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TOWARD CATEGORICAL RISK MEASURE THEORY

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Abstract. We introduce a category that represents varying risk as well as ambiguity. We give a generalized conditional expectation as a presheaf for this category, which not only works as a traditional conditional expectation given a σ-field but also is compatible with change of measure. Then, we reformulate dynamic monetary value measures as a presheaf for the category. We show how some axioms of dynamic monetary value measures in the classical setting are deduced as theorems in the new formulation, which is evidence that the *axioms* are correct. Finally, we point out the possibility of giving a theoretical criteria with which we can pick up appropriate sets of axioms required for monetary value measures to be *good*, using a topology-as-axioms paradigm.

1. Introduction

In everyday activity, financial institutions are trying to manage the risk of their financial positions so that they do not face an undesirable loss. It is crucially important for them to utilize adequate methods of quantifying the risk for the management, where risk measure theory plays a central role.

A financial position is described by the corresponding payoff profile. In the simplest one-period model, it is represented by a real-valued function on a set of possible scenarios. The position has a certain *current* value, and has a random variable instantiating a possible value we will have eventually at a fixed *future* time. In this setting, people are supposed to manage their risk only at the starting (current) time.

A dynamic (multi-period) model is an extension of a single period model, in which people check the risk of the intermediate (random) values of their positions and manage them at multiple (or even infinite) points in the time interval between now and a fixed time horizon. This can be considered as a management along the temporal dimension.

One of the key tools for managing risk is a theory of monetary risk measures. Since the axiomatization of monetary risk measures was initiated by [Artzner et al., 1999], many axioms such as *law invariance* have been presented ([Kusuoka, 2001], [Föllmer and Schied, 2011]). Especially after introducing dynamic versions of monetary risk measures, many researchers have been investigating this axiomatic approach intensively [Artzner

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et al., 2007]. Those investigations are valuable in both theoretical and practical senses. However, it may be expected to have some theoretical criteria for picking appropriate sets of axioms out of them. Thinking about the recent events such as the CDS (credit default swap) hedging failure at JP Morgan Chase where it is doubtful that they excelled at the usage of monetary risk measure, the importance of selecting appropriate axioms of monetary risk measures becomes even bigger than before.

On the other hand, financial industry had been employing techniques relying on the assumption that there exists a unique objective probability to compute risk. And then, they met the Lehman shock. After an excuse saying that it was a 1-in-100-year event, they realized that their assumption might be incorrect, and started thinking that there is no unique probability measure but multiple subjective probability measures that vary along a non-temporal dimension. This new type of uncertainty is called *ambiguity*.

The risk measure theory we are formulating in this note is a theory of dynamic monetary risk measures with ambiguity. We will give a starting point based on which we can provide a theoretical criteria for picking appropriate sets of axioms of dynamic monetary risk measures by reformulating the theory in the language of category theory.

In this note, we will stress three points. First is a categorical method of handling (dynamic) risk as well as ambiguity, which has a potential to develop several stochastic structures with it. Actually, without category theory, it would be hard to integrate these two concepts in a natural single framework. A functorial representation of generalized conditional expectations is a good example showing that the integration works well. Second is how we can formulate some concepts of dynamic risk measure theory in the structure provided in the first point, and show some *axioms* in the classical risk measure theory become *theorems* in our new setting. The result may support the legitimacy of the axioms in the classical setting. Third is the possibility of providing a criteria useful when selecting sets of axioms required for monetary value measures in a sheaf-theoretic point of view.

The remainder of this paper consists of four sections.

In Section 2, we provide an overview of financial risk management for those who are not familiar with it.

In Section 3, we present a base category with which we handle not just a dynamic (temporal) structure but also ambiguity (spacial) structure in the sense that it handles measure change internally. We define a generalized conditional expectation as a contravariant functor from the category.

In Section 4, we give a definition of monetary value measures as contravariant functors from the category defined in Section 3 to the category of sets. Then, we will see the resulting monetary value measures satisfy a time consistency condition and a dynamic programming principle that were introduced as axioms in the classical version of dynamic risk measure theory.

In Section 5, we discuss a possibility of finding an appropriate Grothendieck topology for which monetary value measures satisfying given axioms become sheaves. We also introduce the notion of a complete set of axioms with which we give a method of constructing a monetary value measure satisfying the axiom from any given monetary value measure.

2. An Overview of Financial Risk Management

What is *Risk*? We can say that risk is a failure, an unexpected result. ***Risk*** is the probability that a disaster will happen. We take the latter and formalize the definition of disasters.

We assume two times, 0 (present) and 1 (future). We call a ***one period model***. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space where $\mathcal{G}$ is a σ-field over Ω and $\mathbb{P}$ is a probability measure defined on the measurable space $(\Omega, \mathcal{G})$. Suppose we have a random variable X over the probability space, which means that $X : \Omega \longrightarrow \mathbb{R}$ is a $\mathcal{G}/\mathcal{B}(\mathbb{R})$-measurable function. The random variable X represents an uncertain value at time 1 of some financial value such as a stock price. A ***disaster*** is a situation when $X < d$ for some constant value $d \in \mathbb{R}$.

Now in order to manage the risk, we should inject some capital m which, if added to X and invested into a risk-free asset, keeps the probability of the disaster below a given ***confidence level*** $\alpha \in]0, 1[$, say 0.05.

$$\mathbb{P}(X + m < d) \leq \alpha \tag{1}$$

Without loss of generality, we can redefine the random variable X by $X - d$. Then (1) becomes

$$\mathbb{P}(X + m < 0) \leq \alpha. \tag{2}$$

Naturally, we may prepare the minimum amount of money specified by

$$\mathrm{VaR}_\alpha(X) := \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m < 0) \leq \alpha\} \tag{3}$$

for keeping the risk below α. The value (3) is called ***Value at Risk*** which is a quite important means for managing risk in financial industry these days.

For a σ-field $\mathcal{F} \subset \mathcal{G}$, we denote the set of all bounded $\mathbb{R}$-valued $\mathcal{F}$-measurable functions by $\mathcal{L}^\infty(\Omega, \mathcal{F})$. Let $L^\infty(\Omega, \mathcal{F}, \mathbb{P}|\mathcal{F})$ be the quotient space of $\mathcal{L}^\infty(\Omega, \mathcal{F})$ under the equivalence relation $\sim_\mathbb{P}$ defined by $X \sim_\mathbb{P} Y$ iff $X = Y$ $\mathbb{P}$-a.s.. Then, the space $L^\infty(\Omega, \mathcal{F}, \mathbb{P}|\mathcal{F})$ becomes a Banach space with the usual sup norm.

It is easy to see that $\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\alpha(Y)$ for $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ if $X \sim_\mathbb{P} Y$. Therefore, we can think that the domain of the function VaR_α is $L^\infty(\Omega, \mathcal{G}, \mathbb{P})$. Then, we have the following proposition.

2.1. PROPOSITION. *For* $\alpha, \beta \in]0, 1[$, $X, Y \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ *and* $a \in \mathbb{R}$,

1. $\mathrm{VaR}_\alpha(X + a) = \mathrm{VaR}_\alpha(X) - a$,
2. $X \leq Y \;\Rightarrow\; \mathrm{VaR}_\alpha(X) \geq \mathrm{VaR}_\alpha(Y)$,
3. $\mathrm{VaR}_\alpha(0) = 0$,

4. $\alpha \leq \beta \Rightarrow \mathrm{VaR}_\alpha(X) \geq \mathrm{VaR}_\beta(X)$.

One of the biggest issues in financial risk management is to find a class of good risk measure functions $\rho : L^\infty(\Omega, \mathcal{G}, \mathbb{P}) \longrightarrow \mathbb{R}$ whose value $\rho(X)$ gives a necessary amount of capital which, if added to X, avoids the risk reasonably. The following is a base class of functions for measuring risk, which contains VaR_α.

2.2. Definition. *A **one period monetary risk measure** is a function $\rho : L^\infty(\Omega, \mathcal{G}, \mathbb{P}) \longrightarrow \mathbb{R}$ satisfying the following axioms*

1. *Cash invariance:* $(\forall X)(\forall a \in \mathbb{R})\ \rho(X + a) = \rho(X) - a$,
2. *Monotonicity:* $(\forall X)(\forall Y)\ X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$,
3. *Normalization:* $\rho(0) = 0$.

Here is another example of one period monetary risk measures, called the ***Average Value at Risk***.

$$\mathrm{AVaR}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_u(X) du \tag{4}$$

Instead of fixing a particular confidence level α, we average VaR over all levels $u \leq \alpha$ and thus "look further into the tail" of the distribution of X. AVaR_α has the following properties.

2.3. Proposition. *For $\alpha \in]0, 1[$, $X, Y \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$, $\lambda \in [0, 1]$ and $\xi > 0$,*

1. $\mathrm{AVaR}_\alpha(X) \geq \mathrm{VaR}_\alpha(X)$,
2. *Convexity:* $\mathrm{AVaR}_\alpha(\lambda X + (1 - \lambda)Y) \leq \lambda \mathrm{AVaR}_\alpha(X) + (1 - \lambda)\mathrm{AVaR}_\alpha(Y)$,
3. *Subadditivity:* $\mathrm{AVaR}_\alpha(X + Y) \leq \mathrm{AVaR}_\alpha(X) + \mathrm{AVaR}_\alpha(Y)$,
4. *Positive homogeneity:* $\mathrm{AVaR}_\alpha(\xi X) = \xi \mathrm{AVaR}_\alpha(X)$.

The first property in Proposition 2.3 says that AVaR requires more capital than VaR for managing risk of financial instruments. ***The International regulatory framework for banks*** (***Basel III***) requires banks to use AVaR instead of VaR for their risk management because of the properties described in Proposition 2.3, especially the subadditivity condition that meets our intuition when handling risk.

Since [Artzner et al., 1999], an axiomatic approach for defining monetary risk measures becomes popular in both theoretical and practical aspects of risk management. There are several proposed sets of axioms including ***convex monetary risk measures*** = monetary risk measures satisfying the convexity condition and ***coherent monetary risk measures*** = convex monetary risk measures satisfying the subadditivity and the positive homogeneity conditions. However, we have not yet determined *the* standard set of axioms for appropriate monetary risk measures, which relates to the discussion in Section 5.

We sometimes adopt the practice of using a ***monetary value measure*** φ instead of using a monetary risk measure ρ below by conforming the manner in recent literature such as [Artzner et al., 2007] and [Kusuoka and Morimoto, 2007], where we have a relation $\varphi(X) = -\rho(X)$ for any possible scenario X. In case $\rho = \mathrm{VaR}_\alpha$, its corresponding monetary value measure becomes

$$\varphi(X) = -\mathrm{VaR}_\alpha(X) = \sup\{m \in \mathbb{R} \mid \mathbb{P}(X < m) \leq \alpha\}. \tag{5}$$

Therefore, a value $\varphi(X)$ of a monetary value measure $\varphi : L^\infty(\Omega, \mathcal{G}, \mathbb{P}) \longrightarrow \mathbb{R}$ is a guaranteed value that we can have for X in a reasonable sense. Here is a direct definition of monetary value measures.

2.4. Definition. *A **one period monetary value measure** is a function $\varphi : L^\infty(\Omega, \mathcal{G}, \mathbb{P}) \longrightarrow \mathbb{R}$ satisfying the following axioms*

1. *Cash invariance:* $(\forall X)(\forall a \in \mathbb{R})\ \varphi(X + a) = \varphi(X) + a$,
2. *Monotonicity:* $(\forall X)(\forall Y)\ X \leq Y \ \Rightarrow\ \varphi(X) \leq \varphi(Y)$,
3. *Normalization:* $\varphi(0) = 0$.

Here is another example of one period monetary value measures, called an ***entropic value measure*** defined by

$$\varphi(X) := \lambda^{-1} \log \mathbb{E}^{\mathbb{P}}[e^{\lambda X}] \tag{6}$$

where λ is a positive real number. We will use this in Section 3.

Note that the convexity condition of monetary risk measures indicated in Proposition 2.3 is modified to the following *concavity* condition for monetary value measures.

2.5. Definition. *A monetary value measure $\varphi : L^\infty(\Omega, \mathcal{G}, \mathbb{P}) \longrightarrow \mathbb{R}$ is called **concave** if for $X, Y \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ and $\lambda \in [0, 1]$,*

$$\varphi(\lambda X + (1 - \lambda)Y) \geq \lambda\varphi(X) + (1 - \lambda)\varphi(Y). \tag{7}$$

So far, we only have had two times 0 and 1, which was the one period framework. We only measure the risk of X once at time 0. However, it is more realistic to think the situation that we measure the risk of X several times between present (0) and future (1). We call this a ***dynamic model***.

Now let $T > 0$ be a fixed time, called a ***horizon*** and suppose we have a family of σ-fields, $\mathbb{G} = \{\mathcal{G}_t\}_{t \in [0,T]}$ such that $\mathcal{G}_s \subset \mathcal{G}_t \subset \mathcal{G}$ whenever $s \leq t \leq T$. This type of family of σ-fields is called a ***filtration***. We may measure the risk of the $\mathcal{G}_T$-measurable random variable X at anytime $t \in [0, T]$, which leads to the following definition.

2.6. Definition. *For a σ-field $\mathcal{F} \subset \mathcal{G}$, we write $L(\mathcal{F})$ for $L^\infty(\Omega, \mathcal{F}, \mathbb{P}|\mathcal{F})$. Let $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,T]}$ be a filtration. A* ***dynamic monetary value measure*** *is a collection of functions $\varphi = \{\varphi_t : L(\mathcal{G}_T) \longrightarrow L(\mathcal{G}_t)\}_{t\in[0,T]}$ satisfying*

1. *Cash invariance:* $(\forall X \in L(\mathcal{G}_T))(\forall Z \in L(\mathcal{G}_t))\ \varphi_t(X + Z) = \varphi_t(X) + Z$,
2. *Monotonicity:* $(\forall X \in L(\mathcal{G}_T))(\forall X \in L(\mathcal{G}_T))\ X \leq Y \ \Rightarrow\ \varphi_t(X) \leq \varphi_t(Y)$,
3. *Normalization:* $\varphi_t(0) = 0$.

Since dynamic monetary value measures treat multi-period situations, we may require some extra axioms to regulate them along the temporal dimension. Here are two possible such axioms.

2.7. Axiom. *[Dynamic programming principle] For $0 \leq s \leq t \leq T$, $(\forall X \in L(\mathcal{G}_T))\ \varphi_s(X) = \varphi_s(\varphi_t(X))$.*

2.8. Axiom. *[Time consistency] For $0 \leq s \leq t \leq T$, $(\forall X, \forall Y \in L(\mathcal{G}_T))\ \varphi_t(X) \leq \varphi_t(Y) \ \Rightarrow\ \varphi_s(X) \leq \varphi_s(Y)$.*

Both axioms are quite popular in dynamic risk measure theory. Especially, the axiom of dynamic programming principle is indispensable when we calculate values of φ_t recursively by so-called ***Bellman equations***.

All the discussions we made in this section so far describe the situations under a fixed probability measure $\mathbb{P}$. However, after experiencing some crises including the Lehman shock recently, the financial industry has begun to think the situation where we cannot determine a unique probability to compute the risk. In other words, they started thinking about situations having multiple subjective probabilities. This is called ***ambiguity***. The category χ defined in Section 3 provides an integrated framework for risk and ambiguity. We have not had such an integrated framework in classical financial risk management theory.

Here are some fundamental tools for handling multiple probabilities. Suppose that we have two probability measures $\mathbb{P}$ and $\mathbb{Q}$ defined on $(\Omega, \mathcal{G})$. Then, $\mathbb{Q}$ is called ***absolutely continuous*** to $\mathbb{P}$, denoted by $\mathbb{Q} \ll \mathbb{P}$, if for all $A \in \mathcal{G}$, $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$. $\mathbb{Q}$ is called ***equivalent*** to $\mathbb{P}$, denoted by $\mathbb{Q} \approx \mathbb{P}$, if $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$. Now we have the famous Radon-Nikodym theorem.

2.9. Theorem. *[Radon-Nikodym] Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{G})$ with $\mathbb{Q} \ll \mathbb{P}$. Then, there exists a unique (up to $\mathbb{P}$-a.s.) $\mathbb{P}$-integrable positive random variable Z such that for all $A \in \mathcal{G}$,*

$$\mathbb{Q}(A) = \int_A Z d\mathbb{P}. \tag{8}$$

We write this Z as $\frac{d\mathbb{Q}}{d\mathbb{P}}$ and call it a ***Radon-Nikodym derivative*** *.*

We end our brief introduction to financial risk management here. For those who are interested in more detail about this, please consult financial textbooks such as [Föllmer and Schied, 2011].

3. Generalized Conditional Expectations

We fix a measurable space $(\Omega, \mathcal{G})$ for the rest of this note. Let $\mathcal{F} \subset \mathcal{G}$ be a sub σ-field. A *conditional expectation* given $\mathcal{F}$ under the probability measure $\mathbb{P}$ is a projection from $L^\infty(\Omega, \mathcal{F}, \mathbb{P}|_\mathcal{F})$ to $L^\infty(\Omega, \mathcal{G}, \mathbb{P})$. The concept of conditional expectations is crucially important in probability theory.

In this section, we introduce a category called χ which will be a base category throughout this note, and define a generalized conditional expectation functor on it.

3.1. Definition. *[Category χ] Let $\chi := \chi(\Omega, \mathcal{G})$ be the set of all pairs of the form $(\mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is a sub-σ-field of $\mathcal{G}$ and $\mathbb{P}$ is a probability measure on $\mathcal{G}$. For an element $\mathcal{U} \in \chi$, we denote its σ-field and probability measure by $\mathcal{F}_\mathcal{U}$ and $\mathbb{P}_\mathcal{U}$, respectively. That is, $\mathcal{U} = (\mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U})$.*

Let us introduce a binary relation $\leq_\chi$ on χ by for $\mathcal{U}$ and $\mathcal{V}$ in χ,

$$\mathcal{V} \leq_\chi \mathcal{U} \quad \textit{iff} \quad \mathcal{F}_\mathcal{V} \subset \mathcal{F}_\mathcal{U} \textit{ and } \mathbb{P}_\mathcal{V} \gg \mathbb{P}_\mathcal{U} \tag{9}$$

where $\mathbb{P}_\mathcal{V} \gg \mathbb{P}_\mathcal{U}$ means that $\mathbb{P}_\mathcal{U}$ is absolutely continuous to $\mathbb{P}_\mathcal{V}$. Then, obviously the system $(\chi, \leq_\chi)$ is a preordered set. Hence we can think of χ as a category having exactly one arrow $*_\mathcal{U}^\mathcal{V} : \mathcal{V} \longrightarrow \mathcal{U}$ in χ if and only if $\mathcal{V} \leq_\chi \mathcal{U}$.

We may be able to think of the category χ having two dimensions; one is a temporal dimension or risk dimension that is represented in a horizontal direction in Figure 3, and the other is a spacial dimension or ambiguity dimension representing in a vertical direction.

Note that for objects $\mathcal{U}, \mathcal{V} \in \chi$, $\mathcal{U}$ is isomorphic to $\mathcal{V}$ (we write this by $\mathcal{U} \simeq \mathcal{V}$) if and only if $\mathcal{F}_\mathcal{V} = \mathcal{F}_\mathcal{U}$ and $\mathbb{P}_\mathcal{V} \approx \mathbb{P}_\mathcal{U}$ (equivalent probability measures) .

First, we will make a mapping of an object $\mathcal{U}$ of χ to a Banach space $L^\infty(\Omega, \mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U}|_{\mathcal{F}_\mathcal{U}})$ be a functor. Here is an auxiliary definition used in the subsequent proposition.

3.2. Definition. *For an object $\mathcal{U}$ in χ and $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}_\mathcal{U})$, define a subset $[X]_\mathcal{U} \subset \mathcal{L}^\infty(\Omega, \mathcal{F}_\mathcal{U})$ by*

$$[X]_\mathcal{U} := \{Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}_\mathcal{U}) \mid Y \sim_{\mathbb{P}_\mathcal{U}} X\}. \tag{10}$$

3.3. Proposition. *Suppose that there are arrows $\mathcal{W} \longrightarrow \mathcal{V} \longrightarrow \mathcal{U}$ in χ.*

1. $L^\infty(\Omega, \mathcal{F}_\mathcal{W}, \mathbb{P}_\mathcal{W}|_{\mathcal{F}_\mathcal{W}}) = \{[X]_\mathcal{W} \mid X \in \mathcal{L}^\infty(\Omega, \mathcal{F}_\mathcal{W})\}$.
2. *For $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}_\mathcal{W})$, $[X]_\mathcal{W} \subset [X]_\mathcal{V}$.*
3. *For $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}_\mathcal{W})$, $[X]_\mathcal{W} = [Y]_\mathcal{W}$ implies $[X]_\mathcal{V} = [Y]_\mathcal{V}$.*
4. *For $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}_\mathcal{W})$ and $Z \in [X]_\mathcal{V}$, $[X]_\mathcal{U} = [Z]_\mathcal{U}$.*

$$\begin{tikzcd}
\mathcal{W} \arrow[r] \arrow[rr, bend left] \arrow[loop below] & \mathcal{V} \arrow[r] \arrow[loop below] & \mathcal{U} \arrow[loop below]
\end{tikzcd}$$

Ambiguity $\longrightarrow$

$$\begin{tikzcd}
(\mathcal{F}_{\mathcal{W}}, \mathbb{P}_{\mathcal{W}}) \arrow[r] \arrow[d] \arrow[dr] & (\mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{W}}) \arrow[r] \arrow[d] & (\mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{W}}) \arrow[d] \\
(\mathcal{F}_{\mathcal{W}}, \mathbb{P}_{\mathcal{V}}) \arrow[r] \arrow[d] & (\mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{V}}) \arrow[r] \arrow[d] \arrow[dr] & (\mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{V}}) \arrow[d] \\
(\mathcal{F}_{\mathcal{W}}, \mathbb{P}_{\mathcal{U}}) \arrow[r] & (\mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{U}}) \arrow[r] & (\mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{U}})
\end{tikzcd}$$

Risk $\longrightarrow$

Figure 3.1:

PROOF.

1. For $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}_{\mathcal{W}})$, $\{X = Y\} \in \mathcal{F}_{\mathcal{W}}$. Therefore, $\mathbb{P}_{\mathcal{W}}\{X = Y\} = \mathbb{P}_{\mathcal{W}}|_{\mathcal{F}_{\mathcal{W}}}\{X = Y\}$. Thus, $X \sim_{\mathbb{P}_{\mathcal{W}}|_{\mathcal{F}_{\mathcal{W}}}} Y = X \sim_{\mathbb{P}_{\mathcal{W}}} Y$.
2. Since $\mathcal{F}_{\mathcal{W}} \subset \mathcal{F}_{\mathcal{V}}$, $\mathcal{L}^\infty(\Omega, \mathcal{F}_{\mathcal{W}}) \subset \mathcal{L}^\infty(\Omega, \mathcal{F}_{\mathcal{V}})$. Also since $\mathbb{P}_{\mathcal{V}} \ll \mathbb{P}_{\mathcal{W}}$, $Y \sim_{\mathbb{P}_{\mathcal{W}}} X$ implies $Y \sim_{\mathbb{P}_{\mathcal{V}}} X$.
3. By 2, we have $[X]_{\mathcal{V}} \supset [X]_{\mathcal{W}} = [Y]_{\mathcal{W}} \subset [Y]_{\mathcal{V}}$. Then, since $[X]_{\mathcal{W}}$ is nonempty, two equivalence classes $[X]_{\mathcal{V}}$ and $[Y]_{\mathcal{V}}$ coincide.
4. Since $[Z]_{\mathcal{V}} = [X]_{\mathcal{V}}$, it is an immediate consequence of 3.

■

Proposition 3.3 makes the following definition well-defined.

3.4. DEFINITION. *[Functor L]*
A functor $L : \chi \longrightarrow \mathbf{Set}$ is defined by:

$$\begin{tikzcd}
\mathcal{V} \arrow[r, mapsto, "L"] \arrow[d] & L_{\mathcal{V}} \arrow[d, "L^{\mathcal{V}}_{\mathcal{U}}"] & := & L^\infty(\Omega, \mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{V}}|_{\mathcal{F}_{\mathcal{V}}}) & \ni & {[X]_{\mathcal{V}}} \arrow[d, mapsto, "L^{\mathcal{V}}_{\mathcal{U}}"] \\
\mathcal{U} \arrow[r, mapsto, "L"] & L_{\mathcal{U}} & := & L^\infty(\Omega, \mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{U}}|_{\mathcal{F}_{\mathcal{U}}}) & \ni & {[X]_{\mathcal{U}}}
\end{tikzcd}$$

Now we are ready to develop one of the key functors in this note, a generalized conditional expectation that will be well-defined by the following proposition.

3.5. PROPOSITION. *For $\mathcal{W} \to \mathcal{V} \to \mathcal{U}$ in χ and $X \in L_{\mathcal{U}}$,*

1. $\mathbb{E}^{\mathbb{P}_{\mathcal{U}}}[X|\mathcal{F}_{\mathcal{V}}]\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}} = \mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[X\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{U}}}|\mathcal{F}_{\mathcal{V}}] \quad \mathbb{P}_{\mathcal{V}}$*-a.s.,*

2. $\frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}}|_{\mathcal{F}_{\mathcal{U}}} \times \frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{U}}} = \frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{W}}}|_{\mathcal{F}_{\mathcal{U}}} \quad \mathbb{P}_{\mathcal{U}}$*-a.s..*

PROOF.

1. When $\mathbb{Q} \ll \mathbb{P}$ and $\mathcal{F} \subset \mathcal{G}$, we have

$$\frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}} = \mathbb{E}^P[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}] \quad \mathbb{P}\text{-a.s.} \tag{11}$$

and

$$\mathbb{E}^Q[X \mid \mathcal{F}] = \frac{\mathbb{E}^P[X\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}]}{\mathbb{E}^P[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}]} \quad \mathbb{Q}\text{-a.s.} \tag{12}$$

by Proposition A.11 and Proposition A.12 in [Föllmer and Schied, 2011]. Then, by (11) and since X is $\mathcal{F}_{\mathcal{U}}$-measurable, we have with $\mathbb{P}_{\mathcal{V}}$-a.s.,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[X\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}} |_{\mathcal{F}_{\mathcal{U}}} \mid \mathcal{F}_{\mathcal{V}}] &= \mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[X\mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}} \mid \mathcal{F}_{\mathcal{U}}] \mid \mathcal{F}_{\mathcal{V}}] \\ &= \mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[\mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[X\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}} \mid \mathcal{F}_{\mathcal{U}}] \mid \mathcal{F}_{\mathcal{V}}] \\ &= \mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[X\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}} \mid \mathcal{F}_{\mathcal{V}}].\end{aligned}$$

Therefore, again by (11) and (12), we get the desired equation.

2. By (11), (12) and again by (11), we have with $\mathbb{P}_{\mathcal{U}}$-a.s.,

$$\begin{aligned}\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}} |_{\mathcal{F}_{\mathcal{U}}} &= \mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}} \mid \mathcal{F}_{\mathcal{U}}] = \frac{\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}\frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}} \mid \mathcal{F}_{\mathcal{U}}]}{\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[\frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}} \mid \mathcal{F}_{\mathcal{U}}]} \\ &= \frac{\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{W}}} \mid \mathcal{F}_{\mathcal{U}}]}{\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[\frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}} \mid \mathcal{F}_{\mathcal{U}}]} = \frac{\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{W}}} |_{\mathcal{F}_{\mathcal{U}}}}{\frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}} |_{\mathcal{F}_{\mathcal{U}}}}.\end{aligned}$$

■

3.6. DEFINITION. *[Generalized Conditional Expectation] A* ***generalized conditional expectation*** *is a contravariant functor* $\mathcal{E} : \chi^{op} \longrightarrow \mathbf{Set}$ *defined by for* $\mathcal{V} \longrightarrow \mathcal{U}$ *in* χ*,*

$$\begin{tikzcd} \mathcal{V} \arrow[r, mapsto, "\mathcal{E}"] \arrow[d] & \mathcal{E}(\mathcal{V}) & := & L_{\mathcal{V}} \\ \mathcal{U} \arrow[r, mapsto, "\mathcal{E}"] & \mathcal{E}(\mathcal{U}) \arrow[u, "\mathcal{E}_{\mathcal{U}}^{\mathcal{V}}"'] & := & L_{\mathcal{U}} \end{tikzcd}$$

where

$$\mathcal{E}_{\mathcal{U}}^{\mathcal{V}}(X) := \mathbb{E}^{\mathbb{P}_{\mathcal{U}}}[X|\mathcal{F}_{\mathcal{V}}]\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}} \tag{13}$$

for $X \in L_{\mathcal{U}}$*.*

Note that $\mathcal{E}^{\mathcal{V}}_{\mathcal{U}}$ in Definition 3.6 is well-defined by Proposition 3.5. See also Figure 3.2 and Figure 3.3.

Classical conditional expectations do not accept changes of probability measures within them while our *generalized* conditional expectations are sensitive to the ambiguity dimension as well as the risk dimension, which is a direct result of using the categorical framework χ.

$$\begin{tikzcd}
\mathbb{E}^{\mathbb{P}_{\mathcal{U}}}[X|\mathcal{F}_{\mathcal{V}}]\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}}=\mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[X\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{U}}}|\mathcal{F}_{\mathcal{V}}] \arrow[dr, phantom, sloped, "\in"] & & & X\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{U}}} \arrow[lll, mapsto] \arrow[dl, phantom, sloped, "\ni"] \\
 & L(\mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{V}}) & L(\mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{V}}) \arrow[l] & \\
 & L(\mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{U}}) \arrow[u] & L(\mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{U}}) \arrow[l] \arrow[u] \arrow[ul, "\mathcal{E}^{\mathcal{V}}_{\mathcal{U}}"'] & \\
\mathbb{E}^{\mathbb{P}_{\mathcal{U}}}[X|\mathcal{F}_{\mathcal{V}}] \arrow[uuu, mapsto] \arrow[ur, phantom, sloped, "\in"] & & & X \arrow[lll, mapsto] \arrow[uuu, mapsto] \arrow[ul, phantom, sloped, "\ni"]
\end{tikzcd}$$

Figure 3.2:

4. Monetary Value Measures

In this section, we transplant the concept of dynamic monetary value measures introduced in Definition 2.4 to the category χ as a presheaf, and investigate its properties.

4.1. Definition. *[Monetary Value Measures] A* ***monetary value measure*** *is a contravariant functor*

$$\varphi : \chi^{op} \longrightarrow \mathbf{Set}$$

satisfying the following two conditions:

1. *for $\mathcal{U} \in \chi$, $\varphi(\mathcal{U}) := L_{\mathcal{U}}$,*
2. *for $\mathcal{V} \longrightarrow \mathcal{U}$ in χ, the map $\varphi^{\mathcal{V}}_{\mathcal{U}} := \varphi(\mathcal{V} \longrightarrow \mathcal{U}) : L_{\mathcal{U}} \longrightarrow L_{\mathcal{V}}$ satisfies*
   - *(a) Cash invariance: $(\forall X \in L_{\mathcal{U}})(\forall Z \in L_{\mathcal{V}})\ \varphi^{\mathcal{V}}_{\mathcal{U}}(X + L^{\mathcal{V}}_{\mathcal{U}}(Z)) = \varphi^{\mathcal{V}}_{\mathcal{U}}(X) + Z$ $\mathbb{P}_{\mathcal{V}}$-a.s.,*
   - *(b) Monotonicity: $(\forall X \in L_{\mathcal{U}})(\forall Y \in L_{\mathcal{U}})\ X \leq Y \Rightarrow \varphi^{\mathcal{V}}_{\mathcal{U}}(X) \leq \varphi^{\mathcal{V}}_{\mathcal{U}}(Y)$ $\mathbb{P}_{\mathcal{V}}$-a.s.,*
   - *(c) Normalization: $\varphi^{\mathcal{V}}_{\mathcal{U}}(0_{L_{\mathcal{U}}}) = 0_{L_{\mathcal{V}}}$ $\mathbb{P}_{\mathcal{V}}$-a.s. if $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$.*

At this point, we do not require the monetary value measures to satisfy familiar conditions such as concavity or positive homogeneity. Instead of doing so, we want to see what kind of properties are deduced from this minimal setting.

$$
\begin{tikzcd}
L_{\mathcal{W}} & L_{\mathcal{V}} \arrow[l, "\mathcal{E}_V^{\mathcal{W}}"'] & L_{\mathcal{U}} \arrow[l, "\mathcal{E}_U^{\mathcal{V}}"'] \arrow[ll, bend right, "\mathcal{E}_U^{\mathcal{W}}"']
\end{tikzcd}
$$

Ambiguity $\longrightarrow$

$$
\begin{tikzcd}
L(\mathcal{F}_{\mathcal{W}}, \mathbb{P}_{\mathcal{W}}) & L(\mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{W}}) \arrow[l, "\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[-|\mathcal{F}_{\mathcal{W}}]"'] & L(\mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{W}}) \arrow[l, "\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[-|\mathcal{F}_{\mathcal{V}}]"'] \\
L(\mathcal{F}_{\mathcal{W}}, \mathbb{P}_{\mathcal{V}}) \arrow[u] & L(\mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{V}}) \arrow[l] \arrow[u] \arrow[ul, "\mathcal{E}_V^{\mathcal{W}}"] & L(\mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{V}}) \arrow[l] \arrow[u, "-\times \frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}}|_{\mathcal{F}_{\mathcal{U}}}"'] \\
L(\mathcal{F}_{\mathcal{W}}, \mathbb{P}_{\mathcal{U}}) \arrow[u] & L(\mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{U}}) \arrow[l] \arrow[u] & L(\mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{U}}) \arrow[l] \arrow[u, "-\times \frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{U}}}"'] \arrow[ul, "\mathcal{E}_U^{\mathcal{V}}"']
\end{tikzcd}
$$

Risk $\longrightarrow$

Figure 3.3:

$$
\begin{tikzcd}
\mathcal{W} \arrow[r, mapsto, "\varphi"] \arrow[d] \arrow[dd, bend right] & \varphi(\mathcal{W}) & := & L_{\mathcal{W}} \\
\mathcal{V} \arrow[r, mapsto, "\varphi"] \arrow[d] & \varphi(\mathcal{V}) \arrow[u] & := & L_{\mathcal{V}} \arrow[u, "\varphi_{\mathcal{V}}^{\mathcal{W}}"] \\
\mathcal{U} \arrow[r, mapsto, "\varphi"] & \varphi(\mathcal{U}) \arrow[u] & := & L_{\mathcal{U}} \arrow[u, "\varphi_{\mathcal{U}}^{\mathcal{V}}"] \arrow[uu, bend right, "\varphi_{\mathcal{U}}^{\mathcal{W}}"']
\end{tikzcd}
$$

Figure 4.1:

The most crucial point of Definition 4.1 is that φ does not move only in the direction of time but also moves over several absolutely continuous probability measures *internally*. This means we have a possibility to develop risk measures including ambiguity within this formulation.

Another key point of Definition 4.1 is that φ is a contravariant functor. So, for any triple $\mathcal{W} \longrightarrow \mathcal{V} \longrightarrow \mathcal{U}$ in χ, we have, as seeing in Figure 4.1,

$$\varphi_{\mathcal{U}}^{\mathcal{U}} = 1_{L_{\mathcal{U}}} \text{ and } \varphi_{\mathcal{V}}^{\mathcal{W}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}} = \varphi_{\mathcal{U}}^{\mathcal{W}}. \tag{14}$$

The following example is a variation of (6).

4.2. Example. *[Entropic Value Measure] For a positive real number λ, an **entropic value measure** is a contravariant functor $\varphi : \chi^{op} \longrightarrow$ **Set** defined by*

$$\varphi(\mathcal{U}) := L_{\mathcal{U}} \quad \textit{and} \quad \varphi_{\mathcal{U}}^{\mathcal{V}}(X) := \lambda^{-1} \log \mathcal{E}_{\mathcal{U}}^{\mathcal{V}}(e^{\lambda X}) \tag{15}$$

for $\mathcal{V} \longrightarrow \mathcal{U}$ in χ and $X \in L_{\mathcal{U}}$. Then, it is easy to see that the contravariant functor φ is well-defined and is a monetary value measure.

Now in case $\mathcal{F}_{\mathcal{V}} = \mathcal{F}_{\mathcal{U}}$, we have

$$\varphi^{\mathcal{V}}_{\mathcal{U}}(X) = \lambda^{-1} \log \mathcal{E}^{\mathcal{V}}_{\mathcal{U}}(e^{\lambda X}) = \lambda^{-1} \log \left(e^{\lambda X} \frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}} \right) = X + \lambda^{-1} \log \left(\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}} \right).$$

In particular, we have $\varphi^{\mathcal{V}}_{\mathcal{U}}(0_{L_{\mathcal{U}}}) = \lambda^{-1} \log \left(\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}} \right)$, which is not $0_{L_{\mathcal{V}}}$ unless $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$ on $\mathcal{F}_{\mathcal{V}}$.

This is the reason we require the assumption $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$ in the normalization condition in Definition 4.1.

Here are some properties of monetary value measures.

4.3. Proposition. *Let $\varphi : \chi^{op} \longrightarrow \mathbf{Set}$ be a monetary value measure, and $\mathcal{W} \longrightarrow \mathcal{V} \longrightarrow \mathcal{U}$ be arrows in χ.*

1. *If $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$, we have $\varphi^{\mathcal{V}}_{\mathcal{U}} \circ L^{\mathcal{V}}_{\mathcal{U}} = 1_{L_{\mathcal{V}}}$.*
2. *Idempotence: If $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$, we have $\varphi^{\mathcal{V}}_{\mathcal{U}} \circ L^{\mathcal{V}}_{\mathcal{U}} \circ \varphi^{\mathcal{V}}_{\mathcal{U}} = \varphi^{\mathcal{V}}_{\mathcal{U}}$.*
3. *Local property: $(\forall X \in L_{\mathcal{U}})(\forall Y \in L_{\mathcal{U}})(\forall A \in \mathcal{V})\ \varphi^{\mathcal{V}}_{\mathcal{U}}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) = \mathbb{1}_A \varphi^{\mathcal{V}}_{\mathcal{U}}(X) + \mathbb{1}_{A^c} \varphi^{\mathcal{V}}_{\mathcal{U}}(Y)$.*
4. *Dynamic programming principle: If $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$, we have $\varphi^{\mathcal{W}}_{\mathcal{U}} = \varphi^{\mathcal{W}}_{\mathcal{U}} \circ L^{\mathcal{V}}_{\mathcal{U}} \circ \varphi^{\mathcal{V}}_{\mathcal{U}}$.*
5. *Time consistency: $(\forall X \in L_{\mathcal{U}})(\forall Y \in L_{\mathcal{U}})\ \varphi^{\mathcal{V}}_{\mathcal{U}}(X) \leq \varphi^{\mathcal{V}}_{\mathcal{U}}(Y) \ \Rightarrow\ \varphi^{\mathcal{W}}_{\mathcal{U}}(X) \leq \varphi^{\mathcal{W}}_{\mathcal{U}}(Y)$.*

Proof.

1. For $X \in L_{\mathcal{V}}$, we have by cash invariance and normalization, $\varphi^{\mathcal{V}}_{\mathcal{U}}(L^{\mathcal{V}}_{\mathcal{U}}(X)) = \varphi^{\mathcal{V}}_{\mathcal{U}}(0_{L_{\mathcal{U}}} + L^{\mathcal{V}}_{\mathcal{U}}(X)) = \varphi^{\mathcal{V}}_{\mathcal{U}}(0_{L_{\mathcal{U}}}) + X = X$.
2. Immediate by (1).
3. First, we show that for any $A \in \mathcal{V}$,

$$\mathbb{1}_A \varphi^{\mathcal{V}}_{\mathcal{U}}(X) = \mathbb{1}_A \varphi^{\mathcal{V}}_{\mathcal{U}}(\mathbb{1}_A X). \tag{16}$$

Since $X \in L^\infty(\Omega, \mathcal{U}, \mathbb{P})$, we have $|X| \leq \|X\|_\infty$. Therefore,

$$\mathbb{1}_A X - \mathbb{1}_{A^c} \|X\|_\infty \leq \mathbb{1}_A X + \mathbb{1}_{A^c} X \leq \mathbb{1}_A X + \mathbb{1}_{A^c} \|X\|_\infty.$$

Then, by cash invariance and monotonicity,

$$\begin{aligned} \varphi^{\mathcal{V}}_{\mathcal{U}}(\mathbb{1}_A X) - \mathbb{1}_{A^c} \|X\|_\infty &= \varphi^{\mathcal{V}}_{\mathcal{U}}(\mathbb{1}_A X - \mathbb{1}_{A^c} \|X\|_\infty) \\ &\leq \varphi^{\mathcal{V}}_{\mathcal{U}}(X) \\ &\leq \varphi^{\mathcal{V}}_{\mathcal{U}}(\mathbb{1}_A X + \mathbb{1}_{A^c} \|X\|_\infty) = \varphi^{\mathcal{V}}_{\mathcal{U}}(\mathbb{1}_A X) + \mathbb{1}_{A^c} \|X\|_\infty. \end{aligned}$$

Then,

$$\begin{aligned}\mathbb{1}_A\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_AX) &= \mathbb{1}_A(\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_AX) - \mathbb{1}_{A^c}\|X\|_\infty)\\ &\le \mathbb{1}_A\varphi_{\mathcal{U}}^{\mathcal{V}}(X)\\ &\le \mathbb{1}_A(\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_AX) + \mathbb{1}_{A^c}\|X\|_\infty) = \mathbb{1}_A\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_AX).\end{aligned}$$

Therefore, we get (16).

Next by using (16) twice, we have

$$\begin{aligned}\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_AX + \mathbb{1}_{A^c}Y) &= \mathbb{1}_A\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_AX + \mathbb{1}_{A^c}Y) + \mathbb{1}_{A^c}\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_AX + \mathbb{1}_{A^c}Y)\\ &= \mathbb{1}_A\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A(\mathbb{1}_AX + \mathbb{1}_{A^c}Y)) + \mathbb{1}_{A^c}\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_{A^c}(\mathbb{1}_AX + \mathbb{1}_{A^c}Y))\\ &= \mathbb{1}_A\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_AX) + \mathbb{1}_{A^c}\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_{A^c}Y)\\ &= \mathbb{1}_A\varphi_{\mathcal{U}}^{\mathcal{V}}(X) + \mathbb{1}_{A^c}\varphi_{\mathcal{U}}^{\mathcal{V}}(Y).\end{aligned}$$

4. By (2) and (14), we have

$$\begin{aligned}\varphi_{\mathcal{U}}^{\mathcal{W}} &= \varphi_{\mathcal{V}}^{\mathcal{W}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}} = \varphi_{\mathcal{V}}^{\mathcal{W}} \circ (\varphi_{\mathcal{U}}^{\mathcal{V}} \circ L_{\mathcal{U}}^{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}})\\ &= (\varphi_{\mathcal{V}}^{\mathcal{W}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}}) \circ (L_{\mathcal{U}}^{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}}) = \varphi_{\mathcal{U}}^{\mathcal{W}} \circ L_{\mathcal{U}}^{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}}.\end{aligned}$$

5. Assume $\varphi_{\mathcal{U}}^{\mathcal{V}}(X) \le \varphi_{\mathcal{U}}^{\mathcal{V}}(Y)$. Then, by monotonicity and (14),

$$\varphi_{\mathcal{U}}^{\mathcal{W}}(X) = \varphi_{\mathcal{V}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(X)) \le \varphi_{\mathcal{V}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(Y)) = \varphi_{\mathcal{U}}^{\mathcal{W}}(Y).$$

∎

In Proposition 4.3, two properties, dynamic programming principle and time consistency are usually introduced as axioms ([Detlefsen and Scandolo, 2006]). But, we derive them naturally here from the fact that the monetary value measure is a contravariant functor as a proposition. This may be seen as evidence that Axioms 2.7 and 2.8 are correct in classical settings.

Before ending this section, we mention an interpretation of the Yoneda lemma in our setting.

4.4. Theorem. *[The Yoneda Lemma] For any monetary value measure $\varphi : \chi^{op} \longrightarrow \mathbf{Set}$ and an object $\mathcal{U}$ in χ, there exists a bijective correspondence $y_{\varphi,\mathcal{U}}$ specified by the following diagram:*

$$y_{\varphi,\mathcal{U}} : \mathrm{Nat}(\mathrm{Hom}_\chi(-,\mathcal{U}), \varphi) \xrightarrow{\cong} L_{\mathcal{U}}$$

$$\alpha \longmapsto \alpha_{\mathcal{U}}(*_{\mathcal{U}}^{\mathcal{U}})$$

$$\tilde{X} \longleftarrow\!\shortmid X$$

where $\tilde{X}$ is a natural transformation defined by for any $\mathcal{V} \longrightarrow \mathcal{U}$ in χ, $\tilde{X}_{\mathcal{V}}(*_{\mathcal{U}}^{\mathcal{V}}) := \varphi_{\mathcal{U}}^{\mathcal{V}}(X)$. Moreover, the correspondence is natural in both φ and $\mathcal{U}$.

It makes sense to consider the representable functor $\mathrm{Hom}_\chi(-, \mathcal{U})$ as a generalized *time domain* with time horizon $\mathcal{U}$. Then a natural transformation from $\mathrm{Hom}_\chi(-, \mathcal{U})$ to φ can be seen as a *stochastic process* that is (in a sense) adapted to φ, and its corresponding $\mathcal{F}_{\mathcal{U}}$-measurable random variable represents a terminal value (payoff) at the horizon.

The Yoneda lemma says that we have a bijective correspondence between those stochastic processes and random variables.

5. Monetary Value Measures as Sheaves

As mentioned in Section 1, one of the motivations of our research is to explore some theoretical criteria for selecting appropriate sets of axioms. Since a monetary value measure $\varphi : \chi^{op} \longrightarrow \mathbf{Set}$ is a presheaf, it is natural to consider the possibility to use a Grothendieck topology to characterize a set of monetary value measures that satisfy a given set of axioms.

Now let $\mathcal{A}$ be a given set of axioms for monetary value measures. Then, we have the largest Grothendieck topology $J_{\mathcal{A}}$ on χ such that any monetary value measure $\varphi : \chi^{op} \longrightarrow \mathbf{Set}$ satisfying $\mathcal{A}$ becomes a sheaf [1].

In the following, we write $\mathrm{Pr}(\chi)$ and $\mathrm{Sh}(\chi, J)$ for, respectively, the corresponding categories of presheaves and sheaves where J is a Grothendieck topology on χ. Then, it is well-known that $\mathrm{Sh}(\chi, J)$ is a *reflective subcategory* of $\mathrm{Pr}(\chi)$ and its *reflection* $\pi_J : \mathrm{Pr}(\chi) \longrightarrow \mathrm{Sh}(\chi, J)$ is a *left adjoint* for its inclusion functor, preserving finite limits (See Theorem 3.3.12 in [Borceux, 1994]).

$$\begin{tikzcd} \mathrm{Sh}(\chi, J) \arrow[r, hook, shift left] & \mathrm{Pr}(\chi) \arrow[l, shift left, "\pi_J"] \\[-1em] \Psi & \Psi \\[-1em] \pi_J(\varphi) & \varphi \arrow[l, mapsto] \end{tikzcd} \tag{17}$$

This fact suggests that for an arbitrary monetary value measure, the reflection functor $\pi_{J_{\mathcal{A}}}$ provides one of its closest monetary value measures that *may* satisfy the given set of axioms $\mathcal{A}$. However, in general, the resulting sheaf $\pi_{J_{\mathcal{A}}}(\varphi)$ does not satisfy $\mathcal{A}$. If there is no such case, that is, all sheaves in the form of $\pi_{J_{\mathcal{A}}}(\varphi)$ satisfies $\mathcal{A}$, then we call the set of axioms $\mathcal{A}$ complete. In other words, the set of axioms $\mathcal{A}$ is complete if it has enough members to characterize itself through a corresponding Grothendieck topology. Reminding that the current ways of selecting axioms of risk measures in practice are kind of ad hoc, it would not be so nonsense to have a new regulation for banks that requires their using monetary value measures satisfy some complete set of axioms since at least it guarantees some logical consistency.

Here is a formal definition of completeness of a set of axioms.

[1] For the existence of a largest such Grothendieck topology, see Example 3.2.14.d in [Borceux, 1994].

5.1. Definition. *Let $\mathcal{A}$ be a set of axioms for monetary value measures.*

1. *$\mathcal{M}[\mathcal{A}]$ is the full and faithful subcategory of* $\mathrm{Pr}(\chi)$ *whose objects are all monetary value measures satisfying $\mathcal{A}$.*
2. *$\mathcal{M}_0 := \mathcal{M}[\emptyset]$, that is, the category of all monetary value measures.*
3. *$\mathcal{A}$ is called **complete** if there exists a functor $\eta_{\mathcal{A}} : \mathcal{M}_0 \longrightarrow \mathcal{M}[\mathcal{A}]$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, tail] \arrow[d, "\eta_{\mathcal{A}}"'] & \mathrm{Pr}(\chi) \arrow[d, "\pi_{J_{\mathcal{A}}}"] \\ \mathcal{M}[\mathcal{A}] \arrow[r, tail] & \mathrm{Sh}(\chi, J_{\mathcal{A}}) \end{tikzcd} \tag{18}$$

Note that the existence of the inclusion functor in the bottom of Diagram (18) is guaranteed by the definition of $J_{\mathcal{A}}$. Also the functor $\eta_{\mathcal{A}}$ in Diagram (18) is actually a restriction of $\pi_{J_{\mathcal{A}}}$ to $\mathcal{M}_0$. So, we have the following main result.

5.2. Theorem. *Let $\mathcal{A}$ be a complete set of axioms. Then, for a monetary value measure φ, $\pi_{J_{\mathcal{A}}}(\varphi)$ is the monetary value measure that is the best approximation satisfying axioms $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is complete, for every $\varphi \in \mathcal{M}_0$ we have $\pi_{J_{\mathcal{A}}}(\varphi) = \eta_{\mathcal{A}}(\varphi) \in \mathcal{M}[\mathcal{A}]$. Therefore, $\pi_{J_{\mathcal{A}}}(\varphi)$ is a monetary value measure satisfying $\mathcal{A}$. ∎

Theorem 5.2 is especially important for practitioners since it is sometimes difficult to check whether a monetary value measure at hand is adequate and *safe* to use, in other words, whether it satisfies the given set of axioms. But, Theorem 5.2 tells us that they can get a *closest* safe monetary value measure by remedying the original monetary value measure through the functor $\pi_{J_{\mathcal{A}}}$, in case $\mathcal{A}$ is complete.

We expect that some of the well-known sets of axioms such as those for concave monetary value measures are complete. If we restrict the category χ to the category that is not allowed to vary its probability measures, i.e. no ambiguity version, then we have an example for a quite small Ω with which the axiom set of concave monetary value measures is not complete [Adachi, 2012]. However, we have no significant result so far for the current version of χ that accepts ambiguity.

6. Conclusion

We introduced a category χ that represents varying risk as well as ambiguity. We gave a generalized conditional expectation as a contravariant functor on χ, which works not only in risk direction but also in ambiguity direction.

We specified a concept of monetary value measures as a presheaf for χ. The resulting monetary value measures satisfy naturally so-called time consistency condition as well as dynamic programming principle.

Finally, we discussed a possibility of applying the topology-as-axioms paradigm for getting the best approximation of the monetary value measure that satisfies given axioms from a monetary value measure at hand, which works in case the axioms are complete.

In future work, we will try to formulate a robust representation of concave monetary value measures by using a representation of ambiguity within the category χ, which was originally presented in [Artzner et al., 1999]. We also seek the possibility of representing each individual axiom of monetary value measures as a specific Grothendieck topology which may give us an insight about different aspects of the axioms of monetary value measures, as well as investigating the completeness condition against the important sets of axioms such as those of concave monetary value measures. We may be able to propose a new set of axioms that is complete as a foundation of *safe* monetary value measure theory.

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