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A UNIFIED FRAMEWORK FOR GENERALIZED MULTICATEGORIES

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Abstract. Notions of generalized multicategory have been defined in numerous contexts throughout the literature, and include such diverse examples as symmetric multicategories, globular operads, Lawvere theories, and topological spaces. In each case, generalized multicategories are defined as the "lax algebras" or "Kleisli monoids" relative to a "monad" on a bicategory. However, the meanings of these words differ from author to author, as do the specific bicategories considered. We propose a unified framework: by working with monads on double categories and related structures (rather than bicategories), one can define generalized multicategories in a way that unifies all previous examples, while at the same time simplifying and clarifying much of the theory.

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1. Introduction

A *multicategory* is a generalization of a category, in which the domain of a morphism, rather than being a single object, can be a finite list of objects. A prototypical example is the multicategory **Vect** of vector spaces, in which a morphism $(V_1, \ldots, V_n) \longrightarrow W$ is a

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multilinear map. In fact, any monoidal category gives a multicategory in a canonical way, where the morphisms $(V_1, \ldots, V_n) \longrightarrow W$ are the ordinary morphisms $V_1 \otimes \ldots \otimes V_n \longrightarrow W$. The multicategory **Vect** can be seen as arising in this way, but it is also natural to view its multicategory structure as more basic, with the tensor product then characterized as a representing object for "multimorphisms." This is also the case for many other multicategories; in fact, monoidal categories can be identified with multicategories satisfying a certain representability property (see [Her00] and §9).

In addition to providing an abstract formalization of the passage from "multilinear map" to "tensor product," multicategories provide a convenient way to present certain types of finitary algebraic theories (specifically, *strongly regular* finitary theories, whose axioms involve no duplication, omission, or permutation of variables). Namely, the objects of the multicategories are the *sorts* of the theory, and each morphism $(X_1, \ldots, X_n) \longrightarrow Y$ represents an algebraic operation of the theory. When viewed in this light, multicategories (especially those with one object, which correspond to one-sorted theories) are often called *non-symmetric operads* (see [May72]). The original definition of multicategories in [Lam69] (see also [Lam89]) was also along these lines (a framework for sequent calculus). The two viewpoints are related by the observation that when A is a small multicategory representing an algebraic theory, and $\mathbf{C}$ is a large multicategory such as **Vect**, a *model* of the theory A in $\mathbf{C}$ is simply a *functor* of multicategories $A \longrightarrow \mathbf{C}$. This is a version of the *functorial semantics* of [Law63].

Our concern in this paper is with *generalized multicategories*, a well-known idea which generalizes the basic notion in two ways. Firstly, one allows a change of "base context," thereby including both *internal* multicategories and *enriched* multicategories. Secondly, and more interestingly, one allows the finite lists of objects serving as the domains of morphisms to be replaced by "something else." From the first point of view, this is desirable since there are many other contexts in which one would like to analyze the relationship between structures with coherence axioms (such as monoidal categories) and structures with universal or "representability" properties. From the second point of view, it is desirable since not all algebraic theories are strongly regular.

For example, generalized multicategories include *symmetric* multicategories, in which the finite lists can be arbitrarily permuted. "Representable" symmetric multicategories correspond to symmetric monoidal categories. Enriched symmetric multicategories with one object can be identified with the *operads* of [May72, Kel05, KM95]. These describe algebraic theories in whose axioms variables can be permuted (but not duplicated or omitted). In most applications of operads (see [EM06, BM03] for some recent ones), both symmetry and enrichment are essential.

An obvious variation of symmetric multicategories is *braided* multicategories. If we allow duplication and omission in addition to permutation of inputs, we obtain (multisorted) *Lawvere theories* [Law63]; a slight modification also produces the *clubs* of [Kel72b, Kel92]. There are also important generalizations to "algebraic theories" on more complicated objects; for instance, the *globular operads* of [Bat98, Lei04] describe a certain sort of algebraic theory on globular sets that includes many notions of weak n-category.

A very different route to generalized multicategories begins with the observation of [Bar70] that *topological spaces* can be viewed as a type of generalized multicategory, when finite lists of objects are replaced by ultrafilters, and morphisms are replaced by a convergence relation. Many other sorts of topological structures, such as metric spaces, closure spaces, uniform spaces, and approach spaces, can also be seen as generalized multicategories; see [Law02, CT03, CHT04].

With so many different faces, it is not surprising that generalized multicategories have been independently considered by many authors. They were first studied in generality by [Bur71], but have also been considered by many other authors, including [Lei04], [Lei02], [Her01], [CT03], [CHT04], [Bar70], [Web05], [BD98], [Che04], and [DS03]. While all these authors are clearly doing "the same thing" from an intuitive standpoint, they work in different frameworks at different levels of generality, making the formal definitions difficult to compare. Moreover, all of these approaches share a certain *ad hoc* quality, which, given the naturalness and importance of the notion, ought to be avoidable.

In each case, the authors observe that the "something else" serving as the domain of morphisms in a generalized multicategory should be specified by some sort of *monad*, invariably denoted T. For example, ordinary multicategories appear when T is the "free monoid" monad, globular operads appear when T is the "free strict ω-category" monad, and topological spaces appear when T is the ultrafilter monad. All the difficulties then center around what sort of thing T is a monad *on*.

Leinster [Lei02, Lei04] takes it to be a *cartesian monad* on an ordinary category $\mathbf{C}$, i.e. $\mathbf{C}$ has pullbacks, T preserves them, and the naturality squares for its unit and multiplication are pullback squares. Burroni [Bur71], whose approach is basically the same, is able to deal with any monad on a category with pullbacks. Hermida [Her01] works with a cartesian 2-monad on a suitable 2-category. Barr and Clementino et. al. [Bar70, CT03, CHT04] work with a monad on **Set** equipped with a "lax extension" to the bicategory of matrices in some monoidal category. Weber [Web05] works with a "monoidal pseudo algebra" for a 2-monad on a suitable 2-category. Baez-Dolan [BD98] and Cheng [Che04] (see also [FGHW08]) use a monad on **Cat** extended to the bicategory of profunctors (although they consider only the "free symmetric strict monoidal category" monad).

Inspecting these various definitions and looking for commonalities, we observe that in all cases, the monads involved naturally live on a *bicategory*, be it a bicategory of spans (Burroni, Leinster), two-sided fibrations (Hermida), relations (Barr), matrices (Clementino et. al., Weber), or profunctors (Baez-Dolan, Cheng). What causes problems is that the monads of interest are frequently *lax* (preserving composition only up to a noninvertible transformation), but there is no obvious general notion of lax monad on a bicategory, since there is no good 2-category (or even tricategory) of bicategories that contains lax or oplax functors.

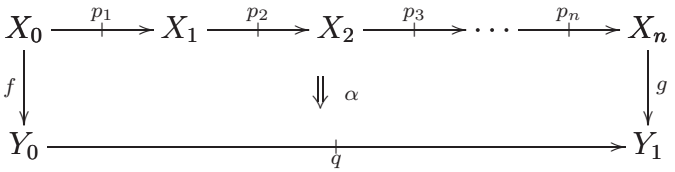
Furthermore, merely knowing the bicategorical monad (however one chooses to formalize this) is insufficient for the theory, and in particular for the definition of functors and transformations between generalized multicategories. Leinster, Burroni, Weber, and Hermida can avoid this problem because their bicategorical monads are induced by monads

on some underlying category or 2-category. Others resolve it by working with an *extension* of a given monad on **Set** or **Cat** to the bicategory of matrices or profunctors, rather than merely the bicategorical monad itself. However, the various definitions of such extensions are tricky to compare and have an *ad hoc* flavor.

Our goal in this paper (and its sequels) is to give a common framework which includes *all* previous approaches to generalized multicategories, and therefore provides a natural context in which to compare them. To do this, instead of considering monads on bicategories, we instead consider monads on types of double categories. This essentially solves both problems mentioned above: on the one hand there is a perfectly good 2-category of double categories and lax functors (allowing us to define monads on a double category), and on the other hand the vertical arrows of the double categories (such as morphisms in the cartesian category $\mathbf{C}$, functions in **Set**, or functors between categories) provide the missing data with which to define functors and transformations of generalized multicategories.

The types of double categories we use are neither strict or pseudo double categories, but instead an even weaker notion, for the following reason. An important intermediate step in the definition of generalized multicategories is the *horizontal Kleisli* construction of a monad T, whose (horizontal) arrows $X \nrightarrow Y$ are arrows $X \nrightarrow TY$. Without strong assumptions on T, such arrows cannot be composed associatively, and hence the horizontal Kleisli construction does not give a pseudo double category or bicategory. It does, however, give a weaker structure, which we call a *virtual double category*.

Intuitively, virtual double categories generalize pseudo double categories in the same way that multicategories generalize monoidal categories. There is no longer a horizontal composition operation, but we have cells of shapes such as the following:



We will give an explicit definition in §2. Virtual double categories have been studied by [Bur71] under the name of *multicatégories* and by [Lei04] under the name of **fc**-*multicategories*, both of whom additionally described a special case of the horizontal Kleisli construction. They are, in fact, the generalized multicategories relative to the "free category" or "free double category" monad (depending on whether one works with spans or profunctors). In [DPP06] virtual double categories were called *lax double categories*, but we believe that name belongs properly to lax algebras for the 2-monad whose strict algebras are double categories. (We will see in Example 9.7 that *oplax* double categories in this "2-monadically correct" sense can be identified with a restricted class of virtual double categories.)

Next, in §§3–4 we will show that for any monad T on a virtual double category $\mathbb{X}$, one can define a notion which we call a T**-monoid**. In fact, we will construct an entire new virtual double category $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$ whose objects are T-monoids, by composing

the "horizontal Kleisli" construction mentioned above with the "monoids and bimodules" construction $\mathbb{M}\mathsf{od}$ described in [Lei04, §5.3], which can be applied to any virtual double category. Then in §6 we will construct from $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$ a 2-category $\mathcal{K}Mon(\mathbb{X}, T)$ of T-monoids, T-monoid functors, and transformations, generalizing the analogous definition in [Lei04, §5.3]. This requires a notion of when a virtual double category has *units*, which we define in §5 along with the parallel notion of when it has *composites*. (These definitions generalize those of [Her00] and can also be found in [DPP06]; they are also a particular case of the "representability" of [Her01] and our §9.)

For particular $\mathbb{X}$ and T, the notion of T-monoid specializes to several previous definitions of generalized multicategories. For example, if $\mathbb{X}$ consists of objects and spans in a cartesian category $\mathbf{C}$ and T is induced from a monad on $\mathbf{C}$, we recover the definitions of Leinster, Kelly, and Burroni. And if $\mathbb{X}$ consists of sets and matrices enriched over some monoidal category $\mathbf{V}$ and T is a "canonical extension" of a taut set-monad to $\mathbb{X}$, then we recover the definitions of Clementino et. al.

However, the other definitions of generalized multicategory cannot quite be identified with T-monoids for any T, but rather with only a restricted class of them. For instance, if $\mathbb{X}$ consists of categories and profunctors, and T extends the "free symmetric monoidal category" monad on $\mathbf{Cat}$ (this is the situation of Baez-Dolan and Cheng), then T-monoids are not quite the same as ordinary symmetric multicategories. Rather, a T-monoid for this T consists of a category A, a symmetric multicategory M, and a bijective-on-objects functor from A to the underlying ordinary category of M. There are two ways to restrict the class of such T-monoids to obtain a notion equivalent to ordinary symmetric multicategories: we can require A to be a discrete category (so that it is simply the set of objects of M), or we can require the functor to also be fully faithful (so that A is simply the underlying ordinary category of M). We call the first type of T-monoid *object-discrete* and the second type *normalized*.

In order to achieve a full unification, therefore, we must give general definitions of these classes of T-monoid and account for their relationship. It turns out that this requires additional structure on our virtual double categories: we need to assume that horizontal arrows can be "restricted" along vertical ones, in a sense made precise in §7. *Pseudo* double categories with this property were called *framed bicategories* in [Shu08], where they were also shown to be equivalent to the *proarrow equipments* of [Woo82] (see also [Ver92]). Accordingly, if a virtual double category $\mathbb{X}$ has this property, as well as all units, we call it a *virtual equipment*.

Our first result in §8, then, is that if T is a well-behaved monad on a virtual equipment, object-discrete and normalized T-monoids are equivalent. However, normalized T-monoids are defined more generally than object-discrete ones, and moreover when T which are insufficiently well-behaved, it is the normalized T-monoids which are of more interest. Thus, we subsequently discard the notion of object-discreteness. (Hermida's generalized multicategories also arise as normalized T-monoids, where $\mathbb{X}$ consists of discrete fibrations in a suitable 2-category $\mathcal{K}$ and T is an extension of a suitable 2-monad on $\mathcal{K}$. Weber's definition is a special case, since as given it really only makes sense for generalized

operads, for which normalization is automatic; see §B.16.) In Table 1 we summarize the meanings of T-monoids and normalized T-monoids for a number of monads T.

Now, what determines whether the "right" notion of generalized multicategory is a plain T-monoid or a normalized one? The obvious thing distinguishing the situations of Leinster, Burroni, and Clementino et. al. from those of Baez-Dolan, Cheng, and Hermida is that in the former case, the objects of $\mathbb{X}$ are "set-like," whereas in the latter, they are "category-like." However, some types of generalized multicategory can be constructed starting from two different monads on two different virtual equipments, one of which belongs to the first group and the other to the second.

For example, observe that an ordinary (non-symmetric) multicategory has an underlying ordinary category, containing the same objects but only the morphisms $(V_1) \longrightarrow W$ with unary source. Thus, such a multicategory can be defined in two ways: either as extra structure on its *set* of objects, or as extra structure on its underlying *category*. In the second case, normalization is the requirement that in the extra added structure, the multimorphisms with unary source do no more than reproduce the originally given category. Thus, ordinary multicategories arise both as T-monoids the "free monoid" monad on sets and spans, and as normalized T-monoids for the "free monoidal category" monad on categories and profunctors.

Our second result in §8 is a generalization of this relationship. We observe that the virtual equipment of categories and profunctors results from applying the "monoids and modules" construction $\mathbb{M}\mathsf{od}$ to the virtual equipment of sets and spans. Thus, we generalize this situation by showing that for any monad T on a virtual equipment, plain T-monoids can be identified with normalized $\mathbb{M}\mathsf{od}(T)$-monoids. That this is so in the examples can be seen by inspection of Table 1. Moreover, it is sensible because application of $\mathbb{M}\mathsf{od}$ takes "set-like" things to "category-like" things.

It follows that the notion of "normalized T-monoid" is actually *more general* than the notion of T-monoid, since arbitrary T-monoids for some T can be identified with the normalized S-monoids for some S (namely $S = \mathbb{M}\mathsf{od}(T)$), whereas normalized S-monoids cannot always be identified with the arbitrary T-monoids for any T. (For instance, this is not the case when S is the "free *symmetric* monoidal category" monad on categories and profunctors.) This motivates us to claim that the "right" notion of generalized multicategory is *a normalized T-monoid*, for some monad T on a virtual equipment.

Having reached this conclusion, we also take the opportunity to propose a new naming system for generalized multicategories which we feel is more convenient and descriptive. Namely, if (pseudo) T-algebras are called *widgets*, then we propose to call normalized T-monoids *virtual widgets*. The term "virtual double category" is of course a special case of this: virtual double categories are the normalized T-monoids for the monad T on $\mathbb{P}\mathsf{rof}(\mathbf{Grph})$ whose algebras are double categories.

Of course, "virtual" used in this way is a "red herring" adjective[1] akin to "pseudo" and "lax", since a virtual widget is not a widget. The converse, however, is true: every widget

[1]The *mathematical red herring principle* states that an object called a "red herring" need not, in general, be either red or a herring.

Monad T	**on**	**T-monoid**	**Normalized T-Monoid**	**Pseudo T -algebra**
Id	$\mathbf{V}$-$\mathbb{M}\mathsf{at}$	$\mathbf{V}$-enriched category	Set	Set
Id	$\mathbb{S}\mathsf{pan}(\mathbf{C})$	Internal category in $\mathbf{C}$	Object of $\mathbf{C}$	Object of $\mathbf{C}$
Id	$\mathbb{R}\mathsf{el}$	Ordered Set	Set	Set
Id	$\overline{\mathbf{R}_+}$-$\mathbb{M}\mathsf{at}$	Metric Space	Set	Set
Powerset	$\mathbb{R}\mathsf{el}$	Closure Space	T_1 Closure Space	Complete Semilattice
$\mathbb{M}\mathsf{od}$(powerset)	$\mathbf{2}$-$\mathbb{P}\mathsf{rof}$	Modular Closure Space	Closure Space	Meet-Complete Preorder
Ultrafilter	$\mathbb{R}\mathsf{el}$	Topological Space	T_1 space	Compact Hausdorff space
$\mathbb{M}\mathsf{od}$(ultrafilter)	$\mathbf{2}$-$\mathbb{P}\mathsf{rof}$	Modular Top. Space	Topological Space	Ordered Compact Hausdorff space
Ultrafilter	$\overline{\mathbf{R}_+}$-$\mathbb{M}\mathsf{at}$	Approach space	?	Compact Hausdorff space
Free monoid	$\mathbb{S}\mathsf{pan}(\mathbf{Set})$	Multicategory	?	Monoid
$\mathbb{M}\mathsf{od}$(free monoid)	$\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$	"Enhanced" multicategory	Multicategory	Monoidal category
Free sym. strict mon. cat.	$\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$	"Enhanced" sym. multicategory	Symmetric multicategory	Symmetric mon. cat.
Free category	$\mathbb{S}\mathsf{pan}(\mathbf{Grph})$	Virtual double category	?	Category
$\mathbb{M}\mathsf{od}$(free category)	$\mathbb{P}\mathsf{rof}(\mathbf{Grph})$	?	Virtual double category	Pseudo double category
Free cat. w/ finite products	$\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$	?	Multi-sorted Lawvere theory	Cat. w/ finite products
Free cat. w/ small products	$\mathbf{Set}$-$\mathbb{P}\mathsf{rof}_{\mathrm{ls}}$	?	Monad on $\mathbf{Set}$	Cat. w/ small products
Free presheaf $\mathbf{S}^{\mathrm{op}} \longrightarrow \mathbf{Set}$	$\mathbb{S}\mathsf{pan}\left(\mathbf{Set}^{\mathrm{ob}(\mathbf{S})}\right)$	Functor $\mathbf{A} \longrightarrow \mathbf{S}$	Functor $\mathbf{A} \longrightarrow \mathbf{S}$ w/ discrete fibers	Functor $\mathbf{S}^{\mathrm{op}} \longrightarrow \mathbf{Set}$
$\mathbb{M}\mathsf{od}$(free presheaf)	$\mathbb{P}\mathsf{rof}(\mathbf{Set}^{\mathrm{ob}(\mathbf{S})})$	?	Functor $\mathbf{A} \longrightarrow \mathbf{S}$	Pseudofunctor $\mathbf{S}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$
Free strict ω-category	$\mathbb{S}\mathsf{pan}(\mathbf{Globset})$	Multi-sorted globular operad	?	Strict ω-category
$\mathbb{M}\mathsf{od}$(free ω-cat.)	$\mathbb{P}\mathsf{rof}(\mathbf{Globset})$	?	Multi-sorted globular operad	Monoidal globular cat.
Free M-set (M a monoid)	$\mathbb{S}\mathsf{pan}(\mathbf{Set})$	M-graded category	?	M-set

Table 1: Examples of generalized multicategories. The boxes marked "?" do not have any established name; in most cases they also do not seem very interesting.

has an underlying virtual widget, so the terminology makes some sense. For example, the observation above that every monoidal category has an underlying multicategory is an instance of this fact. Moreover, it often happens that virtual widgets share many of the same properties as widgets, and many theorems about widgets can easily be extended to virtual widgets. Thus, it is advantageous to use a terminology which stresses the close connection between the two. Another significant advantage of "virtual widget" over "T-multicategory" is that frequently one encounters monads T for which T-algebras have a common name, such as "double category" or "symmetric monoidal category," but T itself has no name less cumbersome than "the free double category monad" or "the free symmetric monoidal category monad." Thus, it makes more sense to name generalized multicategories after the algebras for the monad than after the monad itself.

By the end of §8, therefore, we have unified all existing notions of generalized multicategory under the umbrella of *virtual T-algebras*, where T is a monad on some virtual equipment. Since getting to this point already takes us over 40 pages, we leave to future work most of the development of the theory and its applications, along with more specific comparisons between existing theories (see [CS10a, CS10b]).

However, we do spend some time in §9 on the topic of *representability*. This is a central idea in the theory of generalized multicategories, which states that any pseudo T-algebra (or, in fact, any oplax T-algebra) has an underlying virtual T-algebra. Additionally, one can characterize the virtual T-algebras which arise in this way by a "representability" property. This can then be interpreted as an alternate *definition* of pseudo T-algebra which replaces "coherent algebraic structure" by a "universal property," as advertised in [Her01]. In addition to the identification of monoidal categories with "representable" multicategories, this also includes the fact that compact Hausdorff spaces are T_1 spaces with additional properties, and that fibrations over a category $\mathbf{S}$ are equivalent to pseudofunctors $\mathbf{S}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$. In [CS10b] we will extend more of the theory of representability in [Her01] to our general context.

Finally, the appendices are devoted to showing that all existing notions of generalized multicategory are included in our framework. In Appendix A we prepare the way by giving sufficient conditions for our constructions on virtual double categories to preserve composites, which is important since most existing approaches use bicategories. Then in Appendix B we summarize how each existing theory we are aware of fits into our context.

We have chosen to postpone these comparisons to the end, so that the main body of the paper can present a unified picture of the subject, in a way which is suitable also as an introduction for a reader unfamiliar with any of the existing approaches. It should be noted, though, that we claim no originality for any of the examples or applications, or the ideas of representability in §9. Our goal is to show that all of these examples fall into the same framework, and that this general framework allows for a cleaner development of the theory.

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2. Virtual double categories

The definition of virtual double category may be somewhat imposing, so we begin with some motivation that will hopefully make it seem inevitable. We seek a framework which includes all sorts of generalized multicategories. Since *categories* themselves are a particular sort of generalized multicategory (relative to an identity monad), our framework should in particular include all sorts of generalized categories. In particular, it should include both categories *enriched* in a monoidal category $\mathbf{V}$ and categories *internal* to a category $\mathbf{C}$ with pullbacks, so let us begin by considering how to unify these two situations.

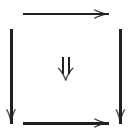
We start by recalling that both $\mathbf{V}$-enriched categories and $\mathbf{C}$-internal categories are particular cases of *monoids in a monoidal category*. On the one hand, if $\mathbf{V}$ is a cocomplete closed monoidal category and $\mathcal{O}$ is a fixed set, then $\mathbf{V}$-enriched categories with object set $\mathcal{O}$ can be identified with monoids in the monoidal category of $\mathcal{O}$-*graphs* in $\mathbf{V}$—i.e. $(\mathcal{O} \times \mathcal{O})$-indexed families of objects of $\mathbf{V}$, with monoidal structure given by "matrix multiplication." On the other hand, if $\mathbf{C}$ is a category with pullbacks and O is an object of $\mathbf{C}$, then $\mathbf{C}$-internal categories with object-of-objects O can be identified with monoids in the monoidal category of O-*spans* in $\mathbf{C}$—i.e. diagrams of the form $O \leftarrow A \rightarrow O$, with monoidal structure given by span composition.

Now, both of these examples share the same defect: they require us to fix the objects (the set $\mathcal{O}$ or object O). In particular, the *morphisms* of monoids in these monoidal categories are functors which are the identity on objects. It is well-known that one can eliminate this fixing of objects by combining all the monoidal categories of graphs and spans, respectively, into a *bicategory*. (In essense, this observation dates all the way back to [Bén67].) In the first case the relevant bicategory $\mathbf{V}$-$\mathcal{M}at$ consists of $\mathbf{V}$-*matrices*: its objects are sets, its arrows from X to Y are $(X \times Y)$-matrices of objects in $\mathbf{V}$, and its composition is by "matrix multiplication." In the second case the relevant bicategory $\mathcal{S}pan(\mathbf{C})$ consists of $\mathbf{C}$-*spans*: its objects are objects of $\mathbf{C}$, its arrows from X to Y are spans $X \leftarrow A \rightarrow Y$ in $\mathbf{C}$, and its composition is by pullback. It is easy to define monoids in a bicategory to generalize monoids in a monoidal category[2].

However, we still have the problem of functors. There is no way to define morphisms between monoids in a bicategory so as to recapture the correct notions of enriched and internal functors in $\mathbf{V}$-$\mathcal{M}at$ and $\mathcal{S}pan(\mathbf{C})$. But we can solve this problem if instead of bicategories we use *(pseudo) double categories*, which come with objects, two different

[2]Monoids in a bicategory are usually called *monads*. However, we avoid that term for these sorts of monoids for two reasons. Firstly, the morphisms of monoids we are interested in are not the same as the usual morphisms of monads (although they are related; see [LS02, §2.3–2.4]). Secondly, there is potential for confusion with the monads *on* bicategories and related structures which play an essential role in the theory we present.

kinds of arrow called "horizontal" and "vertical," and 2-cells in the form of a square:



Both $\mathbf{V}$-$\mathcal{M}at$ and $\mathcal{S}pan(\mathbf{V})$ naturally enlarge to pseudo double categories, interpreting their existing arrows and composition as horizontal and adding new vertical arrows. For $\mathbf{V}$-$\mathcal{M}at$ the new vertical arrows are functions between sets, while for $\mathcal{S}pan(\mathbf{C})$ the new vertical arrows are morphisms in $\mathbf{C}$. We can now define monoids in a double category (relative to the horizontal structure) and morphisms between such monoids (making use of the vertical arrows) so as to recapture the correct notion of functor in both cases (see Definition 2.8).

The final generalization from pseudo double categories to virtual double categories is more difficult to motivate at the moment, but as remarked in the introduction, we will find it essential in §4. Conceptually (and, in fact, formally), a virtual double category is related to a pseudo double category in the same way that a multicategory is related to a monoidal category. Thus, just as one can define monoids in any multicategory, one can likewise do so in any virtual double category.

2.1. Definition. *A **virtual double category** $\mathbb{X}$ consists of the following data.*

- *A category $\mathbf{X}$ (the objects and vertical arrows), with the arrows written vertically:*

$$\begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ Y \end{array}$$

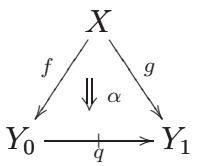
- *For any two objects $X, Y \in \mathbf{X}$, a class of horizontal arrows, written horizontally with a slash through the arrow:*

$$X \xrightarrow{\ p\ } Y$$

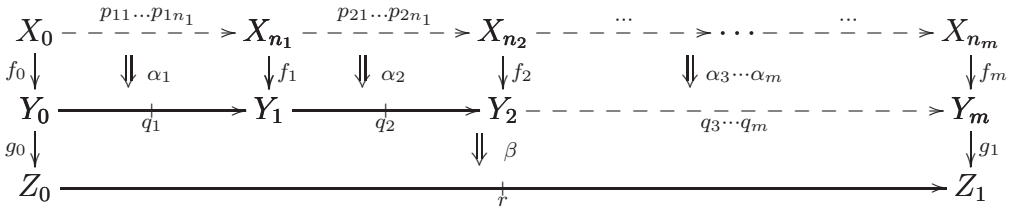
- *Cells, with vertical source and target, and horizontal multi-source and target, written as follows:*

$$\begin{array}{ccccccccc} X_0 & \xrightarrow{p_1} & X_1 & \xrightarrow{p_2} & X_2 & \xrightarrow{p_3} & \cdots & \xrightarrow{p_n} & X_n \\ {\scriptstyle f}\downarrow & & & \Downarrow \alpha & & & & & \downarrow{\scriptstyle g} \\ Y_0 & & & & & & & & Y_1 \\ & & & & & \xrightarrow[q]{} & & & \end{array} \tag{2.2}$$

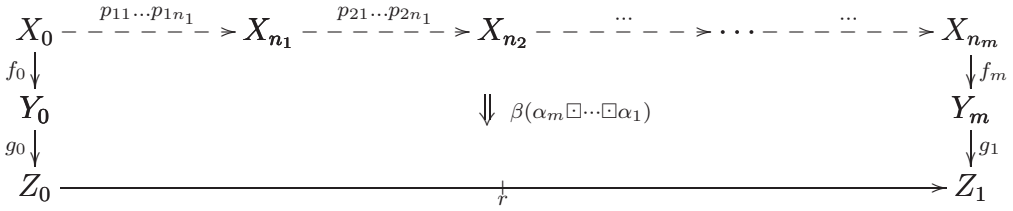
Note that this includes cells with source of length 0, in which case we must have $X_0 = X_n$; such cells are visually represented as follows:



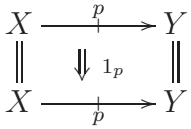
- *For the following configuration of cells,*



a composite cell



- *For each horizontal arrow p, an identity cell*



- *Associativity and identity axioms for cell composition. The associativity axiom states that*

$$\Big(\gamma(\beta_m \boxdot \cdots \boxdot \beta_1)\Big)(\alpha_{mk_m} \boxdot \cdots \boxdot \alpha_{11}) = \gamma\Big(\beta_m(\alpha_{mk_m} \boxdot \cdots \boxdot \alpha_{m1}) \boxdot \cdots \boxdot \beta_1(\alpha_{1k_1} \boxdot \cdots \boxdot \alpha_{11})\Big)$$

while the identity axioms state that

$$\alpha(1_{p_1} \boxdot \cdots \boxdot 1_{p_n}) = \alpha \qquad \textit{and} \qquad 1_q(\alpha_1) = \alpha_1$$

whenever these equations make sense.

2.3. Remark. As mentioned in the introduction, virtual double categories have also been called *fc-multicategories* by Leinster [Lei04] and *multicatégories* by Burroni [Bur71]. Our terminology is chosen to emphasize their close relationship with double categories, and to fit into the general naming scheme of §9.

2.4. Remark. In much of the double-category literature, it is common for the "slashed" arrows (spans, profunctors, etc.) to be the *vertical* arrows. We have chosen the opposite convention purely for economy of space: the cells in a virtual double category fit more conveniently on a page when their multi-source is drawn horizontally.

2.5. Examples. As suggested by the discussion at the beginning of this section, monoidal categories, bicategories, 2-categories, multicategories, and pseudo double categories can each be regarded as examples of virtual double categories, by trivializing the vertical or horizontal structure in various ways; see [Lei02, p. 4] or [Lei04, §5.1] for details.

We now present the two virtual double categories that will serve as initial inputs for most our examples: spans and matrices. (Both are also described in [Lei04, Ch. 5].) For consistency, we name all of our virtual double categories by their horizontal arrows, rather than their vertical arrows or objects.

2.6. Example. Let $(\mathbf{V}, \otimes, I)$ be a monoidal category. The virtual double category $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ is defined as follows: its objects are sets, its vertical arrows are functions, its horizontal arrows $X \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ are families $\{p(y,x)\}_{x\in X, y\in Y}$ of objects of $\mathbf{V}$ (i.e. $(X \times Y)$-matrices), and a cell of the form (2.2) consists of a family of $\mathbf{V}$-arrows

$$p_1(x_1, x_0) \otimes p_2(x_2, x_1) \otimes \cdots \otimes p_n(x_n, x_{n-1}) \xrightarrow{\alpha} q(fx_n, gx_0),$$

one for each tuple $(x_0, \ldots, x_n) \in X_0 \times \cdots \times X_n$. When $n = 0$, of course, the n-ary tensor product on the left is to be interpreted as the unit object of $\mathbf{V}$.

In particular, if $\mathbf{V}$ is the 2-element chain $0 \leq 1$, with $\otimes$ given by $\wedge$, then the horizontal arrows of $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ are relations. In this case we denote $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ by $\mathbb{R}\mathsf{el}$.

It is well-known that $\mathbf{V}$-matrices form a bicategory (and, in fact, a pseudo double category) as long as $\mathbf{V}$ has coproducts preserved by $\otimes$. However, if we merely want a virtual double category, we see that this requirement is unnecessary. (In fact, $\mathbf{V}$ could be merely a multicategory itself.)

2.7. Example. For a category $\mathbf{C}$ with pullbacks, the virtual double category $\mathbb{S}\mathsf{pan}(\mathbf{C})$ is defined as follows: its objects and vertical arrows are those of $\mathbf{C}$, its horizontal arrows $X \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ are spans $X \longleftarrow P \longrightarrow Y$, and a cell of the form (2.2) is a morphism of spans

$$P_1 \times_{X_1} P_2 \times_{X_2} \cdots \times_{X_{n-1}} P_n \xrightarrow{\alpha} Q$$

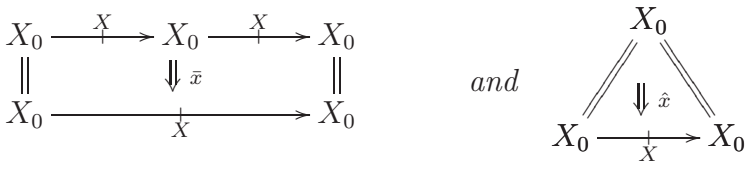
lying over f and g. When $n = 0$, the n-ary span composite in the domain is to be interpreted as the identity span $X_0 \longleftarrow X_0 \longrightarrow X_0$.

Note that in this case, we *do* need to require that $\mathbf{C}$ have pullbacks. If $\mathbf{C}$ does not have pullbacks, a more natural setting would be to consider $\mathbb{S}\mathsf{pan}(\mathbf{C})$ as a "co-virtual double category", in which the horizontal *target* of a cell is a string of horizontal arrows. However, co-virtual double categories do not provide the structure necessary to define generalized multicategories.

We now recall the construction of *monoids and modules* in a virtual double category from [Lei04, §5.3].

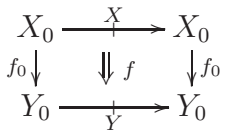
2.8. Definition. *Let $\mathbb{X}$ be a virtual double category. The virtual double category $\mathbb{M}\mathsf{od}(\mathbb{X})$ has the following components:*

- *The objects (**monoids**) consist of four parts $(X_0, X, \bar{x}, \hat{x})$: an object X_0 of $\mathbb{X}$, a horizontal endo-arrow $X_0 \xrightarrow{X} X_0$ in $\mathbb{X}$, and multiplication and unit cells*



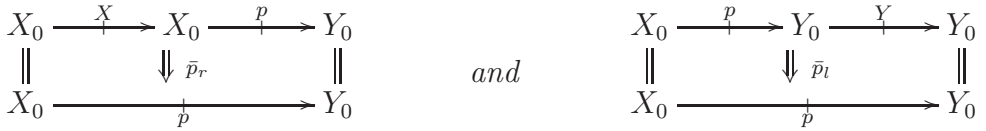
satisfying associativity and identity axioms.

- *The vertical arrows (**monoid homomorphisms**) consist of two parts (f_0, f): a vertical arrow $X_0 \xrightarrow{f_0} Y_0$ in $\mathbb{X}$ and a cell in $\mathbb{X}$:*



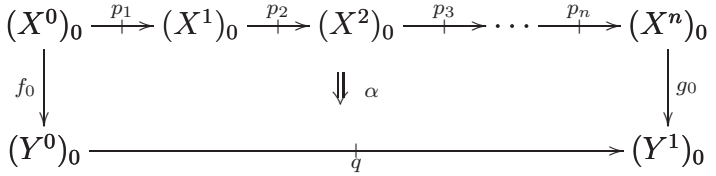
which is compatible with the multiplication and units of X and Y.

- *The horizontal arrows (**modules**) consist of three parts $(p, \bar{p}_r, \bar{p}_l)$: a horizontal arrow $X_0 \xrightarrow{p} Y_0$ in $\mathbb{X}$ and two cells in $\mathbb{X}$:*



which are compatible with the multiplication and units of X and Y.

- *The cells are cells in $\mathbb{X}$:*



which are compatible with the left and right actions of the horizontal cells.

Note that, as observed in [Lei04, §5.3], we can define $\mathbb{M}\mathsf{od}(\mathbb{X})$ without requiring *any* hypotheses on the virtual double category $\mathbb{X}$, in contrast to the situation for bicategories or pseudo double categories.

2.9. Example. We denote the virtual double category $\mathbb{M}\mathsf{od}(\mathbf{V}\text{-}\mathbb{M}\mathsf{at})$ by $\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}$; its objects are $\mathbf{V}$-enriched categories, its vertical arrows are $\mathbf{V}$-functors, its horizontal arrows are $\mathbf{V}$-profunctors, and its cells are a generalization of the "forms" of [DS97] (including, as a special case, natural transformations between profunctors). When $\mathbf{V}$ is closed (hence enriched over itself) and symmetric, $\mathbf{V}$-profunctors $C \xrightarrow{H} D$ can be identified with $\mathbf{V}$-functors $D^{op} \times C \longrightarrow \mathbf{V}$.

Again, note that because we are working with virtual double categories, we do not require that $\mathbf{V}$ have any colimits (in fact, $\mathbf{V}$ could be merely a multicategory).

2.10. Example. Let $\mathbf{C}$ be a category with pullbacks. We denote the virtual double category $\mathbb{M}\mathsf{od}(\mathbb{S}\mathsf{pan}(\mathbf{C}))$ by $\mathbb{P}\mathsf{rof}(\mathbf{C})$; it consists of *internal* categories, functors, profunctors, and transformations in $\mathbf{C}$.

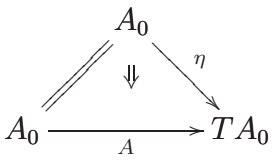
Note that $\mathbf{Set}\text{-}\mathbb{M}\mathsf{at} \cong \mathbb{S}\mathsf{pan}(\mathbf{Set})$ and thus $\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof} \cong \mathbb{P}\mathsf{rof}(\mathbf{Set})$.

3. Monads on a virtual double category

We claimed in §1 that the "inputs" of a generalized multicategory are parametrized by a monad. Why should this be so? Suppose that we have an operation T which, given a set (or object) of objects X, produces a set (or object) TX intended to parametrize such inputs. For "ordinary" multicategories, TX will be the set of finite lists of elements of X.

Now, from the perspective of the previous section, the data of a category includes an object A_0 and a horizontal arrow $A_0 \xrightarrow{A} A_0$ in some virtual double category. For example, if we work in $\mathbf{Set}\text{-}\mathbb{M}\mathsf{at}$, then A is a matrix consisting of the hom-sets $A(x, y)$ for every $x, y \in A_0$. Now, instead, we want to have hom-sets $A(\mathfrak{x}, y)$ whose domain $\mathfrak{x}$ is an element of TA_0. Thus, it makes sense to consider a horizontal arrow $A_0 \xrightarrow{A} TA_0$ as part of the data of a T-multicategory. (We use $A_0 \nrightarrow TA_0$ rather than $TA_0 \nrightarrow A_0$, for consistency with Examples 2.9 and 2.10: the codomain of the horizontal arrow datum of a monoid specifies the *domains* of the arrows *in* that monoid.)

However, we now need to specify the units and composition of our generalized multicategory. The unit should be a cell into A with 0-length domain, but its source and target vertical arrows can no longer both be identities because $A_0 \neq TA_0$. In an ordinary multicategory, the identities are morphisms $(x) \xrightarrow{1_x} x$ whose domain is a singleton list; in terms of $\mathbf{Set}\text{-}\mathbb{M}\mathsf{at}$ this can be described by a cell



where $A_0 \xrightarrow{\eta} TA_0$ is the inclusion of singleton lists.

Regarding composition, in an ordinary multicategory we can compose a morphism $(y_1, \ldots, y_n) \xrightarrow{g} z$ not with a single morphism, but with a *list* of morphisms $(f_1, \ldots, f_n)$ where $(x_{i1}, \ldots, x_{ik_i}) \xrightarrow{f_i} y_i$. In terms of **Set**-$\mathbb{M}$**at** this represents the fact that we cannot ask for a multiplication cell with domain $\xrightarrow{A} \xrightarrow{A}$, since the domain of A does not match its codomain, but instead we can consider a cell with domain $\xrightarrow{A} \xrightarrow{TA}$, where we extend T to act on **Set**-matrices in the obvious way. Now, however, the codomain of TA is T^2A_0; in order to have a cell with codomain A we need to "remove parentheses" from the resulting domain $((x_{11}, \ldots, x_{1k_1}), \ldots, (x_{n1}, \ldots, x_{nk_n}))$ to obtain a single list. Thus the composition should be a cell

$$\begin{array}{ccccc} A_0 & \xrightarrow{A} & TA_0 & \xrightarrow{TA} & T^2A_0 \\ \| & & \Downarrow & & \downarrow \mu \\ A_0 & & \xrightarrow[A]{} & & TA_0 \end{array}$$

where μ is the "remove parentheses" function. Of course, these functions η and μ are the structure maps of the "free monoid" monad on **Set**. Thus we see that in order to define ordinary multicategories, what we require is an "extension" of this monad to **Set**-$\mathbb{M}$**at**.

In order to have a good notion of a *monad on a virtual double category*, we need at least a 2-category of virtual double categories. Since virtual double categories are themselves a special case of generalized multicategories, it suffices to observe that generalized multicategories of any sort form a 2-category. However, since we have not yet defined generalized multicategories in our context, at this point in our exposition it is appropriate to give an explicit description of the 2-category $v\mathcal{D}bl$.

Of course, the objects of $v\mathcal{D}bl$ are virtual double categories, and its 1-morphisms (called *functors* of virtual double categories) are the obvious structure-preserving maps: functions from the objects, vertical and horizontal arrows, and cells of the domain to those of the codomain, preserving all types of source, target, identities, and composition. However, the definition of 2-morphisms in $v\mathcal{D}bl$ is slightly less obvious.

3.1. Definition. *Given functors* $\mathbb{X} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{Y}$ *of virtual double categories, a* ***transformation*** $F \xrightarrow{\theta} G$ *consists of the following data.*

- *For each object* X *in* $\mathbb{X}$*, a vertical arrow* $FX \xrightarrow{\theta_X} GX$*, which form the components of a natural transformation between the vertical parts of* F *and* G*.*
- *For each horizontal arrow* $X \xrightarrow{p} Y$ *in* $\mathbb{X}$*, a cell in* $\mathbb{Y}$*:*

$$\begin{array}{ccc} FX & \xrightarrow{Fp} & FY \\ {\scriptstyle\theta_X}\downarrow & \Downarrow \theta_p & \downarrow{\scriptstyle\theta_Y} \\ GX & \xrightarrow[Gp]{} & GY \end{array} \tag{3.2}$$

- *An axiom asserting that θ is "cell-natural," meaning that*

$$\theta_q(F\alpha) = (G\alpha)(\theta_{p_1} \boxdot \cdots \boxdot \theta_{p_n})$$

whenever this makes sense.

Virtual double categories, functors, and transformations form a 2-category denoted $v\mathcal{D}bl$.

3.3. Example. Any lax monoidal functor $\mathbf{V} \xrightarrow{N} \mathbf{W}$ induces functors $\mathbf{V}\text{-}\mathbb{M}\mathsf{at} \xrightarrow{N_*} \mathbf{W}\text{-}\mathbb{M}\mathsf{at}$ and $\mathbf{V}\text{-}\mathbb{P}\mathsf{rof} \xrightarrow{N_*} \mathbf{W}\text{-}\mathbb{P}\mathsf{rof}$ in an evident way. Moreover, any monoidal natural transformation $N \xrightarrow{\psi} M$ between lax monoidal functors induces transformations $N_* \xrightarrow{\psi_*} M_*$ in both cases. In this way $(-)$-$\mathbb{M}\mathsf{at}$ and $(-)$-$\mathbb{P}\mathsf{rof}$ become 2-functors from the 2-category of monoidal categories to $v\mathcal{D}bl$.

3.4. Example. Similarly, any pullback-preserving functor $\mathbf{C} \xrightarrow{N} \mathbf{D}$ between categories with pullbacks induces functors $\mathbb{S}\mathsf{pan}(\mathbf{C}) \xrightarrow{N_*} \mathbb{S}\mathsf{pan}(\mathbf{D})$ and $\mathbb{P}\mathsf{rof}(\mathbf{C}) \xrightarrow{N_*} \mathbb{P}\mathsf{rof}(\mathbf{D})$, and any natural transformation $N \xrightarrow{\psi} M$ between such functors induces transformations $N_* \xrightarrow{\psi_*} M_*$, thereby making $\mathbb{S}\mathsf{pan}(-)$ and $\mathbb{P}\mathsf{rof}(-)$ into 2-functors as well.

3.5. Example. When restricted to bicategories or pseudo double categories, functors of virtual double categories are equivalent to the usual notions of *lax* functor.

This is a special case of a general fact; see Theorem 9.13. When transformations of virtual double categories are similarly restricted, they become *icons* in the sense of [Lac10] (for bicategories) and *vertical transformations* (for pseudo double categories).

By a *monad on a virtual double category* $\mathbb{X}$, we will mean a monad in the 2-category $v\mathcal{D}bl$. Thus, it consists of a functor $T\colon \mathbb{X} \longrightarrow \mathbb{X}$ and transformations $\eta\colon \mathrm{Id} \longrightarrow T$ and $\mu\colon TT \longrightarrow T$ satisfying the usual axioms. We now give the examples of such monads that we will be interested in.

3.6. Example. Since 2-functors preserve monads, any pullback-preserving monad on a category $\mathbf{C}$ with pullbacks induces monads on $\mathbb{S}\mathsf{pan}(\mathbf{C})$ and $\mathbb{P}\mathsf{rof}(\mathbf{C})$. Examples of such monads include the following.

- The "free monoid" monad on **Set** (or, more generally, on any countably lextensive category).
- The "free M-set" monad $(M \times -)$ on **Set**, for any monoid M (or more generally, for any monoid object in a category with finite limits).
- The monad $(-) + 1$ on any lextensive category.
- The "free category" monad on the category of directed graphs.
- The "free strict ω-category" monad on the category of globular sets.

Many more examples can be found in [Lei04, pp. 103–107]; see also §B.1. By the argument above, each of these monads extends to a monad on a virtual double category of spans.

The assignments $\mathbf{V} \mapsto \mathbf{V}\text{-}\mathbb{M}\mathsf{at}$ and $\mathbf{V} \mapsto \mathbf{V}\text{-}\mathbb{P}\mathsf{rof}$ are also 2-functorial, but the monads we obtain in this way from monads on monoidal categories are not usually interesting for defining multicategories. However, there are some general ways to construct monads on virtual double categories of matrices, at least when $\mathbf{V}$ is a preorder. The following is due to [Sea05], which in turn expands on [CHT04].

3.7. Example. By a **quantale** we mean a closed symmetric monoidal complete lattice. A quantale is **completely distributive** if for any $b \in \mathbf{V}$ we have $b = \bigvee\{a \mid a \prec b\}$, where $a \prec b$ means that whenever $b \leq \bigvee S$ then there is an $s \in S$ with $a \leq s$. (If in this definition S is required to be *directed*, we obtain the weaker notion of a *continuous lattice*.) For us, the two most important completely distributive quantales are the following.

- The two-element chain $\mathbf{2} = (0 \leq 1)$.
- The extended nonnegative reals $\overline{\mathbf{R}_+} = [0, \infty]$ with the reverse of the usual ordering and $\otimes = +$.

A functor said to be **taut** if it preserves pullbacks of monomorphisms (and therefore also preserves monomorphisms). A monad is **taut** if its functor part is taut, and moreover the naturality squares of η and μ for any *monomorphism* are pullbacks. Some important taut monads on **Set** are the identity monad, the powerset monad (whose algebras are complete lattices), the filter monad, and the ultrafilter monad (whose algebras are compact Hausdorff spaces).

Now let $\mathbf{V}$ be a completely distributive quantale and T a taut monad on **Set**. For a $\mathbf{V}$-matrix $X \xrightarrow{p} Y$ and elements $\mathcal{F} \in TX$ and $\mathcal{G} \in TY$, define

$$Tp(\mathcal{G}, \mathcal{F}) = \bigvee \Big\{ v \in \mathbf{V} \;\Big|\; \forall B \subseteq Y : \big(\mathcal{G} \in TB \Rightarrow \mathcal{F} \in T(p_v[B])\big) \Big\},$$

where

$$p_v[B] = \Big\{ x \in X \;\Big|\; \exists y \in B : v \leq p(y, x) \Big\}.$$

It is proven in [Sea05] that this action on horizontal arrows extends T to a monad on $\mathbf{V}\text{-}\mathbb{M}\mathsf{at}$. (Actually, Seal shows that it is a "lax extension of T to $\mathbf{V}\text{-}\mathcal{M}at$ with op-lax unit and counit"; we will show in §B.6 that this is the same as a monad on $\mathbf{V}\text{-}\mathbb{M}\mathsf{at}$.)

In [Sea05] this monad on $\mathbf{V}\text{-}\mathbb{M}\mathsf{at}$ is called the "canonical extension" of T (note, however, that it is written backwards from his definition, as our Kleisli arrows will be $X \mathrel{-\!\!\!\rightarrow} TY$, whereas his are $TX \mathrel{-\!\!\!\rightarrow} Y$). Since $\mathbf{V}\text{-}\mathbb{M}\mathsf{at}$ is isomorphic to its "horizontal opposite," there is also an "op-canonical extension", which is in general distinct (although in some cases, such as for the ultrafilter monad, the two are identical). There are also many other extensions: for more detail, see [SS08].

Another general way of constructing monads on virtual double categories is to apply the construction $\mathbb{M}\mathsf{od}$ from the previous section, which turns out to be a 2-functor. Its 1-functoriality is fairly obvious and was described in [Lei04, §5.3]; its action on 2-morphisms is given as follows.

3.8. Definition. *Let* $\mathbb{X} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{Y}$ *be functors between virtual double categories, and* $F \xrightarrow{\theta} G$ *a transformation. One can define a transformation*

$$\mathbb{M}\mathsf{od}(F) \xrightarrow{\mathbb{M}\mathsf{od}(\theta)} \mathbb{M}\mathsf{od}(G)$$

whose vertical-arrow component at an object $(X_0, X, \bar{x}, \hat{x})$ *is the monoid homomorphism*

$$\begin{array}{ccc} FX_0 & \xrightarrow[F(X)]{} & FX_0 \\ {\scriptstyle \theta_{X_0}}\downarrow & \Downarrow \theta_X & \downarrow{\scriptstyle \theta_{X_0}} \\ GX_0 & \xrightarrow[G(X)]{} & GX_0 \end{array}$$

and whose cell component at a horizontal arrow $(p, \bar{p}_r, \bar{p}_l)$ *is given by*

$$\begin{array}{ccc} FX_0 & \xrightarrow[Fp]{} & FY_0 \\ {\scriptstyle \theta_{X_0}}\downarrow & \Downarrow \theta_p & \downarrow{\scriptstyle \theta_{Y_0}} \\ GX_0 & \xrightarrow[Gp]{} & GY_0 \end{array}$$

3.9. Proposition. *With action on objects, 1-cells, and 2-cells described above,* $\mathbb{M}\mathsf{od}$ *is an endo-2-functor of* $v\mathcal{D}bl$. ∎

Note that the 2-functors $(-)$-$\mathbb{P}\mathsf{rof}$ and $\mathbb{P}\mathsf{rof}(-)$ can now be seen as the composites of $\mathbb{M}\mathsf{od}$ with $(-)$-$\mathbb{M}\mathsf{at}$ and $\mathbb{S}\mathsf{pan}(-)$, respectively.

3.10. Corollary. *A monad* T *on a virtual double category* $\mathbb{X}$ *induces a monad* $\mathbb{M}\mathsf{od}(T)$ *on* $\mathbb{M}\mathsf{od}(\mathbb{X})$. ∎

3.11. Example. Any monad T on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ induces a monad on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$. For instance, this applies to the monads constructed in Example 3.7.

3.12. Example. Let $\mathbf{V}$ be a symmetric monoidal category with an initial object $\emptyset$ preserved by $\otimes$. Then the "free monoid" monad T on $\mathbf{Set}$ extends to a monad on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ as follows: a $\mathbf{V}$-matrix $X \xrightarrow{p} Y$ is sent to the matrix $TX \xrightarrow{Tp} TY$ defined by

$$Tp\Big((y_1, \ldots, y_m), (x_1, \ldots, x_n)\Big) = \begin{cases} p(y_1, x_1) \otimes \ldots \otimes p(y_n, x_n) & \text{if } n = m \\ \emptyset & \text{if } n \neq m \end{cases}$$

Applying $\mathbb{M}\mathsf{od}$, we obtain an extension of the "free strict monoidal $\mathbf{V}$-category" monad from $\mathbf{V}$-$\mathcal{C}at$ to $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$.

3.13. Example. Likewise, any monad T on $\mathbb{S}\mathsf{pan}(\mathbf{C})$ extends to a monad on $\mathbb{P}\mathsf{rof}(\mathbf{C})$. But most interesting monads on $\mathbb{S}\mathsf{pan}(\mathbf{C})$ are induced from $\mathbf{C}$, so this gains us little beyond the observation that $\mathbb{P}\mathsf{rof}(-)$ is a 2-functor.

Not every monad on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ or $\mathbb{P}\mathsf{rof}(\mathbf{C})$ is induced by one on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ or $\mathbb{S}\mathsf{pan}(\mathbf{C})$, however. The following examples are also important.

3.14. Example. Let $\mathbf{V}$ be a symmetric monoidal category with finite colimits preserved by $\otimes$ on both sides. Then there is a "free symmetric strict monoidal $\mathbf{V}$-category" monad T on $\mathbf{V}$-$\mathcal{C}at$, defined by letting the objects of TX be finite lists of objects of X, with

$$TX\Big((x_1,\dots,x_n),(y_1,\dots,y_m)\Big) = \begin{cases} \displaystyle\sum_{\sigma\in S_n} \bigotimes_{1\le i\le n} X(x_{\sigma(i)},y_i) & \text{if } n=m \\ \emptyset & \text{if } n\neq m. \end{cases}$$

A nearly identical-looking definition for profunctors extends this T to a monad on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$. A similar definition applies for braided monoidal $\mathbf{V}$-categories.

3.15. Example. For $\mathbf{V}$ as in Example 3.14, there is also a "free $\mathbf{V}$-category with strictly associative finite products" monad on $\mathbf{V}$-$\mathcal{C}at$. The objects of this TX are again finite lists of objects of X, but now we have

$$TX\Big((x_1,\dots,x_m),(y_1,\dots,y_n)\Big) = \prod_{1\le i\le n}\sum_{1\le j\le m} X(x_j,y_i).$$

If $\mathbf{V}$ is cartesian monoidal, then this can equivalently be written as

$$TX\Big((x_1,\dots,x_m),(y_1,\dots,y_n)\Big) = \sum_{\alpha\colon n\to m}\prod_{1\le i\le n} X(x_{\alpha(i)},y_i).$$

Again, a nearly identical definition for profunctors extends this to a monad on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$.

3.16. Example. Monads that freely adjoin other types of limits and colimits also extend from $\mathbf{V}$-$\mathcal{C}at$ to $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ in a similar way. For instance, if $\mathbf{V}$ is a locally finitely presentable closed monoidal category as in [Kel82], there is a "free $\mathbf{V}$-category with cotensors by finitely presentable objects" monad on $\mathbf{V}$-$\mathcal{C}at$. An object of TX consists of a pair[3] $(v;x)$ where $x\in X$ and $v\in\mathbf{V}$ is finitely presentable. On homs we have

$$TX\big((v;x),(w;y)\big) = \big[w, X(x,y)\otimes v\big].$$

As before, a nearly identical definition extends this to $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$.

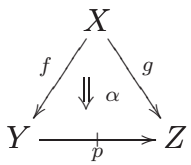
[3]To be precise, this definition only gives a pseudomonad on $\mathbf{V}$-$\mathcal{C}at$. It is, however, easy to modify it to make a strict monad.

4. Generalized multicategories

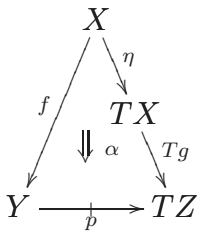
We now lack only one final ingredient for the definition of generalized multicategories. Since multicategories are like categories, we expect them to also be monoids in some virtual double category. However, as we have seen in §3, their underlying data should include a horizontal arrow $A_0 \nrightarrow TA_0$ rather than $A_0 \nrightarrow A_0$. Thus we need to construct, given T and $\mathbb{X}$, a virtual double category in which the horizontal arrows are horizontal arrows of the form $A_0 \nrightarrow TA_0$ in $\mathbb{X}$. This is the purpose of the following definition.

4.1. Definition. *Let T be a monad on a virtual double category $\mathbb{X}$. Define the* ***horizontal Kleisli virtual double category of*** *T, $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$, as follows.*

- *Its vertical category is the same as that of $\mathbb{X}$.*
- *A horizontal arrow $X \xrightarrow{p} Y$ is a horizontal arrow $X \xrightarrow{p} TY$ in $\mathbb{X}$.*
- *A cell with nullary source uses the unit of the monad, so that a cell*

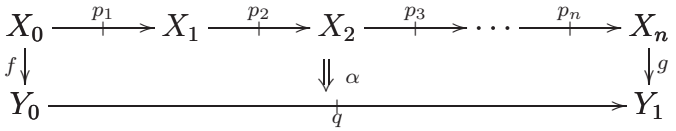


in $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$ is a cell

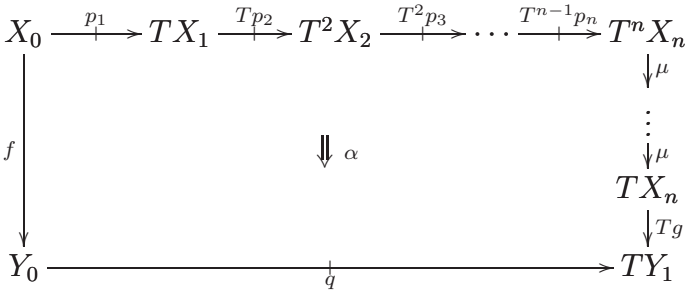


in $\mathbb{X}$ (note that $Tg \circ \eta = \eta \circ g$ by naturality).

- *A cell with non-nullary source uses the multiplication of the monad, so that a cell*

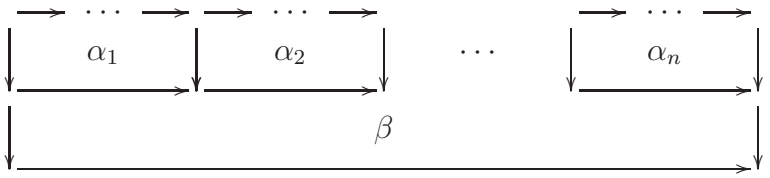


in $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$ *is a cell*

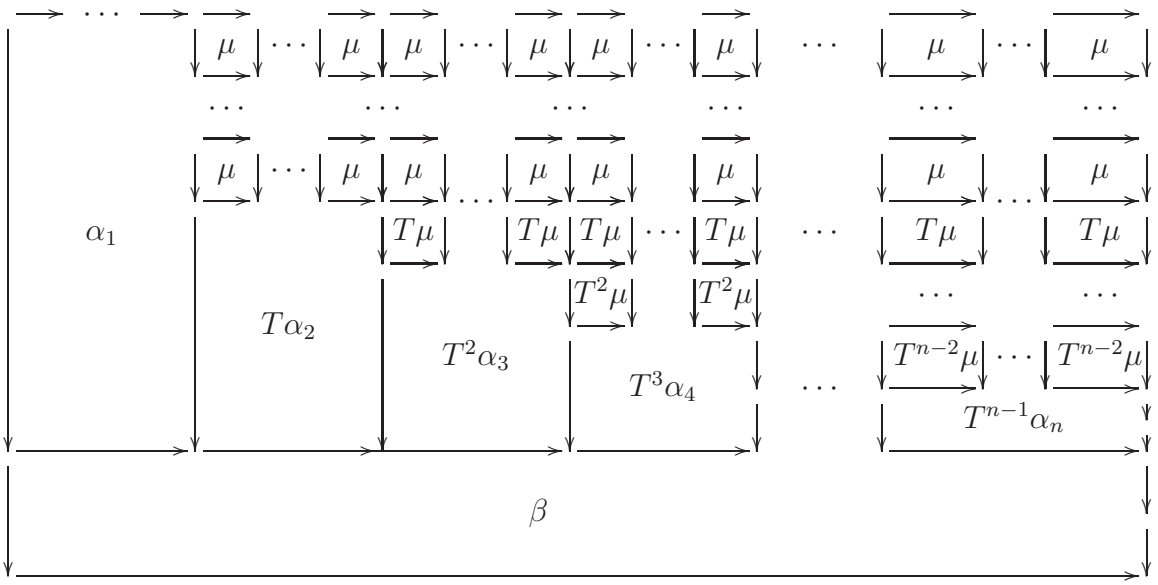


in $\mathbb{X}$ *(note that* $Tg \circ \mu^{n-1} = \mu^{n-1} \circ T^n g$*, by naturality).*

- *The composite of*

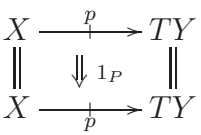


is given by the composite of

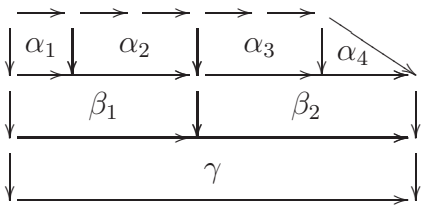


in $\mathbb{X}$*.*

- *Identity cells use those of* $\mathbb{X}$*:*



In general, the associativity for $\mathbb{H}\text{-}\mathsf{KI}(\mathbb{X}, T)$ is shown by using the (cell) naturality of μ and η, as well as the monad axioms. The general associativity is too large a diagram to show here; instead, we will demonstrate a sample associativity calculation, which is representative of the general situation. Consider the following cells in $\mathbb{H}\text{-}\mathsf{KI}(\mathbb{X}, T)$:



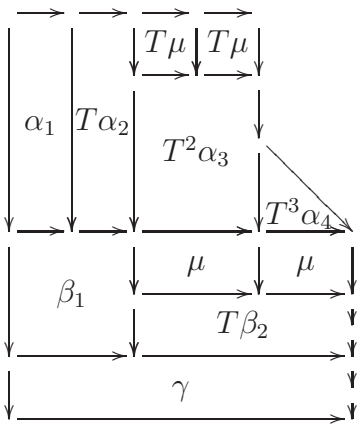
There are two possible ways to compose these cells: either composing the bottom first:

$$(\gamma(\beta_1 \boxdot \beta_2))(\alpha_1 \boxdot \alpha_2 \boxdot \alpha_3 \boxdot \alpha_4)$$

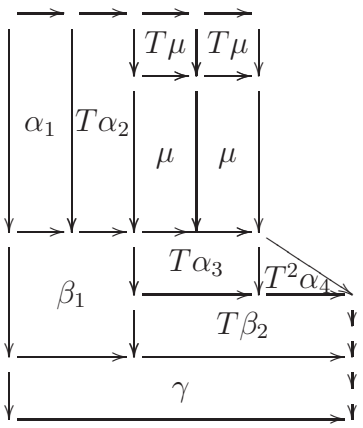
or the top two first, followed by composition with the bottom:

$$\gamma((\beta_1(\alpha_1 \boxdot \alpha_2)) \boxdot (\beta_2(\alpha_3 \boxdot \alpha_4)))$$

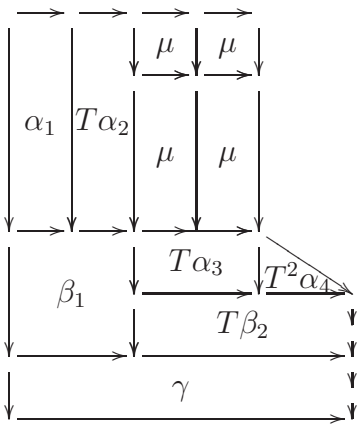
The first composite is given by the following composite in $\mathbb{X}$:



By using cell naturality of μ twice, the above becomes



we then use the monad axiom $T\mu \circ \mu = \mu \circ \mu$ to get



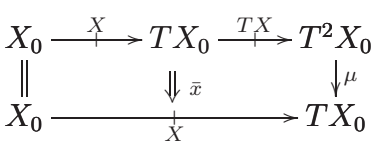
which is the second composite $\gamma((\beta_1(\alpha_1 \boxdot \alpha_2)) \boxdot (\beta_2(\alpha_3 \boxdot \alpha_4)))$.

4.2. Remark. Unfortunately, we do not know of any universal property satisfied by this construction. In particular, $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$ is not a Kleisli object for T in $v\mathcal{D}bl$ in the sense of [Str72a]; the latter would instead contain *vertical* Kleisli arrows. In fact, for general $\mathbb{X}$ there need not even be a canonical functor $\mathbb{X} \longrightarrow \mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$.

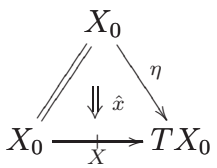
We can now give our first preliminary definition of generalized multicategories relative to a monad T.

4.3. Definition. *Let T be a monad on a virtual double category $\mathbb{X}$. A **T-monoid** is defined to be a monoid in $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$, and likewise for a **$T$-monoid homomorphism**. We denote the virtual double category $\mathbb{M}\mathsf{od}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T))$, whose objects are T-monoids, by $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$.*

As a reference, the data for a T-monoid consists of an object $X_0 \in \mathbb{X}$, a horizontal arrow $X_0 \xrightarrow{X} TX_0$ in $\mathbb{X}$, and cells



and



Note that these cells have precisely the forms we predicted at the beginning of §3.

4.4. Remark. We have seen in §2 that $\mathbb{M}\mathsf{od}$ is a 2-functor. In fact, under suitable hypotheses (involving the notions of restriction and composites to be introduced in §5 and §7), $\mathbb{H}$-Kl is also a (pseudo) functor, and thus so is $\mathbb{K}\mathsf{Mod}$. In fact, $\mathbb{H}$-Kl is a pseudo functor in two different ways, corresponding to the two different kinds of morphisms of monads: lax and colax. This was observed by [Lei04] in his context; we will discuss the functoriality of $\mathbb{H}$-Kl and $\mathbb{K}\mathsf{Mod}$ in our framework in the forthcoming [CS10a].

We now consider some examples.

4.5. Example. Of course, if T is the identity monad on any $\mathbb{X}$, then a T-monoid is just a monoid, and $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T) = \mathbb{M}\mathsf{od}(\mathbb{X})$.

Recall from Example 3.6 that any pullback-preserving monad $\mathbf{C} \xrightarrow{T} \mathbf{C}$ extends to a monad on $\mathbb{S}\mathsf{pan}(\mathbf{C})$.

4.6. Example. If T is the "free monoid" monad extended to $\mathbb{S}\mathsf{pan}(\mathbf{Set}) \cong \mathbf{Set}\text{-}\mathbb{M}\mathsf{at}$, then a T-monoid consists of a set A_0, a $\mathbf{Set}$-matrix $\{A((x_1, \dots, x_n), y)\}_{x_i, y \in A_0}$, and composition and identity functions. It is easy to see that this reproduces the notion of an ordinary multicategory. Likewise, T-monoid homomorphisms are functors between multicategories.

4.7. Example. If T is the "free category" monad on directed graphs, then a T-monoid is a virtual double category. (This is, of course, the origin of the name "fc-multicategory," where **fc** is a name for this monad.) The vertices and edges of the directed graph A_0 are the objects and horizontal arrows, respectively, while in the span $A_0 \leftarrow A \rightarrow TA_0$ the vertices of A are the vertical arrows and its edges are the cells. Likewise, T-monoid homomorphisms are functors between virtual double categories.

4.8. Example. Let M be a monoid and $T = (M \times -)$ the "free M-set" monoid on $\mathbb{S}\mathsf{pan}(\mathbf{Set})$. A T-monoid consists of a set A_0 and a family of sets $\{A(m; x, y)\}_{x,y \in A_0, m \in M}$. The composition and identity functions make it into an *M-graded category*, i.e. a category in which every arrow is labeled by an element of M in a way respecting composition and identities. The case $M = \mathbb{Z}$ may be most familiar.

4.9. Example. Let $\mathbf{C}$ be a lextensive category and T the monad $(-) + 1$ on $\mathbb{S}\mathsf{pan}(\mathbf{C})$. A T-monoid consists of an object A_0 and a span $A_0 \leftarrow A \rightarrow A_0 + 1$; since $\mathbf{S}$ is extensive, A decomposes into two spans $A_0 \leftarrow A_1 \rightarrow A_0$ and $A_0 \leftarrow B \rightarrow 1$. The composition and identity functions then make the first span into an internal category in $\mathbf{C}$ and the second into an internal diagram over this category.

4.10. Example. Let $\mathbf{S}$ be a small category, and let T be the monad on $\mathbf{Set}^{\mathrm{ob}(\mathbf{S})}$ whose algebras are functors $\mathbf{S} \longrightarrow \mathbf{Set}$. Thus, for a family $\{A_x\}_{x \in \mathrm{ob}(\mathbf{S})}$ in $\mathbf{Set}^{\mathrm{ob}(\mathbf{S})}$ we have

$$(TA)_x = \coprod_{y \in \mathrm{ob}(\mathbf{S})} A_y \times \mathbf{S}(x, y).$$

This T preserves pullbacks, so it induces a monad on $\mathbb{S}\mathsf{pan}\left(\mathbf{Set}^{\mathrm{ob}(\mathbf{S})}\right)$. A T-monoid $A \xrightarrow{M} TA$ can be identified with a category over $\mathbf{S}$, i.e a functor $\mathbf{A} \longrightarrow \mathbf{S}$. Namely, the elements of the set A_x are the objects of the fiber of $\mathbf{A}$ over $x \in \mathrm{ob}(\mathbf{S})$, while M can be broken down into a collection of spans

$$A_x \longleftarrow M_{x,y} \longrightarrow A_y \times \mathbf{S}(x, y),$$

which together supply the arrows of $\mathbf{A}$ and their images in $\mathbf{S}$. The morphisms of T-monoids are likewise the functors over $\mathbf{S}$.

4.11. Example. If T is the "free strict ω-category" monad on $\mathbb{S}\mathsf{pan}(\mathbf{Globset})$, then a T-monoid of the form $1 \nrightarrow T1$ can be identified with a *globular operad* in the sense of [Bat98], as described in [Lei04]. General T-monoids are *globular multicategories* (or *many-sorted globular operads*) as considered in [Lei04, p.273–274].

Recall from Example 3.7 that any taut monad T on **Set** (such as the identity monad, the powerset monad, the filter monad, or the ultrafilter monad) extends to a monad on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ for any completely distributive quantale $\mathbf{V}$ (such as $\mathbf{2}$ or $\overline{\mathbf{R}_+}$). We will show in §B.6 that in such a case, our T-monoids are the same as the $(T, \mathbf{V})$-algebras studied by [CT03, CHT04, Sea05], and others; thus we have the following examples.

4.12. Example. If T is the identity monad, then $\mathbb{K}\mathsf{Mod}(\mathbf{V}\text{-}\mathbb{M}\mathsf{at}, T) = \mathbf{V}\text{-}\mathbb{P}\mathsf{rof}$. Thus, for $\mathbf{V} = \mathbf{2}$, T-monoids are preorders; and for $\mathbf{V} = \overline{\mathbf{R}_+}$, T-monoids are metric spaces (in the sense of [Law02]).

4.13. Example. If T is the ultrafilter monad, and $\mathbf{V} = \mathbf{2}$, then a T-monoid consists of a set equipped with a binary relation between ultrafilters and points satisfying unit and composition axioms. If we call this relation "convergence," then the axioms precisely characterize the convergence relation in a topological space; thus T-monoids are topological spaces, and T-monoid homomorphisms are continuous functions. This observation is originally due to [Bar70] (note that although his construction of an ultrafilter monad on $\mathbb{R}\mathsf{el}$ looks quite different, it is in fact the same).

4.14. Example. If T is the powerset monad, and $\mathbf{V} = \mathbf{2}$, then T-monoids are *closure spaces*. A closure space consists of a set A equipped with an operation $c(-)$ on subsets which is:

- extensive: $X \subseteq c(X)$,
- monotone: $Y \subseteq X \Rightarrow c(Y) \subseteq c(X)$, and
- idempotent: $c(c(X)) \subseteq c(X)$.

4.15. Example. If T is the ultrafilter monad and $\mathbf{V} = \overline{\mathbf{R}_+}$, then T-monoids are equivalent to *approach spaces*. An approach space is a set X equipped with a function $d : X \times PX \longrightarrow [0, \infty]$ such that

- $d(x, \{x\}) = 0$,
- $d(x, \emptyset) = \infty$,
- $d(x, A \cup B) = \min\{d(x, A), d(x, B)\}$, and
- $\forall \epsilon \geq 0, d(x, A) \leq d(x, \{x : d(x, A) \leq \epsilon\}) + \epsilon$.

Approach spaces have found applications in approximation theory, products of metric spaces, and measures of non-compactness: for more detail, see [Low88].

Finally, we consider T-monoids relative to the additional examples of monads on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ from the end of §3.

4.16. Example. Let T be the "free symmetric strict monoidal $\mathbf{V}$-category" monad on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ from Example 3.14. If A_0 is a discrete $\mathbf{V}$-category, then a T-monoid $A_0 \xrightarrow{A} TA_0$ is a symmetric $\mathbf{V}$-enriched multicategory (known to some authors as simply a "multicategory"). Likewise, from the "free braided strict monoidal $\mathbf{V}$-category" monad we obtain braided multicategories.

If A_0 is not discrete, then a T-monoid (for $\mathbf{V} = \mathbf{Set}$) is a symmetric multicategory in the sense of [BD98] and [Che04]: in addition to the multi-arrows, there is also another type of arrow between the objects of the multicategory which forms a category, and which acts on the multi-arrows.

4.17. Example. Let T be the "free category with strictly associative finite products" monad on $\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$ from Example 3.15. If A_0 is a one-object discrete category, then a T-monoid $A_0 \xrightarrow{A} TA_0$ is a Lawvere theory, as in [Law63]. If A_0 has more than one object, but is still discrete, then a T-monoid $A_0 \xrightarrow{A} TA_0$ is a "multi-sorted" Lawvere theory.

This is a little different from the more usual definition of Lawvere theory, but the equivalence between the two is easy to see. A Lawvere theory is commonly defined to be a category A with object set $\mathbb{N}$ such that each object n is the n-fold product 1^n. This implies that $A(m,n) \cong A(m,1)^n$, so it is equivalent to give just the collection of sets $A(m,1)$ with suitable additional structure. Since $T1$ has object set $\mathbb{N}$, a T-monoid $1 \xrightarrow{} T1$ also consists of sets $A(m,1)$ for $m \in \mathbb{N}$, and it is then straightforward to verify that the additional structures in the two cases are in bijective correspondence. Note, however, that the *morphisms* between such T-monoids do not correspond to all of the morphisms between theories considered in [Law63], but only those of "degree one;" the others are only visible from the "category with object set $\mathbb{N}$" viewpoint.

The relationship between these two definitions of Lawvere theory is analogous to the way in which an *operad* can also be defined as a certain sort of monoidal category with object set $\mathbb{N}$. In fact, both arise from a very general *adjunction* between T-algebras and T-monoids; see Remark 9.16 and the forthcoming [CS10b].

4.18. Example. If T is the "free $\mathbf{V}$-category with strictly associative finite products" monad on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ from Example 3.15 and A_0 is a one-object discrete $\mathbf{V}$-category, then a T-monoid $A_0 \xrightarrow{A} TA_0$ is a "$\mathbf{V}$-enriched finite product theory." If A_0 is unchanged but T is instead the "free $\mathbf{V}$-category with finitely presentable cotensors" monad from Example 3.16, then a T-monoid $A_0 \xrightarrow{A} TA_0$ is a "Lawvere $\mathbf{V}$-theory" as defined in [Pow99] (with the same caveat as in the previous example). To obtain a "multi-sorted Lawvere $\mathbf{V}$-theory" we need T to adjoin both finite products and finite cotensors.

4.19. Example. If T is any of

- the "free symmetric strict monoidal category" monad,
- the "free category with strictly associative finite products" monad, or
- the "free category with strictly associative finite coproducts" monad,

but now considered as a monad on $\mathbb{S}\mathsf{pan}(\mathbf{Cat})$, then a T-monoid with a discrete category of objects is a *club* in the sense of [Kel72b] and [Kel72a] (relative to P, S, or S^{op}, in Kelly's terminology). See also §B.4.

4.20. Remark. When T is the "free symmetric strict monoidal category" monad on $\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$, the horizontal arrows between discrete categories in $\mathbb{H}$-$\mathsf{Kl}(\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof}, T)$ are the *generalized species of structure* of [FGHW08] (called *structure types* in [BD01]). The *espéces de structures* of [Joy81, Joy86] are the particular case of horizontal arrows $1 \mathrel{+\!\!\!\rightarrow} 1$ in $\mathbb{H}$-$\mathsf{Kl}(\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof}, T)$. Likewise, when T is the analogous monad on $\mathbb{S}\mathsf{pan}(\mathbf{Gpd})$, the horizontal arrows in $\mathbb{H}$-$\mathsf{Kl}(\mathbb{S}\mathsf{pan}(\mathbf{Gpd}), T)$ are the (generalized) *stuff types* of [BD01, Mor06].

We can see from these examples that for virtual double categories whose objects are "category-like," it is often the T-monoids whose objects are discrete which are of particular interest. We will make this notion precise in §8, and propose that often a better solution is to consider "normalized" T-monoids.

First, however, we must develop some additional machinery for virtual double categories. We will describe when horizontal arrows have units and composites, as well as when horizontal arrows can be "restricted" along vertical arrows. With this theory in hand, we can then return to study "object discrete" and "normalized" T-monoids, as well as when such T-monoids are "representable".

4.21. Remark. If $\mathbf{V}$ is a complete and cocomplete closed symmetric monoidal category, then the virtual double category $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ is itself "almost" of the form $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$. We take $\mathbb{X}$ to be the double category $\mathbb{S}\mathsf{q}(\mathbf{V}\text{-}\mathcal{C}at)$, whose objects are $\mathbf{V}$-categories, whose vertical and horizontal arrows are both $\mathbf{V}$-functors, and whose cells are $\mathbf{V}$-natural transformations, and we define $TA = \mathbf{V}^{A^{\mathrm{op}}}$ to be the enriched presheaf category of A. The observation is then that a $\mathbf{V}$-profunctor $A \mathrel{+\!\!\!\rightarrow} B$ can be identified with an ordinary $\mathbf{V}$-functor $A \longrightarrow \mathbf{V}^{B^{\mathrm{op}}}$, so that $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ is almost the same as $\mathbb{H}$-$\mathsf{Kl}(\mathbb{S}\mathsf{q}(\mathbf{V}\text{-}\mathcal{C}at), T)$. This is not quite right, since T is not really a monad due to size issues. But these problems can be dealt with, for instance by using "small presheaves" as in [DL07].

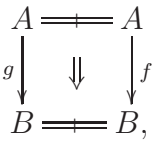
Assuming the functoriality of $\mathbb{H}$-Kl mentioned in Remark 4.4, this observation implies that if S is another monad on $\mathbf{V}$-$\mathcal{C}at$ related to T by a distributive law, or equivalently a monad in the 2-category of monads and monad morphisms (see [Bec69, Str72b]), then S induces a monad on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$, which we can in turn use to define generalized multicategories as S-monoids in $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$. For example, since a symmetric monoidal structure on A extends to TA by Day convolution, the "free symmetric monoidal $\mathbf{V}$-category" monad distributes over T, inducing its extension to $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ considered in Example 3.14. This is the argument used in [FGHW08] to construct the bicategory $\mathcal{H}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}, T))$. Similar arguments apply to the monad from Example 3.15.

5. Composites and units

In §2 we introduced (virtual) double categories as a framework in which one can define monoids and monoid homomorphisms so as to include both enriched and internal cate-

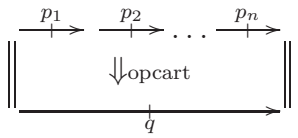
gories with the appropriate notions of functor. However, we would certainly like to be able to recover *natural transformations* as well, but this requires more structure than is present in a virtual double category.

It is not hard to see that the vertical category of any (pseudo) double category can be enriched to a *vertical 2-category*, whose 2-cells $f \Rightarrow g$ are the squares of the form

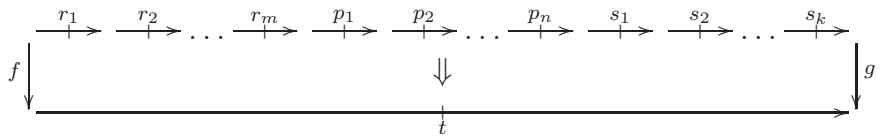


and that in examples such as $\mathbb{P}\mathsf{rof}(\mathbf{C})$ and $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ these 2-cells recover the appropriate notion of natural transformation. In a virtual double category this definition is impossible, since there may not be any horizontal identity arrows. However, it turns out that we can characterize those horizontal identities which *do* exist by means of a universal property. In fact, it is not much harder to characterize arbitrary horizontal composites (viewing identities as 0-ary composites). In this section we study such composites; in the next section we will use them to define vertical 2-categories.

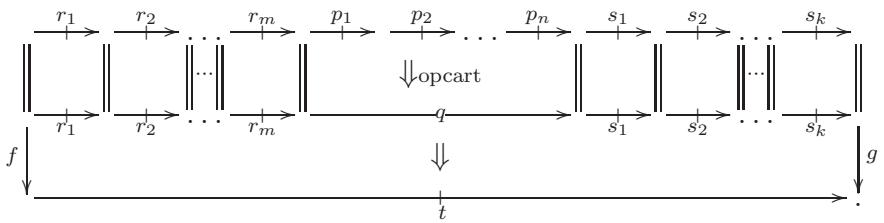
5.1. Definition. *In a virtual double category, a cell*



is said to be ***opcartesian*** *if any cell*

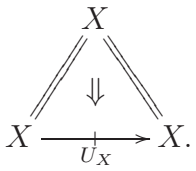


factors through it uniquely as follows:



If a string of composable horizontal arrows is the source of some opcartesian cell, we say that it **has a composite**. We refer to the target q of that cell as a **composite** of the given string and write it as $p_1 \odot \cdots \odot p_n$. Similarly, if $n = 0$ and there is an opcartesian

cell of the form



we say that X **has a unit** U_X.

These universal properties make it easy to show that composites and units, when they exist, behave like composites and units in a pseudo double category. For example, composites and units are unique up to isomorphism: given two opcartesian arrows with the same source, factoring each through the other gives an isomorphism between their targets. Likewise, the composite of opcartesian cells is opcartesian, so composition is associative up to coherent isomorphism whenever all relevant composites exist. More precisely, if $p \odot q$ exists, then $(p \odot q) \odot r$ exists if and only if $p \odot q \odot r$ exists, and in that case they are isomorphic. It follows that if $p \odot q$ and $q \odot r$ exist, then

$$(p \odot q) \odot r \cong p \odot (q \odot r),$$

each existing if the other does. Similarly, any string in which all but one arrow is a unit:

$$X \xrightarrow{U_X} \dots \xrightarrow{U_X} X \xrightarrow{p} Y \xrightarrow{U_Y} \dots \xrightarrow{U_Y} Y$$

has a composite, which is (isomorphic to) p.

The following theorem, which was also observed in [DPP06], is a straightforward generalization of the relationship between monoidal categories and ordinary multicategories described in [Her00]. It is also a special case of the general relationship between pseudo algebras and generalized multicategories, as in [Lei04, §6.6], [Her01], and §9 of the present paper.

5.2. Theorem. *A virtual double category is a pseudo double category if and only if every string of composable horizontal arrows (including zero-length ones) has a composite.*

Proof (sketch). "Only if" is clear, by definition of how a pseudo double category becomes a virtual one. For "if", we use the isomorphisms constructed above; we invoke again the universal property of opcartesian cells to show coherence. ∎

5.3. Example. If $\mathbf{V}$ has an initial object $\emptyset$ which is preserved by $\otimes$ on both sides, then $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ has units: the unit of a set X is the matrix

$$U_X(x, x') = \begin{cases} I & x = x' \\ \emptyset & x \neq x'. \end{cases}$$

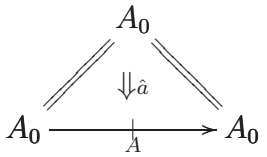
If $\mathbf{V}$ has all small coproducts which are preserved by $\otimes$ on both sides, then $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ has composites given by "matrix multiplication." For instance, the composite of matrices $X \xrightarrow{p} Y$ and $Y \xrightarrow{q} Z$ is

$$(p \odot q)(z, x) = \coprod_{y \in Y} p(y, x) \otimes q(z, y).$$

5.4. Example. Since $\mathbb{S}\mathsf{pan}(\mathbf{C})$ is a pseudo double category, all composites and units always exist. Composites are given by pullback, and the unit of X is the unit span $X \leftarrow X \rightarrow X$.

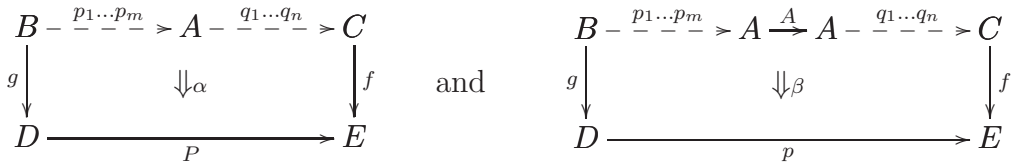
Regarding units in $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ and $\mathbb{P}\mathsf{rof}(\mathbf{C})$, we have the following.

5.5. Proposition. *For any virtual double category $\mathbb{X}$, all units exist in $\mathbb{M}\mathsf{od}(\mathbb{X})$. For any monoid A, its unit cell*



is opcartesian in $\mathbb{M}\mathsf{od}(\mathbb{X})$. Therefore, U_A is A itself, regarded as an A-A-bimodule.

Proof. Firstly, the unit axioms of A show that $\hat{a}$ is, in fact, a cell in $\mathbb{M}\mathsf{od}(\mathbb{X})$. Now we must show that composing with $\hat{a}$ gives a bijection between cells



in $\mathbb{M}\mathsf{od}(\mathbb{X})$. Clearly composing any cell β of the second form with $\hat{a}$ gives a cell α of the first form. Conversely, given α of the first form, there are two cells of the second form defined by letting A act first on p_m from the right and q_1 from the left, respectively. These are equal by one of the axioms for α to be a cell in $\mathbb{M}\mathsf{od}(\mathbb{X})$; we let β be their common value. (In the case when $m = 0$ or $n = 0$, we use the action of f or g instead.) The other axioms for α then carry over to β to show that it is a cell in $\mathbb{M}\mathsf{od}(\mathbb{X})$.

The unit axioms for the action of A on bimodules show that $\alpha \mapsto \beta \mapsto \alpha$ is the identity, while the equivariance axioms for β regarding the two actions of A show that $\beta \mapsto \alpha \mapsto \beta$ is the identity. Thus we have a bijection, as desired. ∎

Therefore, since $\mathbf{V}$-$\mathbb{P}\mathsf{rof} = \mathbb{M}\mathsf{od}(\mathbf{V}\text{-}\mathbb{M}\mathsf{at})$ and $\mathbb{P}\mathsf{rof}(\mathbf{C}) = \mathbb{M}\mathsf{od}(\mathbb{S}\mathsf{pan}(\mathbf{C}))$, they both always have all units. By contrast, extra assumptions on $\mathbb{X}$ are required for composites to exist in $\mathbb{M}\mathsf{od}(\mathbb{X})$; here are the two examples of greatest interest to us.

5.6. Example. If $\mathbf{V}$ has small colimits preserved by $\otimes$ on both sides, then $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ has all composites; the composite of enriched profunctors $A \xrightarrow{p\ \shortmid} B$ and $B \xrightarrow{q\ \shortmid} C$ is given by the coend

$$(p \odot q)(z, x) = \int^{y \in B} p(y, x) \otimes q(z, y).$$

5.7. Example. If $\mathbf{C}$ has coequalizers preserved by pullback, then $\mathbb{P}\mathsf{rof}(\mathbf{C})$ has all composites; the composite of internal profunctors is an "internal coend."

Together with Proposition 5.5, these examples will suffice for the moment. In appendix A we will give general sufficient conditions for composites to exist in $\mathbb{M}\mathsf{od}(\mathbb{X})$, and for composites and units to exist in $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$.

5.8. Remark. If Definition 5.1 is satisfied only for $m = k = 0$, we say that the cell is **weakly opcartesian**. We do not regard a weakly opcartesian cell as exhibiting its target as a composite of its source, since the weak condition is insufficient to prove associativity and unitality. However, a weakly opcartesian cell does suffice to *detect* its target as the composite of its source, if we already know that that composite exists. Furthermore, if any composable string in $\mathbb{X}$ is the source of a weakly opcartesian cell and moreover weakly opcartesian cells are closed under composition, then one can show, as in [Her00], that in fact every weakly opcartesian cell is cartesian; see also §9.

Virtual double categories having only weakly opcartesian cells seem to be fairly rare; one example is $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ where $\mathbf{V}$ has colimits which are not preserved by its tensor product. Note that in this case, $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ need not even have weakly opcartesian cells, since we require $\otimes$ to preserve coequalizers simply to make the composite of two profunctors into a profunctor.

If $\mathbb{X}$ and $\mathbb{Y}$ have units, we say that a functor (or monad) $\mathbb{X} \xrightarrow{F} \mathbb{Y}$ is **normal** if it preserves opcartesian cells with nullary source, which is to say it preserves units in a coherent way. Likewise, if $\mathbb{X}$ and $\mathbb{Y}$ have all units and composites (i.e. are pseudo double categories), we say that $\mathbb{X} \xrightarrow{F} \mathbb{Y}$ is **strong** if it preserves *all* opcartesian cells.

5.9. Example. Any functor $\mathbb{S}\mathsf{pan}(\mathbf{C}) \longrightarrow \mathbb{S}\mathsf{pan}(\mathbf{D})$ induced by a pullback-preserving functor $\mathbf{C} \longrightarrow \mathbf{D}$ is strong, and in particular normal.

5.10. Example. It is also easy to see that $\mathbb{M}\mathsf{od}(F)$ is normal for any functor F, by the construction of units in Proposition 5.5.

5.11. Examples. If $\mathbf{V}$ is a cocomplete symmetric monoidal category in which $\otimes$ preserves colimits on both sides, then $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ has all composites, and the extension of the "free monoid" monad to $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ from Example 3.12 is easily seen to be strong. Since the "free strict monoidal $\mathbf{V}$-category" monad on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ is obtained by applying $\mathbb{M}\mathsf{od}$ to this, it is normal by our above observation. In fact, it is also strong, essentially because the tensor product of reflexive coequalizers is again a reflexive coequalizer (see, for example, [Joh02, A1.2.12]).

5.12. Examples. The "free symmetric strict monoidal $\mathbf{V}$-category" monad on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ from Example 3.14 is also normal, essentially by definition, as are the "free $\mathbf{V}$-category with strictly associative finite products" monad from Example 3.15 and its relatives from Example 3.16. A more involved computation with coequalizers shows that the first is actually strong, and the second is strong whenever $\mathbf{V}$ is cartesian monoidal. However, it seems that the others are not in general strong.

5.13. Examples. The monads on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ constructed in Example 3.7 are not generally strong or even normal. Two notable exceptions are the ultrafilter monads on $\mathbb{R}\mathsf{el}$ and $\overline{\mathbf{R}_+}$-$\mathbb{M}\mathsf{at}$, of which the first is strong and the second is normal.

We write $v\mathcal{D}bl_n$ for the locally full sub-2-category of $v\mathcal{D}bl$ determined by the virtual double categories with units and normal functors between them; thus $\mathbb{M}\mathsf{od}$ is a 2-functor $v\mathcal{D}bl \longrightarrow v\mathcal{D}bl_n$. In fact, we have the following.

5.14. Proposition. *$\mathbb{M}\mathsf{od}$ is right pseudo-adjoint to the forgetful 2-functor $v\mathcal{D}bl_n \longrightarrow v\mathcal{D}bl$.*

Proof. "Pseudo-adjoint" means that we have a pseudonatural η and ϵ that satisfy the triangle identities up to coherent isomorphism. We take $\epsilon_{\mathbb{X}}$ to be the forgetful functor $\mathbb{M}\mathsf{od}(\mathbb{X}) \longrightarrow \mathbb{X}$ which sends a monoid to its underlying object and a module to its underlying horizontal arrow; this is strictly 2-natural. If $\mathbb{X}$ has units, we take $\eta_{\mathbb{X}}$ to be the "unit assigning" functor $\mathbb{X} \longrightarrow \mathbb{M}\mathsf{od}(\mathbb{X})$ which sends X to U_X (which has a unique monoid structure) and $X \xrightarrow{p} Y$ to itself considered as a (U_X, U_Y)-bimodule; this is only pseudonatural since normal functors preserve units only up to isomorphism. But if we choose the units in $\mathbb{M}\mathsf{od}(\mathbb{X})$ according to Proposition 5.5, then the triangle identities are satisfied on the nose. ∎

In particular, if 1 denotes the terminal virtual double category, then the category of *normal* functors $1 \longrightarrow \mathbb{X}$ is equivalent to the vertical category of $\mathbb{X}$. It then follows from Proposition 5.14 that the category of arbitrary functors $1 \longrightarrow \mathbb{X}$ is equivalent to the vertical category of $\mathbb{M}\mathsf{od}(\mathbb{X})$. Thus, Proposition 5.14 generalizes the well-known observation (which dates back to [Bén67]) that monoids in a bicategory $\mathcal{B}$ are equivalent to lax functors $1 \longrightarrow \mathcal{B}$.

5.15. Remark. It follows that $\mathbb{M}\mathsf{od}$ is a *pseudomonad* on the 2-category $v\mathcal{D}bl_n$, and so in particular it has a multiplication

$$\mathbb{M}\mathsf{od}(\mathbb{M}\mathsf{od}(\mathbb{X})) \longrightarrow \mathbb{M}\mathsf{od}(\mathbb{X}). \tag{5.16}$$

Inspection reveals that an object of $\mathbb{M}\mathsf{od}(\mathbb{M}\mathsf{od}(\mathbb{X}))$ consists of an object X of $\mathbb{X}$, two monoids $X \xrightarrow{A} X$ and $X \xrightarrow{M} X$, and a monoid homomorphism $A \longrightarrow M$ whose vertical arrow components are identities. The multiplication (5.16) simply forgets the monoid A. This idea will be further discussed in [CS10b].

5.17. Remark. If $\mathbb{X}$ is a virtual double category in which all units and composites exist (equivalently, it is a pseudo double category), then it has a *horizontal bicategory* $\mathcal{H}(\mathbb{X})$ consisting of its objects, horizontal arrows, and cells of the form

Clearly when $\mathbf{C}$ and $\mathbf{V}$ satisfy the required conditions for all composites to exist in our examples, we recover in this way the usual bicategories of matrices, spans, and enriched

and internal profunctors. Any functor between pseudo double categories likewise induces a lax functor of horizontal bicategories, but this is not true of transformations without additional assumptions; see Remarks 7.26 and A.5.

6. 2-categories of T-monoids

As proposed in the previous section, we now use the notion of units introduced there to define 2-categories of generalized multicategories, generalizing the approach taken in [Lei04, §5.3].

6.1. Proposition. *Let $\mathbb{X}$ be a virtual double category in which all units exist. Then it has a* ***vertical 2-category*** *$\mathcal{V}\mathbb{X}$ whose objects are those of $\mathbb{X}$, whose morphisms are the vertical arrows of $\mathbb{X}$, and whose 2-cells $A \overset{f}{\underset{g}{\Downarrow \alpha}} B$ are the cells*

$$\begin{array}{ccc} A & \xrightarrow[U_A]{} & A \\ {\scriptstyle g}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle f} \\ B & \xrightarrow[U_B]{} & B \end{array}$$

in $\mathbb{X}$.

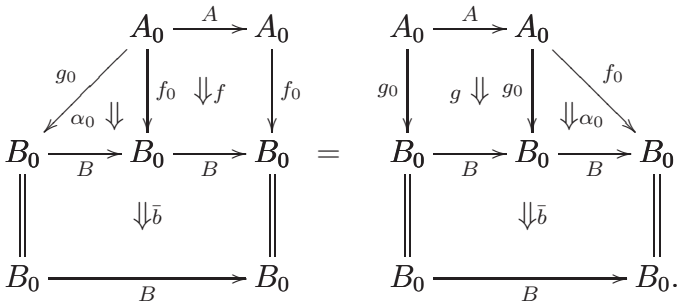
Proof. This is straightforward; note that when composing 2-cells we must use the isomorphisms $U_A \cong U_A \odot U_A$. ∎

In particular, for any $\mathbb{X}$, $\mathbb{M}\mathsf{od}(\mathbb{X})$ has a vertical 2-category, which we denote $\mathcal{M}on(\mathbb{X})$ and call the *2-category of monoids in* $\mathbb{X}$. (This 2-category is closely related to various 2-categories of monads in a bicategory; see [LS02, §2.3–2.4].) It turns out that in this case, the description of the 2-cells of $\mathcal{M}on(\mathbb{X})$ can be rephrased in a way that looks much more like a natural transformation.

6.2. Proposition. *Giving a 2-cell $A \overset{f}{\underset{g}{\Downarrow \alpha}} B$ in $\mathcal{M}on(\mathbb{X})$ is equivalent to giving a cell*

$$\begin{array}{ccc} & A_0 & \\ {\scriptstyle g_0}\swarrow & \Downarrow \alpha_0 & \searrow{\scriptstyle f_0} \\ B_0 & \xrightarrow[B]{} & B_0 \end{array}$$

in $\mathbb{X}$ *such that*



(6.3)

PROOF. This is an instance of Proposition 5.5, where $m = n = 0$, $f = g = 1_A$, and $P = U_A$. ∎

6.4. EXAMPLE. Recall that in $\mathbf{V}$-$\mathbb{P}\mathsf{rof} = \mathbb{M}\mathsf{od}(\mathbf{V}\text{-}\mathbb{M}\mathsf{at})$, the objects are $\mathbf{V}$-enriched categories and the vertical morphisms are $\mathbf{V}$-enriched functors; thus these are the objects and morphisms of $\mathcal{M}on(\mathbf{V}\text{-}\mathbb{M}\mathsf{at})$. Recalling from Example 2.6 the definition of cells in $\mathbf{V}$-$\mathbb{M}\mathsf{at}$, Proposition 6.2 implies that a 2-cell $A \overset{f}{\underset{g}{\Downarrow \alpha}} B$ in $\mathcal{M}on(\mathbf{V}\text{-}\mathbb{M}\mathsf{at})$ is given by a family of morphisms $I \xrightarrow{\alpha_x} B(fx, gx)$ for x an object of A, such that for every x, y the following square commutes:

$$\begin{array}{ccc} A(x,y) & \xrightarrow{\alpha_y \otimes f} & B(fy,gy) \otimes B(fx,fy) \\ {\scriptstyle g\otimes\alpha_x}\downarrow & & \downarrow \\ B(gx,gy) \otimes B(fx,gx) & \longrightarrow & B(fx,gy). \end{array}$$

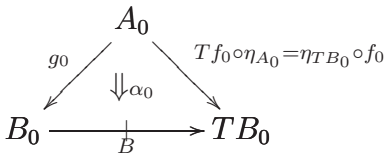
This is precisely the usual definition of a $\mathbf{V}$-enriched natural transformation; thus we have $\mathcal{M}on(\mathbf{V}\text{-}\mathbb{M}\mathsf{at}) \simeq \mathbf{V}\text{-}\mathcal{C}at$.

6.5. EXAMPLE. Likewise, $\mathcal{M}on(\mathbb{S}\mathsf{pan}(\mathbf{C})) \simeq \mathcal{C}at(\mathbf{C})$ is the 2-category of internal categories, functors, and natural transformations in $\mathbf{C}$.

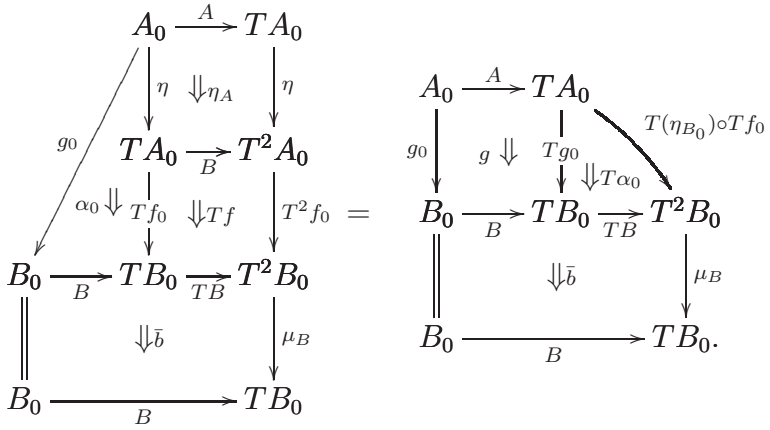
6.6. EXAMPLES. On the other hand, the vertical 2-category of $\mathbb{S}\mathsf{pan}(\mathbf{S})$ is just $\mathbf{S}$, regarded as a 2-category with only identity 2-cells. The vertical 2-category of $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ depends on what $\mathbf{V}$ is, but usually it is not very interesting. Thus, in general, vertical 2-categories are only interesting for virtual double categories whose objects are "category-like" rather than "set-like."

Now, if T is a monad on a virtual double category $\mathbb{X}$, we write $\mathcal{K}Mon(\mathbb{X}, T)$ for the 2-category $\mathcal{V}(\mathbb{K}\mathsf{Mod}(\mathbb{X}, T))$ and call it *the 2-category of* T*-monoids in* $\mathbb{X}$. Its objects are T-monoids, its morphisms are T-monoid homomorphisms, and its 2-cells may be called T-*monoid transformations*. By Proposition 6.2 and the definition of $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$, a T-monoid

transformation $\alpha\colon f \longrightarrow g\colon A \longrightarrow B$ is specified by a cell



such that



Many authors have defined this 2-category $\mathcal{K}Mon(\mathbb{X}, T)$ in seemingly ad-hoc ways, whereas it falls quite naturally out of the framework of virtual double categories. This was also observed in [Lei04, §5.3]; see §B.1.

6.7. Example. Let T be the "free monoid" monad on **Set**-$\mathbb{M}$at, so that T-monoids are ordinary multicategories. If $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are functors, then according to the above, a transformation $f \xrightarrow{\alpha} g$ consists of, for each $x \in A$, a morphism $\alpha_x \in B(\eta(fx), gx)$ (that is to say, a morphism $(fx) \xrightarrow{\alpha_x} gx$ with source of length one) such that for any morphism $\xi\colon (x_1, \ldots, x_n) \longrightarrow y$ in A, we have

$$\alpha_y \circ f(\xi) = g(\xi) \circ (\alpha_{x_1}, \ldots, \alpha_{x_n}).$$

This is the usual notion of transformation for functors between multicategories.

6.8. Example. When T is the "free category" monad on directed graphs, so that T-monoids are virtual double categories, T-monoid transformations are the same as the transformations we defined in 3.1.

6.9. Example. Let U be the ultrafilter monad on $\mathbb{R}$el, so that U-monoids are topological spaces. If $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are continuous maps (i.e. U-monoid homomorphisms), then there exists a transformation $f \longrightarrow g$ (which is necessarily unique) just when for all $x \in A$, the principal ultrafilter η_{fx} converges to gx in B. This is equivalent to saying that $f \geq g$ in

the pointwise ordering induced by the usual *specialization order* on B. The situation for other topological examples is similar.

Any *normal* functor clearly induces a strict 2-functor between vertical 2-categories. In fact, if 2-$\mathcal{C}at$ denotes the 2-category of 2-categories, strict 2-functors, and strict 2-natural transformations, then we have:

6.10. Proposition. *There is a strict 2-functor* $\mathcal{V}\colon v\mathcal{D}bl_n \longrightarrow 2\text{-}\mathcal{C}at$.

In particular, any normal monad T on $\mathbb{X}$ induces a strict 2-monad on $\mathcal{V}(\mathbb{X})$. As we saw in §3, most monads on virtual double categories are "extensions" of a known monad on their vertical categories (or vertical 2-categories), so this construction usually just recovers the familiar monad we started with. In §9, we will show that $\mathcal{V}(T)$-algebras are closely related to T-monoids.

7. Virtual equipments

If we have succeeded in convincing the reader that virtual double categories are inevitable, she may be justified in wondering why they have not been more studied. Certainly, *virtual* double categories involve significant complexity above and beyond *pseudo* double categories, and the latter suffice to describe the important examples $\mathbb{S}\mathsf{pan}(\mathbf{C})$, $\mathbf{V}$-$\mathbb{M}\mathsf{at}$, $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$, and $\mathbb{P}\mathsf{rof}(\mathbf{C})$ as long as $\mathbf{V}$ and $\mathbf{C}$ are suitably cocomplete. However, even pseudo double categories have generally received less publicity than bicategories.

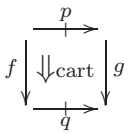
One possible reason for this is that in most of the (pseudo or virtual) double categories arising in practice, the vertical arrows are more tightly coupled to the horizontal arrows that we have heretofore accounted for; in fact they can almost be *identified with* certain horizontal arrows. For example, a $\mathbf{V}$-functor $A \xrightarrow{f} B$ is determined, up to isomorphism, by the $\mathbf{V}$-profunctor $A \xrightarrow{B(1,f)} B$ defined by $B(1,f)(b,a) = B(b,fa)$. Furthermore, this coupling is very important for many applications, such as the formal definition of weighted limits and colimits (see [Str83, Woo82]), so a mere double category (pseudo or virtual) would be insufficient for these purposes. Because of this, many authors have been content to work with bicategories, or bicategories with a collection of horizontal arrows singled out (such as the "proarrow equipments" of [Woo82]).

However, while not all pseudo double categories exhibit this type of coupling, it is possible to characterize those that do (and they turn out to be equivalent to the "proarrow equipments" mentioned above). The basic idea of this dates back to [BS76], but it has recently been revived in various equivalent forms; see for instance [Ver92, GP99, GP04, DPP10, Shu08] and the Notes at the end of [Lei04, Ch. 5]. Since this type of coupling also plays an important role in the theory of generalized multicategories, in this section we give the basic definitions appropriate to the virtual case.

The basic idea is the following. The profunctor $B(1,f)$ considered above can be constructed in two stages: first we consider the hom-profunctor $B(-,-)\colon B \nrightarrow B$, and then we *precompose* it with f on one side. We already know from §5 that hom-profunctors

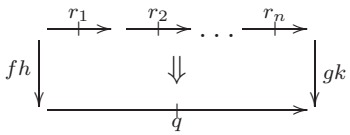
are the *units* in $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$, so it remains only to characterize *precomposition with functors* in terms of $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$. This is accomplished by the following definition.

7.1. Definition. *A cell*

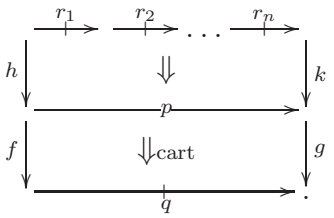


(7.2)

in a virtual double category is ***cartesian*** *if any cell*



factors through it uniquely as follows:



If there exists a cartesian cell (7.2), we say that the p is the **restriction** $q(g, f)$. The notation is intended to suggest precomposition of a profunctor $q(-,-)$ with f and g. When f or g is an identity, we write $q(g, 1)$ or $q(1, f)$, respectively. It is evident from the universal property that restrictions are unique up to isomorphism, and pseudofunctorial; that is, we have $q(1, 1) \cong q$ and $q(1, g)(1, f) \cong q(1, gf)$ coherently.

We say that $\mathbb{X}$ **has restrictions** if $q(g, f)$ exists for all q, f, and g, and that a functor **preserves restrictions** if it takes cartesian cells to cartesian cells. We write $v\mathcal{D}bl_r$ for the sub-2-category of $v\mathcal{D}bl$ determined by the virtual double categories with restrictions and the restriction-preserving functors.

7.3. Examples. The virtual double categories $\mathbf{V}$-$\mathbb{M}\mathsf{at}$, $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$, $\mathbb{S}\mathsf{pan}(\mathbf{C})$, and $\mathbb{P}\mathsf{rof}(\mathbf{C})$ have all restrictions. Restrictions in $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ are given by reindexing matrices, restrictions in $\mathbb{S}\mathsf{pan}(\mathbf{C})$ are given by pullback, and restrictions in $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ and $\mathbb{P}\mathsf{rof}(\mathbf{C})$ are given by precomposing with functors.

Note that the restrictions in $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ and $\mathbb{P}\mathsf{rof}(\mathbf{C})$ are induced by those in $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ and $\mathbb{S}\mathsf{pan}(\mathbf{C})$, in the following general way.

7.4. Proposition. *If $\mathbb{X}$ is a virtual double category with restrictions, then $\mathbb{M}\mathsf{od}(\mathbb{X})$ also has restrictions.*

PROOF. If $B \xrightarrow{p} D$ is a bimodule in $\mathbb{X}$ and

$$\begin{array}{ccc} A_0 & \xrightarrow{A} & A_0 \\ {\scriptstyle f_0}\downarrow & \Downarrow f & \downarrow{\scriptstyle f_0} \\ B_0 & \xrightarrow[B]{} & B_0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C_0 & \xrightarrow{C} & C_0 \\ {\scriptstyle g_0}\downarrow & \Downarrow g & \downarrow{\scriptstyle g_0} \\ D_0 & \xrightarrow[D]{} & D_0 \end{array}$$

are monoid homomorphisms, then the restriction $p(g_0, f_0)$ in $\mathbb{X}$ becomes an (A, C)-bimodule in an obvious way, making it into the restriction $p(g, f)$ in $\mathbb{M}\mathsf{od}(\mathbb{X})$. ∎

The other ingredient in the construction of generalized multicategories also preserves restrictions.

7.5. PROPOSITION. *If $\mathbb{X}$ is a virtual double category with restrictions and T is a monad on $\mathbb{X}$, then $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$ also has restrictions.*

PROOF. Let $A \xrightarrow{p} TB$ be a horizontal arrow in $\mathbb{X}$, regarded as a horizontal arrow $A \nrightarrow B$ in $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$. It is easy to verify that the restriction of p along $C \xrightarrow{f} A$ and $D \xrightarrow{g} B$ in $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$ is given by the restriction $p(Tg, f)$ in $\mathbb{X}$. ∎

Therefore, if $\mathbb{X}$ has restrictions, so does $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$ for any monad T on $\mathbb{X}$. Moreover, by Proposition 5.5, $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$ always also has units. As suggested in the introduction to this section, units and restrictions together are an especially important combination, so we give a special name to this situation.

7.6. DEFINITION. *A **virtual equipment** is a virtual double category in which all units and all restrictions exist.*

We write $v\mathcal{E}quip$ for the locally full sub-2-category of $v\mathcal{D}bl$ determined by the virtual equipments and the normal restriction-preserving functors between them (however, see Theorem 7.24). We can now observe that $\mathbb{M}\mathsf{od}$ is a 2-functor from $v\mathcal{D}bl_r$ to $v\mathcal{E}quip$.

7.7. EXAMPLES. $\mathbb{S}\mathsf{pan}(\mathbf{C})$, $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$, and $\mathbb{P}\mathsf{rof}(\mathbf{C})$ are always virtual equipments, and $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ is a virtual equipment whenever $\mathbf{V}$ has an initial object preserved by $\otimes$. More generally, $\mathbb{M}\mathsf{od}(\mathbb{X})$ and $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$ are virtual equipments whenever $\mathbb{X}$ has restrictions.

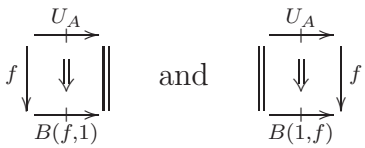
If $A \xrightarrow{f} B$ is a vertical arrow in a virtual equipment, we define its **base change objects** to be

$$B(1, f) = U_B(1, f) \qquad \text{and} \qquad B(f, 1) = U_B(f, 1).$$

These come with cartesian cells

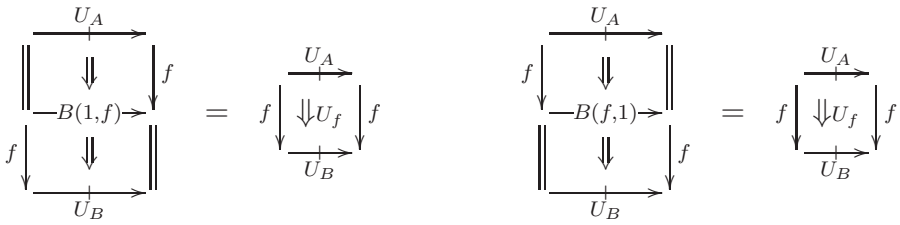
$$\begin{array}{ccc} & \xrightarrow{B(1,f)} & \\ {\scriptstyle f}\downarrow & \Downarrow & \| \\ & \xrightarrow[U_B]{} & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & \xrightarrow{B(f,1)} & \\ \| & \Downarrow & \downarrow{\scriptstyle f} \\ & \xrightarrow[U_B]{} & \end{array} \quad . \tag{7.8}$$

By factoring U_f through these cartesian cells, we obtain two further cells



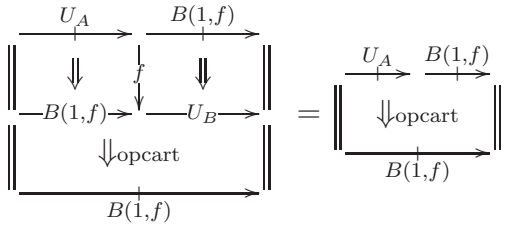
(7.9)

such that the following equations hold:

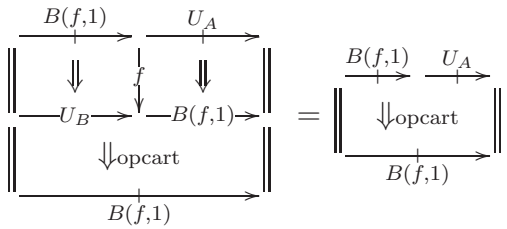


(7.10)

Moreover, the following equations also hold:



(7.11)



(7.12)

We can verify these by postcomposing each side with the appropriate cartesian cell, using the equations (7.10), and invoking the uniqueness of factorizations through cartesian cells. In the terminology of [DPP10], equations (7.10)–(7.12) are said to make $B(1,f)$ and $B(f,1)$ into a **companion** and a **conjoint** of f, respectively.

7.13. Example. In $\mathbb{S}\mathsf{pan}(\mathbf{C})$, the base change objects $B(1,f)$ and $B(f,1)$ are the spans

respectively. These are often called the *graph* of f.

7.14. Example. In $\mathbf{V}$-$\mathbb{M}\mathsf{at}$, for a function $f\colon X \longrightarrow Y$ the base change object $Y(1,f)$ is the matrix

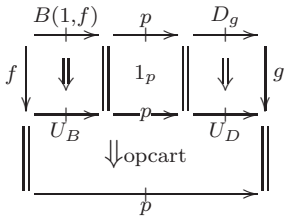
$$Y(1,f)(x,y) = \begin{cases} I & \text{if } f(x) = y \\ \emptyset & \text{otherwise.} \end{cases}$$

7.15. Example. In $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$, the base change objects $B(1,f)$ and $B(f,1)$ are the representable distributors defined by $B(1,f)(b,a) = B(b,fa)$ and $B(f,1)(a,b) = B(fa,b)$. Base change objects in $\mathbb{P}\mathsf{rof}(\mathbf{C})$ are analogous.

At first glance, base change objects may seem only to be a particular special case of restrictions. However, it turns out that *all* restrictions can be recovered by composition with the base change objects (hence the name).

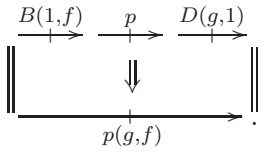
7.16. Theorem. *Let $B \overset{p}{\nrightarrow} D$ be a horizontal arrow and $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$ be vertical arrows in a virtual equipment. Then $B(1,f) \odot p \odot D(g,1)$ exists and is isomorphic to $p(g,f)$.*

Proof. Consider the composite



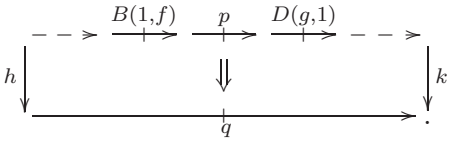
(7.17)

By the universal property of restriction, this factors through the cartesian cell defining $p(g,f)$ to give a canonical cell

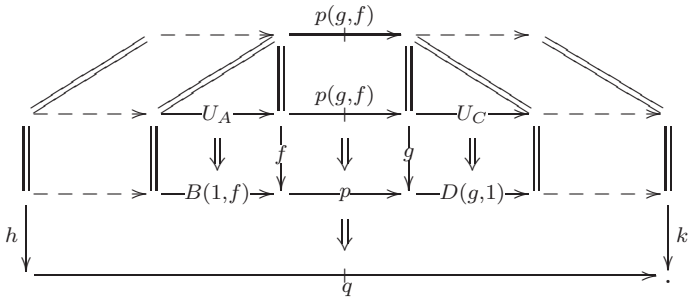


(7.18)

We claim that this cell is opcartesian. To show this, suppose given a cell



We need to factor it uniquely through (7.18). A factorization is given by the composite

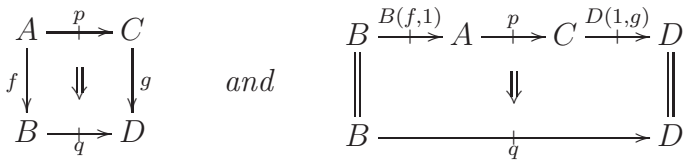


To verify that this is a factorization, and that it is unique, we use the equations (7.10)–(7.12). The details are similar to the proof of [Shu08, Theorem 4.1]. ∎

7.19. Corollary. *For vertical arrows $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$, the composite $B(1,f) \odot C(1,g)$ always exists and is isomorphic to $C(1,gf)$, and dually.* ∎

We also have the following dual result.

7.20. Theorem. *For arrows $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$ in a virtual equipment, we have a bijection between cells of the form*

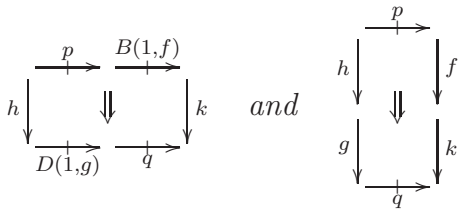


Proof. The inverse bijections are given by composing with the cells (7.8) and (7.9). (Recall that all composites with units exist in any virtual double category.) The fact that they are inverses follows from (7.10)–(7.12). ∎

It follows that in the situation of Theorem 7.20, if the composite $B(f,1) \odot p \odot D(1,g)$ exists, then it is a "corestriction" or "extension" of p along f and g—that is, it satisfies a universal property dual to that of a restriction.

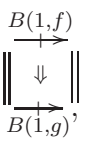
Combining Theorems 7.16 and 7.20, we obtain the following.

7.21. Corollary. *In a virtual equipment, there is a bijection between cells of the form*



Taking p and q to be units and h and k to be identities, we obtain:

7.22. Corollary. *For vertical arrows $f, g\colon A \longrightarrow B$ in a virtual equipment $\mathbb{X}$, there is a bijection between cells $f \longrightarrow g$ in $\mathcal{V}\mathbb{X}$ and cells*



which respects composition. Similarly, we have a bijection between cells $f \longrightarrow g$ and cells $B(g,1) \longrightarrow B(f,1)$. ∎

Now suppose that $\mathbb{X}$ is a virtual equipment which moreover has all composites. Then it has a horizontal bicategory $\mathcal{H}\mathbb{X}$, and Corollaries 7.19 and 7.22 imply that $f \mapsto B(1,f)$ defines a pseudofunctor $\mathcal{V}\mathbb{X} \longrightarrow \mathcal{H}\mathbb{X}$ which is locally full and faithful. Furthermore, it is easy to verify that $B(f,1)$ is right adjoint to $B(1,f)$ in $\mathcal{H}\mathbb{X}$.

This structure—a pseudofunctor which is bijective on objects, locally full and faithful, and which takes each 1-cell to one having a right adjoint—was defined in [Woo82] to constitute an **equipment**. This is the structure we referred to in the introduction to this section, which many authors have used where we would find double categories more natural. In fact, it is not hard to show (see [Shu08, Appendix C]) that an equipment in the sense of [Woo82] is *equivalent* to a virtual equipment which has all composites (this was called a *framed bicategory* in [Shu08]). Therefore, from now on we use *equipment* to mean a virtual equipment having all composites (thereby justifying the terminology "virtual equipment").

7.23. Remark. It was shown in [Shu08] that an equipment can equally well be defined as a pseudo double category in which all restrictions exist, or in which all "extensions" exist (in the sense mentioned after Theorem 7.20), or in which there exist base change objects with cells (7.8) and (7.9) satisfying (7.10)–(7.12). In the virtual case, we have a Goldilocks trifurcation: merely having base change objects is too weak, and having all extensions is too strong, but having restrictions (together with units) is just right.

We now consider how functors and transformations interact with restrictions.

7.24. Theorem. *Any functor F between virtual equipments preserves restriction.*

Proof. The proof given for equipments in [Shu08, Theorem 6.4] applies basically verbatim to virtual equipments. ∎

In particular, $v\mathcal{E}quip$ is in fact a *full* sub-2-category of $v\mathcal{D}bl_n$. Note, though, that an arbitrary functor F between virtual equipments still may not preserve units, so that while we have $F(B(1,f)) \cong FU_B(1,Ff)$, neither need be the same as $FB(1,Ff)$. Of course, they are the same if F is normal.

Now, recall that any transformation $\mathbb{X} \overset{F}{\underset{G}{\Downarrow \alpha}} \mathbb{Y}$ of functors between virtual equipments induces a strictly 2-natural transformation $\mathcal{V}(\alpha)$ of 2-functors between vertical 2-categories. In particular, we have $\alpha_B \circ F(f) = G(f) \circ \alpha_A$ for any vertical arrow $f\colon A \longrightarrow B$ in $\mathbb{X}$. However, we also have the cell component

$$\begin{array}{ccc} & \xrightarrow{F(B(1,f))} & \\ {\scriptstyle \alpha_A}\downarrow & \Downarrow \alpha_{B(1,f)} & \downarrow{\scriptstyle \alpha_B} \\ & \xrightarrow[G(B(1,f))]{} & \end{array}$$

of α. If F and G are normal, so that $F(B(1,f)) \cong FB(1,Ff)$ and $G(B(1,f)) \cong GB(1,Gf)$, then by Corollary 7.21, $\alpha_{B(1,f)}$ induces a 2-cell $\alpha_B \circ F(f) \longrightarrow G(f) \circ \alpha_A$, which seems to be trying to make $\mathcal{V}(\alpha)$ into an *oplax* natural transformation. Fortunately, however, this is an illusion.

7.25. Proposition. *In the above situation, the 2-cell $\alpha_B \circ F(f) \longrightarrow G(f) \circ \alpha_A$ induced by $\alpha_{B(1,f)}$ is an identity.*

PROOF. This follows by inspection of how this 2-cell is constructed, and use of the cell naturality of α. ∎

7.26. REMARK. Recall from Remark 5.17 that any functor $\mathbb{X} \xrightarrow{F} \mathbb{Y}$ between pseudo double categories induces a lax functor $\mathcal{H}(\mathbb{X}) \xrightarrow{\mathcal{H}(F)} \mathcal{H}(\mathbb{Y})$ between horizontal bicategories, but not every transformation $F \longrightarrow G$ induces a transformation $\mathcal{H}(F) \longrightarrow \mathcal{H}(G)$. It is true, however, that if $\mathbb{X}$ and $\mathbb{Y}$ are *equipments*, then any transformation $F \xrightarrow{\alpha} G$ induces an *oplax* transformation $\mathcal{H}(F) \xrightarrow{\mathcal{H}(\alpha)} \mathcal{H}(G)$ whose component at X is $GX(1, \alpha_X)$. Likewise, the components $(GX)(\alpha_X, 1)$ form a lax transformation $\mathcal{H}(G) \longrightarrow \mathcal{H}(F)$. See also Remark A.5.

8. Normalization

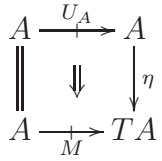
With the notion of virtual equipment under our belt, we now return to the general theory of generalized multicategories. We observed in §4 that for virtual double categories whose objects are "category-like," such as $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ and $\mathbb{P}\mathsf{rof}(\mathbf{C})$ (as opposed to those such as $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ and $\mathbb{S}\mathsf{pan}(\mathbf{C})$, whose objects are "set-like"), general T-monoids often contain too much structure. For instance, if T is the "free strict monoidal category" monad on $\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$, then a T-monoid consists of a category A, a multicategory M and a bijective-on-objects functor from A to the underlying category of M. Usually, the morphisms of A constitute superfluous data which we would like to eliminate. (This is not always true, though: in [Che04] these extra morphisms played an important role.)

The obvious way to eliminate this extra data, which we adopted in describing examples of this sort in §4, is to require A to be a discrete category; this way the extra morphisms simply do not exist. However, a different way to eliminate it is to require the given functor $A \longrightarrow M$ to induce an *isomorphism* between A and the underlying category of M; this way the extra morphisms exist, but are determined uniquely by the rest of the structure. In this section we define general analogues of both approaches, show their equivalence under general hypotheses, and argue that when they are not equivalent it is usually the *second* approach that is more useful. (This second approach was also the one taken in [Her01].)

8.1. DEFINITION. *Let $\mathbb{X}$ be a virtual equipment and let T be a monad on $\mathbb{M}\mathsf{od}(\mathbb{X})$. A T-monoid $A \xrightarrow{M} TA$ is called* ***object-discrete*** *if A is a monoid in $\mathbb{X}$ of the form U_X.*

We write $\mathsf{d}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$ for the full sub-virtual-equipment of $\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$ determined by the object-discrete T-monoids, and $d\mathcal{K}Mon(\mathbb{X}, T)$ for its vertical 2-category. Note that object-discreteness is only defined for a monad on a virtual equipment of the form $\mathbb{M}\mathsf{od}(\mathbb{X})$.

8.2. Definition. *Let T be a monad on a virtual equipment $\mathbb{X}$. A T-monoid $A \xrightarrow{M} TA$ is* ***normalized*** *if its unit cell*

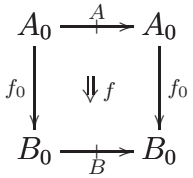


is cartesian in $\mathbb{X}$.

We write $\mathsf{n}\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$ for the full sub-virtual-equipment of $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$ determined by the normalized T-monoids, and $n\mathcal{K}Mon(\mathbb{X}, T)$ for its vertical 2-category. Unlike object-discreteness, normalization is defined for monads on any virtual equipment.

Now, to prove an equivalence between normalization and object-discreteness, we need to introduce the following definitions.

8.3. Definition. *A monoid homomorphism*



in a virtual double category $\mathbb{X}$ is called ***bijective on objects*** *(or* ***b.o.****) if f_0 is an isomorphism. It is called* ***fully faithful*** *(or* ***f.f.****) if the cell f is cartesian.*

8.4. Lemma. *If $A_0 \xrightarrow{A} A_0$ is a monoid in a virtual double category $\mathbb{X}$ with restrictions and $X \xrightarrow{f} A_0$ is any vertical arrow, then $X \xrightarrow{A(f,f)} X$ is also a monoid, and its defining cartesian cell is a monoid homomorphism with f as its vertical part.*

Proof. To obtain a multiplication for $A(f, f)$, we compose two copies of the defining cartesian cell with the multiplication of A, then factor the result through the defining cartesian cell. The unit is similar. ∎

8.5. Lemma. *If $\mathbb{X}$ has restrictions, then (b.o., f.f.) is a factorization system on the category* $\mathbf{Mon}(\mathbb{X})$ *of monoids and monoid homomorphisms in $\mathbb{X}$.*

Proof. Orthogonality is supplied by the universal property of cartesian cells, together with the fact that isomorphisms are orthogonal to anything. Factorizations are given by restriction along the vertical arrow component of a monoid homomorphism, using the previous lemma. ∎

8.6. THEOREM. *Let $\mathbb{X}$ be a virtual equipment, and let T be a monad on $\mathbb{M}\mathsf{od}(\mathbb{X})$ which preserves b.o. morphisms. Then*

(i) $\mathsf{d}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$ *is coreflective in* $\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$ *(that is, its inclusion has a right adjoint in $v\mathcal{E}quip$),*

(ii) $\mathsf{n}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$ *is reflective in* $\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$*, and*

(iii) the induced adjunction

$$\mathsf{d}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T) \rightleftarrows \mathsf{n}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$$

is an adjoint equivalence.

PROOF. We first prove (i). Let $A \xrightarrow{M} TA$ be a T-monoid in $\mathbb{M}\mathsf{od}(\mathbb{X})$, where $A_0 \xrightarrow{A} A_0$ is a monoid in $\mathbb{X}$. The unit of A is a monoid homomorphism $e\colon U_{A_0} \longrightarrow A$, so by Lemma 8.4, $M(Te, e)\colon U_{A_0} \nrightarrow T(U_{A_0})$ is an (object-discrete) T-monoid. Likewise, if $B \xrightarrow{N} TB$ is another T-monoid and $M \xrightarrow{p} TN$ is a horizontal arrow in $\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$, then $p(Te_B, e_A)$ is a horizontal arrow from $M(Te_A, e_A)$ to $N(Te_B, e_B)$. It is straightforward to extend these constructions to a functor $\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T) \longrightarrow \mathsf{d}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$.

Note that $M(Te, e)$ comes with a natural map to M, whose vertical arrow component is e, and likewise for $p(Te_B, e_A)$. This supplies the counit of the desired coreflection. We obtain the unit by observing that if M were already object-discrete, then e would be the identity, so we would have $M(Te, e) \cong M$. The triangle identities are easy to check.

Note that the horizontal arrow in $\mathbb{X}$ underlying $M(Te, e)$ is the restriction of $A_0 \xrightarrow{M} (TA)_0$ along the identity $e_0\colon A_0 = A_0$ and the map $(Te)_0\colon T(U_{A_0})_0 \longrightarrow (TA)_0$. Since e is b.o. and T preserves b.o. morphisms, $(Te)_0$ is an isomorphism; thus the coreflection of M leaves its underlying horizontal arrow in $\mathbb{X}$ essentially unmodified.

We now prove (ii); let A and M be as before. We first observe that the T-monoid M in $\mathbb{M}\mathsf{od}(\mathbb{X})$ has an underlying monoid in $\mathbb{M}\mathsf{od}(\mathbb{X})$, namely $M(\eta, 1)$. (This is a special case of a general functoriality result we will prove in [CS10a].) As noted in Remark 5.15, a monoid in $\mathbb{M}\mathsf{od}(\mathbb{X})$ consists of two monoids in $\mathbb{X}$ and a monoid homomorphism between them whose vertical arrow components are identities. In this case the first monoid is of course A. We denote the second by A' and the monoid homomorphism by $c\colon A \longrightarrow A'$. Note that the underlying horizontal arrow of A' in $\mathbb{X}$ is just $M(\eta, 1)$.

Now since T preserves b.o. morphisms, $TA \xrightarrow{Tc} TA'$ is b.o., hence $(TA)_0 \xrightarrow{(Tc)_0} (TA')_0$ is an isomorphism. By restricting along its inverse and using the identity $(A')_0 = A_0$, from $A_0 \xrightarrow{M} (TA)_0$ we obtain a horizontal arrow $(A')_0 \nrightarrow (TA')_0$. We abuse notation by continuing to denote this M (since restriction along isomorphisms leaves an arrow essentially unchanged). Now A' is a restriction of M, so it acts on M from the left via the multiplication of M. And since T preserves restrictions, TA' is a restriction of TM, so it also acts on M from the right via the multiplication of M. Thus, the horizontal arrow in $\mathbb{X}$ underlying M also admits the structure of a horizontal arrow $A' \nrightarrow TA'$ in $\mathbb{M}\mathsf{od}(\mathbb{X})$,

which we denote M'. Likewise, the multiplication and unit of the T-monoid M induce a multiplication and unit on M', making it also into a T-monoid, and we have a canonical T-monoid homomorphism $M \longrightarrow M'$ which is an isomorphism in $\mathbb{X}$. By definition of A', M' is normalized. There is an analogous construction on horizontal arrows, and together they extend straightforwardly to a functor $\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T) \longrightarrow \mathsf{n}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), T)$.

Now recall that A' came equipped with a b.o. monoid homomorphism $A \longrightarrow A'$. It is straightforward to check that this homomorphism underlies a T-monoid homomorphism $M \longrightarrow M'$; in this way we obtain the unit of the desired reflection. We obtain its counit by observing that if M is already normalized, then $A \cong A'$ and hence $M \cong M'$. The triangle identities are again easy to check.

Finally, to show (iii), we observe that the unit of the reflection and the counit of the coreflection are isomorphisms on the underlying horizontal arrow in $\mathbb{X}$ (since they are restrictions along an isomorphism). Moreover, the reflection and coreflection functors both *invert* morphisms with this property. Statement (iii) then follows formally. ∎

Recall that we began this section by observing that ordinary multicategories can be recovered as either object-discrete or normalized T-monoids, when T is the "free strict monoidal category" monad on $\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$. Since this T preserves b.o. morphisms, this statement is indeed an instance of Theorem 8.6. However, ordinary multicategories can also be obtained as *arbitrary* S-monoids, when S is the free monoid monad on $\mathbf{Set}$-$\mathbb{M}\mathsf{at} = \mathbb{S}\mathsf{pan}(\mathbf{Set})$. Noting that in this case $T = \mathbb{M}\mathsf{od}(S)$, we generalize this statement to the following.

8.7. THEOREM. *Let S be a monad on a virtual equipment $\mathbb{X}$. Then the monad $\mathbb{M}\mathsf{od}(S)$ on $\mathbb{M}\mathsf{od}(\mathbb{X})$ preserves b.o. morphisms, and we have a diagram*

$$\begin{array}{ccccc} \mathsf{d}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S)) & \rightleftarrows & \mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S)) & \rightleftarrows & \mathsf{n}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S)) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{K}\mathsf{Mod}(\mathbb{X}, S) & & \end{array}$$

which serially commutes (up to isomorphism). Moreover, the two diagonal functors

$$\mathsf{d}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S)) \longrightarrow \mathbb{K}\mathsf{Mod}(\mathbb{X}, S) \tag{8.8}$$

$$\mathsf{n}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S)) \longrightarrow \mathbb{K}\mathsf{Mod}(\mathbb{X}, S) \tag{8.9}$$

are equivalences.

PROOF. By definition, $\mathbb{M}\mathsf{od}(S)$ takes a monoid $A_0 \xrightarrow{\ |\ } A_0$ to $SA_0 \xrightarrow{\ |\ } SA_0$, so it preserves b.o. morphisms since S preserves isomorphisms. We define the middle vertical arrow

$$\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S)) \longrightarrow \mathbb{K}\mathsf{Mod}(\mathbb{X}, S) \tag{8.10}$$

by applying the 2-functor $\mathbb{M}\mathsf{od}$ to the functor

$$\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S)) \longrightarrow \mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, S)$$

which takes a monoid $A_0 \xrightarrow{A} A_0$ to its underlying object A_0, and similarly for horizontal arrows. (This is again a special case of the general functorial result of [CS10a].) Thus, (8.10) takes a $\mathbb{M}\mathsf{od}(S)$-monoid $A \xrightarrow{M} \mathbb{M}\mathsf{od}(S)(A)$ to the S-monoid $A_0 \xrightarrow{M} SA_0$.

We define the diagonal functors by composition with this, so that the triangles $\overrightarrow{\searrow\downarrow}$ and $\downarrow\swarrow$ commute by definition. The other triangles commute up to isomorphism because the reflection and coreflection were defined to fix A_0, replace A, and restrict M along an isomorphism, whereas (8.10) simply forgets about A.

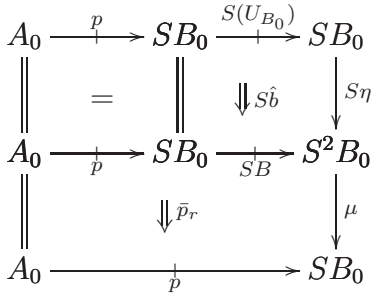
Now, by the 2-out-of-3 property for equivalences, it suffices to show that (8.8) is an equivalence. We will construct an explicit inverse to it. By Proposition 5.14, to construct a normal functor

$$\mathbb{K}\mathsf{Mod}(\mathbb{X}, S) \longrightarrow \mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S)) \tag{8.11}$$

it suffices to construct a not-necessarily-normal functor

$$\mathbb{K}\mathsf{Mod}(\mathbb{X}, S) \longrightarrow \mathbb{H}\text{-}\mathsf{Kl}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S)). \tag{8.12}$$

We define (8.12) on objects by sending an S-monoid $A_0 \xrightarrow{A} SA_0$ to U_{A_0}, and likewise on vertical arrows. A horizontal arrow $A \xrightarrow{p} B$ in $\mathbb{K}\mathsf{Mod}(\mathbb{X}, S)$ has an underlying horizontal arrow $A_0 \xrightarrow{p} SB_0$ in $\mathbb{X}$, which acquires a $S(U_{B_0})$-module structure from the following composite:



This defines (8.12) on horizontal arrows; its action on cells is straightforward. The induced normal functor (8.11) takes an S-monoid $A_0 \xrightarrow{A} SA_0$ to itself, regarded as an $(U_{A_0}, S(U_{A_0}))$-bimodule. Clearly (8.11) followed by the forgetful functor is the identity, while the composite in the other direction is precisely the coreflection functor into $\mathsf{d}\mathbb{K}\mathsf{Mod}(\mathbb{M}\mathsf{od}(\mathbb{X}), \mathbb{M}\mathsf{od}(S))$; this completes the proof. ∎

8.13. Examples. As remarked above, when S is the "free monoid" monad on $\mathbb{S}\mathsf{pan}(\mathbf{Set})$, this shows that ordinary multicategories (i.e. S-monoids) can also be identified with object-discrete or normalized $\mathbb{M}\mathsf{od}(S)$-monoids. Likewise, virtual double categories are S-monoids for the "free category" monad, and thus can also be identified with object-discrete or normalized $\mathbb{M}\mathsf{od}(S)$-monoids.

8.14. Example. Since topological spaces can be identified with U-monoids in $\mathbb{R}\mathsf{el}$, they can also be identified with object-discrete or normalized $\mathbb{M}\mathsf{od}(U)$-monoids in $\mathbf{2}$-$\mathbb{P}\mathsf{rof}$. In the terminology of [Tho09], a $\mathbb{M}\mathsf{od}(U)$-monoid is a *modular topological space*. It is normalized precisely when its order is the specialization order, so that it is equivalent to an ordinary topological space—i.e. a U-monoid, as required by Theorem 8.7.

By no means are all interesting monads on $\mathbb{M}\mathsf{od}(\mathbb{X})$ of the form $\mathbb{M}\mathsf{od}(S)$. However, many of them do preserve b.o. morphisms, so that Theorem 8.6 at least applies.

8.15. Example. The "free symmetric strict monoidal category" monad on $\mathbf{Set}$-$\mathbb{P}\mathsf{rof} = \mathbb{M}\mathsf{od}(\mathbf{Set}\text{-}\mathbb{M}\mathsf{at})$ preserves b.o. morphisms but is not of the form $\mathbb{M}\mathsf{od}(S)$ for any monad S on $\mathbf{Set}$-$\mathbb{M}\mathsf{at}$. We have seen in Example 4.16 that object-discrete T-monoids are symmetric multicategories; hence so are normalized T-monoids.

8.16. Example. The "free category with strictly associative finite products" monad on $\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$ also preserves b.o. morphisms but is not of the form $\mathbb{M}\mathsf{od}(S)$. We have seen in Example 4.17 that object-discrete T-monoids are multi-sorted Lawvere theories; hence so are normalized T-monoids.

8.17. Non-Example. Recall from Example 4.19 that *clubs* are T-monoids in $\mathbb{S}\mathsf{pan}(\mathbf{Cat})$ with a discrete category of objects, where T is a monad like the previous two. However, since $\mathbb{S}\mathsf{pan}(\mathbf{Cat})$ is not of the form $\mathbb{M}\mathsf{od}(\mathbb{X})$, the theory of this section does not apply to clubs. In particular, their "object-discreteness" is not an instance of our definition, and is not the same as normalization.

When T does not preserve b.o. morphisms, however, normalized and discrete T-monoids can be quite different, even on a virtual equipment of the form $\mathbb{M}\mathsf{od}(\mathbb{X})$. Intuitively, saying that T preserves b.o. morphisms says that the possible domains of multimorphisms in a T-multicategory depend only on its objects. If this fails to be true, then how many morphisms are included in the underlying monoid can change what these possible domains are.

8.18. Example. Let T be the "free category with equalizers" monad on $\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$. Then T evidently does not preserve b.o. morphisms, but it is the identity on discrete categories. Therefore, an object-discrete T-monoid is just a category, whereas a normalized T-monoid can have morphisms whose domain is a "formal equalizer" of ordinary morphisms.

More interestingly, normalized T-monoids for the "free category with finite limits" monad (which also does not preserve b.o. morphisms) can be considered a generalization of Lawvere theories to finite-limit logics. We can also continue to generalize to more powerful logics (or "doctrines").

These examples suggest that when T does not preserve b.o. morphisms, it is often the normalized, rather than the object-discrete, T-monoids that better capture the desired notion of T-multicategory. Note also that normalization makes sense for any monad on a virtual equipment, while object-discreteness only makes sense for monads on virtual

equipments of the form $\mathbb{M}\mathsf{od}(\mathbb{X})$. Finally, we will see in the next section that normalized T-monoids are the most natural notion to compare with pseudo T-algebras. This inspires us to take normalized T-monoids as our preferred definition of "generalized multicategory," and to make the following informal definition.

8.19. Definition. *If T is a monad on a virtual equipment for which (possibly pseudo) T-algebras are called* widgets*, then normalized T-monoids are called* ***virtual widgets****.*

The reasons for this definition were summarized in the introduction. In §9 we will prove that any widget has an underlying virtual widget, further justifying the terminology. Of course, we have seen that a number of types of virtual algebras already have their own names, such as "multicategory" and "Lawvere theory." When such common names exist, we of course use them in preference to terms such as "virtual monoidal category" or "category with virtual finite products."

Note that "virtual widget" is, strictly speaking, ambiguous: knowing the notion of widget determines at most the vertical 2-category $\mathcal{V}\mathbb{X}$ and the 2-monad $\mathcal{V}T$, rather than $\mathbb{X}$ and T themselves. However, many 2-categories that arise in practice come with an obvious "natural" extension to a virtual equipment, so in practice there is little ambiguity. (In fact, there is a general construction of an equipment from a well-behaved 2-category; see [CJSV94].) One case of ambiguity is if "widget" is the name for T-algebras in **Set** or **Cat**, but we consider T-monoids in $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ or $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$; in this case we may speak of **$\mathbf{V}$-enriched virtual widgets**.

8.20. Remark. The discussion above suggests that when the objects of $\mathbb{X}$ are category-like, it is the normalized T-monoids (i.e. virtual T-algebras) that are more important, while when the objects of $\mathbb{X}$ are set-like, it is the non-normalized T-monoids (i.e. virtual $\mathbb{M}\mathsf{od}(T)$-algebras) that are more important. This does seem to *usually* be the case, but there are exceptions on both sides, such as the following.

- As we have already remarked, the multicategories of [BD98] and [Che04] are non-normalized T-monoids, when T is the "free symmetric strict monoidal category" monad on **Set**-$\mathbb{P}\mathsf{rof}$ (whose objects are obviously category-like).
- Let U be the ultrafilter monad on $\mathbb{R}\mathsf{el}$, whose objects are set-like. We have seen that a U-monoid is just a topological space, but it is easy to verify that a U-monoid is normalized just when it is a T_1-space—certainly also an important concept.

9. Representability

We now turn to a general version of the comparison between monoidal categories and multicategories. Of course, we first need to identify the analogue of a monoidal category in the general case. We saw in §8 that ordinary multicategories have two different faces in our setup: they are the S-monoids where S is the "free monoid" monad on $\mathbb{S}\mathsf{pan}(\mathbf{Set})$, and also the normalized T-monoids, where $T = \mathbb{M}\mathsf{od}(S)$ is the "free strict monoidal category"

monad on **Set**-$\mathbb{P}\mathsf{rof}$. Monoidal categories, however, are more visible from the second point of view: they are the *pseudo* $\mathcal{V}(T)$*-algebras* in $\mathcal{V}(\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof}) = \mathbf{Cat}$.

Accordingly, in this section we will assume that T is a monad on a virtual equipment $\mathbb{X}$ whose objects are "category-like," and seek to compare (pseudo) $\mathcal{V}(T)$-algebras with (normalized) T-monoids. We will additionally have to assume that T is a *normal* monad as defined in §6, since otherwise it doesn't even induce a 2-monad on $\mathcal{V}(\mathbb{X})$. If we are given instead a monad S on a virtual double category whose objects are "set-like," then in order to apply the theory of this section we simply consider $\mathbb{M}\mathsf{od}(S)$ instead; some examples of this can be found later on. Generalizing the terminology of [Lei04, p. 165], we may call a (pseudo) $\mathcal{V}(\mathbb{M}\mathsf{od}(S))$-algebra an S**-structured monoid**.

Actually, the most natural approach to the comparison turns out to be via *oplax* T-algebras. Recall that for a 2-monad T on a 2-category, an **oplax** T**-algebra** is an object A with a map $a\colon TA \longrightarrow A$ and 2-cells

$$\begin{array}{ccc} T^2A & \xrightarrow{\mu} & TA \\ {\scriptstyle Ta}\downarrow & \swarrow_{\overline{a}} & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\eta} & TA \\ & \searrow^{\widehat{a}} & \downarrow{\scriptstyle a} \\ & & A \end{array} \tag{9.1}$$

satisfying certain straightforward axioms. We call it *normal* if $\widehat{a}$ is an isomorphism, and a *pseudo* T*-algebra* if both $\widehat{a}$ and $\overline{a}$ are isomorphisms. Finally, if T is a monad on a virtual equipment, we will always abuse terminology by saying "T-algebra" (with appropriate prefixes) to mean $\mathcal{V}(T)$-algebra.

9.2. Theorem. *Let T be a normal monad on a virtual equipment $\mathbb{X}$. Then:*

(i) Any oplax T-algebra $TA \xrightarrow{a} A$ in $\mathcal{V}\mathbb{X}$ gives rise to a T-monoid $A \xrightarrow[|]{A(a,1)} TA$, which is normalized if and only if A is normal.

(ii) A T-monoid $A \xrightarrow[|]{M} TA$ arises from an oplax T-algebra if and only if $M \cong A(a,1)$ for some vertical arrow $TA \xrightarrow{a} A$.

Proof. If $a\colon TA \longrightarrow A$ is an oplax T-algebra, then by definition of $A(a,1)$, the 2-cell $\widehat{a}$ induces a unit cell

$$\begin{array}{ccc} A & \xrightarrow[|]{U_A} & A \\ \| & \Downarrow & \downarrow{\scriptstyle \eta} \\ A & \xrightarrow[|]{}_{A(a,1)} & TA. \end{array}$$

Likewise, by the dual of Corollary 7.21, $\overline{a}$ induces a multiplication

$$\begin{array}{ccccc} A & \xrightarrow[|]{A(a,1)} & TA & \xrightarrow[|]{(TA)(Ta,1)} & T^2A \\ \| & & \Downarrow & & \downarrow{\scriptstyle \mu} \\ A & & \xrightarrow[|]{}_{A(a,1)} & & TA \end{array}$$

and using the isomorphism $(TA)(Ta,1) \cong T(A(a,1))$ (since T is normal) we obtain a multiplication cell. The axioms to make $A(a,1)$ into a T-monoid follow directly from the axioms for an oplax T-algebra. To complete (i), we observe that $\widehat{a}$ is an isomorphism if and only if the induced cell $U_A \longrightarrow A(a\eta,1) \cong A(a,1)(\eta,1)$ is an isomorphism, which says precisely that the unit defined above is cartesian. Conversely, if $A \xrightarrow{M} TA$ is a T-monoid and $M \cong A(a,1)$, then the same bijections supply 2-cells $\overline{a}$ and $\widehat{a}$ satisfying the same axioms making $a\colon TA \longrightarrow A$ into an oplax T-algebra; this shows (ii). ∎

The following example may serve to clarify the connection between normality of oplax T-algebras and normalization of T-monoids.

9.3. Example. Let T be the "free strict monoidal category" monad on $\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$. Then an oplax T-algebra is an *oplax monoidal category*: a category A equipped with tensor product functors

$$\begin{aligned} A \times \cdots \times A &\longrightarrow A \\ (x_1, \ldots, x_n) &\mapsto \langle x_1 \otimes \ldots \otimes x_n \rangle \end{aligned}$$

for $n \geq 0$, and transformations

$$\begin{aligned} \langle x \rangle &\longrightarrow x \\ \langle x_{11} \otimes \ldots \otimes x_{nk_n} \rangle &\longrightarrow \langle \langle x_{11} \otimes \ldots \otimes x_{1k_1} \rangle \otimes \ldots \otimes \langle x_{n1} \otimes \ldots \otimes x_{nk_n} \rangle \rangle \end{aligned}$$

satisfying certain evident axioms. Note that the 0-ary tensor product $\langle\rangle$ is a "lax unit" and the 1-ary tensor product $\langle x \rangle$ is not necessarily isomorphic to x, only related by the given unit transformation $\langle x \rangle \longrightarrow x$.

As mentioned previously, a T-monoid consists of a category A, a multicategory M with the same objects, and an identity-on-objects functor from A to the underlying ordinary category of M. Now Theorem 9.2 says that we can make an oplax monoidal category into a T-algebra by defining the multimorphisms in M from $(x_1, \ldots, x_n)$ to y to be the morphisms $\langle x_1 \otimes \ldots \otimes x_n \rangle \longrightarrow y$ in A.

Note that the morphisms from x to y in the underlying ordinary category of M are the morphisms from $\langle x \rangle$ to y in A. The functor $A \longrightarrow M$ is defined by composing with the unit transformation $\langle x \rangle \longrightarrow x$. Clearly this is fully faithful (i.e. the T-monoid is normalized) just when $\langle x \rangle \longrightarrow x$ is an isomorphism (i.e. the oplax T-algebra is normal).

The following characterization of pseudo T-algebras is now obvious.

9.4. Corollary. *A normalized T-monoid $A \xrightarrow{M} TA$ arises from a pseudo T-algebra if and only if*

(i) $M \cong A(a,1)$ for some $TA \xrightarrow{a} A$, and

(ii) the induced 2-cell $\overline{a}$ is an isomorphism.

We say that a normalized T-monoid is **weakly representable** if it satisfies (i), and **representable** if it satisfies both (i) and (ii) (hence is equivalent to a pseudo T-algebra).

9.5. Example. When T is the "free strict monoidal category" monad on $\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$, Corollary 9.4 specializes to the characterization of monoidal categories as representable multicategories, as in [Her00] and [Lei04, §3.3]. We will see in §B.14 that it also includes the general representability notion of [Her01]. The analogue of Theorem 9.2 in the language of [Bur71] can be found in [Pen09], which uses "representable" for what we call "weakly representable" and "lax algebra" for what we call an "oplax algebra."

9.6. Remark. Strictly speaking, the notion of monoidal category obtained in this way is the "unbiased" version, which is equipped with a specified n-ary tensor product for all $n \geq 0$, instead of the usual "biased" version having only a binary and nullary product (see [Lei04, §3.1]). This is generally what happens for pseudoalgebras: if T is a monad whose strict algebras are some strict structure, then pseudo T-algebras are an "unbiased" sort of weak structure. Generally the unbiased version is equivalent to the biased one, but there is real mathematical content in this statement; for instance, the equivalence of biased and unbiased monoidal categories is essentially equivalent to Mac Lane's coherence theorem.

9.7. Example. Recall that virtual double categories can be identified with S-monoids for the "free category" monad S on directed graphs, and hence also with normalized $\mathbb{M}\mathsf{od}(S)$-monoids. In this case it is easy to check that for a normalized $\mathbb{M}\mathsf{od}(S)$-monoid $A \xrightarrow{M} TA$, we have $M \cong A(a, 1)$ iff every composable string of horizontal arrows is the source of a *weakly* opcartesian arrow (see Remark 5.8). Thus, such virtual double categories can be identified with "normal oplax double categories," which are equipped with n-ary composites for all n and comparison maps

$$\langle p_{11} \odot \cdots \odot p_{nk_n} \rangle \longrightarrow \langle \langle p_{11} \odot \cdots \odot p_{1k_1} \rangle \odot \cdots \odot \langle p_{n1} \odot \cdots \odot p_{nk_n} \rangle \rangle$$

and invertible comparison maps

$$\langle p \rangle \xrightarrow{\cong} p$$

satisfying analogous axioms to an oplax monoidal category. Condition (ii) in Corollary 9.4 is then equivalent to requiring weakly opcartesian cells to be closed under composition. As observed in Remark 5.8, this suffices to ensure we have a pseudo double category, i.e. a pseudo $\mathbb{M}\mathsf{od}(S)$-algebra.

9.8. Example. Let U be the ultrafilter monad on $\mathbb{R}\mathsf{el}$. We have seen that a U-monoid is a topological space, and a normalized U-monoid is a T_1-space. A vertical U-algebra (which is automatically strict, since $\mathcal{V}(\mathbb{R}\mathsf{el})$ is locally discrete) is a compact Hausdorff space, and in this case Theorem 9.2 tells us what we already knew: any compact Hausdorff space is, in particular, a T_1 topological space.

Now consider the induced monad $\mathbb{M}\mathsf{od}(U)$ on $\mathbb{M}\mathsf{od}(\mathbb{R}\mathsf{el})$. The objects of $\mathbb{M}\mathsf{od}(\mathbb{R}\mathsf{el}) = \mathbf{2}$-$\mathbb{P}\mathsf{rof}$ are preorders. In the language of [Tho09], a strict $\mathbb{M}\mathsf{od}(U)$-algebra is an *ordered compact Hausdorff space*, whereas by Theorem 8.7 a normalized $\mathbb{M}\mathsf{od}(U)$-monoid is simply a topological space. Thus, Theorem 9.2 tells us that any ordered compact Hausdorff space

X can be equipped with a topology in which an ultrafilter $\mathfrak{F}$ converges to a point y if and only if the (unique) limit of $\mathfrak{F}$ in X is $\leq y$ in the given preorder.

The next three examples can all be found in [Her01] (see B.14 for more on the comparison between our setting and Hermida's).

9.9. Example. Let $\mathbf{S}$ be a small category and T the monad on $\mathbf{C} = \mathbf{Set}^{\mathrm{ob}(\mathbf{S})}$ whose algebras are functors $\mathbf{S} \longrightarrow \mathbf{Set}$, as in Example 4.10, and consider the monad $\mathbb{M}\mathsf{od}(T)$ on $\mathbb{M}\mathsf{od}(\mathbb{S}\mathsf{pan}(\mathbf{C})) = \mathbb{P}\mathsf{rof}(\mathbf{C})$. A strict $\mathbb{M}\mathsf{od}(T)$-algebra is a functor $\mathbf{S} \longrightarrow \mathbf{Cat}$, while a pseudo $\mathbb{M}\mathsf{od}(T)$-algebra is a pseudofunctor $\mathbf{S} \longrightarrow \mathbf{Cat}$. Now by Theorem 8.7, normalized $\mathbb{M}\mathsf{od}(T)$-monoids can be identified with T-monoids, which as we saw can be identified with functors $\mathbf{A} \longrightarrow \mathbf{S}$. It is then easy to verify that a normalized $\mathbb{M}\mathsf{od}(T)$-monoid satisfies 9.4(i) iff the corresponding functor $\mathbf{A} \longrightarrow \mathbf{S}$ admits all weakly opcartesian liftings, and 9.4(ii) iff weakly opcartesian arrows are closed under composition. Thus, in this case Corollary 9.4 specializes to the classical equivalence between pseudofunctors $\mathbf{S} \longrightarrow \mathbf{Cat}$ and opfibrations over $\mathbf{S}$.

9.10. Example. Let T be as in Example 9.9, but now consider T-monoids rather than $\mathbb{M}\mathsf{od}(T)$-monoids. A T-monoid is normalized just when for each $x \in \mathbf{S}$, the induced span $A(x,1) \longleftarrow M \longrightarrow A(x,1)$ is the identity span; i.e. when the fibers of $\mathbf{A} \longrightarrow \mathbf{S}$ are discrete categories. Such a normalized T-monoid satisfies 9.4(i) iff $\mathbf{A} \longrightarrow \mathbf{S}$ admits all weakly opcartesian liftings, which in this case are automatically opcartesian by discreteness. Thus, Corollary 9.4 also specializes to the equivalence of functors $\mathbf{S} \longrightarrow \mathbf{Set}$ and *discrete* opfibrations over $\mathbf{S}$.

9.11. Example. Let T be the "free strict ω-category" monad on $\mathbb{S}\mathsf{pan}(\mathbf{Globset})$, for which we saw in Example 4.11 that T-monoids on 1 are globular multicategories. Pseudo T-algebras are an "unbiased" version of the *monoidal globular categories* of [Bat98]. Thus any monoidal globular category has an underlying globular multicategory, and can be characterized among the latter by a representability property.

The requirement that T be normal in Theorem 9.2 cannot be dispensed with.

9.12. Example. Let P be the extension of the powerset monad to $\mathbb{R}\mathsf{el}$ described in Example 3.7. Since $\mathcal{V}(\mathbb{R}\mathsf{el}) = \mathbf{Set}$ is locally discrete, oplax P-algebras are just P-algebras in $\mathbf{Set}$, which can be identified with complete meet-semilattices (the structure map $PA \longrightarrow A$ takes a subset $X \subseteq A$ to its meet $\bigwedge X$).

Now, we have observed in Example 4.14 that P-monoids can be identified with closure spaces. If we attempt to follow the prescription of Theorem 9.2, starting from a complete join-semilattice we would define the "closure operation" $c(X) = \{\bigwedge X\}$; but this is neither extensive nor monotone.

On the other hand, if we first apply $\mathbb{M}\mathsf{od}$, we obtain a monad $\mathbb{M}\mathsf{od}(P)$ on $\mathbb{M}\mathsf{od}(\mathbb{R}\mathsf{el}) = \mathbf{2}\text{-}\mathbb{P}\mathsf{rof}$, which is normal. By Theorem 8.7, normalized $\mathbb{M}\mathsf{od}(P)$-monoids can be identified with P-monoids, i.e. closure spaces. With a little effort, pseudo $\mathbb{M}\mathsf{od}(P)$-algebras can

be identified with meet-complete preorders (that is, preorders that are complete as categories). Theorem 9.2 then tells us that from a meet-complete preorder we can construct a closure space with $c(X) = \{x \mid \bigwedge X \leq x\}$, which is certainly true.

We can also make the correspondence of Theorem 9.2 functorial. Recall that for any T we have a 2-category $\mathcal{K}Mon(\mathbb{X}, T)$ of T-monoids, defined to be the vertical 2-category of $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$. It turns out that while T-monoids correspond to *oplax* T-algebras, morphisms of T-monoids correspond to *lax* T-algebra morphisms. Recall that a *lax T-morphism* between oplax T-algebras consists of a map $f\colon A \longrightarrow B$ and a 2-cell

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \swarrow_{\overline{f}} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

satisfying certain straightforward axioms. And if $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are two such, a *T-transformation* is a 2-cell $f \xrightarrow{\alpha} g$ such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & \swarrow_{\overline{f}} & \downarrow{\scriptstyle b} \\ A & \overset{f}{\underset{g}{\Downarrow \alpha}} & B \end{array} \quad = \quad \begin{array}{ccc} TA & \overset{Tf}{\underset{Tg}{\Downarrow T\alpha}} & TB \\ {\scriptstyle a}\downarrow & \swarrow_{\overline{f}} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B. \end{array}$$

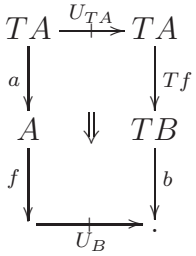
We write $\mathcal{O}plax\text{-}T\text{-}\mathcal{A}lg_l$ for the resulting 2-category.

9.13. Theorem. *Let T be a normal monad on a virtual equipment $\mathbb{X}$. Then there is a strict 2-functor $\mathcal{O}plax\text{-}T\text{-}\mathcal{A}lg_l \longrightarrow \mathcal{K}Mon(\mathbb{X}, T)$, whose underlying 1-functor is fully faithful, and which becomes 2-fully-faithful (that is, an isomorphism on hom-categories) when restricted to* normal *oplax T-algebras.*

Proof. For oplax T-algebras (A, a) and (B, b), regarded as T-monoids $A \xrightarrow{A(a,1)} TA$ and $B \xrightarrow{B(b,1)} TB$, a morphism of T-monoids consists of a vertical arrow $f\colon A \longrightarrow B$ and a 2-cell

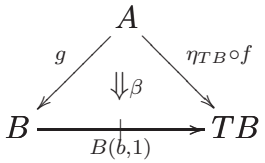
$$\begin{array}{ccc} A & \xrightarrow{A(a,1)} & TA \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle Tf} \\ B & \xrightarrow[B(b,1)]{} & TB \end{array}$$

satisfying certain axioms. But by definition of $B(b,1)$, and by Theorem 7.20 applied to $A(a,1)$, this 2-cell is equivalent to one



This defines a 2-cell $b \circ Tf \longrightarrow f \circ a$ in $\mathcal{V}(\mathbb{X})$, which is precisely the additional data required to make f into a lax T-algebra morphism. It is easy to verify that the axioms of a T-monoid morphism are equivalent to the axioms of a lax T-algebra morphism under this translation, and that composition is preserved.

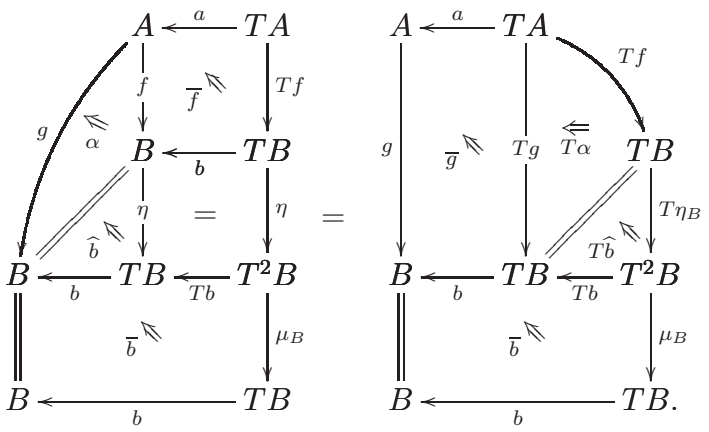
Now let $f, g\colon A \longrightarrow B$ be two such morphisms, and recall from §6 that a T-monoid transformation $\beta\colon f \longrightarrow g$ consists of a cell



satisfying a certain axiom. Equivalently, β is a 2-cell $b\eta f \longrightarrow g$ in $\mathcal{V}(\mathbb{X})$. Thus, given a T-algebra transformation $\alpha\colon f \longrightarrow g$, it is natural to define β to be the composite

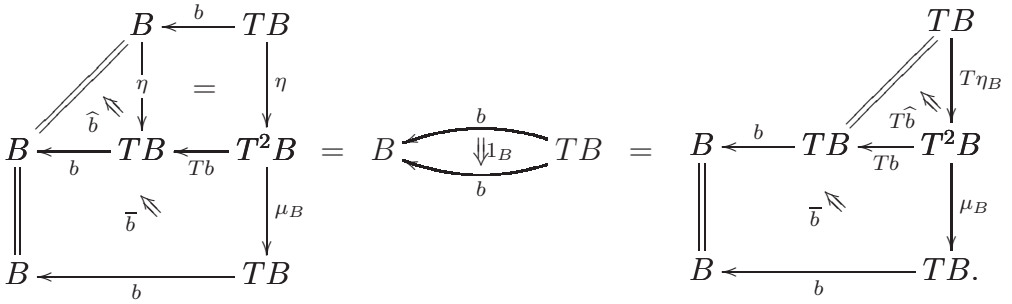
$$b\eta f \xrightarrow{\widehat{b}f} f \xrightarrow{\alpha} g,$$

where $\widehat{b}\colon b\eta \longrightarrow 1$ is the oplax unit map of B. With this definition, the axiom that must be satisfied for β to be a T-monoid transformation becomes the following equality of pasting diagrams in $\mathcal{V}\mathbb{X}$.



(9.14)

The cell marked "=" is an identity by Proposition 7.25 applied to a cell component of η. Now, two of the axioms for an oplax T-algebra say that



After removing these composites from (9.14), what is left is simply the equation for α to be a T-algebra transformation; thus α is such precisely when $\beta = \alpha.\widehat{b}f$ is a T-monoid transformation. And, of course, if B is normal, then $\widehat{b}f$ is an isomorphism, and so α can be recovered uniquely from β. We leave it to the reader to verify that this association preserves both types of 2-cell composition. ∎

The restriction to *normal* oplax algebras in the final statement of Theorem 9.13 cannot be dispensed with either.

9.15. Example. Let A and B be oplax monoidal categories, regarded as T-monoids for the "free strict monoidal category" monad T as in Example 9.3, and let $A \overset{f}{\underset{g}{\rightrightarrows}} B$ be lax monoidal functors. A T-algebra transformation $f \longrightarrow g$ is, in particular, a natural transformation $f \longrightarrow g$, and therefore has components $fx \xrightarrow{\alpha_x} gx$. However, if we unravel the definition of a T-monoid transformation, we see that its components are of the form $\langle fx \rangle \xrightarrow{\beta_x} gx$. Thus, when B is not normal, there can be no bijection in general.

9.16. Remark. Recall from §1 that generalized multicategories can be regarded as "algebraic theories." For instance, ordinary multicategories correspond to strongly regular finitary theories, while Lawvere theories correspond to arbitrary finitary theories. In language introduced by Jon Beck, one may say that the monad T provides the "doctrine" in which the theories are written. (Motivated by this, some authors use the word *doctrine* to mean simply a 2-monad.) If X is a T-monoid, regarded as a theory in the doctrine T, and A is a pseudo T-algebra, then it is natural to define a *model* of X in A to be a T-monoid homomorphism from X to (the underlying T-monoid of) A.

Now, frequently the functor of Theorem 9.13 has a left adjoint F_T when restricted to pseudo T-algebras and morphisms. In such a case, a model of X in A can equally be defined as a T-algebra morphism $F_T X \longrightarrow A$. That is, $F_T X$ is the "free T-algebra containing a model of X." Following [Law63], this is sometimes called the "functorial semantics" of T.

For example, when X is a Lawvere theory, $F_T X$ is the category with finite products that incarnates it. (In fact, as we remarked in Example 4.17, Lawvere theories are often

defined to be certain categories with finite products.) Likewise, when X is a Lawvere $\mathbf{V}$-theory, then $F_T X$ is the $\mathbf{V}$-category with finite cotensors that incarnates it (see [Pow99]) and when X is an ordinary or non-symmetric operad, $F_T X$ is the "PROP" or "PRO" associated to it (see [BV73]).

This adjunction can also be used to characterize representability. It turns out that the strict (resp. pseudo) algebras for the induced 2-monad $T_\dashv$ on $\mathcal{K}Mon(\mathbb{X}, T)$ can be identified with strict (resp. pseudo) T-algebras. Moreover, $T_\dashv$ is a "lax-idempotent" 2-monad in the sense of [KL97], so that A is a pseudo $T_\dashv$-algebra precisely when the unit $A \longrightarrow T_\dashv A$ has a left adjoint. Thus the "structure" imposed by the 2-monad T has been transformed into "property-like structure" imposed by $T_\dashv$. In particular cases, these observations can be found in [Her00, Her01, Pen09]; in [CS10b] we will study them in our general context.

A. Composites in $\mathbb{M}\mathsf{od}$ and $\mathbb{H}$-Kl

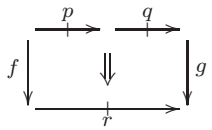
In this appendix we consider the question of when $\mathbb{M}\mathsf{od}(\mathbb{X})$ and $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$ have composites and units, which will be needed for our comparisons with existing theories in the next appendix. The first case is easy; the following was also observed in [Shu08].

A.1. Theorem. *If $\mathbb{X}$ is an equipment in which each category $\mathcal{H}\mathbb{X}(A, B)$ has coequalizers, which are preserved on both sides by $\odot$, then $\mathbb{M}\mathsf{od}(\mathbb{X})$ is also an equipment.*

Proof (Sketch). We have seen already that $\mathbb{M}\mathsf{od}(\mathbb{X})$ always has units, and inherits restrictions from $\mathbb{X}$, so it remains only to construct composites. We define the composite of bimodules $A \overset{p}{\longrightarrow\!\!\!\!\!\!\mid}\; B \overset{q}{\longrightarrow\!\!\!\!\!\!\mid}\; C$ to be the coequalizer of the two actions of B:

$$p \odot B \odot q \rightrightarrows p \odot q \twoheadrightarrow p \odot_B q$$

and similarly for longer composites. Given a cell



in $\mathbb{M}\mathsf{od}(\mathbb{X})$, to factor it through $p \odot_B q$, we first factor it through a cartesian cell to obtain a cell $(p, q) \longrightarrow r(g, f)$ in $\mathcal{H}\mathbb{X}(A, C)$, then factor this through the coequalizer in $\mathcal{H}\mathbb{X}(A, C)$, and finally compose again with the cartesian cell. Thus $(p, q) \longrightarrow p \odot_B q$ is weakly opcartesian; to show that these cells compose we use the fact that $\odot$ preserves coequalizers. ∎

Note that we require $\mathbb{X}$ to have restrictions, as well as composites, in order to show that $\mathbb{M}\mathsf{od}(\mathbb{X})$ has composites. We could instead assume explicitly that the coequalizers in $\mathcal{H}\mathbb{X}(A, B)$ satisfy a universal property relative to all cells, but in practice this generally tends to hold only because of the existence of restrictions.

A.2. EXAMPLE. If $\mathbf{V}$ has small colimits preserved by $\otimes$, then $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ satisfies the hypotheses of Theorem A.1, so (as we have seen) $\mathbf{V}$-$\mathbb{P}\mathsf{rof}$ is an equipment.

A.3. EXAMPLE. If $\mathbf{C}$ has pullbacks and coequalizers preserved by pullback, then $\mathbb{S}\mathsf{pan}(\mathbf{C})$ satisfies the hypotheses of Theorem A.1, so (as we have also seen) $\mathbb{P}\mathsf{rof}(\mathbf{C})$ is an equipment.

We have also already seen that $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$ always inherits restrictions from $\mathbb{X}$. However, to show that it has composites, we require fairly strong conditions not just on $\mathbb{X}$ but on T as well. Recall that a functor between pseudo double categories (and, therefore, also between equipments) is called *strong* if it preserves all composites. We then make the following definition:

A.4. DEFINITION. *A transformation $F \xrightarrow{\alpha} G$ of functors $\mathbb{X} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{Y}$ between equipments is **horizontally strong** if for every horizontal arrow $X \xrightarrow{p} Y$ in $\mathbb{X}$, the cell induced by α_p under the bijection of Corollary 7.21:*

$$\begin{array}{ccccc} FX & \xrightarrow[|]{Fp} & FY & \xrightarrow[|]{GY(1,\alpha_Y)} & GY \\ \| & & \Downarrow & & \| \\ FX & \xrightarrow[|]{GX(1,\alpha_X)} & GX & \xrightarrow[|]{Gp} & GY \end{array}$$

*is an isomorphism. A monad T on an equipment $\mathbb{X}$ is **horizontally strong** if T is a strong functor and μ and η are horizontally strong.*

A.5. REMARK. Recall from Remark 7.26 that any transformation α of functors between equipments induces an oplax transformation $\mathcal{H}(\alpha)$ of functors between horizontal bicategories. Horizontal strength of α is equivalent to requiring $\mathcal{H}(\alpha)$ to be a strong (aka pseudo) transformation (hence the name).

A.6. EXAMPLE. Let $\mathbf{C} \underset{G}{\overset{F}{\Downarrow \alpha}} \mathbf{D}$ be a transformation between pullback-preserving functors between categories with pullbacks. It is not hard to verify that the induced transformation $\mathbb{S}\mathsf{pan}(\alpha)$ is horizontally strong if and only if α is a **cartesian natural transformation**, meaning that all its naturality squares are pullbacks. Therefore, if T is a pullback-preserving monad on $\mathbf{C}$, the monad $\mathbb{S}\mathsf{pan}(\mathbf{T})$ is horizontally strong if and only if μ and η are cartesian natural transformations. Such a T is often called a **cartesian monad**; see §B.1.

A.7. EXAMPLE. We have remarked that most of the monads from Example 3.7 are not even strong functors in general, with the exception of the ultrafilter monad on $\mathbb{R}\mathsf{el}$. One can verify that for this monad, the multiplication transformation is horizontally strong, but the unit transformation is not; hence it is not a horizontally strong monad.

A.8. THEOREM. *If T is a horizontally strong monad on an equipment $\mathbb{X}$, then $\mathbb{H}$-$\mathsf{Kl}(\mathbb{X}, T)$ is also an equipment.*

PROOF (SKETCH). It suffices to show that $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$ has composites. A composable string of horizontal arrows

$$X_0 \xrightarrow{p_1} X_1 \xrightarrow{p_2} \cdots \xrightarrow{p_n} X_n$$

in $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$ consists of horizontal arrows $X_i \xrightarrow{p_{i+1}} TX_{i+1}$ in $\mathbb{X}$. Since $\mathbb{X}$ is an equipment, we can form the composite

$$p_1 \odot T(p_2) \odot \cdots \odot T^{n-1}(p_n) \odot TX_n(1, \mu^{n-1})$$

in $\mathbb{X}$, which clearly supplies a *weakly* opcartesian cell in $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$. Likewise, $TX(1, \eta)$ is a weak unit for X in $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$. The assumptions on T are required to show that these weakly opcartesian cells compose, or equivalently that this composition is associative; rather than write this out in detail we merely compute the 3-fold associativity isomorphism for $X \xrightarrow{p} Y \xrightarrow{q} Z \xrightarrow{r} W$.

$$\begin{aligned}
&(p \odot Tq \odot TZ(1, \mu)) \odot Tr \odot TW(1, \mu) \\
&\qquad \cong p \odot Tq \odot T^2 r \odot T^2 W(1, \mu T) \odot TW(1, \mu) && \text{(strength of } \mu) \\
&\qquad \cong p \odot Tq \odot T^2 r \odot T^2 W(1, T\mu) \odot TW(1, \mu) && \text{(associativity of } \mu) \\
&\qquad \cong p \odot Tq \odot T^2 r \odot T(TW(1, \mu)) \odot TW(1, \mu) && \text{(normality of } T) \\
&\qquad \cong p \odot T(q \odot Tr \odot TW(1, \mu)) \odot TW(1, \mu) && \text{(strength of } T).
\end{aligned}$$

Of course, in the unit isomorphisms η is used instead of μ. ∎

A.9. COROLLARY. *If T is a horizontally strong monad on an equipment $\mathbb{X}$, such that each category $\mathcal{H}\mathbb{X}(A, B)$ has coequalizers that are preserved by $\odot$ on both sides and also preserved by T, then $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$ is also an equipment.*

PROOF. The hypotheses ensure that $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$ satisfies the conditions of Theorem A.1. ∎

A.10. EXAMPLE. If T is a cartesian monad on a category $\mathbf{C}$ with pullbacks, then we have seen that the induced monad on $\mathbb{S}\mathsf{pan}(\mathbf{C})$ is horizontally strong; thus $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{S}\mathsf{pan}(\mathbf{C}), T)$ is an equipment. If furthermore $\mathbf{C}$ has coequalizers that are preserved by pullback and by T, then Corollary A.9 implies that $\mathbb{K}\mathsf{Mod}(\mathbb{S}\mathsf{pan}(\mathbf{C}), T)$ is also an equipment.

For example, the "free M-set" monad $(M \times -)$ on **Set** preserves coequalizers, so we have an equipment of M-graded categories. However, the "free monoid" monad does not preserve coequalizers, and the virtual equipment of ordinary multicategories is not an equipment.

A.11. EXAMPLE. Let $\mathbf{V}$ be a symmetric monoidal category with small coproducts preserved by $\otimes$ on both sides, and let T be the extension of the "free monoid" monad on **Set** to a monad on $\mathbf{V}\text{-}\mathbb{M}\mathsf{at}$ defined in Example 3.12. We have already remarked in Examples 5.11 that T is strong, and an easy calculation shows that it is in fact horizontally strong. Thus, $\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{V}\text{-}\mathbb{M}\mathsf{at}, T)$ is an equipment. However, even if $\mathbf{V}$ is cocomplete, and in particular

has coequalizers preserved by $\otimes$ on both sides, these coequalizers will not in general be preserved by T. Thus, the virtual equipment $\mathbb{K}\mathsf{Mod}(\mathbf{V}\text{-}\mathbb{M}\mathsf{at}, T)$ of $\mathbf{V}$-enriched ordinary multicategories fails to be an equipment. (For $\mathbf{V} = \mathbf{Set}$ we have seen this already in the previous example.)

A.12. Example. Let $\mathbf{V}$ be a cocomplete symmetric monoidal category with small colimits preserved by $\otimes$ on both sides, and let T be the "free symmetric strict monoidal $\mathbf{V}$-category" monad on $\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}$ from Examples 3.14 and 4.16. We remarked in Examples 5.12 that T is strong. In fact, it is easily seen to be horizontally strong, so that $\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}, T)$ is an equipment. As in the previous example, however, T fails to preserve coequalizers in $\mathcal{H}(\mathbf{V}\text{-}\mathbb{P}\mathsf{rof})(A, B)$, so that $\mathbb{K}\mathsf{Mod}(\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}, T)$ is not an equipment even when $\mathbf{V} = \mathbf{Set}$.

When the horizontal arrows in $\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof}, T)$ are identified with the *generalized structure types* of [FGHW08] as in Remark 4.20, their horizontal composites are identified with the *substitution* operation on structure types. In [FGHW08] the bicategory $\mathcal{H}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof}, T))$ was constructed in this way from structure types and substitution.

A.13. Example. Let T be the "free category with strictly associative finite products" monad on $\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}$ from Examples 3.15 and 4.17. We remarked in Examples 5.12 that T is strong if $\mathbf{V}$ is cartesian monoidal. In fact, in this case it can moreover be shown to be horizontally strong, so that $\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}, T)$ is an equipment.

Now we specialize to $\mathbf{V} = \mathbf{Set}$. If 1 denotes the terminal category, then $T1$ is equivalent to $\mathbf{Set}_f^{\mathrm{op}}$, the opposite of the category of finite sets. Thus, a profunctor $1 \nrightarrow T1$ is equivalent to a functor $\mathbf{Set}_f \longrightarrow \mathbf{Set}$, which (since $\mathbf{Set}$ is locally finitely presentable) is equivalent to a finitary endofunctor of $\mathbf{Set}$. It is then not hard to verify that the equivalence

$$\mathcal{H}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof}, T))(1, 1) \simeq [\mathbf{Set}_f, \mathbf{Set}]$$

is actually an equivalence of monoidal categories, and thus induces an equivalence between categories of monoids. But a monoid in $\mathcal{H}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof}, T))(1, 1)$ is a T-monoid $1 \nrightarrow T1$, i.e. a Lawvere theory; thus we recover the classical result of [Law63] that Lawvere theories can be identified with *finitary monads* on $\mathbf{Set}$.

An analogous argument for the "free $\mathbf{V}$-category with finite cotensors" monad on $\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}$ from Examples 3.16 and 4.18 reproduces the result of [Pow99] that Lawvere $\mathbf{V}$-theories can be identified with finitary $\mathbf{V}$-monads on $\mathbf{V}$. In this case, $\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}, T)$ seemingly need not be an equipment, but at least the multicategory $\mathbb{H}\text{-}\mathsf{Kl}(\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}, T)(1, 1)$ is a monoidal category, precisely because it can be identified with the monoidal category of finitary endofunctors of $\mathbf{V}$ under composition.

A.14. Example. Let $\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof}_{\mathrm{ls}}$ denote the virtual double category whose objects are *locally small* categories (that is, large categories with small hom-sets), whose vertical arrows are functors, and whose horizontal arrows are profunctors taking values in *small* sets. Then there is a monad T on $\mathbf{Set}\text{-}\mathbb{P}\mathsf{rof}_{\mathrm{ls}}$ whose algebras are categories with all *small* products, and we have $T1 \simeq \mathbf{Set}^{\mathrm{op}}$. Thus, by analogous reasoning to Example A.13, we see that T-monoids $1 \nrightarrow T1$ for this T can be identified with *arbitrary* monads on

Set. We can likewise obtain monads on any suitable $\mathbf{V}$ by using the monad for arbitrary cotensors on $\mathbf{V}$-$\mathbb{P}\mathsf{rof}_{\mathrm{ls}}$. In [CS10a] we will see that by regarding monads as particular generalized multicategories in this way, we can recover the monad associated to an operad (as originally defined in [May72]) as a particular case of the functoriality of generalized multicategories.

B. Comparisons to previous theories

We now describe the existing approaches to generalized multicategories, and show how they compare to our theory. Most existing approaches turn out to be instances of our theory, applied to a particular sort of monad on a particular sort of virtual equipment. Unsurprisingly, however, often more can be said in such special cases that is not true in general. Thus, in each section below we briefly mention some of the additional results that different authors have obtained in their particular contexts.

B.1. Cartesian Monads. In order to study and define a type of n-category, Leinster developed a theory of cartesian monads and their associated multicategories. This theory was developed in a series of papers, eventually culminating in his book [Lei04]. Recall the following from Example A.6:

B.2. Definition. *Let $\mathbf{E}$ be a category with pullbacks. A monad (T, η, μ) on $\mathbf{E}$ is **cartesian** if T preserves pullbacks, and all naturality squares of η and μ are pullbacks.*

Given a cartesian monad, Leinster constructs a bicategory $\mathbf{E}_{(T)}$ of *T-spans* and defines a *$(\mathbf{E}, T)$-multicategory* (or simply a *T-multicategory*) to be a monad in $\mathbf{E}_{(T)}$. To compare this to our context, recall that whenever $\mathbf{E}$ has pullbacks, $\mathbb{S}\mathsf{pan}(\mathbf{E})$ is an equipment, and any pullback-preserving monad T on such a $\mathbf{E}$ extends to a strong monad on $\mathbb{S}\mathsf{pan}(\mathbf{E})$, which is horizontally strong just when T is cartesian. It is then easy to see:

B.3. Proposition. *For a cartesian monad T on $\mathbf{E}$, Leinster's bicategory $\mathbf{E}_{(T)}$ is isomorphic to $\mathcal{H}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{S}\mathsf{pan}(\mathbf{E}), T))$.*

Therefore, Leinster's category of $(\mathbf{E}, T)$-multicategories is the vertical category of our $\mathbb{K}\mathsf{Mod}(\mathbb{S}\mathsf{pan}(\mathbf{E}), T)$. In particular, what he calls a *T-operad* is a T-monoid $1 \nrightarrow T1$ in $\mathbb{S}\mathsf{pan}(\mathbf{E})$.

Actually, Leinster constructs the whole virtual double category $\mathbb{K}\mathsf{Mod}(\mathbb{S}\mathsf{pan}(\mathbf{E}), T)$ (which he calls an **fc**-multicategory) in [Lei04, §5.3], and uses it to define transformations of his generalized multicategories just as we did in §6. (The notes at the end of his §5.3 also point in the direction of our §§5 and 7.)

Leinster also proves most of the results of our §9 in his context, as well as the functoriality of the construction mentioned in Remark 4.4. Furthermore, he shows that if T is a "suitable" cartesian monad on a "suitable" cartesian category $\mathbf{E}$, then the category of $(\mathbf{E}, T)$-multicategories is itself monadic over a cartesian category of "T-graphs," and this monad is also cartesian. Thus the process can be iterated, leading to a definition of the "opetopes" used in the Baez-Dolan definition of weak n-categories. Finally, Leinster

also studies *algebras* for generalized operads, which are closely related to the horizontal arrows in $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T)$.

B.4. Clubs. Essentially the same theory was developed in [Kel92] for the case of generalized operads (generalized multicategories on 1). Observe that when T is a cartesian monad on a category $\mathbf{E}$ with finite limits, so that $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{S}\mathsf{pan}(\mathbf{E}), T)$ is an equipment, then in particular the category

$$\mathbf{E}/T1 \simeq \mathcal{H}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{S}\mathsf{pan}(\mathbf{E}), T))(1, 1)$$

inherits a monoidal structure. It is easy to verify that this monoidal structure on $\mathbf{E}/T1$ is the same as that constructed by Kelly (see the explicit description in [Kel92, p. 174–175]). Kelly defined a *club over* T to be a monoid for this monoidal structure; thus such clubs can be identified with T-monoids $1 \nrightarrow T1$ in $\mathbb{S}\mathsf{pan}(\mathbf{E})$. He also proved that such clubs are essentially equivalent to cartesian monads equipped with a "cartesian map" to T.

(Actually, in [Kel92] it was assumed that T preserves *certain* pullbacks rather than all pullbacks. This suffices to construct a monoidal structure on $\mathbf{E}/T1$, though not for T to define a monad on all of $\mathbb{S}\mathsf{pan}(\mathbf{E})$.)

B.5. Pseudomonads on Prof. The existing theory which is probably closest to our approach involves the construction of a Kleisli bicategory from a pseudomonad on a bicategory such as $\mathbf{V}\text{-}\mathbb{P}\mathsf{rof}$. A general theory of multicategories based on such pseudomonads does not appear to exist in the literature, but it is implicit in [BD98, Che04, FGHW08, Gar08, DS03] among other places.

The general framework is, however, quite simple to state: from a pseudomonad T on a bicategory $\mathcal{B}$, one can construct the Kleisli bicategory $\mathcal{B}_T$ and consider monads in $\mathcal{B}_T$ as a notion of generalized multicategory. (Leinster's approach is a special case of this for $\mathcal{B} = \mathbb{S}\mathsf{pan}(\mathbf{C})$, as is Hermida's for a different bicategory—see §B.14, below.) The relationship with our theory is that if T is a horizontally strong monad on an equipment $\mathbb{X}$, it gives rise to a pseudomonad $\mathcal{H}(T)$ on the bicategory $\mathcal{H}(\mathbb{X})$, and $\mathcal{H}(\mathbb{X})_{\mathcal{H}(T)} \simeq \mathcal{H}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T))$ so that the resulting notions of multicategory agree.

In the converse direction, of course not every pseudomonad on $\mathcal{H}(\mathbb{X})$ arises from a monad on $\mathbb{X}$ itself, but we have seen that this is true for most monads relative to which one may want to define generalized multicategories. In a few cases, however, the extension to $\mathbb{X}$ may not be vertically strict, necessitating the extension of $v\mathcal{E}quip$ to a tricategory.

Note that if T is also *co-horizontally strong*, in the sense that its horizontal dual is horizontally strong, then it also induces a pseudo-*comonad* $\hat{\mathcal{H}}(T)$ on the bicategory $\mathcal{H}(\mathbb{X})$. From this perspective, T-monoids can be identified with *lax* $\hat{\mathcal{H}}(T)$*-coalgebras*. Of course, if we work in the horizontal dual, then $\hat{\mathcal{H}}(T)$ becomes a pseudomonad and T-monoids are its lax algebras. This is the terminology used by several authors, including Hermida. The authors of [DS03] consider the special case when the bicategory $\mathcal{B}$ is monoidal and T is its free monoid monad, so that T-monoids can be called *lax monoids*. Such lax monoids can also be described directly in terms of $\mathcal{B}$, without the need for cocompleteness hypotheses to ensure that T exists.

B.6. $(T, \mathbf{V})$-ALGEBRAS. Following Barr [Bar70], [CT03] started a series of papers which described the ideas of a "set-monad with lax extension to $\mathbf{V}$-$\mathcal{M}at$", as well as the $(T, \mathbf{V})$-algebras associated to these monads. Barr's original idea showed that the "lax algebras" of the ultrafilter monad are topological spaces; Clementino and Tholen's idea extended this further, developing a framework that eventually included not only topological spaces, but also metric spaces, approach spaces, and closure spaces.

In the work on $(T, \mathbf{V})$-algebras, two definitions of set-monad with lax extension to $\mathbf{V}$-$\mathcal{M}at$ have been proposed. The original version was applicable to all monoidal $\mathbf{V}$. However, it failed to capture all relevant examples, and so a second, slightly different definition was proposed in [Sea05], which captured further examples. However, this definition was only applicable when $\mathbf{V}$ was a preorder. As we shall see, Seal's definition turns out to be equivalent to asking for a monad on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ (in the case when $\mathbf{V}$ is ordered), while the original definition is equivalent to asking for a monad on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ which is normal.

The original definition, given in [CT03], was as follows:

B.7. DEFINITION. *A* ***set monad with lax extension to*** $\mathbf{V}$ *consists of a monad* (T, η, μ) *on* $\mathbf{Set}$*, together with a lax functor* T_M *on* $\mathbf{V}$-$\mathcal{M}at$ *such that:*

- T_M *is the same as* T *on objects and functions (viewed as* $\mathbf{V}$*-matrices),*
- *the comparisons* $(T_M s)(T_M r) \longrightarrow T_M(sr)$ *are isomorphisms when* r *is a function,*
- *when viewed as transformations on* T_M*,* η *and* μ *have op-lax structure.*

In general, however, this definition was found to be too restrictive, as it didn't allow for examples such as extensions of the powerset monad, whose algebras would be closure spaces. To include this type of example, the requirement that T_M was the same as T on functions needed to be removed. Seal's definition, given in [Sea05], was the following:

B.8. DEFINITION. *Suppose that* $\mathbf{V}$ *is a monoidal preorder. A* ***set monad with lax extension to*** $\mathbf{V}$ *consists of a monad* (T, η, μ) *on* $\mathbf{Set}$*, together with a lax functor* T_M *on* $\mathbf{V}$-$\mathcal{M}at$ *which is the same as* T *on objects, and satisfies*

(i) $Tf \leq T_M f$*,*

(ii) $(Tf)^\circ \leq T_M f^\circ$*.*

Here, $()^\circ$ denotes taking the opposite $\mathbf{V}$-matrix. By [Sea05, p. 225] the conditions imply that if q is a $\mathbf{V}$-matrix and f a function, then $T_M(qf) = (T_M q)(Tf)$ and $T_M(f^\circ q) = (Tf)^\circ(T_M q)$.

In [Sea05, p. 203], he also shows that when $\mathbf{V}$ is completely distributive, η and μ have op-lax structure. However, this is not a priori required in his definition. If, however, we include this axiom in his definition, then his notion of set monad with lax extension is equivalent to giving our notion of a monad on the virtual double category $\mathbf{V}$-$\mathbb{M}\mathsf{at}$.

B.9. Proposition. *Suppose that* $\mathbf{V}$ *is a monoidal preorder. If* T *is a set monad with lax extension* T_M *(in the sense of Seal) for which* η *and* μ *are op-lax, then we can define a monad* T *on* $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ *which is* T *on vertical arrows, and* T_M *on horizontal arrows. Conversely, given a monad* T *on* $\mathbf{V}$-$\mathbb{M}\mathsf{at}$*, we can define a set monad with lax extension which is* T *on functions, and uses the horizontal action of* T *to define* T_M*.*

Proof. Suppose that we have a set monad with lax extension to $\mathbf{V}$, in the second sense given above. Define a functor T on the double category $\mathbf{V}$-$\mathbb{M}\mathsf{at}$, which is T on vertical arrows, and T_M on horizontal arrows. Using the η and μ, we get all of the necessary data for a monad on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$, with the exception of checking that

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle g} \\ W & \xrightarrow[q]{} & Z \end{array} \quad \text{implies} \quad \begin{array}{ccc} TX & \xrightarrow{Tp} & TY \\ {\scriptstyle Tf}\downarrow & \Downarrow & \downarrow{\scriptstyle Tg} \\ TW & \xrightarrow[Tq]{} & TZ \end{array}$$

This is equivalent to checking that $p \leq g^\circ qf \Rightarrow T_M(p) \leq (Tg)^\circ T_M q(Tf)$. But this is easy to check by using the two results given after Seal's definition.

$$T_M(p) \leq T_M(g^\circ qf) = (Tg)^\circ T_M(qf) = (Tg)^\circ (T_M q)(Tf)$$

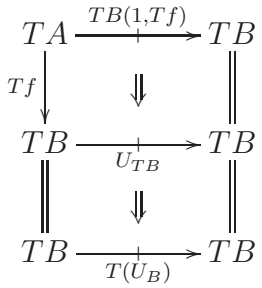
Conversely, suppose that we have a monad T on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$. We would like to define a lax extension of T (considered as a $\mathbf{Set}$-monad) to $\mathbf{V}$-$\mathcal{M}at$. Define T_M on matrices as for T. The only conditions we need to check are $Tf \leq T_M f$ and $(Tf)^\circ \leq T_M f^\circ$. To show the first is equivalent to showing that $TB(1, Tf) \leq T(B(1, f))$. To show this, recall that we have a cartesian cell

$$\begin{array}{ccc} A & \xrightarrow{B(1,f)} & B \\ {\scriptstyle f}\downarrow & \Downarrow & \| \\ B & \xrightarrow[U_B]{} & B \end{array}$$

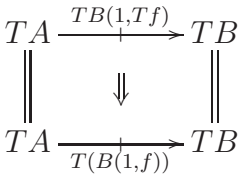
Moreover, since T is a functor, it preserves cartesian cells (Theorem 7.24), and so

$$\begin{array}{ccc} TA & \xrightarrow{T(B(1,f))} & TB \\ {\scriptstyle Tf}\downarrow & \Downarrow & \| \\ TB & \xrightarrow[T(U_B)]{} & TB \end{array}$$

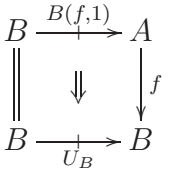
is also cartesian. We can thus factor the cell



through it to get a cell



as required. The second inequality follows similarly, using the cartesian cell



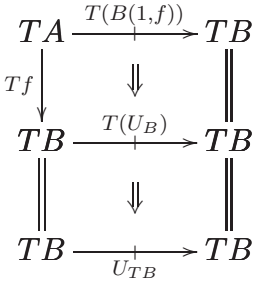
Thus, a monad on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ defines a set-monad with lax extension to $\mathbf{V}$-$\mathcal{M}at$. ∎

For general $\mathbf{V}$, we can also use the above correspondence to recover the first notion of notion of set-monad with lax extension: they are the monads which are normal.

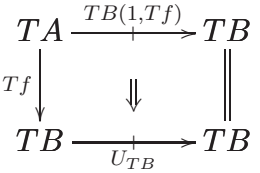
B.10. Proposition. *Using the above correspondence, we get a set-monad with lax extension in the first sense if and only if the monad T on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ is normal.*

Proof. Suppose we have a set-monad T with lax extension T_M in the first sense. Then we have $T_M f \cong Tf$ for all functions f, and we get a monad on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$. Moreover, we also have $T(B(1,f)) \cong TB(1,Tf)$. In particular, we have $T(A(1,1)) \cong TA(1,1)$. But $A(1,1) \cong U_A$, so we have $T(U_A) \cong U_{TA}$. Thus T is normal.

Conversely, suppose that we have a monad T on $\mathbf{V}$-$\mathbb{M}\mathsf{at}$ which is normal. We would like to show that $T(B(1,f)) \cong TB(1,Tf)$. Using the factoring as for the above proposition, we get a cell in one direction. To get the other direction, we factor



through



(note that the bottom cell on the left exists by normality of T). The composites of the two cells are identities by the universal property of the cartesian cells, and so we have $T(B(1,f)) \cong TB(1,Tf)$, as required. We also need to check the condition that the comparison cell be an isomorphism when r is a function. However, this is equivalent to asking that $T(r(1,s)) \cong Tr(1,Ts)$, and this follows from Theorem 7.24. ∎

For a set-monad with lax extension T, the category of $(T,\mathbf{V})$-algebras that Tholen, Clementino and Seal define is exactly the vertical category of the virtual double category $\mathbb{K}\mathsf{Mod}(\mathbf{V}\text{-}\mathbb{M}\mathsf{at},T)$. They also describe $(T,\mathbf{V})$-modules, and these are the horizontal arrows of $\mathbb{K}\mathsf{Mod}(\mathbf{V}\text{-}\mathbb{M}\mathsf{at},T)$.

In addition to providing a more conceptual explanation of the notion of lax extension, and a way to compare $(T,\mathbf{V})$-algebras with other notions of generalized multicategory, our general framework improves the theory of $(T,\mathbf{V})$-algebras in two ways. Firstly, it gives a context in which the horizontal Kleisli construction makes sense; such a construction has been recognized as desirable (see, for example, [Tho07, p. 7]), but is impossible using only bicategories since the monads used in this case are not horizontally strong. Secondly, it provides a general reason for the observation of [Tho07, p. 15] that any set-monad with lax extension to $\mathbf{V}\text{-}\mathcal{M}at$ can also be extended to a monad on $\mathbf{V}\text{-}\mathcal{C}at$ with a lax extension to $\mathbf{V}\text{-}\mathcal{P}rof$ (we simply apply the 2-functor $\mathbb{M}\mathsf{od}$).

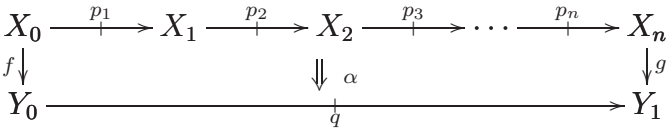
There are, however, other special aspects of the theory of $(T,\mathbf{V})$-algebras which we have not discussed. One is that the category of $(T,\mathbf{V})$-algebras is generally *topological* over **Set**, and has many other similar formal properties. Another is the use of the "pro" construction found in [CHT04]. This is useful to describe additional topological structures; for example, monoids in pro-$\mathbb{R}\mathsf{el}$ are quasi-uniform spaces. In general, given a virtual double category $\mathbb{X}$, one can define a new virtual double category pro-$\mathbb{X}$, and go on to describe "pro-generalized multicategories". Further discussion of this, however, awaits a future paper.

B.11. Non-cartesian monads. The earliest work on generalized multicategories was by Burroni in [Bur71]. His framework is very similar to Leinster's (see §B.1) except that he requires nothing at all about the monad T, not even that it preserve pullbacks (although the category $\mathbf{E}$ must still have pullbacks). This level of generality does not fit into our existing framework, since if T does not preserve pullbacks then it does not induce a functor on $\mathbb{S}\mathsf{pan}(\mathbf{E})$ of the sort we have defined. However, it does induce an *oplax functor* between pseudo double categories.

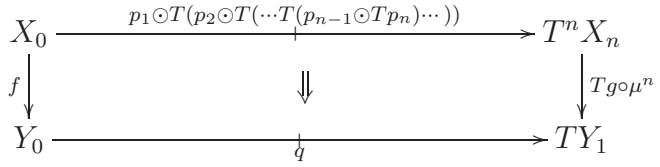
The simplest way for us to define an oplax functor is to say it is a functor between pseudo double categories regarded as co-virtual double categories. Pseudo double categories, oplax functors, and transformations form a 2-category, and we define an *oplax monad* on a pseudo double category to be a monad in this 2-category. Since *any* functor on a category $\mathbf{E}$ with pullbacks induces an oplax functor on $\mathbb{S}\mathsf{pan}(\mathbf{E})$, any monad on such an $\mathbf{E}$ induces an oplax monad on $\mathbb{S}\mathsf{pan}(\mathbf{E})$. Moreover, we can extend our framework to deal with oplax functors as follows.

B.12. Definition. *If T is an oplax monad on a pseudo double category $\mathbb{X}$, the* ***horizontal Kleisli virtual double category*** $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$ *of T is defined as follows.*

- *Its objects, vertical and horizontal arrows, and cells with nullary source are defined as when T is a lax functor.*
- *A cell*

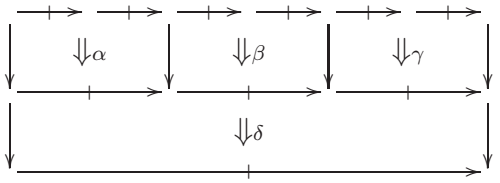


  in $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$ is a cell



  in $\mathbb{X}$.

- *Composition is defined using the multiplication, unit, and oplax structure of T. For example, the composite of*



  is given by the composite

$$\delta \circ (\alpha \odot T(\beta \odot T\gamma)) \circ (1 \odot \mu) \circ (1 \odot T(1 \odot \mu)) \circ T_{\odot}$$

  in $\mathbb{X}$,where $T_{\odot}$ is a composite of the oplax structure maps of T:

$$p_1 \odot T(p_2 \odot T(p_3 \odot T(p_4 \odot T(p_5 \odot T p_6)))) \longrightarrow p_1 \odot T p_2 \odot T^2(p_3 \odot T p_4 \odot T^2(p_5 \odot T p_6))$$

Note that when T is a strong functor, both definitions of $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$ make sense; however, in this case, they are equivalent.

B.13. Definition. *If T is an oplax monad on a pseudo double category $\mathbb{X}$, then a* ***T-monoid*** *is a monoid in $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$, and we write $\mathbb{K}\mathsf{Mod}(\mathbb{X}, T) = \mathbb{M}\mathsf{od}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T))$.*

To compare this to Burroni's definition, note that $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$ clearly has weak composites, and hence is an oplax double category (see Example 9.7). Burroni works instead

with what he calls a *pseudo-category*, and what we would probably call a *lax-biased bicategory*: a bicategory-like structure with units and binary composites and noninvertible comparison maps

$$p \longrightarrow p \odot U_A$$
$$p \longrightarrow U_B \odot p$$
$$p \odot (q \odot r) \longrightarrow (p \odot q) \odot r.$$

satisfying suitable axioms. In fact, of course, Burroni's "pseudo-category of T-spans" extends to a lax-biased *double* category. Moreover, any lax-biased double category defines an oplax double category (and hence a virtual double category), if we take the n-ary composite to be

$$\langle p_1 \odot \cdots \odot p_n \rangle = p_1 \odot (p_2 \odot \cdots (p_{n-1} \odot p_n) \cdots).$$

(Burroni points this out as well; see [Bur71, p. 66, example 3]. He refers to virtual double categories as simply "multicatégories".) In this way, Burroni's $\mathrm{Sp}(T)$ is identified with our $\mathcal{H}(\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{S}\mathsf{pan}(\mathbf{E}), T))$, and thus his "$T$-categories" can be identified with our T-monoids. However, he must move outside the bicategory (or "pseudo-category") framework to define functors of T-categories, whereas they emerge naturally from our setup.

Burroni also constructs the left adjoint F_T from Remark 9.16, and proves the functoriality of his construction under lax morphisms of monads. Working with his language, [Pen09] gives a version of the representability results from §9.

B.14. Cartesian 2-monads. We have described the horizontal arrows $A \nrightarrow B$ in $\mathbf{Set}$-$\mathbb{P}\mathsf{rof}$ as profunctors, i.e. functors $B^{op} \times A \longrightarrow \mathbf{Set}$, but it is well-known that such functors can equivalently be described by *two-sided discrete fibrations*, and that this notion can be internalized to a sufficiently well-behaved 2-category. [Her01] develops a theory of generalized multicategories in such a context.

Let $\mathcal{K}$ be a finitely complete 2-category (see [Str76]). Since its underlying ordinary category $\mathcal{K}_0$ has pullbacks, we can form the virtual equipment $\mathbb{P}\mathsf{rof}(\mathcal{K}_\mathbf{0})$. Now, as observed in [Str74], for any object $A \in \mathcal{K}$ we have an internal category ΦA in $\mathcal{K}$, defined by $(\Phi A)_0 = A$ and $(\Phi A)_1 = A^\mathbf{2}$ (the cotensor with the arrow category $\mathbf{2}$). Similarly, every morphism $f\colon A \longrightarrow B$ defines an internal functor $\Phi f\colon \Phi A \longrightarrow \Phi B$, and the same for 2-cells; thus we have a 2-functor $\mathcal{K} \longrightarrow \mathcal{V}(\mathbb{P}\mathsf{rof}(\mathcal{K}_\mathbf{0}))$. Moreover, this 2-functor is locally full and faithful, i.e. 2-cells $f \longrightarrow g$ are in bijection with internal natural transformations $\Phi f \longrightarrow \Phi g$.

We define an internal profunctor $\Phi A \xrightarrow{H} \Phi B$ to be a **discrete fibration** from A to B if the object $H \in \mathcal{K}/(A \times B)$ is internally discrete, i.e. $(\mathcal{K}/(A \times B))(C, H)$ is a discrete category for any $C \in \mathcal{K}/(A \times B)$. We define $\mathbb{D}\mathsf{Fib}(\mathcal{K})$ to be the sub-virtual double category of $\mathbb{P}\mathsf{rof}(\mathcal{K}_\mathbf{0})$ determined by

- The internal categories of the form ΦA,
- The internal functors of the form Φf,

- The internal profunctors which are discrete fibrations, and
- All cells between these.

Since the pullback of a discrete fibration is a discrete fibration, and $U_{\Phi A}$ is a discrete fibration, $\mathbb{D}\mathsf{Fib}(\mathcal{K})$ is a virtual equipment. Our remarks above show that $\mathcal{K} \simeq \mathcal{V}(\mathbb{D}\mathsf{Fib}(\mathcal{K}))$. Under suitable conditions on $\mathcal{K}$, discrete fibrations can be composed, so that $\mathbb{D}\mathsf{Fib}(\mathcal{K})$ becomes an equipment. (Several authors have tried to isolate these conditions, with varying degrees of success; in addition to [Her01] see [Str80] and [CJSV94]. Our approach sidesteps this issue completely.)

Now suppose that $F\colon \mathcal{K} \longrightarrow \mathcal{L}$ is a 2-functor that preserves pullbacks and comma objects. Then it preserves internal categories, profunctors, the Φ construction, and discrete fibrations, so it induces a normal functor $\mathbb{D}\mathsf{Fib}(F)\colon \mathbb{D}\mathsf{Fib}(\mathcal{K}) \longrightarrow \mathbb{D}\mathsf{Fib}(\mathcal{L})$. Likewise, any 2-natural transformation $F \longrightarrow G$ induces a transformation $\mathbb{D}\mathsf{Fib}(F) \longrightarrow \mathbb{D}\mathsf{Fib}(G)$, so we have a 2-functor $\mathbb{D}\mathsf{Fib}$ from finitely complete 2-categories to $v\mathcal{E}quip$. In particular, any 2-monad T on $\mathcal{K}$ whose functor part preserves pullbacks and comma objects induces a normal monad on $\mathbb{D}\mathsf{Fib}(\mathcal{K})$, so we can talk about $\mathbb{D}\mathsf{Fib}(T)$-monoids.

This is basically the context of [Her01], except that, like most other authors, he works only with bicategories. Thus, he assumes that $\mathcal{K}$ has the structure required to compose discrete fibrations, and that moreover T preserves this structure and that μ and η are cartesian transformations. This ensures that $\mathbb{D}\mathsf{Fib}(T)$ is horizontally strong, so that $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{D}\mathsf{Fib}(\mathcal{K}), \mathbb{D}\mathsf{Fib}(T))$ is an equipment. Under these hypotheses, we have:

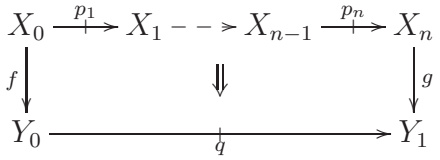
B.15. Theorem. *The 2-category $n\mathcal{K}Mon(\mathbb{D}\mathsf{Fib}(\mathcal{K}), \mathbb{D}\mathsf{Fib}(T))$ of normalized $\mathbb{D}\mathsf{Fib}(T)$-monoids is isomorphic to the 2-category Lax-Bimod(T)-alg defined in [Her01, 4.3 and 4.4].*

Our Theorem 9.2 is also a generalization of results of [Her01]. Hermida proves furthermore that under his hypotheses, the left adjoint F_T from Remark 9.16 exists, the adjunction is monadic when restricted to pseudo T-algebras, and the induced monad $T_\dashv$ on Lax-Bimod(T)-alg is lax-idempotent (see [KL97]). In [CS10b] we will show that an analogous result is true for any monad T on an equipment $\mathbb{X}$ satisfying suitable cocompleteness conditions.

B.16. Monoidal pseudo algebras. In [Web05], Weber gives a definition of generalized operads enriched in *monoidal pseudo algebras*. More precisely, for any 2-monad T on a 2-category $\mathcal{K}$ with finite products, and any pseudo T-algebra A which is also a pseudomonoid in a compatible way, he defines a notion of *T-operad in A*. A general description of the relationship of this theory to ours would take us too far afield, so we will remark only briefly on how such a comparison should go.

For any pseudomonoid A in a 2-category $\mathcal{K}$ with finite products, there is a virtual equipment $A\text{-}\mathbb{M}\mathsf{at}$ defined as follows. Its objects and vertical arrows are the objects and arrows of $\mathcal{K}$. A horizontal arrow from $X \longrightarrow\!\!\!\!\!\!+\;\; Y$ is a morphism $X \times Y \xrightarrow{p} A$ in $\mathcal{K}$, and a

cell



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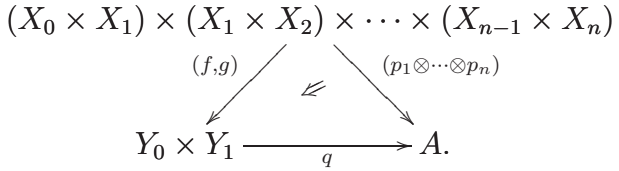
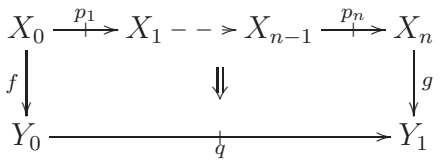


in $\mathcal{K}$, where $\otimes\colon A \times A \longrightarrow A$ is the multiplication of the pseudomonoid A (if $n = 0$ we use its unit instead).

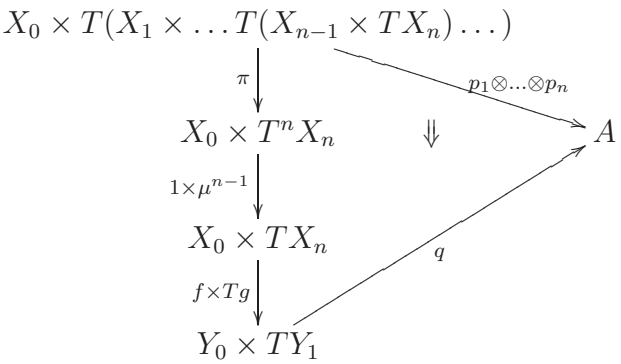
Now, if A is additionally a pseudo T-algebra in a compatible way, we might hope to be able to extend T to a monad on A-$\mathbb{M}\mathsf{at}$. However, given a horizontal arrow $X \xrightarrow{p} Y$, from $X \times Y \xrightarrow{p} A$ we can form the composite

$$T(X \times Y) \xrightarrow{Tp} TA \xrightarrow{a} A,$$

but this is not yet a horizontal arrow $TX \nrightarrow TY$. If we assume that A admits well-behaved *left (Kan) extensions*, then we can define $TX \xrightarrow{Tp} TY$ to be the extension of the above composite along $T(X \times Y) \longrightarrow TX \times TY$. We can then construct $\mathbb{H}$-$\mathsf{Kl}(A$-$\mathbb{M}\mathsf{at}, T)$ and $\mathbb{K}\mathsf{Mod}(A$-$\mathbb{M}\mathsf{at}, T)$ as usual. Moreover, we can give an equivalent characterization of $\mathbb{H}$-$\mathsf{Kl}(A$-$\mathbb{M}\mathsf{at}, T)$ which is valid even in the absence of left extensions: a horizontal arrow $X \nrightarrow Y$ is a morphism $X \times TY \xrightarrow{p} A$ in $\mathcal{K}$, and a cell



is a 2-cell



in $\mathcal{K}$. Thus, we obtain a notion of T-monoid in A for any monoidal pseudo algebra A. Weber only considers the case of *operads*, rather than more general *multicategories*, but it is easy to verify that his T-operads in A coincide with those T-monoids in A whose underlying object in $\mathcal{K}$ is the terminal object 1.

Actually, there is a good reason that Weber considers only operads: T-monoids $X_0 \xrightarrow{X} TX_0$ in this context for which X_0 is not discrete, or at least a groupoid, are not very familiar objects. In familiar cases such as $\mathcal{K} = \mathbf{Cat}$, we would obtain familiar types of A-enriched multicategory only by taking the horizontal arrows $X \nrightarrow Y$ in A-$\mathbb{M}\mathsf{at}$ to be morphisms $X \times Y^{\mathrm{op}} \longrightarrow A$, rather than $X \times Y \longrightarrow A$. To put this in a general context, however, requires a 2-category $\mathcal{K}$ in which "opposites" make sense, such as the "2-toposes" of [Web07].

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