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SNAKE LEMMA IN INCOMPLETE RELATIVE HOMOLOGICAL CATEGORIES

Dedicated to Dominique Bourn on the occasion of his 60th birthday

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ABSTRACT. The purpose of this paper is to prove a new, *incomplete-relative*, version of Non-abelian Snake Lemma, where "relative" refers to a distinguished class of normal epimorphisms in the ground category, and "incomplete" refers to omitting all completeness/cocompleteness assumptions not involving that class.

1. Introduction

The classical Snake Lemma known for abelian categories (see e.g. [5]) has been extended to homological categories by D. Bourn; see F. Borceux and D. Bourn [1], and references there. In [3], we extended it to the context of *relative homological categories*: here "relative" refers to a distinguished class $\mathbf{E}$ of regular epimorphisms in a ground category $\mathbf{C}$ satisfying certain conditions which, in particular, make $(\mathbf{C}, \mathbf{E})$ relative homological when (a) $\mathbf{C}$ is a homological category and $\mathbf{E}$ is the class of all regular epimorphisms in $\mathbf{C}$; (b) $\mathbf{C}$ is a pointed finitely complete category satisfying certain cocompleteness conditions, and $\mathbf{E}$ is the class of all isomorphisms in $\mathbf{C}$. In this paper we drop the completeness/cocompleteness assumption and extend Snake Lemma further to the context of what was called an *incomplete relative homological category* in [4] (see, however, the correction below Definition 2.1).

Let us recall the formulation of Snake Lemma from [1] (see Theorem 4.4.2 of [1]). It

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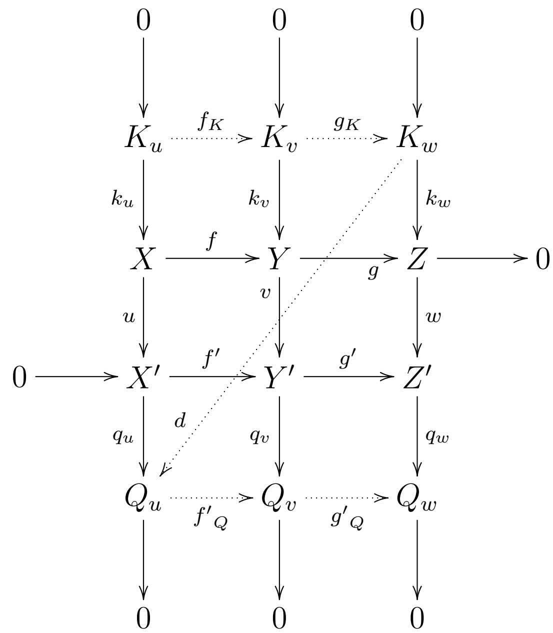
says: "Let **C** be a homological category. Consider the diagram



where all squares of plain arrows are commutative and all sequences of plain arrows are exact. There exists an exact sequence of dotted arrows making all squares commutative".

The proof given in [1] contains explicit constructions of the dotted arrows, which makes the result far more precise. This seems to suggest that these constructions should be included in the formulation of the lemma, but there is a problem: while the constructions of f_K, g_K, f'_Q, and g'_Q as induced morphisms are straightforward, the several-step construction of the "connecting morphism" $d : K_w \to Q_u$ is too long and technical. A natural solution of this problem, well-known in the abelian case, involves partial composition of internal relations in **C** and goes back at least to S. Mac Lane [6] (see also e.g. [2] for the so-called calculus of relations in regular categories): one should simply define d as the composite $q_u f'^{\circ} v g^{\circ} k_w$, where g° is the relation opposite to g, etc. We do not know if the non-abelian version of $d = q_u f'^{\circ} v g^{\circ} k_w$ is mentioned anywhere in the literature, but we use its relative version in our relative Snake Lemma (Theorem 3.1), which is the main result of this paper.

For the reader's convenience we are mostly using the same notation as in [1].

2. Incomplete relative homological categories

Throughout the paper we assume that **C** is a pointed category and **E** is a class of morphisms in **C** containing all isomorphisms. Let us recall from [4]:

2.1. DEFINITION. *A pair* $(\mathbf{C}, \mathbf{E})$ *is said to be an incomplete relative homological category if it satisfies the following conditions:*

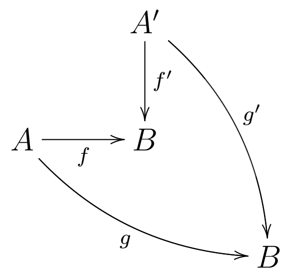
(a) Every morphism in **E** *is a normal epimorphism;*

(b) The class $\mathbf{E}$ *is closed under composition;*

(c) If $f \in \mathbf{E}$ *and* $gf \in \mathbf{E}$ *then* $g \in \mathbf{E}$*;*

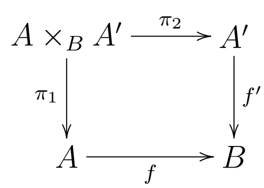
(d) If $f \in \mathbf{E}$ *then* $\ker(f)$ *exists in* $\mathbf{C}$*;*

(e) A diagram of the form



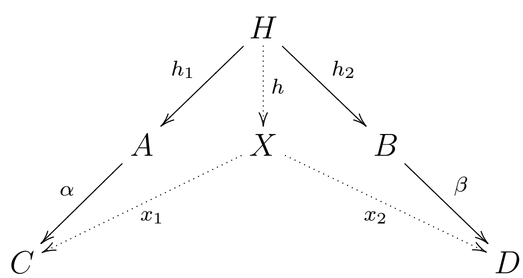
has a limit in $\mathbf{C}$ *provided* f *and* g *are in* $\mathbf{E}$*, and either (i)* $f = g$ *and* $f' = g'$*, or (ii)* f' *and* g' *are in* $\mathbf{E}$*,* (f, g) *and* (f', g') *are reflexive pairs, and* f *and* g *are jointly monic;*

(f) If



is a pullback and f *is in* $\mathbf{E}$*, then* π_2 *is also in* $\mathbf{E}$*;*

(g) If $h_1 : H \to A$ *and* $h_2 : H \to B$ *are jointly monic morphisms in* $\mathbf{C}$ *and if* $\alpha : A \to C$ *and* $\beta : B \to D$ *are morphisms in* $\mathbf{E}$*, then there exists a morphism* $h : H \to X$ *in* $\mathbf{E}$ *and jointly monic morphisms* $x_1 : X \to C$ *and* $x_2 : X \to D$ *in* $\mathbf{C}$ *making the diagram*



commutative (it easily follows from the fact that every morphism in $\mathbf{E}$ *is a normal epimorphism, that such factorization is unique up to an isomorphism);*

(h) *The* **E**-*Short Five Lemma holds in* **C**, *i.e. in every commutative diagram of the form*

$$\begin{array}{ccccc} K & \xrightarrow{k} & A & \xrightarrow{f} & B \\ \| & & \downarrow{\scriptstyle w} & & \| \\ K & \xrightarrow[k']{} & A' & \xrightarrow[f']{} & B \end{array}$$

with f *and* f' *in* **E** *and with* $k = \ker(f)$ *and* $k' = \ker(f')$, *the morphism* w *is an isomorphism;*

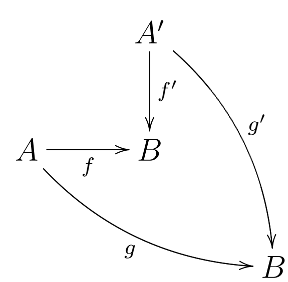
(i) *If in a commutative diagram*

$$\begin{array}{ccccc} K & \xrightarrow{k} & A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} & & \| \\ K' & \xrightarrow[k']{} & A' & \xrightarrow[f']{} & B \end{array}$$

f, f', *and* u *are in* **E**, $k = \ker(f)$ *and* $k' = \ker(f')$, *then there exists a morphism* $e : A \to M$ *in* **E** *and a monomorphism* $m : M \to A'$ *in* **C** *such that* $w = me$.

Let us use this opportunity to make a correction to conditions 2.1(c) and 3.1(c) of [4]. They should be replaced, respectively, with:

(a) A diagram of the form



has a limit in **C** provided f, g, f', and g' are in **E**, and either (i) $f = g$ and $f' = g'$, or (ii) (f, g) and (f', g') are reflexive pairs (that is, $fh = 1_B = gh$ and $f'h' = 1_B = g'h'$ for some h and h'), and f and g are jointly monic.

(b) Condition 2.1(e) of the present paper.

Note that this replacement will not affect any results/arguments of [4], except that without it a pair $(\mathbf{C}, \mathbf{Iso}(\mathbf{C}))$ would be an incomplete relative homological category only when **C** (is pointed and) admits equalizers of isomorphisms.

2.2. Remark. *As easily follows from condition 2.1(g), if a morphism* $f : A \to B$ *in* **C** *factors as* $f = em$ *in which* e *is in* **E** *and* m *is a monomorphism, then it also factors (essentially uniquely) as* $f = m'e'$ *in which* m' *is a monomorphism and* e' *is in* **E**.

2.3. Lemma. *If a pair $(\mathbf{C},\mathbf{E})$ satisfies conditions 2.1(a)-2.1(d), then $(\mathbf{C},\mathbf{E})$ satisfies conditions 2.1(h) and 2.1(i) if and only if in every commutative diagram of the form*

$$\begin{array}{ccccc} K & \xrightarrow{k} & A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} & & \| \\ K' & \xrightarrow[k']{} & A' & \xrightarrow[f']{} & B \end{array} \tag{2.1}$$

with $k = \ker(f)$, $k' = \ker(f')$, and with f, f', and u in $\mathbf{E}$, the morphism w is also in $\mathbf{E}$.

Proof. Suppose $(\mathbf{C},\mathbf{E})$ satisfies conditions 2.1(a)-2.1(d), 2.1(h) and 2.1(i). Consider the commutative diagram (2.1) with $k = \ker(f)$, $k' = \ker(f')$, and with f, f', and u in $\mathbf{E}$. By condition 2.1(i), $w = me$ where $e : A \to M$ is in $\mathbf{E}$ and $m : M \to A'$ is a monomorphism. Consider the commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{u} & K' \\ {\scriptstyle ek}\downarrow & \swarrow{\scriptstyle \bar{m}} & \downarrow{\scriptstyle k'} \\ M & \xrightarrow[m]{} & A' \end{array}$$

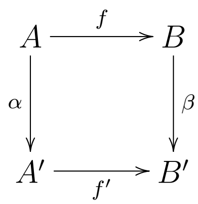
Since u is in $\mathbf{E}$ and m is a monomorphism, by condition 2.1(a) there exists a unique morphism $\bar{m} : K' \to M$ with $\bar{m}u = ek$ and $m\bar{m} = k'$; $\bar{m}$ is a monomorphism since so k'. Since $f'm\bar{m} = 0$, $\bar{m}$ is a monomorphism, and $k' = \ker(f')$, we conclude that $\bar{m} = \ker(f'm)$. By condition 2.1(c), $f'm$ is in $\mathbf{E}$, therefore we can apply the $\mathbf{E}$-Short Five Lemma to the diagram

$$\begin{array}{ccccc} K' & \xrightarrow{\bar{m}} & M & \xrightarrow{f'm} & B \\ \| & & \downarrow{\scriptstyle m} & & \| \\ K' & \xrightarrow[k']{} & A' & \xrightarrow[f']{} & B \end{array}$$

and conclude that m is an isomorphism. Hence, by condition 2.1(b) w is in $\mathbf{E}$, as desired.

Conversely, suppose for every commutative diagram (2.1) with $k = \ker(f)$, $k' = \ker(f')$, and with f and f' in $\mathbf{E}$, if u is in $\mathbf{E}$ then w is also in $\mathbf{E}$. It is a well know fact that under the assumptions of condition 2.1(h), $\ker(w) = 0$; moreover, since $\mathbf{E}$ contains all isomorphisms and f and f' are in $\mathbf{E}$, $w : A \to A'$ is also in $\mathbf{E}$. Since every morphism in $\mathbf{E}$ is a normal epimorphism, we conclude that w is an isomorphism, proving condition 2.1(h). The proof of condition 2.1(i) is trivial. ∎

2.4. Lemma. *Let* $(\mathbf{C}, \mathbf{E})$ *be a pair satisfying conditions 2.1(a)-2.1(d) and 2.1(g). Consider the commutative diagram:*

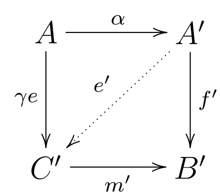


(2.2)

(i) If $\alpha : A \to A'$ *and* $\beta : B \to B'$ *are in* $\mathbf{E}$ *and if* $f : A \to B$ *factors as* $f = me$ *in which* e *is in* $\mathbf{E}$ *and* m *is a monomorphism, then* $f' : A' \to B'$ *also factors as* $f' = m'e'$ *in which* e' *is in* $\mathbf{E}$ *and* m' *is monomorphism.*

(ii) If $\alpha : A \to A'$ *and* $\beta : B \to B'$ *are monomorphisms and if* $f' : A' \to B'$ *factors as* $f' = m'e'$ *in which* e' *is in* $\mathbf{E}$ *and* m' *is a monomorphism, then* $f : A \to B$ *also factors as* $f = me$ *in which* e *is in* $\mathbf{E}$ *and* m *is a monomorphism.*

Proof. (i): Consider the commutative diagram (2.2) and suppose α and β are in $\mathbf{E}$ and $f = me$ in which $e : A \to C$ is in $\mathbf{E}$ and $m : C \to B$ is a monomorphism. Since β is in $\mathbf{E}$ and m is a monomorphism, by Remark 2.2 there exists a morphism $\gamma : C \to C'$ in $\mathbf{E}$ and a monomorphism $m' : C' \to B'$ such that $\beta m = m'\gamma$. Consider the commutative diagram:

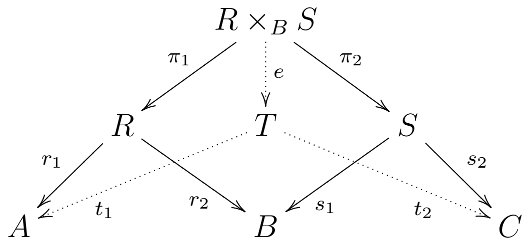


Since α is in $\mathbf{E}$ and m' is a monomorphism, conditions 2.1(a) and 2.1(d) imply the existence of a unique morphism $e' : A' \to C'$ with $e'\alpha = \gamma e$ and $m'e' = f'$. Since α, e, and γ are in $\mathbf{E}$, conditions 2.1(b) and 2.1(c) imply that e' is also in $\mathbf{E}$. Hence, $f' = m'e'$ in which e' is in $\mathbf{E}$ and m' is a monomorphism, as desired.

(ii) can be proved similarly. ∎

Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative homological category. We will need to compose certain relations in $\mathbf{C}$:

Let $R = (R, r_1, r_2) : A \to B$ be a relation from A to B, i.e. a pair of jointly monic morphisms $r_1 : R \to A$ and $r_2 : R \to B$ with the same domain, and let $S = (S, s_1, s_2) : B \to C$ be a relation from B to C. If the pullback $(R \times_B S, \pi_1, \pi_2)$ of r_2 and s_1 exists in $\mathbf{C}$, and if there exists a morphism $e : R \times_B S \to T$ in $\mathbf{E}$ and a jointly monic pair of morphisms $t_1 : T \to A$ and $t_2 : T \to C$ in $\mathbf{C}$ making the diagram



(2.3)

commutative, then we will say that $(T, t_1, t_2) : A \to C$ is the composite of $(R, r_1, r_2) : A \to B$ and $(S, s_1, s_2) : B \to C$. One can similarly define partial composition for three or more relations satisfying a suitable associativity condition. Omitting details, let us just mention that, say, a composite $RR'R''$ might exist even if neither RR' nor $R'R''$ does.

2.5. Convention. *We will say that a relation $R = (R, r_1, r_2) : A \to B$ is a morphism in $\mathbf{C}$ if r_1 is an isomorphism.*

2.6. Definition. *Let $(\mathbf{C}, \mathbf{E})$ be an incomplete relative homological category. A sequence of morphisms*

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots$$

is said to be:

- *(i)* $\mathbf{E}$*-exact at A_i, if the morphism f_{i-1} admits a factorization $f_{i-1} = me$, in which $e \in \mathbf{E}$ and $m = \ker(f_i)$;*
- *(ii) an* $\mathbf{E}$*-exact sequence, if it is* $\mathbf{E}$*-exact at A_i for each i (unless the sequence either begins with A_i or ends with A_i).*

As easily follows from Definition 2.6, the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is $\mathbf{E}$-exact if and only if $f = \ker(g)$; and, if the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is $\mathbf{E}$-exact then $g = \operatorname{coker}(f)$ and g is in $\mathbf{E}$.

In the next section we will often use the following simple fact:

2.7. Lemma. *(Lemma 4.2.4(1) of [1]) If in a commutative diagram*

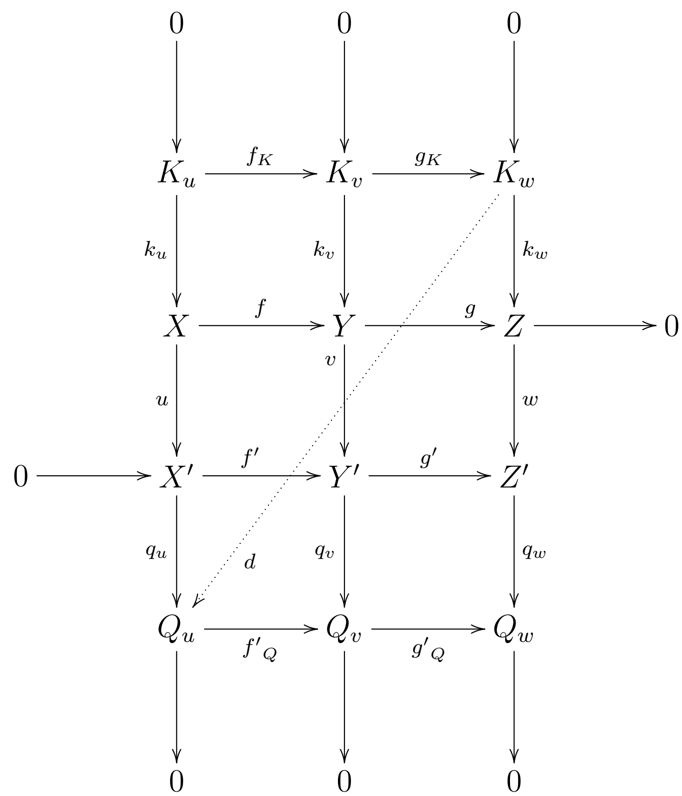
$$\begin{array}{ccccc} K' & \xrightarrow{k'} & A' & \xrightarrow{f'} & B' \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ K & \xrightarrow[k]{} & A & \xrightarrow[f]{} & B \end{array} \tag{2.4}$$

in $\mathbf{C}$, $k = \ker(f)$ and w is a monomorphism, then $k' = \ker(f')$ if and only if the left hand square of the diagram (2.4) is a pullback.

3. The Snake Lemma

This section is devoted to our main result which generalizes Theorem 4.4.2 of [1] and its relative version mentioned in [3]. Formulating it, we use the same notation as in [1].

3.1. THEOREM. [Snake Lemma] *Let* $(\mathbf{C}, \mathbf{E})$ *be an incomplete relative homological category. Consider the commutative diagram*



(3.1)

in which all columns, the second and the third rows are $\mathbf{E}$*-exact sequences. If the morphism* g' *factors as* $g' = g'_2 g'_1$ *in which* g'_1 *is in* $\mathbf{E}$ *and* g'_2 *is a monomorphism, then:*

(i) The composite $d = q_u f'^{\circ} v g^{\circ} k_w : K_w \to Q_u$ *is a morphism in* $\mathbf{C}$*.*

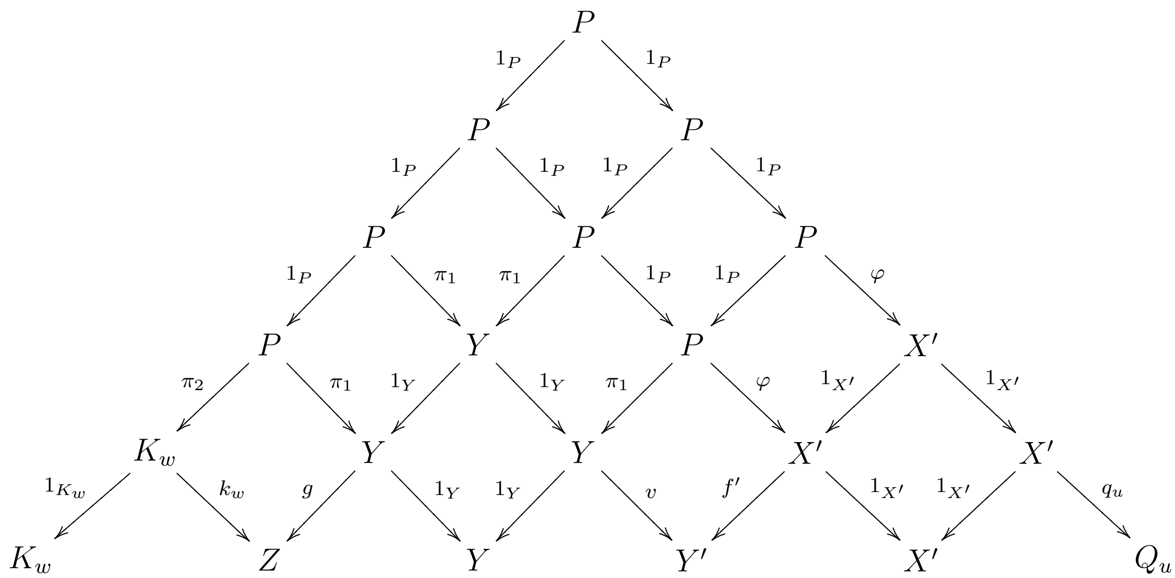
(ii) The sequence

$$K_u \longrightarrow K_v \longrightarrow K_w \xrightarrow{d} Q_u \longrightarrow Q_v \longrightarrow Q_w \qquad (3.2)$$

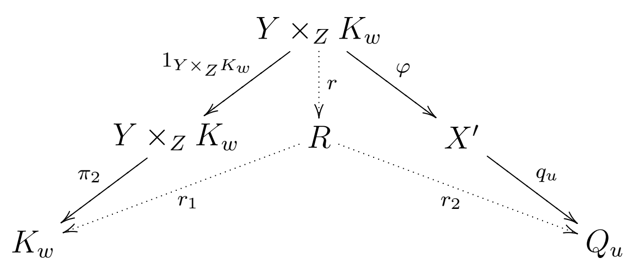
where $d = q_u f'^{\circ} v g^{\circ} k_w$*, is* $\mathbf{E}$*-exact.*

PROOF. (i): Under the assumptions of the theorem, let $(Y \times_Z K_w, \pi_1, \pi_2)$ be the pullback of g and k_w (by condition 2.1(e) this pullback does exist in $\mathbf{C}$); since g is in $\mathbf{E}$, by condition 2.1(f) the morphism π_2 is also in $\mathbf{E}$, and since k_w is a monomorphism so is π_1. Since $f' = \ker(g')$ and $g' v \pi_1 = 0$, there exists a unique morphism $\varphi : Y \times_Z K_w \to X'$ with

$v\pi_1 = f'\varphi$ (see diagram (3.4) below). Using the fact that $(Y \times_Z K_w, \pi_1, \pi_2)$ is the pullback of g and k_w and that $k_w = \ker(w)$, an easy diagram chase proves that $(Y \times_Z K_w, \pi_1, \varphi)$ is the pullback of v and f'. Therefore, we obtain the commutative diagram



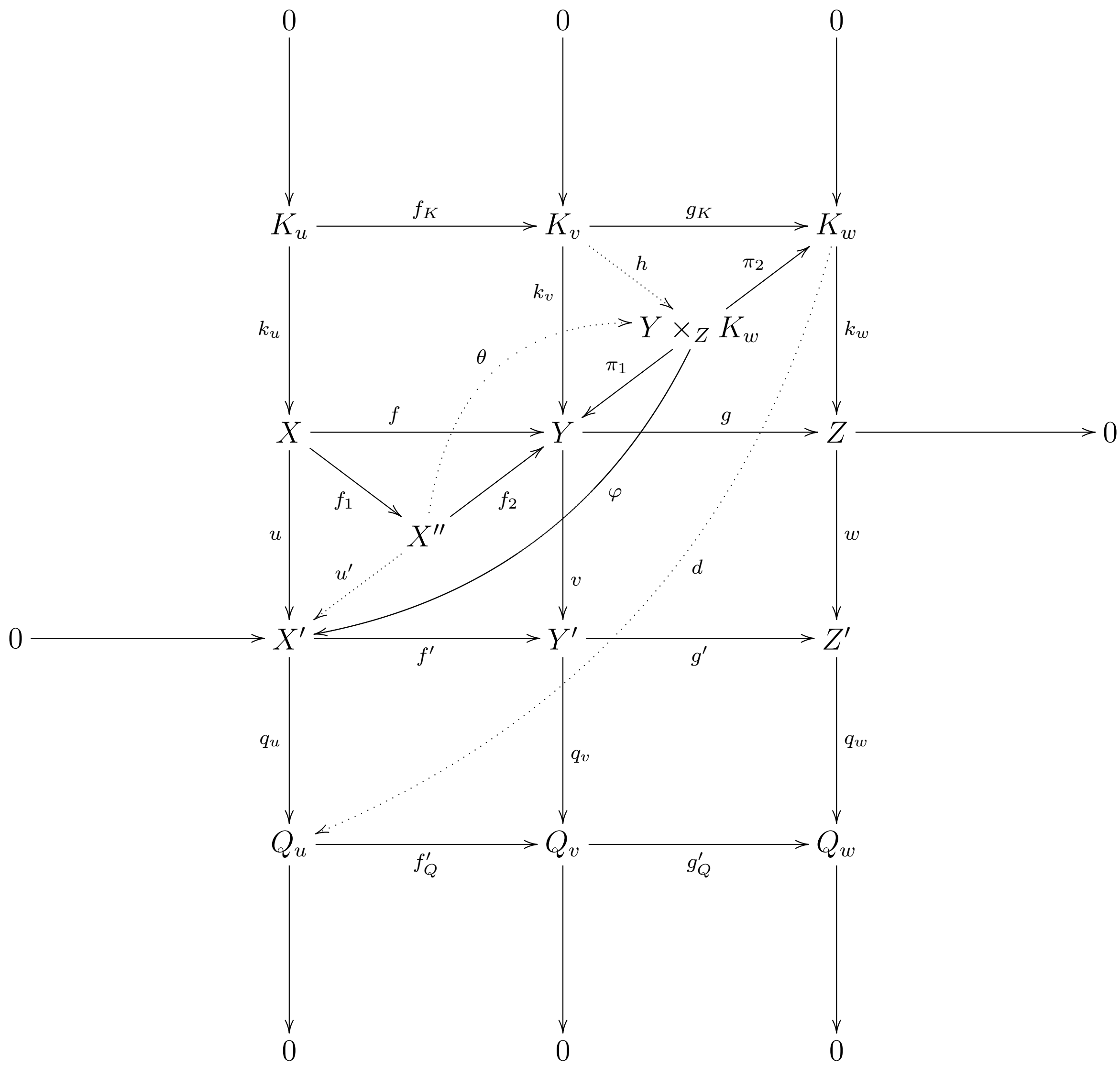
where $P = Y \times_Z K_w$, and all the diamond parts are pullbacks. Since π_2 and q_u are in $\mathbf{E}$, by condition 2.1(g) we have the factorization (unique up to an isomorphism)



(3.3)

where $r : Y \times_Z K_w \to R$ is a morphism in $\mathbf{E}$ and $r_1 : R \to K_w$ and $r_2 : R \to Q_u$ are jointly monic morphisms in $\mathbf{C}$. As follows from the definition of composition of relations, (R, r_1, r_2) is the composite relation $q_u f'^{\circ} v g^{\circ} k_w$ from K_w to Q_u (Note, that since the pullback $(Y \times_Z K_w, \pi_1, \pi_2)$ of k_w and g, and the pullback $(Y \times_Z K_w, \pi_1, \varphi)$ of v and f' exists in $\mathbf{C}$, the composite relations $g^{\circ} k_w : K_w \to Y$ and $f'^{\circ} v : Y \to X'$ also exist. Moreover, since π_2 and q_u are in $\mathbf{E}$, the composite $q_u(f'^{\circ} v)(g^{\circ} k_w)$ of the three relations $g^{\circ} k_w$, $f'^{\circ} v$, and q_u also exists and we have $q_u(f'^{\circ} v)(g^{\circ} k_w) = q_u f'^{\circ} v g^{\circ} k_w$).

To prove that $q_u f'^{\circ} v g^{\circ} k_w : K_w \to Q_u$ is a morphism in $\mathbf{C}$, consider the commutative diagram



(3.4)

in which:

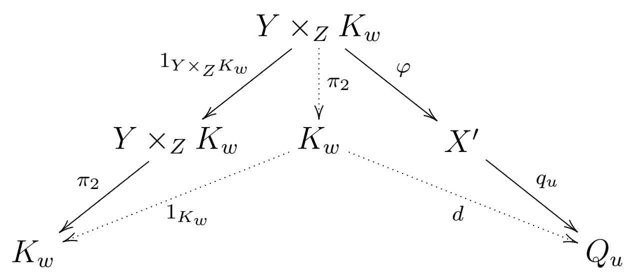
- All the horizontal and vertical arrows are as in diagram (3.1).
- $\pi_1 : Y \times_Z K_w \to Y$, $\pi_2 : Y \times_Z K_w \to K_w$, and $\varphi : Y \times_Z K_w \to X'$ are defined as above, i.e. $(Y \times_Z K_w, \pi_1, \pi_2)$ is the pullback of g and k_w, and $\varphi : Y \times_Z K_w \to X'$ is the unique morphism with $v\pi_1 = f'\varphi$.
- $f = f_2 f_1$ where $f_1 : X \to X''$ is a morphism in **E** and $f_2 : X'' \to Y$ is the kernel of g (such factorization of f does exist in **C** since the second row of diagram (3.1) is **E**-exact).

- Since $(Y \times_Z K_w, \pi_1, \pi_2)$ is the pullback of g and k_w, there exists a unique morphism $\theta : X'' \to Y \times_Z K_w$ with $\pi_1\theta = f_2$ and $\pi_2\theta = 0$. Since π_2 is in $\mathbf{E}$ and $f_2 = \ker(g)$, we conclude that $\pi_2 = \mathrm{coker}(\theta)$.

- Since $f' = \ker(g')$ and $g'vf_2 = 0$, there exists a unique morphism $u' : X'' \to X'$ with $f'u' = vf_2$. It easily follows that $u'f_1 = u$, $q_u = \mathrm{coker}(u')$, and $\varphi\theta = u'$.

- $h : K_v \to Y \times_Z K_w$ is the canonical morphism and an easy diagram chase proves that $h = \ker(\varphi)$ (we do not need this fact now, but we shall need it below when proving (ii)).

Since $q_u\varphi\theta = q_u u' = 0$ and $\pi_2 = \mathrm{coker}(\theta)$, there exists a unique morphism $d : K_w \to Q_u$ with $q_u\varphi = d\pi_2$. We obtain the following factorization:

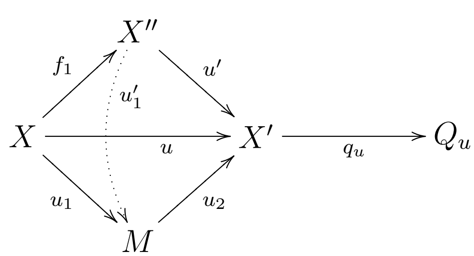


(3.5)

Comparing diagrams (3.3) and (3.5), we conclude that the relation $(K_w, 1_{K_w}, d)$ can be identified with the relation (R, r_1, r_2). Therefore, r_1 is an isomorphism, proving that $q_u f'^{\circ} v g^{\circ} k_w$ is a morphism in $\mathbf{C}$.

(ii): To prove that the sequence (3.2) is $\mathbf{E}$-exact, we need to prove that it is $\mathbf{E}$-exact at K_v, K_w, Q_u, and Q_v.

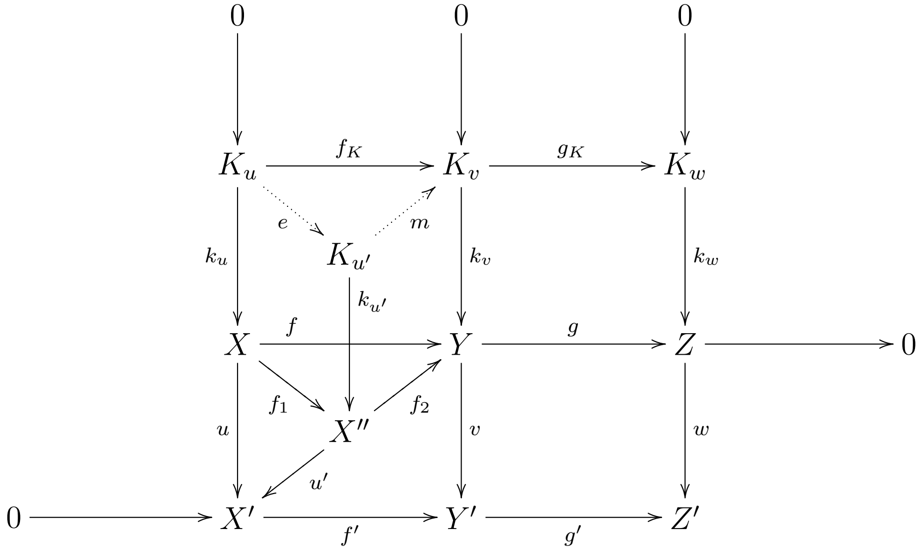
$\mathbf{E}$-*exactness at* K_v: It follows from the fact that the first column of the diagram (3.4) is $\mathbf{E}$-exact at X', that the kernel of u' exists in $\mathbf{C}$. Indeed, consider the commutative diagram



(3.6)

where $u = u_2u_1$ is the factorization of u with $u_2 = \ker(q_u)$ and $u_1 \in \mathbf{E}$ (which does exists since the first column of the diagram (3.4) is $\mathbf{E}$ exact at X'), and u'_1 is the induced morphism. Since f_1 and u_1 are in $\mathbf{E}$, u'_1 is also in $\mathbf{E}$, and therefore the kernel of u'_1 exists in $\mathbf{C}$. Since u_2 is a monomorphism we conclude that $\mathrm{Ker}(u') \approx \mathrm{Ker}(u'_1)$.

Consider the following part of the diagram (3.4)



(3.7)

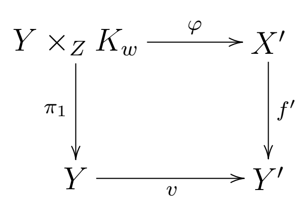
in which:

- $k_{u'} = \ker(u')$;
- Since $k_v = \ker(v)$ and $vf_2k_{u'} = f'u'k_{u'} = 0$, there exists a unique morphism $m : K_{u'} \to K_v$ with $k_v m = f_2 k_{u'}$;
- Since $k_{u'} = \ker(u')$ and $u'f_1k_u = uk_u = 0$, there exists a unique morphism $e : K_u \to K_{u'}$ with $k_{u'}e = f_1k_u$; since k_v is a monomorphism, we conclude that $me = f_K$.

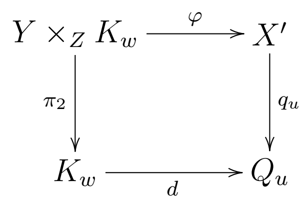
The **E**-exactness at K_v will be proved if we show that $e \in \mathbf{E}$ and $m = \ker(g_K)$. The latter, however, easily follows from Lemma 2.7. Indeed: by Lemma 2.7, the squares $f_1k_u = k_{u'}e$ and $f_2k_{u'} = k_v m$ are pullbacks; therefore, since f_1 is in **E** we conclude that e also is in **E**, and since k_w is a monomorphism, by the same lemma $m = \ker(g_K)$.

E-*exactness at* K_w: Consider the commutative diagram (3.4), we have: $dg_K = d\pi_2 h = q_u\varphi h = 0$. To prove that the sequence (3.2) is **E**-exact at K_w, it suffices to prove that the kernel of d exists in **C** and that the induced morphism from K_v to the kernel of d is in **E**.

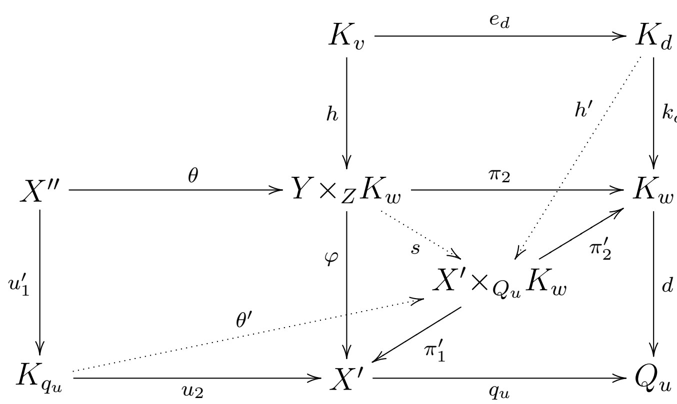
It easily follows from Lemma 2.4 that there exists a factorization $d = d_2d_1$ where d_2 is a monomorphism and d_1 is in **E**. Indeed: since the second column of the Diagram (3.4) is **E**-exact at Y', there exists a factorization $v = v_2v_1$, where $v_2 = \ker(q_v)$ is a monomorphism and v_1 is in **E**. Applying Lemma 2.4(ii) to the diagram



we conclude that $\varphi = \varphi_2\varphi_1$ where φ_2 is a monomorphism and φ_1 is in $\mathbf{E}$, and then applying Lemma 2.4(i) to the diagram



we obtain the desired factorization of d. Since d_1 is in $\mathbf{E}$ and d_2 is a monomorphism, we conclude that the kernel of d exists in $\mathbf{C}$ (precisely, $\mathrm{Ker}(d) \approx \mathrm{Ker}(d_1)$). Let $k_d : K_d \to K_w$ be the kernel of d and let $e_d : K_v \to K_d$ be the induced unique morphism with $k_d e_d = g_K$, it remains to prove that e_d is in $\mathbf{E}$. For, consider the commutative diagram



in which:

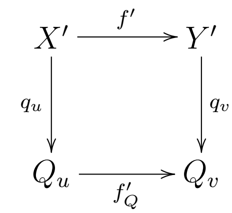
- π'_1, π'_2 are the pullback projections, and $s = \langle\varphi, \pi_2\rangle$, $\theta' = \langle u_2, 0\rangle$, and $h' = \langle 0, k_d\rangle$ are the induced morphisms (the pullback $(X'\times_{Q_u}K_w, \pi'_1, \pi'_2)$ of q_u and d does exist in $\mathbf{C}$ since q_u is in $\mathbf{E}$); since q_u is in $\mathbf{E}$ so is π'_2.
- $u'_1 : X'' \to K_{q_u}$ and $u_2 : K_{q_u} \to X'$ are as in diagram (3.6).
- Since $u_2 = \ker(q_u)$ and $k_d = \ker(d)$, we conclude that $\theta' = \ker(\pi'_2)$ and $h' = \ker(\pi'_1)$.

Since $\theta = \ker(\pi_2)$, $\theta' = \ker(\pi'_2)$, and the morphisms π_2, π'_2, and u'_1 are in $\mathbf{E}$, by Lemma 2.3, $s : Y\times_Z K_w \to X'\times_{Q_u}K_w$ is also in $\mathbf{E}$. Therefore, since $h = \ker(\varphi)$ and $h' = \ker(\pi'_1)$, by Lemma 2.7 and condition 2.1(f) the morphism $e_d : K_v \to K_d$ is also in $\mathbf{E}$, as desired.

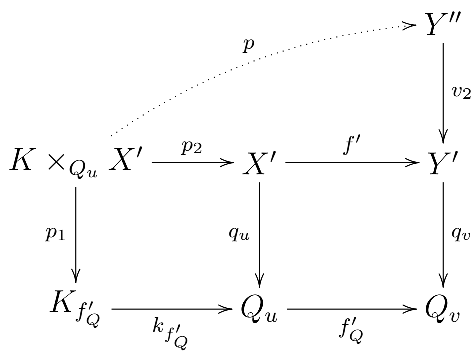
$\mathbf{E}$-*exactness at* Q_u: Consider the commutative diagram (3.4), we have: $f'_Q d\pi_2 = f'_Q q_u\varphi = q_v f'\varphi = q_v v\pi_1 = 0$, and since π_2 is an epimorphism we conclude that $f'_Q d = 0$. To prove that the sequence (3.2) is $\mathbf{E}$-exact at Q_u, it suffices to prove that the kernel of f'_Q exists in $\mathbf{C}$ and that the induced morphism from K_w to the kernel of f'_Q is in $\mathbf{E}$.

It easily follows from Lemma 2.4(i) that there exists a factorization $f'_Q = f'_{Q_2}f'_{Q_1}$ where f'_{Q_2} is a monomorphism and f'_{Q_1} is in $\mathbf{E}$. Indeed: since f' is a monomorphism and

E contains all isomorphisms, applying Lemma 2.4(i) to the diagram

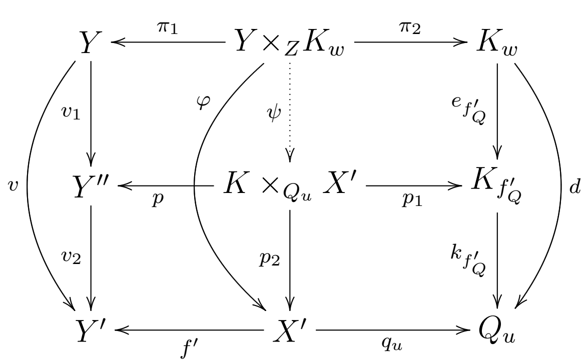


we obtain the desired factorization of f'_Q; since f'_{Q_1} is in **E** and f'_{Q_2} is a monomorphism, we conclude that the kernel of f'_Q exists in **C**. Let $k_{f'_Q} : K_{f'_Q} \to Q_u$ be the kernel of f'_Q and let $e_{f'_Q} : K_w \to K_{f'_Q}$ be the induced unique morphism with $e_{f'_Q} k_{f'_Q} = d$, it remains to prove that $e_{f'_Q}$ is in **E**. Since q_u is in **E**, the pullback $(K \times_{Q_u} X', p_1, p_2)$ of $k_{f'_Q}$ and q_u exists in **C** and p_1 is in **E**; therefore, we have the commutative diagram



in which $v_2 = \ker(q_v)$ (recall, that since the second column of the diagram (3.4) is **E**-exact at Y', we have $v = v_2 v_1$ where $v_1 \in \mathbf{E}$ and $v_2 = \ker(q_v)$) and $p : K \times_{Q_u} X' \to Y''$ is the induced morphism. Using the fact that v_2 and p_2 are monomorphisms and that $k_{f'_Q} = \ker(f'_Q)$, an easy diagram chase proves that the square $f' p_2 = v_2 p$ is the pullback of f' and v_2.

Next, consider the commutative diagram

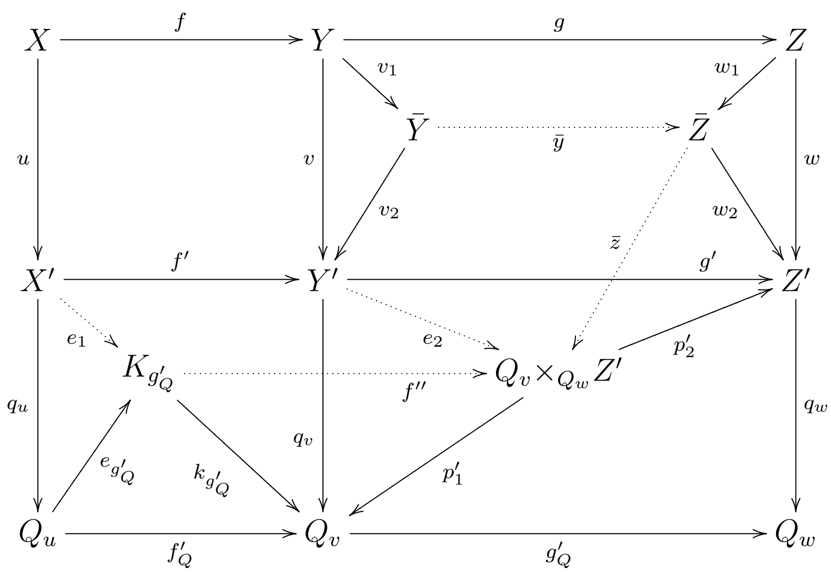


were $\psi = \langle \varphi, v_1 \pi_1 \rangle$; since $k_{f'_Q}$ is a monomorphism and the equalities

$$k_{f'_Q} p_1 \psi = q_u p_2 \psi = q_u \varphi = d\pi_2 = k_{f'_Q} e_{f'_Q} \pi_2$$

hold, we conclude that $p_1\psi = e_{f'_Q}\pi_2$. Recall, that the square $f'\varphi = v\pi_1$ is a pullback (see the proof of (i)), therefore, $(Y\times_Z K_w, \pi_1, \psi)$ is the pullback of p and v_1, yielding that ψ is in $\mathbf{E}$. Since $p_1\psi = e_{f'_Q}\pi_2$, and ψ, p_1, and π_2 are in $\mathbf{E}$, we conclude that $e_{f'_Q}$ is also in $\mathbf{E}$, as desired.

$\mathbf{E}$-*exactness at* Q_v: Consider the commutative diagram (3.4). According to the assumptions of the theorem, we have $g' = g'_2 g'_1$ were g'_1 is a morphism in $\mathbf{E}$ and g'_2 is a monomorphism. Then, by Lemma 2.4(i) there exists a factorization $g'_Q = g'_{Q_2} g'_{Q_1}$ were g'_{Q_1} is a morphism in $\mathbf{E}$ and g'_{Q_2} is a monomorphism, hence, the kernel of g'_Q exists in $\mathbf{C}$. Since $g'_Q f'_Q q_u = q_w g' f' = 0$ and q_u is an epimorphism, we conclude that $g'_Q f'_Q = 0$, therefore, to prove that the sequence (3.2) is $\mathbf{E}$-exact at Q_v it suffices to prove that the induced morphism from Q_u to the kernel of g'_Q is in $\mathbf{E}$. For, consider the commutative diagram



(3.8)

in which:

- $k_{g'_Q} = \ker(g'_Q)$ and $e_{g'_Q} : Q_u \to K_{g'_Q}$ is the induced unique morphism with $k_{g'_Q} e_{g'_Q} = f'_Q$.
- $(Q_v \times_{Q_w} Z', p'_1, p'_2)$ is the pullback of g'_Q and q_w (this pullback does exist since q_w is in $\mathbf{E}$), and $e_2 = \langle q_v, g' \rangle$ and $f'' = \langle k_{g'_Q}, 0 \rangle$ are the canonical morphisms; since $k_{g'_Q} = \ker(g'_Q)$ we conclude that $f'' = \ker(p'_2)$.
- Since $f'' = \ker(p'_2)$ and $p'_2 e_2 f' = 0$ there exists a unique morphism $e_1 : X'' \to K_{g'_Q}$ with $f'' e_1 = e_2 f'$; it easily follows that $e_{g'_Q} q_u = e_1$.
- Since the second and the third columns of the diagram (3.8) are $\mathbf{E}$-exact at Y' and Z' respectively, we have the factorizations $v = v_2 v_1$ and $w = w_2 w_1$, where $v_1, w_1 \in \mathbf{E}$ and $v_2 = \ker(q_v)$ and $w_2 = \ker(q_w)$.

- $\bar{z} = \langle 0, w_2 \rangle$, and since $w_2 = \ker(q_w)$ we conclude that $\bar{z} = \ker(p'_1)$.
- Since $\bar{z} = \ker(p'_1)$ and $p'_1 e_2 v_2 = 0$, there exists a unique morphism $\bar{y} : \bar{Y} \to \bar{Z}$ with $e_2 v_2 = \bar{z}\bar{y}$.

Since w_1, g, and v_1 are in $\mathbf{E}$, we conclude that $\bar{y}$ is in $\mathbf{E}$; therefore, by Lemma 2.3, e_2 is also in $\mathbf{E}$. Then, Lemma 2.7 implies that e_1 is in $\mathbf{E}$, and since $e_{g'_Q} q_u = e_1$ we conclude that $e_{g'_Q}$ is also in $\mathbf{E}$, as desired. ∎

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