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THE URSINI COMMUTATOR AS NORMALIZED SMITH-PEDICCHIO COMMUTATOR

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ABSTRACT. We introduce an intrinsic description of the Ursini commutator [Urs81, GU84] in any ideal determined category and we compare it with the Higgins and Huq commutators. After describing also the Smith-Pedicchio commutator by means of canonical arrows from a coproduct, we compare the two notions, showing that in any exact Mal'tsev normal category the Ursini commutator $[H, K]_U$ of two subobjects H, K of A is the normalization of the Smith-Pedicchio commutator $[R_H, R_K]_{SP}$ of the equivalence relations generated by H and K, extending the result valid for ideal determined varieties given in [Urs81, GU84].

1. Introduction

The problem of knowing when congruences in a variety are determined by ideals was faced by A. Ursini in the context of BIT varieties of [Urs72]. The same varieties were called *ideal determined* in [GU84], where they are shown to give a good setting to describe commutators of congruences in terms of a commutator of ideals introduced by the authors (see also [Urs81]). In [JMTU10] it is shown that a categorical interpretation of BIT varieties can be obtained by removing the so-called Hofmann Axiom from the old-style definition of a semi-abelian category [JMT02]. They called these categories ideal determined.

In this paper, we start working in any ideal determined category, where the images of kernels along regular epimorphisms are again kernels. This fact allows us to formulate the categorical notion of the Ursini commutator of [Urs81, GU84], following the same "formal" method used in [MM10] to introduce the internal version of the Higgins commutator.

The idea is to give an internal interpretation of the *commutator words* of [GU84] by means of a normal subobject of a coproduct and then taking its regular image through the "realization" map. Also in this categorical context, following this approach, it is immediate to see that, given two subobjects H, K of A, this commutator $[H, K]_U$ is always normal in A. Moreover the comparison with the Higgins commutator $[H, K]_H$ and the Huq commutator $[H, K]_Q$ (see [Huq68, MM10]) shows that

$$[H, K]_H \leq [H, K]_Q \leq [H, K]_U$$

and all the relations may be strict, even in a semi-abelian context. Furthermore, in Proposition 4.5, we show a property of the Ursini commutator, namely the invariance

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with respect to normal closures, which is satisfied neither by the Higgins commutator nor by the Huq commutator.

There is another well-known categorical notion of commutator, introduced by M.C. Pedicchio in the fundamental paper [Ped95], where a fruitful categorical generalization of the Smith commutator for congruences ([Smi76]) is given. In [Bou00] a counterexample due to G. Janelidze is quoted which shows that Huq and Higgins commutators are in general not equivalent to the Smith-Pedicchio commutator.

In order to compare the Smith-Pedicchio commutator with the Ursini commutator, we first show that in the case of a Barr-exact Mal'tsev category it is possible to give a description of the Smith-Pedicchio commutator by means of a pushout of canonical arrows from a coproduct (Proposition 4.11).

Finally, we prove that, when the normal category $\mathcal{C}$ is also exact and Mal'tsev, the Ursini commutator $[H, K]_U$ of two subobjects H, K of A is the normalization of the Smith-Pedicchio commutator $[R_H, R_K]_{SP}$ of the equivalence relations generated by H and K, extending the result valid for ideal determined varieties given in [GU84].

2. Preliminaries

Let $\mathbb{C}$ be a finitely complete pointed category with coproducts.

For any object B in $\mathbb{C}$, one can define a functor "ker" from the category of split epimorphisms (points) over B into $\mathbb{C}$

$$\ker : Pt_B(\mathbb{C}) \to \mathbb{C}, \qquad \begin{array}{c} A \\ {\scriptstyle\beta}\uparrow\downarrow{\scriptstyle\alpha} \\ B \end{array} \mapsto \ker(\alpha).$$

This functor has a left adjoint:

$$B{+}(-) : \mathbb{C} \to Pt_B(\mathbb{C}), \qquad A \mapsto \begin{array}{c} B+A \\ {\scriptstyle i_B}\uparrow\downarrow{\scriptstyle [1,0]} \\ B \end{array}.$$

The monad corresponding to this adjunction is denoted by $B\flat(-)$ and $n_{B,A}$ will denote the kernel of $[1,0]$. The $B\flat(-)$-algebras are called internal B-actions in $\mathbb{C}$ (see [BJK05]). Let us observe that in the case of groups, the object $B\flat A$ is the group generated by the *formal conjugates* of elements of A by elements of B, i.e. by the triples of the kind (b, a, b^{-1}) with $b \in B$ and $a \in A$.

For any object A of $\mathbb{C}$, one can define a canonical *conjugation* action χ_A of A on A itself, given by the composition:

$$\chi_A : \ A\flat A \xrightarrow{n_{A,A}} A + A \xrightarrow{[1,1]} A\,.$$

In the category of groups, the morphism χ_A is the internal action associated to the usual conjugation in A: the realization morphism $[1, 1]$ of above makes the formal conjugates of $A\flat A$ computed effectively in A.

Clots A subobject $k : K \rightarrowtail A$ is a clot in A, when there exists a morphism $\chi_| : A\flat K \to K$ such that the diagram

$$\begin{array}{ccc} A\flat K & \xrightarrow{\chi_|} & K \\ {\scriptstyle 1\flat k}\downarrow & & \downarrow{\scriptstyle k} \\ A\flat A & \xrightarrow[\chi_A]{} & A \end{array}$$

commutes. As k is a mono, the morphism $\chi_|$ defined above is unique: namely it is the internal action obtained by restriction of the conjugation in A.

Notice that every normal subobject, i.e. the domain of a kernel, is closed under conjugation, so it is a clot.

Ideals The categorical notion of ideal has been introduced in [JMU09], while the corresponding varietal concept can be found in [Mag67, Urs72, Hig56]. A subobject $k : K \rightarrowtail A$ is an ideal in A when it is the regular image of a clot along a regular epimorphism, i.e. if there exists a commutative diagram

$$\begin{array}{ccc} C & \overset{f'}{\twoheadrightarrow} & K \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle k} \\ B & \underset{f}{\twoheadrightarrow} & A \end{array}$$

with c clot and f, f' regular epimorphisms. It is immediate to observe that, according to this definition, every clot is an ideal subobject.

In a pointed regular category with binary coproducts, since every clot is the regular image of a kernel, ideals are regular images of kernels.

3. Ideal determined categories.

In [JMT02] the authors introduced the definition of a semi-abelian category as a Barr-exact, Bourn-protomodular category with zero object and finite coproducts. In the same paper they state equivalent versions of semi-abelianness, given in terms of some so–called "old" axioms, more commonly used in universal algebra. From there we borrow the following characterization: a pointed category $\mathbb{C}$ with finite limits and colimits is semi-abelian if, and only if, it satisfies A1, A2 and A3 below:

- A1 $\mathbb{C}$ is regular and the classes of regular epimorphisms and normal epimorphisms coincide;
- A2 regular images of kernels are kernels, i.e. if l and l' are regular epimorphisms and x

is a kernel in the diagram below

$$\begin{array}{ccc} X & \xrightarrow{l'} & X' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x'} \\ Y & \xrightarrow[l]{} & Y' \end{array}$$

then x' is also a kernel;

A3 (Hoffmann's axiom) in the diagram below, where l and l' are regular epimorphisms and x' is a kernel, if $\mathrm{Ker}(l) \leq X$ then x is also a kernel.

$$\begin{array}{ccc} X & \xrightarrow{l'} & X' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x'} \\ Y & \xrightarrow[l]{} & Y' \end{array}$$

In this paper we are interested in those categories satisfying only the first two axioms above (A3 is strictly related to protomodularity, as shown in Proposition 3.3. of [JMT02], which gives a characterization in terms of axioms A1 and A3 of homological categories among those with finite limits and zero object, according to the definition due to Borceux and Bourn [BB04]).

Normal categories A pointed category $\mathbb{C}$ with finite limits and colimits where axiom A1 holds is said to be *normal* (see [Jan10]).

Ideal determined categories A normal category where also A2 is valid is called *ideal determined* as in [JMU09], where the authors extend the notion of ideal determined variety introduced by Gumm and Ursini in [GU84] to a categorical context. In this case, all the different notions of kernels, clots and ideals collapse. In the sequel we will often use the following characterization of ideal determined categories (see [MM10, Eve12, GJZU12]):

3.1. Proposition. *For a normal category $\mathbb{C}$, the following conditions are equivalent:*

(i) *$\mathbb{C}$ is ideal determined.*

(ii) *In any pushout of regular epimorphisms*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Z & \xrightarrow[f']{} & W \end{array}$$

the restriction $f_| : K(g) \to K(g')$ of f to kernels of g and g' is a regular epimorphism.

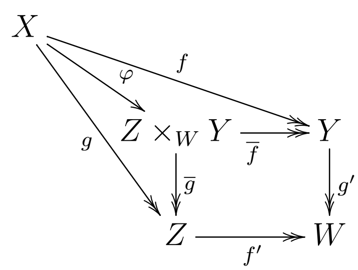
Ideal determined categories are not requested to be Barr-exact, i.e. regular categories where every equivalence relation is effective. In the pointed varietal case, where exactness is given for free, the Mal'tsev condition that every reflexive relation is an equivalence relation does not imply that equivalence relations are determined by ideals (e.g. the Mal'tsev variety of pointed Mal'tsev algebras is not ideal determined) and viceversa the variety of implication algebras is ideal determined, but the Mal'tsev condition fails (see Example 1.11 in [GU84]). But if we add the condition of normality (which is equivalent to 0-regularity for varieties) then any Mal'tsev 0-regular variety is ideal determined. In fact this is true in any normal exact Mal'tsev category and we report here an easy proof (see also Proposition 3.3 in [EGVdL08]):

3.2. Lemma. *Let $\mathbb{C}$ be a normal exact Mal'tsev category with binary coproducts. Then $\mathbb{C}$ is ideal determined.*

Proof. We use the characterization given by Proposition 3.1. We start then with a pushout of regular epimorphisms:

$$\begin{array}{ccc} X & \overset{f}{\twoheadrightarrow} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Z & \underset{f'}{\twoheadrightarrow} & W. \end{array}$$

The comparison arrow $\varphi : X \to Z \times_W Y$ to the pullback of f' along g'



is a regular epimorphism, thanks to the characterization of exact Mal'tsev categories given in [CKP93]. It is easy to see that the induced arrow $\varphi_| : K(g) \to K(\overline{g})$ makes the square

$$\begin{array}{ccc} K(g) & \overset{\varphi_|}{\longrightarrow} & K(\overline{g}) \\ \downarrow & & \downarrow \\ X & \underset{\varphi}{\twoheadrightarrow} & Z \times_W Y \end{array}$$

a pullback square. Since $\mathbb{C}$ is regular, $\varphi_|$ is a regular epimorphism, and so is the restriction $f_| : K(g) \to K(g')$, obtained by composing $\varphi_|$ with the isomorphism $\overline{f}_| : K(\overline{g}) \to K(g')$. ∎

4. The Ursini commutator

Several notions of commutators of two subobjects have been proposed and studied in different algebraic contexts. In this section, we will analyze the definition given by Ursini in the context of ideal determined varieties in [Urs81, GU84], and we will give a categorical definition in the case of an ideal determined category.

First we need to recall the original definitions.

Let $\mathbb{C}$ be an ideal determined variety. A term $m(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ is a *commutator term* in $\overrightarrow{x}$ and in $\overrightarrow{z}$ if it is an ideal term both in $\overrightarrow{x}$ and $\overrightarrow{z}$, that is

$$m(\overrightarrow{0}, \overrightarrow{y}, \overrightarrow{z}) = 0 \qquad \text{and} \qquad m(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{0}) = 0$$

for $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}$ disjoint finite tuples of variables.

Given A in $\mathbb{C}$ and two (normal) subalgebras $h\colon H \rightarrowtail A$ and $k\colon K \rightarrowtail A$, the Ursini commutator $[H, K]_U$ is the set of all $m(\overrightarrow{h}, \overrightarrow{a}, \overrightarrow{k})$ with $\overrightarrow{h} \in H^l$, and $\overrightarrow{a} \in A^n$, $\overrightarrow{k} \in K^p$, m being any commutator word in $\overrightarrow{x}$ and in $\overrightarrow{z}$.

Actually it is possible to describe the Ursini commutator as the image through the canonical map $[h, 1, k] : H+A+K \to A$ of a specific subobject of the coproduct $H+A+K$.

Let us fix some notation. Let $\mathbb{C}$ be a normal category and let $h\colon H \rightarrowtail A$, $k\colon K \rightarrowtail A$ be two given subobjects of A. We denote by $\Omega_{H,K}$ (or simply Ω) the canonical arrow

$$\Omega_{H,K} = \langle [h, 1, 0], [0, 1, 0], [0, 1, k] \rangle = [\langle h, 0, 0\rangle, \langle 1, 1, 1\rangle, \langle 0, 0, k\rangle] : H + A + K \to A \times A \times A.$$

In the category of groups the elements of $H + A + K$ can be represented as reduced formal juxtapositions of elements of H, A and K, say sequences of the kind $(h_1, a_1, k_1, \cdots, h_n, a_n, k_n)$, and Ω acts on sequences of this kind by means of the group operation of A, thus giving the element $(h_1a_1 \cdots h_na_n, a_1 \cdots a_n, a_1k_1 \cdots a_nk_n)$ *computed* in $A \times A \times A$.

Let us observe that in general $\Omega_{H,K}$ is not a regular epimorphism, even in the category of groups, so we can consider the (regular epi, mono) factorization of Ω, given by $H + A + K \xrightarrow{e} R \xrightarrow{m} A \times A \times A$. Of course, the kernels of Ω and e are isomorphic.

4.1. The Ursini commutator in ideal determined categories Let us consider an ideal determined category $\mathbb{C}$ and $h\colon H \rightarrowtail A$, $k\colon K \rightarrowtail A$ subobjects of A.

4.2. Definition. *The* Ursini commutator $[H, K]_U$ *of* H *and* K *in* A *is defined as the regular image of the kernel* $K(\Omega)$ *of* Ω *under the regular epimorphism* $[h, 1, k]$

$$\begin{array}{ccc} K(\Omega) & \twoheadrightarrow & [H,K]_U \\ \downarrow & & \downarrow \\ H + A + K & \xrightarrow{[h,1,k]} & A. \end{array}$$

Since we are in an ideal determined category, we immediately have

4.3. Remark.

(i) $[H,K]_U$ is obtained as a kernel of the pushout of $e: H+A+K \to R$ along $[h,1,k]$:

$$\begin{array}{ccc}
K(\Omega) & \twoheadrightarrow & [H,K]_U = K(q) \\
\downarrow & & \downarrow \\
H+A+K & \xrightarrow[{[h,1,k]}]{} & A \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle q} \\
R & \xrightarrow[r]{} & Q
\end{array}$$

(ii) $[H,K]_U$ is a normal subobject of A, for any pairs H,K of subobjects of A.

This is the first evident difference between the Ursini commutator $[H,K]_U$ and the Higgins commutator $[H,K]_H$, defined in [MM10] as the regular image of $K(\Sigma)$ under the morphism $[h,k]: H+K \to A$:

$$\begin{array}{ccc}
K(\Sigma) & \twoheadrightarrow & [H,K]_H \\
\downarrow & & \downarrow \\
H+K & \xrightarrow[{[h,k]}]{} & A.
\end{array}$$

where

$$\Sigma = \langle [1,0],[0,1] \rangle = [\langle 1,0 \rangle, \langle 0,1 \rangle] : H+K \to H \times K.$$

In fact, $[H,K]_H$ is a normal subobject of $H \vee K$, but in general it is not normal in A, even if H and K are normal in A (see e.g. [MM10]). Its normal closure in A is given by the Huq commutator, when $\mathbb{C}$ is also unital, i.e. when any $\Sigma_{H,K}$ is a regular epimorphism for any choice of H and K. In this case, the Huq commutator $[H,K]_Q$, introduced by S. A. Huq [Huq68] and further developed by D. Bourn [Bou04], can be obtained as the kernel of p in the pushout diagram below (see [MM10]):

$$\begin{array}{ccc}
K(\Sigma) & \longrightarrow & [H,K]_Q \\
\downarrow & & \downarrow \\
H+K & \xrightarrow{[h,k]} & A \\
{\scriptstyle \Sigma}\downarrow & & \downarrow{\scriptstyle p} \\
H \times K & \xrightarrow[m]{} & Q.
\end{array}$$

So if $\mathbb{C}$ is an ideal determined unital category, for H and K subobjects of A, we have that

$$[H,K]_H \leqslant [H,K]_Q \lhd [H,K]_U \lhd A$$

All these relations may be strict, and even for the category of groups the three notions are distinct, if H and K are not normal (see [Cig10], [CM12]). In fact we can now show that, unlike the Higgins and Huq commutators, the Ursini commutator is invariant with respect to normal closure of subobjects.

4.4. Lemma. *In an ideal determined category $\mathbb{C}$, given a subobject K of A, its normal closure $\overline{K}$ in A is obtained by taking the regular image of the kernel $n_{A,K} : A\flat K \to A+K$ along $[1,k]$:*

$$\begin{array}{ccc} A\flat K & \xrightarrow{\chi_{|}} & \overline{K} \\ {\scriptstyle n_{A,K}}\downarrow & & \downarrow{\scriptstyle \overline{k}} \\ A+K & \underset{[1,k]}{\twoheadrightarrow} & A. \end{array}$$

Proof. This is true since the cokernel q_K of $k : K \to A$ can be obtained as the pushout of $[1,0]$ along $[1,k]$:

$$\begin{array}{ccc} A+K & \overset{[1,k]}{\twoheadrightarrow} & A \\ {\scriptstyle [1,0]}\downarrow & & \downarrow{\scriptstyle q_K} \\ A & \underset{q_K}{\twoheadrightarrow} & \frac{A}{K}. \end{array} \tag{1}$$

■

4.5. Proposition. *Let $\mathbb{C}$ be ideal determined. Given two subobjects H and K of A and their normal closures $\overline{H}, \overline{K}$ in A, their Ursini commutators coincide, i.e.*

$$[H,K]_U = [\overline{H}, \overline{K}]_U$$

Proof. Thanks to the previous Lemma 4.4, it is easy to see that the following diagram

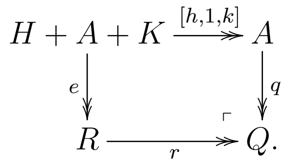
$$\begin{array}{ccccc} A\flat K & & \xrightarrow{\quad\chi_{|}\quad} & & \overline{K} \\ {\scriptstyle n_{A,K}}\downarrow & & & & \\ A+K & & & & \downarrow{\scriptstyle \langle 0,0,\overline{k}\rangle} \\ {\scriptstyle i_{A,K}}\downarrow & & & & \\ H+A+K & \underset{e}{\twoheadrightarrow} & R & \underset{m}{\rightarrowtail} & A\times A\times A \end{array} \tag{2}$$

commutes, so that $\langle 0,0,\overline{k}\rangle$ factorizes through R by $\widetilde{k}$. Analogously $\langle \overline{h},0,0\rangle$ factorizes through R by $\widetilde{h}$ and this means that $m[\widetilde{h},1,\widetilde{k}] = \Omega_{\overline{H},\overline{K}} := \overline{\Omega} : \overline{H}+A+\overline{K} \to A\times A\times A$. Since the restriction of $\overline{\Omega}$ to $H+A+K$ coincides with Ω, we obtain the following commutative diagram:

$$\begin{array}{ccc} H+A+K & \longrightarrow & \overline{H}+A+\overline{K} \\ {\scriptstyle e}\downarrow & \swarrow{\scriptstyle [\tilde{h},1,\tilde{k}]} & \downarrow{\scriptstyle \overline{\Omega}} \\ R & \underset{m}{\rightarrowtail} & A\times A\times A. \end{array}$$

This means that R is the regular image through $\overline{\Omega}$ of $\overline{H}+A+\overline{K}$ in $A\times A\times A$, with $[\widetilde{h},1,\widetilde{k}] := \overline{e}$.

By Remark 4.3 (i), $[H, K]_U$ is obtained as the kernel of the pushout q of e along $[h, 1, k]$:



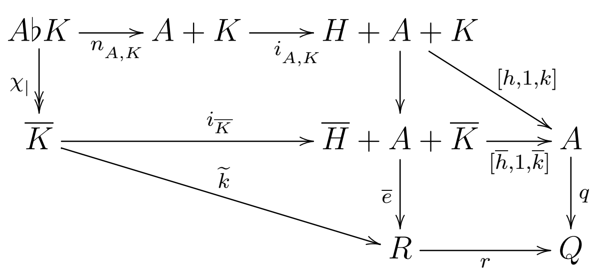
Notice that by pre-composing with coproduct injections into $H + A + K$, we obtain:

$$r\,e\,i_H = q\,h \qquad r\,e\,i_A = q \qquad r\,e\,i_K = q\,k.$$

We want to show that $q : A \to B$ is also the pushout of $\overline{e} : \overline{H} + A + \overline{K} \to R$ along $[\overline{h}, 1, \overline{k}] : \overline{H} + A + \overline{K} \to A$, so that $[H, K]_U = [\overline{H}, \overline{K}]_U$. Since the restrictions to $H + A + K$ of $\overline{e}$ and $[\overline{h}, 1, \overline{k}]$ are respectively e and $[h, 1, k]$, we only need to prove that

$$q[\overline{h}, 1, \overline{k}] = r\overline{e}.$$

The result is obtained by pre-composing with coproduct injections into $\overline{H} + A + \overline{K}$. Indeed, for the injection i_A of A it is trivial; for i_K (and analogously for i_H), by definition of $\widetilde{k}$ as in diagram (2),



we obtain that

$$r\,\overline{e}\,i_{\overline{K}}\,\chi_| = q\,[1, k]\,n_{A,K} = q\,\overline{k}\,\chi_|$$

and since $\chi_|$ is a regular epimorphism, it follows that $r\,\overline{e}\,i_{\overline{K}} = q\,\overline{k} = q\,[\overline{h}, 1, \overline{k}]\,i_{\overline{K}}$.

■

The property of the Ursini commutator described in the previous Proposition highlights another deep difference with the Higgins (and Huq) commutator, even in the category of groups, where Higgins and Huq commutators of two subgroups H and K in A coincide with the usual commutator subgroup $[H, K]$, while $[H, K]_U$ is the usual commutator subgroup $[N(H), N(K)]$ of the normal closures of H and K in A.

Thanks to Proposition 4.5, from now on we can consider H and K **normal** subobjects of A, without loss of generality.

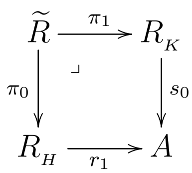
4.6. Remark. *In [GJU12], for two normal subobjects H and K, the same commutator $[H,K]_U$ is obtained as a particular case of the weighted commutator therein introduced (see Corollary 3.5). The approach used in [GJU12] is different, and it is based on the notion of weighted centrality.*

When dealing with normal subobjects, we can exploit the existence in our context of a bijection between kernels and effective equivalence relations (see e.g. [MM10]), moving our attention to the Smith-Pedicchio commutator for equivalence relations, notion introduced by Smith [Smi76] in the context of Mal'tsev varieties and made categorical by Pedicchio [Ped95]. A counter-example due to G. Janelidze in [Bou00] shows that this latter commutator is not equivalent to the Huq commutator even in a varietal context of Ω-groups, in the following sense:

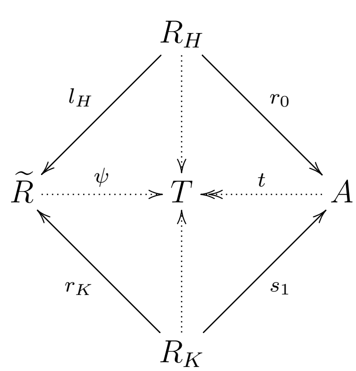
there are normal subobjects H and K with $[H,K]_Q$ strictly smaller than the normal subobject associated to the commutator $[R_H, R_K]_{SP}$ of the equivalence relations R_H, R_K generated by H and K.

It is now quite natural to ask if also in a categorical context the Ursini commutator could be the commutator for ideals equivalent to the Smith-Pedicchio commutator for congruences. In order to make this investigation, starting with two normal subobjects H and K of A, we first need to recall from [BB04] how the Smith-Pedicchio commutator of the two associated equivalence relations R_H and R_K can be defined:

4.7. Definition. *Let $\mathbb{C}$ be a finitely cocomplete regular Mal'tsev category, (R_H, r_0, r_1) and (R_K, s_0, s_1) equivalence relations on A. Consider first the pullback diagram of r_1 along s_0*



and then the colimit T of the solid arrows in the diagram below, where $l_H = \langle 1, 0\rangle$, $r_K = \langle 0, 1\rangle$:

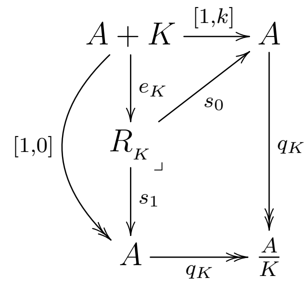


(3)

$[R_H, R_K]_{SP}$ is defined as the kernel pair of the regular epimorphism t in the diagram above, so that its normalization is given by the kernel of t.

We will show that this regular epimorphism t can be obtained as a pushout of a canonical arrow $\varphi : H + A + K \to \widetilde{R}$ and this result will allow us to compare the kernel of t with $[H, K]_U$.

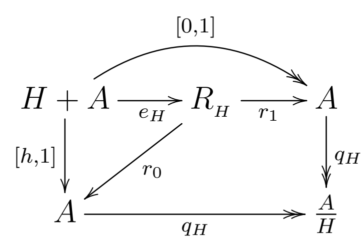
Starting with a normal subobject K of A in a normal category $\mathbb{C}$, the equivalence relation R_K associated to K is given by the kernel pair of the cokernel $q_K : A \to \frac{A}{K}$. From the pushout diagram (1) of Lemma 4.4,



we obtain a comparison arrow e_K (regular epimorphism in the exact Mal'tsev case) such that pre-composing it with the coproduct injections gives:

$$e_K\, i_A = \langle 1, 1\rangle = \Delta_A : A \to R_K, \qquad e_K\, i_K = \langle 0, k\rangle : K \to R_K.$$

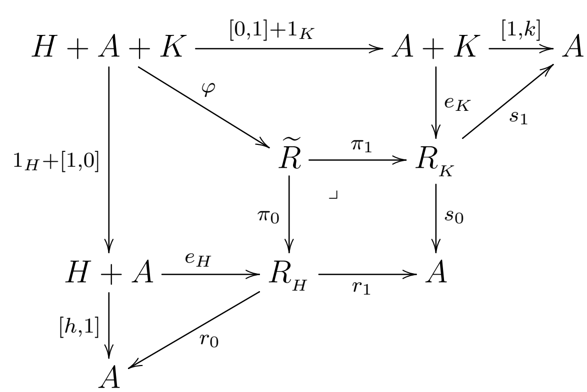
In a symmetric way, for $H \lhd A$, we obtain an arrow e_H:



with

$$e_H\, i_A = \langle 1, 1\rangle = \Delta_A : A \to R_H, \qquad e_H\, i_H = \langle h, 0\rangle : H \to R_H.$$

By the universal property of the pullback, we then obtain a canonical arrow φ : $H + A + K \to \widetilde{R}$

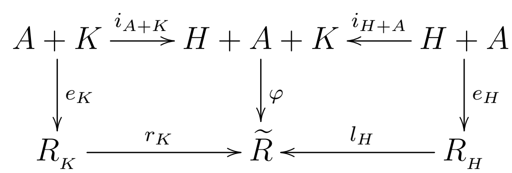


(4)

4.8. LEMMA. *The comparison arrow φ of diagram (4) is such that*

(i) $\langle \widetilde{r_0}, \widetilde{r_1}, \widetilde{r_2} \rangle \varphi = \Omega : H + A + K \to A \times A \times A$, *where* $\widetilde{r_0} = r_0\,\pi_0$, $\widetilde{r_1} = r_1\,\pi_0$ *and* $\widetilde{r_2} = s_1\,\pi_1$.

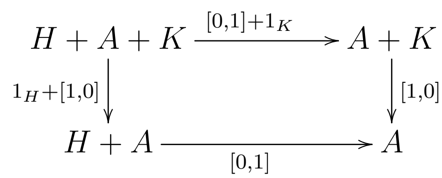
(ii) the following diagram



commutes.

Now we need the following Lemma:

4.9. LEMMA. *The canonical diagram*

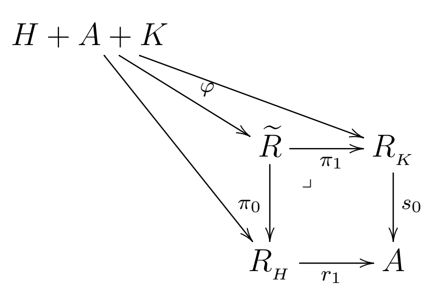


is a pushout diagram.

PROOF. It is canonically commutative. Given $\alpha : A + K \to Z$ and $\beta : H + A \to Z$ with $\alpha([0,1] + 1_K) = \beta(1_H + [1,0])$ by pre-composing both of them with the injection of A into the coproducts we obtain a unique arrow $\gamma : A \to Z$, proving the universal property of the pushout. ∎

4.10. LEMMA. *Let $\mathbb{C}$ be a finitely cocomplete exact Mal'tsev category. The comparison arrow φ is a regular epimorphism and then $\varphi = e : H + A + K \to \widetilde{R} = R$.*

PROOF. In the exact Mal'tsev case, the arrows e_H and e_K in diagram (4) are regular epimorphisms. This implies, together with Lemma 4.9, that also the outer diagram in



is a regular pushout. Consequently the comparison φ is a regular epimorphism and $\varphi = e$, by Lemma 4.8, (i). ∎

Now we can give another description of $[R_H, R_K]_{SP}$ in the exact case (an independent proof of the same description of the Smith-Pedicchio commutator can be found in [GJU12], for the case of 1-weighted commutators therein introduced).

4.11. Proposition. *Let $\mathbb{C}$ be a finitely cocomplete exact Mal'tsev category. The Smith-Pedicchio commutator $[R_H, R_K]_{SP}$ can be obtained as the kernel pair of the pushout t of the comparison arrow φ along $[h, 1, k]$:*

$$\begin{array}{ccc} H + A + K & \xrightarrow{[h,1,k]} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle t} \\ \widetilde{R} & \xrightarrow[\psi]{} & T. \end{array}$$

Proof. We have to prove that $t : A \to T$ is the colimit of the diagram (3) in the Definition 4.7. So we have to show that

$$\psi\, l_H = t\, r_0 \qquad \text{(and symmetrically} \qquad \psi\, r_K = t\, s_1).$$

By Lemma 4.8, (ii), we know that

$$\psi\, l_H\, e_H = \psi\, \varphi\, i_{H,A} = t\, [h, 1] = t\, r_0\, e_H,$$

so that $\psi\, l_H = t\, r_0$, since e_H is an epimorphism. The universal property follows again from Lemma 4.8, (ii), by pre-composing with the coproduct injections. ∎

Now we are ready to state our main result.

4.12. Theorem. *Let $\mathbb{C}$ be a normal exact Mal'tsev category. Given two (normal) subobjects H and K of A, the Ursini commutator $[H, K]_U$ is the normalization of the Smith-Pedicchio commutator $[R_H, R_K]_{SP}$ of their associated equivalence relations.*

Proof. This follows from Remark 4.3, (i), Lemma 4.10 and Proposition 4.11. ∎

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