Theory and Applications of Categories, Vol. 27, No. 10, 2012, pp. 210–221.

DESCENT IN MONOIDAL CATEGORIES

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ABSTRACT. We consider a symmetric monoidal closed category $\mathscr{V} = (\mathscr{V}, \otimes, I, [-,-])$ together with a regular injective object Q such that the functor $[-,Q]\colon \mathscr{V} \to \mathscr{V}^{op}$ is comonadic and prove that in such a category, as in the monoidal category of abelian groups, a morphism of commutative monoids is an effective descent morphism for modules if and only if it is a pure monomorphism. Examples of this kind of monoidal categories are elementary toposes considered as cartesian closed monoidal categories, the module categories over a commutative ring object in a Grothendieck topos and Barr's star-autonomous categories.

1. Introduction

Grothendieck's descent theory for modules in a symmetric monoidal category $\mathscr{V} = (\mathscr{V}, \otimes, I)$ is the study of which morphisms $\iota : \mathbf{A} \to \mathbf{B}$ of commutative $\mathscr{V}$-monoids are effective descent morphisms in the sense that the corresponding extension-of-scalars functor $B \otimes_A - : {}_{\mathbf{A}}\mathscr{V} \to {}_{\mathbf{B}}\mathscr{V}$ from the category of (left) **A**-modules to the category of (left) **B**-modules is comonadic. In [10], [11] and [12], we looked at the case where $\mathscr{V}$ is the monoidal category of abelian groups, or a star-autonomous category in the sense of Barr [1] and proved that a morphism $\iota : \mathbf{A} \to \mathbf{B}$ of commutative $\mathscr{V}$-monoids is an effective descent morphism for modules if and only if it is a pure morphism in ${}_{\mathbf{A}}\mathscr{V}$ (that is, for any **A**-module V, the morphism

$$\iota \otimes_A V \colon V = A \otimes_A V \to B \otimes_A V$$

is a regular monomorphism). The aim of this paper is to provide a unifying categorical approach to these results. Explicitly the setting in which we work is a symmetric monoidal closed category $\mathscr{V} = (\mathscr{V}, \otimes, I, [-,-])$ together with a regular injective object Q such that the functor $[-,Q]\colon \mathscr{V} \to \mathscr{V}^{op}$ is comonadic. Our approach is based on the observation that the proof given in [10] makes heavy use of the description of purity by means of the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : \mathbf{Ab} \to \mathbf{Ab}^{\mathrm{op}}$ which is conservative and preserves all coequalizers, and thus is, in particular, comonadic. In the case of Barr's star-autonomous categories, the corresponding functor is an equivalence of categories.

The work was partially supported by Volkswagen Foundation (Ref.: I/85989) and Shota Rustaveli National Science Foundation Grant DI/12/ 5-103/11.

Received by the editors 2012-03-26 and, in revised form, 2012-10-03.

Published on 2012-10-09 in the volume of articles from CT2011.

2010 Mathematics Subject Classification: 18A20, 18D10, 18D35.

Key words and phrases: symmetric monoidal categories, effective descent morphisms, pure morphisms.

In the first section, we recall some elementary facts about modules in a symmetric monoidal closed category. In section 2, we prove our main result, and in Section 3, we apply it to elementary toposes, the module categories over a commutative ring object in a Grothendieck topos and Barr's star-autonomous categories.

As background to the subject, we refer to S. Mac Lane [7] for generalities on category theory and to G. Janelidze and W. Tholen [3], [4] and [5] for descent theory.

2. Preliminaries

We begin by recalling from [7] and [13] some elementary facts about modules in a symmetric monoidal closed category.

Suppose that $\mathscr{V}$ is a fixed symmetric monoidal closed category with tensor product $\otimes$, unit object I, and internal-hom $[-,-]$; recall that $\mathscr{V}$ is closed means that each functor $V \otimes - : \mathscr{V} \to \mathscr{V}$ has a right adjoint $[V,-] : \mathscr{V} \to \mathscr{V}$. Recall further that the adjunction $V \otimes - \dashv [V,-]$ is internal, in the sense that one has natural isomorphisms

$$[V \otimes W, Y] \simeq [W, [V, Y]] \tag{1}$$

For simplicity of exposition we treat $\otimes$ as strictly associative and I as a strict unit, which is justified by Mac Lane's coherence theorem [7] asserting that every monoidal category is equivalent to a strict one.

A *monoid* in $\mathscr{V}$ (or $\mathscr{V}$-monoid) consists of an object A of $\mathscr{V}$ endowed with a multiplication $m_A : A \otimes A \to A$ and unit morphism $e_A : I \to A$ such that the usual identity and associative conditions are satisfied. A monoid is called *commutative* if the multiplication map is unchanged when composed with the symmetry. We write $\mathbf{Mon}(\mathscr{V})$ for the category of $\mathscr{V}$-monoids.

Recall further that, for any $\mathscr{V}$-monoid $\mathbf{A} = (A, e_A, m_A)$, a *left* $\mathbf{A}$*-module* is a pair (V, ρ_V), where V is an object of $\mathscr{V}$ and $\rho_V : A \otimes V \to V$ is a morphism in $\mathscr{V}$, called the *action* (or the $\mathbf{A}$-*action*) on V, such that $\rho_V(m_A \otimes V) = \rho_V(A \otimes \rho_V)$ and $\rho_V(e_A \otimes V) = 1$.

For a given $\mathscr{V}$-monoid $\mathbf{A}$, the left $\mathbf{A}$-modules are the objects of a category ${}_{\mathbf{A}}\mathscr{V}$. A morphism $f : (V, \rho_V) \to (W, \rho_W)$ is a morphism $f : V \to W$ in $\mathscr{V}$ such that $\rho_W(A \otimes f) = f\rho_V$. Analogously, one has the category $\mathscr{V}_{\mathbf{A}}$ of right $\mathbf{A}$-modules.

The forgetful functor ${}_{\mathbf{A}}U : {}_{\mathbf{A}}\mathscr{V} \to \mathscr{V}$ that takes a left $\mathbf{A}$-module (V, ρ_V) to the object V has a left adjoint ${}_{\mathbf{A}}F : \mathscr{V} \to {}_{\mathbf{A}}\mathscr{V}$ sending an object $V \in \mathscr{V}$ to the "free" $\mathbf{A}$-module $(A \otimes V, A \otimes \rho_V)$.

There is another way of representing the category of left $\mathbf{A}$-modules that involves algebras over the monad associated to the $\mathscr{V}$-monoid $\mathbf{A}$.

Every $\mathscr{V}$-monoid $\mathbf{A} = (A, e_A, m_A)$ defines a monad $\mathcal{L}(\mathbf{A}) = (T, \eta, \mu)$ on $\mathscr{V}$ by

- $T(V) = A \otimes V$,
- $\eta_V = e_A \otimes V : V \to A \otimes V$,
- $\mu_V = m_A \otimes V : A \otimes A \otimes V \to A \otimes V$.

It is well known that the corresponding Eilenberg-Moore category $\mathscr{V}^{\mathcal{L}(\mathbf{A})}$ of $\mathcal{L}(\mathbf{A})$-algebras is exactly the category ${}_{\mathbf{A}}\mathscr{V}$ of left $\mathbf{A}$-modules, and that ${}_{\mathbf{A}}U \dashv {}_{\mathbf{A}}F$ is the familiar forgetful-free adjunction between $\mathscr{V}^{\mathcal{L}(\mathbf{A})}$ and $\mathscr{V}$. This gives in particular that the forgetful functor ${}_{\mathbf{A}}U \colon {}_{\mathbf{A}}\mathscr{V} \to \mathscr{V}$ is monadic. Hence the functor ${}_{\mathbf{A}}U$ creates those limits that exist in $\mathscr{V}$. Moreover, since the functor $A \otimes - : \mathscr{V} \to \mathscr{V}$ admits as a right adjoint the functor $[A, -]- :$ $\mathscr{V} \to \mathscr{V}$, the forgetful functor ${}_{\mathbf{A}}U$ has a right adjoint sending an object $V \in \mathscr{V}$ to the object $[A, V]$, where $[A, V]$ is an object of ${}_{\mathbf{A}}\mathscr{V}$ via the transpose $A \otimes [A, V] \to [A, V]$ of the composite $A \otimes A \otimes [A, V] \xrightarrow{m_A \otimes [A,V]} A \otimes [A, V] \xrightarrow{\mathrm{ev}_V} V$, where $\mathrm{ev}_V : A \otimes [A, V] \to V$ is the V-component of the counit of the adjunction $A \otimes - \dashv [A, -]$. In particular, ${}_{\mathbf{A}}U$ creates those colimits that exist in $\mathscr{V}$.

If $\mathscr{V}$ admits coequalizers, $\mathbf{A}$ is a $\mathscr{V}$-monoid, $(V, \varrho_V) \in \mathscr{V}_{\mathbf{A}}$ a right $\mathbf{A}$-module, and $(W, \rho_W) \in {}_{\mathbf{A}}\mathscr{V}$ a left $\mathbf{A}$-module, then their *tensor product* (*over* $\mathbf{A}$) is the object part of the following coequalizer

$$V \otimes A \otimes W \underset{V \otimes \rho_W}{\overset{\varrho_V \otimes W}{\rightrightarrows}} V \otimes W \longrightarrow V \otimes_A W.$$

When $\mathbf{A}$ is commutative, then for any $(V, \rho_V) \in {}_{\mathbf{A}}\mathscr{V}$, the composite $\rho'_V = \rho_V \tau_{V,A} :$ $V \otimes A \to V$, where τ is the symmetry for $\mathscr{V}$, defines a right $\mathbf{A}$-action on V. Similarly, if $(W, \varrho_V) \in \mathscr{V}_{\mathbf{A}}$, then $\varrho'_W = \varrho_V \tau_{W,A} : W \otimes A \to W$ defines a left $\mathbf{A}$-action on W. These two constructions establish an equivalence between ${}_{\mathbf{A}}\mathscr{V}$ and $\mathscr{V}_{\mathbf{A}}$, and thus we do not have to distinguish between left and right $\mathbf{A}$-modules. In this case, the tensor product of two $\mathbf{A}$-modules is another $\mathbf{A}$-module, and tensoring over $\mathbf{A}$ makes ${}_{\mathbf{A}}\mathscr{V}$ (as well as $\mathscr{V}_{\mathbf{A}}$) into a symmetric monoidal category with unit A. If, in addition, $\mathscr{V}$ admits equalizers, then this monoidal structure on ${}_{\mathbf{A}}\mathscr{V}$ is closed: The internal Hom-object ${}_{\mathbf{A}}[V, W]$ of two $\mathbf{A}$-modules defined to be the equalizer in $\mathscr{V}$ of

$$[V, W] \rightrightarrows [A \otimes V, W],$$

where one of the morphisms is induced by the action of $\mathbf{A}$ on V, and the other is the composition of $A \otimes - : [V, W] \to [A \otimes V, A \otimes W]$ followed by the morphism induced by the action of $\mathbf{A}$ on W.

In what follows, $\mathscr{V}$ denotes a fixed symmetric monoidal closed category with equalizers and coequalizers.

3. Descent theory in monoidal categories

3.1. Let us recall that a morphism in a category $\mathcal{A}$ is a *regular monomorphism* if it is an equalizer of some pair of morphisms. Recall also that a *regular injective object* in $\mathcal{A}$ is an object $X \in \mathcal{A}$ which has the extension property with respect to regular monomorphisms; that is, if every extension problem

$$\begin{array}{ccc} A & \overset{m}{\hookrightarrow} & B \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \bar{f}} & \\ X & & \end{array}$$

with m a regular monomorphism has a solution $\bar{f} : B \to X$ extending f along m, i.e., satisfying $\bar{f}m = f$.

3.2. A *pointed $\mathscr{V}$–endofunctor* on $\mathscr{V}$ is a pair (T, η), where $T : \mathscr{V} \to \mathscr{V}$ is a $\mathscr{V}$–endofunctor on $\mathscr{V}$ and $\eta : 1 \to T$ is a $\mathscr{V}$–natural transformation. Let (T, η) be a pointed $\mathscr{V}$– endofunctor on $\mathscr{V}$. For an object Q of $\mathscr{V}$, we get from T a functor

$$[T(-), Q] \colon \mathscr{V} \to \mathscr{V}^{\mathrm{op}},$$

and we can consider the natural transformation

$$[\eta_-, Q] \colon [T(-), Q] \to [-, Q].$$

3.3. Proposition [12] *The natural transformation $[\eta_-, Q]$ is a split epimorphism if and only if the morphism $\eta_Q \colon Q \to T(Q)$ is a split monomorphism. In particular, if Q is a regular injective object in $\mathscr{V}$, then the natural transformation $[\eta_-, Q]$ is a split epimorphism if and only if η_Q is a regular monomorphism in $\mathscr{V}$.*

Recall [8] that a monad $\mathbf{T}$ on a category $\mathcal{A}$ is called of *descent type* if the free $\mathbf{T}$-algebra functor $F^{\mathbf{T}} : \mathcal{A} \to \mathcal{A}^{\mathbf{T}}$ is precomonadic, and $\mathbf{T}$ is called of *effective descent type* if $F^{\mathbf{T}}$ is comonadic.

3.4. Theorem *Let $\mathscr{V}$ have a regular injective object Q such that the functor*

$$[-, Q] \colon \mathscr{V} \to \mathscr{V}^{op}$$

is comonadic. For any commutative monoid $\mathbf{A} = (A, e_A, m_A)$ in $\mathscr{V}$, the following are equivalent:

(i) the morphism $e_A \colon I \to A$ is pure; that is, for any object $V \in \mathscr{V}$, the morphism

$$e_A \otimes V \colon V \to A \otimes V$$

is a regular monomorphism;

(ii) the morphism $e_A \otimes Q : Q \to A \otimes Q$ is a regular monomorphism;

(iii) the natural transformation $[e_A \otimes -, Q]$ is a split epimorphism;

(iv) the morphism $[e_A, Q] : [A, Q] \to [I, Q]$ is a split epimorphism;

(v) the monad $\mathcal{L}(\mathbf{A})$ is of descent type;

(vi) the monad $\mathcal{L}(\mathbf{A})$ is of effective descent type.

Proof. Clearly, (i) always implies (ii), while (ii) and (iii) are equivalent by Proposition 3.3, since the pair $(A \otimes -, e_A \otimes -)$ is a pointed $\mathscr{V}$–endofunctor on $\mathscr{V}$.

Since the I-component of the natural transformation $[e_A \otimes -, Q]$ is just the morphism $[e_A, Q] : [A, Q] \to [I, Q]$, (iii) implies (iv).

To see that (iv) implies (vi), note first that to say that the natural transformation $[e_A \otimes -, Q]$ is a split epimorphism is to say that the monad $\mathcal{L}(\mathbf{A})$ is $[-, Q]$–separable [8]. Next, observe that the diagram

$$\begin{array}{ccc} \mathscr{V} & \xrightarrow{A\otimes -} & \mathscr{V} \\ {\scriptstyle [-,Q]}\big\downarrow & & \big\downarrow{\scriptstyle [-,Q]} \\ \mathscr{V}^{op} & \xrightarrow[{[A,-]^{op}}]{} & \mathscr{V}^{op} \end{array}$$

commutes up to natural isomorphism by (1). Now, since $\mathscr{V}^{op}$ admits equalizers (and hence is Cauchy complete) and since the functor $\mathscr{V}(-, Q) \colon \mathscr{V} \to \mathscr{V}^{op}$ is comonadic, one can apply [8, Theorem 3.22] to the diagram to conclude that the monad $\mathcal{L}(\mathbf{A})$ is of effective descent type.

(vi) trivially implies (v), while the implication (v) $\Rightarrow$ (i) follows from [8, Theorem 2.3 (i)].

■

3.5. Lemma *Let $\mathscr{V}$ have an object Q such that the functor*

$$\mathscr{V}(-, Q) \colon \mathscr{V} \to \mathscr{V}^{op}$$

is comonadic, and let $\mathbf{A} = (A, e_A, m_A)$ be a commutative monoid in $\mathscr{V}$. Write $Q_{\mathbf{A}}$ for the object $[A, Q]$ of $\mathscr{V}$. Then $Q_{\mathbf{A}} \in {}_{\mathbf{A}}\mathscr{V}$ and the functor

$${}_{\mathbf{A}}[-, Q_{\mathbf{A}}] \colon {}_{\mathbf{A}}\mathscr{V} \to ({}_{\mathbf{A}}\mathscr{V})^{op}$$

is comonadic. Moreover, if Q is regular injective in $\mathscr{V}$, then $Q_{\mathbf{A}}$ is regular injective in ${}_{\mathbf{A}}\mathscr{V}$.

Proof. We have already seen that the functor ${}_{\mathbf{A}}U \colon {}_{\mathbf{A}}\mathscr{V} \to \mathscr{V}$ is left adjoint, with right adjoint $[A, -] \colon \mathscr{V} \to {}_{\mathbf{A}}\mathscr{V}$. Hence, every $[A, V]$ (in particular, $[A, Q]$) is an object of the category ${}_{\mathbf{A}}\mathscr{V}$. Thus $Q_{\mathbf{A}} \in {}_{\mathbf{A}}\mathscr{V}$.

Since for any $V \in {}_{\mathbf{A}}\mathscr{V}$, ${}_{\mathbf{A}}[V, [A, Q]] \simeq [A \otimes_A V, Q] \simeq [V, Q]$ (see, for instance, [13]), one has commutativity (up to isomorphism) in

$$\begin{array}{ccc} {}_{\mathbf{A}}\mathscr{V} & \xrightarrow{{}_{\mathbf{A}}[-,Q_{\mathbf{A}}]} & ({}_{\mathbf{A}}\mathscr{V})^{op} \\ {\scriptstyle {}_{\mathbf{A}}U}\big\downarrow & & \big\downarrow{\scriptstyle ({}_{\mathbf{A}}U)^{op}} \\ \mathscr{V} & \xrightarrow{[-,Q]} & \mathscr{V}^{op} \end{array}$$

and since

- the functor ${}_{\mathbf{A}}[-, Q_{\mathbf{A}}] : {}_{\mathbf{A}}\mathscr{V} \to ({}_{\mathbf{A}}\mathscr{V})^{\mathrm{op}}$ admits as a right adjoint the functor ${}_{\mathbf{A}}[-, Q_{\mathbf{A}}] : ({}_{\mathbf{A}}\mathscr{V})^{\mathrm{op}} \to {}_{\mathbf{A}}\mathscr{V}$,
- the functor ${}_{\mathbf{A}}U \colon {}_{\mathbf{A}}\mathscr{V} \to \mathscr{V}$ preserves all small limits, and thus, in particular, equalizers of ${}_{\mathbf{A}}[-, Q_{\mathbf{A}}]$–split pairs;
- the functor $[-, Q] \colon \mathscr{V} \to \mathscr{V}^{op}$, being comonadic, preserves equalizers of $[-, Q]$–split pairs;
- the functor ${}_{\mathbf{A}}U$ (and hence also $({}_{\mathbf{A}}U)^{op}$) is conservative,

it follows from the dual of [9, Theorem 5.5] that the functor

$${}_{\mathbf{A}}[-, Q_{\mathbf{A}}] \colon {}_{\mathbf{A}}\mathscr{V} \to ({}_{\mathbf{A}}\mathscr{V})^{op}$$

is comonadic.

Now, using that

- the functor $[A, -] \colon \mathscr{V} \to {}_{\mathbf{A}}\mathscr{V}$ is right adjoint to the functor ${}_{\mathbf{A}}U \colon {}_{\mathbf{A}}\mathscr{V} \to \mathscr{V}$;
- ${}_{\mathbf{A}}U$ preserves regular monomorphisms;
- Q is regular injective in $\mathscr{V}$,

it is easy to show that the object $Q_{\mathbf{A}} = [A, Q]$ is regular injective in ${}_{\mathbf{A}}\mathscr{V}$. This completes the proof. ∎

3.6. For any symmetric monoidal closed category $\mathscr{V}$, we denote the category of commutative monoids in $\mathscr{V}$ by $\mathbf{CMon}(\mathscr{V})$. It is well-known that for any commutative $\mathscr{V}$-monoid $\mathbf{A}$, the co-slice category $\mathbf{A}/\mathbf{CMon}(\mathscr{V})$ is equivalent to the category $\mathbf{CMon}({}_{\mathbf{A}}\mathscr{V})$. In other words, to give a commutative monoid $\mathbf{B}$ in the symmetric monoidal closed category ${}_{\mathbf{A}}\mathscr{V}$ is to give a morphism $\mathbf{A} \to \mathbf{B}$ of commutative monoids in $\mathscr{V}$. The latter morphism serves as the unit morphism of the ${}_{\mathbf{A}}\mathscr{V}$-monoid $\mathbf{B}$. If $\iota \colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathbf{CMon}(\mathscr{V})$, then the corresponding commutative monoid in $\mathscr{V}_{\mathbf{A}}$ will be denoted by $\mathbf{B}_\iota$.

One says that a morphism $\iota : \mathbf{A} \to \mathbf{B}$ of commutative $\mathscr{V}$-monoids is an *(effective) descent morphism* if the functor $B \otimes_A - : {}_{\mathbf{A}}\mathscr{V} \to {}_{\mathbf{B}}\mathscr{V}$ is precomonadic (comonadic).

Identifying the morphism $\iota : \mathbf{A} \to \mathbf{B}$ with the monoid $\mathbf{B}_\iota$ in the monoidal category ${}_{\mathbf{A}}\mathscr{V}$ and considering the monad $\mathcal{L}(\mathbf{B}_\iota) = (T_\iota, \eta_\iota, \mu_\iota)$ on ${}_{\mathbf{A}}\mathscr{V}$ induced by $\mathbf{B}_\iota$ (thus, $T_\iota = B \otimes_A -$, $\eta_\iota = \iota \otimes_A -$ and $\mu_\iota = m'_B \otimes_A -$, where $m'_B : B \otimes_A B \to B$ is the unique morphism through which m_B factors), the category ${}_{\mathbf{B}}\mathscr{V}$ can be seen as the Eilenberg-Moore category of $\mathcal{L}(\mathbf{B}_\iota)$-algebras. Hence the category ${}_{\mathbf{B}_\iota}({}_{\mathbf{A}}\mathscr{V})$ can be identified with the category ${}_{\mathbf{B}}\mathscr{V}$. Modulo this identification, the functor $B \otimes_A - : {}_{\mathbf{A}}\mathscr{V} \to {}_{\mathbf{B}}\mathscr{V}$ corresponds to the functor $B_\iota \otimes_A - : {}_{\mathbf{A}}\mathscr{V} \to {}_{\mathbf{B}_\iota}({}_{\mathbf{A}}\mathscr{V})$. Thus the problem of effectiveness of ι is equivalent to the one of the monad $\mathcal{L}(\mathbf{B}_\iota)$. Using this, and the fact that there is a natural isomorphism ${}_{\mathbf{A}}[-, Q_{\mathbf{A}}] \simeq [-, Q]$, we get from Lemma 3.5 and Theorem 3.4:

3.7. Theorem *Let $\mathscr{V}$ have a regular injective object Q such that the functor*

$$[-, Q] \colon \mathscr{V} \to \mathscr{V}^{op}$$

is comonadic, and let $\iota \colon \mathbf{A} \to \mathbf{B}$ be a morphism of commutative monoids in $\mathscr{V}$. The following are equivalent:

(i) $\iota \colon \mathbf{A} \to \mathbf{B}$ is an effective descent morphism;

(ii) $\iota \colon \mathbf{A} \to \mathbf{B}$ is a pure morphism in ${}_{\mathbf{A}}\mathscr{V}$; that is, for any $\mathbf{A}$-module V, the morphism

$$\iota \otimes_A V \colon V = A \otimes_A V \to B \otimes_A V$$

is a regular monomorphism;

(iii) the morphism $[\iota, Q] : [B, Q] \to [A, Q]$ is a split epimorphism in ${}_{\mathbf{A}}\mathscr{V}$;

(iv) the monad $\mathcal{L}(\mathbf{B}_\iota)$ is of descent type;

(v) the monad $\mathcal{L}(\mathbf{B}_\iota)$ is of effective descent type.

4. Applications

4.1. Monoid modules in an elementary topos Let $\mathcal{E}$ be an elementary topos, considered as a cartesian monoidal category. It is well-known [6] that the functor

$$\Omega^{(-)} \colon \mathcal{E}^{op} \to \mathcal{E},$$

where Ω is the subobject classifier for $\mathcal{E}$, is monadic. Hence $\Omega^{(-)}$, seen as a functor $\mathcal{E} \to \mathcal{E}^{\mathrm{op}}$, is comonadic. Moreover, since Ω is an injective object in $\mathcal{E}$ (e.g., [6]) and since in $\mathcal{E}$ regular monomorphisms coincide with monomorphisms, Theorem 3.7 gives the following result:

4.2. Theorem *Let $\mathcal{E}$ be an elementary topos. A morphism $\iota \colon \mathbf{A} \to \mathbf{B}$ of commutative monoids in $\mathcal{E}$ is an effective descent morphism (or, equivalently, the functor $B \otimes_A - : {}_{\mathbf{A}}\mathcal{E} \to {}_{\mathbf{B}}\mathcal{E}$ is comonadic) if and only if $\iota \colon \mathbf{A} \to \mathbf{B}$ is a pure morphism in ${}_{\mathbf{A}}\mathcal{E}$.*

4.3. The case of the topos of sets Specialize now to the case where $\mathcal{E}$ is the topos of sets, $\mathfrak{Set}$, so that $\mathfrak{Set}$-monoids are ordinary monoids, and if $\mathbf{A}$ such a monoid, then (left and right) $\mathbf{A}$-modules are more commonly called $\mathbf{A}$-*actions*. Recall that for any left $\mathbf{A}$–action X, the set $\mathfrak{Set}(X, \mathbf{2})$, where $\mathbf{2} = \{0, 1\}$, is a right $\mathbf{A}$–action under the definition $(f \cdot a)(x) = f(a \cdot x)$ for all $a \in A$, $f \in \mathfrak{Set}(X, \mathbf{2})$ and $x \in X$ (see Section 2). Moreover, for any morphism $f : X \to Y$ of left $\mathbf{A}$–actions, the function

$$\mathfrak{Set}(f, \mathbf{2}) : \mathfrak{Set}(Y, \mathbf{2}) \to \mathfrak{Set}(X, \mathbf{2})$$

is a morphism of right **A**–actions. It is well-known (e.g., [7]) that, when f is injective, then there exists a map

$$\exists_f : \mathfrak{Set}(X, \mathbf{2}) \to \mathfrak{Set}(Y, \mathbf{2})$$

of sets such that $\mathfrak{Set}(f, \mathbf{2}) \cdot \exists_f = 1$. Recall that, for any map $\chi : X \to \mathbf{2}$, the map $\exists_f(\chi) : Y \to \mathbf{2}$ is defined as follows:

$$(\exists_f(\chi))(y) = \begin{cases} 1, & \text{if there is } x \in X \text{ such that } \chi(x) = 1 \text{ and } f(x) = y, \\ 0, & \text{otherwise.} \end{cases}$$

4.4. Proposition *Let* **A** *be a group. Then, for any injective morphism* $f : X \to Y$ *of left* **A**–*actions, the map* $\exists_f : \mathfrak{Set}(X, \mathit{2}) \to \mathfrak{Set}(Y, \mathit{2})$ *is a morphism of right* **A**–*actions.*

Proof. We have to show that $\exists_f(\chi) \cdot a = \exists_f(\chi \cdot a)$ for all $\chi \in \mathfrak{Set}(X, \mathbf{2})$ and all $a \in A$. If $y \in Y$ is an arbitrary element, then $(\exists_f(\chi) \cdot a)(y) = (\exists_f(\chi))(a \cdot y)$, and we have

$$(\exists_f(\chi) \cdot a)(y) = \begin{cases} 1, & \text{if there is } x \in X \text{ such that } \chi(x) = 1 \\ & \text{and } f(x) = a \cdot y, \\ 0, & \text{otherwise.} \end{cases}$$

which, since **A** is a group and since f is a morphism of left **A**–actions, may be written as

$$(\exists_f(\chi) \cdot a)(y) = \begin{cases} 1, & \text{if there is } x \in X \text{ such that } \chi(x) = 1 \\ & \text{and } f(a^{-1} \cdot x) = y, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, we have

$$(\exists_f(\chi \cdot a))(y) = \begin{cases} 1, & \text{if there is } x \in X \text{ such that } (\chi \cdot a)(x) = 1 \\ & \text{and } f(x) = y, \\ 0, & \text{otherwise.} \end{cases}$$

and hence

$$(\exists_f(\chi \cdot a))(y) = \begin{cases} 1, & \text{if there is } x \in X \text{ such that } \chi(a \cdot x) = 1 \\ & \text{and } f(x) = y, \\ 0, & \text{otherwise.} \end{cases}$$

Comparing $(\exists_f(\chi) \cdot a)(y)$ with $(\exists_f(\chi \cdot a))(y)$, we find that they are equal. So $\exists_f(\chi) \cdot a = \exists_f(\chi \cdot a)$, and hence f is a morphism of right **A**–actions. ∎

Since $\mathfrak{Set}(f, \mathbf{2}) \cdot \exists_f = 1$, a corollary follows immediately:

4.5. Corollary *Let* $\mathbf{A}$ *be a group and* $f : X \to Y$ *an injective morphism of left* $\mathbf{A}$*–actions. Then the map*

$$\mathfrak{Set}(f, \mathit{2}) : \mathfrak{Set}(X, \mathit{2}) \to \mathfrak{Set}(Y, \mathit{2})$$

is a split epimorphism of right $\mathbf{A}$*–actions.*

We are now ready to state and prove the following

4.6. Theorem *Let* $\iota : \mathbf{A} \to \mathbf{B}$ *be a morphism of ordinary commutative monoids. If* $\mathbf{A}$ *is an (abelian) group, then* ι *is an effective descent morphism if and only if it is an injective map.*

Proof. One direction is immediate from Theorem 3.7. Conversely, if ι is injective, then the map

$$\mathfrak{Set}(\iota, \mathbf{2}) : \mathfrak{Set}(B, \mathbf{2}) \to \mathfrak{Set}(A, \mathbf{2})$$

is a split epimorphism of right $\mathbf{A}$–actions by Corollary 4.5. But according to Theorem 3.7, ι is an effective descent morphism if and only if the map $\mathfrak{Set}(\iota, \mathbf{2}) : \mathfrak{Set}(B, \mathbf{2}) \to \mathfrak{Set}(A, \mathbf{2})$ is a split epimorphism in ${}_{\mathbf{A}}\mathfrak{Set}$, or equivalently (by the commutativity of $\mathbf{A}$), in $\mathfrak{Set}_{\mathbf{A}}$. Thus, ι is an effective descent morphism. ∎

4.7. The category of internal modules over a Grothendieck topos In order to proceed we need the following easy consequence of a variation of Duskin's theorem (see [2, Theorem 1.3 of Section 9.1]).

4.8. Theorem *A left adjoint additive functor* $F : \mathscr{A} \to \mathscr{B}$ *between abelian categories is comonadic if and only if* F *is conservative and* F *preserves those monomorphisms whose cokernel-pairs are* F*-split.*

A monoidal category $(\mathscr{V}, \otimes, I)$ is called *abelian monoidal* if $\mathscr{V}$ is abelian and the tensor product is an additive bifunctor. An object V of such a category $\mathscr{V}$ is said to be *flat* if the functor $V \otimes - : \mathscr{V} \to \mathscr{V}$ preserves monomorphisms.

4.9. Proposition *Let* $\mathscr{V} = (\mathscr{V}, \otimes, I, \tau, [-,-])$ *be a symmetric monoidal closed abelian category. Suppose* $\mathscr{V}$ *has a generating family formed by flat objects. If* $Q \in \mathscr{V}$ *is an injective cogenerator, then the functor*

$$[-, Q] : \mathscr{V} \to \mathscr{V}^{op}$$

is comonadic.

Proof. We first observe that the functor $[-, Q] : \mathscr{V} \to \mathscr{V}^{\mathrm{op}}$ admits as a right adjoint the functor $[-, Q] : \mathscr{V}^{\mathrm{op}} \to \mathscr{V}$.

Next, if f is a morphism in $\mathscr{V}$ such that the morphism $[f, Q]$ is an isomorphism, then the map $\mathscr{V}(I, [f, Q])$ is bijective. Because of the following chain of bijections $\mathscr{V}(I, [f, Q]) \simeq \mathscr{V}(I \otimes f, Q) \simeq \mathscr{V}(f, Q)$, it follows that the map $\mathscr{V}(f, Q)$ is also bijective. But since Q is an injective cogenerator, the functor $\mathscr{V}(-, Q) : \mathscr{V}^{\mathrm{op}} \to \mathbf{Set}$ is faithful; thus, it reflects epimorphisms and monomorphisms, and hence isomorphisms. Therefore, f is also an isomorphism. Consequently, the functor $[-, Q]$ is conservative.

We next show that functor $[-, Q]$ preserves monomorphisms. Indeed, if $\mathscr{V}$ has a generating family $\{G_\alpha\}$ such that each G_α is flat and if $f : V \to V'$ is a monomorphism in $\mathscr{V}$, then so also is each $G_\alpha \otimes f$. Then, since Q is injective in $\mathscr{V}$, the map

$$\mathscr{V}(G_\alpha \otimes f, Q) : \mathscr{V}(G_\alpha \otimes V', Q) \to \mathscr{V}(G_\alpha \otimes V, Q),$$

and hence also the map

$$\mathscr{V}(G_\alpha, [f, Q]) : \mathscr{V}(G_\alpha, [V', Q]) \to \mathscr{V}(G_\alpha, [V, Q]),$$

is surjective for all α. But $\{G_\alpha\}$ is a generating family for $\mathscr{V}$, i.e. the family of functors $\{\mathscr{V}(G_\alpha, -) : \mathscr{V} \to \mathbf{Set}\}$ is collectively faithful; in particular, this family collectively reflects epimorphisms. Therefore, the morphisms $[f, Q]$ is an epimorphism in $\mathscr{V}$. Applying now the dual of Theorem 4.8 gives that the functor $[-, Q]$ is comonadic. ∎

We now consider the symmetric monoidal closed category $\mathbf{Ab}(\mathcal{E})$ of internal abelian groups in a Grothendieck topos $\mathcal{E}$. It is well-known that (commutative) monoids in $\mathbf{Ab}(\mathcal{E})$ are internal (commutative) rings in $\mathcal{E}$, and that ${}_{\mathbf{A}}\mathbf{Ab}(\mathcal{E})$, $\mathbf{A} \in \mathbf{Mon}(\mathbf{Ab}(\mathcal{E}))$, is the category $\mathbf{Mod}_{\mathbf{A}}(\mathcal{E})$ of internal left $\mathbf{A}$-modules in $\mathcal{E}$. Since $\mathbf{Ab}(\mathcal{E})$ is an Ab5 category with generators and sufficiently many injective objects (e.g., [6]), it also has an injective cogenerator, say, Q (see, for example, [14, Lemma 7.12]). Now, since free abelian groups in $\mathcal{E}$ are flat in $\mathbf{Ab}(\mathcal{E})$ and since $\mathbf{Ab}(\mathcal{E})$ has a generator that is a free abelian group (see, [6]), one can combine Proposition 4.9 with Theorem 3.7 to conclude the following generalization of the main result of [10] (see also [11]):

4.10. Theorem *A morphism $\iota\colon \mathbf{A} \to \mathbf{B}$ of internal commutative rings in a Grothendieck topos $\mathcal{E}$ is an effective descent morphism (or, equivalently, the functor $B \otimes_A - : \mathbf{Mod}_{\mathbf{A}}(\mathcal{E}) \to \mathbf{Mod}_{\mathbf{A}}(\mathcal{E})$ is comonadic) if and only if $\iota\colon \mathbf{A} \to \mathbf{B}$ is a pure morphism of internal (left) $\mathbf{A}$-modules.*

4.11. $\star$-autonomous categories Let now $\mathscr{V}$ be a $*$-autonomous category in the sense of Barr [1]. Then $\mathscr{V}$ is a symmetric monoidal closed category together with a so-called dualizing object Q such that the adjunction

$$[-, Q] \dashv [-, Q]\colon \mathscr{V}^{\mathrm{op}} \to \mathscr{V}$$

is an adjoint equivalence. Quite obviously, the functor $[-, Q]\colon \mathscr{V} \to \mathscr{V}^{\mathrm{op}}$ is then comonadic. Moreover, it is proved in [12] that any dualizing object is regular injective in $\mathscr{V}$ if and only if the tensor unit I is regular projective in $\mathscr{V}$. Thus, when the tensor unit is regular projective in a $*$-autonomous category, Theorem 3.7 applies (see [12], for more on the descent morphisms in $*$-autonomous categories).

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