Theory and Applications of Categories, Vol. 20, No. 18, 2008, pp. 650–738.

FRAMED BICATEGORIES AND MONOIDAL FIBRATIONS

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Abstract. In some bicategories, the 1-cells are 'morphisms' between the 0-cells, such as functors between categories, but in others they are 'objects' over the 0-cells, such as bimodules, spans, distributors, or parametrized spectra. Many bicategorical notions do not work well in these cases, because the 'morphisms between 0-cells', such as ring homomorphisms, are missing. We can include them by using a pseudo double category, but usually these morphisms also induce base change functors acting on the 1-cells. We avoid complicated coherence problems by describing base change 'nonalgebraically', using categorical fibrations. The resulting 'framed bicategories' assemble into 2-categories, with attendant notions of equivalence, adjunction, and so on which are more appropriate for our examples than are the usual bicategorical ones.

We then describe two ways to construct framed bicategories. One is an analogue of rings and bimodules which starts from one framed bicategory and builds another. The other starts from a 'monoidal fibration', meaning a parametrized family of monoidal categories, and produces an analogue of the framed bicategory of spans. Combining the two, we obtain a construction which includes both enriched and internal categories as special cases.

Contents

Received by the editors 2007-06-22 and, in revised form, 2008-12-11.
Transmitted by Robert Paré. Published on 2008-12-17.
2000 Mathematics Subject Classification: 18D05 (Primary), 18D30, 18D10 (Secondary).

1. Introduction

We begin with the observation that there are really two sorts of bicategories (or 2-categories). This fact is well appreciated in 2-categorical circles, but not as widely known as it ought to be. (In fact, there are other sorts of bicategory, but we will only be concerned with two.)

The first sort is exemplified by the 2-category $\mathcal{C}at$ of categories, functors, and natural transformations. Here, the 0-cells are 'objects', the 1-cells are maps between them, and the 2-cells are 'maps between maps.' This sort of bicategory is well-described by the slogan "a bicategory is a category enriched over categories."

The second sort is exemplified by the bicategory $\mathcal{M}od$ of rings, bimodules, and bimodule homomorphisms. Here, the 1-cells are themselves 'objects', the 2-cells are maps between them, and the 0-cells are a different sort of 'object' which play a 'bookkeeping' role in organizing the relationships between the 1-cells. This sort of bicategory is well-described by the slogan "a bicategory is a monoidal category with many objects."

Many notions in bicategory theory work as well for one sort as for the other. For example, the notion of 2-functor (including lax 2-functors as well as pseudo ones) is well-suited to describe morphisms of either sort of bicategory. Other notions, such as that of internal adjunction (or 'dual pair'), are useful in both situations, but their meaning in the two cases is very different.

However, some bicategorical ideas make more sense for one sort of bicategory than for the other, and frequently it is the second sort that gets slighted. A prime example is the notion of equivalence of 0-cells in a bicategory. This specializes in $\mathcal{C}at$ to equivalence of categories, which is unquestionably the fundamental notion of 'sameness' for categories. But in $\mathcal{M}od$ it specializes to Morita equivalence of rings, which, while very interesting, is not the most fundamental sort of 'sameness' for rings; isomorphism is.

This may not seem like such a big deal, since if we want to talk about when two rings are isomorphic, we can use the category of rings instead of the bicategory $\mathcal{M}od$. However, it becomes more acute when we consider the notion of biequivalence of bicategories, which involves pseudo 2-functors F and G, and equivalences $X \simeq GFX$ and $Y \simeq FGY$. This is fine for $\mathcal{C}at$-like bicategories, but for $\mathcal{M}od$-like bicategories, the right notion of equivalence ought to include something corresponding to ring isomorphisms instead. This problem arose in [MS06, 19.3.5], where two $\mathcal{M}od$-like bicategories were clearly 'equivalent', yet the language did not exist to describe what sort of equivalence was meant.

Similar problems arise in many other situations, such as the following.

(i) $\mathcal{C}at$ is a monoidal bicategory in the usual sense, which entails (among other things) natural equivalences $(C \times D) \times E \simeq C \times (D \times E)$. But although $\mathcal{M}od$ is 'morally monoidal' under tensor product of rings, the associativity constraint is really a ring isomorphism $(R \otimes S) \otimes T \cong R \otimes (S \otimes T)$, not an invertible bimodule (although it can be made into one).

(ii) For $\mathcal{C}at$-like bicategories, the notions of pseudonatural transformation and modification, making bicategories into a tricategory, are natural and useful. But for $\mathcal{M}od$-like bicategories, it is significantly less clear what the right sort of higher morphisms are.

(iii) The notion of 'biadjunction' is well-suited to adjunctions between $\mathcal{C}at$-like bicategories, but fails badly for $\mathcal{M}od$-like bicategories. Attempts to solve this problem have resulted in some work, such as [Ver92, CKW91, CKVW98], which is closely related to ours.

These problems all stem from essentially the same source: the bicategory structure does not include the correct 'maps between 0-cells', since the 1-cells of the bicategory are being used for something else. In this paper, we show how to use an abstract structure to deal with this sort of situation by incorporating the maps of 0-cells separately from the 1-cells. This structure forms a pseudo double category with extra properties, which we call a *framed bicategory*.

The first part of this paper is devoted to framed bicategories. In §§2–5 we review basic notions about double categories and fibrations, define framed bicategories, and prove some basic facts about them. Then in §§6–10 we apply framed bicategories to resolve the problems mentioned above. We define lax, oplax, and strong framed functors and framed transformations, and thereby obtain three 2-categories of framed bicategories. We then apply general 2-category theory to obtain useful notions of framed equivalence, framed adjunction, and monoidal framed bicategory.

The second part of the paper, consisting of §§11–17, deals with two important ways of constructing framed bicategories. The first, which we describe in §11, starts with a framed bicategory $\mathbb{D}$ and constructs a new framed bicategory $\mathbb{M}\mathbf{od}(\mathbb{D})$ of monoids and modules in $\mathbb{D}$. The second starts with a different 'parametrized monoidal structure' called a *monoidal fibration*, and is essentially the same as the construction of the bicategory of parametrized spectra in [MS06]. In §§12–13 we introduce monoidal fibrations, and in §14 we explain the connection to framed bicategories. Then in §15, we combine these two constructions and thereby obtain a natural theory of 'categories which are both internal and enriched'. §§16–§17 are devoted to the proofs of the main theorems in §14.

Finally, in the appendices we treat the relationship of framed bicategories to other work. This includes the theory of *connection pairs* and *foldings* in double categories, various parts of pure bicategory theory, and the bicategorical theory of *equipments*. Our conclusion is that they are all, in suitable senses, equivalent, but each has advantages and disadvantages, and we believe that framed bicategories are a better choice for many purposes.

There are two important themes running throughout this paper. One is a preoccupation with defining 2-categories and making constructions 2-functorial. Assembling objects into 2-categories allows us to apply the theory of adjunctions, equivalences, monads, and so on, internal to these 2-categories. Thus, without any extra work, we obtain notions such as framed adjunctions and framed monads, which behave much like ordinary adjunctions and monads. Making various constructions 2-functorial makes it easy to obtain framed adjunctions and monads from more ordinary ones.

We do not use very much 2-category theory in this paper, so a passing acquaintance with it should suffice. Since we are not writing primarily for category theorists, we have attempted to avoid or explain the more esoteric categorical concepts which arise. A classic reference for 2-category theory is [KS74]; a more modern and comprehensive one (going far beyond what we will need) is [Lac07].

The second important theme of this paper is the mixture of 'algebraic' and 'nonalgebraic' structures. A monoidal category is an algebraic structure: the product is a specified operation on objects. On the other hand, a category with cartesian products is a nonalgebraic structure: the products are characterized by a universal property, and merely assumed to exist. We can always make a choice of products to make a category with products into a monoidal category, but there are many possible choices, all isomorphic.

There are many technical advantages to working with nonalgebraic structures. For example, no coherence axioms are required of a category with products, whereas a monoidal category requires several. This advantage becomes more significant as the coherence axioms multiply. On the other hand, when doing concrete work, one often wants to make a specific choice of the structure and work with it algebraically. Moreover, not all algebraic structure satisfies an obvious universal property, and while it can usually be tortured into doing so, frequently it is easier in these cases to stick with the algebraic version.

Framed bicategories are a mixture of algebraic and nonalgebraic notions; the composition of 1-cells is algebraic, while the base change operations are given nonalgebraically, using a 'categorical fibration'. Our experience shows that this mixture is very technically convenient, and we hope to convince the reader of this too. In particular, the proof of Theorem 14.2 is much simpler than it would be if we used fully algebraic definitions. This is to be contrasted with the similar structures we will consider in appendices A and C, which are purely algebraic.

Our intent in this paper is not to present any one particular result, but rather to argue for the general proposition that framed bicategories, and related structures, provide a useful framework for many different kinds of mathematics. Despite the length of this paper, we have only had space in it to lay down the most basic definitions and ideas, and much remains to be said.

The theory of framed bicategories was largely motivated by the desire to find a good categorical structure for the theory of parametrized spectra in [MS06]. The reader familiar with [MS06] should find the idea of a framed bicategory natural; it was realized clearly in [MS06] that existing categorical structures were inadequate to describe the combination of a bicategory with base change operations which arose naturally in that context. Another motivation for this work came from the bicategorical 'shadows' of [Pon07], and a desire to explain in what way they are actually the same as the horizontal composition in the bicategory; we will do this in the forthcoming [PS08].

I would like to thank my advisor, Peter May, as well as Kate Ponto, for many useful discussions about these structures; Tom Fiore, for the idea of using double categories; and Joachim Kock and Stephan Stolz for pointing out problems with the original version of Example 2.6. The term 'framed bicategory' was suggested by Peter May.

2. Double categories

As mentioned in the introduction, most of the problems with $\mathcal{M}od$-like bicategories can be traced to the fact that the 'morphisms' of the 0-cells are missing. Thus, a natural replacement which suggests itself is a *double category*, a structure which is like a 2-category, except that it has two types of 1-cells, called 'vertical' and 'horizontal', and its 2-cells are shaped like squares. Double categories go back originally to Ehresmann in [Ehr63]; a brief introduction can be found in [KS74]. Other references include [BE74, GP99, GP04, Gar07].

In this section, we introduce basic notions of double categories. Our terminology and notation will sometimes be different from that commonly used. For example, usually the term 'double category' refers to a strict object, and the weak version is called a 'pseudo double category'. Since we are primarily interested in the weak version, we will use the term *double category* for these, and add the word 'strict' if necessary.

2.1. Definition. A **double category** $\mathbb{D}$ consists of a 'category of objects' $\mathbb{D}_0$ and a 'category of arrows' $\mathbb{D}_1$, with structure functors

$$\begin{aligned} U\colon \mathbb{D}_0 &\to \mathbb{D}_1 \\ L, R\colon \mathbb{D}_1 &\rightrightarrows \mathbb{D}_0 \\ \odot\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 &\to \mathbb{D}_1 \end{aligned}$$

(where the pullback is over $\mathbb{D}_1 \xrightarrow{R} \mathbb{D}_0 \xleftarrow{L} \mathbb{D}_1$) such that

$$\begin{aligned} L(U_A) &= A \\ R(U_A) &= A \\ L(M \odot N) &= LM \\ R(M \odot N) &= RM \end{aligned}$$

equipped with natural isomorphisms

$$\begin{aligned} &\mathfrak{a} : (M \odot N) \odot P \xrightarrow{\cong} M \odot (N \odot P) \\ &\mathfrak{l} : U_A \odot M \xrightarrow{\cong} M \\ &\mathfrak{r} : M \odot U_B \xrightarrow{\cong} M \end{aligned}$$

such that $L(\mathfrak{a})$, $R(\mathfrak{a})$, $L(\mathfrak{l})$, $R(\mathfrak{l})$, $L(\mathfrak{r})$, and $R(\mathfrak{r})$ are all identities, and such that the standard coherence axioms for a monoidal category or bicategory (such as Mac Lane's pentagon; see [ML98]) are satisfied.

We can think of a double category as an internal category in $\mathcal{C}at$ which is suitably weakened, although this is not strictly true because $\mathcal{C}at$ contains only small categories while we allow $\mathbb{D}_0$ and $\mathbb{D}_1$ to be large categories (but still locally small, that is, having only a set of morphisms between any two objects).

We call the objects of $\mathbb{D}_0$ **objects** or **0-cells**, and we call the morphisms of $\mathbb{D}_0$ **vertical arrows** and write them as $f\colon A \to B$. We call the objects of $\mathbb{D}_1$ **horizontal 1-cells** or just **1-cells**. If M is a horizontal 1-cell with $L(M) = A$ and $R(M) = B$, we write $M\colon A \mathrel{\mapstochar\rightarrow} B$, and say that A is the **left frame** of M and B is the **right frame**. We use this terminology in preference to the more usual 'source' and 'target' because of our philosophy that the horizontal 1-cells are not 'morphisms', but rather objects in their own right which just happen to be 'labeled' by a pair of objects of another type.

A morphism $\alpha\colon M \to N$ of $\mathbb{D}_1$ with $L(\alpha) = f$ and $R(\alpha) = g$ is called a **2-cell**, written $\alpha\colon M \underset{f}{\overset{g}{\Longrightarrow}} N$, or just $M \xrightarrow{\alpha} N$, and drawn as follows:

$$\begin{tikzcd} A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "\shortmid"{marking}] & D \end{tikzcd} . \tag{1}$$

We say that M and N are the **source** and **target** of α, while f and g are its **left frame** and **right frame**. We write the composition of vertical arrows $A \xrightarrow{f} B \xrightarrow{g} C$ and the vertical composition of 2-cells $M \xrightarrow{\alpha} N \xrightarrow{\beta} P$ with juxtaposition, gf or $\beta\alpha$, but we write the horizontal composition of horizontal 1-cells as $M \odot N$ and that of 2-cells as $\alpha \odot \beta$.

We write horizontal composition 'forwards' rather than backwards: for $M\colon A \mathrel{\mapstochar\rightarrow} B$ and $N\colon B \mathrel{\mapstochar\rightarrow} C$, we have $M \odot N\colon A \mathrel{\mapstochar\rightarrow} C$. This is also called 'diagrammatic order' and has several advantages. First, in examples such as that of rings and bimodules (Example 2.2), we can define a horizontal 1-cell $M\colon A \mathrel{\mapstochar\rightarrow} B$ to be an (A, B)-bimodule, rather than a (B, A)-bimodule, and still preserve the order in the definition $M \odot N = M \otimes_B N$ of horizontal composition. It also makes it easier to avoid mistakes in working with 2-cell diagrams; it is easier to compose

$$\begin{tikzcd} A \arrow[r, "M", "\shortmid"{marking}] & B \arrow[r, "N", "\shortmid"{marking}] & C \end{tikzcd}$$

and get

$$\begin{tikzcd} A \arrow[rr, "M \odot N", "\shortmid"{marking}] & & C \end{tikzcd}$$

than to remember to switch the order in which M and N appear every time horizontal 1-cells are composed. Finally, it allows us to say that an adjunction $M \dashv N$ in the horizontal bicategory is the same as a 'dual pair' (M, N) (see §5), with the left adjoint also being the left dual.

Every object A of a double category has a vertical identity 1_A and a horizontal unit U_A, every horizontal 1-cell M has an identity 2-cell 1_M, every vertical arrow f has a horizontal unit U_f, and we have $1_{U_A} = U_{1_A}$ (by the functoriality of U). We will frequently abuse notation by writing A or f instead of U_A or U_f when the context is clear. The important point to remember is that vertical composition is strictly associative and unital, while horizontal composition is associative and unital only up to specified coherent isomorphisms.

Note that if $\mathbb{D}_0$ is the terminal category, then the definition of double category just says that $\mathbb{D}_1$ is a monoidal category. We call such double categories **vertically trivial**.

We call $\mathbb{D}_0$ the **vertical category** of $\mathbb{D}$. We say that two objects are isomorphic if they are isomorphic in $\mathbb{D}_0$, and that two horizontal 1-cells are isomorphic if they are isomorphic in $\mathbb{D}_1$. We will never refer to a horizontal 1-cell as an isomorphism. A 2-cell whose left and right frames are identities is called **globular**. Note that the constraints $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$ are globular isomorphisms, but they are natural with respect to all 2-cells, not just globular ones.

Every double category $\mathbb{D}$ has a **horizontal bicategory** $\mathcal{D}$ consisting of the objects, horizontal 1-cells, and globular 2-cells. If A and B are objects of $\mathbb{D}$, we write $\mathbb{D}(A, B)$ for the *set* of vertical arrows from A to B and $\mathcal{D}(A, B)$ for the *category* of horizontal 1-cells and globular 2-cells from A to B. It is standard in bicategory theory to say that something holds **locally** when it is true of all hom-categories $\mathcal{D}(A, B)$, and we will extend this usage to double categories.

We also write ${}_f\mathbb{D}_g(M, N)$ for the set of 2-cells α of the shape (1). If f and g are identities, we write instead $\mathcal{D}(M, N)$ for the *set* of globular 2-cells from M to N. This may be regarded as shorthand for $\mathcal{D}(A, B)(M, N)$ and is standard in bicategory theory.

We now consider some examples. Note that unlike 1-categories, which we generally name by their objects, we generally name double categories by their horizontal 1-cells.

2.2. Example. Let $\mathbb{M}\mathbf{od}$ be the double category defined as follows. Its objects are (not necessarily commutative) rings and its vertical morphisms are ring homomorphisms. A 1-cell $M : A \nrightarrow B$ is an (A, B)-bimodule, and a 2-cell $\alpha : M \underset{f}{\overset{g}{\Longrightarrow}} N$ is an (f, g)-*bilinear map* $M \to N$, i.e. an abelian group homomorphism $\alpha : M \to N$ such that $\alpha(amb) = f(a)\alpha(m)g(b)$. This is equivalent to saying α is a map of (A, B)-bimodules $M \to {}_fN_g$, where ${}_fN_g$ is N regarded as an (A, B)-bimodule by means of f and g. The horizontal composition of bimodules $M \colon A \nrightarrow B$ and $N \colon B \nrightarrow C$ is given by their tensor product, $M \odot N = M \otimes_B N$. For 2-cells

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "P"] \arrow[d, "g"] \arrow[dr, phantom, "\beta"] & E \arrow[d, "h"] \\ B \arrow[r, "N"'] & D \arrow[r, "Q"'] & F \end{tikzcd}$$

we define $\alpha \odot \beta$ to be the composite

$$M \otimes_C P \xrightarrow{\alpha \otimes \beta} {}_fN_g \otimes_C {}_gQ_h \twoheadrightarrow {}_fN \otimes_D Q_h \cong {}_f(N \otimes_D Q)_h \,.$$

This example may be generalized by replacing **Ab** with any monoidal category $\mathscr{C}$ that has coequalizers preserved by its tensor product, giving the double category $\mathbb{M}\mathbf{od}(\mathscr{C})$ of monoids, monoid homomorphisms, and bimodules in $\mathscr{C}$. If $\mathscr{C} = \mathbf{Mod}_R$ is the category of modules over a commutative ring R, then the resulting double category $\mathbb{M}\mathbf{od}(\mathscr{C}) = \mathbb{M}\mathbf{od}(R)$ is made of R-algebras, R-algebra homomorphisms, and bimodules over R-algebras.

Similarly, we define the double category $\mathbb{CM}\mathbf{od}$ whose objects are commutative rings, and if $\mathscr{C}$ is a symmetric monoidal category, we have $\mathbb{CM}\mathbf{od}(\mathscr{C})$.

2.3. Example. Let $\mathscr{C}$ be a category with pullbacks, and define a double category $\mathbb{S}\mathbf{pan}(\mathscr{C})$ whose vertical category is $\mathscr{C}$, whose 1-cells $A \nrightarrow B$ are spans $A \leftarrow C \rightarrow B$ in $\mathscr{C}$, and whose 2-cells are commuting diagrams:

$$\begin{tikzcd} A \arrow[d] & C \arrow[l] \arrow[r] \arrow[d] & B \arrow[d] \\ D & F \arrow[l] \arrow[r] & E \end{tikzcd}$$

in $\mathscr{C}$. Horizontal composition is by pullback.

2.4. Example. There is a double category of parametrized spectra called $\mathbb{E}\mathbf{x}$, whose construction is essentially contained in [MS06]. The vertical category is a category of (nice) topological spaces, and a 1-cell $A \nrightarrow B$ is a spectrum parametrized over $A \times B$ (or $B \times A$; see the note above about the order of composition).

In [MS06] this structure is described only as a bicategory with 'base change operations', but it is pointed out there that existing categorical structures do not suffice to describe it. We will see in §14 how this sort of structure gives rise, quite generally, not only to a double category, but to a framed bicategory, which supplies the missing categorical structure.

2.5. Example. Let $\mathscr{V}$ be a complete and cocomplete closed symmetric monoidal category, such as **Set**, **Ab**, **Cat**, or a convenient cartesian closed subcategory of topological spaces, and define a double category $\mathbb{D}\mathbf{ist}(\mathscr{V})$ as follows. Its objects are (small) categories enriched over $\mathscr{V}$, or $\mathscr{V}$-categories. Its vertical arrows are $\mathscr{V}$-functors, its 1-cells are *$\mathscr{V}$-distributors*, and its 2-cells are $\mathscr{V}$-natural transformations. (Good references for enriched category theory include [Kel82] and [Dub70].) A $\mathscr{V}$-distributor $H \colon \mathscr{B} \nrightarrow \mathscr{A}$ is simply a $\mathscr{V}$-functor $H \colon \mathscr{A}^{op} \otimes \mathscr{B} \to \mathscr{V}$. When $\mathscr{A}$ and $\mathscr{B}$ have one object, they are just monoids in $\mathscr{V}$, and a distributor between them is a bimodule in $\mathscr{V}$; thus we have an inclusion $\mathbb{M}\mathbf{od}(\mathscr{V}) \hookrightarrow \mathbb{D}\mathbf{ist}(\mathscr{V})$. Horizontal composition of distributors is given by the coend construction, also known as 'tensor product of functors'.

In the bicategorical literature, distributors are often called 'bimodules' or just 'modules', but we prefer to reserve that term for the classical one-object version. The term 'distribut*or*', due to Benabou, is intended to suggest a generalization of 'funct*or*', just as in analysis a 'distribut*ion*' is a generalized 'funct*ion*'. The term 'profunctor' is also used for these objects, but we prefer to avoid it because a distributor is nothing like a pro-object in a functor category.

2.6. Example. We define a double category $n\mathbb{C}\mathbf{ob}$ as follows. Its vertical category consists of oriented $(n-1)$-manifolds without boundary and diffeomorphisms. A 1-cell $M \nrightarrow N$ is a (possibly thin) n-dimensional cobordism from M to N, and a 2-cell is a compatible diffeomorphism. Horizontal composition is given by gluing of cobordisms.

More formally, if A and B are oriented $(n-1)$-manifolds, a horizontal 1-cell $M\colon B \nrightarrow A$ is *either* a diffeomorphism $A \cong B$ (regarded as a 'thin' cobordism from A to B), or an n-manifold with boundary M equipped with a 'collar' map

$$(A^{op} \sqcup B) \times [0,1) \to M$$

which is a diffeomorphism onto its image and restricts to a diffeomorphism

$$A^{op} \sqcup B \cong \partial M.$$

(Here A^{op} means A with the opposite orientation.) The unit is the identity 1_A, regarded as a thin cobordism.

2.7. Example. The following double category is known as $\mathbb{A}\mathbf{dj}$. Its objects are categories, and its *horizontal* 1-cells are functors. Its vertical arrows $C \to D$ are adjoint pairs of functors $f_!\colon C \rightleftarrows D\ : f^*$. We then seem to have two choices for the 2-cells; a 2-cell with boundary

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, shift left, "f_!"] & C \arrow[d, shift left, "g_!"] \\ B \arrow[r, "k"'] \arrow[u, shift left, "f^*"] & D \arrow[u, shift left, "g^*"] \end{tikzcd}$$

could be chosen to be either a natural transformation $g_!h \to kf_!$ or a natural transformation $hf^* \to g^*k$. However, it turns out that there is a natural bijection between natural transformations $g_!h \to kf_!$ and $hf^* \to g^*k$ which respects composition, so it doesn't matter which we pick. Pairs of natural transformations corresponding to each other under this bijection are called **mates**; the mate of a transformation $\alpha\colon hf^* \to g^*k$ is given explicitly as the composite

$$g_!h \xrightarrow{g_!h\eta} g_!hf^*f_! \xrightarrow{g_!\alpha f_!} g_!g^*kf_! \xrightarrow{\varepsilon k f_!} kf_!$$

where η is the unit of the adjunction $f_! \dashv f^*$ and ε is the counit of the adjunction $g_! \dashv g^*$. The inverse construction is dual.

More generally, if $\mathcal{K}$ is any (strict) 2-category, we can define the notion of an *adjunction* internal to $\mathcal{K}$: it consists of morphisms $f\colon A \to B$ and $g\colon B \to A$ together with 2-cells $\eta\colon 1_A \Rightarrow gf$ and $\varepsilon\colon fg \Rightarrow 1_B$ satisfying the usual triangle identities. We can then define a double category $\mathbb{A}\mathbf{dj}(\mathcal{K})$ formed by objects, morphisms, adjunctions, and mate-pairs internal to $\mathcal{K}$.

These double categories have a different flavor than the others introduced above. We mention them partly to point out that double categories have uses other than those we are interested in, and partly because we will need the notion of mates later on. More about mate-pairs in 2-categories and their relationship to $\mathbb{A}\mathbf{dj}$ can be found in [KS74];

one fact we will need is that if h and k are identities, then α is an isomorphism if and only if its mate is an isomorphism.

3. Review of the theory of fibrations

Double categories incorporate both the 1-cells of a $\mathcal{M}od$-like bicategory and the 'morphisms of 0-cells', but there is something missing. An important feature of all our examples is that the 1-cells can be 'base changed' along the vertical arrows. For example, in $\mathbb{M}\mathbf{od}$, we can extend and restrict scalars along a ring homomorphism.

An appropriate abstract structure to describe these base change functors is the well-known categorical notion of a 'fibration'. In this section we will review some of the theory of fibrations, and then in §4 we will apply it to base change functors in double categories. All the material in this section is standard. The theory of fibrations is originally due to Grothendieck and his school; see, for example [sga03, Exposé VI]. Modern references include [Joh02a, B1.3] and [Bor94, Ch. 8]. More abstract versions can be found in the 2-categorical literature, such as [Str80].

3.1. Definition. Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a functor, let $f : A \to C$ be an arrow in $\mathscr{B}$, and let M be an object of $\mathscr{A}$ with $\Phi(M) = C$. An arrow $\phi : f^*M \to M$ in $\mathscr{A}$ is **cartesian over** f if, firstly, $\Phi(\phi) = f$:

$$\begin{tikzcd} f^*M \arrow[r, "\phi"] \arrow[d, mapsto] & M \arrow[d, mapsto] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

and secondly, whenever $\psi : N \to M$ is an arrow in $\mathscr{A}$ and $g : B \to A$ is an arrow in $\mathscr{B}$ such that $\Phi(\psi) = fg$, there is a unique χ such that $\psi = \phi\chi$ and $\Phi(\chi) = g$:

$$\begin{tikzcd} N \arrow[dr, dashed, "\chi"'] \arrow[drr, "\psi"] \arrow[dd, mapsto] & & \\ & f^*M \arrow[r, "\phi"'] \arrow[dd, mapsto] & M \arrow[dd, mapsto] \\ B \arrow[dr, "g"'] \arrow[drr] & & \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

We say that Φ is a **fibration** if for every $f\colon A \to C$ and M with $\Phi(M) = C$, there exists a cartesian arrow $\phi_{f,M} : f^*M \to M$ over f. If Φ is a fibration, a **cleavage** for Φ is a choice, for every f and M, of such a $\phi_{f,M}$. The cleavage is **normal** if $\phi_{1_A,M} = 1_M$; it is **split** if $\phi_{g,M}\phi_{f,g^*M} = \phi_{gf,M}$ for all composable f, g.

For an arrow $f\colon A \to B$, we think of f^* as a 'base change' operation that maps the fiber $\mathscr{A}_B$ (consisting of all objects over B and morphisms over 1_B) to the fiber $\mathscr{A}_A$. We think of $\mathscr{A}$ as 'glued together' from the fiber categories $\mathscr{A}_B$ as B varies, using the base change operations f^*. We think of the whole fibration as 'a category parametrized by $\mathscr{B}$'.

3.2. Example. Let **Ring** be the category of rings, and **Mod** be the category of pairs (R, M) where R is a ring and M is an R-module, with morphisms consisting of a ring homomorphism f and an f-equivariant module map. Then the forgetful functor $\mathtt{Mod}\colon \mathbf{Mod} \to \mathbf{Ring}$, which sends (R, M) to R, is a fibration. If M is an R-module and $f\colon S \to R$ is a ring homomorphism, then if we denote by f^*M the abelian group M regarded as an S-module via f, the identity map of M defines an f-equivariant map $f^*M \to M$, which is a cartesian arrow over f.

Note that the fiber $\mathscr{A}_R$ is the ordinary category of R-modules. Thus we may say that modules form a category parametrized by rings.

3.3. Example. Let $\mathscr{C}$ be a category with pullbacks, let $\mathscr{C}^{\downarrow}$ denote the category of arrows in $\mathscr{C}$ (whose morphisms are commutative squares), and let $\mathtt{Arr}_{\mathscr{C}}\colon \mathscr{C}^{\downarrow} \to \mathscr{C}$ take each arrow to its codomain. Then $\mathtt{Arr}_{\mathscr{C}}$ is a fibration; a commutative square is a cartesian arrow in $\mathscr{C}^{\downarrow}$ precisely when it is a pullback square. This fibration is sometimes referred to as the *self-indexing* of $\mathscr{C}$.

We record some useful facts about fibrations.

3.4. Proposition. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a fibration.*

(i) The composite of cartesian arrows is cartesian.

(ii) If $\phi\colon (fg)^*M \to M$ and $\psi\colon g^*M \to M$ are cartesian over fg and g respectively, and $\chi\colon (fg)^*M \to g^*M$ is the unique factorization of ψ through ϕ lying over f, then χ is cartesian.

(iii) If if $\phi\colon f^*M \to M$ and $\phi'\colon (f^*M)' \to M$ are two cartesian lifts of f, then there is a unique isomorphism $f^*M \cong (f^*M)'$ commuting with ϕ and ϕ'.

(iv) Any isomorphism in $\mathscr{A}$ is cartesian.

(v) If f is an isomorphism in $\mathscr{B}$, then any cartesian lift of f is an isomorphism.

In Example 3.2, there is a 'canonical' choice of a cleavage, but this is not true in Example 3.3, since pullbacks are only defined up to isomorphism. Proposition 3.4(iii) tells us that more generally, cleavages in a fibration are unique up to canonical isomorphism. Thus, a fibration is a 'nonalgebraic' approach to defining base change functors: the operation f^* is characterized by a universal property, and the definition merely stipulates that an object satisfying that property exists, rather than choosing a particular such object as part of the structure. In the terminology of [Mak01], they are *virtual* operations.

The 'algebraic' notion corresponding to a fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a pseudofunctor $P\colon \mathscr{B}^{op} \to \mathcal{C}at$. Given a fibration Φ, if we choose a cleavage, then we obtain, for each

$f: A \to B$ in $\mathscr{B}$, a functor $f^*: \mathscr{A}_B \to \mathscr{A}_A$. If we define $P(A) = \mathscr{A}_A$ and $P(f) = f^*$, the uniqueness-up-to-iso of cartesian lifts makes P into a pseudofunctor. Conversely, given a pseudofunctor $P: \mathscr{B}^{op} \to \mathcal{C}at$, we can build a fibration over $\mathscr{B}$ whose fiber over A is $P(A)$. (This is sometimes called the 'Grothendieck construction'.)

In order to state the full 2-categorical sense in which these constructions are inverse equivalences, we need to introduce the morphisms and transformations between fibrations. Consider a commuting square of functors

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F_1"] \arrow[d, "\Phi"'] & \mathscr{A}' \arrow[d, "\Phi'"] \\ \mathscr{B} \arrow[r, "F_0"'] & \mathscr{B}' \end{tikzcd} \tag{2}$$

where Φ and Φ' are fibrations, and let $\phi: g^*M \to M$ be cartesian in $\mathscr{A}$ over g. Then we have $F_1(\phi): F_1(g^*M) \to F_1M$ in $\mathscr{A}'$ over $F_0(g)$. But since Φ' is a fibration, there is a cartesian arrow $\psi: (F_0g)^*(F_1M) \to F_1M$ over $F_0(g)$, so $F_1(\phi)$ factors uniquely through it, giving a canonical map

$$F_1(g^*M) \longrightarrow (F_0g)^*(F_1M) \tag{3}$$

which is an isomorphism if and only if $F_1(\phi)$ is cartesian.

It should thus be unsurprising that any commuting square (2) gives rise to an oplax natural transformation between the corresponding pseudofunctors. Recall that an *oplax natural transformation* between pseudofunctors $P, Q: \mathscr{B}^{op} \to \mathcal{C}at$ consists of functors $\phi_x: Px \to Qx$ and natural transformations

$$\begin{tikzcd} Px \arrow[r, "Pg"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\swarrow\phi_g"] & Py \arrow[d, "\phi_y"] \\ Qx \arrow[r, "Qg"'] & Qy \end{tikzcd}$$

satisfying appropriate coherence conditions. In a *lax natural transformation*, the 2-cells go the other direction, and in a *pseudo natural transformation* the 2-cells are invertible.

3.5. Definition. Any commuting square of functors (2) is called an **oplax morphism of fibrations**. It is a **strong morphism of fibrations** if whenever ϕ is a cartesian arrow in $\mathscr{A}$ over g, then $F_1(\phi)$ is cartesian in $\mathscr{A}'$ over $F_0(g)$. If F_0 is an identity $\mathscr{B} = \mathscr{B}'$, then we say F_1 is a morphism **over** $\mathscr{B}$.

A **transformation of fibrations** between two (oplax) morphisms of fibrations is just a pair of natural transformations, one lying above the other. If the two morphisms are over $\mathscr{B}$, the transformation is **over** $\mathscr{B}$ if its downstairs component is the identity.

3.6. Proposition. *Let $\mathcal{F}ib_{op\ell,\mathscr{B}}$ denote the 2-category of fibrations over $\mathscr{B}$, oplax morphisms of fibrations over $\mathscr{B}$, and transformations over $\mathscr{B}$, and let $[\mathscr{B}^{op}, \mathcal{C}at]_{op\ell}$ denote* *the 2-category of pseudofunctors $\mathscr{B}^{op} \to \mathcal{C}at$, oplax natural transformations, and modifications. Then the above constructions define an equivalence of 2-categories*

$$\mathcal{F}ib_{op\ell,\mathscr{B}} \simeq [\mathscr{B}^{op}, \mathcal{C}at]_{op\ell}.$$

If we restrict to the strong morphisms of fibrations over $\mathscr{B}$ on the left and the pseudo natural transformations on the right, we again have an equivalence

$$\mathcal{F}ib_{\mathscr{B}} \simeq [\mathscr{B}^{op}, \mathcal{C}at].$$

Compared to pseudofunctors, fibrations have the advantage that they incorporate all the base change functors f^* and all their coherence data automatically. We must remember, however, that the functors f^* are not determined uniquely by the fibration, only up to natural isomorphism.

If Φ is a functor such that $\Phi^{op}\colon \mathscr{A}^{op} \to \mathscr{B}^{op}$ is a fibration, we say that Φ is an **opfibration**. (The term 'cofibration' used to be common, but this carries the wrong intuition for homotopy theorists, since an opfibration is still characterized by a *lifting* property.) The cartesian arrows in $\mathscr{A}^{op}$ are called **opcartesian** arrows in $\mathscr{A}$. A cleavage for an opfibration consists of opcartesian arrows $M \to f_!M$, giving rise to a functor $f_!\colon \mathscr{A}_A \to \mathscr{A}_B$ for each morphism $f\colon A \to B$ in $\mathscr{B}$.

For any opfibration, the collection of functors $f_!$ forms a *covariant* pseudofunctor $\mathscr{B} \to \mathcal{C}at$, and conversely, any covariant pseudofunctor gives rise to an opfibration. A commutative square (2) in which Φ and Φ' are opfibrations is called a **lax morphism of opfibrations**, and it is **strong** if F_1 preserves opcartesian arrows; these correspond to lax and pseudo natural transformations, respectively.

3.7. Proposition. *A fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ is also an opfibration precisely when all the functors f^* have left adjoints $f_!$.*

Proof. By definition of f^*, there is a natural bijection between morphisms $M \to N$ in $\mathscr{A}$ lying over $f\colon A \to B$ and morphisms $M \to f^*N$ in the fiber $\mathscr{A}_A$. But if Φ is also an opfibration, these morphisms are also bijective to morphisms $f_!M \to N$ in $\mathscr{A}_B$, so we have an adjunction $\mathscr{A}_A(M, f^*N) \cong \mathscr{A}_B(f_!M, N)$ as desired. The converse is straightforward. ∎

We will refer to a functor which is both a fibration and an opfibration as a **bifibration**. A square (2) in which Φ and Φ' are bifibrations is called a **lax morphism of bifibrations** if F_1 preserves cartesian arrows, an **oplax morphism of bifibrations** if it preserves opcartesian arrows, and a **strong morphism of bifibrations** if it preserves both.

3.8. Examples. The fibration $\mathtt{Mod}\colon \mathbf{Mod} \to \mathbf{Ring}$ is in fact a bifibration; the left adjoint $f_!$ is given by extension of scalars. For any category $\mathscr{C}$ with pullbacks, the fibration $\mathtt{Arr}_{\mathscr{C}}\colon \mathscr{C}^{\downarrow} \to \mathscr{C}$ is also a bifibration; the left adjoint $f_!$ is given by composing with f.

In many cases, the functors f^* also have *right* adjoints, usually written f_*. These functors are not as conveniently described by a fibrational condition, but we will see in §5 that in a framed bicategory, they can be described in terms of base change objects and a closed structure. We say that a fibration is a **$*$-fibration** if all the functors f^* have right adjoints f_*. Similarly we have a **$*$-bifibration**, in which every morphism f gives rise to an adjoint string $f_! \dashv f^* \dashv f_*$.

3.9. EXAMPLES. `Mod` is a $*$-bifibration; the right adjoints are given by coextension of scalars. $\mathtt{Arr}_{\mathscr{C}}$ is a $*$-bifibration precisely when $\mathscr{C}$ is *locally cartesian closed* (that is, each slice category $\mathscr{C}/X$ is cartesian closed).

Often the mere existence of left or right adjoints is insufficient, and we need to require a commutativity condition. We will explore this further in §16.

4. Framed bicategories

Morally speaking, a framed bicategory is a double category in which the 1-cells can be restricted and extended along the vertical arrows. We will formalize this by saying that L and R are bifibrations. Thus, for any $f\colon A \to B$ in $\mathbb{D}_0$, there will be two different functors which should be called f^*, one arising from L and one from R. We distinguish by writing the first on the left and the second on the right. In other words, f^*M is a horizontal 1-cell equipped with a cartesian 2-cell

$$\begin{tikzcd} A \arrow[r, "f^*M", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{cart}"] & D \arrow[d, equal] \\ B \arrow[r, "M"', "|"{marking}] & D \end{tikzcd}$$

while Mg^* is equipped with a cartesian 2-cell

$$\begin{tikzcd} B \arrow[r, "Mg^*", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{cart}"] & C \arrow[d, "g"] \\ B \arrow[r, "M"', "|"{marking}] & D \end{tikzcd} .$$

A general cartesian arrow in $\mathbb{D}_1$ lying over (f, g) in $\mathbb{D}_0 \times \mathbb{D}_0$ can then be written as $f^*Mg^* \underset{f}{\overset{g}{\Longrightarrow}} M$. We do similarly for opcartesian arrows and the corresponding functors $f_!$. We refer to f^* as **restriction** and to $f_!$ as **extension**. If f^* also has a right adjoint f_*, we refer to it as **coextension**.

It is worth commenting explicitly on what it means for a 2-cell in a double category to be cartesian or opcartesian. Suppose given a 'niche' of the form

$$\begin{tikzcd} A \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "M"', "|"{marking}] & D \end{tikzcd}$$

in a double category $\mathbb{D}$. This corresponds to an object $M \in \mathbb{D}_1$ and a morphism $(f, g)\colon (A, C) \to (B, D) = (L, R)(M)$ in $\mathbb{D}_0 \times \mathbb{D}_0$. A cartesian lifting of this morphism is a 2-cell

$$\begin{tikzcd} A \arrow[r, "f^*Mg^*", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{cart}"] & C \arrow[d, "g"] \\ B \arrow[r, "M"', "\shortmid" marking] & D \end{tikzcd}$$

such that any 2-cell of the form

$$\begin{tikzcd} E \arrow[r, "N", "\shortmid" marking] \arrow[d, "fh"'] \arrow[dr, phantom, "\Downarrow"] & F \arrow[d, "gk"] \\ B \arrow[r, "M"', "\shortmid" marking] & D \end{tikzcd}$$

factors uniquely as follows:

$$\begin{tikzcd} E \arrow[r, "N", "\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow"] & F \arrow[d, "k"] \\ A \arrow[r, "f^*Mg^*" description] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{cart}"] & C \arrow[d, "g"] \\ B \arrow[r, "M"', "\shortmid" marking] & D. \end{tikzcd}$$

In particular, if $h = 1_A$ and $k = 1_C$, this says that any 2-cell

$$\begin{tikzcd} A \arrow[r, "N", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & C \arrow[d, "g"] \\ B \arrow[r, "M"', "\shortmid" marking] & D \end{tikzcd}$$

can be represented by a globular 2-cell

$$\begin{tikzcd} A \arrow[r, "N", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, equal] \\ A \arrow[r, "f^*Mg^*"', "\shortmid" marking] & B. \end{tikzcd}$$

Therefore, 'all the information' about the 2-cells in a framed bicategory will be carried by the globular 2-cells and the base change functors. In particular, we can think of $\mathbb{D}$ as 'the bicategory $\mathcal{D}$ equipped with base change functors'. This can be made precise; see appendix C.

The interaction of fibrational conditions with the double category structure has further implications. It is reasonable to expect that restriction and extension will commute with horizontal composition; thus we will have $f^*(M \odot N)g^* \cong f^*M \odot Ng^*$. This implies, however, that for any 1-cell $M\colon B \to\!\!\!\!\!\shortmid\; C$ and arrow $f\colon A \to B$, we have

$$f^*M \cong f^*(U_B \odot M) \cong f^*U_B \odot M,$$

and hence the base change *functor* f^* can be represented by horizontal composition with the special object f^*U_B, which we call a *base change object*.

In the case of $\mathbb{M}\mathbf{od}$, this is the standard fact that restricting along a ring homomorphism $f\colon A \to B$ is the same as tensoring with the (A, B)-bimodule ${}_fB$, by which we mean B regarded as an (A, B)-bimodule via f on the left. For this reason, we write ${}_fB$ for the base change object f^*U_B in any double category. Similarly, we write B_f for U_Bf^*.

The existence of such base change objects, suitably formalized, turns out to be sufficient to ensure that *all* restrictions exist. This formalization of base change objects can be given in an essentially diagrammatic way, which moreover is self-dual. Thus, it it is also equivalent to the existence of *extensions*. This is the content of the following result.

4.1. THEOREM. *The following conditions on a double category $\mathbb{D}$ are equivalent.*

(i) $(L, R) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ *is a fibration.*

(ii) $(L, R) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ *is an opfibration.*

(iii) For every vertical arrow $f\colon A \to B$, there exist 1-cells ${}_fB\colon A \nrightarrow B$ and $B_f\colon B \nrightarrow A$ together with 2-cells

$$
\begin{tikzcd} {} \arrow[r, "{}_fB", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "U_B"', "|" marking] & {} \end{tikzcd}
,\quad
\begin{tikzcd} {} \arrow[r, "B_f", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, "f"] \\ {} \arrow[r, "U_B"', "|" marking] & {} \end{tikzcd}
,\quad
\begin{tikzcd} {} \arrow[r, "U_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "B_f"', "|" marking] & {} \end{tikzcd}
,\quad \textit{and} \quad
\begin{tikzcd} {} \arrow[r, "U_A", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, "f"] \\ {} \arrow[r, "{}_fB"', "|" marking] & {} \end{tikzcd}
\tag{4}
$$

such that the following equations hold.

$$
\begin{tikzcd} {} \arrow[r, "U_A", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, "f"] \\ {} \arrow[r, "{}_fB" description, "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "U_B"', "|" marking] & {} \end{tikzcd}
=
\begin{tikzcd} {} \arrow[r, "U_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & {} \arrow[d, "f"] \\ {} \arrow[r, "U_B"', "|" marking] & {} \end{tikzcd}
\qquad
\begin{tikzcd} {} \arrow[r, "U_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "B_f" description, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, "f"] \\ {} \arrow[r, "U_B"', "|" marking] & {} \end{tikzcd}
=
\begin{tikzcd} {} \arrow[r, "U_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & {} \arrow[d, "f"] \\ {} \arrow[r, "U_B"', "|" marking] & {} \end{tikzcd}
\tag{5}
$$

$$
\begin{tikzcd} {} \arrow[rr, "{}_fB", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & {} \arrow[d, equal] \\ {} \arrow[r, "U_A", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[r, "{}_fB", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "{}_fB"', "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & {} \arrow[r, "U_B"', "|" marking] & {} \arrow[d, equal] \\ {} \arrow[rr, "{}_fB"', "|" marking] & & {} \end{tikzcd}
=
\begin{tikzcd} {} \arrow[r, "{}_fB", "|" marking] \arrow[d, equal] & {} \arrow[d, equal] \\ {} \arrow[r, "{}_fB"', "|" marking] & {} \end{tikzcd}
\qquad
\begin{tikzcd} {} \arrow[rr, "B_f", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & {} \arrow[d, equal] \\ {} \arrow[r, "{}_fB", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[r, "U_A", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "U_B"', "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & {} \arrow[r, "B_f"', "|" marking] & {} \arrow[d, equal] \\ {} \arrow[rr, "B_f"', "|" marking] & & {} \end{tikzcd}
=
\begin{tikzcd} {} \arrow[r, "B_f", "|" marking] \arrow[d, equal] & {} \arrow[d, equal] \\ {} \arrow[r, "B_f"', "|" marking] & {} \end{tikzcd}
\tag{6}
$$

PROOF. We first show that (i)⇒(iii). As indicated above, if (L, R) is a fibration we define ${}_fB = f^*U_B$ and $B_f = U_Bf^*$, and we let the first two 2-cells in (4) be the cartesian 2-cells characterizing these two restrictions. The unique factorizations of U_f through these two

2-cells then gives us the second two 2-cells in (4) such that the equations (5) are satisfied by definition.

We show that the first equation in (6) is satisfied. If we compose the left side of this equation with the cartesian 2-cell defining ${}_fB$, we obtain

$$
\begin{tikzcd}
\bullet \arrow[rr, "{}_fB"] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "U_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "{}_fB"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "{}_fB"'] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & \bullet \arrow[r, "U_B"'] & \bullet \arrow[d, equal] \\
\bullet \arrow[rr, "{}_fB"] \arrow[d, "f"'] \arrow[drr, phantom, "\Downarrow"] & & \bullet \arrow[d, equal] \\
\bullet \arrow[rr, "U_B"'] & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[rr, "{}_fB"] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "U_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "{}_fB"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "{}_fB"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "U_B"'] \arrow[d, equal] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "U_B"'] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & \bullet \arrow[r, "U_B"'] & \bullet \arrow[d, equal] \\
\bullet \arrow[rr, "U_B"'] & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[rr, "{}_fB"] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "U_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & \bullet \arrow[r, "{}_fB"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "U_B"'] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & \bullet \arrow[r, "U_B"'] & \bullet \arrow[d, equal] \\
\bullet \arrow[rr, "U_B"'] & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "{}_fB"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "U_B"'] & \bullet
\end{tikzcd},
$$

which is once again the cartesian 2-cell defining ${}_fB$. However, we also have

$$
\begin{tikzcd}
\bullet \arrow[r, "{}_fB"] \arrow[d, equal] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "{}_fB"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "U_B"'] & \bullet
\end{tikzcd}, \quad =
\begin{tikzcd}
\bullet \arrow[r, "{}_fB"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "U_B"'] & \bullet
\end{tikzcd},
$$

Thus, the uniqueness of factorizations through cartesian arrows implies that the given 2-cell is equal to the identity, as desired. This shows the first equation in (6); the second is analogous. Thus (i)⇒(iii).

Now assume (iii), and let $M\colon B \nrightarrow D$ be a 1-cell and $f\colon A \to B$ and $g\colon C \to D$ be vertical arrows; we claim that the following composite is cartesian:

$$
\begin{tikzcd}
\bullet \arrow[r, "{}_fB"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "M"] \arrow[d, equal] \arrow[dr, phantom, "1_M"] & \bullet \arrow[r, "D_g"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "U_B"'] \arrow[d, equal] \arrow[drrr, phantom, "\cong"] & \bullet \arrow[r, "M"] & \bullet \arrow[r, "U_D"'] & \bullet \arrow[d, equal] \\
\bullet \arrow[rrr, "M"'] & & & \bullet
\end{tikzcd}
\tag{7}
$$

To show this, suppose that

$$
\begin{tikzcd}
\bullet \arrow[r, "N"] \arrow[d, "fh"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \bullet \arrow[d, "gk"] \\
\bullet \arrow[r, "M"'] & \bullet
\end{tikzcd}
$$

is a 2-cell; we must show that it factors uniquely through (7). Consider the composite

$$\begin{tikzcd}
\bullet \arrow[rrr, "N"] \arrow[d, equal] \arrow[drrr, phantom, "\cong"] & & & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "U_A"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow U_h"] & \bullet \arrow[r, "N"] \arrow[d, "h"] \arrow[ddr, phantom, "\Downarrow\alpha"] & \bullet \arrow[r, "U_C"] \arrow[d, "k"'] \arrow[dr, phantom, "\Downarrow U_k"] & \bullet \arrow[d, "k"] \\
\bullet \arrow[r, "U_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, "f"] & \bullet \arrow[r, "U_C"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "{}_fB"'] & \bullet \arrow[r, "M"'] & \bullet \arrow[r, "D_g"'] & \bullet
\end{tikzcd} . \tag{8}$$

Composing this with (7) and using the equations (5) on each side, we get α back again. Thus, (8) gives a factorization of α through (7). To prove uniqueness, suppose that we had another factorization

$$\begin{tikzcd}
\bullet \arrow[rrr, "N"] \arrow[d, "h"'] \arrow[drrr, phantom, "\Downarrow\beta"] & & & \bullet \arrow[d, "k"] \\
\bullet \arrow[r, "{}_fB"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "M"] \arrow[d, equal] \arrow[dr, phantom, "1_M"] & \bullet \arrow[r, "D_g"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "U_B"'] \arrow[d, equal] \arrow[drrr, phantom, "\cong"] & \bullet \arrow[r, "M"] & \bullet \arrow[r, "U_D"'] & \bullet \arrow[d, equal] \\
\bullet \arrow[rrr, "M"'] & & & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "N"] \arrow[d, "fh"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \bullet \arrow[d, "gk"] \\
\bullet \arrow[r, "M"'] & \bullet
\end{tikzcd} \tag{9}$$

Then if we substitute the left-hand side of (9) for α in (8) and use the equations (6) on the left and right, we see that everything cancels and we just get β. Hence, β is equal to (8), so the factorization is unique. This proves that (7) is cartesian, so (iii)⇒(i). The proof that (ii)⇔(iii) is exactly dual. ∎

4.2. Definition. When the equivalent conditions of Theorem 4.1 are satisfied, we say that $\mathbb{D}$ is a **framed bicategory**.

Thus, a framed bicategory has both restrictions and extensions. By the construction for (iii)⇒(i), we see that in a framed bicategory we have

$$f^*Mg^* \cong {}_fB \odot M \odot D_g. \tag{10}$$

The dual construction for (iii)⇒(ii) shows that

$$f_!Ng_! \cong B_f \odot N \odot {}_gD. \tag{11}$$

In particular, taking $N = U_B$, we see that addition to

$${}_fB \cong f^*U_B \quad \text{and} \quad B_f \cong U_B f^*, \tag{12}$$

we have

$${}_fB \cong U_A f_! \quad \text{and} \quad B_f \cong f_! U_A. \tag{13}$$

More specifically, the first two 2-cells in (4) are always cartesian and the second two are opcartesian. It thus follows that from the uniqueness of cartesian and opcartesian arrows that 1-cells ${}_fB$ and B_f equipped with the data of Theorem 4.1(iii) are unique up to isomorphism. In fact, if ${}_fB$ and $\widetilde{{}_fB}$ are two such 1-cells, the canonical isomorphism ${}_fB \xrightarrow{\cong} \widetilde{{}_fB}$ is given explicitly by the composite

$$
\begin{tikzcd}
{} \arrow[rr, "{}_fB", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & {} \arrow[d, equal] \\
{} \arrow[r, "U_A", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[r, "{}_fB", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\
{} \arrow[r, "\widetilde{{}_fB}"', "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & {} \arrow[r, "U_B"', "|" marking] & {} \arrow[d, equal] \\
{} \arrow[rr, "\widetilde{{}_fB}"', "|" marking] & & {}
\end{tikzcd}
$$

The case of B_f is similar.

We can now prove the expected compatibility between base change and horizontal composition.

4.3. Corollary. *In a framed bicategory, we have*

$$
\begin{aligned}
f^*(M \odot N)g^* &\cong f^*M \odot Ng^* \qquad \textit{and} \\
f_!(M \odot N)g_! &\cong f_!M \odot Ng_!.
\end{aligned}
$$

Proof. Use (10) and (11), together with the associativity of $\odot$. ■

On the other hand, if coextensions exist, we have a canonical morphism

$$f_*M \odot Ng_* \longrightarrow f_*(M \odot N)g_* \tag{14}$$

given by the adjunct of the composite

$$f^*\big(f_*M \odot Ng_*\big)g^* \cong f^*f_*M \odot Ng_*g^* \longrightarrow M \odot N,$$

but it is rarely an isomorphism. In general, coextension is often less well behaved than restriction and extension, which partly justifies our choice to use a formalism in which it is less natural.

4.4. Examples. All of the double categories we introduced in §2 are actually framed bicategories, and many of them have coextensions as well.

- In $\mathbb{M}\mathbf{od}$, if M is an (A, B)-bimodule and $f\colon C \to A$, $g\colon D \to B$ are ring homomorphisms, then the restriction f^*Mg^* is M regarded as a (C, D)-bimodule via f and g. Similarly, $f_!$ is given by extension of scalars and f_* by coextension of scalars. The base change objects ${}_fB$ and B_f are B regarded as an (A, B)-bimodule and (B, A)-bimodule, respectively, via the map f.

- In $\mathbb{S}\mathbf{pan}(\mathscr{C})$, restrictions are given by pullback and extensions are given by composition. The base change objects ${}_fB$ and B_f for a map $f\colon A \to B$ in $\mathscr{C}$ are the spans $A \xleftarrow{1_A} A \xrightarrow{f} B$ and $B \xleftarrow{f} A \xrightarrow{1_A} B$, respectively. These are often known as the *graph* of f. Coextensions exist when $\mathscr{C}$ is locally cartesian closed.

- In $\mathbb{E}\mathbf{x}$, the base change functors are defined in [MS06, §11.4 and §12.6], and the base change objects are a version of the sphere spectrum described in [MS06, §17.2].

- In $\mathbb{D}\mathbf{ist}(\mathscr{V})$, restrictions are given by precomposition, and extensions and coextensions are given by left and right Kan extension, respectively. For a $\mathscr{V}$-functor $f\colon A \to B$, the base change objects ${}_fB$ and B_f are the distributors $B(-, f-)$ and $B(f-, -)$, respectively.

- In $n\mathbb{C}\mathbf{ob}$, restriction, extension, and coextension are all given by composing a diffeomorphism of $(n-1)$-manifolds with the given diffeomorphism onto a collar of the boundary. The base change objects of a diffeomorphism $f\colon A \cong B$ are f and its inverse, regarded as thin cobordisms.

- In $\mathbb{A}\mathbf{dj}$, restriction and extension are given by composing with suitable adjoints. For example, given $h\colon B \to D$ and adjunctions $f_!\colon A \rightleftarrows B\ : f^*$ and $g_!\colon C \rightleftarrows D\ : g^*$, then a cartesian 2-cell is given by the square

$$\begin{tikzcd} A \arrow[r, "g^*hf_!", "|" marking] \arrow[d, "f_!"'] \arrow[dr, phantom, "\swarrow\varepsilon"] & C \arrow[d, "g_!"] \\ B \arrow[r, "h"', "|" marking] & D \end{tikzcd}$$

where ε is the counit of the adjunction $g_! \dashv g^*$. The base change objects for an adjunction $f_! \dashv f^*$ are $f_!$ and f^*, respectively. Coextensions do not generally exist.

We also observe that the base change objects are pseudofunctorial. This is related to, but distinct from, the pseudofunctoriality of the base change *functors*. Pseudofunctoriality of base change functors means that for $A \xrightarrow{f} B \xrightarrow{g} C$, we have

$$f^*(g^*(M)) \cong (gf)^*M$$

coherently, while pseudofunctoriality of base change objects means that we have

$${}_fB \odot {}_gC \cong {}_{gf}C$$

coherently. However, since base change objects represent all base change functors, either implies the other.

4.5. Proposition. *If $\mathbb{D}$ is a framed bicategory with a chosen cleavage, then the operation $f \mapsto {}_fB$ defines a pseudofunctor $\mathbb{D}_0 \to \mathcal{D}$ which is the identity on objects. Similarly, the operation $f \mapsto B_f$ defines a contravariant pseudofunctor ${\mathbb{D}_0}^{op} \to \mathcal{D}$.*

5. Duality theory

We mentioned in Example 2.7 that the notion of an *adjunction* can be defined internal to any 2-category. In fact, the definition can easily be extended to any bicategory: an adjunction in a bicategory $\mathcal{B}$ is a pair of 1-cells $F\colon A \to\!\!\!\!\!\!\;+ B$ and $G\colon B \to\!\!\!\!\!\!\;+ A$ together with 2-cells $\eta\colon U_B \to G \odot F$ and $\varepsilon\colon F \odot G \to U_A$, satisfying the usual triangle identities with appropriate associativity and unit isomorphisms inserted.

An internal adjunction is an example of a formal concept which is useful in both types of bicategories discussed in the introduction, but its *meaning* is very different in the two cases. In $\mathcal{C}at$-like bicategories, adjunctions behave much like ordinary adjoint pairs of functors; in fact, we will use them in this way in §8. In $\mathcal{M}od$-like bicategories, on the other hand, adjunctions encode a notion of *duality*.

In particular, if $\mathscr{C}$ is a monoidal category, considered as a one-object bicategory, an adjunction in $\mathscr{C}$ is better known as a *dual pair* in $\mathscr{C}$, and one speaks of an object Y as being *left* or *right dual* to an object X; see, for example, [May01]. When $\mathscr{C}$ is symmetric monoidal, left duals and right duals coincide.

5.1. Examples. When $\mathscr{C} = \mathbf{Mod}_R$ for a commutative ring R, the dualizable objects are the finitely generated projectives. When $\mathscr{C}$ is the stable homotopy category, the dualizable objects are the finite CW spectra.

The terminology of dual pairs was extended in [MS06] to adjunctions in $\mathcal{M}od$-like bicategories, which behave more like dual pairs in monoidal categories than they do like adjoint pairs of functors. Of course, now the distinction between left and right matters. Explicitly, we have the following.

5.2. Definition. A **dual pair** in a bicategory $\mathcal{D}$ is a pair (M, N), with $M\colon A \to\!\!\!\!\!\!\;+ B$, $N\colon B \to\!\!\!\!\!\!\;+ A$, together with 'evaluation' and 'coevaluation' maps

$$N \odot M \to U_B \qquad \text{and} \qquad U_A \to M \odot N$$

satisfying the triangle identities. We say that N is the **right dual** of M and that M is **right dualizable**, and dually.

The definition of dual pair given in [MS06, 16.4.1] is actually reversed from ours, although it doesn't look it, because of our different conventions about which way to write horizontal composition. But because we also turn around the horizontal 1-cells in all the examples, the terms 'right dualizable' and 'left dualizable' refer to the same actual objects as before. Our convention has the advantage that the right dual is also the right adjoint.

Because a dual pair is formally the same as an adjunction, all formal properties of the latter apply as well to the former. One example is the calculus of mates, as defined in Example 2.7: if (M, N) and (P, Q) are dual pairs, then there is a natural bijection between morphisms $M \to P$ and $Q \to N$.

We define a dual pair in a framed bicategory $\mathbb{D}$ to be just a dual pair in its underlying horizontal bicategory $\mathcal{D}$. In this case, we have natural examples coming from the base change objects.

5.3. Proposition. *If $f: A \to B$ is a vertical arrow in a framed bicategory $\mathbb{D}$, then $({}_fB, B_f)$ is naturally a dual pair.*

Proof. Since the base change *functor* $f_!$ is left adjoint to f^*, we have equivalences

$$\mathcal{D}(M \odot {}_fB, N) \simeq \mathcal{D}(Mf_!, N) \simeq \mathcal{D}(M, Nf^*) \simeq \mathcal{D}(M, N \odot B_f).$$

By the bicategorical Yoneda lemma (see, for example, [Str80]), which applies to dual pairs just as it applies to adjunctions, this implies the desired result.

Alternately, the unit and counit can be constructed directly from the data in Theorem 4.1(iii); the unit is

$$\begin{tikzcd}
{} \arrow[rr, "U_A", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\cong"] && {} \arrow[d, equal] \\
{} \arrow[r, "U_A", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[r, "U_A", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\
{} \arrow[r, "{}_fB"', "|" marking] & {} \arrow[r, "B_f"', "|" marking] & {}
\end{tikzcd}$$

and the counit is

$$\begin{tikzcd}
{} \arrow[r, "B_f", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[r, "{}_fB", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\
{} \arrow[r, "U_B"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & {} \arrow[r, "U_B"', "|" marking] & {} \arrow[d, equal] \\
{} \arrow[rr, "U_B"', "|" marking] && {}
\end{tikzcd}$$

Equations (5) and (6) are then exactly what is needed to prove the triangle identities. ∎

In particular, each of the base change objects ${}_fB$ and B_f determines the other up to isomorphism. Combining these dual pairs with another general fact about adjunctions in a bicategory, we have the following generalization of [MS06, 17.3.3–17.3.4].

5.4. Proposition. *Let (M, N) be a dual pair in a framed bicategory with $M: A \mathrel{\ooalign{$\to$\cr\hidewidth$\shortmid$\hidewidth}} B$, $N: B \mathrel{\ooalign{$\to$\cr\hidewidth$\shortmid$\hidewidth}} A$, and let $f: B \to C$ be a vertical arrow. Then $(Mf_!, f_!N)$ is also a dual pair. Similarly, for any $g: D \to A$, (g^*M, Ng^*) is a dual pair.*

Proof. We have $Mf_! \cong M \odot {}_fB$ and $f_!N \cong B_f \odot N$, so the result follows from the fact that the composite of adjunctions in a bicategory is an adjunction. The other case is analogous. ∎

This implies the following generalization of the calculus of mates.

5.5. Proposition. *Let (M, N) and (P, Q) be dual pairs in a framed bicategory. Then there is a natural bijection between 2-cells of the following forms:*

$$\begin{tikzcd}
A \arrow[r, "M", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "g"] \\
C \arrow[r, "P"', "|" marking] & D
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
B \arrow[r, "N", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\
D \arrow[r, "Q"', "|" marking] & C
\end{tikzcd}.$$

PROOF. A 2-cell of the former shape is equivalent to a globular 2-cell $Mg_! \to f^*P$, and a 2-cell of the latter shape is equivalent to a globular 2-cell $g_!N \to Qf^*$. By Proposition 5.4, we have dual pairs $(Mg_!, g_!N)$ and (f^*P, Qf^*), so the ordinary calculus of mates applies. ■

5.6. EXAMPLES. Dual pairs behave significantly differently in many of our examples.

(i) If R is a not-necessarily commutative ring, then a right R-module $M\colon \mathbb{Z} \nrightarrow R$ is right dualizable in $\mathbb{M}\mathbf{od}$ when it is finitely generated projective.

(ii) The only dual pairs in $\mathbb{S}\mathbf{pan}(\mathscr{C})$ are the base change dual pairs. (This is easy in **Set**, and we can then apply the Yoneda lemma for arbitrary $\mathscr{C}$.)

(iii) If $M\colon A \nrightarrow B$ is a right dualizable distributor in $\mathbb{D}\mathbf{ist}(\mathscr{V})$, and B satisfies a mild cocompleteness condition depending on $\mathscr{V}$ (called 'Cauchy completeness'), then M is necessarily also of the form ${}_fB$ for some $\mathscr{V}$-functor $f\colon A \to B$. When $\mathscr{V} = \mathbf{Set}$, Cauchy completeness just means that every idempotent splits. When $\mathscr{V} = \mathbf{Ab}$, it means that idempotents split and finite coproducts exist. See [Kel82, §5.5] for more about Cauchy completion of enriched categories.

(iv) Dualizable objects in $\mathbb{E}\mathbf{x}$ are studied extensively in [MS06, Ch. 18].

5.7. REMARK. There is also a general notion of *trace* for endomorphisms of a dualizable object in a symmetric monoidal category: if (X, Y) is a dual pair and $f\colon X \to X$, then the trace of f is the composite

$$I \xrightarrow{\eta} X \otimes Y \xrightarrow{f \otimes 1} X \otimes Y \xrightarrow{\cong} Y \otimes X \xrightarrow{\varepsilon} I.$$

Traces were extended to dual pairs in a bicategory in [Pon07], by equipping the bicategory with a suitable structure, called a *shadow*, to take the place of the symmetry isomorphism. In [PS08] we will consider shadows in framed bicategories.

Duality in symmetric monoidal categories is most interesting when the monoidal category is closed. There is also a classical notion of *closed bicategory*, which means that the composition of 1-cells has adjoints on both sides:

$$\mathcal{B}(M \odot N, P) \cong \mathcal{B}(M, N \rhd P) \cong \mathcal{B}(N, P \lhd M).$$

Recall that $\mathcal{B}(M \odot N, P)$ denotes the *set* of globular 2-cells from $M \odot N$ to P. In 2-categorical language, this says that right Kan extensions and right Kan liftings exist in the bicategory $\mathcal{B}$.

It is proven in [MS06, §16.4], extending classical results for symmetric monoidal categories, that when $M\colon A \nrightarrow B$ is right dualizable, its right dual is always (isomorphic to) the 'canonical dual' $D_rM = M \rhd U_B$; and conversely, whenever the canonical map $M \odot D_rM \to M \rhd M$ is an isomorphism, then M is right dualizable. This can also be stated as the generalization to bicategories of the fact (see [ML98, X.7]) that a functor G has an adjoint when the Kan extension of the identity along G exists and is preserved by G, and in that case the Kan extension gives the adjoint.

5.8. DEFINITION. A framed bicategory $\mathbb{D}$ is **closed** just when its underlying horizontal bicategory $\mathcal{D}$ is closed.

5.9. EXAMPLES. Many of our examples of framed bicategories are closed.

- $\mathbb{M}$**od** is closed; its hom-objects are given by

$$\begin{aligned} P \lhd M &= \mathrm{Hom}_C(M, P) \\ N \rhd P &= \mathrm{Hom}_A(N, P). \end{aligned}$$

- As long as $\mathscr{V}$ is closed and complete, then $\mathbb{D}$**ist**$(\mathscr{V})$ is closed; its hom-objects are given by the cotensor product of distributors (the end construction).
- $\mathbb{S}$**pan**$(\mathscr{C})$ is closed precisely when $\mathscr{C}$ is locally cartesian closed.
- $\mathbb{E}$**x** is also closed. This is proven in [MS06, §17.1]; we will describe the general method of proof in §14.

5.10. REMARK. A monoidal category is closed (on both sides) just when its corresponding vertically trivial framed bicategory is closed. On the other hand, if a monoidal category is symmetric, then the left and right internal-homs are isomorphic. In §10 we will prove an analogue of this fact for framed bicategories equipped with an 'involution', which includes all of our examples.

It is not surprising that there is some relationship between closedness and base change.

5.11. PROPOSITION. *Let $\mathbb{D}$ be a closed framed bicategory. Then for any $f : A \to C$, $g : B \to D$, and $M : C \nrightarrow D$, we have*

$$\begin{aligned} f^*Mg^* &\cong ({}_gD \rhd M) \lhd C_f \\ &\cong {}_gD \rhd (M \lhd C_f) \end{aligned}$$

and in particular

$$\begin{aligned} {}_fC &\cong C \lhd C_f \\ D_g &\cong {}_gD \rhd D \end{aligned}$$

PROOF. Straightforward adjunction arguments. ∎

Note that this implies, by uniqueness of adjoints, that the restriction functor f^* can also be described as $f^*N = N \lhd C_f$. Of course there are corresponding versions for composing on the other side.

Moreover, if coextensions exist, then uniqueness of adjoints also implies that we have

$$f_*M \cong M \lhd {}_fC. \tag{15}$$

Conversely, if $\mathbb{D}$ is closed, then (15) defines a right adjoint to f^*; thus coextensions exist in any closed framed bicategory, and also have a natural description in terms of the base change objects.

6. The 2-category of framed bicategories

We now introduce the morphisms between framed bicategories. To begin with, it is easy to define morphisms of double categories by analogy with monoidal categories.

6.1. Definition. Let $\mathbb{D}$ and $\mathbb{E}$ be double categories. A **lax double functor** $F\colon \mathbb{D} \to \mathbb{E}$ consists of the following.

- Functors $F_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1\colon \mathbb{D}_1 \to \mathbb{E}_1$ such that $L\circ F_1 = F_0\circ L$ and $R\circ F_1 = F_0\circ R$.
- Natural transformations $F_\odot\colon F_1M \odot F_1N \to F_1(M \odot N)$ and $F_U\colon U_{F_0A} \to F_1(U_A)$, whose components are globular, and which satisfy the usual coherence axioms for a lax monoidal functor or 2-functor (see, for example, [ML98, §XI.2]).

Dually, we have the definition of an **oplax double functor**, for which $F_\odot$ and F_U go in the opposite direction. A **strong double functor** is a lax double functor for which $F_\odot$ and F_U are (globular) isomorphisms. If just F_U is an isomorphism, we say that F is **normal**.

We occasionally abuse notation by writing just F for either F_0 or F_1. Observe that a lax double functor preserves vertical composition and identities strictly, but preserves horizontal composition and identities only up to constraints. Like the constraints $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$ for a double category, the maps $F_\odot$ and F_U are globular, but must be natural with respect to all 2-cells, not only globular ones.

If $\mathbb{D}$ and $\mathbb{E}$ are just monoidal categories, then a double functor $F\colon \mathbb{D} \to \mathbb{E}$ is the same as a monoidal functor (of whichever sort). The terms 'lax', 'oplax', and 'strong' are chosen to generalize this situation; some authors refer to strong double functors as *pseudo double functors*. Since the monoidal functors which arise in practice are most frequently lax, many authors refer to these simply as 'monoidal functors'. It is also true for framed bicategories that the lax morphisms are often those of most interest, but we will always keep the adjectives for clarity.

6.2. Example. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a lax monoidal functor, where the monoidal categories $\mathscr{C}$ and $\mathscr{D}$ both have coequalizers preserved by $\otimes$. Then it is well known that F preserves monoids, monoid homomorphisms, bimodules, and equivariant maps. Moreover, if $M\colon A \nrightarrow B$ and $N\colon B \nrightarrow C$ are bimodules in $\mathscr{C}$, so that their tensor product is the coequalizer

$$M \otimes B \otimes N \rightrightarrows M \otimes N \longrightarrow M \odot N$$

then we have the commutative diagram

$$\begin{tikzcd}
FM \otimes FB \otimes FN \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & FM \otimes FN \arrow[r] \arrow[d] & FM \odot FN \arrow[d, dotted] \\
F(M \otimes B \otimes N) \arrow[r, shift left] \arrow[r, shift right] & F(M \otimes N) \arrow[r] & F(M \odot N)
\end{tikzcd} \tag{16}$$

in which the top diagram is the coequalizer defining the tensor product of bimodules in $\mathscr{D}$, and hence the dotted map is induced. Moreover, since U_A in $\mathbb{M}\mathbf{od}(\mathscr{C})$ is just A regarded as an (A,A)-bimodule, we have $F(U_A) \cong U_{FA}$. It is straightforward to check that this isomorphism and the dotted map in (16) are the data for a normal lax double functor $\mathbb{M}\mathbf{od}(F) \colon \mathbb{M}\mathbf{od}(\mathscr{C}) \to \mathbb{M}\mathbf{od}(\mathscr{D})$.

Note that F does not need to preserve coequalizers, so the bottom row of (16) need not be a coequalizer diagram. However, if F does preserve coequalizers, and is moreover a strong monoidal functor, so that the left and middle vertical maps are isomorphisms, then so is the right vertical map; hence $\mathbb{M}\mathbf{od}(F)$ is a strong double functor in this case.

In particular, if $\mathscr{C} = \mathbf{Mod}_R$ and $\mathscr{D} = \mathbf{Mod}_S$ for commutative rings R and S and $f \colon R \to S$ is a homomorphism of commutative rings, then the extension-of-scalars functor $f_! \colon \mathbf{Mod}_R \to \mathbf{Mod}_S$ is strong monoidal and preserves coequalizers, hence induces a strong double functor. The restriction-of-scalars functor $f^* \colon \mathbf{Mod}_S \to \mathbf{Mod}_R$, on the other hand, is only lax monoidal, and hence induces a normal lax double functor.

6.3. Example. Let $F \colon \mathscr{C} \to \mathscr{D}$ be any functor between two categories with pullbacks. Then we have an induced normal *oplax* double functor $\mathbb{S}\mathbf{pan}(F) \colon \mathbb{S}\mathbf{pan}(\mathscr{C}) \to \mathbb{S}\mathbf{pan}(\mathscr{D})$. If F preserves pullbacks, then $\mathbb{S}\mathbf{pan}(F)$ is strong.

Now suppose that $\mathbb{D}$ and $\mathbb{E}$ are framed bicategories. Since the characterization of base change objects in Theorem 4.1(iii) only involves horizontal composition with units, any *normal* lax (or oplax) framed functor will preserve base change objects up to isomorphism; that is, $F({}_fB) \cong {}_{Ff}(FB)$. If it is *strong*, then it will also preserve restrictions and extensions, since we have $f^*Mg^* \cong {}_fB \odot M \odot D_g$ and similarly.

More generally, any lax or oplax double functor $F \colon \mathbb{D} \to \mathbb{E}$ between framed bicategories automatically induces comparison 2-cells such as

$$(Ff)_!(FM) \longrightarrow F(f_!M) \qquad \text{and} \tag{17}$$

$$F(f^*N) \longrightarrow (Ff)^*(FN), \tag{18}$$

by unique factorization through cartesian and opcartesian arrows. As remarked in §3, the first of these goes in the 'lax direction' while the second goes in the 'oplax direction'. Thus, for the whole functor to deserve the adjective 'lax', the second of these must be an isomorphism, so that it has an inverse which goes in the lax direction. This happens just when F preserves cartesian 2-cells.

However, it turns out that this is automatic: *any* lax double functor between framed bicategories preserves cartesian 2-cells, so that (18) is always an isomorphism when F is lax. Dually, any oplax double functor preserves opcartesian 2-cells, so that (17) is an isomorphism when F is oplax.

To prove this, we first observe that for any lax double functor $F \colon \mathbb{D} \to \mathbb{E}$ and any arrow $f\colon A \to B$ in $\mathbb{D}$, we have the following diagram of 2-cells in $\mathbb{E}$:

$$\begin{tikzcd}
U_{FA} \arrow[r, "F_U"] \arrow[d, "\mathrm{opcart}"'] & F(U_A) \arrow[d, "F(\mathrm{opcart})"] \\
{}_{Ff}(FB) \arrow[r, dotted, "{}_fF"] \arrow[d, "\mathrm{cart}"'] & F({}_fB) \arrow[d, "F(\mathrm{cart})"] \\
U_{FB} \arrow[r, "F_U"'] & F(U_B).
\end{tikzcd} \tag{19}$$

The dotted arrow, given by unique factorization through the opcartesian one, is the special case of (17) when $M = U_A$; we denote it by ${}_fF$. The upper square in (19) commutes by definition, and the lower square also commutes by uniqueness of the factorization. Similarly, we have a 2-cell $(FB)_{Ff} \xrightarrow{F_f} F(B_f)$.

If F is oplax instead, we have 2-cells in the other direction. If F is strong (or even just normal), the transformations exist in both directions, are inverse isomorphisms, and each is the mate of the inverse of the other.

6.4. Proposition. *Any lax double functor between framed bicategories preserves cartesian 2-cells, and any oplax double functor between framed bicategories preserves opcartesian 2-cells.*

Proof. Let $M\colon B \nrightarrow D$ in $\mathbb{D}$ and $f\colon A \to B$, $g\colon C \to D$. Then the following composite is cartesian in $\mathbb{D}$:

$$\begin{tikzcd}
A \arrow[r, "{}_fB", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{cart}"] & B \arrow[r, "M", "\shortmid"{marking}] \arrow[d, equal] & D \arrow[r, "D_g", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{cart}"] & C \arrow[d, "g"] \\
B \arrow[r, "U_B"', "\shortmid"{marking}] & B \arrow[r, "M"', "\shortmid"{marking}] & D \arrow[r, "U_D"', "\shortmid"{marking}] & D
\end{tikzcd} \tag{20}$$

and the following composite is cartesian in $\mathbb{E}$:

$$\begin{tikzcd}
FA \arrow[r, "{}_{Ff}(FB)", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "\mathrm{cart}"] & FB \arrow[r, "FM", "\shortmid"{marking}] \arrow[d, equal] & FD \arrow[r, "(FD)_{Fg}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{cart}"] & FC \arrow[d, "Fg"] \\
FB \arrow[r, "U_{FB}"', "\shortmid"{marking}] & FB \arrow[r, "FM"', "\shortmid"{marking}] & FD \arrow[r, "U_{FD}"', "\shortmid"{marking}] & FD.
\end{tikzcd} \tag{21}$$

Applying F to (20) and factoring the result through (21), we obtain a comparison map

$$F({}_fB \odot M \odot D_g) \longrightarrow {}_{Ff}(FB) \odot FM \odot (FD)_{Fg}, \tag{22}$$

which we want to show to be an isomorphism. We have an obvious candidate for its inverse, namely the following composite.

$$\begin{tikzcd}
{} \arrow[r, "{}_{Ff}(FB)", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "{}_fF\Downarrow"] & {} \arrow[r, "FM", "|" marking] \arrow[d, equal] & {} \arrow[r, "(FD)_{Fg}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_f"] & {} \arrow[d, equal] \\
{} \arrow[r, "F({}_fB)"', "|" marking] \arrow[d, equal] & {} \arrow[r, "FM", "|" marking] \arrow[d, phantom, "\Downarrow F_\odot"] & {} \arrow[r, "F(D_g)"', "|" marking] & {} \arrow[d, equal] \\
{} \arrow[rrr, "F({}_fB\odot M\odot D_g)"', "|" marking] & {} & & {}
\end{tikzcd} \tag{23}$$

Consider first the composite of (23) followed by (22):

$$\begin{tikzcd}
{} \arrow[r, "{}_{Ff}(FB)", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "{}_fF\Downarrow"] & {} \arrow[r, "FM", "|" marking] \arrow[d, equal] & {} \arrow[r, "(FD)_{Fg}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_f"] & {} \arrow[d, equal] \\
{} \arrow[r, "F({}_fB)"', "|" marking] \arrow[d, equal] & {} \arrow[r, "FM", "|" marking] \arrow[d, phantom, "\Downarrow F_\odot"] & {} \arrow[r, "F(D_g)"', "|" marking] & {} \arrow[d, equal] \\
{} \arrow[rrr, "F({}_fB\odot M\odot D_g)", "|" marking] \arrow[d, equal] & {} \arrow[d, phantom, "\Downarrow"] & & {} \arrow[d, equal] \\
{} \arrow[r, "{}_{Ff}(FB)"', "|" marking] & {} \arrow[r, "FM"', "|" marking] & {} \arrow[r, "(FD)_{Fg}"', "|" marking] & {}
\end{tikzcd}$$

If we postcompose this with (21), then by definition of (22), we obtain

$$\begin{tikzcd}
{} \arrow[r, "{}_{Ff}(FB)", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "{}_fF\Downarrow"] & {} \arrow[r, "FM", "|" marking] \arrow[d, equal] & {} \arrow[r, "(FD)_{Fg}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_f"] & {} \arrow[d, equal] \\
{} \arrow[r, "F({}_fB)"', "|" marking] \arrow[d, equal] & {} \arrow[r, "FM", "|" marking] \arrow[d, phantom, "\Downarrow F_\odot"] & {} \arrow[r, "F(D_g)"', "|" marking] & {} \arrow[d, equal] \\
{} \arrow[rrr, "F({}_fB\odot M\odot D_g)", "|" marking] \arrow[d, "Ff"'] & {} \arrow[d, phantom, "F(\mathrm{cart})"] & & {} \arrow[d, "Fg"] \\
{} \arrow[rrr, "FM"', "|" marking] & {} & & {}
\end{tikzcd}.$$

By naturality of the lax constraint for F, this is equal to

$$\begin{tikzcd}
{} \arrow[r, "{}_{Ff}(FB)", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "{}_fF\Downarrow"] & {} \arrow[r, "FM", "|" marking] \arrow[d, equal] & {} \arrow[r, "(FD)_{Fg}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_f"] & {} \arrow[d, equal] \\
{} \arrow[r, "F({}_fB)", "|" marking] \arrow[d, "Ff"'] \arrow[dr, phantom, "F(\mathrm{cart})"] & {} \arrow[r, "FM", "|" marking] \arrow[d, equal] & {} \arrow[r, "F(D_g)", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "F(\mathrm{cart})"] & {} \arrow[d, "Fg"] \\
{} \arrow[r, "F(U_B)"', "|" marking] \arrow[d, equal] & {} \arrow[r, "FM", "|" marking] \arrow[d, phantom, "\Downarrow F_\odot"] & {} \arrow[r, "F(U_D)"', "|" marking] & {} \arrow[d, equal] \\
{} \arrow[rrr, "FM"', "|" marking] & {} & & {}
\end{tikzcd}$$

Because the lower square in (19) commutes, this is equal to

$$\begin{tikzcd}
\cdot \arrow[r, "{}_{Ff}(FB)", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "\mathrm{cart}"] & \cdot \arrow[r, "FM", "\shortmid"{marking}] \arrow[d, equal] & \cdot \arrow[r, "(FD)_{Fg}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{cart}"] & \cdot \arrow[d, "Fg"] \\
\cdot \arrow[r, "U_{FB}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_U"] & \cdot \arrow[r, "FM", "\shortmid"{marking}] \arrow[d, equal] & \cdot \arrow[r, "U_{FD}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_U"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "F(U_B)"', "\shortmid"{marking}] \arrow[d, equal] & \cdot \arrow[r, "FM", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow F_{\odot}"] & \cdot \arrow[r, "F(U_D)"', "\shortmid"{marking}] & \cdot \arrow[d, equal] \\
\cdot \arrow[rrr, "FM"', "\shortmid"{marking}] & & & \cdot
\end{tikzcd}$$

which is equal to (21), by the coherence axioms for F. Thus, by unique factorization through (21), we conclude that (23) followed by (22) is the identity.

Now consider the composite of (22) followed by (23). By the construction of factorizations in Theorem 4.1, (22) can be computed by composing horizontally with opcartesian 2-cells; thus our desired composite is

$$\begin{tikzcd}
\cdot \arrow[r, "U_{FA}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{opcart}"] & \cdot \arrow[r, "F({}_fB\odot M\odot D_g)", "\shortmid"{marking}] \arrow[d, "Ff"] \arrow[dr, phantom, "F(\mathrm{cart})"] & \cdot \arrow[r, "U_{FC}", "\shortmid"{marking}] \arrow[d, "Fg"'] \arrow[dr, phantom, "\mathrm{opcart}"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "{}_{Ff}(FB)"] \arrow[d, equal] \arrow[dr, phantom, "{}_fF\Downarrow"] & \cdot \arrow[r, "FM"] \arrow[d, equal] & \cdot \arrow[r, "(FD)_{Fg}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_f"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "F({}_fB)"', "\shortmid"{marking}] \arrow[d, equal] & \cdot \arrow[r, "FM", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow F_{\odot}"] & \cdot \arrow[r, "F(D_g)"', "\shortmid"{marking}] & \cdot \arrow[d, equal] \\
\cdot \arrow[rrr, "F({}_fB\odot M\odot D_g)"', "\shortmid"{marking}] & & & \cdot
\end{tikzcd}.$$

By definition of ${}_fF$ and F_f, this is equal to

$$\begin{tikzcd}
\cdot \arrow[r, "U_{FA}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_U"] & \cdot \arrow[r, "F({}_fB\odot M\odot D_g)", "\shortmid"{marking}] \arrow[d, equal] & \cdot \arrow[r, "U_{FC}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_U"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "F(U_A)"] \arrow[d, equal] \arrow[dr, phantom, "F(\mathrm{opcart})"] & \cdot \arrow[r, "F({}_fB\odot M\odot D_g)", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "F(\mathrm{cart})"] & \cdot \arrow[r, "F(U_C)"] \arrow[d] \arrow[dr, phantom, "F(\mathrm{opcart})"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "F({}_fB)"', "\shortmid"{marking}] \arrow[d, equal] & \cdot \arrow[r, "FM"] \arrow[d, phantom, "\Downarrow F_{\odot}"] & \cdot \arrow[r, "F(D_g)"', "\shortmid"{marking}] & \cdot \arrow[d, equal] \\
\cdot \arrow[rrr, "F({}_fB\odot M\odot D_g)"', "\shortmid"{marking}] & & & \cdot
\end{tikzcd}.$$

By naturality of $F_\odot$, this is equal to

$$\begin{tikzcd}
{} \arrow[r, "U_{FA}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_U"] & {} \arrow[r, "F({}_fB\odot M\odot D_g)"] \arrow[d, equal] & {} \arrow[r, "U_{FC}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_U"] & {} \arrow[d, equal] \\
{} \arrow[r, "F(U_A)"'] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow F_\odot"] & {} \arrow[r, "F({}_fB\odot M\odot D_g)"] & {} \arrow[r, "F(U_C)"'] & {} \arrow[d, equal] \\
{} \arrow[rrr, "F({}_fB\odot M\odot D_g)"] \arrow[d, equal] \arrow[drrr, phantom, "F(\text{stuff})"] & & & {} \arrow[d, equal] \\
{} \arrow[rrr, "F({}_fB\odot M\odot D_g)"'] & & & {}
\end{tikzcd} \tag{24}$$

where 'stuff' is the composite

$$\begin{tikzcd}
{} \arrow[r, "U_A"] \arrow[d, equal] \arrow[dr, phantom, "\text{opcart}"] & {} \arrow[r, "{}_fB"] \arrow[d, "f"] \arrow[dr, phantom, "\text{cart}"] & {} \arrow[r, "M"] \arrow[d, equal] & {} \arrow[r, "D_g"] \arrow[d, equal] \arrow[dr, phantom, "\text{cart}"] & {} \arrow[r, "U_C"] \arrow[d, "g"] \arrow[dr, phantom, "\text{opcart}"] & {} \arrow[d, equal] \\
{} \arrow[r, "{}_fB"'] & {} \arrow[r, "U_B"'] & {} \arrow[r, "M"'] & {} \arrow[r, "U_D"'] & {} \arrow[r, "D_g"'] & {}
\end{tikzcd}$$

which is equal (modulo constraints) to the identity on ${}_fB \odot M \odot D_g$. Thus, applying the coherence axioms for F again, (24) reduces to the identity of $F({}_fB \odot M \odot D_g)$. Therefore, (23) is a two-sided inverse for (22), so the latter is an isomorphism; hence F preserves cartesian 2-cells. The oplax case is dual. ∎

Here we see again the advantage of using fibrations rather than introducing base change functors explicitly: since fibrations are 'non-algebraic', all their constraints and coherence come for free. This leads us to the following definition.

6.5. Definition. A **lax framed functor** is a lax double functor between framed bicategories. Similarly, an **oplax** or **strong framed functor** is a double functor of the appropriate type between framed bicategories.

We observed in §1 that while 2-functors give a good notion of morphism between both sorts of bicategories, the right notion of transformation for $\mathcal{M}od$-like bicategories is rather murkier. Once we include the vertical arrows to get a framed bicategory, however, it becomes much clearer what the transformations should be.

6.6. Definition. A **double transformation** between two lax double functors $\alpha : F \to G : \mathbb{D} \to \mathbb{E}$ consists of natural transformations $\alpha_0 \colon F_0 \to G_0$ and $\alpha_1 \colon F_1 \to G_1$ (both usually written as α), such that $L(\alpha_M) = \alpha_{LM}$ and $R(\alpha_M) = \alpha_{RM}$, and such that

$$\begin{tikzcd}
FA \arrow[r, "FM"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow F_\odot"] & FB \arrow[r, "FN"] & FC \arrow[d, equal] \\
FA \arrow[rr, "F(M\odot N)"] \arrow[d, "\alpha_A"'] \arrow[drr, phantom, "\Downarrow \alpha_{M\odot N}"] & & FC \arrow[d, "\alpha_C"] \\
GA \arrow[rr, "G(M\odot N)"'] & & GC
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, "FM"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\Downarrow \alpha_M"] & FB \arrow[r, "FN"] \arrow[d, "\alpha_B"] \arrow[dr, phantom, "\Downarrow \alpha_N"] & FC \arrow[d, "\alpha_C"] \\
GA \arrow[r, "GM"'] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow G_\odot"] & GB \arrow[r, "GN"'] & GC \arrow[d, equal] \\
GA \arrow[rr, "G(M\odot N)"'] & & GC
\end{tikzcd}$$

and

$$\begin{tikzcd} FA \arrow[r, "U_{FA}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_0"] & FA \arrow[d, equal] \\ FA \arrow[r, "F(U_A)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\Downarrow \alpha_{U_A}"] & FA \arrow[d, "\alpha_A"] \\ GA \arrow[r, "G(U_A)"', "|" marking] & GA \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "U_{FA}", "|" marking] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\Downarrow U_{\alpha_A}"] & FA \arrow[d, "\alpha_A"] \\ GA \arrow[r, "U_{GA}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow F_0"] & GA \arrow[d, equal] \\ GA \arrow[r, "G(U_A)"', "|" marking] & GA \end{tikzcd}.$$

The framed version of this definition requires no modification at all.

6.7. Definition. A **framed transformation** between two lax framed functors is simply a double transformation between their underlying lax double functors.

We leave it to the reader to define transformations between oplax functors. In the case of ordinary bicategories, there is also a notion of 'modification', or morphism between transformations, but with framed bicategories we usually have no need for these. Thus, the framed bicategories, framed functors, and framed transformations form a $\mathcal{C}at$-like bicategory, which is in fact a strict 2-category.

6.8. Proposition. *Small framed bicategories, lax framed functors, and framed transformations form a strict 2-category $\mathcal{F}r\mathcal{B}i_\ell$. If we restrict to strong framed functors, we obtain a 2-category $\mathcal{F}r\mathcal{B}i$, and if we use oplax framed functors instead, we obtain a 2-category $\mathcal{F}r\mathcal{B}i_{op\ell}$.*

Of course, double categories, double functors, and double transformations also form larger 2-categories $\mathcal{D}bl_\ell$, $\mathcal{D}bl$, and $\mathcal{D}bl_{op\ell}$.

6.9. Remark. Recall that we can regard a monoidal category as a framed bicategory whose vertical category is trivial, and that the framed functors between vertically trivial framed bicategories are precisely the monoidal functors (whether lax, oplax, or strong). It is easy to check that framed transformations are also the same as monoidal transformations; thus $\mathcal{M}on\mathcal{C}at$ is equivalent to a full sub-2-category of $\mathcal{F}r\mathcal{B}i$.

This is to be contrasted with the situation for ordinary 'unframed' bicategories. We can also consider monoidal categories to be bicategories with just one 0-cell, and 2-functors between such bicategories do also correspond to monoidal functors, but most transformations between such 2-functors do not give rise to anything resembling a monoidal transformation; see [CG06]. Thus, framed bicategories are a better generalization of monoidal categories than ordinary bicategories are.

6.10. Example. Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories with coequalizers preserved by $\otimes$, and let $\alpha\colon F \Rightarrow G\colon \mathscr{C} \to \mathscr{D}$ be a monoidal natural transformation between lax monoidal functors. We have already seen that F and G give rise to lax framed functors. Moreover, the fact that α is a monoidal transformation implies that if A is a monoid in $\mathscr{C}$, $\alpha_A\colon FA \to GA$ is a monoid homomorphism in $\mathscr{D}$, and similarly for bimodules. Therefore, we have an induced framed transformation

$$\mathbb{M}\mathbf{od}(\alpha)\colon \mathbb{M}\mathbf{od}(F) \to \mathbb{M}\mathbf{od}(G).$$

This makes $\mathbb{M}\mathbf{od}(-)$ into a strict 2-functor. Its domain is the 2-category of monoidal categories with coequalizers preserved by $\otimes$, lax monoidal functors, and monoidal transformations, and its codomain is $\mathcal{F}r\mathcal{B}i_\ell$. If we restrict the domain to strong monoidal functors which preserve coequalizers, the image lies in $\mathcal{F}r\mathcal{B}i$.

6.11. Example. Let $\mathscr{C}, \mathscr{D}$ be categories with pullbacks and $\alpha\colon F \Rightarrow G\colon \mathscr{C} \to \mathscr{D}$ a natural transformation. Then α induces a framed transformation

$$\mathbb{S}\mathbf{pan}(\alpha)\colon \mathbb{S}\mathbf{pan}(F) \to \mathbb{S}\mathbf{pan}(G)$$

in an obvious way. This makes $\mathbb{S}\mathbf{pan}$ into a strict 2-functor from the 2-category of categories with pullbacks, all functors, and all natural transformations, to $\mathcal{F}r\mathcal{B}i_{op\ell}$. If we restrict the domain to functors which preserve pullbacks, the image lies in $\mathcal{F}r\mathcal{B}i$.

6.12. Remark. It is easy to see that any framed functor induces a 2-functor of the appropriate type between horizontal bicategories, but the situation for framed transformations is less clear. We will consider this further in appendix B.

7. Framed equivalences

All the usual notions of 2-category theory apply to the study of framed bicategories via the 2-categories $\mathcal{F}r\mathcal{B}i_\ell$, $\mathcal{F}r\mathcal{B}i$, and $\mathcal{F}r\mathcal{B}i_{op\ell}$, and generally reduce to elementary notions when expressed explicitly. Since, as remarked above, the lax framed functors are often those of most interest, we work most frequently in $\mathcal{F}r\mathcal{B}i_\ell$, but analogous results are always true for the other two cases.

One important 2-categorical notion is that of *internal equivalence*. This is defined to be a pair of morphisms $F\colon D \to E$ and $G\colon E \to D$ with 2-cell isomorphisms $FG \cong \mathrm{Id}$ and $GF \cong \mathrm{Id}$. The notion of equivalence for framed bicategories we obtain in this way solves another of the problems raised in §1.

7.1. Definition. A **framed equivalence** is an internal equivalence in $\mathcal{F}r\mathcal{B}i_\ell$.

Thus, a framed equivalence consists of lax framed functors $F : \mathbb{D} \rightleftarrows \mathbb{E} : G$ with framed natural isomorphisms $\eta : \mathrm{Id}_{\mathbb{D}} \cong GF$ and $\varepsilon : FG \cong \mathrm{Id}_{\mathbb{E}}$. It might seem strange not to require F and G to be *strong* framed functors in this definition, but in fact this is automatic.

7.2. Proposition. *In a framed equivalence as above, F and G are automatically strong framed functors (hence give an equivalence in $\mathcal{F}r\mathcal{B}i$).*

We will prove this in the next section as Corollary 8.3.

Since strict 2-functors preserve internal equivalences, our 2-functorial ways of constructing framed bicategories give us a ready supply of framed equivalences. For example, any monoidal equivalence $\mathscr{C} \simeq \mathscr{D}$ of monoidal categories with coequalizers preserved by $\otimes$ induces a framed equivalence $\mathbb{M}\mathbf{od}(\mathscr{C}) \simeq \mathbb{M}\mathbf{od}(\mathscr{D})$. Similarly, any equivalence of categories with pullbacks induces a framed equivalence between framed bicategories of spans.

As for ordinary categories, we can characterize the framed functors which are equivalences as those which are 'full, faithful, and essentially surjective'. First we introduce the terminology, beginning with double categories. Recall that we write ${}_f\mathbb{D}_g(M,N)$ for the set of 2-cells of the form

$$\begin{tikzcd} A \arrow[r, "M", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "|"{marking}] & D. \end{tikzcd}$$

7.3. Definition. A lax or oplax double functor is **full** (resp. **faithful**) if it is full (resp. faithful) on vertical categories and each map

$$F\colon {}_g\mathbb{D}_f(M,N) \longrightarrow {}_{Fg}\mathbb{E}_{Ff}(FM,FN) \tag{25}$$

is surjective (resp. injective).

In the case of a *framed* functor, however, the notions simplify somewhat.

7.4. Proposition. *A lax or oplax* framed *functor $F\colon \mathbb{D} \to \mathbb{E}$ is full (resp. faithful) in the sense of Definition 7.3 if and only if it is full (resp. faithful) on vertical categories and each functor $\mathcal{D}(A,B) \to \mathcal{E}(FA,FB)$ is full (resp. faithful).*

Proof. Definition 7.3 clearly implies the given condition. Conversely, suppose that $F\colon \mathbb{D} \to \mathbb{E}$ is a lax framed functor. We have a natural bijection

$${}_g\mathbb{D}_f(M,N) \cong \mathcal{D}(M, f^*Ng^*)$$

which is preserved by F, since it preserves restriction. In other words, the diagram

$$\begin{tikzcd} {}_g\mathbb{D}_f(M,N) \arrow[rr, "\cong"] \arrow[d, "F"'] & & \mathcal{D}(M, f^*Ng^*) \arrow[d, "F"] \\ {}_{Fg}\mathbb{E}_{Ff}(FM,FN) \arrow[drr, "\cong"] & & \mathcal{E}(FM, F(f^*Ng^*)) \arrow[d, "\cong"] \\ & & \mathcal{E}(FM, (Ff)^*(FN)(Fg)^*) \end{tikzcd}$$

commutes. Thus, if the right-hand map is surjective (resp. injective), so is the left-hand map. An analogous argument works for an oplax framed functor, using extension instead of restriction. ∎

This is yet another expression of the fact that in a framed bicategory, the globular 2-cells carry the information about all the 2-cells. A similar thing happens for essential surjectivity.

7.5. DEFINITION. A lax or oplax double functor $F : \mathbb{D} \to \mathbb{E}$ is **essentially surjective** if we can simultaneously make the following choices:

- For each object C of $\mathbb{E}$, an object A_C of $\mathbb{D}$ and a vertical isomorphism $\alpha_C : F(A_C) \cong C$, and
- For each horizontal 1-cell $N : C \nrightarrow D$ in $\mathbb{E}$, a horizontal arrow $M_N\colon A_C \nrightarrow A_D$ in $\mathbb{E}$ and a 2-cell isomorphism

$$\begin{tikzcd} F(A_C) \arrow[r, "F(M_N)"] \arrow[d, "\alpha_C"'] \arrow[dr, phantom, "\alpha_M \cong"] & F(A_D) \arrow[d, "\alpha_D"] \\ C \arrow[r, "N"', "|" marking] & D \end{tikzcd}.$$

7.6. PROPOSITION. *A lax or oplax* framed *functor is essentially surjective, in the sense of Definition 7.5, if and only if it is essentially surjective on vertical categories and each functor $\mathcal{D}(A, B) \to \mathcal{E}(FA, FB)$ is essentially surjective.*

PROOF. Clearly Definition 7.5 implies the given condition. Conversely, suppose that F satisfies the given condition. Choose isomorphisms $\alpha_C : F(A_C) \cong C$ for each object C of $\mathbb{E}$, which exist because F is essentially surjective on vertical categories. Then given $N : C \nrightarrow D$, we have $\alpha_C^* N \alpha_D^*\colon F(A_C) \nrightarrow F(A_D)$, so since $F\colon \mathcal{D}(A_C, A_D) \to \mathcal{E}(F(A_C), F(A_D))$ is essentially surjective, we have an $M_N\colon A_C \to A_D$ and a globular isomorphism $F(M_N) \cong \alpha_C^* N \alpha_D^*$. Composing this with the cartesian 2-cell defining $\alpha_C^* N \alpha_D^*$, we obtain the desired α_M. ■

The following theorem and its corollary are the main points of this section. Of course, we define a **double equivalence** to be an internal equivalence in $\mathcal{D}bl_\ell$.

7.7. THEOREM. *A strong double functor $F : \mathbb{D} \to \mathbb{E}$ is part of a double equivalence if and only if it is full, faithful, and essentially surjective.*

PROOF. We sketch a construction of an inverse equivalence $G : \mathbb{E} \to \mathbb{D}$ for F. Make choices as in Definition 7.5, and define $GC = A_C$ and $GN = M_N$. Define G on vertical arrows and 2-cells by composing with the chosen isomorphisms; vertical functoriality follows from F being full and faithful. We produce the constraint cells for G by composing these isomorphisms with the *inverses* of the constraint cells for F and using that F is full and faithful; this is why we need F to be strong.

The choices from the definition of essentially surjective then give directly a double natural isomorphism $FG \cong \mathrm{Id}_{\mathbb{E}}$, and we can produce a double natural isomorphism $GF \cong \mathrm{Id}_{\mathbb{D}}$ by reflecting identity maps in $\mathbb{E}$. Thus G and F form a double equivalence. ■

7.8. Corollary. *A strong framed functor $F\colon \mathbb{D} \to \mathbb{E}$ is part of a framed equivalence precisely when*

- *It induces an equivalence $F_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ on vertical categories, and*
- *Each functor $F\colon \mathcal{D}(A,B) \to \mathcal{E}(FA,FB)$ is an equivalence of categories.*

Proof. Combine Proposition 7.4 and Proposition 7.6 with Theorem 7.7 to see that F has an inverse which is a strong double functor, hence also a strong framed functor by Proposition 6.4. ∎

A framed equivalence $F\colon \mathbb{D} \rightleftarrows \mathbb{E} :G$ clearly includes an equivalence $F_0\colon \mathbb{D}_0 \rightleftarrows \mathbb{E}_0 :G_0$ of vertical categories. It is less clear that it induces a biequivalence $\mathcal{D} \simeq \mathcal{E}$ of horizontal bicategories. We will see in appendix B, however, that this is true, though not trivial. This lack of triviality, in the following example, was one of the original motivations for this work.

7.9. Example. There are a number of framed bicategories related to $\mathbb{E}\mathbf{x}$, such as a fiberwise version $\mathbb{E}\mathbf{x}_B$ where the objects are already parametrized over some space B, and an equivariant version $G\mathbb{E}\mathbf{x}$ in which everything carries an action by some fixed group G. In [MS06, 19.3.5] it was observed (essentially) that $G\mathbb{E}\mathbf{x}_{G/H}$, the framed bicategory of G-equivariant parametrized spectra all over the coset space G/H, and $H\mathbb{E}\mathbf{x}$, the framed bicategory of H-equivariant parametrized spectra, are equivalent.

However, as observed in [MS06], the language of bicategories does not really suffice to describe this fact. On objects, the equivalence goes as follows: if X is a G-space over G/H, the fiber X_e is an H-space; while if Y is an H-space, $G \times_H Y$ is a G-space over G/H. But the composites in either direction are only homeomorphic, not equal, whereas the bicategory $\mathcal{E}x$ described in [MS06] does not include any information about homeomorphisms of base spaces.

8. Framed adjunctions

Adjunctions are one of the most important tools of category theory. Thus, from a categorical point of view, one of the most serious problems with $\mathcal{M}od$-like bicategories is the lack of a good notion of adjunction between them. For example, Ross Street wrote the following in a review of [CKW91]:

> Nearly two decades after J. W. Gray's work [Gra74], the most useful general notion of adjointness for morphisms between 2-categories has still not emerged. Perhaps the good notion should depend on the kind of 2-categories in mind; 2-categories whose arrows are functions or functors are of a different nature from those whose arrows are relations or profunctors.

In fact, motivated by the desire for a good notion of adjunction, [CKW91] and related papers such as [Ver92, CKVW98] come very close to our definition of framed bicategory. In appendix C we will make a formal comparison; for now we simply develop the theory of framed adjunctions.

8.1. Definition. A **framed adjunction** $F \dashv G$ is an internal adjunction in the 2-category $\mathcal{F}r\mathcal{B}i_\ell$. Explicitly, it consists of lax framed functors $F\colon \mathbb{D} \to \mathbb{E}$ and $G\colon \mathbb{E} \to \mathbb{D}$, together with framed transformations $\eta\colon \mathrm{Id}_{\mathbb{D}} \to GF$ and $\varepsilon\colon FG \to \mathrm{Id}_{\mathbb{E}}$ satisfying the usual triangle identities. Similarly, an **op-framed adjunction** is an internal adjunction in $\mathcal{F}r\mathcal{B}i_{op\ell}$.

Experience shows that adjunctions in $\mathcal{F}r\mathcal{B}i_\ell$ arise more frequently than the other two types, hence deserve the unadorned name. However, we have the following fundamental result.

8.2. Proposition. *In any framed adjunction $F \dashv G$, the left adjoint F is always a* strong *framed functor.*

Sketch of Proof. This actually follows formally from a general 2-categorical result known as 'doctrinal adjunction'; see [Kel74]. For the non-2-categorically inclined reader we sketch a more concrete version of the proof. We first show that the following composite is an inverse to $F_\odot\colon FM \odot FN \to F(M \odot N)$:

$$F(M \odot N) \xrightarrow{F(\eta\odot\eta)} F(GFM \odot GFN) \xrightarrow{G_\odot} FG(FM \odot FN) \xrightarrow{\varepsilon} FM \odot FN \qquad (26)$$

For example, the following diagram shows that the composite in one direction is the identity.

$$\begin{tikzcd}
F(GFM \odot GFN) \arrow[r, "G_\odot"] & FG(FM \odot FN) \arrow[r, "\varepsilon"] \arrow[d, "FG(F_\odot)"] & FM \odot FN \arrow[d, "F_\odot"] \\
F(M \odot N) \arrow[u, "F(\eta\odot\eta)"] \arrow[r, "F(\eta)"] \arrow[rr, bend right, "\mathrm{id}"'] & FGF(M \odot N) \arrow[r, "\varepsilon"] & F(M \odot N).
\end{tikzcd}$$

The right-hand square commutes by naturality of ε, the left-hand square commutes because η is a framed transformation, and the lower triangle is one of the triangle identities. The other direction is analogous.

Similarly, we show that the following composite is an inverse to $F_U\colon U_{FA} \to F(U_A)$:

$$F(U_A) \xrightarrow{F(U_\eta)} FU_{GFA} \xrightarrow{G_U} FGU_{FA} \xrightarrow{\varepsilon} U_{FA}, \qquad (27)$$

so that F is strong. ■

The similarity between (26) and (27) is obvious. In fact, these composites are the mates of the constraint cells for G under an adjunction in a suitable 2-category; the reader may consult [Kel74] for details. Of course, in an op-framed adjunction, the right adjoint is strong.

We can now prove Proposition 7.2.

8.3. Corollary. *Both functors in a framed equivalence are strong framed functors.*

Proof. It is well-known that any classical equivalence of categories can be improved to an 'adjoint equivalence', meaning an equivalence in which the isomorphisms $FG \cong \mathrm{Id}$ and $\mathrm{Id} \cong GF$ are also the unit and counit of an adjunction $F \dashv G$, and hence their inverses are the unit and counit of an adjunction $G \dashv F$. This fact can easily be 'internalized' to any 2-category, such as $\mathcal{F}r\mathcal{B}i_\ell$. Thus, any lax framed functor which is part of a framed equivalence is a framed left adjoint, and hence by Proposition 8.2 is strong. ∎

As is the case for categories, we can also characterize framed adjunctions using universal arrows. A similar result for double categories was given in [Gar07].

Recall that given a functor $G\colon \mathscr{E} \to \mathscr{D}$, a *universal arrow* to G is an arrow $\eta\colon A \to GFA$ in $\mathscr{D}$, for some object $FA \in \mathscr{E}$, such that any other arrow $A \to GY$ factors through η via a unique map $FA \to Y$ in $\mathscr{E}$. Similarly, if $G\colon \mathbb{E} \to \mathbb{D}$ is a framed functor, we define a *universal 2-cell* to be a 2-cell $\eta\colon M \to GFM$ in $\mathbb{D}$, not in general globular, whose left and right frames are universal arrows in $\mathbb{D}_0$, and such that any 2-cell $M \to GN$ factors through η via a unique 2-cell $FM \to N$ in $\mathbb{E}$.

8.4. Proposition. *Let $G\colon \mathbb{E} \to \mathbb{D}$ be a lax framed functor. Then G has a framed left adjoint if and only if the following are true.*

(i) For every object A in $\mathbb{D}$, there is a universal arrow $A \to GFA$.

(ii) For every horizontal 1-cell $M\colon A \,\mathrel{-\!\!\!\mapsto}\, B$ in $\mathbb{D}$, there is a universal 2-cell $M \to GFM$, as described above.

(iii) If $M \to GFM$ and $N \to GFN$ are universal 2-cells, then so is the composite

$$\begin{tikzcd}
{} \arrow[r, "M"] \arrow[d] \arrow[dr, phantom, "\text{univ}"] & {} \arrow[r, "N"] \arrow[d] \arrow[dr, phantom, "\text{univ}"] & {} \arrow[d] \\
{} \arrow[r, "GFM"] \arrow[d, equal] & {} \arrow[r, "GFN"] \arrow[d, phantom, "\Downarrow G_\odot"] & {} \arrow[d, equal] \\
{} \arrow[rr, "G(FM\odot FN)"'] & & {}
\end{tikzcd}$$

(iv) If $A \to GFA$ is universal, then so is the composite

$$\begin{tikzcd}
{} \arrow[r, "U_A"] \arrow[d, "\text{univ}"'] \arrow[dr, phantom, "\Downarrow U_{\text{univ}}"] & {} \arrow[d, "\text{univ}"] \\
{} \arrow[r, "U_{GFA}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow G_0"] & {} \arrow[d, equal] \\
{} \arrow[r, "G(U_{FA})"'] & {}
\end{tikzcd}$$

If G is strong, then (iii) simplifies to 'the horizontal composite of universal 2-cells is universal' and (iv) simplifies to 'the horizontal unit of a universal arrow is a universal 2-cell'.

Sketch of Proof. It is straightforward to show that if G has a left adjoint, then the conditions are satisfied. Conversely, conditions (i) and (ii) clearly guarantee that G_0 and G_1 both have left adjoints F_0 and F_1, and that $LF_1 \cong F_0L$ and $RF_1 \cong F_0R$. Since $\mathbb{E}$ is a framed bicategory, we can redefine F_1 by restricting along these isomorphisms to ensure that $LF_1 = F_0L$ and $RF_1 = F_0R$.

Conditions (iii) and (iv) then supply the constraints to make F into a strong framed functor. The universal cells give a double transformation $\eta\colon \mathrm{Id} \to GF$ and the counit $\varepsilon\colon FG \to \mathrm{Id}$ is constructed as usual. The last statement follows because anything isomorphic to a universal arrow is universal. ∎

Since strict 2-functors preserve internal adjunctions, our 2-functorial ways of constructing framed bicategories also give us a ready supply of framed adjunctions.

8.5. Example. Since $\mathbb{M}\mathbf{od}$ is a 2-functor, any monoidal adjunction between monoidal categories with coequalizers preserved by $\otimes$ gives rise to a framed adjunction. Here by a *monoidal adjunction* we mean an adjunction in the 2-category $\mathcal{M}on\mathcal{C}at_\ell$ of monoidal categories and lax monoidal functors.

For example, if $f\colon R \to S$ is a homomorphism of commutative rings, we have an induced monoidal adjunction

$$f_!\colon \mathbf{Mod}_R \rightleftarrows \mathbf{Mod}_S \;:f^*$$

and therefore a framed adjunction

$$\mathbb{M}\mathbf{od}(f_!)\colon \mathbb{M}\mathbf{od}(R) \rightleftarrows \mathbb{M}\mathbf{od}(S) \;:\mathbb{M}\mathbf{od}(f^*).$$

8.6. Example. Since $\mathbb{S}\mathbf{pan}$ is a strict 2-functor, any adjunction $f^*\colon \mathscr{E} \rightleftarrows \mathscr{F} \;: f_*$ between categories with pullbacks gives rise to an op-framed adjunction $\mathbb{S}\mathbf{pan}(\mathscr{E}) \rightleftarrows \mathbb{S}\mathbf{pan}(\mathscr{F})$. If f^* also preserves pullbacks, then this adjunction lies in $\mathcal{F}r\mathcal{B}i$, hence is also a framed adjunction.

9. Monoidal framed bicategories

Most of our examples also have an 'external' monoidal structure. For example, if M is an (A, B)-bimodule and N is a (C, D)-bimodule, we can form the $(A \otimes C, B \otimes D)$-bimodule $M \otimes N$. The definition of a 'monoidal bicategory' involves many coherence axioms (see [GPS95, Gur06]), but for *framed* bicategories we can simply invoke general 2-category theory once again.

In any 2-category with finite products, we have the notion of a *pseudo-monoid*: this is an object A equipped with multiplication $A \times A \to A$ and unit $1 \to A$ satisfying the usual monoid axioms up to coherent isomorphism. A pseudo-monoid in $\mathcal{C}at$ is precisely an ordinary monoidal category. Thus, it makes sense to define a monoidal framed bicategory to be a pseudo-monoid in $\mathcal{F}r\mathcal{B}i$. What this means is essentially the following.

9.1. Definition. A **monoidal framed bicategory** is a framed bicategory equipped with a strong framed functor $\otimes\colon \mathbb{D}\times\mathbb{D}\to\mathbb{D}$, a unit $I\in\mathbb{D}_0$, and framed natural constraint isomorphisms satisfying the usual axioms.

If we unravel this definition more explicitly, it says the following.

(i) $\mathbb{D}_0$ and $\mathbb{D}_1$ are both monoidal categories.

(ii) I is the monoidal unit of $\mathbb{D}_0$ and U_I is the monoidal unit of $\mathbb{D}_1$.

(iii) The functors L and R are strict monoidal.

(iv) We have an 'interchange' isomorphism

$$\mathfrak{x}\colon (M\otimes P)\odot(N\otimes Q)\cong(M\odot N)\otimes(P\odot Q)$$

and a unit isomorphism

$$\mathfrak{u}\colon U_{A\otimes B}\cong(U_A\otimes U_B)$$

satisfying appropriate axioms (these arise from the constraint data for the strong framed functor $\otimes$).

(v) The associativity and unit isomorphisms for $\otimes$ are framed transformations.

As we saw in §6, a strong framed functor such as $\otimes$ preserves cartesian and opcartesian arrows. Thus we automatically have isomorphisms such as $f^*M\otimes g^*N\cong(f\otimes g)^*(M\otimes N)$.

9.2. Examples. Many of our examples of framed bicategories are in fact monoidal.

- The framed bicategory $\mathbb{M}\mathbf{od}$, and more generally $\mathbb{M}\mathbf{od}(\mathscr{C})$ for a symmetric monoidal $\mathscr{C}$, is monoidal under the tensor product of rings and bimodules. Note that the tensor product of bimodules referred to here is 'external': if M is an (R,S)-bimodule and N is a (T,V)-bimodule, then $M\otimes N$ is an $(R\otimes T,S\otimes V)$-bimodule.
- If $\mathscr{C}$ has finite limits, then $\mathbb{S}\mathbf{pan}(\mathscr{C})$ is a monoidal framed bicategory under the cartesian product of objects and spans.
- $\mathbb{E}\mathbf{x}$ is monoidal under the cartesian product of spaces and the 'external smash product' $\overline{\wedge}$ of parametrized spectra.
- $n\mathbb{C}\mathbf{ob}$ is monoidal under disjoint union of manifolds and cobordisms.
- $\mathbb{D}\mathbf{ist}(\mathscr{V})$ is monoidal under the tensor product of $\mathscr{V}$-categories (see [Kel82, §1.4]).

9.3. Example. Recalling that monoidal categories can be identified with vertically trivial framed bicategories, it is easy to check that a vertically trivial monoidal framed bicategory is the same as a category with two interchanging monoidal structures. More generally, if $\mathbb{D}$ is any monoidal framed bicategory, then the category $\mathcal{D}(I, I)$ inherits two interchanging monoidal structures $\odot$ and $\otimes$. By the Eckmann-Hilton argument, any two such interchanging monoidal structures agree up to isomorphism and are braided.

We emphasize that the associativity and unit constraints are *vertical* isomorphisms. For example, in the monoidal framed bicategory $\mathbb{M}\mathbf{od}$, the associativity constraint on objects is the ring isomorphism $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$. This is to be contrasted with the classical notion of 'monoidal bicategory' in which the constraints are 1-cells, which would correspond to bimodules in this case. So while a framed bicategory obviously has an underlying horizontal bicategory, it requires proof that a monoidal framed bicategory has an underlying monoidal bicategory; see appendix B.

We observe, in passing, that an external monoidal structure automatically preserves dual pairs.

9.4. Proposition. *If (M, N) and (P, Q) are dual pairs in a monoidal framed bicategory, then so is $(M \otimes P, N \otimes Q)$.*

Proof. It is easy to see that any strong framed functor preserves dual pairs, and $\otimes$ is a strong framed functor. ∎

Now, just as an ordinary monoidal category can be braided or symmetric, so can a pseudo-monoid in an arbitrary 2-category with products. We define a **braided** or **symmetric** monoidal framed bicategory to be essentially a braided or symmetric pseudo-monoid in $\mathcal{F}r\mathcal{B}i$.

More explicitly, a braided monoidal framed bicategory is a monoidal framed bicategory such that $\mathbb{D}_0$ and $\mathbb{D}_1$ are braided monoidal with braidings $\mathfrak{s}$, the functors L and R are braided monoidal, and the following diagrams commute:

$$\begin{tikzcd}
(M \odot N) \otimes (P \odot Q) \arrow[r, "\mathfrak{s}"] \arrow[d, "\mathfrak{x}"'] & (P \odot Q) \otimes (M \odot N) \arrow[d, "\mathfrak{x}"] \\
(M \otimes P) \odot (N \otimes Q) \arrow[r, "\mathfrak{s}"'] & (P \otimes M) \odot (Q \otimes N)
\end{tikzcd}$$

$$\begin{tikzcd}
(U_A \otimes U_B) \arrow[r, "\mathfrak{u}"] \arrow[d, "\mathfrak{s}"'] & U_{A\otimes B} \arrow[d, "U_{\mathfrak{s}}"] \\
U_B \otimes U_A \arrow[r, "\mathfrak{u}"'] & U_{B\otimes A}
\end{tikzcd} .$$

A symmetric monoidal framed bicategory is a braided monoidal framed bicategory such that $\mathbb{D}_0$ and $\mathbb{D}_1$ are symmetric.

9.5. Examples. All the examples of monoidal framed bicategories given in Examples 9.2 are in fact symmetric monoidal.

9.6. Example. If $\mathbb{D}$ is a braided or symmetric monoidal framed bicategory, then $\mathcal{D}(I, I)$ inherits two interchanging monoidal structures, one of which is braided, and therefore it is essentially a symmetric monoidal category. Conversely, the vertically trivial monoidal framed bicategory corresponding to any symmetric monoidal category is a symmetric monoidal framed bicategory.

We now define the morphisms between monoidal framed bicategories. As usual, these come in three flavors.

9.7. Definition. A **lax monoidal framed functor** between monoidal framed bicategories $\mathbb{D}, \mathbb{E}$ consists of the following structure and properties.

- A lax framed functor $F\colon \mathbb{D} \to \mathbb{E}$.
- The structure of a lax monoidal functor on F_0 and F_1.
- Equalities $LF_1 = F_0L$ and $RF_1 = F_0R$ of lax monoidal functors.
- The composition constraints for the lax framed functor F are monoidal natural transformations.

It is **strong** if F is a strong framed functor and F_0 and F_1 are strong monoidal functors. If $\mathbb{D}$ and $\mathbb{E}$ are braided (resp. symmetric), then F is **braided** (resp. **symmetric**) if F_0 and F_1 are. We have a dual definition of **oplax monoidal framed functor**. A **monoidal framed transformation** is a framed transformation such that α_0 and α_1 are monoidal transformations.

These definitions give various 2-categories, each of which has its own attendant notion of equivalence and adjunction. We will not spell these out explicitly.

9.8. Examples. The 2-functor $\mathbb{M}\mathbf{od}$ lifts to a 2-functor from symmetric monoidal categories with coequalizers preserved by $\otimes$ to symmetric monoidal framed bicategories. Similarly, $\mathbb{S}\mathbf{pan}$ lifts to a 2-functor landing in symmetric monoidal framed bicategories.

Finally, we consider what it means for a framed bicategory to be 'closed monoidal'.

9.9. Definition. A monoidal framed bicategory $\mathbb{D}$ is **externally closed** if for any objects A, B, C, D, the functor

$$\otimes\colon \mathcal{D}(A, C) \times \mathcal{D}(B, D) \longrightarrow \mathcal{D}(A \otimes B, C \otimes D)$$

has right adjoints in each variable, which we write $\overline{\lhd}$ and $\overline{\rhd}$.

Explicitly, this means that for horizontal 1-cells $M\colon A \nrightarrow C$, $N\colon B \nrightarrow D$, and $P\colon A \otimes B \nrightarrow C \otimes D$, there are 1-cells $N \mathbin{\overline{\rhd}} P$ and $P \mathbin{\overline{\lhd}} M$ and bijections

$$\mathcal{D}(M \otimes N, P) \cong \mathcal{D}(M, N \mathbin{\overline{\rhd}} P) \cong \mathcal{D}(N, P \mathbin{\overline{\lhd}} M).$$

Of course, if $\mathbb{D}$ is symmetric, then $\overline{\lhd}$ and $\overline{\rhd}$ agree, modulo suitable isomorphisms.

9.10. Examples. The monoidal framed bicategory $\mathbb{M}\mathbf{od}$ is externally closed, as are $\mathbb{E}\mathbf{x}$ and $\mathbb{D}\mathbf{ist}(\mathscr{V})$. If $\mathscr{C}$ is locally cartesian closed, then $\mathbb{S}\mathbf{pan}(\mathscr{C})$ is also externally closed.

10. Involutions

In most of our examples, the 'directionality' of the horizontal 1-cells is to some extent arbitrary. For example, an (A, B)-bimodule could just as well be regarded as a (B^{op}, A^{op})-bimodule. We now define a structure which encodes this fact formally.

If $\mathbb{D}$ is a framed bicategory, we write $\mathbb{D}^{h\cdot op}$ for its 'horizontal dual': $\mathbb{D}^{h\cdot op}$ has the same vertical category as $\mathbb{D}$, but a horizontal 1-cell from A to B in $\mathbb{D}^{h\cdot op}$ is a horizontal 1-cell from B to A in $\mathbb{D}$, and the 2-cells are similarly flipped horizontally.

10.1. Definition. An **involution** on a framed bicategory $\mathbb{D}$ consists of the following.

(i) A strong framed functor $(-)^{op} : \mathbb{D}^{h\cdot op} \to \mathbb{D}$.

(ii) A framed natural isomorphism $\xi : ((-)^{op})^{op} \cong \mathrm{Id}_{\mathbb{D}}$ such that $(\xi_A)^{op} = \xi_{A^{op}}$; thus ξ and ξ^{-1} make $(-)^{op}$ into an adjoint equivalence.

We say an involution is **vertically strict** if the vertical arrow components of ξ are identities. If $\mathbb{D}$, $(-)^{op}$, and ξ are all monoidal (resp. symmetric monoidal), we say that the involution is **monoidal** (resp. **symmetric monoidal**).

The strong functoriality of $(-)^{op}$ implies that we have

$$\begin{aligned}
(U_A)^{op} &\cong U_{A^{op}} \\
(M \odot N)^{op} &\cong N^{op} \odot M^{op}. \\
(f^* M g^*)^{op} &\cong (g^{op})^*(M^{op})(f^{op})^* \\
(f_! M g_!)^{op} &\cong (g^{op})_!(M^{op})(f^{op})_!.
\end{aligned}$$

In particular, we have $(A_f)^{op} \cong {}_{f^{op}}(A^{op})$ and dually. If the involution is monoidal, we also have

$$\begin{aligned}
(A \otimes B)^{op} &\cong A^{op} \otimes B^{op} \\
I^{op} &\cong I.
\end{aligned}$$

10.2. Examples. Most of our examples are equipped with vertically strict symmetric monoidal involutions.

- The involution on $\mathbb{M}\mathbf{od}$ takes a ring A to the opposite ring A^{op}, and an (A, B)-bimodule to the same abelian group regarded as a (B^{op}, A^{op})-bimodule.
- The involution on $\mathbb{D}\mathbf{ist}(\mathscr{V})$ takes a $\mathscr{V}$-category to its opposite and reverses distributors in an obvious way.

- The involution on $\mathbb{S}\mathbf{pan}(\mathscr{C})$ takes each object to itself, and a span $A \xleftarrow{f} X \xrightarrow{g} B$ to the span $B \xleftarrow{g} X \xrightarrow{f} A$.
- The involution on $n\mathbb{C}\mathbf{ob}$ takes a manifold M to the manifold M^{op} with the opposite orientation, and reverses the direction of cobordisms in an obvious way.

In all these cases, the 2-cell components of ξ can also be chosen to be identities, but this is not true for all involutions, even vertically strict ones.

- The involution on $\mathbb{E}\mathbf{x}$ takes each space to itself, but takes a spectrum E parametrized over $B\times A$ to the pullback $\mathfrak{s}^*E$ over $A\times B$, where $\mathfrak{s}$ is the symmetry isomorphism $A\times B \cong B\times A$. Here $\mathfrak{s}^*\mathfrak{s}^*E$ is only canonically isomorphic to E, by pseudofunctoriality.

In [MS06, 16.2.1] an involution on a *bicategory* was defined to be essentially a pseudofunctor $(-)^{op}\colon \mathcal{B}^{op} \to \mathcal{B}$ equipped with a pseudonatural transformation $\xi\colon ((-)^{op})^{op} \cong \mathrm{Id}_{\mathcal{B}}$ whose 1-cell components are identities (although the unit axiom for ξ was omitted). It is easy to see that any vertically strict involution on $\mathbb{D}$ gives rise to an involution on $\mathcal{D}$. All the above examples are vertically strict, but in §11 and §15 we will see examples which are not.

Any symmetric monoidal category, considered as a vertically trivial framed bicategory, has a canonical involution. The functor $(-)^{op}$ is the identity on 1-cells (the objects of the monoidal category), and its composition constraint is the symmetry isomorphism:

$$(A \odot B)^{op} = A \odot B \xrightarrow{\cong} B \odot A = B^{op} \odot A^{op}.$$

All the components of ξ are identities. In fact, to give an involution on a vertically trivial framed bicategory which is the identity on 1-cells and for which ξ is an identity is essentially to give a symmetry for the corresponding monoidal category. Thus, we may view an involution on a framed bicategory as a generalization of a symmetry on a monoidal category.

One consequence of a monoidal category's being symmetric is that if it is closed, then the left and right internal-homs are isomorphic. The original motivation in [MS06] for introducing involutions was to obtain a similar result for closed bicategories; see [MS06, 16.3.5]. Of course, this is also true for framed bicategories.

10.3. Proposition. *If $\mathbb{D}$ is a closed framed bicategory equipped with an involution, then we have*

$$M \rhd N \cong (N^{op} \lhd M^{op})^{op}.$$

PROOF. Since $(-)^{op}$ is a framed equivalence, it is locally full and faithful. Thus, if $M\colon A \nrightarrow B$, $N\colon C \nrightarrow B$, and $P\colon C \nrightarrow A$, we have

$$\begin{aligned} \mathcal{D}(C,A)(P, M \rhd N) &\cong \mathcal{D}(C,B)(P \odot M, N) \\ &\cong \mathcal{D}(B^{op}, C^{op})((P \odot M)^{op}, N^{op}) \\ &\cong \mathcal{D}(B^{op}, C^{op})(M^{op} \odot P^{op}, N^{op}) \\ &\cong \mathcal{D}(A^{op}, C^{op})(P^{op}, N^{op} \lhd M^{op}) \\ &\cong \mathcal{D}\Big((C^{op})^{op}, (A^{op})^{op}\Big)\Big((P^{op})^{op}, (N^{op} \lhd M^{op})^{op}\Big) \\ &\cong \mathcal{D}(C,A)(P, (N^{op} \lhd M^{op})^{op}) \end{aligned}$$

so the result follows by the Yoneda lemma. ∎

11. Monoids and modules

In most of our examples of monoidal framed bicategories, the external monoidal structure and the horizontal composition are more closely related than is captured by the interchange isomorphism: namely, the horizontal composition $M \odot N$ is a subobject or quotient of the external product $M \otimes N$. For example, in $\mathbb{M}\mathbf{od}$ the tensor product $M \otimes_R N$ is a quotient of the external product $M \otimes N$, while in $\mathbb{S}\mathbf{pan}$ the pullback $M \times_B N$ is a subobject of the external product $M \times N$. An analogous relationship holds between the bicategorical homs $\lhd, \rhd$ and the external homs $\overline{\lhd}$, $\overline{\rhd}$.

In this section we will generalize the construction of the framed bicategory $\mathbb{M}\mathbf{od}(\mathscr{C})$ of monoids and modules from Example 2.2, replacing the monoidal category $\mathscr{C}$ with a framed bicategory $\mathbb{D}$. This describes one general class of examples in which the horizontal composition of 'bimodules' is defined as a coequalizer. In §§12–14, we will investigate framed bicategories constructed in a way analogous to $\mathbb{S}\mathbf{pan}$. We will then combine these two constructions in §15 to define framed bicategories of internal and enriched categories.

11.1. DEFINITION. Let $\mathbb{D}$ be a framed bicategory.

- A **monoid** in $\mathbb{D}$ consists of an object R, a horizontal 1-cell $A\colon R \nrightarrow R$, and globular 2-cells $e\colon R \to A$ and $m\colon A \odot A \to A$ called 'unit' and 'multiplication' such that the standard diagrams commute. Thus it is just a monoid in the ordinary monoidal category $\mathcal{D}(R,R)$.

- A **monoid homomorphism** $(R,A) \to (S,B)$ consists of a vertical arrow $f\colon R \to S$ and a 2-cell $\phi\colon A \overset{f}{\underset{f}{\Longrightarrow}} B$ such that $\phi \circ e = e$ and $\phi \circ m = m \circ (\phi \odot \phi)$.

- A **bimodule** from a monoid (R,A) to a monoid (S,B) is a horizontal 1-cell $M\colon R \nrightarrow S$ together with action maps $a_\ell\colon A \odot M \to M$ and $a_r\colon M \odot B \to M$ obeying the obvious compatibility axioms.

- Let $(f,\phi)\colon (R,A)\to(S,B)$ and $(g,\psi)\colon (T,C)\to(U,D)$ be monoid homomorphisms and $M\colon (R,A)\nrightarrow(T,C)$, $N\colon (S,B)\nrightarrow(U,D)$ be bimodules. A **(ϕ,ψ)-equivariant map** is a 2-cell $\alpha\colon M \xRightarrow[f]{g} N$ such that $a_\ell(\phi\odot\alpha)=\alpha a_\ell$ and $a_r(\alpha\odot\psi)=\alpha a_r$.

- Let $M\colon R\nrightarrow S$ be an (A,B)-bimodule and $N\colon S\nrightarrow T$ be a (B,C)-bimodule. Their **tensor product** is the following coequalizer in $\mathcal{D}(A,C)$, if it exists:

$$M\odot B\odot N \rightrightarrows M\odot N \to M\odot_B N.$$

Of course, if $\mathbb{D}$ is a monoidal category, these notions reduce to the usual ones.

11.2. Example. If $\mathscr{C}$ has pullbacks, then a monoid in $\mathbb{S}\mathbf{pan}(\mathscr{C})$ is an internal category in $\mathscr{C}$, and a monoid homomorphism is an internal functor. A bimodule in $\mathbb{S}\mathbf{pan}(\mathscr{C})$ is an 'internal distributor'.

11.3. Example. A monoid in $\mathbb{M}\mathbf{od}$ consists of a ring R together with an R-algebra A, and a monoid homomorphism $(R,A)\to(S,B)$ consists of a ring homomorphism $f\colon R\to S$ and an f-equivariant algebra map $A\to B$. A bimodule in $\mathbb{M}\mathbf{od}$ is just a bimodule for the algebras.

In order to define a framed bicategory of monoids and bimodules in $\mathbb{D}$, we need to know that coequalizers exist and are well-behaved.

11.4. Definition. A framed bicategory $\mathbb{D}$ has **local coequalizers** if each category $\mathcal{D}(A,B)$ has coequalizers and $\odot$ preserves coequalizers in each variable. We introduce the following notations.

- $\mathcal{F}r\mathcal{B}i_\ell^q$ denotes the *full* sub-2-category of $\mathcal{F}r\mathcal{B}i_\ell$ determined by the framed bicategories with local coequalizers.

- $\mathcal{F}r\mathcal{B}i_{\ell,n}^q$ denotes the locally full sub-2-category of $\mathcal{F}r\mathcal{B}i_\ell$ determined by the framed bicategories with local coequalizers and the *normal* lax framed functors.

- $\mathcal{F}r\mathcal{B}i^q$ denotes the locally full sub-2-category of $\mathcal{F}r\mathcal{B}i$ determined by the framed bicategories with local coequalizers and the strong framed functors which preserve local coequalizers.

Note that if $\mathbb{D}$ is closed, as defined in §5, then $\odot$ preserves all colimits since it is a left adjoint. The following omnibus theorem combines all our results about monoids and modules in framed bicategories.

11.5. Theorem. *Let $\mathbb{D}$ be a framed bicategory with local coequalizers. Then there is a framed bicategory $\mathbb{M}\mathbf{od}(\mathbb{D})$ of monoids, monoid homomorphisms, bimodules, and equivariant maps in $\mathbb{D}$. Moreover:*

- $\mathbb{M}\mathbf{od}(\mathbb{D})$ *also has local coequalizers.*

- *If $\mathbb{D}$ is closed and each category $\mathcal{D}(A,B)$ has equalizers, then* $\mathbb{M}\mathbf{od}(\mathbb{D})$ *is closed.*
- *If $\mathbb{D}$ is monoidal and its external product $\otimes$ preserves local coequalizers, then* $\mathbb{M}\mathbf{od}(\mathbb{D})$ *has both of these properties. If $\mathbb{D}$ is symmetric, so is* $\mathbb{M}\mathbf{od}(\mathbb{D})$*. If $\mathbb{D}$ is externally closed and each category $\mathcal{D}(A,B)$ has equalizers, then* $\mathbb{M}\mathbf{od}(\mathbb{D})$ *is externally closed.*
- *If $\mathbb{D}$ is equipped with an involution, so is* $\mathbb{M}\mathbf{od}(\mathbb{D})$*. If the involution of $\mathbb{D}$ is monoidal or symmetric monoidal, so is that of* $\mathbb{M}\mathbf{od}(\mathbb{D})$*.*
- $\mathbb{M}\mathbf{od}$ *defines 2-functors $\mathcal{F}r\mathcal{B}i^q_\ell \to \mathcal{F}r\mathcal{B}i^q_{\ell,n}$ and $\mathcal{F}r\mathcal{B}i^q \to \mathcal{F}r\mathcal{B}i^q$, and similarly for the monoidal versions.*

Even if F is a strong framed functor, $\mathbb{M}\mathbf{od}(F)$ is only lax unless F preserves local coequalizers. If F is oplax, we cannot even define $\mathbb{M}\mathbf{od}(F)$. Of course, there is a dual construction $\mathbb{C}\mathbf{omod}$, but it arises much less frequently in practice.

11.6. EXAMPLE. If $\mathscr{C}$ is a monoidal category with coequalizers preserved by $\otimes$, then $\mathbb{M}\mathbf{od}(\mathscr{C})$ has local coequalizers, so we have a framed bicategory $\mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{od}(\mathscr{C}))$ of algebras and bimodules in $\mathscr{C}$.

11.7. EXAMPLE. If $\mathscr{C}$ is a category with pullbacks and coequalizers preserved by pullback, then $\mathbb{S}\mathbf{pan}(\mathscr{C})$ has local coequalizers, so we have a framed bicategory $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathscr{C}))$ of internal categories and distributors in $\mathscr{C}$.

11.8. EXAMPLE. When $\mathscr{V}$ is a cocomplete closed monoidal category, we can also construct the framed bicategory $\mathbb{D}\mathbf{ist}(\mathscr{V})$ of enriched categories and distributors in this way. We first define the framed bicategory $\mathbb{M}\mathbf{at}(\mathscr{V})$ as follows: its vertical category is **Set**, and the category $\mathcal{M}at(\mathscr{V})(A,B)$ is the category of $A\times B$ matrices $(M_{ab})_{a\in A, b\in B}$ of objects of $\mathscr{V}$. Composition is by 'matrix multiplication'. It is then easy to check that $\mathbb{M}\mathbf{at}(\mathscr{V})$ has local coequalizers and that $\mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{at}(\mathscr{V})) \cong \mathbb{D}\mathbf{ist}(\mathscr{V})$. The monoidal category $\mathcal{M}at(\mathscr{V})(A,A)$ is also called the category of *$\mathscr{V}$-graphs* with object set A.

11.9. EXAMPLE. Unlike these examples, $\mathbb{E}\mathbf{x}$ does not have local coequalizers. We will see a replacement for '$\mathbb{M}\mathbf{od}(\mathbb{E}\mathbf{x})$' in §15.

The rest of this section is devoted to the proof of Theorem 11.5, breaking it up into a series of propositions for clarity. Although long, the proof is routine and follow-your-nose, so it can easily be skipped.

11.10. PROPOSITION. *If $\mathbb{D}$ is a framed bicategory with local coequalizers, then there is a framed bicategory* $\mathbb{M}\mathbf{od}(\mathbb{D})$ *of monoids, monoid homomorphisms, bimodules, and equivariant maps in $\mathbb{D}$, and it also has local coequalizers.*

Proof. The proof that $\mathbb{M}\mathbf{od}(\mathbb{D})$ is a double category is similar to the case of a monoidal category. For example, we need the fact that $\odot$ preserves coequalizers to show that $M \odot N$ is a bimodule and that the tensor product is associative. To define the horizontal composite of bimodule maps $\alpha\colon M \overset{\psi}{\underset{\phi}{\Longrightarrow}} N$ and $\beta\colon P \overset{\chi}{\underset{\psi}{\Longrightarrow}} Q$ (where $\phi\colon A \overset{f}{\underset{f}{\Longrightarrow}} D$, $\psi\colon B \overset{g}{\underset{g}{\Longrightarrow}} E$, and $\chi\colon C \overset{h}{\underset{h}{\Longrightarrow}} F$ are monoid homomorphisms), we start with the composite

$$\begin{tikzcd}
R \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & S \arrow[r, "P"] \arrow[d, "g"] \arrow[dr, phantom, "\beta"] & T \arrow[d, "h"] \\
U \arrow[r, "N"] \arrow[d, equal] & V \arrow[r, "Q"] \arrow[d, phantom, "\mathrm{coeq}"] & W \arrow[d, equal] \\
U \arrow[rr, "N \odot_E Q"'] & & W
\end{tikzcd} \tag{28}$$

which we would like to factor through the coequalizer defining $M \odot_B P$. However, that coequalizer lives in $\mathcal{D}(R,T)$, whereas (28) is not globular. But since $\mathbb{D}$ is a framed bicategory, we can factor (28) through a cartesian arrow to get a map

$$M \odot P \to f^*(N \odot_E Q)g^*$$

in $\mathcal{D}(R,T)$, and then apply the universal property of the coequalizer to get a map $M \odot_B P \to f^*(N \odot_E Q)g^*$, and hence $M \odot_B P \overset{g}{\underset{f}{\Longrightarrow}} (N \odot_E Q)$. This defines a (ϕ,χ)-equivariant map which we call the horizontal composite $\alpha \odot_\psi \beta$. The axioms for a double category follow directly.

We now show that $\mathbb{M}\mathbf{od}(\mathbb{D})$ is a framed bicategory. By Theorem 4.1, it suffices to show that it has restrictions. Thus, suppose that $A\colon R \nrightarrow R$, $B\colon S \nrightarrow S$, $C\colon T \nrightarrow T$, and $D\colon U \nrightarrow U$ are monoids in $\mathbb{D}$, $M\colon S \nrightarrow U$ is a (B,D)-bimodule, and $\phi\colon A \overset{f}{\underset{f}{\Longrightarrow}} B$ and $\psi\colon C \overset{g}{\underset{g}{\Longrightarrow}} D$ are monoid homomorphisms. We then have the restriction $f^*Mg^*\colon R \nrightarrow T$ in $\mathbb{D}$. By composing the cartesian arrow in $\mathbb{D}$ with ϕ or ψ and using the actions of B and D on M, then factoring through the cartesian arrow, we obtain actions of A and C on f^*Mg^*. For example, the action of A on f^*Mg^* is determined by the equality

$$\begin{tikzcd}
R \arrow[r, "A"] \arrow[d, "f"'] \arrow[dr, phantom, "\phi"] & R \arrow[r, "f^*Mg^*"] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{cart}"] & T \arrow[d, "g"] \\
S \arrow[r, "B"] \arrow[d, equal] & S \arrow[r, "M"'] \arrow[d, phantom, "\mathrm{act}"] & U \arrow[d, equal] \\
S \arrow[rr, "M"'] & & U
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
R \arrow[r, "A"] \arrow[d, equal] & R \arrow[r, "f^*Mg^*"] \arrow[d, phantom, "\mathrm{act}"] & T \arrow[d, equal] \\
R \arrow[rr, "f^*Mg^*"] \arrow[d, "f"'] & {} \arrow[d, phantom, "\mathrm{cart}"] & T \arrow[d, "g"] \\
S \arrow[rr, "M"'] & & U
\end{tikzcd}$$

It is straightforward to check that with this structure, the cartesian arrow $f^*Mg^* \overset{g}{\underset{f}{\Longrightarrow}} M$ in $\mathbb{D}$ defines a cartesian arrow $\phi^*M\psi^* \overset{\psi}{\underset{\phi}{\Longrightarrow}} M$ in $\mathbb{M}\mathbf{od}(\mathbb{D})$. ∎

11.11. PROPOSITION. $\mathbb{M}\mathbf{od}$ *defines a 2-functor* $\mathcal{F}r\mathcal{B}i^q_\ell \to \mathcal{F}r\mathcal{B}i^q_{\ell,n}$*, which restricts to a 2-functor* $\mathcal{F}r\mathcal{B}i^q \to \mathcal{F}r\mathcal{B}i^q$*.*

PROOF. Let $\mathbb{D}, \mathbb{E} \in \mathcal{F}r\mathcal{B}i^q_\ell$ and let $F\colon \mathbb{D} \to \mathbb{E}$ be a lax framed functor. Then F preserves monoids, monoid homomorphisms, bimodules, and equivariant maps, for the same reasons that lax monoidal functors do. We define the unit constraint for $\mathbb{M}\mathbf{od}(F)$ to be the identity on FA, and the composition constraint to be the result of factoring the composite

$$FM \odot FN \xrightarrow{F_\odot} F(M \odot N) \longrightarrow F(M \odot_B N) \tag{29}$$

through the coequalizer

$$FM \odot FN \to FM \odot_{FB} FN \tag{30}$$

It is straightforward to check that this makes $\mathbb{M}\mathbf{od}(F)$ into a normal lax double functor. Similarly, the components of a framed transformation $F \to G$ define a framed transformation $\mathbb{M}\mathbf{od}(F) \to \mathbb{M}\mathbf{od}(G)$.

Finally, if F is strong and preserves local coequalizers, then (29) is a coequalizer of the same maps that (30) is. Hence the induced composition constraint is an isomorphism, so $\mathbb{M}\mathbf{od}(F)$ is strong. It is easy to see that $\mathbb{M}\mathbf{od}(F)$ also preserves local coequalizers, so that it lies in $\mathcal{F}r\mathcal{B}i^q$. ∎

11.12. PROPOSITION. *If* $\mathbb{D}$ *is a monoidal framed bicategory with local coequalizers preserved by* $\otimes$*, then so is* $\mathbb{M}\mathbf{od}(\mathbb{D})$*. If* $\mathbb{D}$ *is symmetric, so is* $\mathbb{M}\mathbf{od}(\mathbb{D})$*.*

PROOF. It is easy to check that the 2-functor $\mathbb{M}\mathbf{od}\colon \mathcal{F}r\mathcal{B}i^q \to \mathcal{F}r\mathcal{B}i^q$ preserves products, so it must preserve pseudo-monoids and symmetric pseudo-monoids. ∎

11.13. PROPOSITION. *Suppose that* $\mathbb{D}$ *has local coequalizers and each category* $\mathcal{D}(A, B)$ *has equalizers. If* $\mathbb{D}$ *is closed, then* $\mathbb{M}\mathbf{od}(\mathbb{D})$ *is closed. If* $\mathbb{D}$ *is monoidal and externally closed with local coequalizers preserved by* $\otimes$*, then* $\mathbb{M}\mathbf{od}(\mathbb{D})$ *is externally closed.*

PROOF. Just as for monoidal categories. ∎

11.14. PROPOSITION. *If* $\mathbb{D}$ *has local coequalizers and is equipped with an involution, so is* $\mathbb{M}\mathbf{od}(\mathbb{D})$*. If* $\mathbb{D}$*,* $\mathbb{M}\mathbf{od}(\mathbb{D})$*, and the involution on* $\mathbb{D}$ *are monoidal or symmetric monoidal, so is the involution on* $\mathbb{M}\mathbf{od}(\mathbb{D})$*.*

PROOF. It is easy to see that $\mathbb{M}\mathbf{od}(\mathbb{D}^{h\cdot op}) \simeq \mathbb{M}\mathbf{od}(\mathbb{D})^{h\cdot op}$, so we can simply apply the 2-functor $\mathbb{M}\mathbf{od}$ to $(-)^{op}$ and ξ. ∎

Note, however, that since the vertical arrow components of ξ in $\mathbb{M}\mathbf{od}(\mathbb{D})$ are defined from the *2-cell* components of ξ in $\mathbb{D}$, the involution of $\mathbb{M}\mathbf{od}(\mathbb{D})$ may not be vertically strict even if the involution of $\mathbb{D}$ is so.

12. Monoidal fibrations

The generalized $\mathbb{M}\mathbf{od}$ construction from §11 defines a horizontal composition from an external product via a coequalizer. In §14 we will explain how in a cartesian situation, horizontal compositions can be constructed using a pullback or equalizer-type construction instead. The basic input for this construction is a structure called a 'monoidal fibration', which includes base change operations and an external product, but *a priori* no horizontal composition.

12.1. Definition. A **monoidal fibration** is a functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ such that

(i) $\mathscr{A}$ and $\mathscr{B}$ are monoidal categories;

(ii) Φ is a fibration and a strict monoidal functor; and

(iii) The tensor product $\otimes$ of $\mathscr{A}$ preserves cartesian arrows.

If Φ is also an opfibration and $\otimes$ preserves opcartesian arrows, we say that Φ is a **monoidal bifibration**. We say that Φ is **braided** (resp. **symmetric**) if $\mathscr{A}$, $\mathscr{B}$, and the functor Φ are braided (resp. symmetric).

We will also speak of 'monoidal $*$-fibrations' and 'monoidal $*$-bifibrations', but without implying any compatibility between the monoidal structure and the right adjoints f_*. This is because in most cases there is no such compatibility.

12.2. Example. Let $\mathscr{C}$ be a category with finite limits. Recall that if $\mathscr{C}^{\downarrow}$ denotes the category of arrows in $\mathscr{C}$, the codomain functor gives a bifibration $\mathtt{Arr}_{\mathscr{C}}\colon \mathscr{C}^{\downarrow} \to \mathscr{C}$ called the 'self-indexing' of $\mathscr{C}$. It is easy to see that $\mathtt{Arr}_{\mathscr{C}}$ is a monoidal bifibration when $\mathscr{C}$ and $\mathscr{C}^{\downarrow}$ are equipped with their cartesian products.

12.3. Example. If $\mathbb{D}$ is a monoidal framed bicategory, then $(L, R)\colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a monoidal bifibration. If $\mathbb{D}$ is braided or symmetric, so is (L, R).

12.4. Example. The fibration $\mathtt{Mod}\colon \mathbf{Mod} \to \mathbf{Ring}$ is a monoidal $*$-bifibration under the tensor product of rings and the 'external' tensor product of modules.

For most of our applications, such as Theorem 12.7 below and the construction of framed bicategories in §14, we will require the base category $\mathscr{B}$ to be cartesian or cocartesian monoidal. However, we see from Examples 12.3 and 12.4 that this is not always the case, and the general notion of monoidal fibration is interesting in its own right.

Recall from Proposition 3.6 that the 2-category of fibrations $\Phi\colon \mathscr{A} \to \mathscr{B}$ is equivalent to the 2-category of pseudofunctors $\mathscr{B}^{op} \to \mathcal{C}at$. We intend to prove an analogous result for monoidal fibrations over cartesian base categories, but first we must define the 2-category of monoidal fibrations.

12.5. Definition. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ and $\Phi'\colon \mathscr{A}' \to \mathscr{B}'$ be monoidal fibrations.

- An **oplax monoidal morphism of fibrations** is a commuting square

$$\begin{tikzcd} \mathscr{A}' \arrow[r, "F_1"] \arrow[d, "\Phi'"'] & \mathscr{A} \arrow[d, "\Phi"] \\ \mathscr{B}' \arrow[r, "F_0"'] & \mathscr{B} \end{tikzcd} \tag{31}$$

  (that is, an oplax morphism of fibrations) together with the data of oplax monoidal functors on F_0 and F_1 such that the identity $\Phi F_1 = F_0\Phi'$ is a monoidal natural transformation.

- An oplax morphism is **strong** if F_0 and F_1 are strong monoidal functors and F_1 preserves cartesian arrows.

- A **lax** morphism is a square (31) such that F_0 and F_1 are lax monoidal functors, F_1 preserves cartesian arrows, and the equality $\Phi F_1 = F_0\Phi'$ is a monoidal transformation.

Any sort of morphism is **over** $\mathscr{B}$ if F_0 is an identity $\mathscr{B}' = \mathscr{B}$. If Φ and Φ' are braided (resp. symmetric), then any sort of monoidal morphism is braided (resp. symmetric) if the functors F_0 and F_1 and the equality $\Phi F_1 = F_0\Phi'$ are braided (resp. symmetric).

If Φ and Φ' are monoidal bifibrations, then a **lax monoidal morphism of bifibrations** is just a lax monoidal morphism of fibrations, while an **oplax** (resp. **strong**) **monoidal morphism of bifibrations** is an oplax (resp. strong) monoidal morphism of fibrations which also preserves opcartesian arrows.

A **monoidal transformation of fibrations**, or of bifibrations, is a transformation of fibrations whose components are monoidal natural transformations. If the two morphisms are over $\mathscr{B}$, then the transformation is **over** $\mathscr{B}$ if its downstairs component is an identity.

12.6. Notations. Let $\mathcal{MF}_{op\ell}$ (resp. $\mathcal{MF}$, $\mathcal{MF}_\ell$) be the 2-category of monoidal fibrations, oplax (resp. strong, lax) monoidal morphisms of fibrations, and monoidal transformations of fibrations. We write $\mathcal{BMF}$ and $\mathcal{SMF}$ for the braided and symmetric versions. Let $\mathcal{MF}_{\mathscr{B}}$ denote the sub-2-category of $\mathcal{MF}$ consisting of fibrations, morphisms, and transformations over $\mathscr{B}$, and so on. Finally, we write $\mathcal{M}on\mathcal{C}at$ for the 2-category of monoidal categories, strong monoidal functors, and monoidal natural transformations, and similarly $\mathcal{B}r\mathcal{M}on\mathcal{C}at$ and $\mathcal{S}ym\mathcal{M}on\mathcal{C}at$.

12.7. Theorem. *If $\mathscr{B}$ is cartesian monoidal, the equivalence of Proposition 3.6 lifts to equivalences of 2-categories*

$$\mathcal{MF}_{\mathscr{B}} \simeq [\mathscr{B}^{op}, \mathcal{M}on\mathcal{C}at]$$
$$\mathcal{BMF}_{\mathscr{B}} \simeq [\mathscr{B}^{op}, \mathcal{B}r\mathcal{M}on\mathcal{C}at]$$
$$\mathcal{SMF}_{\mathscr{B}} \simeq [\mathscr{B}^{op}, \mathcal{S}ym\mathcal{M}on\mathcal{C}at].$$

This means that, in particular, in a monoidal fibration with cartesian base, each fiber is monoidal and each transition functor f^* is strong monoidal. We call the monoidal structure on $\mathscr{A}$ the **external** monoidal structure, and the monoidal structures on fibers the **internal** monoidal structures.

In many cases, the internal monoidal structures on the fibers are more familiar and predate the external monoidal structure. For example, in $\mathtt{Arr}_{\mathscr{C}}$, the fiber over B is the slice category $\mathscr{C}/B$, and the internal monoidal structure is the fiber product over B.

It is crucial that $\mathscr{B}$ be *cartesian* monoidal for Theorem 12.7 to be true. For example, the fiber of $\mathtt{Mod}$ over a noncommutative ring R is the category $\mathbf{Mod}_R$ of R-modules, which does not in general have an internal tensor product. But if we restrict to the monoidal fibration $\mathtt{CMod}$ of modules over *commutative* rings, the tensor product in **CRing** becomes the coproduct, so we can apply the dual result, obtaining the familiar tensor product on $\mathbf{Mod}_R$ in the commutative case.

12.8. Notation. In a cartesian monoidal category $\mathscr{B}$, we write π_B for any map which projects B out of a product; thus we have $\pi_B \colon B \to 1$, but also $\pi_B \colon A \times B \times C \to A \times C$. We also write $\Delta_B \colon B \to B \times B$ for the diagonal, and other maps constructed from it such as $A \times B \times C \to A \times B \times B \times C$.

Proof of Theorem 12.7. Let $\Phi \colon \mathscr{A} \to \mathscr{B}$ be a monoidal fibration with a chosen cleavage, and let $B \in \mathscr{B}$. We define a monoidal structure on the fiber $\mathscr{A}_B$ as follows. The unit object is $I_B = \pi_B^* I$, and the product is given by

$$M \boxtimes N = \Delta_B^*(M \otimes N) \tag{32}$$

where $M, N \in \mathscr{A}_B$ and $\otimes$ is the monoidal structure of $\mathscr{A}$. To obtain the associativity isomorphism, we tensor the cartesian arrow

$$M \boxtimes N \longrightarrow M \otimes N$$

(which lives over Δ_B) with Q to get an arrow

$$(M \boxtimes N) \otimes Q \longrightarrow (M \otimes N) \otimes Q$$

which is cartesian since $\otimes$ preserves cartesian arrows. We then compose with another cartesian arrow over Δ_B to obtain a composite cartesian arrow

$$(M \boxtimes N) \boxtimes Q \longrightarrow (M \otimes N) \otimes Q.$$

We do the same on the other side to get a cartesian arrow

$$M \boxtimes (N \boxtimes Q) \longrightarrow M \otimes (N \otimes Q)$$

and the unique factorization of

$$\mathfrak{a} \colon (M \otimes N) \otimes Q \cong M \otimes (N \otimes Q)$$

through these cartesian arrows gives an associativity isomorphism

$$(M \boxtimes N) \boxtimes Q \cong M \boxtimes (N \boxtimes Q).$$

for $\mathscr{A}_B$. The pentagon axiom follows from unique factorization through cartesian arrows and the pentagon axiom for $\mathscr{A}$. The unit constraints and axioms are analogous, using the fact that $\pi_B \Delta_B = 1_B$, as is the braiding when Φ is braided or symmetric.

Now consider a map $f \colon A \to B$; we show that f^* is strong monoidal. We have the composite cartesian arrows

$$f^*M \boxtimes f^*N \longrightarrow f^*M \otimes f^*N \longrightarrow M \otimes N$$

and

$$f^*(M \boxtimes N) \longrightarrow M \boxtimes N \longrightarrow M \otimes N,$$

both lying over $\Delta_B f = (f \times f)\Delta_A$; hence we obtain a canonical isomorphism

$$f^*M \boxtimes f^*N \cong f^*(M \boxtimes N).$$

The unit constraint is similar and, as before, the coherence of these constraints follows from the uniqueness of factorization through cartesian arrows, as does the fact that the isomorphisms $(fg)^* \cong f^*g^*$ and $(1_B)^* \cong \mathrm{Id}$ are monoidal. Therefore, we have constructed a pseudofunctor $\mathscr{B}^{op} \to \mathcal{M}on\mathcal{C}at$ from a monoidal fibration. It is straightforward to extend this construction to give 2-functors

$$\begin{aligned} \mathcal{MF}_{\mathscr{B}} &\longrightarrow [\mathscr{B}^{op}, \mathcal{M}on\mathcal{C}at] \\ \mathcal{BMF}_{\mathscr{B}} &\longrightarrow [\mathscr{B}^{op}, \mathcal{B}r\mathcal{M}on\mathcal{C}at] \\ \mathcal{SMF}_{\mathscr{B}} &\longrightarrow [\mathscr{B}^{op}, \mathcal{S}ym\mathcal{M}on\mathcal{C}at]. \end{aligned}$$

Uniqueness of factorization again gives the coherence to show that the resulting pseudonatural transformations are pointwise monoidal.

Conversely, given a pseudofunctor $\mathscr{B}^{op} \to \mathcal{M}on\mathcal{C}at$, we define a fibration over $\mathscr{B}$ in the usual way, and define an external product as follows: given M, N over A, B respectively, let

$$M \otimes N = \pi_B^* M \boxtimes \pi_A^* N. \tag{33}$$

The external unit is I_1, the internal unit in the fiber over 1. For an associativity isomorphism we use

$$\begin{aligned} (M \otimes N) \otimes Q &= \pi_C^*(\pi_B^* M \boxtimes \pi_A^* N) \boxtimes \pi_{AB}^* Q \\ &\cong (\pi_{BC}^* M \boxtimes \pi_{AC}^* N) \boxtimes \pi_{AB}^* Q \\ &\cong \pi_{BC}^* M \boxtimes (\pi_{AC}^* N \boxtimes \pi_{AB}^* Q) \\ &\cong \pi_{BC}^* M \boxtimes \pi_A^*(\pi_C^* N \boxtimes \pi_B^* Q) \\ &= M \otimes (N \otimes Q) \end{aligned}$$

using the monoidal constraints for the strong monoidal functors π^*, the composition constraints for the pseudofunctor, and the associativity for the internal products. It is straightforward, if tedious, to check that this isomorphism satisfies the pentagon axiom. Similarly, we have a unit constraint

$$\begin{aligned} M \otimes I_1 &= \pi_1^* M \boxtimes \pi_A^* I_1 \\ &\cong M \boxtimes I_A \\ &\cong M \end{aligned}$$

which can be checked to be coherent; thus $\mathscr{A}$ is monoidal, and Φ is strict monoidal by definition. It is obvious how to define a braiding in the braided or symmetric case making $\mathscr{A}$ and Φ braided or symmetric. Finally, using the composition constraints and monoidal constraints, we have:

$$\begin{aligned} f^*M \otimes g^*N &= \pi^* f^* M \boxtimes \pi^* g^* N \\ &\cong (f \times g)^* \pi^* M \boxtimes (f \times g)^* \pi^* N \\ &\cong (f \times g)^* \big(\pi^* M \boxtimes \pi^* N \big) \\ &= (f \times g)^* (M \otimes N), \end{aligned}$$

which we can then use to verify that $\otimes$ preserves cartesian arrows. Thus we have constructed a monoidal fibration of the desired type. It is straightforward to extend this to a 2-functor and verify that these constructions are inverse equivalences.

12.9. Remark. Under the above equivalence, pseudofunctors which land in *cartesian* monoidal categories correspond to fibrations where the total category $\mathscr{A}$ is cartesian monoidal.

We end this section by introducing a few new examples of monoidal fibrations.

12.10. Example. Let $\mathscr{C}$ be a category with finite limits and colimits, and assume that pullbacks in $\mathscr{C}$ preserve finite colimits. (For example, $\mathscr{C}$ could be locally cartesian closed.) Let $\mathrm{Retr}(\mathscr{C})$ be the category of *retractions* in $\mathscr{C}$. That is, an object of $\mathrm{Retr}(\mathscr{C})$ is a pair of maps $A \xrightarrow{s} X \xrightarrow{r} A$ such that $rs = 1_A$. This is also known as an object X 'parametrized' over A, in which case s is called the 'section'. We define $\mathtt{Retr}_{\mathscr{C}}\colon \mathrm{Retr}(\mathscr{C}) \to \mathscr{C}$ to take the above retraction to A. It is easy to check that pullback and pushout make Φ into a bifibration, which is a $*$-bifibration if $\mathscr{C}$ is locally cartesian closed.

The fiber over $B \in \mathscr{C}$ is the category $\mathscr{C}_B$ of objects parametrized over B. It has finite products, given by pullback over B, but usually the relevant monoidal structure is not the cartesian product but the *fiberwise smash product*, defined as the pushout

$$\begin{tikzcd} X \sqcup_B Y \arrow[r] \arrow[d] & X \times_B Y \arrow[d] \\ B \arrow[r] & X \wedge_B Y. \end{tikzcd}$$

The unit is $B \sqcup B \to B$ with section given by one of the coprojections. Under the assumption that pullbacks preserve finite colimits, this defines a symmetric monoidal structure on $\mathscr{C}_B$, all the functors f^* are strong symmetric monoidal, and the coherence isomorphisms are also monoidal. Thus by Theorem 12.7, $\mathtt{Retr}_{\mathscr{C}}$ is a symmetric monoidal fibration, and it is easy to check that it is actually a monoidal bifibration. The external monoidal structure on $\mathrm{Retr}(\mathscr{C})$ is called the *external smash product* $\overline{\wedge}$.

12.11. Example. Suppose that $\mathscr{C}$ has finite limits and colimits, and *not* all pullbacks preserve finite colimits, but there is some full subcategory $\mathscr{B}$ of $\mathscr{C}$ such that pullbacks along morphisms in $\mathscr{B}$ do preserve finite colimits. Then we can repeat the construction of Example 12.10 using parametrized objects whose base objects are restricted to lie in $\mathscr{B}$. This is what is done in [MS06, §2.5], with $\mathscr{C} = \mathscr{K}$ the category of k-spaces and $\mathscr{B} = \mathscr{U}$ the category of compactly generated spaces. By a slight abuse of notation, we call the resulting monoidal $*$-bifibration $\mathtt{Retr}_{\mathbf{Top}}$, since we have only been prevented from considering all retractions in **Top** by point-set technicalities. The objects of $\mathrm{Retr}(\mathbf{Top})$ are called *ex-spaces*.

12.12. Example. For each space $B \in \mathscr{U}$, a category $\mathscr{S}_B$ of orthogonal spectra parametrized over B is defined in [MS06, Ch. 11]. A map $f\colon A \to B$ of spaces gives rise to a string of adjoints $f_! \dashv f^* \dashv f_*$ which are pseudofunctorial in f. Each category $\mathscr{S}_B$ is closed symmetric monoidal under an internal smash product $\wedge_B$, each functor f^* is closed symmetric monoidal, and so are the composition constraints. Thus, by Theorem 12.7, we obtain a symmetric monoidal fibration which we denote $\mathtt{Sp}$. The external smash product $\overline{\wedge}$ is defined in [MS06, 11.4.10] just as we have done in (33).

To show that $\mathtt{Sp}$ is in fact a monoidal $*$-bifibration, one can check directly that $\overline{\wedge}$ preserves opcartesian arrows. However, this will also follow from Proposition 13.17 below.

12.13. Example. Let $\mathscr{B} = \mathscr{U}$ as in Example 12.12, but instead of $\mathscr{S}_B$ we use its homotopy category $\mathrm{Ho}\,\mathscr{S}_B$. It is proven in [MS06, 12.6.7] that $f_! \dashv f^*$ is a Quillen adjunction, for a suitable choice of model structures on $\mathscr{S}_B$, hence it descends to an adjunction on homotopy categories which is still pseudofunctorial; thus we obtain another functor $\mathrm{Ho}(\mathtt{Sp})\colon \mathscr{A} \to \mathscr{B}$ which is a bifibration.

The external smash product $\overline{\wedge}$ is proven to be a Quillen left adjoint in [MS06, 12.6.6]; thus it descends to homotopy categories to make $\mathscr{A}$ symmetric monoidal. Since [MS06, 13.7.2] shows that $\overline{\wedge}$ preserves cartesian arrows, $\mathrm{Ho}(\mathtt{Sp})$ is a symmetric monoidal fibration, and the same methods as in Example 12.12 show that it is a monoidal bifibration. The derived functors f^* also have right adjoints, although these are constructed in [MS06, 13.1.18] using Brown representability rather than by deriving the point-set level right adjoints; thus $\mathrm{Ho}(\mathtt{Sp})$ is a monoidal $*$-bifibration.

13. Closed monoidal fibrations

We now consider two different notions of when a monoidal fibration is ‘closed’. To fix terminology and notation, we say an ordinary monoidal category $\mathscr{C}$ with product $\boxtimes$ is *closed* if the functors $(M \boxtimes -)$ and $(- \boxtimes N)$ have right adjoints $(- \blacktriangleleft M)$ and $(N \blacktriangleright -)$, respectively, for all M, N. Of course, if $\mathscr{C}$ is symmetric, then $P \blacktriangleleft M \cong M \blacktriangleright P$. If $\mathscr{C}$ and $\mathscr{D}$ are closed monoidal categories and $f^*\colon \mathscr{C} \to \mathscr{D}$ is a strong monoidal functor, then there are canonical natural transformations

$$f^*(N \blacktriangleright P) \longrightarrow f^*N \blacktriangleright f^*P \tag{34}$$
$$f^*(P \blacktriangleleft N) \longrightarrow f^*P \blacktriangleleft f^*N. \tag{35}$$

When these transformations are isomorphisms, we say that f^* is *closed monoidal.* Of course, in the symmetric case, (34) is an isomorphism if and only if (35) is.

13.1. Definition. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a monoidal fibration where $\mathscr{B}$ is cartesian monoidal (so that each fiber is a monoidal category). We say Φ is **internally closed** if each fiber $\mathscr{A}_B$ is closed monoidal and each functor f^* is closed monoidal.

However, in any monoidal fibration, we can also ask whether the external product

$$\otimes\colon \mathscr{A}_A \times \mathscr{A}_B \to \mathscr{A}_{A\otimes B}$$

has adjoints $\overline{\lhd}$, $\overline{\rhd}$, with defining isomorphisms

$$\mathscr{A}_{A\otimes B}(M \otimes N, P) \cong \mathscr{A}_A(M, N \mathbin{\overline{\rhd}} P) \cong \mathscr{A}_B(N, P \mathbin{\overline{\lhd}} M).$$

If so, then for any $f\colon C \to A$ and $g\colon D \to B$ there are canonical transformations

$$f^*(N \mathbin{\overline{\rhd}} P) \longrightarrow N \mathbin{\overline{\rhd}} (f\otimes 1)^*P \tag{36}$$
$$g^*(P \mathbin{\overline{\lhd}} M) \longrightarrow (1\otimes g)^*P \mathbin{\overline{\lhd}} M \tag{37}$$

defined analogously to (34) and (35). For example, (36) is the adjunct of the composite

$$f^*(N \mathbin{\overline{\rhd}} P) \otimes N \xrightarrow{\cong} (f\otimes 1)^*\big((N \mathbin{\overline{\rhd}} P) \otimes N\big) \longrightarrow (f\otimes 1)^*P.$$

13.2. Definition. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a monoidal fibration. We say that Φ is **externally closed** if the adjoints $\overline{\lhd}$, $\overline{\rhd}$ exist and the maps (36) and (37) are isomorphisms for all f, g.

13.3. Examples. If $\mathscr{C}$ is locally cartesian closed, then $\mathtt{Arr}_{\mathscr{C}}$ is internally and externally closed. If $\mathscr{C}$ also has finite colimits, then $\mathtt{Retr}_{\mathscr{C}}$ is internally and externally closed.

13.4. Example. The fibration $\mathtt{Sp}$ of parametrized orthogonal spectra over spaces is internally and externally closed; its internal homs are defined in [MS06, 11.2.5] and the base change functors are shown to be closed in [MS06, 11.4.1]. We postpone consideration of $\mathrm{Ho}(\mathtt{Sp})$ until later.

13.5. Example. The fibration $\mathtt{Mod}\colon \mathbf{Mod} \to \mathbf{Ring}$ is externally closed. If N is a B-module and P is an $A\otimes B$-module, the external-hom $N \mathbin{\overline{\rhd}} P$ is $\mathrm{Hom}_B(N, P)$, which retains the A-module structure from P. In this case, internal closure makes no sense because the fibers are not even monoidal.

13.6. EXAMPLE. The monoidal $*$-bifibration `CMod` of modules over *commutative* rings is also externally closed. In this case the fibers $\mathbf{Mod}_R$ are closed monoidal, but neither $f_!$ nor f^* is a closed monoidal functor.

13.7. EXAMPLE. If $\mathbb{D}$ is a monoidal framed bicategory, then the monoidal bifibration (L, R) is externally closed just when $\mathbb{D}$ is externally closed in the sense of §9. The fact that (36) and (37) are isomorphisms in this case will follow from Proposition 13.17, below.

13.8. REMARK. Contrary to what one might expect (see, for example, [MS06, §2.4]), external closedness does not imply that the monoidal category $\mathscr{A}$ is closed in its own right. For one thing, $N \overline{\rhd} P$ is only defined when $N \in \mathscr{A}_B$ and $P \in \mathscr{A}_{A\times B}$. But even when defined, $N \overline{\rhd} P$ is not an internal-hom for $\mathscr{A}$: if $M \in \mathscr{A}_C$, then the morphisms $M \to N \overline{\rhd} P$ in $\mathscr{A}$ are bijective not to all morphisms $M \otimes N \to P$, but only those lying over $f \times 1$ for some $f\colon C \to A$.

In the rest of this section, we will prove that under mild hypotheses, internal and external closedness are equivalent, and give useful dual versions of the maps (34), (35), (36), and (37). We begin by comparing the internal and external homs.

13.9. PROPOSITION. *Let Φ be either*

(i) a monoidal $*$-fibration in which $\mathscr{B}$ is cartesian monoidal, or

(ii) a monoidal bifibration in which $\mathscr{B}$ is cocartesian monoidal.

Then the right adjoints $\overline{\lhd}$, $\overline{\rhd}$ exist if and only if Φ has closed fibers (i.e. the right adjoints $\blacktriangleleft, \blacktriangleright$ exist).

PROOF. Suppose first that $\mathscr{B}$ is cartesian and each fiber is closed. Then for $N \in \mathscr{A}_B$ and $Q \in \mathscr{A}_{A\times B}$ we define

$$N \overline{\rhd} Q = (\pi_B)_*\big(\pi_A^* N \blacktriangleright Q\big). \tag{38}$$

and similarly for $\overline{\lhd}$. Conversely, if $\overline{\lhd}$, $\overline{\rhd}$ exist, then for $N, Q \in \mathscr{A}_A$ we define

$$N \blacktriangleright Q = N \overline{\rhd} ((\Delta_A)_* Q) \tag{39}$$

and similarly for $\blacktriangleleft$. It is easy to check, using the relationships between $\otimes$ and $\boxtimes$ established in Theorem 12.7, that these definitions suffice.

In the cocartesian case, these relationships become

$$M \boxtimes N \cong \nabla_!(M \otimes N)$$
$$M \otimes N \cong \eta_! M \boxtimes \eta_! N,$$

where $\nabla_A\colon A \sqcup A \to A$ denotes the 'fold' or codiagonal, and $\eta\colon \emptyset \to A$ is the unique map from the initial object. Therefore, the analogous definitions:

$$M \blacktriangleright N = M \overline{\rhd} (\nabla^* N)$$
$$M \overline{\rhd} N = \eta^*\big(\eta_! M \blacktriangleright N\big).$$

allow us to pass back and forth between internal and external closedness. ∎

This equivalence is valuable because sometimes one of the two types of right adjoints is much easier to construct than the other.

13.10. Example. The homotopy-level fibration Ho(Sp) has the adjoints $\overline{\lhd}$, $\overline{\rhd}$, since the adjunction between $\overline{\wedge}$ and $\overline{\rhd}$ in Sp is Quillen (see [MS06, 12.6.6]). This then implies, by Proposition 13.9, that the fibers of Ho(Sp) are all closed monoidal. This would be difficult to prove directly, since we have no homotopical control over the internal monoidal structures in Sp.

In order to prove a full equivalence of local and external closedness, we need to assume an extra condition on the commutativity of right and left adjoints. Suppose that $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a fibration and that the square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd} \tag{40}$$

commutes in $\mathscr{B}$. Thus we obtain a square

$$\begin{tikzcd} \mathscr{A}_A & \mathscr{A}_B \arrow[l, "h^*"'] \\ \mathscr{A}_C \arrow[u, "k^*"] \arrow[ur, phantom, "\cong"] & \mathscr{A}_D \arrow[l, "f^*"] \arrow[u, "g^*"'] \end{tikzcd} \tag{41}$$

which commutes up to canonical isomorphism. If Φ is a bifibration, there is a canonical natural transformation

$$k_! h^* \longrightarrow f^* g_!, \tag{42}$$

namely the 'mate' of the isomorphism (41). Explicitly, it is the composite

$$k_! h^* \xrightarrow{\eta} k_! h^* g^* g_! \cong k_! k^* f^* g_! \xrightarrow{\varepsilon} f^* g_!.$$

Similarly, if Φ is a $*$-fibration, there is a canonical transformation

$$g^* f_* \longrightarrow h_* k^*. \tag{43}$$

13.11. Definition. If Φ is a bifibration (resp. a $*$-fibration), we say that the square (40) satisfies the **Beck-Chevalley condition** if the natural transformation (42) (resp. (43)) is an isomorphism. We say that Φ is **strongly BC** if this condition is satisfied by every pullback square, and **weakly BC** if it is satisfied by every pullback square in which one of the legs (f or g, above) is a product projection. If instead all *pushout* squares satisfy the Beck-Chevalley condition, we say that Φ is **strongly co-BC**.

If Φ is a $*$-bifibration, then (42) and (43) are mates under the composite adjunctions $f^* g_! \dashv g^* f_*$ and $k_! h^* \dashv h_* k^*$, so that one is an isomorphism if and only if the other is. Thus, a $*$-bifibration is strongly or weakly BC as a bifibration if and only if it is so as a $*$-fibration.

13.12. Examples. The monoidal bifibrations $\mathtt{Arr}_{\mathscr{C}}$ and $\mathtt{Retr}_{\mathscr{C}}$ are always strongly BC, as is the monoidal $*$-bifibration $\mathtt{Sp}$ (see [MS06, 11.4.8]).

13.13. Example. The monoidal $*$-bifibration $\mathtt{CMod}$, whose base is cocartesian monoidal, is strongly co-BC.

13.14. Example. The homotopy-level monoidal $*$-bifibration $\mathrm{Ho}(\mathtt{Sp})$ is only weakly BC; it is proven in [MS06, 13.7.7] that the Beck-Chevalley condition is satisfied for pullback squares one of whose legs is a fibration in the topological sense (which includes product projections, of course). It does *not* satisfy the Beck-Chevalley condition for arbitrary pullback squares; a concrete counterexample is given in [MS06, 0.0.1]. One intuitive reason for this is that since $\mathtt{Sp}$ also incorporates 'homotopical' information about the base spaces, we should only expect the derived operations to be well-behaved on *homotopy* pullback squares. This is our main motivation for introducing the notion of 'weakly BC'.

Of course, the idea of commuting adjoints is older than the term 'Beck-Chevalley condition'. In the theory of fibered categories, what we call a 'strongly BC bifibration' is referred to as a 'fibration with indexed coproducts'.

We will eventually use Beck-Chevalley conditions in our construction of a framed bicategory from a monoidal fibration (Theorem 14.2), but we mention them in this section for the purposes of the following result.

13.15. Proposition. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a monoidal $*$-fibration in which $\mathscr{B}$ is cartesian monoidal. Then*

(i) if Φ is internally closed and weakly BC, then it is externally closed, and

(ii) if Φ is externally closed and strongly BC, then it is internally closed.

In particular, a strongly BC monoidal $*$-fibration is internally closed if and only if it is externally closed.

Sketch of Proof. Under the equivalences (38) and (39), each of the maps (34) and (36) is equal to the composite of the other with a Beck-Chevalley transformation, and similarly for (35) and (37). It turns out that both of these transformations come from pullback squares; thus since π appears in (38) but Δ appears in (39), the weak condition is good enough in one case but not the other. ■

Now, if f^* is strong monoidal and has a left adjoint $f_!$, there is a canonical map

$$f_!(M \boxtimes f^*N) \longrightarrow f_!M \boxtimes N. \tag{44}$$

When the monoidal categories in question are closed, this is the mate of (34), so one is an isomorphism if and only if the other is. In particular, if Φ is a monoidal bifibration with cartesian monoidal base and closed fibers, then Φ is internally closed if and only if the maps (44), together with the analogous maps

$$f_!(f^*N \boxtimes M) \longrightarrow N \boxtimes f_!M, \tag{45}$$

are all isomorphisms. This dual condition is sometimes easier to check.

13.16. Example. Topological arguments involving excellent prespectra are used in [MS06, 13.7.6] to show that the derived maps (44) are isomorphisms, and therefore Ho($\mathtt{Sp}$) is internally closed. Since it is weakly BC, we can then conclude, by Proposition 13.15(i), that it is externally closed as well.

In a similar way, if Φ is any monoidal bifibration with right adjoints $\overline{\triangleleft}$, $\overline{\triangleright}$, then (36) has a mate

$$(f \times 1)_!(M \otimes N) \longrightarrow f_! M \otimes N \tag{46}$$

which is an isomorphism if and only if (36) is. But (46) is an isomorphism just when $- \otimes N$ preserves the opcartesian arrow $M \to f_! M$, so we have the following.

13.17. Proposition. *Let Φ be a monoidal fibration which is also an opfibration and such that the right adjoints $\overline{\triangleleft}$, $\overline{\triangleright}$ exist. Then $\otimes$ preserves opcartesian arrows (that is, Φ is a monoidal bifibration) if and only if Φ is externally closed.*

13.18. Example. As remarked earlier, this implies that a monoidal framed bicategory $\mathbb{D}$ is externally closed in the sense of §9 if and only if the monoidal bifibration (L, R) is externally closed in the sense of this section.

13.19. Example. In the converse direction, Proposition 13.17 can be used to show that Ho($\mathtt{Sp}$) and $\mathtt{Sp}$ are monoidal bifibrations, since we know that they are externally closed. This could also be shown directly.

13.20. Corollary. *Let Φ be a strongly BC monoidal $*$-bifibration over a cartesian base and having closed fibers. Then Φ is internally and externally closed.*

Proof. Since Φ is a $*$-fibration, by Proposition 13.9 it also has right adjoints $\overline{\triangleleft}$, $\overline{\triangleright}$. Then, since it is a monoidal bifibration, it is externally closed by Proposition 13.17. But since it is strongly BC, Proposition 13.15(ii) then implies that it is also internally closed. ■

14. From fibrations to framed bicategories

We now prove that any well-behaved monoidal bifibration gives rise to a framed bicategory. The reader may not be too surprised that there is some relationship, since many of our examples of monoidal bifibrations look very similar to our examples of framed bicategories. In this section we state our results; the proofs will be given in §§16–17 after we consider an important class of examples in §15.

To motivate the precise construction, consider the relationship between the framed bicategory $\mathbb{S}\mathbf{pan}(\mathscr{C})$ and the monoidal bifibration $\mathtt{Arr}_{\mathscr{C}} \colon \mathscr{C}^{\downarrow} \to \mathscr{C}$. A horizontal 1-cell $M \colon A \nrightarrow B$ in $\mathbb{S}\mathbf{pan}(\mathscr{C})$ is a span $A \leftarrow M \to B$, which can also be considered as an arrow $M \to A \times B$, and hence an object of $\mathscr{C}^{\downarrow}$ over $A \times B$. The horizontal composition of $M \colon A \nrightarrow B$ and $N \colon B \nrightarrow C$ is given by pulling back along the maps to B, then remembering only the maps to A and C:

$$\begin{tikzcd}[column sep=small] & & M \times_B N \arrow[dl] \arrow[dr] & & \\ & M \arrow[dl] \arrow[dr] & & N \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

But this can also be phrased in terms of the maps $M \to A \times B$ and $N \to B \times C$ by taking the product map

$$M \times N \to A \times B \times B \times C,$$

pulling back along the diagonal Δ_B:

$$\begin{tikzcd} M \times_B N \arrow[r] \arrow[d] & M \times N \arrow[d] \\ A \times B \times C \arrow[r] & A \times B \times B \times C \end{tikzcd}$$

and then composing with the projection $\pi_B \colon A \times B \times C \to A \times C$. In terms of the monoidal bifibration $\mathtt{Arr}_{\mathscr{C}}$, this can be written as

$$M \times_B N = (\pi_B)_! \Delta_B^*(M \times N). \tag{47}$$

Similarly, the unit object U_A in $\mathbb{S}\mathbf{pan}(\mathscr{C})$ is the span $A \leftarrow A \to A$, alternatively viewed as the diagonal map $A \to A \times A$. This can be obtained (in a somewhat perverse way) by pulling back the terminal object 1 along the map $\pi_A \colon A \to 1$, then composing with the diagonal $\Delta_A \colon A \to A \times A$. In the language of $\mathtt{Arr}_{\mathscr{C}}$, we have

$$U_A = (\Delta_A)_! \pi_A^* 1. \tag{48}$$

We now observe that the expressions (47) and (48) can easily be generalized to any monoidal bifibration in which the base is cartesian monoidal, so that we have diagonals and projections. This may help to motivate the following result.

14.1. Definition. We say that a monoidal bifibration $\Phi \colon \mathscr{A} \to \mathscr{B}$ is **frameable** if $\mathscr{B}$ is cartesian monoidal and Φ is either

(i) strongly BC or

(ii) weakly BC and internally closed.

14.2. Theorem. *Let $\Phi \colon \mathscr{A} \to \mathscr{B}$ be a frameable monoidal bifibration. Then there is a framed bicategory $\mathbb{F}\mathbf{r}(\Phi)$ with a vertically strict involution, defined as follows.*

(i) $\mathbb{F}\mathbf{r}(\Phi)_0 = \mathscr{B}$.

(ii) $\mathbb{F}\mathbf{r}(\Phi)_1$, *L, and R are defined by the following pullback square.*

$$\begin{tikzcd} \mathbb{F}\mathbf{r}(\Phi)_1 \arrow[r] \arrow[d, "{(L,R)}"'] & \mathscr{A} \arrow[d, "\Phi"] \\ \mathscr{B} \times \mathscr{B} \arrow[r, "\times"'] & \mathscr{B}. \end{tikzcd}$$

Thus the horizontal 1-cells $A \nrightarrow B$ *are the objects of* $\mathscr{A}$ *over* $A \times B$*, and the 2-cells* $M \underset{f}{\overset{g}{\Longrightarrow}} N$ *are the arrows of* $\mathscr{A}$ *over* $f \times g$*.*

(iii) The horizontal composition of $M\colon A \nrightarrow B$ *and* $N\colon B \nrightarrow C$ *is*

$$M \odot N = (\pi_B)_! \Delta_B^*(M \otimes N),$$

and similarly for 2-cells.

(iv) The horizontal unit of A is

$$U_A = (\Delta_A)_! \pi_A^* I.$$

(v) The involution is the identity on objects and we have $M^{op} = \mathfrak{s}^* M$*, where* $\mathfrak{s}$ *is the symmetry isomorphism.*

If Φ *is externally closed and a* $*$*-bifibration, then* $\mathbb{F}\mathbf{r}(\Phi)$ *is closed in the sense of §5. If* Φ *is symmetric, then* $\mathbb{F}\mathbf{r}(\Phi)$ *is symmetric monoidal in the sense of §9 and its involution is also symmetric monoidal.*

14.3. Examples. As alluded to above, if $\mathscr{C}$ has finite limits, the symmetric monoidal bifibration $\mathtt{Arr}_{\mathscr{C}}$ gives rise to the symmetric monoidal framed bicategory $\mathbb{S}\mathbf{pan}(\mathscr{C})$, which is closed if $\mathscr{C}$ is locally cartesian closed.

14.4. Example. If $\mathscr{C}$ has finite limits and colimits preserved by pullback, then the monoidal bifibration $\mathtt{Retr}_{\mathscr{C}}$ gives rise to a symmetric monoidal framed bicategory of *parametrized objects*, which we denote $\mathbb{E}\mathbf{x}(\mathscr{C})$. It is also closed if $\mathscr{C}$ is locally cartesian closed.

Applied to the monoidal $*$-bifibration $\mathtt{Retr}_{\mathbf{Top}}$ of ex-spaces from Example 12.11, we obtain a framed bicategory $\mathbb{E}\mathbf{x}(\mathbf{Top})$ of parametrized spaces which is both symmetric monoidal and closed.

14.5. Example. The monoidal $*$-bifibration $\mathtt{Sp}$ of parametrized orthogonal spectra gives rise to a point-set level framed bicategory of parametrized spectra, which we may denote $\mathbb{S}\mathbf{p}$. It is symmetric monoidal and closed.

14.6. Example. The homotopy-category monoidal $*$-bifibration $\mathrm{Ho}(\mathtt{Sp})$, which is weakly BC and internally closed, gives rise to a framed bicategory $\mathrm{Ho}(\mathbb{S}\mathbf{p})$. This is the same as the framed bicategory we have been calling $\mathbb{E}\mathbf{x}$ ever since §2. Similarly, $\mathrm{Ho}(\mathtt{Retr}_{\mathbf{Top}})$ gives rise to a homotopy-level framed bicategory $\mathrm{Ho}(\mathbb{E}\mathbf{x}(\mathbf{Top}))$ of parametrized spaces. Both of these framed bicategories are symmetric monoidal and closed. These are the only ones of our examples which are weakly rather than strongly BC.

The dual version of Theorem 14.2 says the following.

14.7. THEOREM. *If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a strongly co-BC monoidal bifibration where $\mathscr{B}$ is cocartesian monoidal, then there is a framed bicategory $\mathbb{F}\mathbf{r}(\Phi)$ with a vertically strict involution, defined as in Theorem 14.2, except that composition is given by*

$$M \odot N = \eta^* \nabla_!(M \otimes N)$$

and units are given by

$$U_A = \nabla^* \eta_! I.$$

If Φ is externally closed and is a $*$-bifibration, then $\mathbb{F}\mathbf{r}(\Phi)$ is closed. If Φ is symmetric, then $\mathbb{F}\mathbf{r}(\Phi)$ and its involution are symmetric monoidal.

14.8. EXAMPLE. The monoidal $*$-bifibration `CMod` gives rise to the framed bicategory $\mathbb{CM}\mathbf{od}$, which is symmetric monoidal and closed. However, $\mathbb{M}\mathbf{od}$ cannot be constructed in this way, because the category of noncommutative rings is not cocartesian monoidal.

Like our other ways of constructing framed bicategories, these results are 2-functorial. To state this precisely, we need to define the right 2-categories. In §12 we defined lax, strong, and oplax monoidal morphisms of bifibrations to be those that preserve both the monoidal structure and the base change in appropriate ways. These morphisms, together with the monoidal transformations defined there, give us 2-categories $\mathcal{M}bi\mathcal{F}$, $\mathcal{M}bi\mathcal{F}$, and $\mathcal{M}bi\mathcal{F}_\ell$. Let $\mathcal{MF}^{\mathrm{fr}}$ denote the full sub-2-category of $\mathcal{M}bi\mathcal{F}$ spanned by the frameable monoidal bifibrations, and similarly for $\mathcal{MF}^{\mathrm{fr}}_{op\ell}$ and $\mathcal{MF}^{\mathrm{fr}}_{\ell}$.

14.9. THEOREM. *The construction of Theorem 14.2 extends to a 2-functor*

$$\mathbb{F}\mathbf{r}\colon \mathcal{MF}^{\mathrm{fr}} \longrightarrow \mathcal{F}r\mathcal{B}i$$

and similarly for oplax and lax morphisms.

14.10. EXAMPLE. The 2-functor $\mathbb{S}\mathbf{pan}\colon \mathcal{C}art \to \mathcal{F}r\mathcal{B}i$ clearly factors through $\mathbb{F}\mathbf{r}$ via a 2-functor $\mathtt{Arr}\colon \mathcal{C}art \to \mathcal{MF}^{\mathrm{fr}}$.

14.11. EXAMPLE. Let $bi\mathcal{C}art^{\mathrm{psc}}$ denote the 2-category of categories with finite limits and finite colimits preserved by pullback, functors which preserve finite limits and colimits, and natural transformations. Then we have a 2-functor $\mathtt{Retr}\colon bi\mathcal{C}art^{\mathrm{psc}} \to \mathcal{MF}^{\mathrm{fr}}$. Composing this with $\mathbb{F}\mathbf{r}$ defines a 2-functor $\mathbb{E}\mathbf{x}\colon bi\mathcal{C}art^{\mathrm{psc}} \to \mathcal{F}r\mathcal{B}i$.

As with our other 2-categories, we automatically obtain notions of equivalence and adjunction between monoidal bifibrations, and these are preserved by 2-functors such as $\mathbb{F}\mathbf{r}$. As usual, we can also characterize these more explicitly; we omit the proof of the following.

14.12. PROPOSITION. *An adjunction $F \dashv G$ in $\mathcal{MF}^{\mathrm{fr}}_{\ell}$ between $\Phi\colon \mathscr{A} \to \mathscr{B}$ and $\Phi'\colon \mathscr{A}' \to \mathscr{B}'$ consists of the following properties and structure.*

(i) F is a strong monoidal morphism of bifibrations and G is a lax monoidal morphism of bifibrations;

(ii) We have monoidal adjunctions $F_0\colon \mathscr{B} \rightleftarrows \mathscr{B}' :G_0$ *and* $F_1\colon \mathscr{A} \rightleftarrows \mathscr{A}' :G_1$*;*

(iii) We have equalities $\Phi' F_1 = F_0 \Phi$ *and* $\Phi G_1 = G_0 \Phi'$ *which are monoidal transformations; and*

(iv) The adjunction $F_1 \dashv G_1$ *'lies over'* $F_0 \dashv G_0$ *in the sense that the following square commutes:*

$$\begin{tikzcd} \mathscr{A}(M, G_1 N) \arrow[r, "\cong"] \arrow[d, "\Phi"'] & \mathscr{A}'(F_1 M, N) \arrow[d, "\Phi'"] \\ \mathscr{B}(A, G_0 B) \arrow[r, "\cong"'] & \mathscr{B}'(F_0 A, B). \end{tikzcd}$$

14.13. Remark. In fact, these conditions are somewhat redundant. For example, left adjoints automatically preserve opcartesian arrows and right adjoints automatically preserve cartesian ones, and the right adjoint of a strong monoidal functor is always lax monoidal. These are consequences of 'doctrinal adjunction' (see Proposition 8.2 and [Kel74]) and a property called 'lax-idempotence' (see [KL97]).

In many cases, F_0 and G_0 are the identity, and the entire adjunction is 'over $\mathscr{B}$' in the sense introduced in Definition 12.5.

14.14. Example. Let $\mathscr{C}$ have finite limits and finite colimits preserved by pullback. Then there is a forgetful lax monoidal morphism $\mathtt{Retr}_{\mathscr{C}} \to \mathtt{Arr}_{\mathscr{C}}$ lying over $\mathscr{C}$. Cartesian arrows are given by pullback in both cases, and hence are preserved strongly, but opcartesian arrows are given by pushout in $\mathtt{Retr}_{\mathscr{C}}$ and mere composition in $\mathtt{Arr}_{\mathscr{C}}$, hence are preserved only laxly. The lax monoidal constraint is given by the quotient map $M \times N \to M \barwedge N$.

This forgetful morphism has a left adjoint

$$(-)_+ \colon \mathtt{Arr}_{\mathscr{C}} \longrightarrow \mathtt{Retr}_{\mathscr{C}}. \tag{49}$$

which takes an object $X \to A$ over A to the retraction

$$A \longrightarrow X_+ = X \sqcup A \longrightarrow A.$$

We say that the functor $(-)_+$ *adjoins a disjoint section.* It is straightforward to check that $(-)_+$ is a strong monoidal morphism of bifibrations and the pair satisfies Proposition 14.12.

As $\mathscr{C}$ varies, the forgetful morphisms define a 2-natural transformation from the 2-functor $\mathtt{Retr}$ to the 2-functor $\mathtt{Arr}$, while the morphisms $(-)_+$ form an oplax natural transformation $\mathtt{Arr} \to \mathtt{Retr}$. This remains true upon composing with the 2-functor $\mathbb{F}\mathbf{r}$, so we obtain framed adjunctions

$$(-)_+ \colon \mathbb{S}\mathbf{pan}(\mathscr{C}) \rightleftarrows \mathbb{E}\mathbf{x}(\mathscr{C}) \ :U$$

where the right adjoint is 2-natural in $\mathscr{C}$ and the left adjoint is oplax natural in $\mathscr{C}$.

14.15. EXAMPLE. It is essentially shown in [MS06, Ch. 11] that we have an adjunction

$$\Sigma^\infty \colon \mathtt{Retr}_{\mathbf{Top}} \rightleftarrows \mathtt{Sp} : \Omega^\infty. \tag{50}$$

of monoidal bifibrations lying over $\mathscr{U}$. Applying $\mathbb{F}\mathbf{r}$, we obtain a framed adjunction

$$\Sigma^\infty \colon \mathbb{E}\mathbf{x}(\mathbf{Top}) \rightleftarrows \mathbb{S}\mathbf{p} : \Omega^\infty. \tag{51}$$

The fiber adjunctions are shown to be Quillen in [MS06, 12.6.2], so passing to homotopy categories of horizontal 1-cells, we obtain a framed adjunction

$$\Sigma^\infty \colon \operatorname{Ho}(\mathbb{E}\mathbf{x}(\mathbf{Top})) \rightleftarrows \mathbb{E}\mathbf{x} : \Omega^\infty. \tag{52}$$

15. Monoids in monoidal fibrations and examples

We have seen that well-behaved monoidal bifibrations give rise to framed bicategories, and that framed bicategories with local coequalizers admit the $\mathbb{M}\mathbf{od}$ construction, so it is natural to ask what conditions on a monoidal fibration ensure that the resulting framed bicategory has local coequalizers.

15.1. DEFINITION. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a fibration.

- We say that Φ has **fiberwise coequalizers** if each fiber $\mathscr{A}_B$ has coequalizers and the functors f^* preserve coequalizers. Note that this latter condition is automatic if Φ is a $*$-fibration, since then f^* is a left adjoint.
- Similarly, we say that Φ has **fiberwise equalizers** if each fiber $\mathscr{A}_B$ has equalizers and f^* preserves equalizers, the second condition being automatic if Φ is a bifibration.
- If Φ is a monoidal fibration with fiberwise coequalizers, we say these coequalizers are **preserved by** $\otimes$ if the functors $\otimes\colon \mathscr{A}_A \times \mathscr{A}_B \longrightarrow \mathscr{A}_{A\otimes B}$ all preserve coequalizers in each variable. This is automatic if the right adjoints $\overline{\lhd}$, $\overline{\rhd}$ exist.

15.2. PROPOSITION. *Let Φ be a frameable monoidal bifibration with fiberwise coequalizers preserved by $\otimes$. Then:*

(i) The framed bicategory $\mathbb{F}\mathbf{r}(\Phi)$ has local coequalizers, so there is a framed bicategory $\mathbb{M}\mathbf{od}(\mathbb{F}\mathbf{r}(\Phi))$.

(ii) If Φ is symmetric, then $\mathbb{F}\mathbf{r}(\Phi)$ is a monoidal framed bicategory with local coequalizers preserved by $\otimes$; hence $\mathbb{M}\mathbf{od}(\mathbb{F}\mathbf{r}(\Phi))$ is also monoidal.

(iii) If Φ is externally closed, a $*$-fibration, and has fiberwise equalizers, then $\mathbb{F}\mathbf{r}(\Phi)$ is closed and its hom-categories have equalizers; hence $\mathbb{M}\mathbf{od}(\mathbb{F}\mathbf{r}(\Phi))$ is also closed.

Proof. Since the hom-category $\mathcal{F}r(\Phi)(A, B)$ is just the fiber of Φ over $A \times B$, it has coequalizers. And since $M \odot N = \pi_! \Delta^*(M \otimes N)$, where $\otimes$ and Δ^* preserve coequalizers by assumption and $\pi_!$ preserves all colimits as it is a left adjoint, these coequalizers are preserved by $\odot$; thus $\mathbb{F}\mathbf{r}(\Phi)$ has local coequalizers. Item (i) then follows from Theorem 11.5.

For (ii), we need to know what the external monoidal structure of $\mathbb{F}\mathbf{r}(\Phi)$ is. When we prove in Proposition 17.1 that $\mathbb{F}\mathbf{r}(\Phi)$ is monoidal, we will define this external product to be essentially that of Φ, but with a slight twist. Namely, if $\Phi(M) = A \times B$ and $\Phi(N) = C \times D$, then we have $\Phi(M \otimes N) = (A \times B) \times (C \times D)$, whereas their product in $\mathbb{F}\mathbf{r}(\Phi)$ should lie over $(A \times C) \times (B \times D)$. Thus we define the external product $M \otimes' N$ of $\mathbb{F}\mathbf{r}(\Phi)$ to be the base change of $M \otimes N$ along the constraint isomorphism

$$(A \times C) \times (B \times D) \cong (A \times B) \times (C \times D).$$

In particular, we have $M \otimes' N \cong M \otimes N$; thus $\otimes'$ preserves coequalizers because $\otimes$ does. It then follows from Theorem 11.5 that $\mathbb{M}\mathbf{od}(\mathbb{F}\mathbf{r}(\Phi))$ is monoidal.

Finally, (iii) follows directly from Theorems 14.2 and 11.5. ∎

15.3. Example. If $\mathscr{C}$ has finite limits and coequalizers preserved by pullback, then its self-indexing $\mathtt{Arr}_{\mathscr{C}}$ satisfies the conditions of Proposition 15.2, and we obtain the framed bicategory $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathscr{C}))$ of internal categories and distributors which we mentioned in Example 11.7.

However, we can also obtain *enriched* categories and distributors, by starting with a different monoidal bifibration.

15.4. Example. Given any ordinary category $\mathscr{V}$, let $\mathbf{Fam}(\mathscr{V})$ be the category of *families* of objects of $\mathscr{V}$. That is, an object of $\mathbf{Fam}(\mathscr{V})$ is a set X together with an X-indexed family $\{A_x\}_{x\in X}$ of objects in $\mathscr{V}$. Then there is a fibration $\mathtt{Fam}_{\mathscr{V}} \colon \mathbf{Fam}(\mathscr{V}) \to \mathbf{Set}$ which is sometimes called the *naive indexing* of $\mathscr{V}$; its fiber over a set X is the category $\mathscr{V}^X$. The reader may check the following.

- If $\mathscr{V}$ is a monoidal category, then $\mathtt{Fam}_{\mathscr{V}}$ is a monoidal fibration; the external product of $\{A_x\}_{x\in X}$ and $\{B_y\}_{y\in Y}$ is $\{A_x \otimes B_y\}_{(x,y)\in X\times Y}$. The fiberwise monoidal structure is the obvious one. If $\mathscr{V}$ is braided or symmetric, then so is $\mathtt{Fam}_{\mathscr{V}}$.

- If $\mathscr{V}$ has small coproducts (resp. products), then $\mathtt{Fam}_{\mathscr{V}}$ is a strongly BC bifibration (resp. $*$-fibration). If $\mathscr{V}$ is also monoidal and $\otimes$ preserves coproducts, then $\mathtt{Fam}_{\mathscr{V}}$ is a monoidal bifibration.

- If $\mathscr{V}$ has coequalizers preserved by $\otimes$, then $\mathtt{Fam}_{\mathscr{V}}$ has fiberwise coequalizers preserved by $\otimes$. If $\mathscr{V}$ has equalizers, then $\mathtt{Fam}_{\mathscr{V}}$ has fiberwise equalizers.

- If $\mathscr{V}$ is closed, then $\mathtt{Fam}_{\mathscr{V}}$ is internally closed. Thus, by Corollary 13.20, if $\mathscr{V}$ also has small products, then $\mathtt{Fam}_{\mathscr{V}}$ is externally closed.

In particular, when $\mathscr{V}$ is monoidal and has colimits preserved by $\otimes$, $\mathtt{Fam}_{\mathscr{V}}$ is frameable and $\mathbb{F}\mathbf{r}(\mathtt{Fam}_{\mathscr{V}})$ has local coequalizers, so we can define $\mathbb{M}\mathbf{od}(\mathbb{F}\mathbf{r}(\mathtt{Fam}_{\mathscr{V}}))$. It is easy to see that $\mathbb{F}\mathbf{r}(\mathtt{Fam}_{\mathscr{V}})$ is equivalent to the framed bicategory $\mathbb{M}\mathbf{at}(\mathscr{V})$ defined in Example 11.8, where we observed that $\mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{at}(\mathscr{V})) \simeq \mathbb{D}\mathbf{ist}(\mathscr{V})$.

This suggests that we can view a frameable closed symmetric monoidal $*$-bifibration Φ with fiberwise equalizers and coequalizers as a 'parametrized' version of a complete and cocomplete closed symmetric monoidal category $\mathscr{V}$, and that we can view the associated framed bicategory $\mathbb{M}\mathbf{od}(\mathbb{F}\mathbf{r}(\Phi))$ as a parametrized version of $\mathbb{D}\mathbf{ist}(\mathscr{V})$. The other example of $\mathtt{Arr}_{\mathscr{C}}$ seems to bear this out.

In fact, we can view the monoidal bifibrations $\mathtt{Fam}_{\mathscr{V}}$ and $\mathtt{Arr}_{\mathscr{C}}$ as living at opposite ends of a continuum. In $\mathtt{Fam}_{\mathscr{V}}$, the base category **Set** is fairly uninteresting, while all the interesting things happen in the fibers. On the other hand, in $\mathtt{Arr}_{\mathscr{C}}$, the base category $\mathscr{C}$ can be interesting, but the fibers carry essentially no new information, being determined by the base. Other monoidal bifibrations will fall somewhere in between the two, and monoids in the resulting framed bicategories can be thought of as 'categories which are both internal and enriched'.

15.5. Example. If $\mathscr{C}$ has finite limits and finite colimits preserved by pullback, then Proposition 15.2 applies to the monoidal bifibration $\mathtt{Retr}_{\mathscr{C}}$, so that $\mathbb{M}\mathbf{od}(\mathbb{E}\mathbf{x}(\mathscr{C}))$ is a symmetric monoidal and closed framed bicategory. A monoid in $\mathbb{E}\mathbf{x}(\mathscr{C})$ may be thought of as a 'pointed internal category' in $\mathscr{C}$. For example, a monoid in $\mathbb{E}\mathbf{x}(\mathbf{Set})$ is a small category enriched over the category $\mathbf{Set}_*$ of pointed sets with smash product, meaning that each hom-set has a chosen basepoint and composition preserves basepoints. Similarly, a monoid in $\mathbb{E}\mathbf{x}(\mathbf{Top})$ is a 'based topological category'. If its space of objects is discrete, then it is just a small category enriched over based topological spaces, but in general it will be 'both internal and enriched'.

Applying $\mathbb{M}\mathbf{od}$ to the disjoint-sections functor $(-)_+$ from Example 14.14, we obtain a framed functor $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathscr{C})) \to \mathbb{M}\mathbf{od}(\mathbb{E}\mathbf{x}(\mathscr{C}))$. Thus, any internal category can be made into a pointed internal category by 'adjoining disjoint basepoints to hom-objects'.

15.6. Example. Proposition 15.2 applies to the point-set fibration $\mathtt{Sp}$ of parametrized spectra, so $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{p})$ is a symmetric monoidal and closed framed bicategory. A monoid in $\mathbb{S}\mathbf{p}$ can be viewed as a category 'internal to spaces and enriched over spectra'; if its space of objects is discrete, then it is just a small category enriched over orthogonal spectra.

To obtain other examples, we can apply $\mathbb{M}\mathbf{od}(\Sigma^\infty)$ to any based topological category as in Example 15.5, and thereby to any internal category in **Top** with a disjoint section adjoined. Certain monoids in $\mathbb{S}\mathbf{p}$ arising in this way from the topologized fundamental groupoid ΠM or path-groupoid $\mathcal{P}M$ of a space M play an important role in [Pon07].

A good case can be made (see [MS06]) that a monoid in $\mathbb{S}\mathbf{p}$ is the right parametrized analogue of a classical ring spectrum, since when its space of objects is a point, it reduces to an orthogonal ring spectrum. The more naive notion of a monoid in $\mathbf{Sp}_B$ with respect to the internal smash product $\wedge_B$ is poorly behaved because, unlike the situation for the external smash product $\bar{\wedge}$, we have no homotopical control over $\wedge_B$.

The above two examples give framed bicategories with involutions which are not vertically strict, since the 2-cell components of ξ in $\mathbb{F}\mathbf{r}(\Phi)$ are not identities.

15.7. Example. Proposition 15.2 does *not* apply to the homotopy-level monoidal fibration $\mathrm{Ho}(\mathtt{Sp})$, since the stable homotopy categories of parametrized spectra do not in general admit coequalizers. Rather than $\mathbb{M}\mathbf{od}(\mathbb{E}\mathbf{x}) = \mathbb{M}\mathbf{od}(\mathrm{Ho}(\mathbb{S}\mathbf{p}))$, the correct thing to consider is '$\mathrm{Ho}(\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{p}))$'. Here the objects are honest monoids in $\mathbb{S}\mathbf{p}$, whose multiplication is associative and unital on the point-set level, just like in $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{p})$, but we pass to homotopy categories of horizontal 1-cells. We then need to use a 'homotopy tensor product' to define the derived horizontal composition, as was done in [Pon07]. We hope to investigate the homotopy theory of framed bicategories more fully in a later paper.

15.8. Example. We can construct various monoidal fibrations over any '**Set**-like' category $\mathscr{E}$ by mimicking the constructions of classical **Set**-based monoidal categories. For example, we have a fibration $\mathtt{Ab}_{\mathscr{E}}$ over $\mathscr{E}$ whose fiber over B is the category $\mathbf{Ab}(\mathscr{E}/B)$ of abelian group objects in $\mathscr{E}/B$. If $\mathscr{E}$ is locally cartesian closed, has finite colimits, and the forgetful functors $\mathbf{Ab}(\mathscr{E}/B) \to \mathscr{E}/B$ have left adjoints, then this is a strongly BC $*$-bifibration (see [Joh02b, D5.3.2]). If $\mathscr{E}$ is cocomplete, the tensor product of abelian group objects can be defined internally and makes $\mathtt{Ab}_{\mathscr{E}}$ a monoidal bifibration; monoids in the corresponding framed bicategory $\mathbb{A}\mathbf{b}(\mathscr{E})$ give a notion of '**Ab**-category in $\mathscr{E}$'.

For example, if $\mathscr{E}$ is a category of topological spaces, then any vector bundle over a space B gives an object of $\mathbf{Ab}(\mathscr{E}/B)$. One might argue, analogously to Example 15.6, that monoids in $\mathbb{A}\mathbf{b}(\mathscr{E})$ give a good notion of a 'bundle of rings'.

The theory of such *relative enriched categories* appears to be fairly unexplored; the only references we know are [GG76] and [Prz07]. We will explore this theory more extensively in a later paper; in many ways, it is very similar to classical enriched category theory. We end with one further example of this phenomenon.

If $\mathscr{V}$ is an ordinary monoidal category with coproducts preserved by $\otimes$, then any small unenriched category C gives rise to a 'free' $\mathscr{V}$-category $\mathscr{V}[C]$ whose hom-objects are given by copowers of the unit object:

$$\mathscr{V}[C](x,y) = \coprod_{C(x,y)} I. \tag{53}$$

For a monoidal fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$, the analogue of an unenriched category is an internal category in $\mathscr{B}$. The following is an analogue of this construction in our general context.

15.9. Proposition. *Let $\mathscr{B}$ have finite limits and let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a strongly BC monoidal bifibration. Then there is a canonical strong monoidal morphism of bifibrations*

$$\mathtt{Arr}_{\mathscr{B}} \longrightarrow \Phi \tag{54}$$

which takes an object $X \xrightarrow{f} B$ of $\mathscr{B}/B$ to the object $f_!\pi_X^* I$ of $\mathscr{A}_B$. Consequently, there is a canonical framed functor

$$\mathbb{S}\mathbf{pan}(\mathscr{B}) \longrightarrow \mathbb{F}\mathbf{r}(\Phi) \tag{55}$$

and thus, if Φ has fiberwise coequalizers preserved by $\otimes$ and $\mathscr{B}$ has coequalizers preserved by pullback, a framed functor

$$\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathscr{B})) \longrightarrow \mathbb{M}\mathbf{od}(\mathbb{F}\mathbf{r}(\Phi)). \tag{56}$$

15.10. EXAMPLE. When $\Phi = \mathtt{Fam}_{\mathscr{V}}$ for an ordinary monoidal category $\mathscr{V}$, the morphism $\mathtt{Arr}_{\mathscr{B}} \to \mathtt{Fam}_{\mathscr{V}}$ takes a set A to $\coprod_A I$. Thus the induced framed functor $\mathbb{D}\mathbf{ist}(\mathbf{Set}) \to \mathbb{D}\mathbf{ist}(\mathscr{V})$ is exactly the 'free $\mathscr{V}$-category' operation (53) described above.

15.11. EXAMPLES. More interestingly, when Φ is $\mathtt{Retr}_{\mathscr{C}}$, the morphism $\mathtt{Arr}_{\mathscr{C}} \to \mathtt{Retr}_{\mathscr{C}}$ is the disjoint-section operation described in Example 14.14. And when Φ is the monoidal fibration $\mathtt{Sp}$ of orthogonal spectra, the morphism $\mathtt{Arr}_{\mathscr{U}} \to \mathtt{Sp}$ first adjoins a disjoint section, then applies the parametrized suspension-spectrum functor Σ^∞ from Example 14.15. Therefore, if C is an internal category in topological spaces, the 'topologically internal and spectrally enriched category' $\Sigma^\infty C_+$ considered in Example 15.6 is in fact 'freely generated by C' in this canonical way.

16. Two technical lemmas

In preparation for our proof of Theorem 14.2 in §17, in this section we reformulate the Beck-Chevalley condition and internal closedness in terms of cartesian arrows.

16.1. LEMMA. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a bifibration. Then a commuting square*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd} \tag{57}$$

in $\mathscr{B}$ satisfies the Beck-Chevalley condition if and only if for every $M \in \mathscr{A}_B$, the square (57) lifts to some commutative square

$$\begin{tikzcd} M' \arrow[r, "\phi"] \arrow[d, "\chi"'] & M \arrow[d, "\xi"] \\ M''' \arrow[r, "\psi"'] & M'' \end{tikzcd} \tag{58}$$

in $\mathscr{A}$ in which ϕ and ψ are cartesian and χ and ξ are opcartesian.

Note that given ϕ, χ, ξ lifting h, k, g with χ opcartesian, there is exactly one ψ lifting f which makes (58) commute. Thus the condition can also be stated as "Given any cartesian ϕ and opcartesian χ, ξ, the unique morphism ψ over f making (58) commute is cartesian".

Proof. Choose a cleavage. Then by the universal properties of cartesian and opcartesian arrows, there is a unique dotted arrow living over 1_C which makes the following pentagon commute:

$$
\begin{tikzcd}
& h^*M \arrow[r, "\mathrm{cart}"] \arrow[dl, "\mathrm{opcart}"'] & M \arrow[d, "\mathrm{opcart}"] \\
k_!h^*M \arrow[dr, dotted] & & g_!M \\
& f^*g_!M. \arrow[ur, "\mathrm{cart}"'] &
\end{tikzcd}
$$

We claim that in fact this dotted arrow is the component of the Beck-Chevalley natural transformation (42) at M. To see this, we fill out the diagram as follows.

$$
\begin{tikzcd}
h^*M \arrow[rrrr, "\mathrm{cart}"] \arrow[dr, dashed, "h^*\eta"'] \arrow[ddd, "\mathrm{opcart}"'] & & & & M \arrow[dl, dashed, "\eta"] \arrow[dddd, "\mathrm{opcart}"] \\
& h^*g^*g_!M \arrow[rr, "\mathrm{cart}"'] \arrow[dr, dashed, "\cong"] \arrow[dd, "\mathrm{opcart}"] & & g^*g_!M \arrow[dddr, "\mathrm{cart}"'] & \\
& & k^*f^*g_!M \arrow[d, "\mathrm{opcart}"'] \arrow[ddr, bend left, "\mathrm{cart}"] & & \\
k_!h^*M \arrow[r, dashed, "k_!h^*\eta"] \arrow[drrr, dotted] & k_!h^*g^*g_!M \arrow[r, dashed, "\cong"] & k_!k^*f^*g_!M \arrow[dr, dashed, "\varepsilon f^*g_!"] & & \\
& & & f^*g_!M \arrow[r, "\mathrm{cart}"'] & g_!M.
\end{tikzcd}
$$

Here the dashed arrows are unique factorizations through (op)cartesian arrows. This exhibits the dotted arrow as the composite of a unit, canonical isomorphism, and counit, which is the definition of the transformation (42). This proves our claim.

Therefore, if (42) is an isomorphism, the composite $k_!h^*M \cong f^*g_!M \to g_!M$ is cartesian, and hence we have a commuting square of cartesian and opcartesian arrows as desired. Conversely, if we have such a commuting square, then clearly for some choice of cleavage, the dotted arrow is the identity; hence it is an isomorphism for all cleavages. ∎

As always, it simplifies our life greatly to work with cartesian arrows rather than chosen cleavages. For example, we can now easily show the following.

16.2. Corollary. *If $\mathbb{D}$ is a monoidal framed bicategory, then the square*

$$
\begin{tikzcd}
A \otimes C \arrow[r, "f\otimes 1"] \arrow[d, "1\otimes g"'] & B \otimes C \arrow[d, "1\otimes g"] \\
A \otimes D \arrow[r, "f\otimes 1"'] & B \otimes D
\end{tikzcd}
\tag{59}
$$

in $\mathbb{D}_0$ satisfies the Beck-Chevalley condition for the bifibrations L and R.

PROOF. Let $M\colon B\otimes C \nrightarrow E$ be a horizontal 1-cell and consider the following diagram in $R^{-1}(E)$.

$$\begin{tikzcd}
({}_fB\otimes C)\odot M \arrow[r, "\mathrm{cart}"] \arrow[d, "\mathrm{opcart}"'] & (B\otimes C)\odot M \arrow[r, "\cong"] \arrow[d, "\mathrm{opcart}"] & M \\
({}_fB\otimes D_g)\odot M \arrow[r, "\mathrm{cart}"'] & (B\otimes D_g)\odot M
\end{tikzcd}$$

The arrows labeled cartesian or opcartesian are obtained from the cartesian ${}_fB \to B$ and opcartesian $C \to D_g$ via $\otimes$ and $\odot$. The square commutes by functoriality of $\otimes$, so the result follows from Lemma 16.1. ∎

16.3. COROLLARY. *If $\mathbb{D}$ is a monoidal framed bicategory in which $\mathbb{D}_0$ is cartesian monoidal, then the monoidal bifibration (L, R) is weakly BC.*

PROOF. Taking $D = 1$ in the square (59) shows immediately that L and R are weakly BC; an analogous square in $\mathbb{D}_0 \times \mathbb{D}_0$ applies to (L, R). ∎

To deal with the 'weakly BC and internally closed' case of Theorem 14.2, we also need a statement about cartesian arrows that makes use of the closed structure. Recall that if f^* is strong monoidal and has a left adjoint $f_!$, then f^* is closed monoidal if and only if the dual maps

$$f_!(M \boxtimes f^*N) \longrightarrow f_!M \boxtimes N \tag{60}$$

$$f_!(f^*N \boxtimes M) \longrightarrow N \boxtimes f_!M \tag{61}$$

are isomorphisms. This latter condition is amenable to restatement in fibrational terms, using the characterization of $\boxtimes$ in terms of $\otimes$ and Δ^*.

16.4. LEMMA. *Let $\Phi\colon \mathscr{E} \to \mathscr{B}$ be an internally closed monoidal bifibration, where $\mathscr{B}$ is cartesian monoidal. Then for any $f\colon A \to B$ in $\mathscr{B}$, and any M, N in $\mathscr{E}$ with $\Phi(M) = A, \Phi(N) = B$, the square*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\Delta_A"'] & B \arrow[d, "\Delta_B"] \\
A\times A \arrow[r, "f\times f"'] & B\times B
\end{tikzcd} \tag{62}$$

in $\mathscr{B}$ lifts to a square

$$\begin{tikzcd}
\Delta_A^*(M\otimes f^*N) \arrow[r, "\mathrm{opcart}"] \arrow[d, "\mathrm{cart}"'] & \Delta_B^*(f_!M\otimes N) \arrow[d, "\mathrm{cart}"] \\
M\otimes f^*N \arrow[r, "\mathrm{opcart}\otimes\mathrm{cart}"'] & f_!M\otimes N
\end{tikzcd} \tag{63}$$

in $\mathscr{A}$, and dually.

PROOF. A cleavage gives us an opcartesian $M \to f_! M$, a cartesian $f^*N \to N$, and cartesian arrows on the left and right, inducing a unique arrow on the top which lifts f. We then observe that $\Delta_A^*(M \otimes f^*N) \cong M \boxtimes f^*N$ and $\Delta_B^*(f_!M \otimes N) \cong f_!M \boxtimes N$, so factoring the top arrow through an opcartesian arrow gives us precisely (60). Since Φ is internally closed, this is an isomorphism, so the top arrow must be opcartesian. ■

17. Proofs of Theorems 14.2 and 14.9

This section is devoted to the proofs of Theorems 14.2 and 14.9 (and the dual version Theorem 14.7). To make the proofs more manageable, we split them up into several propositions.

17.1. PROPOSITION. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a strongly BC monoidal bifibration, where $\mathscr{B}$ is cartesian monoidal. Then there is a framed bicategory $\mathbb{F}\mathbf{r}(\Phi)$ defined as follows.*

(i) $\mathbb{F}\mathbf{r}(\Phi)_0 = \mathscr{B}$.

(ii) $\mathbb{F}\mathbf{r}(\Phi)_1$*, L, and R are defined by the following pullback square.*

$$\begin{tikzcd}
\mathbb{F}\mathbf{r}(\Phi)_1 \arrow[r] \arrow[d, "{(L,R)}"'] & \mathscr{A} \arrow[d, "\Phi"] \\
\mathscr{B} \times \mathscr{B} \arrow[r, "\times"'] & \mathscr{B}.
\end{tikzcd}$$

Thus the horizontal 1-cells $A \to\!\!\!\!\!\!+ B$ are the objects of $\mathscr{A}$ over $A \times B$, and the 2-cells $M \overset{g}{\underset{f}{\Longrightarrow}} N$ are the arrows of $\mathscr{A}$ over $f \times g$.

(iii) The horizontal composition of $M\colon A \to\!\!\!\!\!\!+ B$ and $N\colon B \to\!\!\!\!\!\!+ C$ is

$$M \odot N = (\pi_B)_! \Delta_B^*(M \otimes N),$$

and similarly for 2-cells.

(iv) The horizontal unit of A is

$$U_A = (\Delta_A)_! \pi_A^* I.$$

If Φ is symmetric, then $\mathbb{F}\mathbf{r}(\Phi)$ is symmetric monoidal.

PROOF. Throughout the proof, we will write $\mathbb{P} = \mathbb{F}\mathbf{r}(\Phi)$ for brevity. Since we intend to construct an algebraic structure (a framed bicategory), we choose once and for all a cleavage (and opcleavage) on Φ, and reserve the notations f^*, $f_!$ and so on for the functors given by this cleavage. However, we will still use (op)cartesian arrows which are not in this cleavage in order to construct the constraints and coherence.

We have the structure and operations, at least, of a double category essentially already defined, except for the functoriality of $\odot$ and U. It is easy to see that $\odot$ is a functor, since

$\otimes$, Δ^*, and $\pi_!$ are functors. The functoriality of U is similar, but perhaps not as obvious since it is a functor $\mathbb{P}_0 \to \mathbb{P}_1$. Its action on an arrow $f\colon A \to B$ is given by the unique factorization U_f as follows.

$$\begin{tikzcd}
I & \pi_A^* I \arrow[l, "\mathrm{cart}"'] \arrow[r, "\mathrm{opcart}"] \arrow[d, dashed] & U_A \arrow[d, dashed, "U_f"] \\
& \pi_B^* \arrow[ul, "\mathrm{cart}"] \arrow[r, "\mathrm{opcart}"'] & U_B
\end{tikzcd}$$

Thus, to show that $\mathbb{P}$ is a double category, it suffices to construct coherent associativity and unit constraints. The following arguments should remind the reader of the proof of Theorem 12.7, although they are more complicated.

Note first that for horizontal 1-cells $M\colon A \to\!\!\!\!\!\!+\ B$ and $N\colon B \to\!\!\!\!\!\!+\ C$ in $\mathbb{P}$, we have $\Phi(M) = A \times B$ and $\Phi(N) = B \times C$, and the chosen cleavage gives us canonical morphisms

$$M \otimes N \xleftarrow{\mathrm{cart}} \Delta_B^*(M \otimes N) \xrightarrow{\mathrm{opcart}} (\pi_B)_! \Delta_B^*(M \otimes N) = M \odot N. \tag{64}$$

We begin with the associativity isomorphism. So suppose in addition to M, N we have $Q\colon C \to\!\!\!\!\!\!+\ D$. Then since $\otimes$ preserves (op)cartesian arrows, we can construct the following diagram.

$$\begin{tikzcd}[column sep=small]
& & (M \odot N) \otimes Q & & \\
(M \otimes N) \otimes Q & \Delta_B^*(M \otimes N) \otimes Q \arrow[l, "\mathrm{cart}"'] \arrow[ur, "\mathrm{opcart}"] & & \Delta_C^*\big((M \odot N) \otimes Q\big) \arrow[ul, "\mathrm{cart}"'] \arrow[r, "\mathrm{opcart}"] & (M \odot N) \odot Q. \\
& & \Delta_{BC}^*\big((M \otimes N) \otimes Q\big) \arrow[ul, dashed, "\mathrm{cart}"] \arrow[ur, dotted, "\mathrm{opcart}"'] \arrow[ull, bend left=10, "\mathrm{cart}"] & &
\end{tikzcd}$$

Here the solid arrows are part of the chosen cleavage. The dashed arrow is a unique factorization, which is cartesian by Proposition 3.4(ii). The dotted arrow, also a unique factorization, is opcartesian by the Beck-Chevalley condition (Lemma 16.1), because the square in question lifts the pullback square

$$\begin{tikzcd}
A \times B \times C \times D \arrow[r, "\Delta_C"] \arrow[d, "\pi_B"'] & A \times B \times C \times C \times D \arrow[d, "\pi_B"] \\
A \times C \times D \arrow[r, "\Delta_C"'] & A \times C \times C \times D.
\end{tikzcd}$$

Composing the two opcartesian arrows on the right, we obtain a span

$$(M \otimes N) \otimes Q \xleftarrow{\mathrm{cart}} \Delta_{BC}^*\big((M \otimes N) \otimes Q\big) \xrightarrow{\mathrm{opcart}} (M \odot N) \odot Q. \tag{65}$$

We perform an analogous construction for $M \odot (N \odot Q)$, then factor the associativity isomorphism for $\otimes$ through these cartesian and opcartesian arrows to obtain an associativity isomorphism for $\odot$:

$$\begin{tikzcd}
(M\otimes N)\otimes Q \arrow[d, "\cong"'] & \Delta^*_{BC}\big((M\otimes N)\otimes Q\big) \arrow[l, "\mathrm{cart}"'] \arrow[r, "\mathrm{opcart}"] \arrow[d, dotted, "\cong"'] & (M\odot N)\odot Q \arrow[d, dotted, "\cong"] \\
M\otimes (N\otimes Q) & \Delta^*_{BC}\big(M\otimes (N\otimes Q)\big) \arrow[l, "\mathrm{cart}"] \arrow[r, "\mathrm{opcart}"'] & M\odot (N\odot Q).
\end{tikzcd} \tag{66}$$

This isomorphism is natural because it is defined by unique factorization. The proof that it satisfies the pentagon axiom is similar to its construction: we tensor (65) with $R\colon D \nrightarrow E$, then use the Beck-Chevalley condition again for the square

$$\begin{tikzcd}
ABCDE \arrow[r] \arrow[d] & ABCDDE \arrow[d] \\
ADE \arrow[r] & ADDE
\end{tikzcd}$$

(where we omit the symbol $\times$) to obtain a span

$$((M\otimes N)\otimes Q)\otimes R \xleftarrow{\mathrm{cart}} \Delta^*_{BCD}\big(((M\otimes N)\otimes Q)\otimes R\big) \xrightarrow{\mathrm{opcart}} ((M\odot N)\odot Q)\odot R. \tag{67}$$

By uniqueness of factorizations, the isomorphism

$$((M\odot N)\odot Q)\odot R \cong (M\odot (N\odot Q))\odot R,$$

obtained by applying the functor $-\odot R$ to (66), is the same as the isomorphism obtained by factoring the isomorphism

$$((M\otimes N)\otimes Q)\otimes R \cong (M\otimes (N\otimes Q))\otimes R$$

through the span (67). Therefore, by inspecting the following diagram and using unique factorization again, we see that the pentagon axiom for $\otimes$ implies the pentagon axiom for $\odot$.

$$\begin{tikzcd}
 & & & ((M\odot N)\odot Q)\odot R \arrow[dlll, "\cong"'] \arrow[drrr, "\cong"] & & & \\
(M\odot N)\odot (Q\odot R) \arrow[ddddd, "\cong"] & & & {} \arrow[u] \arrow[d] \arrow[dll, "\cong"'] \arrow[drr, "\cong"] & & & (M\odot (N\odot Q))\odot R \arrow[ddddd, "\cong"] \\
 & {} \arrow[ul] \arrow[dr] \arrow[dddd, "\cong"] & & ((M\otimes N)\otimes Q)\otimes R \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & {} \arrow[ur] \arrow[dl] \arrow[dddd, "\cong"] & \\
 & & (M\otimes N)\otimes (Q\otimes R) \arrow[d, "\cong"] & & (M\otimes (N\otimes Q))\otimes R \arrow[d, "\cong"] & & \\
 & & M\otimes (N\otimes (Q\otimes R)) & & M\otimes ((N\otimes Q)\otimes R) \arrow[ll, "\cong"'] & & \\
 & {} \arrow[ur] \arrow[dl] & & & & {} \arrow[ul] \arrow[llll, "\cong"] \arrow[dr] & \\
M\odot (N\odot (Q\odot R)) & & & & & & M\odot ((N\odot Q)\odot R) \arrow[llllll, "\cong"]
\end{tikzcd}$$

Now we consider the left unit transformation. Let $M\colon A \nrightarrow B$ and recall that $U_A = (\Delta_A)_! \pi_A^* I$, so that we have

$$I \xleftarrow{\text{cart}} \pi_A^* I \xrightarrow{\text{opcart}} U_A\ .$$

Tensoring this with M and adding the arrows from (64) for $U_A \odot M$, we have

$$\begin{tikzcd}
 & & U_A \otimes M & & \\
I \otimes M & \pi_A^* I \otimes M \arrow[l, "\text{cart}"'] \arrow[ur, "\text{opcart}"] & & \Delta_A^*(U_A \otimes M) \arrow[ul, "\text{cart}"'] \arrow[r, "\text{opcart}"] & U_A \odot M. \\
 & & M \arrow[ull, bend left, "\cong", "\mathfrak{l}"'] \arrow[ul, dashed, "\text{cart}"'] \arrow[ur, dotted, "\text{opcart}"'] & &
\end{tikzcd} \tag{68}$$

The solid arrows marked cartesian or opcartesian are part of the chosen cleavage. The other solid arrow is the left unit constraint for $\otimes$, which is an isomorphism, hence also cartesian. The dashed arrow is cartesian by Proposition 3.4(ii), and the dotted arrow is opcartesian by the Beck-Chevalley condition for the pullback square

$$\begin{tikzcd}
A \times B \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \times A \times B \arrow[d, "\Delta \times 1"] \\
A \times A \times B \arrow[r, "1 \times \Delta"'] & A \times A \times A \times B.
\end{tikzcd} \tag{69}$$

Since the composite of the two opcartesian arrows on the right is opcartesian and lies over $1_{A\times B}$, it is an isomorphism

$$M \cong U_A \odot M$$

which we take as the left unit isomorphism for $\odot$. Its naturality follows, as before, from unique factorization. The right unit isomorphism is analogous.

We now show the unit axiom. We tensor the diagram

$$I \otimes M \xleftarrow{\cong} M \xrightarrow{\cong} U_A \odot M$$

with N and compose with the defining cartesian and opcartesian arrows for $\odot$ to obtain the following diagram.

$$\begin{tikzcd}
 & & N \otimes (U_A \odot M) & & \\
N \otimes (I \otimes M) & N \otimes M \arrow[l, "\cong"'] \arrow[ur, "\cong"] & & \Delta_A^*(N \otimes (U_A \odot M)) \arrow[ul, "\text{cart}"'] \arrow[r, "\text{opcart}"] & N \odot (U_A \odot M). \\
 & & \Delta_A^*(N \otimes (I \otimes M)) \arrow[ul, dashed, "\text{cart}"] \arrow[ur, dotted, "\cong", "\text{opcart}"'] & &
\end{tikzcd} \tag{70}$$

We do the same for $(N \odot U_A) \odot M$. By universal factorization, the two unit isomorphisms

$$N \odot (U_A \odot M) \cong N \odot M$$
$$(N \odot U_A) \odot M \cong N \odot M$$

are given by factorization through these (op)cartesian arrows, as is the associativity isomorphism

$$(N \odot U_A) \odot M \cong N \odot (U_A \odot M).$$

Thus, as for the pentagon, the unit axiom for $\otimes$ implies the unit axiom for $\odot$.

This shows that $\mathbb{P}$ is a double category. Since the pullback of a bifibration is a bifibration, (L, R) is a bifibration. Thus, by Theorem 4.1, $\mathbb{P}$ is a framed bicategory.

We now assume that Φ is symmetric and show that $\mathbb{P}$ is a symmetric monoidal framed bicategory. Since $\mathbb{P}_0 = \mathscr{B}$, it is already (cartesian) symmetric monoidal. The monoidal structure of $\mathbb{P}_1$ is almost the same as that of $\mathscr{A}$, but with a slight twist. If $M\colon A \to\!\!\!\!\!+ B$ and $N\colon C \to\!\!\!\!\!+ D$, so that $\Phi(M) = A \times B$ and $\Phi(N) = C \times D$, then we have

$$\Phi(M \otimes N) = (A \times B) \times (C \times D)$$

whereas the product of M and N in $\mathbb{P}$ should be an object of $\mathscr{A}$ lying over $(A\times C)\times(B\times D)$. But the chosen cleavage gives us a cartesian arrow ending at $M \otimes N$ lying over the unique constraint

$$(A \times C) \times (B \times D) \cong (A \times B) \times (C \times D),$$

and we call its domain $M \otimes' N$. Since cartesian arrows over isomorphisms are isomorphisms, we have $M \otimes' N \cong M \otimes N$. Similarly, the unit for $\mathscr{A}$ should be $U_1 = (\Delta_1)_!(\pi_1)^* I$, and since $\pi_1 = 1_1$ and Δ_1 is the unique isomorphism $1 \cong 1 \times 1$ we have $U_1 \cong I$; we define $I' = U_1$. The constraints and coherence axioms for $\otimes$ and I pass across these isomorphisms to make $\mathbb{P}_1$ a symmetric monoidal category under $\otimes'$, with (L, R) a strict symmetric monoidal functor.

Thus, to make $\mathbb{P}$ a symmetric monoidal framed bicategory, it remains to construct coherent interchange and unit isomorphisms and show that the monoidal associativity and unit constraints are framed transformations. Our by-now familiar procedure gives the following diagram for the interchange isomorphism.

$$\begin{tikzcd}
(M \otimes' P) \otimes (N \otimes' Q) \arrow[d, "\cong"'] & \Delta^*\big((M \otimes' P) \otimes (N \otimes' Q)\big) \arrow[l, "\text{cart}"'] \arrow[r, "\text{opcart}"] \arrow[d, "\cong"'] & (M \otimes' P) \odot (N \otimes' Q) \arrow[d, "\cong"] \\
(M \otimes N) \otimes' (P \otimes Q) & \Delta^*(M \otimes N) \otimes' \Delta^*(P \otimes Q) \arrow[l, "\text{cart}"] \arrow[r, "\text{opcart}"'] & (M \odot N) \otimes (P \odot Q).
\end{tikzcd}$$

For the the unit isomorphism we have

$$\begin{tikzcd}
I \arrow[d, "\cong"'] & \pi^*_{A\times B} I \arrow[l, "\text{cart}"'] \arrow[r, "\text{opcart}"] \arrow[d, "\cong"'] & U_{A\times B} \arrow[d, "\cong"] \\
I \otimes I & \pi^*_A I \otimes \pi^*_B I \arrow[l, "\text{cart}"] \arrow[r, "\text{opcart}"'] & U_A \otimes' U_B.
\end{tikzcd}$$

As before, by factoring known commuting diagrams through cartesian and opcartesian arrows, we can show that these constraints are framed transformations and satisfy the monoidal coherence axioms. ∎

17.2. Corollary. *If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a strongly co-BC monoidal bifibration where $\mathscr{B}$ is cocartesian monoidal, then there is a framed bicategory $\mathbb{F}\mathbf{r}(\Phi)$ defined as in Proposition 17.1, except that composition is given by*

$$M \odot N = \eta^* \nabla_!(M \otimes N),$$

units are given by

$$U_A = \nabla^* \eta_! I,$$

and similarly for the other data. If Φ is symmetric, then $\mathbb{F}\mathbf{r}(\Phi)$ is symmetric monoidal.

Proof. Simply apply Proposition 17.1 to the strongly BC monoidal bifibration $\Phi^{op}\colon \mathscr{A}^{op} \to \mathscr{B}^{op}$, since $\mathscr{B}^{op}$ is cartesian monoidal. ∎

We now consider the case when Φ is only weakly BC. Most of the pullback squares for which we used the Beck-Chevalley condition in Proposition 17.1 had one of their legs a product projection, so those parts of the proof carry over with no problem. However, there was one which involved only diagonal maps, and this is the problem that Lemma 16.4 was designed to solve. This is essentially the same method as that used in [MS06, Ch. 17] for the case of Ho($\mathtt{Sp}$).

17.3. Proposition. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a weakly BC and internally closed monoidal bifibration, where $\mathscr{B}$ is cartesian monoidal. Then the same definitions as in Proposition 17.1 give a framed bicategory, which is symmetric monoidal if Φ is.*

Proof. There is only one place in the proof of Theorem 14.2 where we used a Beck-Chevalley property for a 'bad' square: in proving that the unit transformation is an isomorphism, using the square (69). In this case, the dotted arrow in (68) which we want to be opcartesian is defined by unique factorization from a square of the form

$$\begin{tikzcd} M \arrow[r, dotted] \arrow[d, "\text{cart}"'] & \Delta_A^*(U_A \otimes M) \arrow[d, "\text{cart}"] \\ \pi_A^* I \otimes M \arrow[r, "\text{opcart}\otimes 1"'] & U_A \otimes M. \end{tikzcd} \tag{71}$$

This is almost of the form (63), but not quite, since it lies over the square

$$\begin{tikzcd} AB \arrow[r, "\Delta_A B"] \arrow[d, "\Delta_A B"'] & AAB \arrow[d, "A\Delta_A B"] \\ AAB \arrow[r, "\Delta_A AB"'] & AAAB \end{tikzcd}$$

which is not of the form (62). But we can decompose it into another pair of squares:

$$\begin{tikzcd}
AB \arrow[r, "\Delta_A B"] \arrow[d, "\Delta_{AB}"] & AAB \arrow[d, "\Delta_{AAB}"'] \\
ABAB \arrow[r, "\Delta_A B \Delta_A B"] \arrow[d, "\pi_B"] & AABAAB \arrow[d, "\pi_A \pi_B"'] \\
AAB \arrow[r, "\Delta_A AB"'] & AAAB
\end{tikzcd}$$

Here the top square is of the form (62), where $f = \Delta_A B$. If we then construct (71) by lifting in stages, we obtain

$$\begin{tikzcd}
M \arrow[r, dotted] \arrow[d, "\mathrm{cart}"] & \Delta_A^*(U_A \otimes M) \arrow[d, "\mathrm{cart}"'] \\
\pi_{AB}^* I \otimes M \arrow[r, dashed] \arrow[d, "\mathrm{cart}"] & \pi_B^* U_A \otimes \pi_A^* M \arrow[d, "\mathrm{cart}"'] \\
\pi_A^* I \otimes M \arrow[r, "\mathrm{opcart}\otimes 1"'] & U_A \otimes M.
\end{tikzcd} \tag{72}$$

where the outer rectangle is the same as (71). We can then obtain the bottom square as the product of a square

$$\begin{tikzcd}
\pi_{AB}^* I \arrow[r, dashed] \arrow[d, "\mathrm{cart}"'] & \pi^* U_A \arrow[d, "\mathrm{cart}"] \\
\pi_A^* I \arrow[r, "\mathrm{opcart}"'] & U_A,
\end{tikzcd}$$

where the dashed arrow is opcartesian by the Beck-Chevalley condition, and a square

$$\begin{tikzcd}
M \arrow[r, dashed] \arrow[d, equal] & \pi_A^* M \arrow[d, "\mathrm{cart}"] \\
M \arrow[r, equal] & M.
\end{tikzcd}$$

where the dashed arrow is cartesian by Proposition 3.4(ii). Thus the dashed arrow in (72) is of the form opcart $\otimes$ cart, so by Lemma 16.4, the dotted arrow is opcartesian as desired. Since the unit transformation that we have just shown to be an isomorphism is the same as the transformation defined in the proof of Theorem 14.2, the same proof of the coherence axioms applies. ∎

Note that Corollary 16.3 shows that any monoidal framed bicategory with cartesian base is weakly BC, so being weakly BC is a *necessary* condition for the construction of Theorem 14.2 to give a framed bicategory. We do not know whether being weakly BC is sufficient for frameability without closedness, but we suspect not.

17.4. Proposition. *If Φ is a frameable monoidal bifibration, then $\mathbb{F}\mathbf{r}(\Phi)$ has a vertically strict involution given by the identity on objects and $M^{op} = \mathfrak{s}^* M$ on 1-cells. If Φ is symmetric, this involution is symmetric monoidal.*

Proof. Left to the reader. ∎

17.5. Proposition. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a frameable monoidal $*$-bifibration which is externally closed. Then the resulting framed bicategory $\mathbb{F}\mathbf{r}(\Phi)$ is closed.*

Proof. Define $N \rhd P = N \mathrel{\overline{\rhd}} \Delta_* \pi^* P$. Writing $\mathcal{D}$ for the horizontal bicategory of $\mathbb{F}\mathbf{r}(\Phi)$, we have

$$\begin{aligned}
\mathcal{D}(M \odot N, P) &= \mathcal{D}(\pi_! \Delta^*(M \otimes N), P)\\
&\cong \mathcal{D}(\Delta^*(M \otimes N), \pi^* P)\\
&\cong \mathcal{D}(M \otimes N, \Delta_* \pi^* P)\\
&\cong \mathcal{D}(M, N \mathrel{\overline{\rhd}} \Delta_* \pi^* P)\\
&= \mathcal{D}(M, N \rhd P).
\end{aligned}$$

The construction of $\lhd$ is similar. ∎

17.6. Proposition. *Let Φ be an externally closed and strongly BC monoidal $*$-bifibration in which $\mathscr{B}$ is cocartesian monoidal. Then the resulting framed bicategory $\mathbb{F}\mathbf{r}(\Phi)$ is closed.*

Proof. Define $N \rhd P = N \mathrel{\overline{\rhd}} \nabla^* \eta_* P$. Again writing $\mathcal{D}$ for the horizontal bicategory of $\mathbb{F}\mathbf{r}(\Phi)$, we have

$$\begin{aligned}
\mathcal{D}(M \odot N, P) &= \mathcal{D}(\eta^* \nabla_!(M \otimes N), P)\\
&\cong \mathcal{D}(\nabla_!(M \otimes N), \eta_* P)\\
&\cong \mathcal{D}(M \otimes N, \nabla^* \eta_* P)\\
&\cong \mathcal{D}(M, N \mathrel{\overline{\rhd}} \nabla^* \eta_* P)\\
&= \mathcal{D}(M, N \rhd P).
\end{aligned}$$

The construction of $\lhd$ is similar. ∎

Finally, we sketch the proof of Theorem 14.9.

17.7. Proposition. *The construction of Theorem 14.2 extends to a 2-functor*

$$\mathbb{F}\mathbf{r}\colon \mathcal{MF}^{\mathrm{fr}} \longrightarrow \mathcal{F}r\mathcal{B}i$$

and similarly for oplax and lax morphisms.

Sketch of Proof. Let $F\colon \Phi \to \Psi$ be a morphism in the appropriate domain category. We define $\mathbb{F}\mathbf{r}(F)$ to be F_0 on vertical categories. If $M\colon A \nrightarrow B$, so that $\Phi(M) = A \times B$ and thus $\Psi(F_1(M)) = F_0(A \times B)$, we let $\mathbb{F}\mathbf{r}(F)(M) = (F_\times)_! F_1(M)$, where $F_\times\colon F_0(A \times B) \to F_0 A \times F_0 B$ is the unique oplax constraint downstairs (which is an isomorphism if F is strong or lax).

The horizontal composition and units are built out of the monoidal structure and the functors f^* and $f_!$, so the lax or oplax constraints for these induce lax or oplax constraints for a strong double functor. For example, suppose $F\colon \Phi \to \Psi$ is a lax monoidal morphism of fibrations and that $M\colon A \nrightarrow B$ and $N\colon B \nrightarrow C$ are horizontal 1-cells in $\mathbb{F}\mathbf{r}(\Phi)$. Then $M \odot N$ comes with a diagram

$$M \otimes N \xleftarrow{\text{cart}} \Delta^*(M \otimes N) \xrightarrow{\text{opcart}} M \odot N \tag{73}$$

lying over

$$A \times B \times B \times B \longleftarrow A \times B \times C \longrightarrow A \times C. \tag{74}$$

Applying F to (74), we obtain the following diagram (omitting the symbol $\times$).

$$\begin{tikzcd}
F(ABBC) & F(ABC) \arrow[l] \arrow[r] & F(AC) \\
(FA)(FB)(FB)(FC) \arrow[u, "\cong"'] & (FA)(FB)(FC) \arrow[l] \arrow[r] \arrow[u, "\cong"'] & (FA)(FC). \arrow[u, "\cong"']
\end{tikzcd} \tag{75}$$

Applying F to (73), and adding the defining arrows for $FM \odot FN$, we obtain

$$\begin{tikzcd}
F(M \otimes N) & F(\Delta^*(M \otimes N)) \arrow[l, "\text{cart}"'] \arrow[r] & F(M \odot N) \\
FM \otimes FN \arrow[u] & \Delta^*(FM \otimes FN) \arrow[l, "\text{cart}"] \arrow[r, "\text{opcart}"'] \arrow[u, dashed] & FM \odot FN \arrow[u, dotted]
\end{tikzcd}$$

The dashed and dotted arrows follow by factoring the lax constraint of F through the given cartesian and opcartesian arrows. Since F does not preserve opcartesian arrows, the top-right solid arrow is not necessarily opcartesian, but this does not matter. The unit constraint is similar.

Finally, the oplax case is dual to this; the only difference is that all the vertical arrows go the other way, and in (75) they are no longer isomorphisms. ∎

A. Connection pairs

As mentioned in §1, the questions which led us to framed bicategories have been addressed by others in several ways. In this section we explain how framed bicategories are related to connection pairs on a double category; in the other appendices we consider their relationship to various parts of bicategory theory.

For further detail on connection pairs, we refer the reader to [BS76, BM99] and also to [Fio06], which proved that connection pairs are equivalent to 'foldings'. Our presentation of the theory differs from the usual one because we focus on the pseudo case, which turns out to *simplify* the definition greatly. The following terminology is from [GP04, PPD06].

A.1. Definition. Let $\mathbb{D}$ be a double category and $f\colon A \to B$ a vertical arrow. A **companion** for f is a horizontal 1-cell ${}_fB\colon A \nrightarrow B$ together with 2-cells

$$\begin{tikzcd} {} \arrow[r, "{{}_fB}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "U_B"', "\shortmid"{marking}] & {} \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} {} \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, "f"] \\ {} \arrow[r, "{{}_fB}"', "\shortmid"{marking}] & {} \end{tikzcd}$$

such that the following equations hold.

$$\begin{tikzcd} {} \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, "f"] \\ {} \arrow[r, "{{}_fB}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "U_B"', "\shortmid"{marking}] & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & {} \arrow[d, "f"] \\ {} \arrow[r, "U_B"', "\shortmid"{marking}] & {} \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[rr, "{{}_fB}", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & {} \arrow[d, equal] \\ {} \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[r, "{{}_fB}", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "{{}_fB}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & {} \arrow[r, "U_B"', "\shortmid"{marking}] & {} \arrow[d, equal] \\ {} \arrow[rr, "{{}_fB}"', "\shortmid"{marking}] & & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, "{{}_fB}", "\shortmid"{marking}] \arrow[d, equal] & {} \arrow[d, equal] \\ {} \arrow[r, "{{}_fB}"', "\shortmid"{marking}] & {} \end{tikzcd}$$

A **conjoint** for f is a horizontal 1-cell $B_f\colon B \nrightarrow A$ together with 2-cells

$$\begin{tikzcd} {} \arrow[r, "B_f", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, "f"] \\ {} \arrow[r, "U_B"', "\shortmid"{marking}] & {} \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} {} \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "B_f"', "\shortmid"{marking}] & {} \end{tikzcd}$$

such that the following equations hold.

$$\begin{tikzcd} {} \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "B_f", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, "f"] \\ {} \arrow[r, "U_B"', "\shortmid"{marking}] & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & {} \arrow[d, "f"] \\ {} \arrow[r, "U_B"', "\shortmid"{marking}] & {} \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[rr, "B_f", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & {} \arrow[d, equal] \\ {} \arrow[r, "{{}_fB}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[r, "U_A", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d, equal] \\ {} \arrow[r, "U_B"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & {} \arrow[r, "B_f"', "\shortmid"{marking}] & {} \arrow[d, equal] \\ {} \arrow[rr, "B_f"', "\shortmid"{marking}] & & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, "B_f", "\shortmid"{marking}] \arrow[d, equal] & {} \arrow[d, equal] \\ {} \arrow[r, "B_f"', "\shortmid"{marking}] & {} \end{tikzcd}$$

Comparing this definition with Theorem 4.1(iii), the following becomes evident.

A.2. Theorem. *A double category is a framed bicategory exactly when every vertical arrow has both a companion and a conjoint.*

One can prove in general that companions and conjoints are unique up to canonical isomorphism, that $({}_fB, B_f)$ is a dual pair if both are defined, and that the operations $f \mapsto {}_fB$ and $f \mapsto B_f$ are pseudofunctorial insofar as they are defined.

The following definition is then easily seen to be equivalent to those given in [BS76, BM99, Fio06]. Because it was originally motivated by double categories like the 'quintets' of a 2-category, it includes only companions and not conjoints.

A.3. Definition. Let $\mathbb{D}$ be a strict double category. A **connection pair** on $\mathbb{D}$ is a choice of a companion ${}_fB$ each vertical arrow f such that the pseudofunctor $f \mapsto {}_fB$ is a strict 2-functor.

Thus, an arbitrary choice of companions on a non-strict double category may be called a 'pseudo connection pair', and a choice of conjoints may be called a 'pseudo op-connection pair'. Theorem A.2 then states that a double category is a framed bicategory precisely when it admits both a pseudo connection pair and a pseudo op-connection pair.

B. Biequivalences, biadjunctions, and monoidal bicategories

We now consider the question of how much of the structure of a framed bicategory $\mathbb{D}$ is reflected in its underlying bicategory $\mathcal{D}$. Note that any bicategory may be considered as a framed bicategory with only identity vertical arrows; we call such framed bicategories **vertically discrete**. If $\mathcal{F}r\mathcal{B}i_0$ denotes the underlying 1-category of $\mathcal{F}r\mathcal{B}i$ and **Bicat** denotes the 1-category of bicategories and pseudo 2-functors, we have an adjunction

$$\mathbf{Bicat} \rightleftarrows \mathcal{F}r\mathcal{B}i_0. \tag{76}$$

in which the left adjoint considers a bicategory as a vertically discrete framed bicategory, while the right adjoint takes a framed bicategory to its underlying horizontal bicategory.

The left adjoint $\mathbf{Bicat} \to \mathcal{F}r\mathcal{B}i_0$ does not extend to a 2-functor or 3-functor, but in the other direction, any framed transformation $\alpha\colon F \to G\colon \mathbb{D} \to \mathbb{E}$ can be 'lifted' to an oplax transformation between the underlying pseudofunctors as follows. For an object $A \in \mathbb{D}$, we define

$$\widetilde{\alpha}_A = (\alpha_A)^*(U_{GA})\colon FA \mathrel{\mapstochar\rightarrow} GA.$$

For a horizontal 1-cell $M\colon A \mathrel{\mapstochar\rightarrow} B$, we define

$$\widetilde{\alpha}_M\colon FM \odot \widetilde{\alpha}_B \cong (FM)(\alpha_B)_! \longrightarrow (\alpha_A)^*(GM) \cong \widetilde{\alpha}_A \odot GM$$

to be the globular 2-cell corresponding to $\alpha_M\colon FM \underset{\alpha_A}{\overset{\alpha_B}{\Longrightarrow}} GM$. It is easy to check that this defines an oplax natural transformation between the pseudo 2-functors induced by F and G.

If $\mathcal{D}$ and $\mathcal{E}$ are bicategories, we write $\mathfrak{Bicat}_{op\ell}(\mathcal{D},\mathcal{E})$ for the bicategory of pseudo 2-functors, oplax natural transformations, and modifications from $\mathcal{D}$ to $\mathcal{E}$. By the pseudo-functoriality of base change, Proposition 4.5, the above construction defines a pseudo-functor

$$\mathcal{F}r\mathcal{B}i(\mathbb{D},\mathbb{E}) \longrightarrow \mathfrak{Bicat}_{op\ell}(\mathcal{D},\mathcal{E}). \tag{77}$$

We can also allow lax or oplax functors on both sides. Note, however, that framed transformations always give rise to *oplax* natural transformations.

We would like to say that this construction extends to a functor from $\mathcal{F}r\mathcal{B}i$ to the tricategory of bicategories, but unfortunately there is no tricategory of bicategories which includes oplax natural transformations, since the composition operation

$$\mathfrak{Bicat}_{op\ell}(\mathcal{F},\mathcal{E}) \times \mathfrak{Bicat}_{op\ell}(\mathcal{D},\mathcal{E}) \to \mathfrak{Bicat}_{op\ell}(\mathcal{D},\mathcal{F})$$

would be only an oplax 2-functor. We could allow the codomain to be a sort of 'oplax tricategory', such as the 'bicategory op-enriched categories' of [Ver92, 1.3], but this would take us too far afield. Instead, we merely observe that if α happens to be a framed natural isomorphism, then $\widetilde{\alpha}$ is a pseudo natural equivalence. This suffices to prove the following.

B.1. PROPOSITION. *An equivalence of framed bicategories induces a biequivalence of horizontal bicategories.*

PROOF. If F, G are inverse equivalences in $\mathcal{F}r\mathcal{B}i$, then they give rise to pseudofunctors, and by the above observation, the framed natural isomorphisms $FG \cong \mathrm{Id}$ and $\mathrm{Id} \cong GF$ give rise to pseudo natural equivalences. ∎

For example, this implies that in Example 7.9, the horizontal bicategories $G\mathcal{E}x_{G/H}$ and $H\mathcal{E}x$ actually *are* biequivalent. However, we believe the equivalence is more naturally stated, and easier to work with, in $\mathcal{F}r\mathcal{B}i$.

In a similar way, we can lift a monoidal structure on a framed bicategory to a monoidal structure on its horizontal bicategory. Many examples of monoidal bicategories actually arise from monoidal framed bicategories. This is useful, because monoidal bicategories are complicated 'tricategorical' objects, whereas monoidal framed bicategories are much easier to get a handle on. See [GPS95, Gur06, CG07] for a definition of monoidal bicategory.

B.2. THEOREM. *If $\mathbb{D}$ is a monoidal framed bicategory, then any cleavage for $\mathbb{D}$ makes $\mathcal{D}$ into a monoidal bicategory in a canonical way.*

SKETCH OF PROOF. $\mathcal{D}$ already has a product and a unit object induced from $\mathbb{D}$, so it suffices to construct the constraints and coherence. We consider the associativity constraints, leaving the unit constraints to the reader. Since $\mathbb{D}$ is a monoidal double category, it has a *vertical* associativity constraint

$$\mathfrak{a}\colon (A\otimes B)\otimes C \cong A\otimes(B\otimes C).$$

But since $\mathbb{D}$ is a framed bicategory, this vertical isomorphism can be 'lifted' to an equivalence in $\mathcal{D}$:

$$\widetilde{\mathfrak{a}} = \mathfrak{a}^*\big((A\otimes B)\otimes C\big)\colon (A\otimes B)\otimes C \twoheadrightarrow A\otimes(B\otimes C)$$

with adjoint inverse $\big((A \otimes B) \otimes C\big)\mathfrak{a}^*$; this will be the associativity equivalence for the monoidal bicategory $\mathcal{D}$. We need further a 'pentagonator' 2-isomorphism

$$\widetilde{\mathfrak{a}} \circ \widetilde{\mathfrak{a}} \circ \widetilde{\mathfrak{a}} \cong \widetilde{\mathfrak{a}} \circ \widetilde{\mathfrak{a}}.$$

But the coherence pentagon for the vertical isomorphism $\mathfrak{a}$ tells us that

$$\mathfrak{a} \circ \mathfrak{a} \circ \mathfrak{a} = \mathfrak{a} \circ \mathfrak{a}$$

and since base change objects are pseudofunctorial by Proposition 4.5, this *equality* in $\mathbb{D}_0$ becomes a canonical *isomorphism* in $\mathcal{D}$, which we take as the pentagonator.

It remains to check that this pentagonator satisfies the 'cocycle equation' for relations between quintuple products. However, since all the pentagonators are defined by universal properties (being canonical isomorphisms between two cartesian arrows), both sides of the cocycle equation are also characterized by the same universal property, and therefore must be equal. ∎

In contrast to these well-behaved cases, a framed adjunction $F \colon \mathbb{D} \rightleftarrows \mathbb{E} : G$ does not generally give rise to a biadjunction $\mathcal{D} \rightleftarrows \mathcal{E}$. It does, however, give rise to a *local adjunction* in the sense of [BP88]; this consists of an oplax 2-functor $F \colon \mathcal{D} \to \mathcal{E}$, a lax 2-functor $G \colon \mathcal{E} \to \mathcal{D}$, and an adjunction

$$\mathcal{D}(A, GB) \rightleftarrows \mathcal{E}(FA, B). \tag{78}$$

In a biadjunction, F and G would be pseudo 2-functors and (78) would be an equivalence.

When F and G arise from a framed or op-framed adjunction, a local adjunction (78) is given by

$$\mathcal{D}(A, GB) \underset{\eta^*(G-)}{\overset{(F-)\varepsilon_!}{\rightleftarrows}} \mathcal{E}(FA, B)\,.$$

Of course, in a framed adjunction F is strong, while in an op-framed adjunction G is strong. A bit more 2-category theory than we have discussed here (see [Kel74]) gives us a notion of 'lax/oplax' framed adjunction, in which the left adjoint is oplax and the right adjoint is lax; these also give rise to local adjunctions between horizontal bicategories.

In this way, practically any framed-bicategorical notion gives rise to a counterpart on the purely bicategorical level. For example, by a process similar to that in Theorem B.2, any involution on $\mathbb{D}$ gives rise to a 'bicategorical involution' on $\mathcal{D}$.

Of course, we can also define monoids and bimodules in any bicategory; in this context monoids are often called *monads*, since in $\mathcal{C}at$ they reduce to the usual notion of monad. The fact that both internal and enriched categories are monoids in appropriate bicategories is well-known, and bicategory theorists have studied categories enriched in a bicategory as a generalization of categories enriched in a monoidal category; see [Wal81, Str81, Str83a, Str83b, CJSV94, KLSS02].

However, pure bicategory theory usually starts to break down whenever we need to use vertical arrows. For example, it is harder to get a handle on internal or enriched *functors* purely bicategorically. In the next appendix we introduce a structure called an *equipment* which is sometimes used for this purpose, for example in [LS02].

C. Equipments

For the theory of equipments we refer the reader to [Woo82, Woo85, CJSV94, Ver92]. From our point of view, it is natural to introduce them by asking how the vertical arrows of $\mathbb{D}$ are reflected in $\mathcal{D}$. We know that there is a pseudofunctor $\mathbb{D}_0 \to \mathcal{D}$ sending $f\colon A \to B$ to the base change object ${}_fB$; this pseudofunctor is bijective on objects and each ${}_fB$ has a right adjoint in $\mathcal{D}$. Thus we almost have an instance of the following structure.

C.1. DEFINITION. A **proarrow equipment** is a pseudo 2-functor $\overline{(-)}\colon \mathcal{K} \to \mathcal{M}$ between bicategories such that

(i) $\mathcal{K}$ and $\mathcal{M}$ have the same objects and $\overline{(-)}$ is the identity on objects;

(ii) For every arrow f in $\mathcal{K}$, $\overline{f}$ has a right adjoint $\widetilde{f}$ in $\mathcal{M}$; and

(iii) $\overline{(-)}$ is locally full and faithful.

The only difference is that in an equipment, $\mathcal{K}$ is a bicategory rather than the 1-category $\mathbb{D}_0$, but condition (iii) means that the 2-cells in $\mathcal{K}$ are determined by those in $\mathcal{M}$ anyway. Thus, given a framed bicategory $\mathbb{D}$, we can factor the base-change object pseudofunctor $\mathbb{D}_0 \to \mathcal{D}$ as

$$\mathbb{D}_0 \xrightarrow{\;i\;} \mathcal{K} \xrightarrow{\overline{(-)}} \mathcal{D}$$

where i is bijective on objects and 1-cells and $\overline{(-)}$ is locally full and faithful. The objects and morphisms of $\mathcal{K}$ are those of $\mathbb{D}_0$, and its 2-cells from $f \to g$ are the 2-cells ${}_fB \to {}_gB$ in $\mathcal{D}$. We have proven the following.

C.2. PROPOSITION. *If $\mathbb{D}$ is a framed bicategory, then the above pseudofunctor $\overline{(-)}$ is a proarrow equipment.*

Note that in the proarrow equipment arising from a framed bicategory, the bicategory $\mathcal{K}$ is actually a strict 2-category. However, this is essentially the only restriction on the equipments which arise in this way.

C.3. PROPOSITION. *Let $\overline{(-)}\colon \mathcal{K} \to \mathcal{M}$ be a proarrow equipment such that $\mathcal{K}$ is a strict 2-category. Define a double category $\mathbb{D}$ whose*

- *Objects are those of $\mathcal{K}$ (and $\mathcal{M}$);*
- *Vertical arrows are the arrows of $\mathcal{K}$;*
- *Horizontal 1-cells are the arrows of $\mathcal{M}$; and*
- *2-cells*

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "N"'] & D \end{tikzcd}$$

are the 2-cells $\alpha\colon M \odot \overline{g} \longrightarrow \overline{f} \odot N$ in $\mathcal{M}$.

Then $\mathbb{D}$ is a framed bicategory.

SKETCH OF PROOF. First we show that $\mathbb{D}$ is a double category. The vertical composite

$$\begin{tikzcd}
\cdot \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "N"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & \cdot \arrow[d, "k"] \\
\cdot \arrow[r, "P"'] & \cdot
\end{tikzcd}$$

is defined to be the composite

$$\begin{aligned}
M \odot \overline{kg} &\cong M \odot (\overline{g} \odot \overline{k}) \\
&\cong (M \odot \overline{g}) \odot \overline{k} \\
&\xrightarrow{\alpha} (\overline{f} \odot N) \odot \overline{k} \\
&\cong \overline{f} \odot (N \odot \overline{k}) \\
&\xrightarrow{\beta} \overline{f} \odot (\overline{h} \odot P) \\
&\cong (\overline{f} \odot \overline{h}) \odot P \\
&\cong \overline{hf} \odot P.
\end{aligned}$$

The coherence theorems for bicategories and pseudofunctors imply that this is vertically associative and unital. Horizontal composition of 1-cells is defined as in $\mathcal{M}$, and horizontal composition of 2-cells is defined analogously to their vertical composition. The constraints come from those of $\mathcal{M}$.

Finally, for an arrow $f\colon A \to B$ in $\mathcal{K}$, and $\widetilde{f}$ the right adjoint of $\overline{f}$, it is easy to check that the 2-cells

$$\begin{tikzcd}
\cdot \arrow[r, "\overline{f}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow 1_{\overline{f}}"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "U_B"'] & \cdot
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
\cdot \arrow[r, "\widetilde{f}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\varepsilon"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "U_B"'] & \cdot
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
\cdot \arrow[r, "U_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta"] & \cdot \arrow[d, equal] \\
\cdot \arrow[r, "\widetilde{f}"'] & \cdot
\end{tikzcd}
\quad , \quad \text{and} \quad
\begin{tikzcd}
\cdot \arrow[r, "U_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow 1_{\overline{f}}"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "\overline{f}"'] & \cdot
\end{tikzcd}
\quad ,$$

defined by identities and by the unit and counit of the adjunction $\overline{f} \dashv \widetilde{f}$, satisfy the equations of Theorem 4.1(iii). Thus $\mathbb{D}$ is a framed bicategory. ∎

At the level of objects, it is easy to show that the two constructions are inverses up to isomorphism. In order to state this as an equivalence of 2-categories, however, we would need morphisms and especially transformations between equipments, and it is not immediately obvious how to define these.

The approach to constructing a 2-category of equipments taken in [Ver92] is essentially to first make equipments into double categories, as we have done, and *define* morphisms and transformations of equipments to be morphisms between the corresponding double categories. This makes our desired equivalence true by definition. Actually, [Ver92] uses 'doubly weak' double categories to deal with equipments where $\mathcal{K}$ is not a strict 2-category, and thus obtains a tricategory rather than a 2-category, but the idea is the same.

Thus, framed bicategories can be regarded as a characterization of the double categories which arise from equipments. However, since the correct notions of morphism and transformation are apparent only from the side of double categories, we believe it is more natural to work directly with framed bicategories.

C.4. Remark. The authors of [CKW91, CKVW98] consider a related notion of 'equipment' where $\mathcal{K}$ is replaced by a 1-category but the horizontal composition is forgotten. If $\mathbb{D}$ is a framed bicategory, then the span

$$\mathbb{D}_0 \xleftarrow{L} \mathbb{D}_1 \xrightarrow{R} \mathbb{D}_0 \tag{79}$$

has the property that L is a fibration, R is an opfibration, and the two types of base change commute, making it into a 'two-sided fibration' from $\mathbb{D}_0$ to $\mathbb{D}_0$ in the sense of [Str80]; these are essentially what [CKVW98] studies under the name 'equipment'. The fact that L is also an opfibration, and R a fibration, in a commuting way, make (79) into what they call a *starred pointed equipment*. This structure incorporates less of the structure of a framed bicategory, but it was sufficient in [CKW91, CKVW98] to obtain a 2-category or tricategory of equipments and a notion of equipment adjunction. It is easy to check that any framed adjunction gives rise to an equipment adjunction in their sense.

D. Epilogue: framed bicategories versus bicategories

We end with some more philosophical remarks about the relationship of framed bicategories to pure bicategory theory. For any bicategory $\mathcal{B}$, there is a canonical proarrow equipment $\overline{(-)}\colon \mathcal{K} \to \mathcal{B}$, where $\mathcal{K}$ is the bicategory of adjunctions $\overline{f} \dashv \widetilde{f}$ in $\mathcal{B}$. When $\mathcal{B}$ is a strict 2-category, so is $\mathcal{K}$, and the resulting framed bicategory is what we called $\mathbb{A}\mathbf{dj}(\mathcal{B})$ in Example 2.7. In general, we obtain a 'doubly weak' framed bicategory which we also call $\mathbb{A}\mathbf{dj}(\mathcal{B})$.

Thus, we can regard the theory of framed bicategories, or of equipments, as a generalization of the theory of bicategories in which we specify which adjunctions are the base change objects, rather than using all of them. A certain amount of pure bicategory theory can be regarded as implicitly working with the framed bicategory $\mathbb{A}\mathbf{dj}(\mathcal{B})$; frequently 1-cells with right adjoints are called *maps* and take on a special role. See, for example, [Str81] and [CKW87].

This purely bicategorical approach works well in bicategories like $\mathcal{D}ist(\mathscr{V})$, because, as we mentioned in Example 5.6(iii), the mild condition of 'Cauchy completeness' on the $\mathscr{V}$-categories involved is sufficient to ensure that any distributor with a right adjoint is isomorphic to a base change object. However, in other framed bicategories, such as $\mathbb{M}\mathbf{od}$ and $\mathbb{E}\mathbf{x}$, there will not be a good supply of 'Cauchy complete' objects, so framed bicategories or equipments are necessary. Moreover, even when working with $\mathcal{D}ist(\mathscr{V})$, framed bicategories are implicit in some of the bicategorical literature, such as the 'calculus of modules' for enriched categories; see, for example, [SW78, Woo82, CKW87, Str83a].

Finally, framed bicategories are much easier to work with than ordinary bicategories or equipments, because the vertical arrows form a strict 1-category rather than a weak bicategory. In situations where this fails, we can still use 'doubly weak' framed bicategories, as in [Ver92], but a good deal of simplicity is lost. However, in almost all examples, this strictness property does hold, and the virtue of framed bicategories is that they take advantage of this fact. For example, this is what enables us to define the strict 2-category $\mathcal{F}r\mathcal{B}i$ and apply the powerful methods of 2-category theory, rather than having to delve into the waters of tricategories.

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