iterated tangent bundle, of M, is replaced by consideration of the first and second neighbourhood of the diagonal. This is in the spirit of [4], [6], [7], [8], and [5] I.18 (but unlike [11], [9], or most of [5]). In particular, we want to describe the notion of pseudo-Riemannian metric in these terms.

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Roughly, this description of the notion of metric, and the subsequent description of parallelograms, go lie this. The metric g associates to a pair of second-order neighbours x, z a number $g(x, z) \in R$, thought of as the squared distance of x and z. (Thus we require $g(x, z) = 0$ if x and z are first order neighbours.) Assuming a (standard) non-degeneracy property of g, we can prove the existence of a well defined *geodesic midpoint* y of such x and z; namely y is determined by a variational principle: it is a critical value for the "energy", $g(x, y) + g(y, z)$. Finally, having such midpoint formation, we can construct parallelograms by a piece of truly synthetic, and well known, Euclidean geometry, relating *parallelogram* and *midpoint* formation.

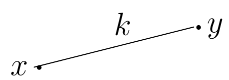
We prove that the parallelogram notion thus arrived at is indeed a connection. By coordinate calculations, not particularly elegant, we establish that it is the Levi-Civita connection. This would also follow if we could prove directly that it is compatible with the metric; one would hope that such compatibility property could be proved by geometric means, but we haven't yet been able to do so.

I want to thank Prakash Panangaden, who explained to me the importance of paths "with critical energy", in the context of Discrete Analytical Mechanics [1]. The idea of considering variation of energy rather than variation of arc length (which is ill behaved for 1- and 2-neighbours) is crucial to our approach here. (Classically, geodesics can be characterized by either of these variational principles, cf. e.g. [2] Ch. 9.)

1. Reminders

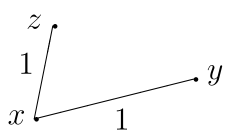
Recall that if R is a commutative ring, we may consider the subset $D_k(n) \subseteq R^n$ (affine n-space over R): it is the set of $(x_1, \ldots, x_n) \in R^n$ such that the product of any $k+1$ of the x's (repetitions allowed) is 0. We call two "vectors" $\underline{x}$ and $\underline{y}$ in R^n *k-neighbours* if $\underline{x} - \underline{y} \in D_k(n)$; we often write this as $\underline{x} \sim_k \underline{y}$. This is clearly a symmetric and reflexive relation. It is not transitive, but a simple binomial expansion proves that $\underline{x} \sim_k \underline{y}$ and $\underline{y} \sim_l \underline{z}$ implies $\underline{x} \sim_{k+l} \underline{z}$. Also $\underline{x} \sim_k \underline{y}$ implies $\underline{x} \sim_{k+1} \underline{y}$.

This applies also when R is a "ring of line type" in a model of Synthetic Differential Geometry (cf. e.g. [5] or [9]), which we shall henceforth assume, so R is to be thought of as the number line, R^n as affine n-space. Any (locally defined) map $R^n \to R^m$ preserves the relation $\sim_k$. An n-dimensional manifold is an M which is locally diffeomorphic to R^n. It follows that the relation $\sim_k$ can be defined on M from a chart $R^n \to M$, but is independent of the choice of the chart. The subset of pairs $x, y \in M$ with $x \sim_k y$ is called *the k'th neighbourhood of the diagonal* and denoted $M_{[k]} \subseteq M \times M$. In pictures, when we want to state that x and y are k-neighbours, we will connect them with an edge and write the number k on that edge:

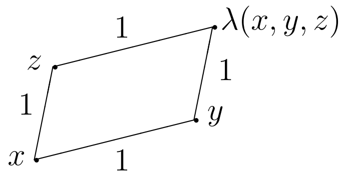


For $k = 1$, the edge itself has a geometric meaning, since we can canonically form affine combinations of 1-neighbours, see Theorem 2.2 below. For $k = 2$, some further structure on the manifold is needed to give geometric meaning ("geodesic") to such edge, see the last remarks of Section 3 below.

Recall from [7] that an *affine connection* λ on a manifold M is a law λ which allows one to complete any configuration (with $x \sim_1 y, x \sim_1 z$))



into a configuration



(with $z \sim_1 \lambda(x, y, z) \sim_1 y$), and configurations arising this way are called λ-parallelograms. There is only one axiom assumed:

$$\lambda(x, x, z) = z;\ \lambda(x, y, x) = y.$$

If $\lambda(x, y, z) = \lambda(x, z, y)$ for all $x \sim_1 y$, $x \sim_1 z$, we call the connection *symmetric*; these are the only ones we shall consider in the present note. For $M = R^n$, an example of a (symmetric) connection is given by

$$\lambda(x, y, z) = y + z - x,$$

and any connection λ on R^n is of form

$$\lambda(x, y, z) = y + z - x - \Gamma(x; y - x, z - x), \tag{1}$$

where for each $x \in R^n$, $\Gamma(x; -, -) : R^n \times R^n \to R^n$ is a bilinear map (symmetric iff λ is). (The bilinear map $\Gamma(x; -, -) : R^n \times R^n \to R^n$ is given by the Christoffel symbols Γ^k_{ij} of the connection.)

If besides $x \sim_1 y$ and $x \sim_1 z$, we also have $y \sim_1 z$ (so x, y, z form an *infinitesimal 2-simplex* in the terminology of the next Section), then there is no information contained in $\lambda(x, y, z)$, see Proposition 2.3 below.

2. Infinitesimal affine structure

It is a well known idea that a manifold is a space which is infinitesimally like an affine space (of finite dimension, say $\mathbf{R}^m$). We intend here to make this seed of truth grow a little more into a plant.

Recall that an *affine space* is a set equipped with an algebraic structure allowing one to form linear combinations where the sum of the coefficients is 1. Any vector space is an affine space. An affine space is *finite dimensional* if it isomorphic to some finite dimensional vector space, hence to some $\mathbf{R}^m$. Contrasting the finite dimensional affine spaces, we shall right away encounter some affine spaces which are not of this kind (which is not to say, though, that they are "infinite dimensional", in any reasonable sense). In fact, the affine space ensuing from the following result, is an example.

Recall from [8] that an *infinitesimal n-simplex* in a manifold M is an $n+1$-tuple of points $z_0, \ldots, z_n \in M$ with $z_i \sim_1 z_j$ for all $i, j = 0, \ldots, n$.

2.1. Lemma. *Let $z_0, z_1, \ldots, z_n$ be an infinitesimal n-simplex in a finite dimensional affine space W. Then the set of affine combinations of the z_i's form an affine subspace, and any two points in it are 1-neighbours.*

Proof. Clearly, the subset described is an affine subspace. Consider two arbitrary points in it, say $\sum_{i=0}^n t_i z_i$ and $\sum_{i=0}^n s_i z_i$. We want to prove that they are 1-neighbours. Without loss of generality, we may assume that W is a vector space, and that $z_0 = 0 \in W$. Then we have to prove that $(\sum_{i=1}^n t_i z_i - \sum_{i=1}^n s_i z_i) \sim_1 0$. In fact we prove that $\sum_{i=1}^n r_i z_i \sim_1 0$ for any n-tuple of scalars $r_1, \ldots, r_n$. This we do by calculating in coordinates, assuming that $W = \mathbf{R}^m$ (by the finite-dimensionality assumption). Let the p'th coordinate of z_i be denoted z_i^p. Then we have to prove that the product of the p'th and q'th coordinate of the linear combination $\sum r_i z_i$ is 0, for any pair of indices $p, q = 1, \ldots, m$. This product is

$$(\sum_i r_i z_i^p)(\sum_j r_j z_j^q) = \sum_{i<j} r_i r_j z_i^p z_j^q + \sum_i r_i r_i z_i^p z_i^q + \sum_{i>j} r_i r_j z_i^p z_j^q.$$

The middle sum is zero because $z_i \sim_1 0$. The $ij = kl$ term of the first sum $(k < l)$ cancels the $ij = lk$ term of the last sum because

$$z_l^p z_k^q = -z_k^p z_l^q;$$

this is a fundamental consequence of $z_k \sim_1 0$, $z_l \sim_1 0$ and $z_k \sim_1 z_l$,(cf. [5] I.16, or calculate). ∎

We can now prove

2.2. Theorem. *Let $x_0, x_1, \ldots x_n$ be an infinitesimal n-simplex in a manifold M (of dimension m, say). Then any formal affine combination of the vertices x_i can be canonically given a value in M. The subset span$(x_0, x_1, \ldots x_n)$ of M consisting of points that can be obtained in this way is canonically an affine space; and any two points of span$(x_0, x_1, \ldots x_n)$ are 1-neighbours.*

Proof/Construction. Pick a chart ϕ, i.e. a diffeomorphism from an open subset of a finite dimensional vector space V to an open subset of M containing the x_i's. For ease of notation, we shall write $\phi : V \to M$, although ϕ may not be everywhere defined on V. Let y_i be the unique element in V with $\phi(y_i) = x_i$. Let $t_0, t_1, \ldots, t_n$ be scalars with sum 1. Then we put

$$\sum_{i=0}^{n} t_i x_i := \phi(\sum_{i=0}^{n} t_i y_i).$$

If $\psi : W \to M$ is another such chart, with $\psi(z_i) = x_i$, there is a (locally defined) diffeomorphism $f : W \to V$ with $\phi \circ f = \psi$ and hence $f(z_i) = y_i$ $(i = 0, \ldots, n)$. Since $x_i \sim_1 x_0$, $z_i \sim_1 z_0$. The restriction of $f : W \to V$ to $\mathcal{M}_1(z_0)$ (=the set of 1-neighbours of z_0, being a set $\cong D_1(n)$) extends by the fundamental axioms of Synthetic Differential Geometry ("Kock-Lawvere axiom"), (cf. [5] or [9]) to a unique affine map $F : W \to V$. Now we calculate:

$$f(\sum_{i=0}^{n} t_i z_i) = F(\sum_{i=0}^{n} t_i z_i) = \sum_{i=0}^{n} t_i F(z_i) = \sum_{i=0}^{n} t_i f(z_i),$$

using that (by the Lemma) $\sum_{i=0}^{n} t_i z_i \in \mathcal{M}_1(z_0)$ on which f agrees with F, and the fact that F commutes with affine combinations.

From this, and from $\psi = f \circ \phi$, the well definedness immediately follows. The fact that any two points thus obtained as canonical affine combinations of the x_i's are 1-neighbours, is immediate from the similar fact, proved in the lemma, about affine combinations of the z_i's.

∎

2.3. Proposition. *Let λ be a symmetric affine connection on the manifold M. If x, y, z form an infinitesimal 2-simplex, then $\lambda(x, y, z)$ equals the affine combination $y - x + z$.*

Proof. If we pick a chart $\phi : V \to M$ with V an affine space, the data of λ can be expressed (for fixed x, and omitting ϕ from notation) as

$$\lambda(x, y, z) = y - x + z - \Gamma(y - x, z - x)$$

with $\Gamma : V \times V \to V$ bilinear, and symmetric, since λ is assumed to be so. But if $u \sim 0$, $v \sim 0$, and $u \sim v$ in a finite dimensional vector space V, then any symmetric bilinear form $\Gamma(-, -)$ vanishes on (u, v), by the calculations of [8]. ∎

3. Metrics

Although we shall not need here the notion of differential form, expressed in point-terms (see [8] for a recent account), we shall, for comparison, recall that a 1-form on a manifold M is a map $\omega : M_{[1]} \to R$, vanishing on $M_{[0]}$, i.e. with $\omega(x, x) = 0$ for all x; it is automatically alternating, $\omega(x, y) = -\omega(y, x)$.

Now we pose our version of pseudo-Riemannian metric on a manifold M: it is a *quadratic differential form* on M, with a non-degeneracy condition (to be made explicit later), where

3.1. Definition. *A quadratic differential form g on a manifold M is a map*

$$g : M_{[2]} \to R,$$

vanishing on $M_{[1]}$.

3.2. Proposition. *Let $g : M_{[2]} \to R$ be a quadratic differential form on M; then for all $x \sim_2 y$, we have*

$$g(x, y) = g(y, x).$$

Proof. Since the question is local, we may assume that M is the n-dimensional vector space R^n; then $M_{[2]} \cong R^n \times D_2(n)$ canonically, and under this isomorphism $g(x, y)$ corresponds to $G(x, y - x)$, with $G : R^n \times D_2(n) \to R$. The assumption that g vanishes on $M_{(1)}$ means that for each x, $G(x, -) : D_2(n) \to R$ vanishes on $D_1(n)$. By the fundamental axiom of Synthetic Differential Geometry, this implies that, in coordinates, $G(x, -)$ is given by a homogeneous degree 2 polynomial $R^n \to R$, so $G(x, d) = G(x, -d)$ for any $d \in D_2(n)$. Furthermore, $G(x, -d) = G(x + d, -d)$, since the Taylor expansion of the latter from x yields $G(x, -d)$ plus terms which are of degree 3 in the coordinates of $d \in D_2(n)$, and therefore vanish. Translating $G(x, d) = G(x + d, -d)$ back into g-terms yields $g(x, y) = g(y, x)$. ∎

The following "unique extension" result will be a fundamental tool in our construction:

3.3. Theorem. *Given a quadratic differential form $g : M_{[2]} \to R$ on M, there is a unique map $\overline{g} : M_{[3]} \to R$ which extends g and which is symmetric, $\overline{g}(x, z) = \overline{g}(z, x)$ for all $x \sim_3 z$.*

Proof/Construction. We first prove uniqueness; this will also provide the right formula. Again, we work in coordinates, assuming $M = R^n$. We use notation as in the proof of the Proposition above. Let $x \in M$ be fixed. Then $G(x, -) : R^n \to R$ is a homogeneous polynomial of degree 2; an extension of g to $M_{[3]}$ at x amounts to: adding to $G(x, -) : R^n \to R$ a polynomial $T(x, -) : R^n \to R$, homogeneous of degree 3. So such T we assume given for each x. Now let $x \sim_3 z$. The assumed symmetry $\overline{g}(x, z) = \overline{g}(z, x)$ reads, in terms of G and T:

$$G(x, z - x) + T(x, z - x) = G(z, x - z) + T(z, x - z).$$

We Taylor expand the right hand side here from x, in the first variable. In the T term, this has the effect of just writing $T(x, x - z)$ instead of $T(z, x - z)$, since the expression is already cubic in $x - z$, so that the correction term is killed. For the G term, $G(z, x - z) = G(x, x - z) + D_{z-x}G(x, x - z)$, where D_v means derivative along v (applied in the first variable), i.e. the differential of G applied to v. We call this the "directional derivative", even though v is not a unit vector. Thus the displayed equation becomes

$$G(x, z - x) + T(x, z - x) = G(x, x - z) + D_{z-x}G(x, x - z) + T(x, x - z).$$

(Here, D_v again refers to directional derivative in the first variable, and the whole formula refers to some coordinate chart, where it makes sense to extend g to a quadratic polynomial, etc.) But $T(x, x-z) = -T(x, z-x)$, since T is an odd function of its second argument (being homogeneous of degree 3). Similarly $G(x, z-x) = G(x, x-z)$. This term may be removed from both sides of the equality sign, and rearranging the remaining terms gives

$$2T(x, z-x) = D_{z-x}G(x, x-z),$$

from which the uniqueness of $T(x, -)$ follows. So in coordinates,

$$\overline{g}(x, z) = G(x, z-x) + \frac{1}{2}D_{z-x}G(x, z-x).$$

∎

If we rewrite the homogeneous degree 2 polynomial $G(x, -)$ in terms of a symmetric bilinear form $A(x; -, -)$ with $A(x; v, v) = G(x, v)$ for all v, then we get as coordinate expression for $\overline{g}$:

$$\overline{g}(x, z) = A(x; z-x, z-x) + \frac{1}{2}D_{z-x}A(x; z-x, z-x). \tag{2}$$

A quadratic differential form g on M is called *non-degenerate* if for every $x \in M$, and for one (hence every) coordinate system, the symmetric bilinear form $R^n \times R^n \to R$ to which the homogeneous degree 2 polynomial $G(x, -) : R^n \to R$ polarizes, is non-degenerate, i.e. has invertible determinant. A non-degenerate quadratic differential form on a manifold is called a *pseudo-Riemannian metric*; this is standard (except for the underlying notion of quadratic differential form); this is why we don't give it a displayed definition number. Our aim is to derive some synthetic geometry out of a pseudo-Riemannian metric, notably geometric notions like *parallelism* and *geodesics*. In classical differential geometry, this transition from the quantitative aspect, the metric, on the one hand, to the qualitative one, the synthetic geometry, on the other, is furnished by Levi-Civita's construction of a *connection* or *parallelism* out of the metric. This derivation is classically made as an analytic/algebraic calculation. The description we give is more geometric.

3.4. Definition. *Let $x \sim_2 z$ in a manifold M with a pseudo-Riemannian metric g. We say that y_0 is* geodesic midpoint *(relative to g) of x and z if $y_0 \sim_2 x$, $y_0 \sim_2 z$ and for any $y \sim_1 y_0$, we have $\overline{g}(x, y) + \overline{g}(z, y) = g(x, y_0) + g(z, y_0)$.*

Unique existence of geodesic midpoints will be proved below. Note that $y \sim_1 y_0$ implies $x \sim_3 y \sim_3 z$, but not, in general $x \sim_2 y$, or $y \sim_2 z$, so that we need the extension $\overline{g}$ to $M_{[3]}$, whose unique existence is asserted in Theorem 3.3.

The condition in the Definition may be viewed as a variational condition, in fact as an infinitesimal aspect of 'geodesics as critical (or stationary) for kinetic energy'.

3.5. Theorem. *For any $x \sim_2 z$ in a manifold M with a pseudo-Riemannian metric, there is a unique geodesic midpoint y_0 of x and z.*

We will deduce this by putting $t = \frac{1}{2}$ in a more general result:

3.6. Theorem. *For any $x \sim_2 z$ in a manifold M with a pseudo-Riemannian metric, and for any $t \in R$, there is a unique y_0 with $y_0 \sim_2 x$, and which is a critical point for the function of y*

$$t\overline{g}(x,y) + (1-t)\overline{g}(z,y). \tag{3}$$

Proof. We are going to make an appeal to a version of the implicit function theorem: locally, a certain implicitly defined function η will define y_0 as a function of z where $z \sim_2 x$ (keeping x fixed). To state a version of the implicit function theorem in the synthetic context (and hence to investigate its validity in the models) requires that one makes explicit the use of the word 'local'. The version we need here is the cheapest possible, where we take 'local' to mean "in the *infinitesimal* neighbourhood of x", i.e. the set of y's with $y \sim_k x$ for some k. With this form of 'local', the implicit function theorem holds in *all* models for SDG (assuming that the model contains the rational numbers), since the required implicitly defined function is (uniquely) given by a formal power series, whose coefficients are successively calculated.

So let us use an affine chart around x, and use (2) to rewrite the function (3). We get

$$t\overline{g}(x,y) + (1-t)\overline{g}(z,y)$$

$$= tA(x; y-x, y-x) + \frac{t}{2}D_{y-x}A(x; y-x, y-x)$$

$$+(1-t)A(z; y-z, y-z) + \frac{1-t}{2}D_{y-z}A(z; y-z, y-z).$$

We shall replace terms involving $A(z; -, -)$ by $A(x; -, -)$ by a Taylor expansion from x (taking only the linear and quadratic terms in $z - x$, since $z \sim_2 x$). We get

$$tA(x; y-x, y-x) + (1-t)A(x; y-z, y-z)$$

plus terms which contain $y-x$, $z-x$, and $y-z$ in a trilinear way, e.g. $(1-t)D_{z-x}A(x; y-z, y-z)$, and using $z-x = (y-x)-(y-z)$, these terms can be rewritten as terms which are of total degree 3 in $y-x$ and $y-z$. Let us write briefly

$$t\overline{g}(x,y) + (1-t)\overline{g}(z,y)$$

$$= tA(x; y-x, y-x) + (1-t)A(x; y-z, y-z) + C(x, z; y-x, y-z), \tag{4}$$

with C of total degree 3 in the arguments after the semicolon. The condition on y that it is a critical value for this expression is that its directional derivative D_v is zero, for any direction v (viewing the expression as a function of y alone, keeping x, z fixed). Calculating this directional derivative, we get the following condition for criticalness of y for (3)

$$0 = 2tA(x; y-x, v) + 2(1-t)A(x; y-z, v) + \overline{C}(x, z; y-x, y-z, v), \tag{5}$$

where $\overline{C}(x,z;y-x,y-z,v)$ is linear in v and of total degree 2 in the two other arguments after the semicolon. We simplify the expression here into

$$2A(x;y-(tx+(1-t)z),v)+\overline{C}(x,z;y-x,y-z,v); \tag{6}$$

call this expression $\phi(z,y,v)$ (x is fixed, so we omit it from the notation). We reconsider it as a function $\Phi : V\times V\to V^*$

$$(z,y)\mapsto [v\mapsto \phi(z,y,v)].$$

We clearly have $\Phi(x,x)=0$. Also the differential of Φ with respect to its second variable y can be calculated from (6); the $\overline{C}$-term yields zero, since it is of total degree 2 in $y-x, y-z$, and the first term yields the linear map $V\to V^*$ given by

$$u\mapsto [v\mapsto 2A(x;u,v)]$$

which is an invertible linear map $V\to V^*$, by the assumption of non-degeneracy of A. Thus, for each x, we get by the Implicit Function Theorem (in the version given by, say, [12] p. 380-381, with 'local' as explained above) a function $\eta : U\to V$ (defined in the infinitesimal neighbourhood U of x) with $\eta(x)=x$, and for each z picking out the unique $y=\eta(z)\in U$ with

$$\Phi(z,\eta(z))=0.$$

For $z\sim_2 x$, we have $\eta(z)\sim_2\eta(x)=x$, since the function η preserves $\sim_k$. But also the function $z\mapsto\eta(z)-z$ preserves $\sim_k$, so for $z\sim_2 x$,

$$\eta(z)-z\sim_2\eta(x)-x=0,$$

so $\eta(z)\sim_2 z$. Since for such points y, $\Phi(z,y)=0$ was the criterion for criticalness of y for the function (3), we get the unique existence, as claimed. So $y_0=\eta(z)$ for the given z. The theorem is proved. ∎

The Theorem, with its appeal to the implicit function theorem, does not provide us with an explicit formula for geodesic midpoints; such formula will be given later (Lemma 4.1 below).

3.7. Proposition. *Let $x\sim_1 z$, and let $t\in R$. The unique y_0 asserted in Theorem 3.6 is the point given as the affine combination $tx+(1-t)z$.*

(The affine combination here makes canonical sense, by the results of Section 2, because $x\sim_1 z$.)

Proof. Since $y_0\sim_1 x$ and $y_0\sim_1 z$, by Theorem 2.2, the 1-neighbours y of y_0 needed to state that y_0 is critical, have $y\sim_2 x$ and $y\sim_2 z$. This means that the expression (3) involves g only, not its extension $\overline{g}$, and it yields, again by a Taylor expansion of $A(z;-,-)$ from x,

$$tA(x;y-x,y-x)+(1-t)A(x;y-z,y-z),$$

whose directional derivative in direction v is easily calculated to be $2A(x;y-(tx+(1-t)z,v)$, and is therefore zero for all v iff $y=tx+(1-t)z$. ∎

For $x \sim_2 z$, the function $f : R \to M$ taking $t \in R$ to the critical point of (3) has the property that $f(0) = z, f(1) = x$, and, by the Proposition, reduces to the canonical affine line spanned by z, x if $x \sim_1 z$.

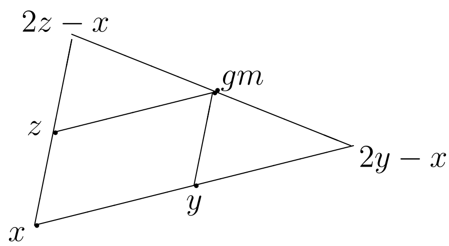
4. Parallelism from metric

Given a manifold M with a pseudo-Riemannian metric g, we would like to describe a notion of parallelism, i.e. an affine connection λ, cf. Section 1.

We construct such λ now, out of the construction of geodesic midpoints from the previous section (recall that the latter notion depends on the metric g). Write $gm(u, v)$ for the geodesic midpoint of $u \sim_2 v \in M$. Recall also that for $x \sim_1 y$, we may form affine combinations; in particular, we may form $2y - x$. Then the construction of a connection λ is achieved by the following formula:

$$\lambda(x, y, z) = gm(2y - x, 2z - x); \tag{7}$$

the geometry of this is given in the picture (and summarized in the statement of Theorem 4.2 below):



We shall prove that this λ is indeed a connection. Consider $\lambda(x, x, z)$ (where $z \sim_1 x$). We have $\lambda(x, x, z) = gm(2x - x, 2z - x) = gm(x, 2z - x)$, but since $2z - x \sim_1 x$, by Theorem 2.2, it follows from Proposition 3.7 that their geodesic midpoint agrees with their affine midpoint which is just z. Similarly $\lambda(x, y, x) = y$. As stated in Section 1, this is the only thing needed in order to have a connection λ. In fact, the connection constructed is clearly symmetric, since the notion of geodesic midpoint is so.

We would like to argue that the connection thus described is the Levi-Civita connection; the desirable proof would be to argue geometrically that it is compatible with the metric. Alternatively, we may calculate its Christoffel symbols, and compare them with Levi-Civita's. We haven't been able to carry out the first, so we shall be content with doing the latter. It may be of some interest in its own right, though, since it gives a variational status to the "Christoffel symbols of the first kind".

Let us work in coordinates, with metric $g(x, z)$ given by $A(x; z - x, z - x)$, as above, and let a connection λ be given by the analytic expression $\lambda(x, y, z) = y + z - x - \Gamma(x; y - x, z - x)$, where $\Gamma(x; -, -)$ is a symmetric bilinear form. For simplicity, we shall state the following result for the case where the "origin" of the parallelograms considered is 0, and rename the y into x, and $\Gamma(x;-,-) = \Gamma(0;-,)$ into $\Gamma(-,-)$; so we are investigating $\lambda(0,x,z)$ (which thus is $x+z-\Gamma(x,z)$). Since $x \sim_1 0$, $2x$ makes unique sense (as an affine combination of 0 and x), by the results of Section 2; similarly for $2z$. Recall that $\overline{g}$ denotes the unique symmetric extension of g to the third neighbourhood of the diagonal, cf. Theorem 3.3, and that it is given in coordinates, in terms of A, by the expression (2), $\overline{g}(x,z) = A(x;z-x,z-x) + \frac{1}{2}D_{z-x}A(x;z-x,z-x)$.

4.1. Lemma. *In order that $y := x+z-\Gamma(x,z)$ be a critical point for $\overline{g}(2x,y)+\overline{g}(2z,y)$, it is necessary and sufficient that for all v*

$$A(0;\Gamma(x,z),v) = \frac{1}{2}\{D_zA(0;x,v) + D_x(0;z,v) - D_vA(0;x,z)\}.$$

Proof. This is a rather long calculation: Let B be any vector with the property that for any bilinear ϕ, $\phi(B,x)=0$, $\phi(x,B)=0$, $\phi(B,z)=0$, and $\phi(z,B)=0$ for the given x,z, and also $\phi(B,B)=0$. This will surely be the case for $B=\Gamma(x,z)$ with $\Gamma(-,-)$ bilinear. Also assume v is a vector $\sim_1 0$, so $\phi(v,v)=0$ for any bilinear ϕ. Now we calculate the variation of $\overline{g}(2x,y)+\overline{g}(2z,y)$ from $y=z+x-B$ in the direction of v, i.e. the difference

$$\overline{g}(2z,z+x-B+v)+\overline{g}(2x,z+x-B+v)-[\overline{g}(2z,z+x-B)+\overline{g}(2x,z+x-B)].$$

(In the square bracket, we might write g rather than $\overline{g}$, since $2z \sim_2 z+x-B$, and similarly for $2x$.) This expression we rewrite, using (2), and then we calculate. We first expand, using bilinearity of $A(2x;-,-)$ and $A(2z;-,-)$; next we Taylor expand $A(2x;-,-)$ or $A(2z;-,-)$ from 0 (the relevant 'series' has only two terms, since $2x \sim_1 0$ and similarly for $2z$); and then we expand again, using bilinearity of $A(0;-,-)$ or trilinearity of its directional derivatives (i.e. trilinearity of $D_uA(0;w_1,w_2)$ in u,w_1,w_2). Many terms disappear during this expansion, e.g. those that contain x twice in linear positions, or contain x and B in linear positions, etc. We end up with

$$-4A(0;B,v)+2D_zA(0;x,v)+2D_xA(0;z,v)-2D_vA(0;x,z).$$

So the variation of $\overline{g}(2x,y)+\overline{g}(2z,y)$ (with $y=x+x-B$) is 0 for all directions v iff this expression vanishes for all v. This proves the Lemma. ∎

Remark. The unique y provided by Lemma 4.1 solves the variational problem defining the notion of geodesic midpoint, so one may be tempted to think that Theorem 3.5/Theorem 3.6, asserting the unique existence of such, were subsumed. However, the y that Lemma 4.1 provides is unique *under the assumption* that it is of a special form. The description of what kind of form this is, depends not only on x and z, but on a third point as well (in the calculation chosen to be the 0 of the coordinate system); essentially, the y provided by Lemma 4.1 does suffice to complete $0, \frac{1}{2}x, \frac{1}{2}z$ into a parallelogram. Thus it does suffice to give a variational description of the fourth point y of this parallelogram, but for a notion of geodesic midpoint, the independence of y on the third point 0 must be argued, and I could not do this except by an appeal to the uniqueness assertion of Theorem 3.5.

If $A(0;-,-)$ is a non-degenerate bilinear form, we can, by the Lemma, express the unique $B(x,z)$ that must be subtracted from $x+z$ in order to get the critical point for energy, i.e. our $\lambda(0,x,z)$, namely

$$A^{-1}(\frac{1}{2}\{D_zA(0;x,v)+D_x(0;z,v)-D_vA(0;x,z)\}).$$

This is the classical "Christoffel symbols of the second kind" for the connection derived from the metric, cf. e.g. [2] Ch. 2 formula (10).

This proves that the connection we have constructed from the metric agrees with the classical Levi-Civita parallelism (and of course, we would have been surprised if it did not). If we agree to write g also for the symmetric extension $\overline{g}$ of g to the third neighbourhood of the diagonal, we may summarize our result as follows. (To say that m is infinitesimally close to x is here meant: $m \sim_k x$ for some k.)

4.2. Theorem. *Let g be a pseudo-Riemannian metric on a manifold. Then the Levi-Civita parallelism can be described geometrically as the following procedure for forming infinitesimal parallelograms $x,y,z,\lambda(x,y,z)$: take $\lambda(x,y,z)$ to be the geodesic midpoint of $2y-x$ and $2z-x$, i.e. the unique critical point (infinitesimally close to x and z) m for the function $g(2y-x,m)+g(2z-x,m)$.*

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