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INTERNAL CATEGORIES, ANAFUNCTORS AND LOCALISATIONS

In memory of Luanne Palmer (1965-2011)

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Abstract.

In this article we review the theory of anafunctors introduced by Makkai and Bartels, and show that given a subcanonical site S, one can form a bicategorical localisation of various 2-categories of internal categories or groupoids at weak equivalences using anafunctors as 1-arrows. This unifies a number of proofs throughout the literature, using the fewest assumptions possible on S.

1. Introduction

It is a well-known classical result of category theory that a functor is an equivalence (that is, in the 2-category of categories) if and only if it is fully faithful and essentially surjective. This fact is equivalent to the axiom of choice. It is therefore *not* true if one is working with categories internal to a category S which doesn't satisfy the (external) axiom of choice. This is may fail even in a category very much like the category of sets, such as a well-pointed boolean topos, or even the category of sets in constructive foundations. As internal categories are the objects of a 2-category $\mathbf{Cat}(S)$ we can talk about internal equivalences, and even fully faithful functors. In the case S has a singleton pretopology J (i.e. covering families consist of single maps) we can define an analogue of essentially surjective functors. Internal functors which are fully faithful and essentially surjective are called *weak equivalences* in the literature, going back to [Bunge-Paré 1979]. We shall call them J-equivalences for clarity. We can recover the classical result mentioned above if we localise the 2-category $\mathbf{Cat}(S)$ at the class W_J of J-equivalences.

We are not just interested in localising $\mathbf{Cat}(S)$, but various full sub-2-categories $C \hookrightarrow \mathbf{Cat}(S)$ which arise in the study of presentable stacks, for example algebraic, topological, differentiable, etc. stacks. As such it is necessary to ask for a compatibility condition between the pretopology on S and the sub-2-category we are interested in. We call this condition existence of *base change* for covers of the pretopology, and demand that for any cover $p\colon U \longrightarrow\!\!\!\!\!\rightarrow X_0$ (in S) of the object of objects of $X \in C$, there is a fully faithful functor

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in C with object component p.

1.1. THEOREM. *Let S be a category with singleton pretopology J and let C be a full sub-2-category of $\mathbf{Cat}(S)$ which admits base change along arrows in J. Then C admits a calculus of fractions for the J-equivalences.*

Pronk gives us the appropriate notion of a calculus of fractions for a 2-category in [Pronk 1996] as a generalisation of the usual construction for categories [Gabriel-Zisman 1967]. In her construction, 1-arrows are spans and 2-arrows are equivalence classes of bicategorical spans of spans. This construction, while canonical, can be a little unwieldy so we look for a simpler construction of the localisation.

We find this in the notion of *anafunctor*, introduced by Makkai for plain small categories [Makkai 1996] (Kelly described them briefly in [Kelly 1964] but did not develop the concept further). In his setting an anafunctor is a span of functors such that the left (or source) leg is a surjective-on-objects, fully faithful functor.[1] For a general category S with a *subcanonical* singleton pretopology J [Bartels 2006], the analogon is a span with left leg a fully faithful functor with object component a cover. Composition of anafunctors is given by composition of spans in the usual way, and there are 2-arrows between anafunctors (a certain sort of span of spans) that give us a bicategory $\mathbf{Cat}_{\mathrm{ana}}(S, J)$ with objects internal categories and 1-arrows anafunctors. We can also define the full sub-bicategory $C_{\mathrm{ana}}(J) \hookrightarrow \mathbf{Cat}_{\mathrm{ana}}(S, J)$ analogous to C, and there is a strict inclusion 2-functor $C \hookrightarrow C_{\mathrm{ana}}(J)$. This gives us our second main theorem.

1.2. THEOREM. *Let S be a category with subcanonical singleton pretopology J and let C be a full sub-2-category of $\mathbf{Cat}(S)$ which admits base change along arrows in J, Then $C \hookrightarrow C_{\mathrm{ana}}(J)$ is a localisation of C at the class of J-equivalences.*

So far we haven't mentioned the issue of size, which usually is important when constructing localisations. If the site (S, J) is locally small, then C is locally small, in the sense that the hom-categories are small. This also implies that $C_{\mathrm{ana}}(J)$ and hence any $C[W_J^{-1}]$ has *locally* small hom-categories i.e. has only a set of 2-arrows between any pair of 1-arrows. To prove that the localisation is locally essentially small (that is, hom-categories are equivalent to small categories), we need to assume a size restriction axiom on the pretopology J, called WISC (Weakly Initial Sets of Covers).

WISC can be seen as an extremely weak choice principle, weaker than the existence of enough projectives, and states that for every object A of S, there is a set of J-covers of A which is cofinal in all J-covers of A. It is automatically satisfied if the pretopology is specified as an assignment of a *set* of covers to each object.

1.3. THEOREM. *Let S be a category with subcanonical singleton pretopology J satisfying WISC, and let C be a full sub-2-category of $\mathbf{Cat}(S)$ which admits base change along arrows in J. Then any localisation of C at the class of J-equivalences is locally essentially small.*

[1] Anafunctors were so named by Makkai, on the suggestion of Pavlovic, after profunctors, in analogy with the pair of terms anaphase/prophase from biology. For more on the relationship between anafunctors and profunctors, see below.

Since a singleton pretopology can be conveniently defined as a certain wide subcategory, this is not a vacuous statement for large sites, such as **Top** or **Grp**(E) (group objects in a topos E). In fact WISC is independent of the Zermelo-Fraenkel axioms (without Choice) [van den Berg 2012, Roberts 2013]. It is thus possible to have the theorem fail for the topos $S = \mathbf{Set}_{\neg AC}$ with surjections as covers.

Since there have been many very closely related approaches to localisation of 2-categories of internal categories and groupoids, we give a brief sketch in the following section. Sections 3 to 6 of this article then give necessary background and notation on sites, internal categories, anafunctors and bicategories of fractions respectively. Section 7 contains our main results, while section 8 shows examples from the literature that are covered by the theorems from section 7. A short appendix detailing superextensive sites is included, as this material does not appear to be well-known (they were discussed in the recent [Shulman 2012], Example 11.12).

This article started out based on the first chapter of the author's PhD thesis, which only dealt with groupoids in the site of topological spaces and open covers. Many thanks are due to Michael Murray, Mathai Varghese and Jim Stasheff, supervisors to the author. The patrons of the n-Category Café and nLab, especially Mike Shulman and Toby Bartels, provided helpful input and feedback. Steve Lack suggested a number of improvements, and the referee asked for a complete rewrite of this article, which has greatly improved the theorems, proofs, and hopefully also the exposition. Any delays in publication are due entirely to the author.

2. Anafunctors in context

The theme of giving 2-categories of internal categories or groupoids more equivalences has been approached in several different ways over the decades. We sketch a few of them, without necessarily finding the original references, to give an idea of how widely the results of this paper apply. We give some more detailed examples of this applicability in section 8.

Perhaps the oldest related construction is the distributors of Bénabou, also known as modules or profunctors [Bénabou 1973] (see [Johnstone 2002] for a detailed treatment of internal profunctors, as the original article is difficult to source). Bénabou pointed out [Bénabou 2011], after a preprint of this article was released, that in the case of the category **Set** (and more generally in a finitely complete site with reflexive coequalisers that are stable under pullback, see [MMV 2012]), the bicategory of small (resp. internal) categories with representable profunctors as 1-arrows is equivalent to the bicategory of small categories with anafunctors as 1-arrows. In fact this was discussed by Baez and Makkai [Baez-Makkai 1997], where the latter pointed out that representable profunctors correspond to *saturated* anafunctors in his setting. The author's preference for anafunctors lies in the fact they can be defined with weaker assumptions on the site (S, J), and in fact in the sequel [Roberts B], do not require the 2-category to have objects which are internal categories. In a sense this is analogous to [Street 1980], where the formal bicategorical

approach to profunctors between objects of a bicategory is given, albeit still requiring more colimits to exist than anafunctors do.

Bénabou has pointed out in private communication that he has an unpublished distributor-like construction that does not rely on existence of reflexive coequalisers; the author has not seen any details of this and is curious to see how it compares to anafunctors.

Related to this is the original work of Bunge and Paré [Bunge-Paré 1979], where they consider functors between indexed categories associated to internal categories, that is, the *externalisation* of an internal category and stack completions thereof. This was one motivation for considering weak equivalences in the first place, in that a pair of internal categories have equivalent stack completed externalisations if and only if they are connected by a span of internal functors which are weak equivalences.

Another approach is constructing bicategories of fractions à la Pronk [Pronk 1996]. This has been followed by a number of authors, usually followed up by an explicit construction of a localisation simplifying the canonical one. Our work here sits at the more general end of this spectrum, as others have tailored their constructions to take advantage of the structure of the site they are interested in. For example, *butterflies* (originally called papillons) have been used for the category of groups [Noohi 2005b, Aldrovandi-Noohi 2009, Aldrovandi-Noohi 2010], abelian categories [Breckes 2009] and semiabelian categories [AMMV 2010, MMV 2012]. These are similar to the meromorphisms of [Pradines 1989], introduced in the context of the site of smooth manifolds; though these only use a 1-categorical approach to localisation.

Vitale [Vitale 2010], after first showing that the 2-category of groupoids in a regular category has a bicategory of fractions, then shows that for protomodular regular categories one can generalise the pullback congruences of Bénabou in [Bénabou 1989] to discuss bicategorical localisation. This approach can be applied to internal categories, as long as one restricts to invertible 2-arrows. Similarly, [MMV 2012] give a construction of what they call *fractors* between internal groupoids in a Mal'tsev category, and show that in an efficiently regular category (e.g. a Barr-exact category) fractors are 1-arrows in a localisation of the 2-category of internal groupoids. The proof also works for internal categories if one considers only invertible 2-arrows.

Other authors, in dealing with internal groupoids, have adopted the approach pioneered by Hilsum and Skandalis [Hilsum-Skandalis 1987], which has gone by various names including Hilsum-Skandalis morphisms, Morita morphisms, bimodules, bibundles, right principal bibundles and so on. All of these are very closely related to saturated anafunctors, but in fact no published definition of a saturated anafunctor in a site other than **Set** ([Makkai 1996]) has appeared, except in the guise of internal profunctors (e.g. [Johnstone 2002], section B2.7). Note also that this approach has only been applied to internal groupoids. The review [Lerman 2010] covers the case of Lie groupoids, and in particular orbifolds, while [Mrčun 2001] treats bimodules between groupoids in the category of affine schemes, but from the point of view of Hopf algebroids.

The link between localisation at weak equivalences and presentable stacks is considered in (of course) [Pronk 1996], as well as more recently in [Carchedi 2012], [Schäppi 2012], in

the cases of topological and algebraic stacks respectively, and for example [TXL-G 2004] in the case of differentiable stacks.

A third approach is by considering a model category structure on the 1-category of internal categories. This is considered in [Joyal-Tierney 1991] for categories in a topos, and in [EKvdL 2005] for categories in a finitely complete subcanonical site (S, J). In the latter case the authors show when it is possible to construct a Quillen model category structure on $\mathbf{Cat}(S)$ where the weak equivalences are the weak equivalences from this paper. Sufficient conditions on S include being a topos with nno, being locally finitely presentable or being finitely complete regular Mal'tsev – and additionally having enough J-projective objects. If one is willing to consider other model-category-like structures, then these assumptions can be dropped. The proof from [EKvdL 2005] can be adapted to show that for a finitely complete site (S, J), the category of groupoids with source and target maps restricted to be J-covers has the structure of a category of fibrant objects, with the same weak equivalences. We note that [Colman-Costoya 2009] gives a Quillen model structure for the category of orbifolds, which are there defined to be proper topological groupoids with discrete hom-spaces.

In a similar vein, one could consider a localisation using *hammock* localisation [Dwyer-Kan 1980a] of a category of internal categories, which puts one squarely in the realm of $(\infty, 1)$-categories. Alternatively, one could work with the $(\infty, 1)$-category arising from a 2-category of internal categories, functors and natural *isomorphisms* and consider a localisation of this as given in, say [Lurie 2009a]. However, to deal with general 2-categories of internal categories in this way, one needs to pass to $(\infty, 2)$-categories to handle the non-invertible 2-arrows. The theory here is not so well-developed, however, and one could see the results of the current paper as giving toy examples with which one could work. This is one motivation for making sure the results shown in this paper apply to not just 2-categories of groupoids. Another is extending the theory of presentable stacks from stacks of groupoids to stacks of categories [Roberts A].

3. Sites

The idea of *surjectivity* is a necessary ingredient when talking about equivalences of categories—in the guise of just essential surjectivity—but it doesn't generalise in a straightforward way from the category **Set**. The necessary properties of the class of surjective maps are encoded in the definition of a Grothendieck pretopology, in particular a singleton pretopology. This section gathers definitions and notations for later use.

3.1. Definition. A *Grothendieck pretopology* (or simply *pretopology*) on a category S is a collection J of families

$$\{(U_i \longrightarrow A)_{i \in I}\}_{A \in \mathrm{Obj}(S)}$$

of morphisms for each object $A \in S$ satisfying the following properties

1. $(A' \xrightarrow{\sim} A)$ is in J for every isomorphism $A' \simeq A$.

2. Given a map $B \longrightarrow A$, for every $(U_i \longrightarrow A)_{i\in I}$ in J the pullbacks $B \times_A A_i$ exist and $(B \times_A A_i \longrightarrow B)_{i\in I}$ is in J.

3. For every $(U_i \longrightarrow A)_{i\in I}$ in J and for a collection $(V_k^i \longrightarrow U_i)_{k\in K_i}$ from J for each $i \in I$, the family of composites

$$(V_k^i \longrightarrow A)_{k\in K_i, i\in I}$$

are in J.

Families in J are called *covering families*. We call a category S equipped with a pretopology J a *site*, denoted (S, J) (note that often one sees a site defined as a category equipped with a Grothendieck *topology*).

The pretopology J is called a *singleton* pretopology if every covering family consists of a single arrow $(U \longrightarrow A)$. In this case a covering family is called a *cover* and we call (S, J) a *unary* site.

Very often, one sees the definition of a pretopology as being an assignment of a *set* covering families to each object. We do not require this, as one can define a singleton pretopology as a subcategory with certain properties, and there is not necessarily then a set of covers for each object. One example is the category of groups with surjective homomorphisms as covers. This distinction will be important later.

One thing we will require is that sites come with *specified* pullbacks of covering families. If one does not mind applying the axiom of choice (resp. axiom of choice for classes) then any small site (resp. large site) can be so equipped. But often sites that arise in practice have more or less canonical choices for pullbacks, such as the category of ZF-sets.

3.2. Example. The prototypical example is the pretopology $\mathcal{O}$ on **Top**, where a covering family is an open cover. The class of numerable open covers (i.e. those that admit a subordinate partition of unity [Dold 1963]) also forms a pretopology on **Top**. Much of traditional bundle theory is carried out using this site; for example the Milnor classifying space classifies bundles which are locally trivial over numerable covers.

3.3. Definition. A covering family $(U_i \longrightarrow A)_{i\in I}$ is called *effective* if A is the colimit of the following diagram: the objects are the U_i and the pullbacks $U_i \times_A U_j$, and the arrows are the projections

$$U_i \leftarrow U_i \times_A U_j \longrightarrow U_j.$$

If the covering family consists of a single arrow $(U \longrightarrow A)$, this is the same as saying $U \longrightarrow A$ is a regular epimorphism.

3.4. Definition. A site is called *subcanonical* if every covering family is effective.

3.5. Example. On **Top**, the usual pretopology $\mathcal{O}$ of opens, the pretopology of numerable covers and that of open surjections are subcanonical.

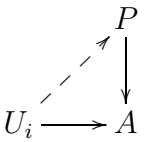
3.6. Example. In a regular category, the class of regular epimorphisms forms a subcanonical singleton pretopology.

In fact we can make the following definition.

3.7. Definition. For a category S, the largest class of regular epimorphisms of which all pullbacks exist, and which is stable under pullback, is called the *canonical singleton pretopology* and denoted $\underline{c}$.

This is a to be contrasted to the canonical *topology* on a category, which consists of covering sieves rather than covers. The canonical singleton pretopology is the largest subcanonical singleton pretopology on a category.

3.8. Definition. Let (S, J) be a site. An arrow $P \longrightarrow A$ in S is called a *J-epimorphism* if there is a covering family $(U_i \longrightarrow A)_{i \in I}$ and a lift



for every $i \in I$. A J-epimorphism is called *universal* if its pullback along an arbitrary map exists. We denote the singleton pretopology of universal J-epimorphisms by J_{un}.

This definition of J-epimorphism is equivalent to the definition in III.7.5 in [Mac Lane-Moerdijk 1992]. The dotted maps in the above definition are called local sections, after the case of the usual open cover pretopology on **Top**. If J is a singleton pretopology, it is clear that $J \subset J_{un}$.

3.9. Example. The universal $\mathcal{O}$-epimorphisms for the pretopology $\mathcal{O}$ of open covers on **Diff** form $Subm$, the pretopology of surjective submersions.

3.10. Example. In a finitely complete category the universal *triv*-epimorphisms are the split epimorphisms, where *triv* is the *trivial pretopology* where all covering families consist of a single isomorphism. In **Set** with the axiom of choice there are all the epimorphisms.

Note that for a finitely complete site (S, J), J_{un} contains $triv_{un}$, hence all the split epimorphisms.

Although we will not assume that all sites we consider are finitely complete, results similar to ours have, and so in that case we can say a little more, given stronger properties on the pretopology.

3.11. Definition. A singleton pretopology J is called *saturated* if whenever the composite $A \xrightarrow{h} B \xrightarrow{g} C$ is in J, then $g \in J$.

The concept of a saturated pretopology was introduced by Bénabou under the name *calibration* [Bénabou 1975]. It follows from the definition that a saturated singleton pretopology contains the split epimorphisms (take h to be a section of the epimorphism g).

3.12. Example. The canonical singleton pretopology $\underline{c}$ in a regular category (e.g. a topos) is saturated.

3.13. Example. Given a pretopology J on a finitely complete category, J_{un} is saturated.

Sometimes a pretopology J contains a smaller pretopology that still has enough covers to compute the same J-epimorphisms.

3.14. Definition. If J and K are two singleton pretopologies with $J \subset K$, such that $K \subset J_{un}$, then J is said to be *cofinal* in K.

Clearly J is cofinal in J_{un} for any singleton pretopology J.

3.15. Lemma. *If J is cofinal in K, then $J_{un} = K_{un}$.*

We have the following lemma, which is essentially proved in [Johnstone 2002], C2.1.6.

3.16. Lemma. *If a pretopology J is subcanonical, then so any pretopology in which it is cofinal. In particular, J subcanonical implies J_{un} subcanonical.*

As mentioned earlier, one may be given a singleton pretopology such that each object has more than a set's worth of covers. If such a pretopology contains a cofinal pretopology with set-many covers for each object, then we can pass to the smaller pretopology and recover the same results (in a way that will be made precise later). In fact, we can get away with something weaker: one could ask only that the category of all covers of an object (see definition 3.18 below) has a set of weakly initial objects, and such set may not form a pretopology. This is the content of the axiom WISC below. We first give some more precise definitions.

3.17. Definition. A category C has a *weakly initial set* $\mathcal{I}$ of objects if for every object A of C there is an arrow $O \longrightarrow A$ from some object $O \in \mathcal{I}$.

For example the large category **Fields** of fields has a weakly initial set, consisting of the prime fields $\{\mathbb{Q}, \mathbb{F}_p | p \text{ prime}\}$. To contrast, the category of sets with surjections for arrows doesn't have a weakly initial set of objects. Every small category has a weakly initial set, namely its set of objects.

We pause only to remark that the statement of the adjoint functor theorem can be expressed in terms of weakly initial sets.

3.18. Definition. Let (S, J) be a site. For any object A, the *category of covers of* A, denoted J/A has as objects the covering families $(U_i \longrightarrow A)_{i \in I}$ and as morphisms $(U_i \longrightarrow A)_{i \in I} \longrightarrow (V_j \longrightarrow A)_{j \in J}$ tuples consisting of a function $r \colon I \longrightarrow J$ and arrows $U_i \longrightarrow V_{r(i)}$ in S/A.

When J is a singleton pretopology this is simply a full subcategory of S/A. We now define the axiom WISC (Weakly Initial Set of Covers), due independently to Mike Shulman and Thomas Streicher.

3.19. Definition. A site (S, J) is said to *satisfy WISC* if for every object A of S, the category J/A has a weakly initial set of objects.

A site satisfying WISC is in some sense constrained by a small amount of data for each object. Any small site satisfies WISC, for example, the usual site of finite-dimensional smooth manifolds and open covers. Any pretopology J containing a cofinal pretopology K such that K/A is small for every object A satisfies WISC.

3.20. Example. Any regular category (for example a topos) with enough projectives, equipped with the canonical singleton pretopology, satisfies WISC. In the case of **Set** 'enough projectives' is the Presentation Axiom (PAx), studied, for instance, by Aczel [Aczel 1978] in the context of constructive set theory.

3.21. Example. [Shulman] $(\mathbf{Top}, \mathcal{O})$ satisfies WISC, using AC in **Set**.

Choice may be more than is necessary here; it would be interesting to see if weaker choice principles in the site $(\mathbf{Set}, surjections)$ are enough to prove WISC for $(\mathbf{Top}, \mathcal{O})$ or other concrete sites.

3.22. Lemma. *If (S, J) satisfies WISC, then so does (S, J_{un}).*

It is instructive to consider an example where WISC fails in a non-artificial way. The category of sets and surjections with all arrows covers clearly doesn't satisfy WISC, but is contrived and not a 'useful' sort of category. For the moment, assume the existence of a Grothendieck universe $\mathbb{U}$ with cardinality λ, and let $\mathrm{Set}_{\mathbb{U}}$ refer to the category of $\mathbb{U}$-small sets. Clearly we can define WISC relative to $\mathbb{U}$, call it $\mathrm{WISC}_{\mathbb{U}}$. Let G be a $\mathbb{U}$-large group and $\mathbf{B}G$ the $\mathbb{U}$-large groupoid with one object associated to G. The boolean topos $\mathrm{Set}_{\mathbb{U}}^{\mathbf{B}G}$ of $\mathbb{U}$-small G-sets is a unary site with the class *epi* of epimorphisms for covers. One could consider this topos as being an exotic sort of forcing construction.

3.23. Proposition. *If G has at least λ-many conjugacy classes of subgroups, then the site $(\mathrm{Set}_{\mathbb{U}}^{\mathbf{B}G}, epi)$ does not satisfy $WISC_{\mathbb{U}}$.*

Alternatively, one could work in foundations where it is legitimate to discuss a proper class-sized group, and then consider the topos of sets with an action by this group. If there is a proper class of conjugacy classes of subgroups, then this topos with its canonical singleton pretopology will fail to satisfy WISC. Simple examples of such groups are $\mathbb{Z}^{\mathbb{U}}$ (given a universe $\mathbb{U}$) and $\mathbb{Z}^K$ (for some proper class K).

Recently, [van den Berg 2012] (relative to a large cardinal axiom) and [Roberts 2013] (with no large cardinals) have shown that the category of sets may fail to satisfy WISC. The models constructed in [Karaglia 2012] are also conjectured to not satisfy WISC.

Perhaps of independent interest is a form of WISC with a bound: the weakly initial set for each category J/A has cardinality less than some cardinal κ (call this WISC_{κ}). Then one could consider, for example, sites where each object has a weakly initial finite or countable set of covers. Note that the condition 'enough projectives' is the case $\kappa = 2$.

4. Internal categories

Internal categories were introduced in [Ehresmann 1963], starting with differentiable and topological categories (i.e. internal to **Diff** and **Top** respectively). We collect here the necessary definitions, terminology and notation. For a thorough recent account, see [Baez-Lauda 2004] or the encyclopedic [Johnstone 2002].

Fix a category S, referred to as the *ambient category*.

4.1. Definition. An *internal category* X in a category S is a diagram

$$X_1 \times_{X_0} X_1 \times_{X_0} X_1 \rightrightarrows X_1 \times_{X_0} X_1 \xrightarrow{m} X_1 \overset{s,t}{\rightrightarrows} X_0 \xrightarrow{e} X_1$$

in S such that the *multiplication* m is associative (we demand the limits in the diagram exist), the *unit map* e is a two-sided unit for m and s and t are the usual *source* and *target*. An *internal groupoid* is an internal category with an involution

$$(-)^{-1} \colon X_1 \longrightarrow X_1$$

satisfying the usual diagrams for an inverse.

Since multiplication is associative, there is a well-defined map $X_1 \times_{X_0} X_1 \times_{X_0} X_1 \longrightarrow X_1$, which will also be denoted by m. The pullback in the diagram in definition 4.1 is

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle s} \\ X_1 & \xrightarrow[t]{} & X_0 \,. \end{array}$$

and the double pullback is the limit of $X_1 \xrightarrow{t} X_0 \xleftarrow{s} X_1 \xrightarrow{t} X_0 \xleftarrow{s} X_0$. These, and pullbacks like these (where source is pulled back along target), will occur often. If confusion can arise, the maps in question will be explicitly written, as in $X_1 \times_{s,X_0,t} X_1$. One usually sees the requirement that S is finitely complete in order to define internal categories. This is not strictly necessary, and not true in the well-studied case of $S = \mathbf{Diff}$, the category of smooth manifolds.

Often an internal category will be denoted $X_1 \rightrightarrows X_0$, the arrows m, s, t, e (and $(-)^{-1}$) will be referred to as *structure maps* and X_1 and X_0 called the object of arrows and the object of objects respectively. For example, if $S = \mathbf{Top}$, we have the space of arrows and the space of objects, for $S = \mathbf{Grp}$ we have the group of arrows and so on.

4.2. Example. If $X \longrightarrow Y$ is an arrow in S admitting iterated kernel pairs, there is an internal groupoid $\check{C}(X)$ with $\check{C}(X)_0 = X$, $\check{C}(X)_1 = X \times_Y X$, source and target are projection on first and second factor, and the multiplication is projecting out the middle factor in $X \times_Y X \times_Y X$. This groupoid is called the *Čech groupoid* of the map $X \longrightarrow Y$. The origin of the name is that in **Top**, for maps of the form $\coprod_I U_i \longrightarrow Y$ (arising from an open cover), the Čech groupoid $\check{C}(\coprod_I U_i)$ appears in the definition of Čech cohomology.

4.3. Example. Let S be a category with binary products. For each object $A \in S$ there is an internal groupoid $\mathrm{disc}(A)$ which has $\mathrm{disc}(A)_1 = \mathrm{disc}(A)_0 = A$ and all structure maps equal to id_A. Such a category is called *discrete*. There is also an internal groupoid $\mathrm{codisc}(A)$ with

$$\mathrm{codisc}(A)_0 = A,\ \mathrm{codisc}(A)_1 = A \times A$$

and where source and target are projections on the first and second factor respectively. Such a groupoid is called *codiscrete*.

4.4. Definition. Given internal categories X and Y in S, an *internal functor* $f : X \longrightarrow Y$ is a pair of maps

$$f_0 \colon X_0 \longrightarrow Y_0 \quad \text{and} \quad f_1 \colon X_1 \longrightarrow Y_1$$

called the object and arrow component respectively. Both components are required to commute with all the structure maps.

4.5. Example. If $A \longrightarrow C$ and $B \longrightarrow C$ are maps admitting iterated kernel pairs, and $A \longrightarrow B$ is a map over C, there is a functor $\check{C}(A) \longrightarrow \check{C}(B)$.

4.6. Example. If (S, J) is a subcanonical unary site, and $U \longrightarrow A$ is a cover, a functor $\check{C}(U) \longrightarrow \mathrm{disc}(B)$ gives a unique arrow $A \longrightarrow B$. This follows immediately from the fact A is the colimit of the diagram underlying $\check{C}(U)$.

4.7. Definition. Given internal categories X, Y and internal functors $f, g \colon X \longrightarrow Y$, an *internal natural transformation* (or simply *transformation*)

$$a \colon f \Rightarrow g$$

is a map $a \colon X_0 \longrightarrow Y_1$ such that $s \circ a = f_0,\ t \circ a = g_0$ and the following diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{(g_1, a \circ s)} & Y_1 \times_{Y_0} Y_1 \\ {\scriptstyle (a \circ t, f_1)}\downarrow & & \downarrow{\scriptstyle m} \\ Y_1 \times_{Y_0} Y_1 & \xrightarrow{m} & Y_1 \end{array} \tag{1}$$

expressing the naturality of a.

Internal categories (resp. groupoids), functors and transformations in a locally small category S form a locally small 2-category $\mathbf{Cat}(S)$ (resp. $\mathbf{Gpd}(S)$) [Ehresmann 1963]. There is clearly an inclusion 2-functor $\mathbf{Gpd}(S) \longrightarrow \mathbf{Cat}(S)$. Also, disc and codisc, described in example 4.3, are 2-functors $S \longrightarrow \mathbf{Gpd}(S)$, whose underlying functors are left and right adjoint to the functor

$$\mathrm{Obj} \colon \mathbf{Cat}(S)_{\leq 1} \longrightarrow S, \qquad (X_1 \rightrightarrows X_0) \mapsto X_0.$$

Here $\mathbf{Cat}(S)_{\leq 1}$ is the 1-category underlying the 2-category $\mathbf{Cat}(S)$. Hence for an internal category X in S, there are functors $\mathrm{disc}(X_0) \longrightarrow X$ and $X \longrightarrow \mathrm{codisc}(X_0)$, the arrow component of the latter being $(s, t) : X_1 \longrightarrow X_0^2$.

We say a natural transformation is a *natural isomorphism* if it has an inverse with respect to vertical composition. Clearly there is no distinction between natural transformations and natural isomorphisms when the codomain of the functors is an internal groupoid. We can reformulate the naturality diagram (1) in the case that a is a natural isomorphism. Denote by $-a$ the inverse of a. Then the diagram (1) commutes if and only if the diagram

$$\begin{array}{ccc} X_0 \times_{X_0} X_1 \times_{X_0} X_0 & \xrightarrow{-a\times f_1\times a} & Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1 \\ \simeq\downarrow & & \downarrow m \\ X_1 & \xrightarrow[g_1]{} & Y_1 \end{array} \tag{2}$$

commutes, a fact we will use several times.

4.8. Example. If X is a category in S, A is an object of S and $f, g : X \longrightarrow \mathrm{codisc}(A)$ are functors, there is a unique natural isomorphism $f \xRightarrow{\sim} g$.

4.9. Definition. An *internal* or *strong equivalence* of internal categories is an equivalence in the 2-category of internal categories. That is, an internal functor $f\colon X \longrightarrow Y$ such that there is a functor $f'\colon Y \longrightarrow X$ and natural isomorphisms $f \circ f' \Rightarrow \mathrm{id}_Y$, $f' \circ f \Rightarrow \mathrm{id}_X$.

4.10. Definition. For an internal category X and a map $p : M \longrightarrow X_0$ in S the *base change of X along p* is any category $X[M]$ with object of objects M and object of arrows given by the pullback

$$\begin{array}{ccc} M^2 \times_{X_0^2} X_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ M^2 & \xrightarrow[p^2]{} & X_0^2 \end{array}$$

If $C \subset \mathbf{Cat}(S)$ denotes a full sub-2-category and if the base change along any map in a given class K of maps exists in C for all objects of C, then we say C *admits base change along maps in K*, or simply *admits base change for K*.

4.11. Remark. In all that follows, 'category' will mean object of C and similarly for 'functor' and 'natural transformation/isomorphism'.

The strict pullback of internal categories

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

when it exists, is the internal category with objects $X_0 \times_{Y_0} Z_0$, arrows $X_1 \times_{Y_1} Z_1$, and all structure maps given componentwise by those of X and Z. Often we will be able to prove that certain pullbacks exist because of conditions on various component maps in S. We do not assume that all strict pullbacks of internal categories exists in our chosen C.

It follows immediately from definition 4.10 that given maps $N \longrightarrow M$ and $M \longrightarrow X_0$, there is a canonical isomorphism

$$X[M][N] \simeq X[N]. \tag{3}$$

with object component the identity map, when these base changes exist.

4.12. Remark. If we agree to follow the convention that $M \times_N N = M$ is the pullback along the identity arrow id_N, then $X[X_0] = X$. This also simplifies other results of this paper, so will be adopted from now on.

One consequence of this assumption is that the iterated fibre product

$$M \times_M M \times_M \ldots \times_M M,$$

bracketed in any order, is *equal* to M. We cannot, however, equate two bracketings of a general iterated fibred product; they are only canonically isomorphic.

4.13. Lemma. *Let $Y \longrightarrow X$ be a functor in S and $j_0 \colon U \longrightarrow X_0$ a map. If the base change along j_0 exists, the following square is a strict pullback*

$$\begin{array}{ccc} Y[Y_0 \times_{X_0} U] & \longrightarrow & X[U] \\ \downarrow & & \downarrow{\scriptstyle j} \\ Y & \longrightarrow & X \end{array}$$

assuming it exists.

Proof. Since base change along j_0 exists, we know that we have the functor $Y[Y_0 \times_{X_0} U] \longrightarrow Y$, we just need to show it is a strict pullback of j. On the level of objects this is clear, and on the level of arrows, we have

$$\begin{aligned} (Y_0 \times_{X_0} U)^2 \times_{Y_0^2} Y_1 &\simeq U^2 \times_{X_0^2} Y_1 \\ &\simeq (U^2 \times_{X_0^2} X_1) \times_{X_1} Y_1 \\ &\simeq X[U]_1 \times_{X_1} Y_1 \end{aligned}$$

so the square is a pullback. ∎

We are interested in 2-categories C which admits base change for a given pretopology J on S, which we shall cover in more detail in section 8.

Equivalences in **Cat**—assuming the axiom of choice—are precisely the fully faithful, essentially surjective functors. For internal categories, however, this is not the case. In addition, we need to make use of a pretopology to make the 'surjective' part of 'essentially surjective' meaningful.

4.14. Definition. Let (S, J) be a unary site. An internal functor $f : X \longrightarrow Y$ in S is called

1. *fully faithful* if

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ X_0 \times X_0 & \xrightarrow[f_0 \times f_0]{} & Y_0 \times Y_0 \end{array}$$

   is a pullback diagram;

2. *J-locally split* if there is a J-cover $U \longrightarrow Y_0$ and a diagram

$$\begin{array}{ccc} Y[U] & & \\ {\scriptstyle \bar{f}}\downarrow & \overset{u}{\searrow} & \\ X & \xrightarrow[f]{} & Y \end{array}$$

   commuting up to a natural isomorphism;

3. a *J-equivalence* if it is fully faithful and J-locally split.

The class of J-equivalences will be denoted W_J. If mention of J is suppressed, they will be called *weak equivalences*.

4.15. Remark. There is another definition of full faithfulness for internal categories, namely that of a functor $f\colon Z \longrightarrow Y$ being *representably fully faithful*. This means that for all categories Z, the functor

$$f_*\colon \mathbf{Cat}(S)(Z, X) \longrightarrow \mathbf{Cat}(S)(Z, Y)$$

is fully faithful. It is a well-known result that these two notions coincide, so we shall use either characterisation as needed.

4.16. Lemma. *If $f : X \longrightarrow Y$ is a fully faithful functor such that f_0 is in J, then f is J-locally split.*

That is, the canonical functor $X[U] \longrightarrow X$ is a J-equivalence whenever the base change exists. Also, we do not require that J is subcanonical. We record here a useful lemma.

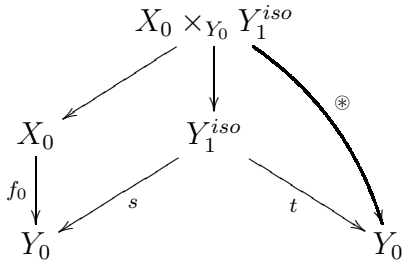
4.17. Lemma. *Given a fully faithful functor $f\colon X \longrightarrow Y$ in C and a natural isomorphism $f \Rightarrow g$, the functor g is also fully faithful. In particular, an internal equivalence is fully faithful.*

Proof. This is a simple application of the definition of representable full faithfulness and the fact that the result is true in **Cat**. ∎

The first definition of weak equivalence of internal categories along the lines we are considering appeared in [Bunge-Paré 1979] for S a regular category, and J the class of regular epimorphisms (i.e. $\underline{c}$), in the context of stacks and indexed categories. This was later generalised in [EKvdL 2005] to more general finitely complete sites to discuss model structures on the category of internal categories. Both work only with saturated singleton pretopologies.

Note that when S is finitely complete, the object $X_1^{iso} \hookrightarrow X_1$ of isomorphisms of a category X can be constructed as a finite limit [Bunge-Paré 1979], and in the case when X is a groupoid we have $X_1^{iso} \simeq X_1$.

4.18. Definition. [Bunge-Paré 1979, EKvdL 2005] For a finitely complete unary site (S, J) with J saturated, a functor f is called *essentially J-surjective* if the arrow labelled $\circledast$ below is in J.



A functor is called a *Bunge-Paré J-equivalence* if it is fully faithful and essentially J-surjective. Denote the class of such maps by W_J^{BP}.

Definition 4.14 is equivalent to the one in [Bunge-Paré 1979, EKvdL 2005] in the sites they consider but seems more appropriate for sites without all finite limits. Also, definition 4.14 makes sense in 2-categories other than $\mathbf{Cat}(S)$ or sub-2-categories thereof.

4.19. Proposition. *Let (S, J) be a finitely complete unary site with J saturated. Then a functor is a J-equivalence if and only if it is a Bunge-Paré J-equivalence.*

Proof. Let $f\colon X \longrightarrow Y$ be a Bunge-Paré J-equivalence, and consider the J-cover given by the map $U := X_0 \times_{Y_0} Y_1^{iso} \longrightarrow Y_0$. Denote by $\iota\colon U \longrightarrow Y_1^{iso}$ the projection on the second factor, by $-\iota$ the composite of ι with the inversion map $(-)^{-1}$ and by $s_0\colon U \longrightarrow X_0$ the projection on the first factor. The arrow s_0 will be the object component of a functor $s\colon Y[U] \longrightarrow X$, we need to define the arrow component s_1. Consider the composite

$$\begin{aligned} Y[U]_1 \simeq U \times_{Y_0} Y_1 \times_{Y_0} U &\xrightarrow{(s,\iota)\times \mathrm{id} \times (-\iota, s)} (X_0 \times_{Y_0} Y_1^{iso}) \times_{Y_0} Y_1 \times_{Y_0} (Y_1^{iso} \times_{Y_0} X_0) \\ &\hookrightarrow X_0 \times_{Y_0} Y_3 \times_{Y_0} X_0 \xrightarrow{\mathrm{id} \times m \times \mathrm{id}} X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0 \simeq X_1 \end{aligned}$$

where the last isomorphism arises from f being fully faithful. It is clear that this commutes with source and target, because these are given by projection on the first and last factor at each step. To see that it respects identities and composition, one can use generalised elements and the fact that the ι component will cancel with the $-\iota = (-)^{-1} \circ \iota$ component.

We define the natural isomorphism $f \circ s \Rightarrow j$ (here $j\colon Y[U] \longrightarrow Y$ is the canonical functor) to have component ι as denoted above. Notice that the composite $f_1 \circ s_1$ is just

$$Y[U]_1 \simeq U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{\iota \times \mathrm{id} \times -\iota} Y_1^{iso} \times_{Y_0} Y_1 \times_{Y_0} Y_1^{iso} \hookrightarrow Y_3 \xrightarrow{m} Y_1.$$

Since the arrow component of $Y[U] \longrightarrow Y$ is $U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{\mathrm{pr}_2} Y_1$, ι is indeed a natural isomorphism using the diagram (2). Thus a Bunge-Paré J-equivalence is a J-equivalence.

In the other direction, given a J-equivalence $f\colon X \longrightarrow Y$, we have a J-cover $j\colon U \longrightarrow Y_0$ and a map $(\overline{f}, a)\colon U \longrightarrow X_0 \times Y_1^{iso}$ such that $j = (t \circ pr_2) \circ (\overline{f}, a)$. Since J is saturated, $(t \circ pr_2) \in J$ and hence f is a Bunge-Paré J-equivalence. ∎

We can thus use definition 4.14 as we like, and it will still refer to the same sorts of weak equivalences that appear in the literature.

5. Anafunctors

We now let J be a *subcanonical* singleton pretopology on the ambient category S. In this section we assume that $C \hookrightarrow \mathbf{Cat}(S)$ admits base change along arrows in the given pretopology J. This is a slight generalisation of what is considered in [Bartels 2006], where only $C = \mathbf{Cat}(S)$ is considered.

5.1. Definition. [Makkai 1996, Bartels 2006] An *anafunctor* in (S, J) from a category X to a category Y consists of a J-cover $(U \longrightarrow X_0)$ and an internal functor

$$f\colon X[U] \longrightarrow Y.$$

Since $X[U]$ is an object of C, an anafunctor is a span in C, and can be denoted

$$(U, f)\colon X \nrightarrow Y.$$

5.2. Example. For an internal functor $f\colon X \longrightarrow Y$ in S, define the anafunctor $(X_0, f)\colon X \nrightarrow Y$ as the following span

$$X \xleftarrow{=} X[X_0] \xrightarrow{f} Y.$$

We will blur the distinction between these two descriptions. If $f = id\colon X \longrightarrow X$, then (X_0, id) will be denoted simply by id_X.

5.3. Example. If $U \longrightarrow A$ is a cover in (S, J) and $\mathbf{B}G$ is a groupoid with one object in S (i.e. a group in S), an anafunctor $(U, g)\colon \mathrm{disc}(A) \nrightarrow \mathbf{B}G$ is the same thing as a Čech cocycle.

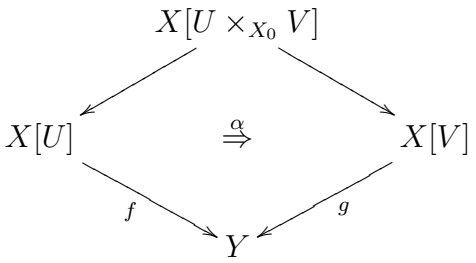
5.4. Definition. [Makkai 1996, Bartels 2006] Let (S, J) be a site and let

$$(U, f), (V, g) \colon X \nrightarrow Y$$

be anafunctors in S. A *transformation*

$$\alpha \colon (U, f) \Rightarrow (V, g)$$

from (U, f) to (V, g) is a natural transformation



If α is a natural isomorphism, then α will be called an *isotransformation*. In that case we say (U, f) is isomorphic to (V, g). Clearly all transformations between anafunctors between internal groupoids are isotransformations.

5.5. Example. Given functors $f, g \colon X \longrightarrow Y$ between categories in S, and a natural transformation $a \colon f \Rightarrow g$, there is a transformation $a \colon (X_0, f) \Rightarrow (X_0, g)$ of anafunctors, given by the component $X_0 \times_{X_0} X_0 = X_0 \xrightarrow{a} Y_1$.

5.6. Example. If $(U, g), (V, h) \colon \mathrm{disc}(A) \nrightarrow \mathbf{B}G$ are two Čech cocycles, a transformation between them is a coboundary on the cover $U \times_A V \longrightarrow A$.

5.7. Example. Let $(U, f) \colon X \nrightarrow Y$ be an anafunctor in S. There is an isotransformation $1_{(U,f)} \colon (U, f) \Rightarrow (U, f)$ called the *identity transformation*, given by the natural transformation with component

$$U \times_{X_0} U \simeq (U \times U) \times_{X_0^2} X_0 \xrightarrow{id_U^2 \times e} X[U]_1 \xrightarrow{f_1} Y_1 \tag{4}$$

5.8. Example. [Makkai 1996] Given anafunctors $(U, f) \colon X \longrightarrow Y$ and $(V, f \circ k) \colon X \longrightarrow Y$ where $k \colon V \longrightarrow U$ is a cover (over X_0), a *renaming transformation*

$$(U, f) \Rightarrow (V, f \circ k)$$

is an isotransformation with component

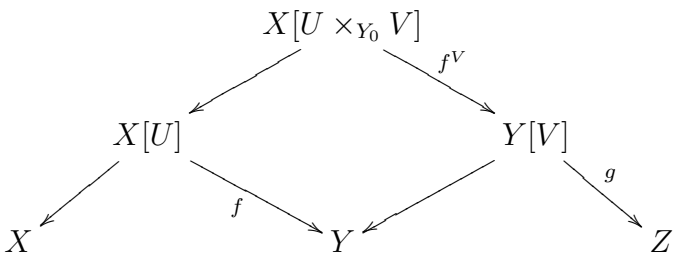
$$1_{(U,f)} \circ (k \times \mathrm{id}) : V \times_{X_0} U \longrightarrow U \times_{X_0} U \longrightarrow Y_1.$$

(We also call its inverse for vertical composition a renaming transformation.) If k is an isomorphism, then it will itself be referred to as a *renaming isomorphism*.

We define (following [Bartels 2006]) the composition of anafunctors as follows. Let

$$(U, f)\colon X \nrightarrow Y \quad \text{and} \quad (V, g)\colon Y \nrightarrow Z$$

be anafunctors in the site (S, J). Their composite $(V, g) \circ (U, f)$ is the composite span defined in the usual way. It is again a span in C:



The square is a pullback by lemma 4.13 (which exists because $V \longrightarrow Y_0$ is a cover), and the resulting span is an anafunctor because $V \longrightarrow Y_0$, hence $U \times_{Y_0} V \longrightarrow X_0$, are covers, and using the isomorphism (3). We will sometimes denote the composite by $(U \times_{Y_0} V, g \circ f^V)$.

Here we are using the fact we have specified pullbacks of covers in S. Without this we would not end up with a bicategory (see theorem 5.16), but what [Makkai 1996] calls an *anabicategory*. This is similar to a bicategory, but composition and other structural maps are only anafunctors, not functors.

Consider the special case when $V = Y_0$, so that (Y_0, g) is just an ordinary functor. Then there is a renaming transformation (the identity transformation!) $(Y_0, g) \circ (U, f) \Rightarrow (U, g \circ f)$, using the equality $U \times_{Y_0} Y_0 = U$ (by remark 4.12). If we let $g = \mathrm{id}_Y$, then we see that (Y_0, id_Y) is a strict unit on the left for anafunctor composition. Similarly, considering $(V, g) \circ (Y_0, \mathrm{id})$, we see that (Y_0, id_Y) is a two-sided strict unit for anafunctor composition. In fact, we have also proved

5.9. Lemma. *Given two functors $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ in S, their composition as anafunctors is equal to their composition as functors:*

$$(Y_0, g) \circ (X_0, f) = (X_0, g \circ f).$$

As a concrete and relevant example of a renaming transformation we can consider the triple composition of anafunctors

$$\begin{aligned}(U, f)\colon X &\nrightarrow Y,\\ (V, g)\colon Y &\nrightarrow Z,\\ (W, h)\colon Z &\nrightarrow A.\end{aligned}$$

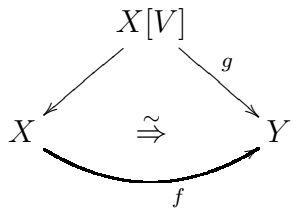
The two possibilities of composing these are

$$\left((U \times_{Y_0} V) \times_{Z_0} W, h \circ (g f^V)^W\right) \quad \text{and} \quad \left(U \times_{Y_0} (V \times_{Z_0} W), h \circ g^W \circ f^{V \times_{Z_0} W}\right).$$

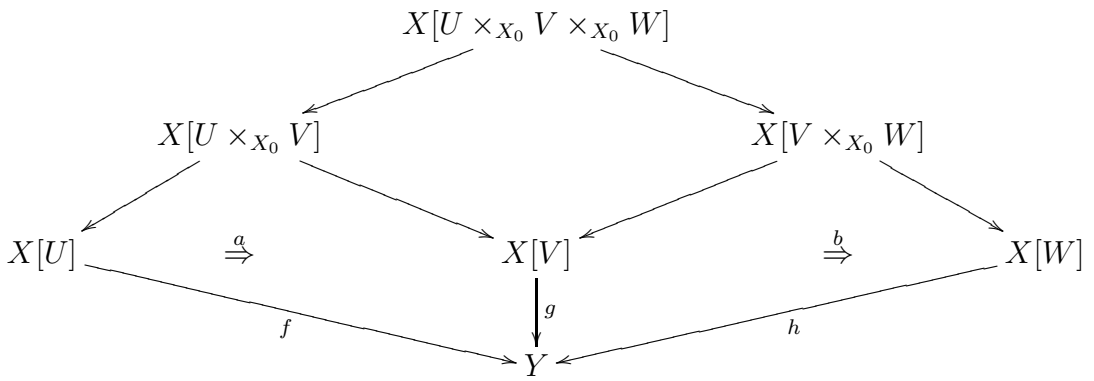
5.10. Lemma. *The unique isomorphism* $(U \times_{Y_0} V) \times_{Z_0} W \simeq U \times_{Y_0} (V \times_{Z_0} W)$ *commuting with the various projections is a renaming isomorphism. The isotransformation arising from this renaming transformation is called the* associator.

A simple but useful criterion for describing isotransformations where one of the anafunctors involved is a functor is as follows.

5.11. Lemma. *An anafunctor* $(V, g)\colon X \nrightarrow Y$ *is isomorphic to a functor* $(X_0, f)\colon X \nrightarrow Y$ *if and only if there is a natural isomorphism*



Just as there is a vertical composition of natural transformations between internal functors, there is a vertical composition of transformations between internal anafunctors [Bartels 2006]. This is where the subcanonicity of J will be used in order to construct a map locally over some cover. Consider the following diagram



We can form a natural transformation between the leftmost and the rightmost composites as functors in S. This will have as its component the arrow

$$\widetilde{ba}\colon U \times_{X_0} V \times_{X_0} W \xrightarrow{\mathrm{id}\times\Delta\times\mathrm{id}} U \times_{X_0} V \times_{X_0} V \times_{X_0} W \xrightarrow{a\times b} Y_1 \times_{Y_0} Y_1 \xrightarrow{m} Y_1$$

in S. Notice that the Čech groupoid of the cover

$$U \times_{X_0} V \times_{X_0} W \longrightarrow U \times_{X_0} W \tag{5}$$

is

$$U \times_{X_0} V \times_{X_0} V \times_{X_0} W \rightrightarrows U \times_{X_0} V \times_{X_0} W,$$

with source and target arising from the two projections $V \times_{X_0} V \longrightarrow V$. Denote this pair of parallel arrows by $s, t\colon UV^2W \rightrightarrows UVW$ for brevity. In [Bartels 2006], section 2.2.3, we find the commuting diagram

$$\begin{array}{ccc} UV^2W & \xrightarrow{t} & UVW \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \widetilde{ba}} \\ UVW & \xrightarrow[\widetilde{ba}]{} & Y_1 \end{array} \tag{6}$$

(this can be checked by using generalised elements) and so we have a functor

$$\check{C}(U \times_{X_0} V \times_{X_0} W) \longrightarrow \mathrm{disc}(Y_1).$$

Our pretopology J is assumed to be subcanonical, so example 4.6 gives us a unique arrow $ba\colon U \times_{X_0} W \longrightarrow Y_1$, which is the data for the composite of a and b.

5.12. Remark. In the special case that $U \times_{X_0} V \times_{X_0} W \longrightarrow U \times_{X_0} W$ is split (e.g. is an isomorphism), the composite transformation has

$$U \times_{X_0} W \longrightarrow U \times_{X_0} V \times_{X_0} W \xrightarrow{\widetilde{ba}} Y_1$$

as its component arrow. In particular, this is the case if one of a or b is a renaming transformation.

5.13. Example. Let $(U, f) : X \mathrel{-\!\!\!\rightarrow} Y$ be an anafunctor and $U'' \xrightarrow{j'} U' \xrightarrow{j} U$ successive refinements of $U \longrightarrow X_0$ (e.g isomorphisms). Let $(U', f_{U'})$ and $(U'', f_{U''})$ denote the composites of f with $X[U'] \longrightarrow X[U]$ and $X[U''] \longrightarrow X[U]$ respectively. The arrow

$$U \times_{X_0} U'' \xrightarrow{\mathrm{id}_U \times j \circ j'} U \times_{X_0} U \longrightarrow Y_1$$

is the component for the composition of the isotransformations $(U, f) \Rightarrow (U', f_{U'}), \Rightarrow (U'', f_{U''})$ described in example 5.8. Thus we can see that the composite of renaming transformations associated to isomorphisms ϕ_1, ϕ_2 is simply the renaming transformation associated to their composite $\phi_1 \circ \phi_2$.

This can be used to show that the associator satisfies the necessary coherence conditions.

5.14. Example. If $a\colon f \Rightarrow g,\ b\colon g \Rightarrow h$ are natural transformations between functors $f, g, h\colon X \longrightarrow Y$ in S, their composite as transformations between anafunctors

$$(X_0, f), (X_0, g), (X_0, h)\colon X \mathrel{-\!\!\!\rightarrow} Y.$$

is just their composite as natural transformations. This uses the equality

$$X_0 \times_{X_0} X_0 \times_{X_0} X_0 = X_0 \times_{X_0} X_0 = X_0,$$

which is due to our choice in remark 4.12 of canonical pullbacks.

Even though we don't have pseudoinverses for weak equivalences of internal categories, one might guess that the local splitting guaranteed to exist by definition is actually more than just a splitting of sorts. This is in fact the case, if we use anafunctors.

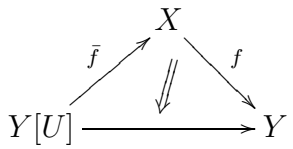
5.15. Lemma. *Let $f\colon X \longrightarrow Y$ be a J-equivalence in S. There is an anafunctor*

$$(U, \bar{f})\colon Y \nrightarrow X$$

and isotransformations

$$\iota\colon (X_0, f) \circ (U, \bar{f}) \Rightarrow id_Y$$
$$\epsilon\colon (U, \bar{f}) \circ (X_0, f) \Rightarrow id_X$$

Proof. We have the anafunctor $(U, \bar{f})$ by definition as f is J-locally split. Since the anafunctors id_X, id_Y are actually functors, we can use lemma 5.11. Using the special case of anafunctor composition when the second is a functor, this tells us that ι will be given by a natural isomorphism



with component $\iota\colon U \longrightarrow Y_1$. Notice that the composite $f_1 \circ \bar{f}_1$ is just

$$Y[U]_1 \simeq U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{\iota \times \mathrm{id} \times -\iota} Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1 \hookrightarrow Y_3 \xrightarrow{m} Y_1.$$

Since the arrow component of $Y[U] \longrightarrow Y$ is $U \times_{Y_0} Y_1 \times_{Y_0} U \xrightarrow{\mathrm{pr}_2} Y_1$, ι is indeed a natural isomorphism using the diagram (2).

The other isotransformation ϵ is between $(X_0 \times_{Y_0} U, \bar{f} \circ \mathrm{pr}_2)$ and (X_0, id_X), and is given by the component

$$\epsilon\colon X_0 \times_{X_0} X_0 \times_{Y_0} U = X_0 \times_{Y_0} U \xrightarrow{\mathrm{id} \times (\bar{f}_0, \iota)} X_0 \times_{Y_0} (X_0 \times_{Y_0} Y_1) \simeq X_0^2 \times_{Y_0^2} Y_1 \simeq X_1$$

The diagram

$$\begin{array}{ccc}
(X_0 \times_{Y_0^2} U)^2 \times_{X_0^2} X_1 & \xrightarrow{\quad\mathrm{pr}_2\quad} & X_1 \\
\simeq \downarrow & & \\
U \times_{Y_0} X_1 \times_{Y_0} U & & \downarrow \simeq \\
{\scriptstyle -\iota \times f \times \iota} \downarrow & & \\
(X_0 \times_{Y_0} Y_1) \times_{Y_0} Y_1 \times_{Y_0} (Y_1 \times_{Y_0} X_0) & \xrightarrow[\mathrm{id} \times m \times \mathrm{id}]{} & X_0 \times_{Y_0} Y_1 \times_{Y_0} X_0
\end{array}$$

commutes (a fact which can be checked using generalised elements), and using (2) we see that ϵ is natural. ∎

The first half of the following theorem is proposition 12 in [Bartels 2006], and the second half follows because all the constructions of categories involved in dealing with anafunctors outlined above are still objects of C.

5.16. Theorem. *[Bartels 2006] For a site (S, J) where J is a subcanonical singleton pretopology, internal categories, anafunctors and transformations form a bicategory called $\mathbf{Cat}_{\mathrm{ana}}(S, J)$. If we restrict attention to a full sub-2-category C which admits base change for arrows in J, we have an analogous full sub-bicategory $C_{\mathrm{ana}}(J)$.*

In fact the bicategory $C_{ana}(J)$ fails to be a strict 2-category only in the sense that the associator is given by the non-identity isotransformation from lemma 5.10. All the other structure is strict.

There is a strict 2-functor $C_{\mathrm{ana}}(J) \longrightarrow \mathbf{Cat}_{\mathrm{ana}}(S, J)$ which is an inclusion on objects and fully faithful in the strictest sense, namely being the identity functor on hom-categories. The following is the main result of this section, and allows us to relate anafunctors to the localisations considered in the next section.

5.17. Proposition. *There is a strict, identity-on-objects 2-functor*

$$\alpha_J \colon C \longrightarrow C_{\mathrm{ana}}(J)$$

sending J-equivalences to equivalences, and commuting with the respective inclusions into $\mathbf{Cat}(S)$ and $\mathbf{Cat}_{\mathrm{ana}}(S, J)$.

Proof. We define α_J to be the identity on objects, and as described in examples 5.2, 5.5 on 1-arrows and 2-arrows (i.e. functors and transformations). We need first to show that this gives a functor $C(X, Y) \longrightarrow C_{\mathrm{ana}}(J)(X, Y)$. This is precisely the content of example 5.14. Since the identity 1-cell on a category X in $C_{\mathrm{ana}}(J)$ is the image of the identity functor on S in C, α_J respects identity 1-cells. Also, lemma 5.9 tells us that α_J respects composition. That α_J sends J-equivalences to equivalences is the content of lemma 5.15. ∎

The 2-category C is locally small (i.e. enriched in small categories) if S itself is locally small (i.e. enriched in sets), but *a priori* the collection of anafunctors $X \nrightarrow Y$ do not constitute a set for S a large category.

5.18. Proposition. *Let (S, J) be a locally small, subcanonical unary site satisfying WISC and let C admit base change along arrows in J. Then $C_{\mathrm{ana}}(J)$ is locally essentially small.*

Proof. Given an object A of S, let $I(A)$ be a weakly initial set for J/A. Consider the locally full sub-2-category of $C_{\mathrm{ana}}(J)$ with the same objects, and arrows those anafunctors $(U, f) : X \nrightarrow Y$ such that $U \longrightarrow X_0$ is in $I(X_0)$. Every anafunctor is then isomorphic, by example 5.8, to one in this sub-2-category. The collection of anafunctors $(U, f) : X \nrightarrow Y$ for a fixed U forms a set, by local smallness of C, and similarly the collection of transformations between a pair of anafunctors forms a set by local smallness of S. ∎

Examples of locally small sites (S, J) where $C_{\text{ana}}(J)$ is not known to be locally essentially small are the category of sets from the model of ZF used in [van den Berg 2012], the model of ZF constructed in [Roberts 2013] and the topos from proposition 3.23. We note that local essential smallness of $C_{\text{ana}}(J)$ seems to be a condition just slightly weaker than WISC.

6. Localising bicategories at a class of 1-cells

Ultimately we are interesting in inverting all J-equivalences in C and so need to discuss what it means to add the formal pseudoinverses to a class of 1-cells in a 2-category – a process known as *localisation*. This was done in [Pronk 1996] for the more general case of a class of 1-cells in a bicategory, where the resulting bicategory is constructed and its universal properties examined. The application in *loc. cit.* is to show the equivalence of various bicategories of stacks to localisations of 2-categories of smooth, topological and algebraic groupoids. The results of this article can be seen as one-half of a generalisation of these results to more general sites.

6.1. Definition. [Pronk 1996] Let B be a bicategory and $W \subset B_1$ a class of 1-cells. A *localisation of B with respect to W* is a bicategory $B[W^{-1}]$ and a weak 2-functor

$$U\colon B \longrightarrow B[W^{-1}]$$

such that U sends elements of W to equivalences, and is universal with this property i.e. precomposition with U gives an equivalence of bicategories

$$U^*\colon Hom(B[W^{-1}], D) \longrightarrow Hom_W(B, D),$$

where Hom_W denotes the sub-bicategory of weak 2-functors that send elements of W to equivalences (call these W*-inverting*, abusing notation slightly).

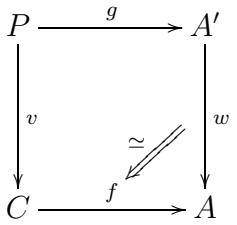
The universal property means that W-inverting weak 2-functors $F\colon B \longrightarrow D$ factor, up to an equivalence, through $B[W^{-1}]$, inducing an essentially unique weak 2-functor $\widetilde{F}\colon B[W^{-1}] \longrightarrow D$.

6.2. Definition. [Pronk 1996] Let B be a bicategory with a class W of 1-cells. W is said to *admit a right calculus of fractions* if it satisfies the following conditions

2CF1. W contains all equivalences

2CF2. a) W is closed under composition
b) If $a \in W$ and there is an isomorphism $a \xRightarrow{\sim} b$ then $b \in W$

2CF3. For all $w\colon A' \longrightarrow A$, $f\colon C \longrightarrow A$ with $w \in W$ there exists a 2-commutative square



with $v \in W$.

2CF4. If $\alpha\colon w \circ f \Rightarrow w \circ g$ is a 2-arrow and $w \in W$ there is a 1-cell $v \in W$ and a 2-arrow $\beta\colon f \circ v \Rightarrow g \circ v$ such that $\alpha \circ v = w \circ \beta$. Moreover: when α is an isomorphism, we require β to be an isomorphism too; when v' and β' form another such pair, there exist 1-cells u, u' such that $v \circ u$ and $v' \circ u'$ are in W, and an isomorphism $\epsilon\colon v \circ u \Rightarrow v' \circ u'$ such that the following diagram commutes:

$$\begin{array}{ccc} f \circ v \circ u & \xRightarrow{\beta \circ u} & g \circ v \circ u \\ {\scriptstyle f\circ\epsilon}\Downarrow\simeq & & \simeq\Downarrow{\scriptstyle g\circ\epsilon} \\ f \circ v' \circ u' & \xRightarrow[\beta' \circ u']{} & g \circ v' \circ u' \end{array} \tag{7}$$

For a bicategory B with a calculus of right fractions, [Pronk 1996] constructs a localisation of B as a bicategory of fractions; the 1-arrows are spans and the 2-arrows are equivalence classes of bicategorical spans-of-spans diagrams.

From now on we shall refer to a calculus of right fractions as simply a calculus of fractions, and the resulting localisation constructed by Pronk as a bicategory of fractions. Since $B[W^{-1}]$ is defined only up to equivalence, it is of great interest to know when a bicategory D, in which elements of W are sent to equivalences by a 2-functor $B \longrightarrow D$, is equivalent to $B[W^{-1}]$. In particular, one might be interested in finding such an equivalent bicategory with a simpler description than that which appears in [Pronk 1996].

6.3. Proposition. *[Pronk 1996] A weak 2-functor $F : B \longrightarrow D$ which sends elements of W to equivalences induces an equivalence of bicategories*

$$\widetilde{F}\colon B[W^{-1}] \xrightarrow{\sim} D$$

if the following conditions hold

EF1. F is essentially surjective,

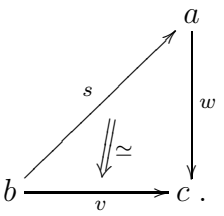
EF2. For every 1-cell $f \in D_1$ there are 1-cells $w \in W$ and $g \in B_1$ such that $Fg \overset{\sim}{\Rightarrow} f \circ Fw$,

EF3. F is locally fully faithful.

Thanks are due to Matthieu Dupont for pointing out (in personal communication) that proposition 6.3 actually only holds in the one direction, not in both, as claimed in *loc. cit.*

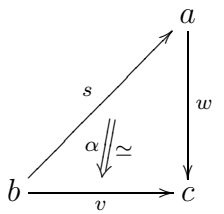
The following is useful in showing a weak 2-functor sends weak equivalences to equivalences, because this condition only needs to be checked on a class that is in some sense cofinal in the weak equivalences.

6.4. Proposition. *Let $V \subset W$ be two classes of 1-cells in a bicategory B such that for all $w \in W$, there exists $v \in V$ and $s \in W$ and an invertible 2-cell*



Then a weak 2-functor $F\colon B \longrightarrow D$ that sends elements of V to equivalences also sends elements of W to equivalences.

Proof. In the following the coherence arrows will be present, but unlabelled. It is enough to prove that if in a bicategory D with a class of maps M (in our case $M = F(W)$) such that for all $w \in M$ there is an equivalence v and an isomorphism α,

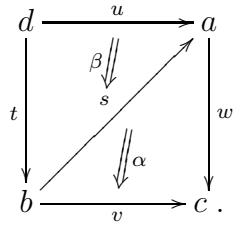


where $s \in M$, then all elements of M are also equivalences.

Let $\bar{v}$ be a pseudoinverse for v and let $j = s \circ \bar{v}$. Then there is sequence of isomorphisms

$$w \circ j \Rightarrow (w \circ s) \circ \bar{v} \Rightarrow v \circ \bar{v} \Rightarrow I.$$

Since $s \in M$, there is an equivalence u, $t \in M$ and an isomorphism β giving the following diagram



Let $\bar{u}$ be a pseudoinverse of u. We know from the first part of the proof that we have a pseudosection $k = t \circ \bar{u}$ of s, with an isomorphism $s \circ k \Rightarrow I$. We then have the following sequence of isomorphisms:

$$j \circ w = (s \circ \bar{v}) \circ w \Rightarrow ((s \circ \bar{v}) \circ w) \circ (s \circ k) \Rightarrow s \circ ((\bar{v} \circ v) \circ (t \circ \bar{u})) \Rightarrow (s \circ t) \circ u \Rightarrow \bar{u} \circ u \Rightarrow I.$$

Thus all elements of M are equivalences. ∎

7. 2-categories of internal categories admit bicategories of fractions

In this section we prove the result that $C \hookrightarrow \mathbf{Cat}(S)$ admits a calculus of fractions for the J-equivalences, where J is a singleton pretopology on S.

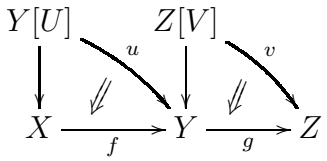
The following is the first main theorem of the paper, and subsumes a number of other, similar theorems throughout the literature (see section 8 for details).

7.1. THEOREM. *Let S be a category with a singleton pretopology J. Assume the full sub-2-category $C \hookrightarrow \mathbf{Cat}(S)$ admits base change along maps in J. Then C admits a right calculus of fractions for the class W_J of J-equivalences.*

PROOF. We show the conditions of definition 6.2 hold.

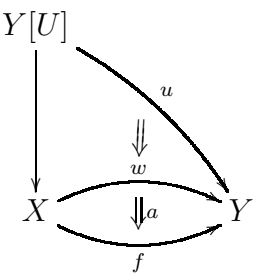
2CF1. An internal equivalence is clearly J-locally split. Lemma 4.17 gives us the rest.

2CF2. a) That the composition of fully faithful functors is again fully faithful is trivial. Consider the composition $g \circ f$ of two J-locally split functors,



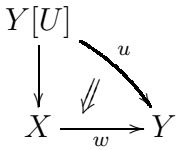
By lemma 4.13 the functor u pulls back to a functor $Z[U \times_{Y_0} V] \longrightarrow Z[V]$. The composite $Z[U \times_{Y_0} V] \longrightarrow Z$ is fully faithful with object component in J, hence $g \circ f$ is J-locally split.

b) Lemma 4.17 tells us that fully faithful functors are closed under isomorphism, so we just need to show J-locally split functors are closed under isomorphism. Let $w, f \colon X \longrightarrow Y$ be functors and $a \colon w \Rightarrow f$ be a natural isomorphism. First, let w be J-locally split. It is immediate from the diagram

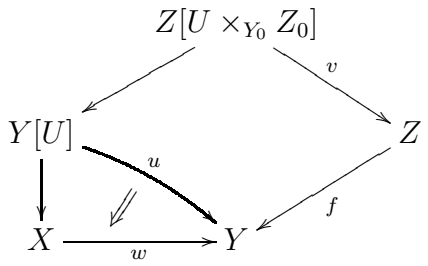


that f is also J-locally split.

2CF3. Let $w\colon X \longrightarrow Y$ be a J-equivalence, and let $f\colon Z \longrightarrow Y$ be a functor. From the definition of J-locally split, we have the diagram

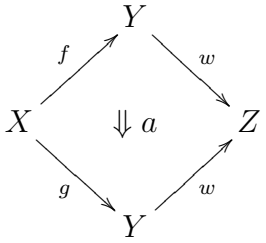


We can use lemma 4.13 to pull u back along f to get a 2-commuting diagram

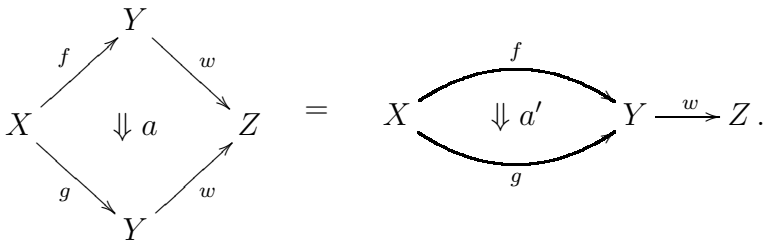


with $v \in W_J$ as required.

2CF4. Since J-equivalences are representably fully faithful, given

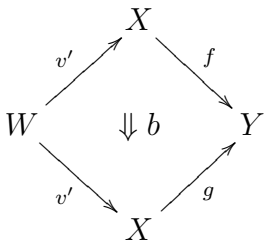


where $w \in W_J$, there is a unique $a'\colon f \Rightarrow g$ such that

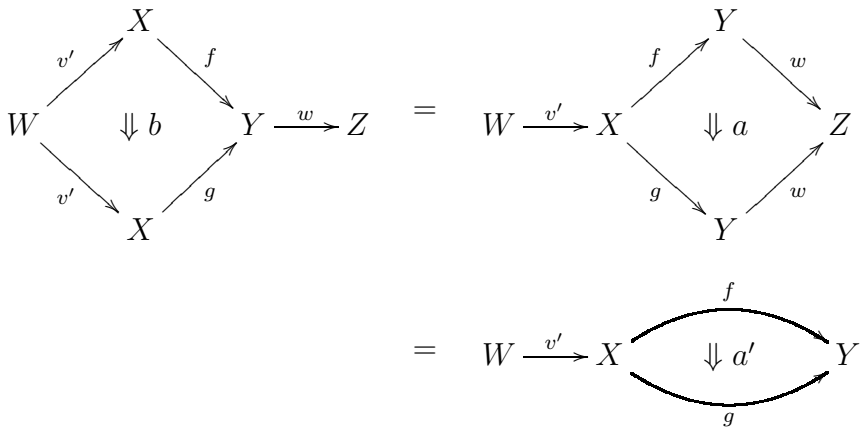


The existence of a' is the first half of 2CF4, where $v = \mathrm{id}_X$. Note that if a is an isomorphism, so if a', since w is representably fully faithful. Given $v'\colon W \longrightarrow X \in W_J$

such that there is a transformation

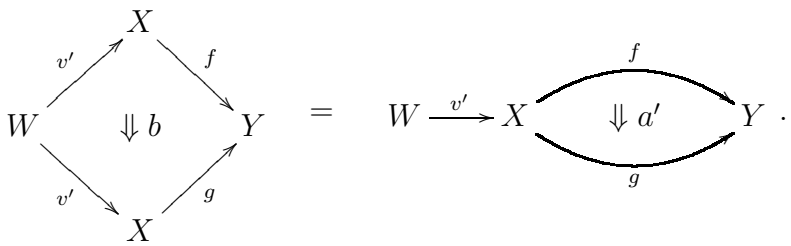


satisfying



(8)

then uniqueness of a', together with equation (8) gives us



This is precisely the diagram (7) with $v = \mathrm{id}_X$, $u = v'$, $u' = \mathrm{id}_W$ and ϵ the identity 2-arrow. Hence 2CF4 holds. ∎

The proof of theorem 7.1 is written using only the language of 2-categories, so can be generalised from C to other 2-categories. This approach will be taken up in [Roberts B].

The second main result of the paper is that we want to know when this bicategory of fractions is equivalent to a bicategory of anafunctors, as the latter bicategory has a much simpler construction.

7.2. Theorem. *Let (S, J) be a subcanonical unary site and let the full sub-2-category $C \hookrightarrow \mathbf{Cat}(S)$ admit base change along arrows in J. Then there is an equivalence of bicategories*

$$C_{\mathrm{ana}}(J) \simeq C[W_J^{-1}]$$

under C.

PROOF. Let us show the conditions in proposition 6.3 hold. To begin with, the 2-functor $\alpha_J \colon C \longrightarrow C_{ana}(J)$ sends J-equivalences to equivalences by proposition 5.17.

EF1. α_J is the identity on 0-cells, and hence surjective on objects.

EF2. This is equivalent to showing that for any anafunctor $(U, f) \colon X \multimap Y$ there are functors w, g such that w is in W_J and

$$(U, f) \stackrel{\sim}{\Rightarrow} \alpha_J(g) \circ \alpha_J(w)^{-1}$$

where $\alpha_J(w)^{-1}$ is some pseudoinverse for $\alpha_J(w)$.

Let w be the functor $X[U] \longrightarrow X$ and let $g = f \colon X[U] \longrightarrow Y$. First, note that

$$\begin{array}{ccc} & X[U] & \\ \swarrow & & \searrow^{=} \\ X & & X[U] \end{array}$$

is a pseudoinverse for

$$\alpha_J(w) \quad = \quad \left(\begin{array}{ccc} & X[U][U] & \\ \swarrow & =\ \searrow & \\ X[U] & & X \end{array} \right) .$$

Then the composition $\alpha_J(f) \circ \alpha_J(w)^{-1}$ is

$$\begin{array}{ccc} & X[U \times_U U \times_U U] & \\ \swarrow & & \searrow \\ X & & Y \end{array} ,$$

which is just (U, f) (recall we have the equality $U \times_U U \times_U U = U$ by remark 4.12).

EF3. If $a \colon (X_0, f) \Rightarrow (X_0, g)$ is a transformation of anafunctors for functors $f, g \colon X \longrightarrow Y$, it is given by a natural transformation

$$f \Rightarrow g \colon X = X[X_0 \times_{X_0} X_0] \longrightarrow Y.$$

Hence we get a unique natural transformation $a \colon f \Rightarrow g$ such that a is the image of a' under α_J. ∎

We now give a series of results following from this theorem, using basic properties of pretopologies from section 3.

7.3. Corollary. *When J and K are two subcanonical singleton pretopologies on S such that $J_{un} = K_{un}$, for example J cofinal in K, there is an equivalence of bicategories*

$$C_{\mathrm{ana}}(J) \simeq C_{\mathrm{ana}}(K).$$

The class of maps in **Top** of the form $\coprod U_i \longrightarrow X$ for an open cover $\{U_i\}$ of X form a singleton pretopology. This is because $\mathcal{O}$ is a *superextensive* pretopology (see the appendix). Given a site with a superextensive pretopology J, we have the following result which is useful when J is not a singleton pretopology (the *singleton* pretopology $\amalg J$ is defined analogously to the case of **Top**, details are in the appendix).

7.4. Corollary. *Let (S, J) be a superextensive site where J is a subcanonical pretopology. Then*

$$C[W_{J_{un}}^{-1}] \simeq C_{\mathrm{ana}}(\amalg J).$$

Proof. This essentially follows by lemma A.9. ∎

Obviously this can be combined with previous results, for example if K is cofinal in $\amalg J$, for J a non-singleton pretopology, K-anafunctors localise C at the class of J_{un}-equivalences.

Finally, given WISC we have a bound on the size of the hom-categories, up to equivalence.

7.5. Theorem. *Let (S, J) be a subcanonical unary site satisfying WISC with S locally small and let $C \hookrightarrow \mathbf{Cat}(S)$ admit base change along arrows in J. Then any localisation $C[W_J^{-1}]$ is locally essentially small.*

Recall that this localisation can be chosen such that the class of objects is the same as the class of objects of C, and so it is not necessary to consider additional set-theoretic mechanisms for dealing with large (2-)categories here.

We note that the issue of size of localisations is not touched on in [Pronk 1996]. even though such issues are commonly addressed in localisation of 1-categories. If we have a specified bound on the hom-sets of S and also know that some WISC_κ holds, then we can put specific bounds on the size of the hom-categories of the localisation. This is important if examining fine size requirements or implications for localisation theorems such as these, for example higher versions of locally presentable categories.

8. Examples

The simplest example is when we take the trivial singleton pretopology *triv*, where covering families are just single isomorphisms: *triv*-equivalences are internal equivalences and, up to equivalence, localisation at W_{triv} does nothing. It is worth pointing out that if we localise at $W_{triv_{un}}$, which is equivalent to considering anafunctors with source leg having

a split epimorphism for its object component, then by corollary 7.3 this is equivalent to localising at W_{triv}, so $C_{ana}(triv_{un}) \simeq C_{ana}(triv) \simeq C$.

The first non-trivial case is that of a regular category with the canonical singleton pretopology $\underline{c}$. This is the setting of [Bunge-Paré 1979]. Recall that W_J^{BP} is the class of Bunge-Paré J-equivalences (definition 4.18). For now, let C denote either $\mathbf{Cat}(S)$ or $\mathbf{Gpd}(S)$.

8.1. Proposition. *Let (S, J) be a finitely complete unary site with J saturated. Then we have*

$$C[(W_J^{BP})^{-1}] \simeq C[W_J^{-1}]$$

This is merely a restatement of the fact Bunge-Paré J-equivalences and ordinary J-equivalences coincide in this case.

8.2. Corollary. *The canonical singleton pretopology $\underline{c}$ on a finitely complete category S is saturated. Hence $W_{\underline{c}}^{BP} = W_{\underline{c}}$ for this site, and*

$$C[(W_{\underline{c}}^{BP})^{-1}] \simeq C[W_{\underline{c}}^{-1}] \simeq C_{\mathrm{ana}}(\underline{c})$$

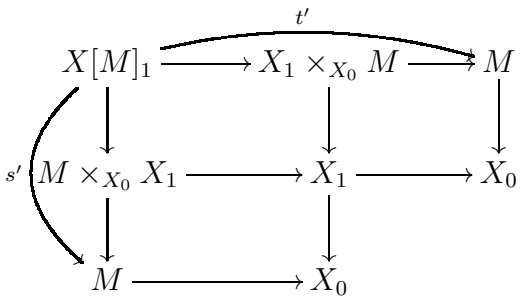
We can combine this corollary with corollary 7.3 so that the localisation of either $\mathbf{Cat}(S)$ or $\mathbf{Gpd}(S)$ at the Bunge-Paré weak equivalences can be calculated using J-anafunctors for J cofinal in $\underline{c}$. We note that $\underline{c}$ does not satisfy WISC in general (see proposition 3.23 and the comments following), so the localisation might not be locally essentially small.

The previous corollaries deal with the case when we are interested in the 2-categories consisting of all of the internal categories or groupoids in a site. However, for many applications of internal categories/groupoids it is not sufficient to take all of $\mathbf{Cat}(S)$ or $\mathbf{Gpd}(S)$. One widely used example is that of Lie groupoids, which are groupoids internal to the category of (finite-dimensional) smooth manifolds such that source and target maps are submersions (more on these below). Other examples are used in the theory of algebraic stacks, namely groupoids internal to schemes or algebraic spaces. Other types of such *presentable* stacks use groupoids internal to some site with specified conditions on the source and target maps. Although it is not covered explicitly in the literature, it is possible to consider presentable stacks of categories, and this will be taken up in future work [Roberts A].

We thus need to furnish examples of sub-2-categories C, specified by restricting the sort of maps that are allowed for source and target, that admit base change along some class of arrows. The following lemma gives a sufficiency condition for this to be so.

8.3. Lemma. *Let $\mathbf{Cat}^{\mathcal{M}}(S)$ be defined as the full sub-2-category of $\mathbf{Cat}(S)$ with objects those categories such that the source and target maps belong to a singleton pretopology $\mathcal{M}$. Then $\mathbf{Cat}^{\mathcal{M}}(S)$ admits base change along arrows in $\mathcal{M}$, as does the corresponding 2-category $\mathbf{Gpd}^{\mathcal{M}}(S)$ of groupoids.*

PROOF. Let X be an object of $\mathbf{Cat}^{\mathcal{M}}(S)$ and $f\colon M \longrightarrow X_0 \in \mathcal{M}$. In the following diagram, all the squares are pullbacks and all arrows are in $\mathcal{M}$.



The maps marked s', t' are the source and target maps for the base change along f, so $X[M]$ is in $\mathbf{Cat}^{\mathcal{M}}(S)$. The same argument holds for groupoids verbatim. ∎

In practice one often only wants base change along a subclass of $\mathcal{M}$, such as the class of open covers sitting inside the class of open maps in **Top**. We can then apply theorems 7.1 and 7.2 to the 2-categories $\mathbf{Cat}^{\mathcal{M}}(S)$ and $\mathbf{Gpd}^{\mathcal{M}}(S)$ with the classes of $\mathcal{M}$-equivalences, and indeed to sub-2-categories of these, as we shall in the examples below.

We shall focus of a few concrete cases to show how the results of this paper subsume similar results in the literature proved for specific sites.

The category of smooth manifolds is not finitely complete so the localisation results in this section so far do not apply to it. There are two ways around this. The first is to expand the category of manifolds to a category of smooth spaces which *is* finitely complete (or even cartesian closed). In that case all the results one has for finitely complete sites can be applied. The other is to take careful note of which finite limits are actually needed, and show that all constructions work in the original category of manifolds. There is then a hybrid approach, which is to work in the expanded category, but point out which results/constructions actually fall inside the original category of manifolds. Here we shall take the second approach. First, let us pin down some definitions.

8.4. DEFINITION. Let **Diff** be the category of smooth, finite-dimensional manifolds. A *Lie category* is a category internal to **Diff** where the source and target maps are submersions (and hence the required pullbacks exist). A *Lie groupoid* is a Lie category which is a groupoid. A *proper* Lie groupoid is one where the map $(s,t)\colon X_1 \longrightarrow X_0 \times X_0$ is proper. An *étale* Lie groupoid is one where the source and target maps are local diffeomorphisms.

By lemma 8.3 the 2-categories of Lie categories, Lie groupoids and proper Lie groupoids admit base change along any of the following classes of maps: open covers ($\amalg\mathcal{O}$), surjective local diffeomorphisms ($\acute{e}t$), surjective submersions ($Subm$). The 2-categories of étale Lie groupoids and proper étale Lie groupoids admit base change along arrows in $\acute{e}t$ and $Subm$. We should note that we have $\amalg\mathcal{O}$ cofinal in $\acute{e}t$, which is cofinal in $Subm$.

We can thus apply the main results of this paper to the sites $(\mathbf{Diff}, \mathcal{O})$, $(\mathbf{Diff}, \amalg\mathcal{O})$, $(\mathbf{Diff}, \acute{e}t)$ and $(\mathbf{Diff}, Subm)$ and the 2-categories of Lie categories, Lie groupoids, proper Lie groupoids and so on. However, the definition of weak equivalence we have here,

involving J-locally split functors, is not one that appears in the Lie groupoid literature, which is actually Bunge-Paré $Subm$-equivalence. However, we have the following result:

8.5. Proposition. *A functor $f\colon X \longrightarrow Y$ between Lie categories is a Subm-equivalence if and only if it is a Bunge-Paré Subm-equivalence.*

Before we prove this, we need a lemma proved by Ehresmann.

8.6. Lemma. *[Ehresmann 1959] For any Lie category X, the subset of invertible arrows, $X_1^{iso} \hookrightarrow X_1$ is an open submanifold.*

Hence there is a Lie groupoid X^{iso} and an identity-on-objects functor $X^{iso} \longrightarrow X$ which is universal for functors from Lie groupoids. In particular, a natural isomorphism between functors with codomain X is given by a component map that factors through X_1^{iso}, and the induced source and target maps $X_1^{iso} \longrightarrow X_0$ are submersions.

Proof. (proposition 8.5) Full faithfulness is the same for both definitions, so we just need to show that f is $Subm$-locally split if and only if it is essentially $Subm$-surjective. We first show the forward implication.

The special case of a $\amalg\mathcal{O}$-equivalence between Lie groupoids is a small generalisation of the proof of proposition 5.5 in [Moerdijk-Mrčun 2003], which states than an internal equivalence of Lie groupoids is a Bunge-Paré $Subm$-equivalence. Since $\amalg\mathcal{O}$ is cofinal in $Subm$, a $Subm$-equivalence is a $\amalg\mathcal{O}$-equivalence, hence a Bunge-Paré $Subm$-equivalence.

For the case when X and Y are Lie categories, we use the fact that we can define $X_0 \times_{Y_0} Y_1^{iso}$ and that the local sections constructed in Moerdijk-Mrčun's proof factor through this manifold to set up the proof as in the groupoid case.

For the reverse implication, the construction in the first half of the proof of proposition 4.19 goes through verbatim, as all the pullbacks used involve submersions. ∎

The need to localise the category of Lie groupoids at W_{Subm} was perhaps first noted in [Pradines 1989], where it was noted that something other than the standard construction of a category of fractions was needed. However Pradines lacked the necessary 2-categorical localisation results. Pronk considered the sub-2-category of étale Lie groupoids, also localised at W_{Subm}, in order to relate these groupoids to differentiable étendues [Pronk 1996]. Lerman discusses the 2-category of orbifolds *qua* stacks [Lerman 2010] and argues that it should be a localisation of the 2-category of proper étale Lie groupoids (again at W_{Subm}). These three cases use different constructions of the 2-categorical localisation: Pradines used what he called *meromorphisms*, which are equivalence classes of butterfly-like diagrams and are related to Hilsum-Skandalis morphisms, Pronk introduces the techniques outlined in this paper, and Lerman uses Hilsum-Skandalis morphisms, also known as right principal bibundles.

Interestingly, [Colman 2010] considers this localisation of the 2-category of Lie groupoids then considers a further localisation, not given by the results of this paper.[2] Colman in

[2]In fact this is the only 2-categorical localisation result involving internal categories or groupoids known to the author to *not* be covered by theorem 7.1 or its sequel [Roberts B].

essence shows that the full sub-2-category of topologically discrete groupoids, i.e. ordinary small groupoids, is a localisation at those internal functors which induce an equivalence on fundamental groupoids.

Our next example is that of topological groupoids, which correspond to various flavours of stacks on the category **Top**. The idea of weak equivalences of topological groupoids predates the case of Lie groupoids, and [Pradines 1989] credits it to Haefliger, van Est, and [Hilsum-Skandalis 1987]. In particular the first two were ultimately interested in defining the fundamental group of a foliation, that is to say, of the topological groupoid associated to a foliation, considered up to weak equivalence.

However more recent examples have focussed on topological stacks, or variants thereon. In particular, in parallel with the algebraic and differentiable cases, the topological stacks for which there is a good theory correspond to those topological groupoids with conditions on their source and target maps. Aside from étale topological groupoids (which were considered by [Pronk 1996] in relation to étendues), the real advances here have come from work of Noohi, starting with [Noohi 2005a], who axiomatised the concept of *local fibration* and asked that the source and target maps of topological groupoids are local fibrations.

8.7. Definition. A singleton pretopology LF in **Top** is called a class of *local fibrations* if the following conditions hold:[3]

1. LF contains the open embeddings
2. LF is stable under coproducts, in the sense that $\coprod_{i\in I} X_i \longrightarrow Y$ is in LF if each $X_i \longrightarrow Y$ is in LF
3. LF is local on the target for the open cover pretopology. That is, if the pullback of a map $f\colon X \longrightarrow Y$ along an open cover of Y is in LF, then f is in LF.

Conditions 1. and 2. tell us that $\amalg\mathcal{O} \subset LF$, and that LF is $\amalg J$ for some superextensive pretopology J containing the open embeddings as singleton 'covering' families (beware the misleading terminology here: covering families are not assumed to be jointly surjective). Note that LF will not be subcanonical, by condition 1. As an example, given any of the following pretopologies K:

- Serre fibrations,
- Hurewicz fibrations,
- open maps,
- split maps,
- projections out of a cartesian product,

[3]We have packaged the conditions in a way slightly different to [Noohi 2005a], but the definition is in fact identical.

- isomorphisms;

one can define a class of local fibrations by choosing those maps which are in K on pulling back to an open cover of the codomain. Such maps are then called *local* K. As an example of the usefulness of this concept, the topological stacks corresponding to topological groupoids with local Hurewicz fibrations as source and target have a nicely behaved homotopy theory. The case of étale groupoids corresponds to the last named class of maps, which give us local isomorphisms, i.e. étale maps. We can then apply lemma 8.3 and theorem 7.1 to the 2-category $\mathbf{Grp}^{LF}(\mathbf{Top})$ to localise at the class $W_{\amalg\mathcal{O}}$ (as $\amalg\mathcal{O} \subset LF$), or any other singleton pretopology contained in LF, using anafunctors whenever this pretopology is subcanonical. Note that if C satisfies WISC, so will the corresponding LF, although this is probably not necessary to consider in the presence of full AC.

A slightly different approach is taken in [Carchedi 2012], where the author introduces a new pretopology on the category CGH of compactly generated Hausdorff spaces. We give a definition equivalent to the one in *loc cit.*

8.8. Definition. A (not necessarily open) cover $\{V_i \hookrightarrow X\}_{i\in I}$ is called a $\mathcal{CG}$-cover if for any map $K \longrightarrow X$ from a compact space K, there is a finite open cover $\{U_j \hookrightarrow K\}$ which refines the cover $\{V_i \times_X K \longrightarrow K\}_{i\in I}$. $\mathcal{CG}$-covers form a pretopology $\mathcal{CG}$ on CGH.

Compactly generated stacks then correspond to groupoids in CGH such that source and target maps are in the pretopology $\mathcal{CG}_{un}$. Again, we can localise $\mathbf{Gpd}^{\mathcal{CG}}(CGH)$ at $W_{\mathcal{CG}_{un}}$ using lemma 8.3 and theorem 7.1, and anafunctors can be again pressed into service.

We now arrive at the more involved case of algebraic stacks (cf. the continually growing [Stacks project] for the extent of the theory of algebraic stacks), which were the first presentable stacks to be defined. There are some subtleties about the site of definition for algebraic stacks, and powerful representability theorems, but we can restrict to three main cases: groupoids in the category of affine schemes $\mathbf{Aff} = \mathbf{Ring}^{op}$; groupoids in the category **Sch** of schemes; and groupoids in the category **AlgSp** of algebraic spaces. Algebraic spaces reduce to algebraic stacks on **Sch** represented by groupoids with trivial automorphism groups, and the category of schemes is a subcategory of $Sh(\mathbf{Aff})$, so we shall just consider the case when our ambient category is **Aff**. In any case, all the special properties of classes of maps in all three sites are ultimately defined in terms of properties of ring homomorphisms. Note that groupoids in **Aff** are exactly the same thing as cogroupoid objects in **Ring**, which are more commonly known as *Hopf algebroids*.

Despite the possibly unfamiliar language used by algebraic geometry, algebraic stacks reduce to the following semiformal definition. We fix three singleton pretopologies on our site **Aff**: J, E and D such that E and D are local on the target for the pretopology J. An algebraic stack then is a stack on **Aff** for the pretopology J which 'corresponds' to a groupoid X in **Aff** such that source and target maps belong to E and $(s,t)\colon X_1 \longrightarrow X_0^2$ belongs to D. We recover the algebraic stacks by localising the 2-category of such groupoids

at W_E (this claim of course needs substantiating, something we will not do here for reasons of space, referring rather to [Pronk 1996, Schäppi 2012] and the forthcoming [Roberts A]).

In practice, D can be something like closed maps (to recover Hausdorff-like conditions) or all maps, and E consists of either smooth or étale maps, corresponding to Artin and Deligne-Mumford stacks respectively. J is then something like the étale topology (or rather, the singleton pretopology associated to it, as the étale topology is superextensive), and we can apply lemma 8.3 to see that base change exists along J, along with the fact that asking for $(s,t) \in D$ is automatically stable under forming the base change. In practice, a variety of combinations of J, E and D are used, as well as passing from **Aff** to **Sch** and **AlgSp**, so there are various compatibilities to check in order to know one can apply theorem 7.1.

A final application we shall consider is when our ambient category consists of algebraic objects. As mentioned in section 2, a number of authors have considered localising groupoids in Mal'tsev, or Barr-exact, or protomodular, or semi-abelian categories, which are hallmarks of categories of algebraic objects rather than spatial ones, as we have been considering so far.

In the case of groupoids in **Grp** (which, as in any Mal'tsev category, coincide with the internal categories) it is a well-known result that they can be described using *crossed modules*.

8.9. Definition. A *crossed module* (in **Grp**) is a homomorphism $t\colon G \longrightarrow H$ together with a homomorphism $\alpha\colon H \longrightarrow \mathrm{Aut}(G)$ such that t is H-equivariant (using the conjugation action of H on itself), and such that the composition $\alpha \circ t\colon G \longrightarrow \mathrm{Aut}(G)$ is the action of G on itself by conjugation. A crossed module is often denoted, when no confusion will arise, by $(G \longrightarrow H)$. A morphism $(G \longrightarrow H) \longrightarrow (K \longrightarrow L)$ of crossed modules is a pair of maps $G \longrightarrow K$ and $H \longrightarrow L$ making the obvious square commute, and commuting with all the action maps.

Similar definitions hold for groups internal to cartesian closed categories, and even just finite-product categories if one replaces $H \longrightarrow \mathrm{Aut}(G)$ with its transpose $H \times G \longrightarrow G$. Ultimately of course there is a definition for crossed modules in semiabelian categories (e.g. [AMMV 2010]), but we shall consider just groups. There is a natural definition of 2-arrow between maps of crossed modules, but the specifics are not important for the present purposes, so we refer to [Noohi 2005c, definition 8.5] for details. The 2-categories of groupoids internal to **Grp** and crossed modules are equivalent, so we shall just work with the terminology of the latter.

Given the result that crossed modules correspond to pointed, connected homotopy 2-types, it is natural to ask if all maps of such arise from maps between crossed modules. The answer is, perhaps unsurprisingly, no, as one needs maps which only *weakly* preserve the group structure. One can either write down the definition of some generalised form of map ([Noohi 2005c, definition 8.4]), or localise the 2-category of crossed modules ([Noohi 2005c] considers a model structure on the category of crossed modules). To localise the 2-category of crossed modules we can consider the singleton pretopology *epi* on **Grp** consisting of the epimorphisms, and localise $\mathbf{Gpd}(\mathbf{Grp})$ at W_{epi}.

There are potentially interesting sub-2-categories of crossed modules that one might want to consider, for example, the one corresponding to *nilpotent* pointed connected 2-types. These are crossed modules $t\colon G \longrightarrow H$ where the cokernel of t is a nilpotent group and the (canonical) action of $\operatorname{coker} t$ on $\ker t$ is nilpotent. The correspondence between such crossed modules and the corresponding internal groupoids is a nice exercise, as well as seeing that this 2-category admits base change for the pretopology *epi*.

A. Superextensive sites

The usual sites of topological spaces, manifolds and schemes all share a common property: one can (generally) take coproducts of covering families and end up with a cover. In this appendix we gather some results that generalise this fact, none of which are especially deep, but help provide examples of bicategories of anafunctors. Another reference for superextensive sites is [Shulman 2012].

A.1. Definition. [CLW 1993] A *finitary* (resp. *infinitary*) *extensive* category is a category with finite (resp. small) coproducts such that the following condition holds: let I be a a finite set (resp. any set), then, given a collection of commuting diagrams

$$\begin{array}{ccc} X_i & \longrightarrow & Z \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & \coprod_{i\in I} A_i \end{array},$$

one for each $i \in I$, the squares are all pullbacks if and only if the collection $\{X_i \longrightarrow Z\}_{i\in I}$ forms a coproduct diagram.

In such a category there is a strict initial object: given a map $A \longrightarrow 0$ with 0 initial, we have $A \simeq 0$.

A.2. Example. **Top** is infinitary extensive. $\mathbf{Ring}^{op}$, the category of affine schemes, is finitary extensive.

In **Top** we can take an open cover $\{U_i\}_I$ of a space X and replace it with the single map $\coprod_I U_i \longrightarrow X$, and work just as before using this new sort of cover, using the fact **Top** is extensive. The sort of sites that mimic this behaviour are called *superextensive*.

A.4. Definition. (Bartels-Shulman) A *superextensive site* is an extensive category S equipped with a pretopology J containing the families

$$(U_i \longrightarrow \coprod_I U_i)_{i\in I}$$

and such that all covering families are bounded; this means that for a finitely extensive site, the families are finite, and for an infinitary site, the families are small. The pretopology in this instance will also be called superextensive.

A.5. Example. Given an extensive category S, the *extensive pretopology* has as covering families the bounded collections $(U_i \longrightarrow \coprod_I U_i)_{i\in I}$. The pretopology on any superextensive site contains the extensive pretopology.

A.6. Example. The category **Top** with its usual pretopology of open covers is a superextensive site.

A.7. Example. An elementary topos with the coherent pretopology is finitary superextensive, and a Grothendieck topos with the canonical pretopology is infinitary superextensive.

Given a superextensive site (S, J), one can form the class $\amalg J$ of arrows of the form $\coprod_I U_i \longrightarrow A$ for covering families $\{U_i \longrightarrow A\}_{i\in I}$ in J (more precisely, all arrows isomorphic in S/A to such arrows).

A.8. Proposition. *The class* $\amalg J$ *is a singleton pretopology, and is subcanonical if and only if* J *is.*

Proof. Since isomorphisms are covers for J they are covers for $\amalg J$. The pullback of a $\amalg J$-cover $\coprod_I U_i \longrightarrow A$ along $B \longrightarrow A$ is a $\amalg J$-cover as coproducts and pullbacks commute by definition of an extensive category. Now for the third condition we use the fact that in an extensive category a map

$$f\colon B \longrightarrow \coprod_I A_i$$

implies that $B \simeq \coprod_I B_i$ and $f = \coprod_i f_i$. Given $\amalg J$-covers $\coprod_I U_i \longrightarrow A$ and $\coprod_J V_j \longrightarrow (\coprod_I U_i)$, we see that $\coprod_J V_j \simeq \coprod_I W_i$ for some objects W_i. By the previous point, the pullback

$$\coprod_I U_k \times_{\coprod_I U_{i'}} W_i$$

is a $\amalg J$-cover of U_i, and hence $(U_k \times_{\coprod_I U_{i'}} W_i \longrightarrow U_k)_{i\in I}$ is a J-covering family for each $k \in I$. Thus

$$(U_k \times_{\coprod_I U_{i'}} W_i \longrightarrow A)_{i,k\in I}$$

is a J-covering family, and so

$$\coprod_J V_j \simeq \coprod_{k\in I} \left(\coprod_{i\in I} U_k \times_{\coprod_I U_{i'}} W_i \right) \longrightarrow A$$

is a $\amalg J$-cover.

The map $\coprod_I U_i \longrightarrow A$ is the coequaliser of $\coprod_{I\times I} U_i \times_A U_j \rightrightarrows \coprod_I U_i$ if and only if A is the colimit of the diagram in definition 3.3. Hence $(\coprod_I U_i \longrightarrow A)$ is effective if and only if $(U_i \longrightarrow A)_{i\in I}$ is effective ∎

Notice that the original superextensive pretopology J is generated by the union of $\amalg J$ and the extensive pretopology.

One reason we are interested in superextensive sites is the following.

A.9. Lemma. *In a superextensive site (S, J), we have $J_{un} = (\amalg J)_{un}$.*

This means we can replace the singleton pretopology J_{un} (e.g. local-section-admitting maps of topological spaces) with the singleton pretopology $\amalg J$ (e.g. disjoint unions of open covers) when defining anafunctors. This makes for much smaller pretopologies in practice.

One class of extensive categories which are of particular interest is those that also have finite/small limits. These are called *lextensive*. For example, **Top** is infinitary lextensive, as is a Grothendieck topos. In contrast, an elementary topos is in general only finitary lextensive. We end with a lemma about WISC.

A.10. Lemma. *If (S, J) is a superextensive site, (S, J) satisfies WISC if and only if $(S, \amalg J)$ does.*

One reason for why superextensive sites are so useful is the following result from [Schäppi 2012].

A.11. Proposition. [Schäppi 2012] *Let (S, J) be a superextensive site, and F a stack for the extensive topology on S. Then the associated stack $\widetilde{F}$ on the site $(S, \amalg J)$ is also the associated stack for the site (S, J).*

As a corollary, since every weak 2-functor $F\colon S \longrightarrow \mathbf{Gpd}$ for extensive S represented by an internal groupoid is automatically a stack for the extensive topology, we see that we only need to stackify F with respect to a singleton pretopology on S. This will be applied in [Roberts A].

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