

THE WEAKLY GLOBULAR DOUBLE CATEGORY OF FRACTIONS OF A CATEGORY

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ABSTRACT. This paper introduces the construction of a weakly globular double category of fractions for a category and studies its universal properties. It shows that this double category is locally small and considers a couple of concrete examples.

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1. Introduction

Localization of categories or bicategories is the process of freely adding inverses or pseudo inverses to a class of arrows to turn them into isomorphisms or internal equivalences respectively. This has turned out to be important in homotopy theory where one only considers maps up to homotopy. In doing this we need to add inverses to those maps that are equivalences up to homotopy, but don't have a continuous inverse. And in geometry, it can be useful to view some well-known categories as localizations of simpler categories. For instance, when objects are defined using an atlas of local charts, we can define maps between these objects as (smooth) maps defined on charts, also called 'atlas maps', but we need to add inverses to those atlas maps that correspond to the identity on the underlying objects and may not be invertible as atlas maps. As a further example, the homotopy theory of orbifolds and stacks is best studied in terms of groupoid representations [22, 1, 23], but in order to obtain the correct notion of maps between orbifolds we need to invert the Morita equivalences between groupoids (which in general do not have inverse maps as smooth/continuous groupoid homomorphisms).

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Localizing a category can be done in various ways. The simplest way to construct a localization is to simply add the required inverse arrows to the category and take the free category of paths modulo the required inversions plus the old composition structure. However, this has the draw-back that the equivalence relation is rather unwieldy and one does not know a priori how long the paths need to be in order to have a representative for all arrows in the localized category. The first way to deal with this is to consider classes of arrows that satisfy the conditions of a left or right category of fractions, as defined by Gabriel and Zisman [12], or its bicategorical generalization, the bicategory of fractions [26]. In this situation we only need to consider spans (or cospans) of arrows where the left-hand (right-hand) side of the span (cospan) is an arrow that needed to be inverted. The equivalence relation becomes a lot easier to describe as well in this case. However, there is still an issue left: we do not have a guarantee that the resulting category of fractions will have small hom-sets, unless the original category had only a set of objects.

In practice, we see that besides the canonical category of fractions there are often other concrete models of the localized category that clearly do have small hom-sets, and this guarantees then that the category of fractions has small hom-sets as well. For instance, we can view the bicategory of orbifolds as the bicategory of fractions of the category of orbifold groupoids with respect to essential equivalences. However, orbifolds can also be viewed as étendues (a special kind of toposes) with a proper diagonal, and orbifold morphisms correspond then to geometric morphisms between étendues. For more details on this, see [26]. Another example is that of 2-groups or crossed modules, where the arrows in the homotopy category with respect to weak equivalences can be described in terms of so-called butterflies [23].

As a way to systematically deal with the size-problem Quillen model structures have been introduced [29]. However, one does not always have the extra structure of fibrations and cofibrations to be able to use this theory. Also, generalizing the theory of Quillen model structures to higher categories is highly non-trivial. An attempt that provides the fibrant half of a Quillen model structure for a bicategory was given in [28]. To obtain a simplicially enriched localization, the hammock localization [8] was developed by Dwyer and Kan.

In this paper we want to take a different approach, and use an alternate model for a weak 2-categorical structure to study localizations in terms of categories of fractions. In [24] we introduced the notion of weakly globular double category to obtain a new model for weak 2-categories that is a subcategory of the category of double categories. The main idea behind the definition of weakly globular double category is that we want to weaken the globularity condition in the definition of a 2-category rather than the unit or associativity conditions. The globularity condition says that in each dimension the collection of cells is *discrete*, i.e., a set or class. One way to describe a double category is as an internal category in the category of categories, i.e. a diagram

$$\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \xrightarrow{m} \mathbb{X}_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{u} \\ \xrightarrow{t} \end{array} \mathbb{X}_0. \tag{1}$$

So a double category \mathbb{X} has a category (rather than a set) \mathbb{X}_i in each dimension. For weakly

globular double categories, we require that \mathbb{X}_0 be a posetal groupoid (or, equivalence relation) rather than a set. This is what we call our weak globularity condition. This comes then with additional requirements on \mathbb{X}_1 , which are spelled out in Definition 2.9 below. In [24] we obtained a biequivalence of 2-categories

$$\mathbf{Dbl}: \mathbf{Bicat}_{\text{icon}} \rightleftarrows \mathbf{WGDb}_{\text{ps},v} : \mathbf{Bic}.$$

(The functors involved in this biequivalence are reviewed in Sections 2.11 and 2.12 below.) Here, $\mathbf{Bicat}_{\text{icon}}$ is the 2-category of bicategories, homomorphisms and icons, whereas $\mathbf{WGDb}_{\text{ps},v}$ is the 2-category of weakly globular double categories, pseudo-functors and vertical transformations.

In this paper (in Section 5) we will construct $\mathbf{C}\{W\}$, the weakly globular double category of fractions for a category \mathbf{C} with respect to a class W of arrows. We will consider the category \mathbf{C} here as a horizontal double category, which means that in the diagram (1), \mathbb{X}_0 is the discrete category on the objects of \mathbf{C} and \mathbb{X}_1 is the discrete category on the arrows of \mathbf{C} . We denote this double category by $H\mathbf{C}$. The weakly globular double category $\mathbf{C}\{W\}$ has small hom-sets of horizontal arrows, vertical arrows and cells, so there are no size-issues here. This can be explained by the fact that this construction works by adding additional objects to the category that represent the arrows in W and the fact that in some sense the new inverses are being added in the vertical direction. For instance, for the category of atlases and atlas maps, the new objects correspond to atlas refinements $\mathcal{U}' \xrightarrow{\nu} \mathcal{U}$. The horizontal arrows are given by atlas maps between the domains of these refinements and there is a vertical map between two refinements of the same atlas if they have a common refinement. There is a double cell precisely when two atlas maps become the same when restricted to some common refinements for the domain and codomain.

Aside from this size property, what makes the weakly globular double category of fractions interesting is its universal property. For the category of fractions $\mathbf{C}[W^{-1}]$, composition with the inclusion functor $I_W: \mathbf{C} \rightarrow \mathbf{C}[W^{-1}]$ gives an equivalence of categories,

$$\mathbf{Cat}(\mathbf{C}[W^{-1}], \mathbf{D}) \simeq \mathbf{Cat}_W(\mathbf{C}, \mathbf{D}),$$

where $\mathbf{Cat}_W(\mathbf{C}, \mathbf{D})$ is the category of functors and natural transformations that send arrows in W to isomorphisms. This determines the category of fractions up to an equivalence of categories.

For the bicategory of fractions $\mathbf{C}(W^{-1})$, we want an equivalence of categories or 2-categories between appropriate hom categories or hom 2-categories. Note that the hom-category $\mathbf{Bicat}(\mathcal{B}, \mathcal{C})$ of two bicategories \mathcal{B} and \mathcal{C} can be viewed as a 2-category in various ways, depending on the type of morphisms and transformations one is interested in. In particular, one may take homomorphisms of bicategories (which preserve units and composition both up to coherent isomorphisms) as objects and (op)lax transformations as morphisms. When one considers these transformations as homomorphisms into the cylinder bicategory on \mathcal{C} , this leads the way to the next level of cells: the modifications which can be viewed as transformations between the first-level transformations when considered

as homomorphisms. One can also restrict oneself to pseudo natural transformations, or to icons. And one can restrict the objects to only consider strict functors or relax them to consider lax or oplax functors.

For a bicategory \mathcal{C} with a class of arrows W satisfying the conditions to form a bicategory of fractions, the universal property of the bicategory of fractions $\mathbf{C}(W^{-1})$ is stated as follows.

1.1. THEOREM. *Composition with the homomorphism $J_W: \mathbf{C} \rightarrow \mathbf{C}(W^{-1})$ induces an equivalence of bicategories*

$$\text{Bicat}(\mathbf{C}(W^{-1}), \mathcal{D}) \simeq \text{Bicat}_W(\mathbf{C}, \mathcal{D}),$$

for any bicategory \mathcal{D} .

In [26] this universal property was given in terms of homomorphisms and oplax transformations, but it can be restricted to homomorphisms with pseudo transformations or to homomorphisms with icons. There is also an analogous result for homomorphisms with lax transformations.

For weakly globular double categories \mathbf{C} and \mathbb{D} , the hom object $\text{WGDbI}(\mathbf{C}, \mathbb{D})$ can be given the structure of a category in two ways; namely, by using horizontal or vertical transformations. We show that $\mathbf{C}\{W\}$ has a universal property with respect to both structures. The inclusion of weakly globular double categories, $\mathcal{J}_W: HC \rightarrow \mathbf{C}\{W\}$, induces two equivalences of categories,

$$\text{WGDbI}_{\text{ps},v}(\mathbf{C}\{W\}, \mathbb{D}) \simeq \text{WGDbI}_{\text{ps},v,W}(HC, \mathbb{D})$$

and

$$\text{WGDbI}_{\text{st},h}(\mathbf{C}\{W\}, \mathbb{D}) \simeq \text{WGDbI}_{\text{st},h,W}(HC, \mathbb{D}).$$

Note that the first equivalence, also called the *vertical universal property*, is with respect to pseudo-functors, whereas the second universal property, also called the *horizontal universal property*, is with respect to strict functors. The vertical universal property determines $\mathbf{C}\{W\}$ up to vertical equivalence and the horizontal universal property determines $\mathbf{C}\{W\}$ up to horizontal equivalence. The two notions of equivalence are generally unrelated, so it is important to have both.

One of the main purposes of this paper is to establish what the objects and arrows of the categories $\text{WGDbI}_{\text{ps},v,W}(HC, \mathbb{D})$ and $\text{WGDbI}_{\text{st},h,W}(HC, \mathbb{D})$ are. One way to do this is to consider the image of $J_W: \mathbf{C} \rightarrow \mathbf{C}(W^{-1})$ under the 2-functor \mathbf{DbI} . However, this functor is only defined on icons, not on any other transformations (since it needs to have a 2-category of bicategories as its domain). So we need apply it to the universal property expressed in terms of icons. This translates then into a universal property for weakly globular double categories in terms of pseudo-functors and vertical transformations. Composition with $\mathbf{DbI}(J_W)$ gives rise to an equivalence of categories,

$$\text{WGDbI}_{\text{ps},v}(\mathbf{DbI}(\mathbf{C}(W^{-1})), \mathbf{DbI} \mathcal{D}) \simeq \text{WGDbI}_{\text{ps},v,W}(\mathbf{DbI}(\mathbf{C}), \mathbf{DbI} \mathcal{D}).$$

Note that in this equation, $\mathbf{WGDb}_{\text{ps},v,W}(\mathbf{Db}\mathbf{l}(\mathbf{C}), \mathbf{Db}\mathbf{l}\mathcal{D})$ is really a different expression for

$$\mathbf{Db}\mathbf{l}(\mathbf{Bicat}_{\text{icon},W}(\mathbf{C}, \mathcal{D})).$$

In order to find a description of $\mathbf{WGDb}_{\text{ps},v,W}(\mathbf{Db}\mathbf{l}\mathbf{C}, \mathbf{Db}\mathbf{l}\mathcal{D}) \subseteq \mathbf{WGDb}_{\text{ps},v}(\mathbf{Db}\mathbf{l}\mathbf{C}, \mathbf{Db}\mathbf{l}\mathcal{D})$ we need to describe the special properties of horizontal arrows in weakly globular double categories that correspond to internal equivalences in bicategories (since this notion is used to describe the universal property of a bicategory of fractions). To find this we first study for a bicategory \mathcal{B} what type of horizontal arrows in the weakly globular double category $\mathbf{Db}\mathbf{l}(\mathcal{B})$ correspond to quasi units in \mathcal{B} (where we call an arrow $f: A \rightarrow A$ in a bicategory a quasi unit when $f \cong 1_A$). We show that these are precisely the horizontal arrows in $\mathbf{Db}\mathbf{l}(\mathcal{B})$ that have a vertical companion as defined in [15]. Furthermore, we also show that for a weakly globular double category \mathbb{X} , horizontal arrows that have vertical companions in \mathbb{X} correspond to quasi units in $\mathbf{Bic}(\mathbb{X})$.

Corresponding to the bicategorical notion of internal equivalence, we define the notion of *precompanion* for horizontal arrows in a weakly globular double category. And we prove that the objects of $\mathbf{WGDb}_{\text{ps},v,W}(\mathbf{Db}\mathbf{l}\mathbf{C}, \mathbf{Db}\mathbf{l}\mathcal{D})$ are indeed the pseudo-functors that send the arrows in the image of W to precompanions. The arrows in this category are the so-called W -transformations that respect the precompanion structure in an appropriate way (see Definition 4.11). Since every weakly globular double category is vertically equivalent to one of the form $\mathbf{Db}\mathbf{l}(\mathcal{B})$ for a bicategory \mathcal{B} , the notion of W -pseudo-functor and W -transformation can be extended to pseudo-functors with an arbitrary weakly globular double category as domain, and the weakly globular double category $\mathbf{Db}\mathbf{l}(\mathbf{C}(W^{-1}))$ has the following (vertical) universal property.

1.2. THEOREM. *Composition with the functor $\mathbf{Db}\mathbf{l}(J): \mathbf{Db}\mathbf{l}(\mathbf{C}) \rightarrow \mathbf{Db}\mathbf{l}(\mathbf{C}(W^{-1}))$ induces an equivalence of categories*

$$\mathbf{WGDb}_{\text{ps},v}(\mathbf{Db}\mathbf{l}(\mathbf{C}(W^{-1})), \mathbb{D}) \simeq \mathbf{WGDb}_{\text{ps},v,W}(\mathbf{Db}\mathbf{l}(\mathbf{C}), \mathbb{D}),$$

where the objects of $\mathbf{WGDb}_{\text{ps},v,W}(\mathbf{Db}\mathbf{l}(\mathbf{C}), \mathbb{D})$ are pseudo-functors that send arrows related to those in W (as specified in Definition 4.7 below) to precompanions.

So this seems to be the right vertical universal property for a weakly globular double category of fractions. However, this is as far as $\mathbf{Db}\mathbf{l}(J_W): \mathbf{Db}\mathbf{l}(\mathbf{C}) \rightarrow \mathbf{Db}\mathbf{l}(\mathbf{C}(W^{-1}))$ can help us. This functor is strict, but it has no obvious universal property in terms of horizontal transformations. This is not surprising, since $\mathbf{Db}\mathbf{l}$ is part of an equivalence that involves only the pseudo-functors and vertical transformations. So the functor $\mathbf{Db}\mathbf{l}(J_W): \mathbf{Db}\mathbf{l}(\mathbf{C}) \rightarrow \mathbf{Db}\mathbf{l}(\mathbf{C}(W^{-1}))$ cannot be our prototypical example of a universal arrow into a weakly globular double category of fractions. Now we could try to just adjust its codomain - as long as the new codomain is vertically equivalent to $\mathbf{Db}\mathbf{l}(\mathbf{C}(W^{-1}))$ it will have the correct vertical universal property. However, as an example of a weakly globular double category with a class of horizontal arrows W , $\mathbf{Db}\mathbf{l}(\mathbf{C})$ does not look general enough (the arrows corresponding to those in W occur in many different guises, namely

as composites of paths that are not necessarily in W , and we do need to take all of them into account when constructing the category of fractions).

Now note that there is another embedding of categories into weakly globular double categories (which does not extend to all bicategories), namely the horizontal embedding $H: \mathbf{Cat} \rightarrow \mathbf{WGDbI}_{st,h}$, which sends a category \mathbf{C} to the weakly globular double category $H(\mathbf{C})$ with the objects of \mathbf{C} as objects, the arrows of \mathbf{C} as horizontal arrows, only identity vertical arrows, and only vertical identity cells on the arrows of \mathbf{C} as double cells. There is a vertical weak equivalence between the weakly globular double categories $H(\mathbf{C})$ and $\mathbf{DbI}(\mathbf{C})$, with a strict double functor $\mathbf{DbI}(\mathbf{C}) \rightarrow H(\mathbf{C})$, but its vertical pseudo-inverse is only a pseudo-functor. As a consequence, the composed pseudo-functor $H(\mathbf{C}) \rightarrow \mathbf{DbI}(\mathbf{C}(W^{-1}))$ has still the correct vertical universal property for the weakly globular double category of fractions.

So we are looking for a strict double functor $\mathcal{J}_W: H(\mathbf{C}) \rightarrow \mathbf{C}\{W\}$ such that $\mathbf{C}\{W\}$ is vertically weakly equivalent to $\mathbf{DbI}(\mathbf{C}(W^{-1}))$. It turns out that this can be done by turning the arrows in W into precompanions in a very specific way. The resulting weakly globular double category has both a horizontal universal property and a vertical universal property.

1.3. THEOREM. *The double functor $\mathcal{J}_W: H(\mathbf{C}) \rightarrow \mathbf{C}\{W\}$ induces equivalences of categories*

$$\mathbf{WGDbI}_{ps,v}(\mathbf{C}\{W\}, \mathbb{D}) \simeq \mathbf{WGDbI}_{ps,v,W}(H\mathbf{C}, \mathbb{D})$$

and

$$\mathbf{WGDbI}_{st,h}(\mathbf{C}\{W\}, \mathbb{D}) \simeq \mathbf{WGDbI}_{st,h,W}(H\mathbf{C}, \mathbb{D}),$$

where $\mathbf{WGDbI}_{ps,v,W}(H\mathbf{C}, \mathbb{D})$ consists of pseudo-functors that send arrows in W to precompanions and vertical transformations that respect this structure, and $\mathbf{WGDbI}_{st,h,W}(H\mathbf{C}, \mathbb{D})$ consists of strict functors and horizontal transformations with a W -friendly structure (as described in Definitions 6.11 and 6.13).

The 2-functor H is defined for all 2-categories, so one might wonder whether we can extend this to 2-categories with a class of arrows W . This is indeed possible, but the technicalities related to the 2-cells obscure the ideas of this construction. So we will leave this case for a sequel to this paper [25]. We will then show that the additional ideas needed to handle arbitrary 2-categories are exactly what we need to define the weakly globular double category of fractions for an arbitrary weakly globular double category and a class of horizontal arrows with the appropriate properties.

The current paper is organized as follows. In Section 2 we give the background material on double categories and the equivalence between bicategories and weakly globular double categories. We also explicitly describe pseudo-functors, and vertical and horizontal transformations between weakly globular double categories. In Section 3 we describe companions and precompanions and show how they are related to quasi units and internal equivalences respectively. In Section 4 we describe the vertical universal property of $\mathbf{DbI}(\mathbf{C}(W^{-1}))$. In Section 5 we describe the weakly globular double category of fractions

$\mathbf{C}\{W\}$ and in Section 6 we give its universal properties. Section 7 consists of concluding remarks and questions for further research.

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2. Double categories and bicategories

In this section we review the definition of weakly globular double category and the concepts and notation related to double categories and bicategories used in this paper. We begin by introducing our notation for double categories.

2.1. DOUBLE CATEGORIES. A double category \mathbb{X} is an internal category in \mathbf{Cat} , i.e., a diagram of the form

$$\mathbb{X} = \left(\mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \xrightarrow{-m\rightarrow} \mathbb{X}_1 \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{s} \\ \xrightarrow{d_1} \end{array} \mathbb{X}_0 \right). \quad (2)$$

The elements of \mathbb{X}_{00} , i.e., the objects of the category \mathbb{X}_0 , are the *objects* of the double category. The elements of \mathbb{X}_{01} , i.e., the arrows of the category \mathbb{X}_0 , are the *vertical arrows* of the double category. Their (vertical) domains, codomains, identities, and composition in the double category \mathbb{X} are as in the category \mathbb{X}_0 . For objects $A, B \in \mathbb{X}_{00}$, we write

$$\mathbb{X}_v(A, B) = \mathbb{X}_0(A, B)$$

for the set of vertical arrows from A to B . We denote a vertical identity arrow by 1_A and write \cdot for vertical composition. The elements of \mathbb{X}_{10} , i.e., the objects of the category \mathbb{X}_1 , are the *horizontal arrows* of the double category, and their domain, codomain, identities and composition are determined by d_0 , d_1 , s , and m in (2). For objects $A, B \in \mathbb{X}_{00}$, we write

$$\mathbb{X}_h(A, B) = \{f \in \mathbb{X}_{10} \mid d_0(f) = A \text{ and } d_1(f) = B\}.$$

We use \circ for the composition of horizontal arrows, $g \circ f = m(g, f)$. In order to make a notational distinction between horizontal and vertical arrows, we denote the vertical arrows by \dashrightarrow and the horizontal arrows by \longrightarrow . The elements of \mathbb{X}_{11} , i.e., the arrows of the category \mathbb{X}_1 , are the *double cells* of the double category. An element $\alpha \in \mathbb{X}_{11}$ has a vertical domain and codomain in \mathbb{X}_{10} (since \mathbb{X}_1 is a category), which are horizontal arrows, say h and k respectively. The cell α also has a horizontal domain, $d_0(\alpha)$, and a horizontal codomain, $d_1(\alpha)$, which are elements of \mathbb{X}_{01} , i.e., vertical arrows. Furthermore,

the horizontal and vertical domains and codomains of these arrows match up in such a way that all this data fits together in a diagram

$$\begin{array}{ccc}
 & \xrightarrow{h} & \\
 d_0(\alpha) \bullet & \alpha & \bullet d_1(\alpha) \\
 & \xrightarrow{k} &
 \end{array}
 .$$

These double cells can be composed vertically by composition in \mathbb{X}_1 (again written as \cdot) and horizontally by using m , and written as $\alpha_1 \circ \alpha_2 = m(\alpha_1, \alpha_2)$. The identities in \mathbb{X}_1 give us vertical identity cells, denoted by

$$\begin{array}{ccc}
 A \xrightarrow{f} B & & A \xrightarrow{f} B \\
 1_A \downarrow \quad 1_f \downarrow 1_B & \text{OR} & \parallel \quad 1_f \quad \parallel \\
 A \xrightarrow{f} B & & A \xrightarrow{f} B .
 \end{array}$$

The image of s gives us horizontal identity cells, denoted by

$$\begin{array}{ccc}
 A \xrightarrow{\text{Id}_A} A & & A \equiv A \\
 v \downarrow \quad \text{id}_v \downarrow v & \text{OR} & v \downarrow \quad \text{id}_v \quad v \\
 B \xrightarrow{\text{Id}_B} B & & B \equiv B ,
 \end{array}$$

where $\text{Id}_A = s(A)$ and $\text{id}_v = s(v)$. Composition of squares satisfies horizontal and vertical associativity laws and functoriality of m is equivalent to the middle four interchange law. Further, $\text{id}_{1_A} = 1_{\text{Id}_A}$ and we will denote this cell by ι_A ,

$$\begin{array}{ccc}
 A \xrightarrow{\text{Id}_A} A & & \\
 1_A \downarrow \quad \iota_A \downarrow 1_A & & \\
 A \xrightarrow{\text{Id}_A} A & & .
 \end{array}$$

For any double category \mathbb{X} , the *horizontal nerve* $N_h\mathbb{X}$ is defined to be the functor $N_h\mathbb{X}: \Delta^{\text{op}} \rightarrow \text{Cat}$ such that $(N_h\mathbb{X})_0 = \mathbb{X}_0$, $(N_h\mathbb{X})_1 = \mathbb{X}_1$ and $(N_h\mathbb{X})_k = \mathbb{X}_1 \times_{\mathbb{X}_0} \cdots \times_{\mathbb{X}_0} \mathbb{X}_1$ for $k \geq 2$. So $N_h\mathbb{X}$ is given by the diagram

$$\cdots \quad \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 \xrightleftharpoons[\pi_2]{\pi_1} \mathbb{X}_1 \xrightleftharpoons[d_1]{d_0} \mathbb{X}_0$$

The objects at level k are composable paths of horizontal arrows of length k and the arrows at level k are horizontally composable paths of double cells,

$$\begin{array}{ccccccc}
 A_0 & \xrightarrow{h_1} & A_1 & \xrightarrow{h_2} & A_2 & \cdots & A_{k-1} & \xrightarrow{h_k} & A_k \\
 v_0 \downarrow & & v_1 \downarrow & & v_2 \downarrow & & & & v_k \downarrow \\
 \alpha_1 & & \alpha_2 & & & & \alpha_k & & \\
 A'_0 & \xrightarrow{h'_1} & A'_1 & \xrightarrow{h'_2} & A'_2 & \cdots & A'_{k-1} & \xrightarrow{h'_k} & A'_k .
 \end{array}$$

2.2. PSEUDO-FUNCTORS AND STRICT FUNCTORS. As maps between (weakly globular) double categories we consider those functors that correspond to natural transformations between their horizontal nerves.

2.3. DEFINITION.

1. A *strict functor* $F: \mathbb{X} \rightarrow \mathbb{Y}$ between double categories is given by a natural transformation

$$F: N_h \mathbb{X} \Rightarrow N_h \mathbb{Y}: \Delta^{\text{op}} \rightarrow \text{Cat}.$$

2. A *pseudo-functor* $F: \mathbb{X} \rightarrow \mathbb{Y}$ between double categories is given by a pseudo-natural transformation

$$F: N_h \mathbb{X} \Rightarrow N_h \mathbb{Y}: \Delta^{\text{op}} \rightarrow \text{Cat}.$$

So strict functors send objects to objects, horizontal arrows to horizontal arrows, vertical arrows to vertical arrows, and double cells to double cells, and preserve domains, codomains, identities and horizontal and vertical composition strictly. Pseudo-functors preserve all of these up to coherent isomorphisms. Note that this notion of pseudo-functor between double categories is different from what is described in [7], for instance. The pseudo-functors of [7] preserve domains and codomains strictly.

A pseudo-functor $(F, (\varphi_i), \sigma, \mu, (\theta_i)): \mathbb{X} \rightarrow \mathbb{Y}$ consists of functors $F_0: \mathbb{X}_0 \rightarrow \mathbb{Y}_0, F_1: \mathbb{X}_1 \rightarrow \mathbb{Y}_1$ and $F_k: \mathbb{X}_1 \times_{\mathbb{X}_0} \cdots \times_{\mathbb{X}_0} \mathbb{X}_1 \rightarrow \mathbb{Y}_1 \times_{\mathbb{Y}_0} \cdots \times_{\mathbb{Y}_0} \mathbb{Y}_1$ for $k \geq 2$, together with invertible natural transformations,

$$\begin{array}{ccc}
 \mathbb{X}_1 & \xrightarrow{F_1} & \mathbb{Y}_1 \\
 d_i \downarrow & \varphi_i \Downarrow & \downarrow d_i \\
 \mathbb{X}_0 & \xrightarrow{F_0} & \mathbb{Y}_0
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{X}_0 & \xrightarrow{F_0} & \mathbb{Y}_0 \\
 s \downarrow & \sigma \Downarrow & \downarrow s \\
 \mathbb{X}_1 & \xrightarrow{F_1} & \mathbb{Y}_1
 \end{array}
 \tag{3}$$

$$\begin{array}{ccc}
 \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 & \xrightarrow{F_2} & \mathbb{Y}_1 \times_{\mathbb{Y}_0} \mathbb{Y}_1 \\
 m \downarrow & \mu \Downarrow & \downarrow m \\
 \mathbb{X}_1 & \xrightarrow{F_1} & \mathbb{Y}_1
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{X}_1 \times_{\mathbb{X}_0} \mathbb{X}_1 & \xrightarrow{F_2} & \mathbb{Y}_1 \times_{\mathbb{Y}_0} \mathbb{Y}_1 \\
 \pi_i \downarrow & \theta_i \Downarrow & \downarrow \pi_i \\
 \mathbb{X}_1 & \xrightarrow{F_1} & \mathbb{Y}_1,
 \end{array}$$

(where $i = 1, 2$), and analogously for F_k with $k \geq 2$. These satisfy the usual naturality and coherence conditions, that can be derived from their simplicial description. We will list the ones that we will use in the remainder of this paper. For instance,

$$\begin{array}{ccc}
 \mathbb{X}_0 & \xrightarrow{F_0} & \mathbb{Y}_0 \\
 s \downarrow & \sigma \Downarrow & \downarrow s \\
 \mathbb{X}_1 & \xrightarrow{F_1} & \mathbb{Y}_1 \\
 d_i \downarrow & \varphi_i \Downarrow & \downarrow d_i \\
 \mathbb{X}_0 & \xrightarrow{F_0} & \mathbb{Y}_0
 \end{array}
 =
 \begin{array}{ccc}
 \mathbb{X}_0 & \xrightarrow{F_0} & \mathbb{Y}_0 \\
 \parallel & & \parallel \\
 \mathbb{X}_0 & \xrightarrow{1_{F_0}} & \mathbb{Y}_0 \\
 \parallel & & \parallel \\
 \mathbb{X}_0 & \xrightarrow{F_0} & \mathbb{Y}_0
 \end{array}
 \qquad \text{for } i = 1, 2.
 \tag{4}$$

This means that vertical composition and domains and codomains are preserved strictly, but horizontal composition and domains and codomains are only preserved up to a vertical isomorphism. For the pseudo-morphisms we consider in this paper the typical image of a horizontal arrow $A \xrightarrow{f} B$ under F corresponds to the following diagram:

$$\begin{array}{ccc} d_0 F_1 f & \xrightarrow{F_1 f} & d_1 F_1 f \\ (\varphi_0)_f \downarrow \sim & & \sim \downarrow (\varphi_1)_f \\ F_0 A & & F_0 B \end{array}$$

where $(\varphi_0)_f$ and $(\varphi_1)_f$ are components of φ_0 and φ_1 given in (3). The equation (4) means that the components of σ have the following shape:

$$\begin{array}{ccc} F_0 A & \xrightarrow{\text{Id}_{F_0 A}} & F_0 A \\ (\varphi_0)_{\text{Id}_A}^{-1} \downarrow & \sigma_A & \downarrow (\varphi_1)_{\text{Id}_A}^{-1} \\ d_0 F_1(\text{Id}_A) & \xrightarrow{F_1(\text{Id}_A)} & d_1 F_1(\text{Id}_A) \end{array}$$

Naturality of μ means that for any pair of horizontally composable double cells in \mathbb{X} ,

$$\begin{array}{ccccc} A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 \\ u \downarrow & \alpha & v_1 \downarrow & \beta & \downarrow w \\ A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2, \end{array}$$

the following equation holds:

$$\begin{array}{c} \begin{array}{ccc} \xrightarrow{F_1(g_1 \circ f_1)} & & \\ \sim \downarrow & \xrightarrow{\mu_{g_1, f_1}} & \downarrow \sim \\ \xrightarrow{\pi_2 F_2(g_1, f_1)} & & \xrightarrow{\pi_1 F_2(g_1, f_1)} \\ \sim \downarrow & & \downarrow \sim \\ (\theta_2)_{g_1, f_1} & & (\theta_1)_{g_1, f_1} \\ \sim \downarrow & & \downarrow \sim \\ \xrightarrow{F_1 f_1} & & \xrightarrow{F_1 g_1} \\ \sim \downarrow & & \downarrow \sim \\ F_1 \alpha & = & F_1 \beta \\ \sim \downarrow & & \downarrow \sim \\ \xrightarrow{F_1 f_2} & & \xrightarrow{F_1 g_2} \\ \sim \downarrow & & \downarrow \sim \\ (\theta_2)_{g_2, f_2}^{-1} & & (\theta_1)_{g_2, f_2}^{-1} \\ \sim \downarrow & & \downarrow \sim \\ \xrightarrow{\pi_2 F_2(g_2, f_2)} & & \xrightarrow{\pi_1 F_2(g_2, f_2)} \\ \sim \downarrow & \xrightarrow{\mu_{g_2, f_2}^{-1}} & \downarrow \sim \\ \xrightarrow{F_1(g_2 \circ f_2)} & & \end{array} & = & \begin{array}{ccc} \xrightarrow{F_1(g_1 \circ f_1)} & & \\ \sim \downarrow & \xrightarrow{F_1(\beta \circ \alpha)} & \downarrow \sim \\ \xrightarrow{F_1(g_2 \circ f_2)} & & \end{array} \end{array} \tag{5}$$

The identity coherence axioms for m and s state that for a horizontal arrow $A \xrightarrow{f} B$,

$$\begin{array}{ccc}
 \begin{array}{c}
 \xrightarrow{F_1(\text{Id}_B \circ f)} \\
 \downarrow \pi_2(F_2(\text{Id}_B, f)) \quad \mu_{\text{Id}_B, f} \quad \downarrow \pi_1(F_2(\text{Id}_B, f)) \\
 \begin{array}{c}
 \downarrow (\theta_2)_{\text{Id}_B, f} \\
 \downarrow F(\text{Id}_B) \\
 \downarrow \sigma_B^{-1} \\
 \downarrow \text{Id}_{F_0 B} \\
 \downarrow \text{id} \\
 \xrightarrow{F_1(f)}
 \end{array} \\
 \xrightarrow{\text{Id}_{d_1 F_1(f)}}
 \end{array}
 & = &
 \begin{array}{c}
 \xrightarrow{F_1(f)} \\
 \downarrow \text{id}_f \\
 \xrightarrow{F_1(f)}
 \end{array}
 \end{array} \tag{6}$$

and

$$\begin{array}{ccc}
 \begin{array}{c}
 \xrightarrow{F_1(f \circ \text{Id}_A)} \\
 \downarrow \pi_2(F_2(f, \text{Id})) \quad \mu_{f, \text{Id}_A} \quad \downarrow \pi_1(F_2(f, \text{Id}_A)) \\
 \begin{array}{c}
 \downarrow (\theta_2)_{f, \text{Id}_A} \\
 \downarrow F_1(\text{Id}_A) \\
 \downarrow \sigma_A^{-1} \\
 \downarrow \text{Id}_{F_0 A} \\
 \downarrow \text{id} \\
 \xrightarrow{\text{Id}_{d_0 F_1(f)}}
 \end{array} \\
 \xrightarrow{F_1(f)}
 \end{array}
 & = &
 \begin{array}{c}
 \xrightarrow{F_1(f)} \\
 \downarrow \text{id}_f \\
 \xrightarrow{F_1(f)}
 \end{array}
 \end{array} \tag{7}$$

2.4. TRANSFORMATIONS. Since double categories have two types of arrows, there are two possible choices for types of transformations between maps of double functors: vertical and horizontal transformations. Vertical transformations correspond to modifications between natural transformations of functors from Δ^{op} into Cat .

2.5. DEFINITION. A *vertical transformation* $\gamma: F \Rightarrow G: \mathbb{X} \rightrightarrows \mathbb{Y}$ between strict double functors has components vertical arrows $\gamma_A: FA \rightarrow GA$ indexed by the objects of \mathbb{X} and for each horizontal arrow $A \xrightarrow{h} B$ in \mathbb{X} , a double cell

$$\begin{array}{ccc}
 FA & \xrightarrow{Fh} & FB \\
 \gamma_A \downarrow & \gamma_h & \downarrow \gamma_B \\
 GA & \xrightarrow{Gh} & GB,
 \end{array}$$

such that γ is strictly functorial in the horizontal direction, i.e., $\gamma_{h_2 \circ h_1} = \gamma_{h_2} \circ \gamma_{h_1}$, and natural in the vertical direction, i.e.,

$$\begin{array}{ccc}
 \begin{array}{ccc}
 FA & \xrightarrow{Fh} & FB \\
 Fv \downarrow & F\zeta & \downarrow Fw \\
 FC & \xrightarrow{Fk} & FD \\
 \gamma_C \downarrow & \gamma_k & \downarrow \gamma_D \\
 GC & \xrightarrow{Gk} & GD
 \end{array} & \equiv & \begin{array}{ccc}
 FA & \xrightarrow{Fh} & FB \\
 \gamma_A \downarrow & \gamma_h & \downarrow \gamma_B \\
 GA & \xrightarrow{Gh} & GB \\
 Gv \downarrow & G\zeta & \downarrow Gw \\
 GC & \xrightarrow{Gk} & GD,
 \end{array}
 \end{array} \tag{8}$$

for any double cell ζ in \mathbb{X} .

To give a vertical transformation between pseudo-functors of double categories, we need to require that the data above fits together with the structure cells of the pseudo-functors.

2.6. DEFINITION. A vertical transformation $\gamma: F \Rightarrow G: \mathbb{X} \rightrightarrows \mathbb{Y}$ between pseudo double functors corresponds to a modification between their representations as simplicial natural transformations between the horizontal nerves of the double categories. It has components vertical arrows $\gamma_A: FA \twoheadrightarrow GA$ indexed by the objects of \mathbb{X} and for each horizontal arrow $A \xrightarrow{h} B$ in \mathbb{X} , a double cell

$$\begin{array}{ccc}
 d_0 Fh & \xrightarrow{Fh} & d_1 Fh \\
 d_0 \gamma_h \downarrow & \gamma_h & \downarrow d_1 \gamma_h \\
 d_0 Gh & \xrightarrow{Gh} & d_1 Gh,
 \end{array}$$

such that the following squares of vertical arrows commute:

$$\begin{array}{ccc}
 F_0 A \twoheadrightarrow d_0 Fh & & F_0 B \twoheadrightarrow d_1 F_1 h \\
 \gamma_A \downarrow & \downarrow d_0 \gamma_h & \downarrow d_1 \gamma_h \\
 G_0 A \twoheadrightarrow d_0 G_1 h & & G_0 B \twoheadrightarrow d_1 G_1 h
 \end{array}$$

where the unlabeled arrows are the structure isomorphisms corresponding to F and G .

We require that γ is natural in the vertical direction, in the sense that the following square of vertical arrows commutes for a vertical arrow $A \xrightarrow{v} B$ in \mathbb{X} ,

$$\begin{array}{ccc}
 F_0 A & \xrightarrow{F_1 v} & F_0 B \\
 \gamma_A \downarrow & & \downarrow \gamma_B \\
 G_0 A & \xrightarrow{G_1 v} & G_0 B
 \end{array}$$

and furthermore, for any double cell ζ in \mathbb{X} ,

$$\begin{array}{ccc}
 d_0 F_1 h \xrightarrow{Fh} d_1 F_1 h & & d_0 F_1 h \xrightarrow{Fh} d_1 F_1 h \\
 d_0 F_1 \zeta \downarrow \quad F_1 \zeta \quad \downarrow d_1 F_1 \zeta & & d_0 \gamma_h \downarrow \quad \gamma_h \quad \downarrow d_1 \gamma_h \\
 d_0 F_1 k \xrightarrow{Fk} d_1 F_1 k & \equiv & d_0 G_1 h \xrightarrow{Gh} d_1 G_1 h \\
 d_0 \gamma_k \downarrow \quad \gamma_k \quad \downarrow d_1 \gamma_k & & d_0 G_1 \zeta \downarrow \quad G_1 \zeta \quad \downarrow d_1 G_1 \zeta \\
 d_0 G_1 k \xrightarrow{Gk} d_1 G_1 k & & d_0 G_1 k \xrightarrow{Gk} d_1 G_1 k .
 \end{array}$$

In the horizontal direction we require pseudo-functoriality, which means that

$$\begin{array}{ccc}
 \begin{array}{c}
 \xrightarrow{F_1(gf)} \\
 \downarrow \bullet \\
 \xrightarrow{\pi_2 F_2(g,f)} \quad \mu_{g,f}^F \quad \xrightarrow{\pi_1 F_2(g,f)} \\
 \downarrow \bullet \quad \downarrow \bullet \\
 (\theta_2)_{g,f} \quad \downarrow \quad (\theta_1)_{g,f} \\
 \downarrow \bullet \quad \downarrow \bullet \\
 \xrightarrow{F_1 f} \quad \downarrow \quad \xrightarrow{F_1 g} \\
 \downarrow \bullet \quad \downarrow \bullet \\
 d_0 \gamma_f \quad \downarrow \quad \gamma_g \quad \downarrow \quad d_1 \gamma_g \\
 \downarrow \bullet \quad \downarrow \bullet \\
 \xrightarrow{G_1 f} \quad \downarrow \quad \xrightarrow{G_1 g} \\
 \downarrow \bullet \quad \downarrow \bullet \\
 (\theta_2)_{g,f}^{-1} \quad \downarrow \quad (\theta_1)_{g,f}^{-1} \\
 \downarrow \bullet \quad \downarrow \bullet \\
 \xrightarrow{\pi_2 G_2(g,f)} \quad (\mu_{g,f}^G)^{-1} \quad \xrightarrow{\pi_1 G_2(g,f)} \\
 \downarrow \bullet \quad \downarrow \bullet \\
 \xrightarrow{G_1(gf)}
 \end{array}
 & = &
 \begin{array}{c}
 \xrightarrow{F_1(gf)} \\
 \downarrow \bullet \\
 \xrightarrow{\quad \quad \quad} \\
 \downarrow \bullet \\
 \xrightarrow{\quad \quad \quad} \\
 \downarrow \bullet \\
 \xrightarrow{\quad \quad \quad} \\
 \downarrow \bullet \\
 \xrightarrow{G_1(gf)}
 \end{array}
 \end{array}$$

In order to state the universal property in Section 6 below, we also need to consider horizontal transformations between strict functors of double categories. The definition of a horizontal transformation is dual to that of a vertical transformation in that all mentions of vertical and horizontal have been exchanged.

2.7. DEFINITION. A horizontal transformation $a: G \Rightarrow K: \mathbb{D} \rightrightarrows \mathbb{E}$ between strict functors of (weakly globular) double categories has components horizontal arrows $GX \xrightarrow{a_X} KX$ indexed by the objects of \mathbb{D} and for each vertical arrow $X \xrightarrow{v} Y$ in \mathbb{D} , a double cell

$$\begin{array}{ccc}
 GX & \xrightarrow{a_X} & KX \\
 Gv \downarrow & a_v & \downarrow Kv \\
 GY & \xrightarrow{a_Y} & KY,
 \end{array}$$

such that a is strictly functorial in the vertical direction, i.e., $a_{v_1 \cdot v_2} = a_{v_1} \cdot a_{v_2}$ and natural in the horizontal direction, i.e., the composition of

$$\begin{array}{ccccc}
 GX & \xrightarrow{Gf} & GX' & \xrightarrow{a_{X'}} & KX' \\
 \downarrow Gv & & \downarrow Gv' & \xrightarrow{a_{v'}} & \downarrow Kv' \\
 GY & \xrightarrow{Gg} & GY' & \xrightarrow{a_{Y'}} & KY'
 \end{array}$$

is equal to the composition of

$$\begin{array}{ccccc}
 GX & \xrightarrow{a_X} & KX & \xrightarrow{Kf} & KX' \\
 \downarrow Gv & & \downarrow Kv & \xrightarrow{K\zeta} & \downarrow Kv' \\
 GY & \xrightarrow{a_Y} & KY & \xrightarrow{Kg} & KY'
 \end{array}$$

for any double cell ζ in \mathbb{D} .

We write $\mathbf{DbCat}_{\text{st},v}$, respectively $\mathbf{DbCat}_{\text{st},h}$, for the 2-categories of double categories, strict functors, and vertical transformations, respectively horizontal transformations. We will write $\mathbf{DbCat}_{\text{ps},v}$ for the 2-category of double categories, pseudo-functors, and vertical transformations of pseudo-functors.

There are 2-functors $H: \mathbf{2-Cat} \rightarrow \mathbf{DbCat}_{\text{st},h}$ and $V: \mathbf{2-Cat} \rightarrow \mathbf{DbCat}_{\text{st},v}$. H sends a 2-category \mathcal{C} to a double category HC with $HC_0 = \mathcal{C}_0$, the discrete category on the objects of \mathcal{C} and HC_1 has the arrows of \mathcal{C} as objects and the 2-cells of \mathcal{C} as arrows (and composition in HC_1 corresponds to vertical composition in \mathcal{C}). The horizontal structure of HC corresponds to (horizontal) composition in \mathcal{C} . V sends the 2-category \mathcal{C} to a double category VC with the objects of \mathcal{C} as objects, the arrows of \mathcal{C} as vertical arrows, only identity horizontal arrows, and the double cells of \mathcal{C} are the 2-cells of HC . H has a right adjoint $h: \mathbf{DbCat}_{\text{st},h} \rightarrow \mathbf{2-Cat}$ which sends a double category to its 2-category of horizontal arrows and special cells with identity arrows as vertical arrows. Analogously, V has a right adjoint $v: \mathbf{DbCat}_{\text{st},v} \rightarrow \mathbf{2-Cat}$ which send a double category to its 2-category of vertical arrows and special cells with identity arrows as horizontal arrows.

2.8. WEAKLY GLOBULAR DOUBLE CATEGORIES. In [24] we introduced the notion of weakly globular double category. Weakly globular double categories were defined in such a way that there is a biequivalence of 2-categories,

$$\mathbf{Bic} : \mathbf{WGDb}_{\text{ps},v} \simeq \mathbf{Bicat}_{\text{icon}} : \mathbf{Db}.$$

Here $\mathbf{Bicat}_{\text{icon}}$ denotes the 2-category of bicategories, homomorphisms and icons.

The adjunction $H \dashv h$ described above restricts to an adjoint equivalence between the 2-category of double categories \mathbb{D} such that \mathbb{D}_0 is discrete and the 2-category of 2-categories. And a similar result is true when we take pseudo double functors as in [7] for instance and homomorphisms of 2-categories with oplax transformations. The

requirement that the vertical category \mathbb{D}_0 be discrete can be viewed as a globularity condition: we require that the cells have vertical arrows that are identities, so they are globular in shape.

To obtain a kind of double categories that can model bicategories we choose to weaken this globularity condition to obtain the first condition in the definition below. This means that rather than having a set of objects we have a set with an equivalence relation, i.e., a category that is equivalent to a discrete one, a posetal groupoid. In order to make sure that this interacts well with the structure of the horizontal arrows, we need the second condition in the definition below. To obtain an understanding of this condition, consider some arrangement of horizontal and vertical arrows of the form

$$\begin{array}{ccccccc}
 \xrightarrow{f_1} & & \xrightarrow{f_3} & & \dots & & \xrightarrow{f_n} \\
 \downarrow v_1 & & \downarrow v_2 & & \downarrow v_{n-2} & & \downarrow v_{n-1} \\
 \xrightarrow{f_2} & & \xrightarrow{f_4} & & \dots & & \xrightarrow{f_{n-1}}
 \end{array} \tag{9}$$

We will refer to such an arrangement of alternating horizontal and vertical arrows as a *staircase path*. Since we think of the vertical arrows as just indicating that two objects are equivalent, we require that this diagram can be completed to

$$\begin{array}{ccccccc}
 \xrightarrow{f'_1} & \xrightarrow{f'_2} & \xrightarrow{f'_3} & \xrightarrow{f'_4} & \dots & \xrightarrow{f'_{n-1}} & \xrightarrow{f'_n} \\
 \downarrow u_0 & \varphi_1 & \downarrow u_1 & \varphi_2 & \downarrow u_2 & \varphi_3 & \downarrow u_3 & \varphi_4 & \dots & \downarrow u_{n-2} & \varphi_{n-1} & \downarrow u_{n-1} & \varphi_n & \downarrow u_n \\
 \xrightarrow{f_1} & & \xrightarrow{f_2} & & \dots & & \xrightarrow{f_{n-1}} & & & & & & & & \\
 \downarrow v_1 & & \downarrow v_2 & & \downarrow v_3 & & \dots & & \downarrow v_{n-2} & & \downarrow v_{n-1} & & & & \\
 \xrightarrow{f_2} & & \xrightarrow{f_3} & & \dots & & \xrightarrow{f_{n-1}} & & & & & & & &
 \end{array}$$

where all the double cells are vertically invertible. (And so we can think of the top path as the corresponding path that is horizontally composable.) Furthermore, we require that this correspondence is part of an equivalence of categories.

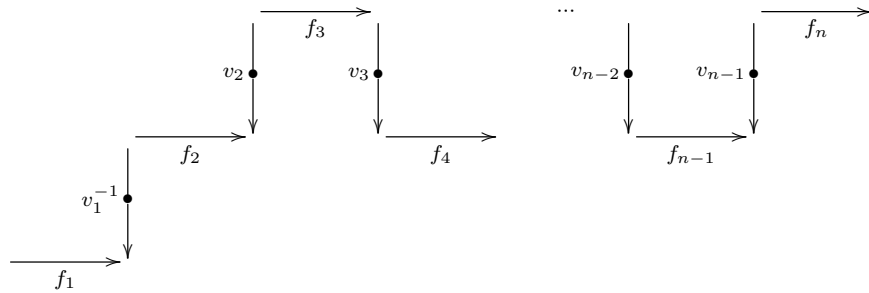
2.9. DEFINITION. A *weakly globular double category* \mathbb{X} is a double category that satisfies the following two conditions

- (the weak globularity condition) there is an equivalence of categories $\gamma : \mathbb{X}_0 \rightarrow \mathbb{X}_0^d$, where \mathbb{X}_0^d is the discrete category of the path components of \mathbb{X}_0 ;
- (the induced Segal maps condition) γ induces an equivalence of categories, for all $n \geq 2$,

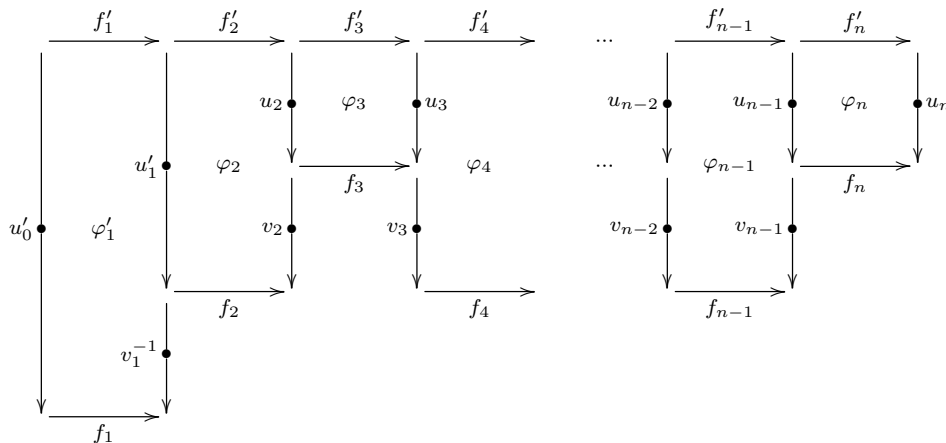
$$\mathbb{X}_1 \times_{\mathbb{X}_0} \cdots \times_{\mathbb{X}_0} \mathbb{X}_1 \simeq \mathbb{X}_1 \times_{\mathbb{X}_0^d} \cdots \times_{\mathbb{X}_0^d} \mathbb{X}_1 . \tag{10}$$

Analogously to the notation introduced above we write $\mathbf{WGDbI}_{\text{ps},v}$ for the 2-category of weakly globular double categories, pseudo-functors, and vertical transformations between them. We also write $\mathbf{WGDbI}_{\text{st},v}$ and $\mathbf{WGDbI}_{\text{st},h}$ for the 2-categories with strict functors and vertical, respectively horizontal, transformations.

2.10. REMARK. Since the vertical arrows in a weakly globular double category are unique and invertible, any object $(f_1, f_2, \dots, f_n) \in \mathbb{X}_1 \times_{\mathbb{X}_0^d} \dots \times_{\mathbb{X}_0^d} \mathbb{X}_1$ corresponds uniquely to a diagram as in (9). However, there are many other ways to represent this object as a staircase path. For instance,



The induced Segal maps condition would then give us a diagram of vertically invertible double cells that looks slightly different:



However, note that the composable path of horizontal arrows will still be the same, since this is the path assigned to (f_1, \dots, f_n) by the equivalence. But then we realize that the composites of the vertical arrows also need to be the same, so $u_0 = u'_0$ and $v_1 \cdot u_1 = v_1^{-1} \cdot u'_1$ and furthermore, $\varphi'_1 = \varphi_1$. So we conclude that the choice of cells and horizontal arrows is completely determined by the equivalence in the induced Segal maps condition, but the way they are arranged can vary depending on which vertical arrows are used in the first staircase path.

In the next two subsections, we briefly review the explicit descriptions of the 2-functors involved in the biequivalence of 2-categories

$$\mathbf{Bic} : \mathbf{WGDbI}_{\text{ps},v} \simeq \mathbf{Bicat}_{\text{icon}} : \mathbf{DbI}.$$

For more details on these functors, see also [24].

2.11. THE FUNDAMENTAL BICATEGORY. Let \mathbb{X} be a weakly globular double category. The *objects* of $\mathbf{Bic}\mathbb{X}$ are obtained as the connected components $\pi_0\mathbb{X}_0$ of the vertical arrow category \mathbb{X}_0 . When A is an object of \mathbb{X} , i.e., an element of \mathbb{X}_{00} , we write \bar{A} for the corresponding object in $\mathbf{Bic}\mathbb{X}$. Note that $\bar{A} = \bar{B}$ if and only if there is a (unique) vertical arrow $v: A \rightarrow B$ in \mathbb{X} (since the vertical arrow category \mathbb{X}_0 is posetal and groupoidal).

For any two objects \bar{A} and \bar{B} in \mathbb{X} , the *set of arrows*, $\mathbf{Bic}\mathbb{X}(\bar{A}, \bar{B})$ is obtained as a disjoint union of horizontal hom-sets in \mathbb{X} ,

$$\mathbf{Bic}\mathbb{X}(\bar{A}, \bar{B}) = \coprod_{\substack{\bar{A}' = \bar{A} \\ \bar{B}' = \bar{B}}} \mathbb{X}_h(A', B').$$

Note that we do not put an equivalence relation on the horizontal arrows of \mathbb{X} to obtain the arrows of the fundamental bicategory; we will therefore use the same symbol to denote a horizontal arrow in \mathbb{X} and the corresponding arrow in $\mathbf{Bic}(\mathbb{X})$.

For any two arrows $\bar{A} \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} \bar{B}$ in $\mathbf{Bic}\mathbb{X}$ represented by horizontal arrows $A_1 \xrightarrow{f} B_1$ and $A_2 \xrightarrow{g} B_2$ in \mathbb{X} , the *2-cells* from f to g correspond to double cells of the form

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & B_1 \\ v \downarrow & \alpha & \downarrow w \\ A_2 & \xrightarrow{g} & B_2 \end{array} .$$

Since v and w are unique, we will denote the corresponding 2-cell in $\mathbf{Bic}\mathbb{X}$ by $\alpha: f \Rightarrow g$.

Let $f: A_1 \rightarrow B_1$ and $g: B_2 \rightarrow C_2$ be horizontal arrows such that there is an invertible vertical arrow $v: B_2 \rightarrow B_1$. Then the induced Segal maps condition (see Definition 2.9) gives rise to a diagram

$$\begin{array}{ccccc} A_3 & \xrightarrow{f_3} & B_3 & \xrightarrow{g_3} & C_3 \\ \downarrow & & \downarrow y & \varphi_{g_3, g} & \downarrow z \\ x \bullet & \varphi_{f_3, f} & B_2 & \xrightarrow{g} & C_2 \\ \downarrow & & \downarrow v & & \\ A_1 & \xrightarrow{f} & B_1 & & \end{array} .$$

Then the composition of $f: \bar{A}_1 \rightarrow \bar{B}_1$ and $g: \bar{B}_2 \rightarrow \bar{C}_2$ (where $\bar{B}_1 = \bar{B}_2$) in $\mathbf{Bic}\mathbb{X}$ is defined to be the horizontal composite $g_3 \circ f_3: \bar{A}_1 = \bar{A}_3 \rightarrow \bar{C}_3 = \bar{C}_2$.

The horizontal composition of 2-cells is defined as follows: Let

$$\begin{array}{ccccc} \bar{A} & \xrightarrow{f} & \bar{B} & \xrightarrow{h} & \bar{C} \\ \downarrow \alpha & & \downarrow \beta & & \\ \bar{A} & \xrightarrow{g} & \bar{B} & \xrightarrow{k} & \bar{C} \end{array}$$

be a diagram of arrows and cells in the fundamental bicategory, represented by double cells in \mathbb{X} ,

$$\begin{array}{ccc}
 A_2 & \xrightarrow{f} & B_2 \\
 u_{21} \downarrow & \alpha & \downarrow v_{21} \\
 A_1 & \xrightarrow{g} & B_1
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 B_4 & \xrightarrow{h} & C_4 \\
 v_{43} \downarrow & \beta & \downarrow w_{43} \\
 B_3 & \xrightarrow{k} & C_3.
 \end{array}$$

Let the composite of g and k in $\mathbf{Bic}\mathbb{X}$ be the arrow $k_5 \circ g_5$ with a corresponding diagram (using the induced Segal maps condition from Definition 2.9 again),

$$\begin{array}{ccccc}
 A_5 & \xrightarrow{g_5} & B_5 & \xrightarrow{k_5} & C_5 \\
 \downarrow & & v_{53} \downarrow & \varphi_{k_5,k} & \downarrow w_{53} \\
 u_{51} \downarrow & \varphi_{g_5,g} & B_3 & \xrightarrow{k} & C_3 \\
 & & \downarrow v_{31} & & \\
 A_1 & \xrightarrow{g} & B_1 & &
 \end{array}$$

and let the composite of f and h be the arrow $h_6 \circ f_6$ as in the diagram

$$\begin{array}{ccccc}
 A_6 & \xrightarrow{f_6} & B_6 & \xrightarrow{h_6} & C_6 \\
 \downarrow & & v_{64} \downarrow & \varphi_{h_6,h} & \downarrow w_{64} \\
 u_{62} \downarrow & \varphi_{f_6,f} & B_4 & \xrightarrow{h} & C_4 \\
 & & \downarrow v_{42} & & \\
 A_2 & \xrightarrow{f} & B_2 & &
 \end{array}$$

Then the horizontal composition of α and β is represented by the following pasting of double cells:

$$\begin{array}{ccccc}
 A_6 & \xrightarrow{f_6} & B_6 & \xrightarrow{h_6} & C_6 \\
 \downarrow & & v_{64} \downarrow & \varphi_{h_6,h} & \downarrow w_{64} \\
 & & B_4 & \xrightarrow{h} & C_4 \\
 u_{62} \downarrow & \varphi_{f_6,f} & \downarrow v_{43} & \beta & \downarrow w_{43} \\
 & & B_3 & \xrightarrow{k} & C_3 \\
 & & \downarrow v_{32} & & \\
 A_2 & \xrightarrow{f} & B_2 & & \\
 u_{21} \downarrow & \alpha & v_{21} \downarrow & \varphi_{k_5,k}^{-1} & \downarrow v_{35} \\
 A_1 & \xrightarrow{g} & B_1 & & \\
 u_{15} \downarrow & \varphi_{g_5,g}^{-1} & \downarrow v_{15} & & \\
 A_5 & \xrightarrow{g_5} & B_5 & \xrightarrow{k_5} & C_5.
 \end{array}$$

(Here, $u_{ij} = u_{ji}^{-1}$ and $u_{jk} \cdot u_{ij} = u_{ik}$, and analogous for v and w , since the vertical category is groupoidal posetal. Furthermore, the same holds for the cells, because they are components of a vertical transformation.)

The units for the composition are obtained from the functor $\mu_0: \mathbb{X}_0^d \rightarrow \mathbb{X}_0$ that is part of the equivalence of categories, $\mathbb{X}_0^d \simeq \mathbb{X}_0$. For an object \bar{A} in $\mathbf{Bic}\mathbb{X}$, $1_{\bar{A}}$ is the horizontal arrow $\text{Id}_{\mu_0(\bar{A})}$.

There are associativity and unit isomorphisms for this composition that satisfy the usual coherence conditions by the results in [33] and [18].

2.12. THE WEAKLY GLOBULAR DOUBLE CATEGORY OF PATHS. Let \mathcal{B} be a bicategory. Before we begin the construction of $\mathbf{Db}\ell(\mathcal{B})$, we first choose a composite $\varphi_{f_1, \dots, f_n}$ for each finite path $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n$ of (composable) arrows in \mathcal{B} . If the path is empty, we take $\varphi_{A_0} = 1_{A_0}$. For each path of such paths,

$$\left(\xrightarrow{f_{i_1}} \dots \xrightarrow{f_{i_{n_1}}} \right) \left(\xrightarrow{f_{i_{n_1+1}}} \dots \xrightarrow{f_{i_{n_2}}} \right) \dots \left(\xrightarrow{f_{i_{n_{m-1}+1}}} \dots \xrightarrow{f_{i_{n_m}}} \right)$$

the associativity and unit cells give rise to unique invertible comparison 2-cells, which we denote by

$$\begin{array}{c} \varphi_{\varphi_{f_{i_1}, \dots, f_{i_{n_1}}}, \varphi_{f_{i_{n_1+1}}, \dots, f_{i_{n_2}}}, \dots, \varphi_{f_{i_{n_{m-1}+1}}, \dots, f_{i_{n_m}}}} \\ \Phi_{f_{i_1}, \dots, f_{i_{n_1}}, f_{i_{n_1+1}}, \dots, f_{i_{n_2}}, \dots, f_{i_{n_{m-1}+1}}, \dots, f_{i_{n_m}}} \\ \varphi_{f_{i_1}, \dots, f_{i_{n_m}}} \end{array} \quad .$$

(The uniqueness of these cells follows from the associativity and unit coherence axioms.)

The *objects* of $\mathbf{Db}\ell(\mathcal{B})$ are given as pairs of an arrow $\psi: [0] \rightarrow [n]$ in Δ with a path, $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n$, of length n in \mathcal{B} , for all n . Since the arrow ψ is determined by its image $i_0 = \psi(0) \in [n]$, we will denote this object in $\mathbf{Db}\ell(\mathcal{B})$ by

$$\left(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n ; i_0 \right)$$

and think of A_{i_0} as a marked object along the path. So we will also use the notation

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_n} A_n .$$

There is a unique *vertical arrow* from $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_n} A_n$ to $B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} \dots \xrightarrow{g_{j_0}} [B_{j_0}] \xrightarrow{g_{j_0+1}} \dots \xrightarrow{g_m} B_m$ if and only if $A_{i_0} = B_{j_0}$. In diagrams we will include this vertical arrow in the following way

$$\begin{array}{c} A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_n} A_n \\ \parallel \\ B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} \dots \xrightarrow{g_{j_0}} [B_{j_0}] \xrightarrow{g_{j_0+1}} \dots \xrightarrow{g_m} B_m \end{array}$$

Horizontal arrows in $\mathbf{Db}(\mathcal{B})$ are given as pairs of an arrow $\psi: [1] \rightarrow [n]$ in Δ with a path of length n in \mathcal{B} , for all n . Analogous to what we did for objects we denote horizontal arrows by

$$(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n; i_0, i_1) \quad \text{with} \quad i_0 \leq i_1,$$

or by

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \dots \xrightarrow{f_n} A_n.$$

The domain of $(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n; i_0, i_1)$ is $(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n; i_0)$ and the codomain is $(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n; i_1)$. For a horizontal identity arrow,

$$(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n; i_0, i_0)$$

we will also use the notation

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_{i_0}} [[A_{i_0}]] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_n} A_n$$

or

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \equiv [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_n} A_n$$

when this makes it easier to fit such an arrow into a diagram representing a double cell as shown below.

A *double cell* consists of two horizontal arrows

$$(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n; i_0, i_1) \quad \text{and} \quad (B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} \dots \xrightarrow{g_m} B_m; j_0, j_1)$$

(for the vertical domain and codomain respectively), such that $A_{i_0} = B_{j_0}$ and $A_{i_1} = B_{j_1}$ (such that there are unique vertical arrows between the domains of these arrows and between the codomains of these arrows), together with a 2-cell in \mathcal{B} between the chosen composites,

$$\begin{array}{ccc} \varphi_{f_{i_0+1}, \dots, f_{i_1}} & & \\ \curvearrowright & & \\ \Downarrow \alpha & & \\ \curvearrowleft & & \\ \varphi_{g_{j_0+1}, \dots, g_{j_1}} & & \end{array} .$$

We combine all this information together in the following diagram

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_1} & \dots & \xrightarrow{f_{i_0}} & [A_{i_0}] & \xrightarrow{f_{i_0+1}} & \dots & \xrightarrow{f_{i_1}} & [A_{i_1}] & \xrightarrow{f_{i_1+1}} & \dots & \xrightarrow{f_n} & A_n \\ & & & & \parallel & \xrightarrow{\varphi_{f_{i_0+1}, \dots, f_{i_1}}} & & & \parallel & & & & \\ & & & & & \alpha & & & & & & & \\ & & & & \parallel & \xrightarrow{\varphi_{g_{j_0+1}, \dots, g_{j_1}}} & & & \parallel & & & & \\ B_0 & \xrightarrow{g_1} & \dots & \xrightarrow{g_{j_0}} & [B_{j_0}] & \xrightarrow{g_{j_0+1}} & \dots & \xrightarrow{g_{j_1}} & [B_{j_1}] & \xrightarrow{g_{j_1+1}} & \dots & \xrightarrow{g_m} & B_m \end{array}$$

So this represents a double cell in $\mathbf{DbI}(\mathcal{B})$.

Two horizontal arrows,

$$(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n; i_0, i_1) \text{ and } (B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} \dots \xrightarrow{g_m} B_m; j_0, j_1),$$

are composable if and only if the two paths are the same, i.e., $m = n$, $A_i = B_i$ for $i = 0, \dots, n$, and $f_i = g_i$ for $i = 1, \dots, n$, and furthermore, $i_1 = j_0$. In that case, the *horizontal composition* of these arrows is given by $(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n; i_0, j_1)$.

The *horizontal composition of double cells*

$$\begin{array}{c} A_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \dots \xrightarrow{f_n} A_n \\ \parallel \quad \quad \quad \varphi_{f_{i_0+1}, \dots, f_{i_1}} \quad \quad \quad \parallel \\ \alpha \\ \varphi_{g_{j_0+1}, \dots, g_{j_1}} \\ \parallel \quad \quad \quad \varphi_{g_{j_0+1}} \xrightarrow{\dots} \varphi_{g_{j_1}} \quad \quad \quad \parallel \\ B_0 \xrightarrow{g_1} \dots \xrightarrow{g_{j_0}} [B_{j_0}] \xrightarrow{g_{j_0+1}} \dots \xrightarrow{g_{j_1}} [B_{j_1}] \xrightarrow{g_{j_1+1}} \dots \xrightarrow{g_m} B_m \end{array}$$

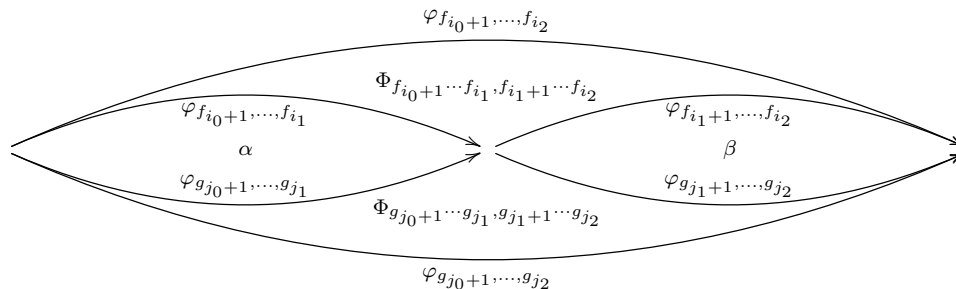
and

$$\begin{array}{c} A_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \dots \xrightarrow{f_{i_2}} [A_{i_2}] \xrightarrow{f_{i_2+1}} \dots \xrightarrow{f_n} A_n \\ \parallel \quad \quad \quad \varphi_{f_{i_1+1}, \dots, f_{i_2}} \quad \quad \quad \parallel \\ \beta \\ \varphi_{g_{j_1+1}, \dots, g_{j_2}} \\ \parallel \quad \quad \quad \varphi_{g_{j_1+1}} \xrightarrow{\dots} \varphi_{g_{j_2}} \quad \quad \quad \parallel \\ B_0 \xrightarrow{g_1} \dots \xrightarrow{g_{j_1}} [B_{j_1}] \xrightarrow{g_{j_1+1}} \dots \xrightarrow{g_{j_2}} [B_{j_2}] \xrightarrow{g_{j_2+1}} \dots \xrightarrow{g_m} B_m \end{array}$$

is defined to be

$$\begin{array}{c} A_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_{i_2}} [A_{i_2}] \xrightarrow{f_{i_2+1}} \dots \xrightarrow{f_n} A_n \\ \parallel \quad \quad \quad \varphi_{f_{i_0+1}, \dots, f_{i_2}} \quad \quad \quad \parallel \\ \alpha \otimes \beta \\ \varphi_{g_{j_0+1}, \dots, g_{j_2}} \\ \parallel \quad \quad \quad \varphi_{g_{j_0+1}} \xrightarrow{\dots} \varphi_{g_{j_2}} \quad \quad \quad \parallel \\ B_0 \xrightarrow{g_1} \dots \xrightarrow{g_{j_0}} [B_{j_0}] \xrightarrow{g_{j_0+1}} \dots \xrightarrow{g_{j_2}} [B_{j_2}] \xrightarrow{g_{j_2+1}} \dots \xrightarrow{g_m} B_m \end{array}$$

where $\alpha \otimes \beta$ is the 2-cell in \mathcal{B} given by the following pasting diagram



2.13. REMARKS.

1. Note that both the category of horizontal arrows and the category of vertical arrows of $\mathbf{DbI}(\mathcal{B})$ are posetal.
2. We call $\mathbf{DbI}(\mathcal{B})$ the *double category of marked paths in \mathcal{B}* .
3. For a 2-category \mathcal{C} , the double category $\mathbf{DbI}(\mathcal{C})$ is not isomorphic to the double category HC , but it is vertically equivalent to it. And the same is true for a category \mathbf{C} : $HC \not\cong \mathbf{DbI}(\mathbf{C})$, but $HC \simeq_2 \mathbf{DbI}(\mathbf{C})$.

3. Companions and precompanions

In general double categories, the notions of companion and conjoint have been recognized as important concepts related to the notion of adjoint. While adjoint arrows have to be of the same kind, i.e., both horizontal or both vertical, the relations of companionship and conjointship are for arrows of different types. The notions of companion and conjoint were first introduced by Ehresmann in [14], but companions in symmetric double categories (where the horizontal and vertical arrow categories are the same) were studied by Brown and Mosa [5] under the name ‘connections’. Connection pairs were first introduced by Spencer in [32]. The existence of companions and conjoints for the vertical arrows in a double category is related to Shulman’s notion of an anchored bicategory [31], also called a gregarious double category in [7]. We will show that for weakly globular double categories horizontal companions are related to so-called quasi units, i.e., arrows with an invertible 2-cell to a unit arrow, in their associated bicategories. Then we will introduce a slightly weaker notion, that of a *precompanion*. We will show that precompanions in weakly globular double categories correspond to equivalences in bicategories. In Section 6 we will show that both companions and precompanions play an important role in the description of the universal properties of a weakly globular double category of fractions.

We begin this section by repeating the definitions of companion and conjoint from [14] to set our notation, and then we will discuss their relationship to quasi units. Then we will introduce both a category and a double category of companions. And in the last part we will study precompanions and their properties.

3.1. COMPANIONS AND CONJOINTS. Recall the definitions of companion and conjoint.

3.2. DEFINITION. Let \mathbb{D} be a double category and consider horizontal morphisms $f: A \rightarrow B$ and $u: B \rightarrow A$ and a vertical morphism $v: A \rightarrow B$. We say that f and v are *companions* if there exist *binding cells*

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \parallel & \psi & \downarrow v \\
 A & \xrightarrow{f} & B
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 v \downarrow & x & \parallel \\
 B & \xlongequal{\quad} & B,
 \end{array}$$

such that

$$\begin{array}{ccc}
 \begin{array}{c} A \rightrightarrows A \xrightarrow{f} B \\ \parallel \quad \downarrow \psi \quad \downarrow v \quad \chi \quad \parallel \\ A \xrightarrow{f} B \rightrightarrows B \end{array} & = & \begin{array}{c} A \xrightarrow{f} B \\ \parallel \quad 1_f \quad \parallel \\ A \xrightarrow{f} B \end{array} \\
 \text{and} & & \begin{array}{c} A \rightrightarrows A \\ \parallel \quad \downarrow \psi \quad \downarrow v \\ A \xrightarrow{f} B \\ \downarrow v \quad \chi \quad \parallel \\ B \rightrightarrows B \end{array} = \begin{array}{c} A \rightrightarrows A \\ \downarrow v \quad \text{id}_v \quad \downarrow v \\ B \rightrightarrows B \end{array} .
 \end{array} \tag{11}$$

Dually, u and v are *conjoint*s if there exist *binding cells*

$$\begin{array}{c} A \rightrightarrows A \\ \downarrow v \quad \alpha \quad \parallel \\ B \xrightarrow{u} A \end{array} \quad \text{and} \quad \begin{array}{c} B \xrightarrow{u} A \\ \parallel \quad \beta \quad \downarrow v \\ B \rightrightarrows B \end{array} ,$$

such that

$$\begin{array}{ccc}
 \begin{array}{c} B \xrightarrow{u} A \rightrightarrows A \\ \parallel \quad \beta \quad \downarrow v \quad \alpha \quad \parallel \\ B \rightrightarrows B \xrightarrow{u} A \end{array} & = & \begin{array}{c} B \xrightarrow{u} A \\ \parallel \quad 1_u \quad \parallel \\ B \xrightarrow{u} A \end{array} \\
 \text{and} & & \begin{array}{c} A \rightrightarrows A \\ \downarrow v \quad \alpha \quad \parallel \\ B \xrightarrow{u} A \\ \parallel \quad \beta \quad \downarrow v \\ B \rightrightarrows B \end{array} = \begin{array}{c} A \rightrightarrows A \\ \downarrow v \quad \text{id}_v \quad \downarrow v \\ B \rightrightarrows B \end{array} .
 \end{array} \tag{12}$$

3.3. REMARK. Since the vertical arrow category in a weakly globular double category is a posetal groupoid, it follows from Proposition 3.5 in [7] that the inverse of a vertical arrow that is a companion is a conjoint, and the binding cells for the conjoint pair can be taken to be the vertical inverses of the binding cells for the companion pair. So a horizontal arrow in a weakly globular double category has a vertical conjoint if and only if it has a vertical companion. So it will often be sufficient to focus on the companion pairs.

3.4. UNITS. The units in a bicategory are weak units in the sense that there are invertible 2-cells

$$\lambda_f: 1_B \circ f \xrightarrow{\cong} f \quad \text{and} \quad \rho_f: f \circ 1_A \xrightarrow{\cong} f \tag{13}$$

for any arrow $f: A \rightarrow B$ (and these need to satisfy the coherence conditions). Such units are not necessarily unique: any arrow $g: A \rightarrow A$ with an invertible 2-cell $g \xrightarrow{\cong} 1_A$ would satisfy (13). In this paper we want to consider the property for an arrow to behave like a weak unit without necessarily being part of any coherent family of weak units. To stress this fact, we give them a different name.

3.5. DEFINITION. An endomorphism $f: A \rightarrow A$ in a bicategory is a *quasi unit* if it satisfies the following equivalent conditions:

1. $f \cong 1_A$;
2. $f \circ g \cong g$ for all $g: C \rightarrow A$ and $h \circ f \cong h$ for all $h: A \rightarrow B$.

We want to characterize the quasi units in the fundamental bicategory $\mathbf{Bic}\mathbb{X}$.

3.6. LEMMA. *Every arrow of the form $\text{Id}_A: \bar{A} \rightarrow \bar{A}$ is a quasi unit in $\mathbf{Bic}\mathbb{X}$.*

PROOF. Recall from the end of Section 2.11 that the identity arrow on \bar{A} is $\text{Id}_{\mu_0(\bar{A})}$ where $\mu_0: \mathbb{X}_0^d \rightarrow \mathbb{X}_0$ is part of the equivalence of categories in the weak globularity condition. There is an invertible 2-cell $\text{Id}_A \Rightarrow \text{Id}_{\mu_0(\bar{A})}$ given by the vertically invertible double cell

$$\begin{array}{ccc} A & \xrightarrow{\text{Id}_A} & A \\ x \downarrow & \text{id}_x & \downarrow x \\ \mu_0(\bar{A}) & \xrightarrow{\text{Id}_{\mu_0(\bar{A})}} & \mu_0(\bar{A}) \end{array}$$

(with vertical inverse $\text{id}_{x^{-1}}$). ■

We can now characterize the quasi units in $\mathbf{Bic}\mathbb{X}$ as those horizontal arrows in \mathbb{X} that have a companion.

3.7. PROPOSITION. *Let $w: A \rightarrow B$ be a horizontal arrow in a weakly globular double category \mathbb{X} . Then $w: \bar{A} \rightarrow \bar{B}$ is a quasi unit in $\mathbf{Bic}\mathbb{X}$ if and only if $w: A \rightarrow B$ has a companion $w_*: A \bullet \rightarrow B$ in \mathbb{X} .*

PROOF. If w is a quasi unit, then $\bar{A} = \bar{B}$ and there is an invertible 2-cell $\zeta: w \Rightarrow 1_{\bar{A}}$ in $\mathbf{Bic}\mathbb{X}$, given by a vertically invertible double cell in \mathbb{X} ,

$$\begin{array}{ccc} A & \xrightarrow{w} & B \\ x \downarrow & \zeta & \downarrow y \\ \theta(\bar{A}) & \xrightarrow{\text{Id}_{\theta(\bar{A})}} & \theta(\bar{A}) \end{array}$$

We can compose this with the horizontal identity cell on y^{-1} to obtain a double cell

$$\begin{array}{ccc} A & \xrightarrow{w} & B \\ x \downarrow & \zeta & \downarrow y \\ \theta(\bar{A}) & \xrightarrow{\text{Id}_{\theta(\bar{A})}} & \theta(\bar{A}) \\ y^{-1} \downarrow & \text{id}_{y^{-1}} & \downarrow y^{-1} \\ B & \xrightarrow{\text{Id}_B} & B \end{array} = \begin{array}{ccc} A & \xrightarrow{w} & B \\ y^{-1} \cdot x \downarrow & \text{id}_{y^{-1}} \cdot \zeta & \downarrow 1_B \\ B & \xrightarrow{\text{Id}_B} & B \end{array}$$

Furthermore, precomposing ζ^{-1} (the vertical inverse of ζ) with the horizontal identity cell on x gives

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xrightarrow{\text{Id}_A} & A \\
 \downarrow x & \text{id}_x & \downarrow x \\
 \theta(\bar{A}) & \xrightarrow{\text{Id}_{\theta(\bar{A})}} & \theta(\bar{A}) \\
 \downarrow x^{-1} & \zeta^{-1} & \downarrow y^{-1} \\
 A & \xrightarrow{w} & B
 \end{array} & = & \begin{array}{ccc}
 A & \xrightarrow{\text{Id}_A} & A \\
 \downarrow 1_A & \zeta^{-1} \cdot \text{id}_x & \downarrow y^{-1} \cdot x \\
 A & \xrightarrow{w} & B
 \end{array}
 \end{array}$$

So we can define $w_* = y^{-1} \cdot x$ with binding cells $\chi_w = \text{id}_{y^{-1}} \cdot \zeta$ and $\psi_w = \zeta^{-1} \cdot \text{id}_x$. (It is straightforward to verify that these double cells satisfy the binding cell equations (11) for companions.)

Conversely, let $w: A \rightarrow B$ be a horizontal arrow with a companion $w_*: A \bullet \rightarrow B$ and binding cells

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xrightarrow{w} & B \\
 \downarrow w_* & \chi_w & \downarrow 1_B \\
 B & \xrightarrow{\text{Id}_B} & B
 \end{array} & \text{and} & \begin{array}{ccc}
 A & \xrightarrow{\text{Id}_A} & A \\
 \downarrow 1_A & \psi_w & \downarrow w_* \\
 A & \xrightarrow{w} & B.
 \end{array}
 \end{array}$$

By the previous lemma it is sufficient to show that there is an invertible 2-cell $\text{Id}_A \Rightarrow w$ in **Bic** \mathbb{X} . Such a 2-cell is provided by the binding cell ψ_w . ■

Now we start with a bicategory \mathcal{B} and consider its associated double category $\mathbf{Db}(\mathcal{B})$. The following proposition describes the relationship between quasi units in \mathcal{B} and companions in $\mathbf{Db}(\mathcal{B})$.

3.8. PROPOSITION. *A horizontal arrow*

$$A_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \dots \xrightarrow{f_n} A_n$$

in $\mathbf{Db}(\mathcal{B})$ has a companion if and only if $A_{i_0} = A_{i_1}$ and the chosen composition $\varphi_{f_{i_0+1} \dots f_{i_1}}$ is a quasi unit.

PROOF. Suppose that

$$A_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \dots \xrightarrow{f_n} A_n$$

has a companion. Then there are double cells of the form

$$\begin{array}{ccc}
 A_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i_0}} [A_{i_0}] & \xlongequal{\quad \psi \quad} & [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_n} A_n \\
 \parallel & \varphi_{f_{i_0+1} \dots f_{i_1}} & \parallel \\
 A_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i_0}} [A_{i_0}] & \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_{i_1}} [A_{i_1}] & \xrightarrow{f_{i_1+1}} \dots \xrightarrow{f_n} A_n
 \end{array}$$

$$\begin{array}{c}
 A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n \\
 \parallel \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \parallel \\
 \varphi_{f_{i_0+1} \cdots f_{i_1}} \\
 \chi \\
 \parallel \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \parallel \\
 A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n.
 \end{array}$$

So $A_{i_0} = A_{i_1}$ and there are 2-cells $\psi: \text{Id}_{A_{i_0}} \Rightarrow \varphi_{f_{i_0+1} \cdots f_{i_1}}$ and $\chi: \varphi_{f_{i_0+1} \cdots f_{i_1}} \Rightarrow \text{Id}_{A_{i_1}}$ in \mathcal{B} . The binding cell equations are equivalent to stating that these two 2-cells are inverse to each other, so $\varphi_{f_{i_0+1} \cdots f_{i_1}}$ is a quasi unit.

Conversely, if $A_{i_0} = A_{i_1}$ and $\varphi_{f_{i_0+1} \cdots f_{i_1}}$ is a quasi unit with an invertible 2-cell $\theta: \text{Id}_{A_{i_0}} \Rightarrow \varphi_{f_{i_0+1} \cdots f_{i_1}}$, then

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n$$

has a companion

$$\begin{array}{c}
 A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_n} A_n \\
 \parallel \\
 A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n
 \end{array}$$

with binding cells

$$\begin{array}{c}
 A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_n} A_n \\
 \parallel \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \parallel \\
 \theta \\
 \varphi_{f_{i_0+1} \cdots f_{i_1}} \\
 A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n \\
 \parallel \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \parallel \\
 A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n \\
 \parallel \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \parallel \\
 \varphi_{f_{i_0+1} \cdots f_{i_1}} \\
 \theta^{-1} \\
 \parallel \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \parallel \\
 A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n
 \end{array}$$

■

3.9. PRESERVATION OF COMPANIONS. It is clear that strict functors between double categories preserve companions and conjoinants. It is not immediately obvious that the same is true for pseudo-functors, given the fact that horizontal identities and domains and codomains are not preserved strictly.

However, since companions in weakly globular double categories correspond to quasi units in bicategories, and we know that homomorphisms of bicategories preserve quasi units, pseudo-functors between weakly globular double categories should preserve companions. Here we give a direct proof of this result in terms of companions and binding cells.

3.10. PROPOSITION. *Let $F: \mathbb{X} \rightarrow \mathbb{Y}$ be a pseudo-functor of weakly globular double categories. If a horizontal arrow f in \mathbb{X} has a companion then so does $F(f)$ in \mathbb{Y} .*

PROOF. Let $A \xrightarrow{f} B$ be a horizontal arrow with vertical companion $A \xrightarrow{v} B$ and binding cells

$$\begin{array}{ccc} A \xlongequal{\quad} A & & A \xrightarrow{f} B \\ \parallel & \psi & \downarrow v \\ A \xrightarrow{f} B & & \downarrow v \\ & & B \xlongequal{\quad} B \end{array} \quad .$$

Consider the following pastings of double cells in \mathbb{Y} :

$$\psi_{F_1 f} := \begin{array}{ccc} d_0 F_1 f \xrightarrow{\text{Id}_{d_0 F_1 f}} d_0 F_1 f & & \\ (\varphi_0)_{\text{Id}_A} \cdot (d_0 F_1 \psi)^{-1} \downarrow & \text{id} & \downarrow (\varphi_0)_{\text{Id}_A} \cdot (d_0 F_1 \psi)^{-1} \\ F_0 A \xrightarrow{\text{Id}_{F_0 A}} F_0 A & & \\ (\varphi_0)_{\text{Id}_A}^{-1} \downarrow & \sigma_A & \downarrow (\varphi_1)_{\text{Id}_A}^{-1} \\ d_0 F_1(\text{Id}_A) \xrightarrow{F_1(\text{Id}_A)} d_1 F_1(\text{Id}_A) & & \\ d_0 F_1 \psi \downarrow & F_1 \psi & \downarrow d_1 F_1 \psi \\ d_0 F_1 f \xrightarrow{F_1 f} d_1 F_1 f & & \end{array}$$

and

$$\chi_{F_1 f} := \begin{array}{ccc} d_0 F_1 f \xrightarrow{F_1 f} d_1 F_1 f & & \\ d_0 F_1 \chi \downarrow & F_1 \chi & \downarrow d_1 F_1 \chi \\ d_0 F_1(\text{Id}_B) \xrightarrow{F_1(\text{Id}_B)} d_1 F_1(\text{Id}_B) & & \\ (\varphi_0)_{\text{Id}_B} \downarrow & \sigma_B^{-1} & \downarrow (\varphi_1)_{\text{Id}_B} \\ F_0 B \xrightarrow{\text{Id}_{F_0 B}} F_0 B & & \\ (d_1 F_1 \chi)^{-1} \cdot (\varphi_1)_{\text{Id}_B}^{-1} \downarrow & \text{id} & \downarrow (d_1 F_1 \chi)^{-1} \cdot (\varphi_1)_{\text{Id}_B}^{-1} \\ d_1 F_1 f \xrightarrow{\text{Id}_{d_1 F_1 f}} d_1 F_1 f & & \end{array} \quad .$$

Note that $d_1 F_1 \psi \cdot (\varphi_1)_{\text{Id}_A}^{-1} \cdot (\varphi_0)_{\text{Id}_A} \cdot (d_0 F_1 \psi)^{-1} = (d_1 F_1 \chi)^{-1} \cdot (\varphi_1)_{\text{Id}_B}^{-1} \cdot (\varphi_0)_{\text{Id}_B} \cdot d_0 F_1 \chi$ since the vertical arrow category is posetal, and we claim that this arrow is the vertical companion of $F_1 f$ with binding cells $\psi_{F_1 f}$ and $\chi_{F_1 f}$. So we need to check that the equations in (11) are satisfied. It is relatively easy to check that the second equation, involving the vertical composition of these cells, is satisfied. For the first equation we need to refer to the coherence conditions in (5), (6), and (7). The horizontal composition considered in the

binding cell equation is

$$\begin{array}{ccccc}
 d_0 F_1 f & \xrightarrow{\text{Id}_{d_0 F_1 f}} & d_0 F_1 f & \xlongequal{\quad} & d_0 F_1 f & \xrightarrow{F_1 f} & d_0 F_1 f \\
 \downarrow (\varphi_0)_{\text{Id}_A} \cdot (d_0 F_1 \psi)^{-1} & & \downarrow (\varphi_0)_{\text{Id}_A} \cdot (d_0 F_1 \psi)^{-1} & & \downarrow d_0 F_1 \chi & & \downarrow d_1 F_1 \chi \\
 F A & \xrightarrow{\text{Id}_{F_0 A}} & F A & & d_0 F_1(\text{Id}_B) & \xrightarrow{-F_1(\text{Id}_B)} & d_1 F_1(\text{Id}_B) \\
 \downarrow (\varphi_0)_{\text{Id}_A}^{-1} & & \downarrow (\varphi_1)_{\text{Id}_A}^{-1} & = & \downarrow (\varphi_0)_{\text{Id}_B} & & \downarrow (\varphi_1)_{\text{Id}_B} \\
 d_0 F_1(\text{Id}_A) & \xrightarrow{-F_1(\text{Id}_A)} & d_1 F_1(\text{Id}_A) & & F B & \xrightarrow{\text{Id}_{F_0 B}} & F B \\
 \downarrow d_0 F_1 \psi & & \downarrow d_1 F_1 \psi & & \downarrow (d_1 F_1 \chi)^{-1} \cdot (\varphi_1)_{\text{Id}_B}^{-1} & & \downarrow (d_1 F_1 \chi)^{-1} \cdot (\varphi_1)_{\text{Id}_B}^{-1} \\
 d_0 F_1 f & \xrightarrow{F_1 f} & d_1 F_1 f & \xlongequal{\quad} & d_1 F_1 f & \xrightarrow{\text{Id}_{d_1 F_1 f}} & d_1 F_1 f
 \end{array}$$

By (6), and (7) this is equal to

$$\begin{array}{ccc}
 & \xrightarrow{F_1(f \circ \text{Id}_A)} & \\
 \downarrow & \xrightarrow{\pi_2(F_2(f, \text{Id}))} & \xrightarrow{\mu_{f, \text{Id}_A}} & \xrightarrow{\pi_1(F_2(f, \text{Id}_A))} & \downarrow \\
 \downarrow & \xrightarrow{(\theta_2)_{f, \text{Id}_A}} & & & \downarrow \\
 \downarrow & \xrightarrow{F_1(\text{Id}_A)} & = & & \downarrow \\
 \downarrow & \xrightarrow{\sigma_A^{-1}} & & & \downarrow \\
 \downarrow & \xrightarrow{\text{Id}_{F_0 A}} & & & \downarrow \\
 \downarrow & \xrightarrow{\text{Id}_{d_0 F_1 f}} & & & \downarrow \\
 \downarrow & \xrightarrow{\text{Id}_{F_0 A}} & & & \downarrow \\
 \downarrow & \xrightarrow{\sigma_A} & = & & \downarrow \\
 \downarrow & \xrightarrow{F_1(\text{Id}_A)} & & & \downarrow \\
 \downarrow & \xrightarrow{F_1 \psi} & & & \downarrow \\
 \downarrow & \xrightarrow{F_1 f} & & & \downarrow \\
 \downarrow & & & & \downarrow \\
 \downarrow & \xrightarrow{(\theta_2)_{\text{Id}_B, f}^{-1}} & = & & \downarrow \\
 \downarrow & \xrightarrow{\pi_2 F_2(\text{Id}_B, f)} & \xrightarrow{\mu_{\text{Id}_B, f}^{-1}} & \xrightarrow{\pi_1 F_2(\text{Id}_B, f)} & \downarrow \\
 & \xrightarrow{F_1(f)} & & &
 \end{array}$$

Cancellation of vertical inverses gives that this is equal to

$$\begin{array}{ccc}
 \begin{array}{c}
 \xrightarrow{F_1(f \circ \text{Id}_A)} \\
 \downarrow \pi_2 F_2(f, \text{Id}_A) \quad \xrightarrow{\mu_{f, \text{Id}_A}} \quad \downarrow \pi_1 F_2(f, \text{Id}_A) \\
 \downarrow (\theta_2)_{f, \text{Id}_A} \quad \sim \quad \downarrow (\theta_1)_{f, \text{Id}_A} \\
 \downarrow F_1 \text{Id}_A \quad \xrightarrow{F_1 f} \quad \downarrow F_1 f \\
 \downarrow F_1 \psi \quad = \quad \downarrow F_1 \chi \\
 \downarrow F_1 f \quad \xrightarrow{F_1 \text{Id}_B} \quad \downarrow F_1 \text{Id}_B \\
 \downarrow (\theta_2)_{\text{Id}_B, f}^{-1} \quad \sim \quad \downarrow (\theta_1)_{\text{Id}_B, f}^{-1} \\
 \downarrow \pi_2 F_2(\text{Id}_B, f) \quad \xrightarrow{\mu_{\text{Id}_B, f}^{-1}} \quad \downarrow \pi_1 F_2(\text{Id}_B, f) \\
 \xrightarrow{F_1(\text{Id}_B \circ f)}
 \end{array}
 & = &
 \begin{array}{c}
 \xrightarrow{F_1(f)} \\
 \downarrow F_1(\chi \circ \psi) \\
 \xrightarrow{F_1(\text{Id}_B \circ f)}
 \end{array}
 \end{array}$$

where the last equality is by (5). This gives us the required identity double cell. ■

3.11. REMARK. The proof of this proposition uses the fact that we are working with weakly globular double categories rather than arbitrary double categories in an essential way. In fact, the result is not true in this generality for arbitrary double categories: it was shown in [7] that pseudo-functors between arbitrary double categories preserve companions if and only if they are normal.

3.12. COMPANION CATEGORIES. The family of horizontal/vertical arrows that have a vertical/horizontal companion is closed under horizontal/vertical composition and the binding cells of the composite can be expressed as composites of the binding cells of the individual arrows. So for an arbitrary double category we can construct a category of companions. This category can furthermore be viewed as the category of vertical arrows of a double category that has the same horizontal arrows as the original double category, but its cells are only those that interact well with the binding cells of the companion pairs. This is made precise in the following definition.

3.13. DEFINITION. Let \mathbb{D} be an arbitrary double category. Then $\text{Comp}(\mathbb{D})$, the *double category of companions in \mathbb{D}* , is defined as follows. It has the same objects and horizontal arrows as \mathbb{D} , but its vertical arrows are companion pairs (with their binding cells) in \mathbb{D} . So a vertical arrow $\theta: A \bullet \rightarrow B$ in $\text{Comp}(\mathbb{D})$ is given by a quadruple $\theta = (h_\theta, v_\theta, \psi_\theta, \chi_\theta)$, where v_θ and h_θ are companions with binding cells

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xrightarrow{\text{Id}_A} & A \\
 \downarrow 1_A & \psi_\theta & \downarrow v_\theta \\
 A & \xrightarrow{h_\theta} & B
 \end{array}
 & \text{and} &
 \begin{array}{ccc}
 A & \xrightarrow{h_\theta} & B \\
 \downarrow v_\theta & \chi_\theta & \downarrow 1_A \\
 B & \xrightarrow{\text{Id}_B} & B
 \end{array}
 \end{array}$$

The vertical identity arrow 1_A is given by $(\text{Id}_A, 1_A, \iota_A, \iota_A)$. Vertical composition is defined by $\theta' \cdot \theta = (h_{\theta'} h_\theta, v_{\theta'} \cdot v_\theta, (\psi_{\theta'} \circ 1_{h_\theta}) \cdot (\text{id}_{v_{\theta'}} \circ \psi_\theta), (\chi_{\theta'} \circ \text{id}_{v_\theta}) \cdot (1_{h_{\theta'}} \circ \chi_\theta))$. A double cell

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \theta \downarrow & \Theta & \downarrow \theta' \\ B & \xrightarrow{g} & B' \end{array}$$

in $\text{Comp}(\mathbb{D})$ consists of a double cell

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ v_\theta \downarrow & \Theta & \downarrow v_{\theta'} \\ B & \xrightarrow{g} & B' \end{array}$$

in \mathbb{D} (involving just the vertical parts of the companion pairs in the cell we are defining) that has the following properties:

1. the square of horizontal arrows

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ h_\theta \downarrow & & \downarrow h_{\theta'} \\ B & \xrightarrow{g} & B' \end{array}$$

commutes in \mathbb{D} ;

- 2.

$$\begin{array}{ccccc} A & \xrightarrow{f} & A' & \xrightarrow{h_{\theta'}} & B' \\ v_\theta \downarrow & \Theta & \downarrow v_{\theta'} & \chi_{\theta'} & \parallel \\ B & \xrightarrow{g} & B' & \xrightarrow{1_{B'}} & B' \end{array} = \begin{array}{ccccc} A & \xrightarrow{h_\theta} & B & \xrightarrow{g} & B' \\ v_\theta \downarrow & \chi_\theta & \parallel & 1_g & \parallel \\ B & \xrightarrow{g} & B & \xrightarrow{g} & B' \end{array}$$

- 3.

$$\begin{array}{ccccc} A & \xrightarrow{1_A} & A & \xrightarrow{f} & A' \\ \parallel & \psi_\theta & \downarrow v_\theta & \Theta & \downarrow v_{\theta'} \\ A & \xrightarrow{h_\theta} & B & \xrightarrow{g} & B' \end{array} = \begin{array}{ccccc} A & \xrightarrow{f} & A' & \xrightarrow{1_{A'}} & B' \\ \parallel & 1_f & \parallel & \psi_\theta & \downarrow v_{\theta'} \\ A & \xrightarrow{f} & A' & \xrightarrow{h_{\theta'}} & B' \end{array}$$

3.14. REMARKS.

1. We write $\text{Comp}(\mathbb{D})$ for the category of vertical arrows, $v\text{Comp}(\mathbb{D})$, i.e., the category of companion pairs (with chosen binding cells).
2. If we have a functorial choice of companions and binding cells and we only use the horizontal arrows that are taken in these choices in the construction of $\text{Comp}(\mathbb{D})$, the result is a double category with a thin structure as defined in [5].

3.15. **PRECOMPANIONS.** We saw before that companions in weakly globular double categories correspond to quasi units in bicategories. We now introduce a class of horizontal arrows in weakly globular double categories that will correspond to internal equivalences in bicategories.

3.16. **DEFINITION.** Let \mathbb{X} be a weakly globular double category.

1. A horizontal arrow $A \xrightarrow{f} B$ in \mathbb{X} is a *left precompanion* if there are horizontal arrows $A' \xrightarrow{f'} B'$ and $B' \xrightarrow{r_f} C$ with a vertically invertible double cell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \varphi & \downarrow \\ A' & \xrightarrow{f'} & B' \end{array}$$

such that $r_f \circ f'$ is a companion in \mathbb{X} .

2. Dually, a horizontal arrow $A \xrightarrow{f} B$ in \mathbb{X} is a *right precompanion* if there are horizontal arrows $A'' \xrightarrow{f''} B''$ and $D \xrightarrow{l_f} A''$ with a vertically invertible double cell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \varphi' & \downarrow \\ A'' & \xrightarrow{f''} & B'' \end{array}$$

such that $f'' \circ l_f$ is a companion in \mathbb{X} .

3. A horizontal arrow $A \xrightarrow{f} B$ in \mathbb{X} is a *precompanion* if it is both a left and a right precompanion.

3.17. **EXAMPLE.** Every horizontal isomorphism $f: A \rightarrow B$ is a precompanion with $f' = f = f''$ and $l_f = r_f = f^{-1}$.

We first prove some basic properties of precompanions.

3.18. **LEMMA.** *If a horizontal arrow $A \xrightarrow{f} B$ is a precompanion in a weakly globular double category, then there exist horizontal arrows $D \xrightarrow{\bar{l}_f} A'$, $A' \xrightarrow{\bar{f}} B'$ and $B' \xrightarrow{\bar{r}_f} C$ with a vertically invertible double cell*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \psi & \downarrow \\ A' & \xrightarrow{\bar{f}} & B' \end{array}$$

such that $\bar{r}_f \circ \bar{f}$ is a companion and $\bar{f} \circ \bar{l}_f$ is a companion.

PROOF. Since f is both a left and a right precompanion there are horizontal arrows l_f , r_f , f' and f'' with double cells φ and φ' as in Definition 3.16. These can be combined in the following diagram.

$$\begin{array}{ccccc}
 & A' & \xrightarrow{f'} & B' & \xrightarrow{r_f} & C \\
 & \downarrow & & \downarrow & & \\
 & \bullet & & \bullet & & \\
 & \downarrow & \varphi^{-1} & \downarrow & & \\
 & A & \xrightarrow{f} & B & & \\
 & \downarrow & & \downarrow & & \\
 & \bullet & & \bullet & & \\
 & \downarrow & \varphi' & \downarrow & & \\
 D & \xrightarrow{l_f} & A'' & \xrightarrow{f''} & B'' &
 \end{array}$$

By the weak globularity condition, we can complete this staircase diagram with arrows and vertically invertible cells as in the following diagram,

$$\begin{array}{ccccccc}
 \bar{D} & \xrightarrow{\bar{l}_f} & \bar{A} & \xrightarrow{\bar{f}} & \bar{B} & \xrightarrow{\bar{r}_f} & \bar{C} \\
 \downarrow & & \downarrow & \gamma_2 & \downarrow & \gamma_3 & \downarrow \\
 & & A' & \xrightarrow{f'} & B' & \xrightarrow{r_f} & C \\
 \downarrow & \gamma_1 & \downarrow & \varphi^{-1} & \downarrow & & \\
 & & A & \xrightarrow{f} & B & & \\
 \downarrow & & \downarrow & \varphi' & \downarrow & & \\
 D & \xrightarrow{l_f} & A'' & \xrightarrow{f''} & B'' & &
 \end{array}$$

Now the composites $\bar{f} \circ \bar{l}_f$ and $\bar{r}_f \circ \bar{f}$ are both companions, since they are vertically isomorphic to companions. We will give an explicit proof in terms of binding cells for one of them. Let the binding cells for $f' \circ r_f$ with its companion be as in the following diagram.

$$\begin{array}{ccc}
 A' & \xlongequal{\quad} & A' \\
 \parallel & \psi_f^r & \downarrow \\
 A' & \xrightarrow{f'} & B' \xrightarrow{r_f} C \\
 & & \downarrow \\
 & & C
 \end{array}
 \qquad
 \begin{array}{ccc}
 A' & \xrightarrow{f'} & B' \xrightarrow{r_f} C \\
 \downarrow & & \downarrow \\
 C & \xlongequal{\quad} & C \\
 & & \parallel
 \end{array}$$

Then the binding cells for $\bar{f} \circ \bar{r}_f$ with its companion are

$$\begin{array}{ccc}
 \bar{A} & \xlongequal{\quad} & \bar{A} \\
 \downarrow & \text{id} & \downarrow \\
 A' & \xlongequal{\quad} & A' \\
 \parallel & \psi_f^r & \downarrow \\
 A' & \xrightarrow{f'} B' \xrightarrow{r_f} C & \\
 \downarrow & \gamma_2^{-1} \downarrow \gamma_3^{-1} & \downarrow \\
 \bar{A} & \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{r}_f} \bar{C} & \\
 & & \downarrow \\
 & & \bar{C}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \bar{A} & \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{r}_f} \bar{C} \\
 \downarrow & \gamma_2 \downarrow \gamma_3 & \downarrow \\
 A' & \xrightarrow{f'} B' \xrightarrow{r_f} C \\
 \downarrow & \chi_f^r & \parallel \\
 C & \xlongequal{\quad} & C \\
 \downarrow & \text{id} & \downarrow \\
 \bar{C} & \xlongequal{\quad} & \bar{C}
 \end{array}$$

The rest of the details are left to the reader. ■

3.19. LEMMA. *If f is a precompanion with arrows l_f and r_f as in Definition 3.16, then there is a vertically invertible double cell of the form*

$$\begin{array}{ccc}
 & \xrightarrow{r_f} & \\
 \bullet & \nu_f & \bullet \\
 & \xrightarrow{l_f} &
 \end{array}
 .$$

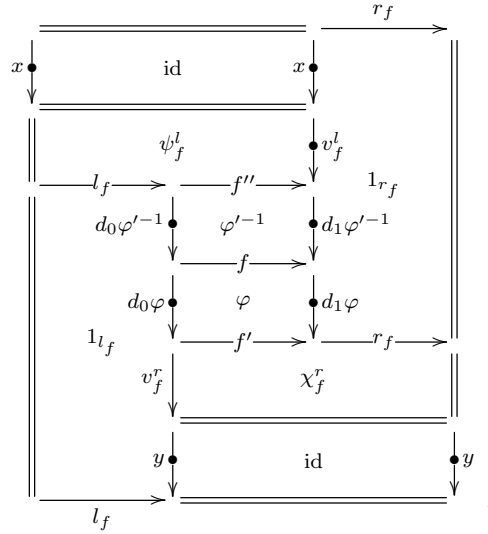
PROOF. Let $r_f \circ f'$ have vertical companion v_f^r with binding cells

$$\left\| \begin{array}{ccc} & \xrightarrow{f'} & \\ \psi_f^r & & \\ & \xrightarrow{r_f} & \end{array} \right\| \downarrow v_f^r \quad \text{and} \quad v_f^r \downarrow \left\| \begin{array}{ccc} & \xrightarrow{f'} & \xrightarrow{r_f} \\ & \chi_f^r & \\ & \xrightarrow{r_f} & \end{array} \right\| ,$$

and let $f'' \circ l_f$ have vertical companion v_f^l with binding cells

$$\left\| \begin{array}{ccc} & \xrightarrow{l_f} & \\ \psi_f^l & & \\ & \xrightarrow{f''} & \end{array} \right\| \downarrow v_f^l \quad \text{and} \quad v_f^l \downarrow \left\| \begin{array}{ccc} & \xrightarrow{l_f} & \xrightarrow{f''} \\ & \chi_f^l & \\ & \xrightarrow{f''} & \end{array} \right\| .$$

Further, let $x = (v_f^l)^{-1} \cdot d_1\varphi' \cdot (d_1\varphi)^{-1}$ and $y = (d_0\varphi') \cdot (d_0\varphi)^{-1} \cdot (v_f^r)^{-1}$. Then ν_f can be obtained as the following pasting of double cells:

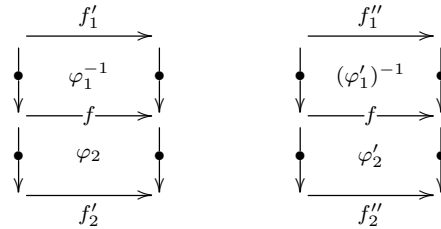


■

It is well-known that companions, just like adjoints, are unique up to special invertible double cells. A similar result applies to precompanions and the proof is a 2-dimensional version of the proof that the pseudo-inverse of an equivalence in a bicategory is unique up to invertible 2-cell.

3.20. LEMMA. *For any precompanion f in a weakly globular double category, the precompanion structure given in Definition 3.16 is unique up to vertically invertible double cells.*

PROOF. Suppose that $\varphi_i, r_{f,i}, v_{f,i}^r, f'_i, \psi_{f,i}^r,$ and $\chi_{f,i}^r$ give two right precompanion structures for $i = 1, 2,$ and $\varphi'_i, l_{f,i}, f''_i, v_{f,i}^l, \psi_{f,i}^l$ and $\chi_{f,i}^l$ give two left precompanion structures for $i = 1, 2.$ Then there are vertically invertible double cells as follows:



According to Lemma 3.19 there are vertically invertible double cells

$$\begin{array}{c} \xrightarrow{r_{f,1i}} \\ \bullet \nu_{f,ij} \bullet \\ \xrightarrow{l_{f,j}} \end{array} \text{ for } i, j \in \{1, 2\}.$$

So we obtain vertically invertible cells

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{r_{f,1}} \\ \bullet \downarrow \nu_{f,21}^{-1} \cdot \nu_{f,11} \downarrow \bullet \\ \xrightarrow{r_{f,2}} \end{array} & \text{and} & \begin{array}{c} \xrightarrow{l_{f,1}} \\ \bullet \downarrow \nu_{f,22} \cdot \nu_{f,21}^{-1} \downarrow \bullet \\ \xrightarrow{l_{f,2}} \end{array}
 \end{array}$$

■

The next two propositions establish the relationship between precompanions in weakly globular double categories and equivalences in bicategories.

3.21. PROPOSITION. *Let \mathcal{B} be a bicategory. A horizontal arrow*

$$A_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \dots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \dots \xrightarrow{f_n} A_n \tag{14}$$

in $\mathbf{Db}(\mathcal{B})$ is a precompanion if and only if the chosen composition $\varphi_{f_{i_0+1} \dots f_{i_1}}$ is an equivalence in \mathcal{B} .

PROOF. Suppose that the chosen composition $\varphi_{f_{i_0+1} \dots f_{i_1}}$ is an equivalence in \mathcal{B} . Denote the arrow (14) in $\mathbf{Db}(\mathcal{B})$ by f . Let $g: A_{i_1} \rightarrow A_{i_0}$ be a pseudo-inverse of $\varphi_{f_{i_0+1} \dots f_{i_1}}$ in \mathcal{B} . Then we may take f' in Definition 3.16 to be the arrow

$$[A_{i_0}] \xrightarrow{\varphi_{f_{i_0+1} \dots f_{i_1}}} [A_{i_1}] \xrightarrow{g} A_{i_0}$$

with vertically invertible cell

$$\begin{array}{ccccccc}
 A_0 & \xrightarrow{f_1} & \dots & \xrightarrow{f_{i_0}} & [A_{i_0}] & \xrightarrow{f_{i_0+1}} & \dots & \xrightarrow{f_{i_1}} & [A_{i_1}] & \xrightarrow{f_{i_1+1}} & \dots & \xrightarrow{f_n} & A_n \\
 & & & & \parallel & \searrow \varphi_{f_{i_0+1} \dots f_{i_1}} & & & \parallel & & & & \\
 & & & & [A_{i_0}] & \xrightarrow{=} & [A_{i_1}] & \xrightarrow{g} & A_{i_0} & & & &
 \end{array}$$

and we may take r_f in Definition 3.16 to be the arrow

$$A_{i_0} \xrightarrow{\varphi_{f_{i_0+1} \dots f_{i_1}}} [A_{i_1}] \xrightarrow{g} [A_{i_0}].$$

The horizontal composition $r_f \circ f'$ is given by

$$[A_{i_0}] \xrightarrow{\varphi_{f_{i_0+1} \dots f_{i_1}}} A_{i_1} \xrightarrow{g} [A_{i_0}],$$

and this is a companion by Proposition 3.8, since $g \circ \varphi_{f_{i_0+1} \dots f_{i_1}}$ is a quasi unit in \mathcal{B} . So (14) is a left precompanion. The proof that it is a right precompanion goes similarly.

Now suppose that (14) is a precompanion in $\mathbf{Dbl}(\mathcal{B})$. So there are diagrams with vertically invertible double cells of the form

$$\begin{array}{ccccccc}
 A_0 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{i_0}} & [A_{i_0}] & \xrightarrow{f_{i_0+1}} & \cdots & \xrightarrow{f_{i_1}} & [A_{i_1}] & \xrightarrow{f_{i_1+1}} & \cdots & \xrightarrow{f_n} & A_n \\
 & & & & \parallel & \searrow^{\varphi_{f_{i_0+1} \cdots f_{i_1}}} & & \searrow^{\varphi} & \parallel & & & & \\
 B_0 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_{j_0}} & [A_{i_0}] & \xrightarrow{g_{j_0+1}} & \cdots & \xrightarrow{g_{j_1}} & [A_{i_1}] & \xrightarrow{g_{j_1+1}} & \cdots & \xrightarrow{g_{j_2}} & C & \xrightarrow{g_{j_2+1}} & \cdots & \xrightarrow{g_m} & B_m
 \end{array}$$

and

$$\begin{array}{ccccccc}
 A_0 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{i_0}} & [A_{i_0}] & \xrightarrow{f_{i_0+1}} & \cdots & \xrightarrow{f_{i_1}} & [A_{i_1}] & \xrightarrow{f_{i_1+1}} & \cdots & \xrightarrow{f_n} & A_n \\
 & & & & \parallel & \searrow^{\varphi_{f_{i_0+1} \cdots f_{i_1}}} & & \searrow^{\varphi'} & \parallel & & & & \\
 B'_0 & \xrightarrow{h_1} & \cdots & \xrightarrow{h_{k_0}} & D & \xrightarrow{h_{k_0+1}} & \cdots & \xrightarrow{h_{k_1}} & [A_{i_0}] & \xrightarrow{h_{k_1+1}} & \cdots & \xrightarrow{h_{k_2}} & [A_{i_1}] & \xrightarrow{h_{k_2+1}} & \cdots & \xrightarrow{h_p} & B'_p
 \end{array}$$

such that the arrows

$$B_0 \xrightarrow{g_1} \cdots \xrightarrow{g_{j_0}} [A_{i_0}] \xrightarrow{g_{j_0+1}} \cdots \xrightarrow{g_{j_1}} [A_{i_1}] \xrightarrow{g_{j_1+1}} \cdots \xrightarrow{g_{j_2}} [C] \xrightarrow{g_{j_2+1}} \cdots \xrightarrow{g_m} B_m$$

and

$$B'_0 \xrightarrow{h_1} \cdots \xrightarrow{h_{k_0}} [D] \xrightarrow{h_{k_0+1}} \cdots \xrightarrow{h_{k_1}} [A_{i_0}] \xrightarrow{h_{k_1+1}} \cdots \xrightarrow{h_{k_2}} [A_{i_1}] \xrightarrow{h_{k_2+1}} \cdots \xrightarrow{h_p} B'_p$$

have companions in $\mathbf{Dbl}(\mathcal{B})$. By Proposition 3.8 this implies that $A_{i_0} = C$, $D = A_{i_1}$ and both chosen composites $\varphi_{g_{j_0+1} \cdots g_{j_2}}$ and $\varphi_{h_{k_0+1} \cdots h_{k_2}}$ are quasi units in \mathcal{B} . So the chosen composite $\varphi_{g_{j_1+1} \cdots g_{j_2}}$ is a pseudo-inverse for $\varphi_{f_{i_0+1} \cdots f_{i_1}}$, making this arrow an equivalence in \mathcal{B} . ■

3.22. PROPOSITION. *Let \mathbb{X} be a weakly globular double category with a horizontal arrow $f: A \rightarrow B$. Then the arrow $f: \bar{A} \rightarrow \bar{B}$ in $\mathbf{Bic}(\mathbb{X})$ is an equivalence if and only if f is a precompanion in \mathbb{X} .*

PROOF. Suppose that $f: \bar{A} \rightarrow \bar{B}$ has pseudo inverse $g: \bar{B} \rightarrow \bar{A}$. The two compositions of these arrows in $\mathbf{Bic}(\mathbb{X})$ are given by the horizontal compositions $\tilde{g} \circ \tilde{f}$ and $\hat{f} \circ \hat{g}$ as in the following diagrams of double cells

$$\begin{array}{ccccc}
 \tilde{A} & \xrightarrow{\tilde{f}} & \tilde{B} & \xrightarrow{\tilde{g}} & \tilde{A}' \\
 \downarrow & & \downarrow & \cong & \downarrow \\
 \bullet & \cong & B' & \xrightarrow{g} & A' \\
 \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{f} & B & &
 \end{array} \tag{15}$$

and

$$\begin{array}{ccccc}
 \hat{B}' & \xrightarrow{\hat{g}} & \hat{A} & \xrightarrow{\hat{f}} & \hat{B} \\
 \downarrow & & \downarrow & \cong & \downarrow \\
 \bullet & & A & \xrightarrow{f} & B \\
 \downarrow & \cong & \downarrow & & \downarrow \\
 B' & \xrightarrow{g} & A' & & .
 \end{array} \tag{16}$$

By Proposition 3.7, the compositions $\tilde{g} \circ \tilde{f}$ and $\hat{f} \circ \hat{g}$ are companions in \mathbb{X} and the diagrams (15) and (16) show that f is a precompanion with $f' = \tilde{f}$, $r_f = \tilde{g}$, $f'' = \hat{f}$ and $l_f = \hat{g}$.

Now suppose that f is a precompanion in \mathbb{X} with additional arrows and cells as in Definition 3.16:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{f} & \\
 \downarrow & \varphi & \downarrow \\
 & \xrightarrow{f'} & \xrightarrow{r_f}
 \end{array} & &
 \begin{array}{ccc}
 & \xrightarrow{f} & \\
 \downarrow & \varphi' & \downarrow \\
 & \xrightarrow{l_f} & \xrightarrow{f''}
 \end{array} .
 \end{array}$$

By Proposition 3.7, $r_f \circ f'$ and $f'' \circ l_f$ are quasi units in $\mathbf{Bic}(\mathbb{X})$, say with invertible 2-cells $\alpha: r_f \circ f' \Rightarrow \text{Id}_{\bar{A}}$ and $\beta: f'' \circ l_f \Rightarrow \text{Id}_{\bar{B}}$. So we have the composites, $r_f f \xrightarrow{r_f \varphi} r_f \circ f' \xrightarrow{\alpha} \text{Id}_{\bar{A}}$ and $f l_f \xrightarrow{\varphi' l_f} f'' \circ l_f \xrightarrow{\beta} \text{Id}_{\bar{B}}$. All these 2-cells are invertible and this implies that $f: \bar{A} \rightarrow \bar{B}$ in $\mathbf{Bic}(\mathbb{X})$ is an equivalence. ■

3.23. PROPOSITION. *A pseudo-functor between weakly globular double categories preserves precompanions.*

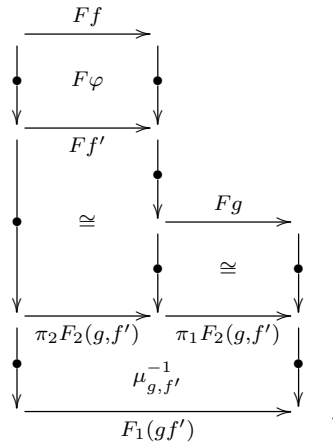
PROOF. Let $F: \mathbb{X} \rightarrow \mathbb{Y}$ be a pseudo-functor between two weakly globular double categories \mathbb{X} and \mathbb{Y} , and let $A \xrightarrow{f} B$ be a left precompanion in \mathbb{X} , with

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & \varphi & \downarrow \\
 A' & \xrightarrow{f'} & B' \xrightarrow{g} C
 \end{array}$$

as in Definition 3.16, companion v_f , and binding cells

$$\begin{array}{ccc}
 A' \xlongequal{\quad} A' & & A' \xrightarrow{f'} B' \xrightarrow{g} C \\
 \parallel & \psi_f & \downarrow v_f \\
 A' \xrightarrow{f'} B' \xrightarrow{g} C & \text{and} & v_f \downarrow \quad \chi_f \quad \parallel \\
 & & C \xlongequal{\quad} C .
 \end{array}$$

Then we have the following diagram in \mathbb{Y}



By Proposition 3.10, $F_1(gf')$ is a companion in \mathbb{Y} , so $\pi_1 F_2(g, f') \circ \pi_2 F_2(g, f')$ is a companion as well.

The fact that pseudo-functors preserve right precompanions goes similarly. ■

4. The weakly globular double category $\mathbf{DbI}(\mathbf{C}(W^{-1}))$

Given a category \mathbf{C} with a class W of arrows that satisfies the conditions to define a category of fractions $\mathbf{C}[W^{-1}]$ as given by Gabriel and Zisman (and repeated in Section 4.1 below), it is possible to not just define the category of fractions $\mathbf{C}[W^{-1}]$, but also the bicategory of fractions denoted by $\mathbf{C}(W^{-1})$. Furthermore, the bicategory of fractions is defined precisely when the category of fractions is defined. The bicategory of fractions has a universal property in terms of hom categories rather than just hom sets, but the fundamental category of this bicategory is the original category of fractions and the bicategory of fractions is even biequivalent to (as bicategories) the category of fractions. This shows us that the hom categories in the bicategory of fractions will just consist of the equivalence relation on the arrows of the category of fractions, as described in [12], i.e., there is 2-cell between any two parallel arrows in the bicategory of fractions precisely when the two arrows are equivalent in the category of fractions and this 2-cell is unique. In the remainder of this paper we want to study the weakly globular double categories that can be used as a counterpart to the bicategory of fractions $\mathbf{C}(W^{-1})$. We will describe their universal properties and finally find a weakly globular double category that can be viewed as the weakly globular double category of fractions. The absence of non-identity 2-dimensional cells in \mathbf{C} simplifies the calculations and helps us to see what the basic properties of such weakly globular double categories are. In the sequel [25] to this paper we will show how the cells of a bicategory or weakly globular double category can be added to this construction in a very natural way to give the weakly globular double category of fractions of a weakly globular double category with respect to a class of horizontal arrows.

4.1. CATEGORIES OF FRACTIONS. Let \mathbf{C} be a category with a class $W \subseteq \mathbf{C}_1$ of arrows satisfying the conditions for a calculus of fractions given by Gabriel and Zisman [12] (which imply the existence of a category of fractions $\mathbf{C}[W^{-1}]$):

- **CF1** W contains all isomorphisms in \mathbf{C} and is closed under composition;
- **CF2** For any diagram $C \xrightarrow{f} B \xleftarrow{w} A$ in \mathbf{C} with $w \in W$, there exist arrows $D \xrightarrow{\bar{f}} A$ and $D \xrightarrow{\bar{w}} C$ with $\bar{w} \in W$ such that

$$\begin{array}{ccc} D & \xrightarrow{\bar{f}} & A \\ \bar{w} \downarrow & & \downarrow w \\ C & \xrightarrow{f} & B \end{array}$$

commutes;

- **CF3** For any diagram $A \xrightarrow[g]{f} B \xrightarrow{w} C$ with $w \in W$, such that $wf = wg$, there exists an arrow $X \xrightarrow{\tilde{w}} A$ in W such that $f\tilde{w} = g\tilde{w}$.

Note that the conditions on a class of arrows W in a bicategory \mathbf{C} to form a bicategory of fractions coincide with these conditions when \mathbf{C} is a category (cf. [26]). So when a class W of arrows satisfies these conditions, both the category $\mathbf{C}[W^{-1}]$ and the bicategory $\mathbf{C}(W^{-1})$ are defined (and the former can be obtained as a quotient of the latter).

4.2. EXAMPLE. *Manifold Atlases* Consider manifolds with atlases such that the intersection of two atlas charts is either itself a chart or covered by charts. For any smooth map between two manifolds $f: M \rightarrow N$ there are such atlases \mathcal{U} for M and \mathcal{V} for N such that for each chart $U \in \mathcal{U}$, there is a chart $V \in \mathcal{V}$ with a smooth map $f_U: U \rightarrow V$ and such that for any pair of charts with a chart embedding $\lambda: U_1 \hookrightarrow U_2$, there are charts V_1 and V_2 in \mathcal{V} with a chart embedding $\mu: V_1 \hookrightarrow V_2$ and such that $f_{U_i}: U_i \rightarrow V_i$ for $i = 1, 2$ and the diagram

$$\begin{array}{ccc} U_1 & \xrightarrow{f_{U_1}} & V_1 \\ \lambda \downarrow & & \downarrow \mu \\ U_2 & \xrightarrow{f_{U_2}} & V_2 \end{array}$$

commutes. We may choose to start with manifolds represented by such atlases (without explicitly keeping track of the underlying spaces) and consider atlas maps to be such families of smooth maps that are locally compatible as described above. Two such atlas maps $(f_U)_{U \in \mathcal{U}}$ and $(g_U)_{U \in \mathcal{U}}$ are equivalent when for each $U \in \mathcal{U}$ with $f_U: U \rightarrow V_{f,U}$ and $g_U: U \rightarrow V_{g,U}$, there is a chart $V \in \mathcal{V}$ with a smooth map $h_U: U \rightarrow V$ and chart

embeddings $\lambda_f: V \hookrightarrow V_{f,U}$ and $\lambda_g: V \hookrightarrow V_{g,U}$ such that the following diagram commutes,

$$\begin{array}{ccc}
 & & V_{f,U} \\
 & \nearrow f_U & \uparrow \lambda_f \\
 U & \xrightarrow{h_U} & V \\
 & \searrow g_U & \downarrow \lambda_g \\
 & & V_{g,U} .
 \end{array}$$

The category **Atlases** consists of atlases and equivalence classes of atlas maps.

To obtain the usual category **Mfds** of manifolds from **Atlases** we do the following. First, note that if \mathcal{U} and \mathcal{U}' are two atlases for the same manifold and \mathcal{U}' is a refinement of \mathcal{U} , then there is a further refinement \mathcal{U}'' of \mathcal{U}' with an atlas map $\mathcal{U}'' \rightarrow \mathcal{U}$. We will call this a refinement atlas map. Furthermore, the class W of (equivalence classes of) refinement atlas maps satisfies the conditions **CF1**, **CF2** and **CF3** above. The corresponding category of fractions $\mathbf{Atlases}[W^{-1}]$ is categorically equivalent to **Mfds**. Arrows in this category can be thought of as first taking an atlas refinement and then mapping out. In the category of fractions we consider two such maps to be the same if there is a common refinement on which they would become the same (and this is the case precisely when they induce the same maps on the underlying manifolds.) If we instead consider the bicategory of fractions, we consider two maps defined by different refinements as distinct but there is a unique invertible 2-cell between two such maps if there is a common refinement where they become the same.

4.3. UNIVERSAL PROPERTIES. Both the category and bicategory of fractions have a universal property as a (weak) coinverter [17]. Specifically, a version of the universal property of the bicategory of fractions is given in the following theorem.

4.4. THEOREM. *For a bicategory \mathcal{B} and a class W of arrows satisfying the conditions to admit a bicategory of fractions, there is a homomorphism of bicategories $J_W: \mathcal{B} \rightarrow \mathcal{B}(W^{-1})$ such that composition with J_W induces an equivalence of categories*

$$\mathbf{Bicat}(\mathcal{B}(W^{-1}), \mathcal{D}) \simeq \mathbf{Bicat}_W(\mathcal{B}, \mathcal{D}),$$

for any bicategory \mathcal{D} .

4.5. REMARKS.

1. In this theorem, we may take $\mathbf{Bicat}(\mathcal{B}, \mathcal{D})$ to be the category of homomorphisms and oplax transformations, and $\mathbf{Bicat}_W(\mathcal{B}, \mathcal{D})$ the 2-category of homomorphism and oplax transformations that send elements of W to internal equivalences. We will also call these W -homomorphisms and W -transformations. (Note that we can make sense of this statement by representing a transformation in $\mathbf{Bicat}(\mathcal{B}, \mathcal{D})$ by a homomorphism $\mathcal{B} \rightarrow \mathbf{Cyl}(\mathcal{D})$.) There is also a version for lax transformations.

- Note that [26] gives this universal property as an equivalence of 2-categories, which involves also the modifications. However, the proof given there also shows that this restricts to an equivalence of categories when we just ignore the modifications.

So when we consider the notion of weakly globular double category of fractions, we would like to get a similar universal property in a 2-category of weakly globular double categories. However, there are two choices for such a 2-category, $\mathbf{WGDb}_{\text{st},h}$ and $\mathbf{WGDb}_{\text{ps},v}$, which are generally very different. Ideally, we would like to have a weakly globular double category of fractions that has a universal property in both of these 2-categories. We will begin by applying the 2-functor \mathbf{Db} to the bicategory of fractions $\mathbf{C}(W^{-1})$ of a category \mathbf{C} and study the universal property of the resulting weakly globular double category.

In order to be able to translate the universal property in Theorem 4.4 into a universal property involving weakly globular bicategories, we note that the equivalence of categories given there restricts to the categories with icons as arrows instead of general oplax transformations,

$$\mathbf{Bicat}_{\text{icon}}(\mathbf{C}(W^{-1}), \mathcal{D}) \simeq \mathbf{Bicat}_{\text{icon},W}(\mathbf{C}, \mathcal{D}). \tag{17}$$

When we apply the 2-functor \mathbf{Db} to this equivalence, we obtain

$$\mathbf{WGDb}_{\text{ps},v}(\mathbf{Db}(\mathbf{C}(W^{-1})), \mathbf{Db}(\mathcal{D})) \simeq \mathbf{WGDb}_{\text{ps},v,W}(\mathbf{Db}(\mathbf{C}), \mathbf{Db}(\mathcal{D})),$$

where the righthand side is for now just defined as the image of $\mathbf{Bicat}_{\text{icon},W}(\mathbf{C}, \mathcal{D})$ under the 2-functor \mathbf{Db} . In the next sections we will study $\mathbf{Db}(\mathbf{C}(W^{-1}))$ in more detail and give a description of the pseudo-functors and vertical transformations in $\mathbf{WGDb}_{\text{ps},v,W}(\mathbf{Db}(\mathbf{C}), \mathbf{Db}(\mathcal{D}))$, so that we can extend this to an equivalence

$$\mathbf{WGDb}_{\text{ps},v}(\mathbf{Db}(\mathbf{C}(W^{-1})), \mathbb{X}) \simeq \mathbf{WGDb}_{\text{ps},v,W}(\mathbf{Db}(\mathbf{C}), \mathbb{X}),$$

for an arbitrary weakly globular double category \mathbb{X} .

4.6. THE WEAKLY GLOBULAR DOUBLE CATEGORY $\mathbf{Db}(\mathbf{C}(W^{-1}))$. The objects of the weakly globular double category $\mathbf{Db}(\mathbf{C}(W^{-1}))$ are of the form

$$\leftarrow \begin{array}{c} q_1 \\ \hline \end{array} \begin{array}{c} p_1 \\ \hline \end{array} \rightarrow \dots \leftarrow \begin{array}{c} q_{i_0} \\ \hline \end{array} \begin{array}{c} p_{i_0} \\ \hline \end{array} \rightarrow [A_{i_0}] \leftarrow \begin{array}{c} q_{i_0+1} \\ \hline \end{array} \begin{array}{c} p_{i_0+1} \\ \hline \end{array} \rightarrow \dots \leftarrow \begin{array}{c} q_n \\ \hline \end{array} \begin{array}{c} p_n \\ \hline \end{array} \rightarrow$$

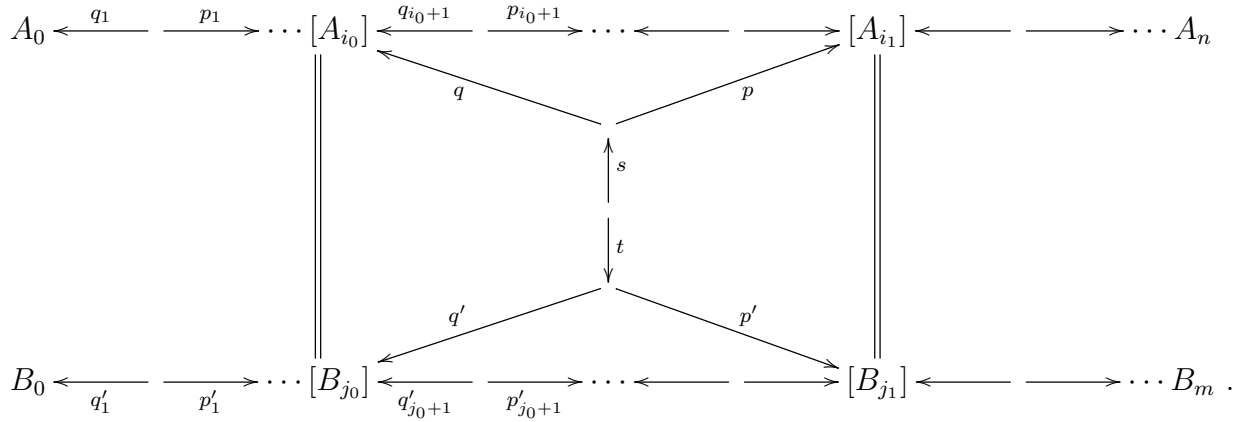
and the arrows are of the form

$$\leftarrow \begin{array}{c} q_1 \\ \hline \end{array} \begin{array}{c} p_1 \\ \hline \end{array} \rightarrow \dots \leftarrow \begin{array}{c} q_{i_0} \\ \hline \end{array} \begin{array}{c} p_{i_0} \\ \hline \end{array} \rightarrow [A_{i_0}] \dots \leftarrow \begin{array}{c} q_{i_1} \\ \hline \end{array} \begin{array}{c} p_{i_1} \\ \hline \end{array} \rightarrow [A_{i_1}] \dots \leftarrow \begin{array}{c} q_n \\ \hline \end{array} \begin{array}{c} p_n \\ \hline \end{array} \rightarrow$$

To define 2-cells we choose a composite $\leftarrow \begin{array}{c} q \\ \hline \end{array} \begin{array}{c} p \\ \hline \end{array} \rightarrow$ for each path

$$\leftarrow \begin{array}{c} q_{i_0+1} \\ \hline \end{array} \begin{array}{c} p_{i_0+1} \\ \hline \end{array} \rightarrow \dots \leftarrow \begin{array}{c} q_{i_1} \\ \hline \end{array} \begin{array}{c} p_{i_1} \\ \hline \end{array} \rightarrow .$$

Then 2-cells have the form



We introduce special terminology for the arrows in $\mathbf{Db}l(\mathcal{C})$ that are related to the elements of W .

4.7. DEFINITION. A horizontal arrow in $\mathbf{Db}l(\mathcal{C})$ is called a W -doubly marked path if it is of the form

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0}} [A_{i_0}] \xrightarrow{f_{i_0+1}} \cdots \xrightarrow{f_{i_1}} [A_{i_1}] \xrightarrow{f_{i_1+1}} \cdots \xrightarrow{f_n} A_n$$

with $f_{i_1} \circ \cdots \circ f_{i_0+1} \in W$.

Since precompanions in weakly globular double categories correspond to equivalences in bicategories as shown in Proposition 3.7 and Proposition 3.8, we can characterize the arrows in the image of the W -homomorphisms under $\mathbf{Db}l$ as follows.

4.8. LEMMA. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a homomorphism in \mathbf{Bicat} . Then F is a W -homomorphism if and only if $\mathbf{Db}l(F)$ sends a W -doubly marked path in $\mathbf{Db}l(\mathcal{C})$ to a precompanion in $\mathbf{Db}l(\mathcal{D})$.

Although this lemma follows immediately from the results mentioned, it is illuminating to consider the precompanion structures required in the proof of the left to right implication, so we include it here.

PROOF. (of Lemma 4.8) Suppose that F is a W -homomorphism. Let f be the horizontal arrow

$$FA_0 \xrightarrow{Ff_1} \cdots \xrightarrow{Ff_{i_0}} [FA_{i_0}] \xrightarrow{Ff_{i_0+1}} \cdots \xrightarrow{Ff_{i_1}} [FA_{i_1}] \xrightarrow{Ff_{i_1+1}} \cdots \xrightarrow{Ff_n} FA_n$$

in $\mathbf{Db}l(\mathcal{D})$ with $w = f_{i_1} \circ \cdots \circ f_{i_0+1}$ an arrow in W . Let $g \in \mathcal{D}$ be the chosen composite for $F(f_{i_1}) \circ \cdots \circ F(f_{i_0+1})$, and $\phi: g \Rightarrow Fw$, the invertible coherence cell from F . Furthermore, let $(Fw)^*$ be a chosen pseudo-inverse for Fw (which exists since F is a W -homomorphism) with invertible unit and counit cells $\varepsilon_f: (Fw)^* \circ Fw \Rightarrow \text{Id}_{FA_{i_1}}$ and $\eta_f: \text{Id}_{FA_{i_0}} \Rightarrow Fw \circ (Fw)^*$. Then we take f' in Definition 3.16 to be the horizontal arrow

$$[FA_{i_0}] \xrightarrow{Fw} [FA_{i_1}] \xrightarrow{(Fw)^*} F_{i_0}$$

and r_f to be the horizontal arrow

$$FA_{i_0} \xrightarrow{Fw} [FA_{i_1}] \xrightarrow{(Fw)^*} [F_{i_0}].$$

Then the double cell φ in Definition 3.16, Part 1, is

$$\begin{array}{ccccccc}
 FA_0 & \xrightarrow{Ff_1} & \dots & \xrightarrow{Ff_{i_0}} & [FA_{i_0}] & \xrightarrow{Ff_{i_0+1}} & \dots & \xrightarrow{Ff_{i_1}} & [FA_{i_1}] & \xrightarrow{Ff_{i_1+1}} & \dots & \xrightarrow{Ff_{i_n}} & FA_n \\
 & & & & \parallel & & & & \parallel & & & & \\
 & & & & & \xrightarrow{\quad g \quad} & & & & & & & \\
 & & & & & \phi & & & & & & & \\
 & & & & \parallel & & & & \parallel & & & & \\
 [FA_{i_0}] & \xrightarrow{Fw} & & & [FA_{i_1}] & \xrightarrow{(Fw)^*} & & & F_{i_0} & & & &
 \end{array}$$

Furthermore, the composition $r_f \circ f'$ is the horizontal arrow

$$[FA_{i_0}] \xrightarrow{Fw} FA_{i_1} \xrightarrow{(Fw)^*} [F_{i_0}],$$

and it is straightforward to check that ε_f and its inverse give rise to the cells to make this a companion.

Analogously, f'' can be taken to be the horizontal arrow

$$FA_{i_1} \xrightarrow{(Fw)^*} [FA_{i_0}] \xrightarrow{Fw} [FA_{i_1}]$$

with l_f the arrow,

$$[FA_{i_1}] \xrightarrow{(Fw)^*} [FA_{i_0}] \xrightarrow{Fw} FA_{i_1}.$$

■

Since every weakly globular double category is vertically equivalent to one of the form $\mathbf{DbI}(\mathcal{D})$ for some bicategory \mathcal{D} and pseudo-functors of weakly globular double categories preserve precompanions, this leads us to the following definition.

4.9. DEFINITION. A pseudo-functor $F: \mathbf{DbI}(\mathbf{C}) \rightarrow \mathbb{X}$ is called a W -pseudo-functor when it sends all W -doubly marked paths in $\mathbf{DbI}(\mathbf{C})$ to precompanions in \mathbb{X} .

4.10. REMARK. It is clear that we have some choice for the f' and f'' in the proof of Lemma 4.8. However, once they are chosen, r_f and l_f are fixed, since they need to be composable with f' and f'' respectively. We saw in Lemma 3.18 already that it is possible to choose f' and f'' as the same horizontal arrow, \bar{f} . One of the questions one may ask about the precompanion structure is whether one could require l_f and r_f to be equal as well. The case of the proof above shows that one cannot ask this much for arbitrary precompanions. We see here that we could have made \bar{f} to be equal to

$$FA_{i_1} \xrightarrow{(Fw)^*} [FA_{i_0}] \xrightarrow{Fw} [FA_{i_1}] \xrightarrow{(Fw)^*} FA_{i_0}.$$

However, that would have caused us to define r_f to be

$$FA_{i_1} \xrightarrow{(Fw)^*} FA_{i_0} \xrightarrow{Fw} [FA_{i_1}] \xrightarrow{(Fw)^*} [FA_{i_0}]$$

and l_f to be

$$[FA_{i_1}] \xrightarrow{(Fw)^*} [FA_{i_0}] \xrightarrow{Fw} FA_{i_1} \xrightarrow{(Fw)^*} FA_{i_0}$$

which are not equal.

We also need a description of the vertical transformations in the image of the W -transformations under \mathbf{DbI} . We claim that when $\mathbb{X} = \mathbf{DbI}(\mathcal{D})$ they are the following class of vertical transformations.

4.11. DEFINITION. Let W in \mathbf{C} be a class of arrows to be inverted which satisfies the Gabriel Zisman conditions. Let $F, G: \mathbf{DbI}(\mathbf{C}) \rightrightarrows \mathbb{X}$ be W -pseudo-functors and for any W -doubly marked path w in $\mathbf{DbI}(\mathbf{C})$, choose precompanion structures in \mathbb{X} ,

$$\begin{array}{ccc} & \xrightarrow{Fw} & \\ \downarrow & \varphi_w^l & \downarrow \\ \xrightarrow{l_{Fw}} & \xrightarrow{(Fw)'} & \xrightarrow{r_{Fw}} \\ \downarrow & \varphi_w^r & \downarrow \\ & \xrightarrow{(Fw)''} & \end{array}$$

and

$$\begin{array}{ccc} & \xrightarrow{Gw} & \\ \downarrow & \psi_w^l & \downarrow \\ \xrightarrow{l_{Gw}} & \xrightarrow{(Gw)'} & \xrightarrow{r_{Gw}} \\ \downarrow & \psi_w^r & \downarrow \\ & \xrightarrow{(Gw)''} & \end{array} .$$

For a vertical transformation $\alpha: F \rightrightarrows G$ we write

$$(\alpha_w)' = \begin{array}{ccc} & \xrightarrow{(Fw)'} & \\ \downarrow & (\varphi_w^l)^{-1} & \downarrow \\ & \xrightarrow{Fw} & \\ \downarrow & \alpha_w & \downarrow \\ & \xrightarrow{Gw} & \\ \downarrow & \psi_w^l & \downarrow \\ & \xrightarrow{(Gw)'} & \end{array} \quad \text{and} \quad (\alpha_w)'' = \begin{array}{ccc} & \xrightarrow{(Fw)''} & \\ \downarrow & (\varphi_w^r)^{-1} & \downarrow \\ & \xrightarrow{Fw} & \\ \downarrow & \alpha_w & \downarrow \\ & \xrightarrow{Gw} & \\ \downarrow & \psi_w^r & \downarrow \\ & \xrightarrow{(Gw)''} & \end{array}$$

Then α is a (*vertical*) W -transformation if there are double cells

$$\begin{array}{ccc} \xrightarrow{l_{Fw}} & & \xrightarrow{r_{Fw}} \\ \downarrow & \alpha_w^l & \downarrow \\ \xrightarrow{l_{Gw}} & & \xrightarrow{r_{Gw}} \end{array} \quad \text{and} \quad \begin{array}{ccc} \xrightarrow{r_{Fw}} & & \xrightarrow{r_{Gw}} \\ \downarrow & \alpha_w^r & \downarrow \\ \xrightarrow{r_{Gw}} & & \end{array} \tag{18}$$

that are compatible with the companion cells in the sense that the following four equations hold:

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccc}
 \xrightarrow{l_{Fw}} & (Fw)' & \\
 \downarrow \alpha_w^l & \downarrow (\alpha_w)' & \downarrow d_1 \alpha_w^l \\
 \xrightarrow{l_{Gw}} & (Gw)' & \\
 \downarrow \chi_{Gw}^l & & \\
 \downarrow v_{Gw}^l & & \\
 \hline
 \end{array} & = & \begin{array}{c}
 \begin{array}{ccc}
 \xrightarrow{l_{Fw}} & (Fw)' & \\
 \downarrow \chi_{Fw}^l & & \\
 \downarrow v_{Fw}^l & & \\
 \hline
 \end{array} \\
 \begin{array}{ccc}
 \downarrow d_1(\alpha_w)' & \text{id} & \downarrow d_1(\alpha_w)' \\
 \hline
 \end{array}
 \end{array} \\
 \\
 \begin{array}{c}
 \begin{array}{ccc}
 \downarrow d_0 \alpha_w^l & \text{id} & \downarrow d_0 \alpha_w^l \\
 \hline
 \end{array} \\
 \begin{array}{ccc}
 \downarrow \psi_{Gw}^l & & \downarrow v_{Gw}^l \\
 \hline
 \end{array} \\
 \begin{array}{ccc}
 \xrightarrow{l_{Gw}} & Gw & \\
 \hline
 \end{array}
 \end{array} & = & \begin{array}{c}
 \begin{array}{ccc}
 \downarrow \psi_{Fw}^l & & \downarrow v_{Fw}^l \\
 \hline
 \end{array} \\
 \begin{array}{ccc}
 \xrightarrow{l_{Fw}} & (Fw)' & \\
 \downarrow \alpha_w^l & \downarrow (\alpha_w)' & \\
 \xrightarrow{l_{Gw}} & (Gw)' & \\
 \hline
 \end{array}
 \end{array} \\
 \\
 \begin{array}{c}
 \begin{array}{ccc}
 (Fw)' & r_{Fw} & \\
 \downarrow (\alpha_w)' & \downarrow \alpha_w^r & \downarrow d_1 \alpha_w^r \\
 \xrightarrow{(Gw)'} & r_{Gw} & \\
 \downarrow \chi_{Gw}^r & & \\
 \downarrow v_{Gw}^r & & \\
 \hline
 \end{array} & = & \begin{array}{c}
 \begin{array}{ccc}
 (Fw)' & r_{Fw} & \\
 \downarrow \chi_{Fw}^r & & \\
 \downarrow v_{Fw}^r & & \\
 \hline
 \end{array} \\
 \begin{array}{ccc}
 \downarrow d_1 \alpha_w^r & \text{id} & \downarrow d_1 \alpha_w^r \\
 \hline
 \end{array}
 \end{array} \\
 \\
 \begin{array}{c}
 \begin{array}{ccc}
 \downarrow \psi_{Fw}^r & & \downarrow v_{Fw}^r \\
 \hline
 \end{array} \\
 \begin{array}{ccc}
 (Fw)' & r_{Fw} & \\
 \downarrow (\alpha_w)' & \downarrow \alpha_w^r & \\
 (Gw)' & r_{Gw} & \\
 \downarrow \psi_{Gw}^r & & \downarrow v_{Gw}^r \\
 \hline
 \end{array}
 \end{array} & = & \begin{array}{c}
 \begin{array}{ccc}
 \downarrow d_0(\alpha_w)' & \text{id} & \downarrow d_0(\alpha_w)' \\
 \hline
 \end{array} \\
 \begin{array}{ccc}
 \downarrow \psi_{Gw}^r & & \downarrow v_{Gw}^r \\
 \hline
 \end{array} \\
 \begin{array}{ccc}
 (Gw)' & r_{Gw} & \\
 \hline
 \end{array}
 \end{array}
 \end{array}
 \end{array}$$

4.12. REMARK. Note that since the precompanion structures for a given horizontal arrow are unique up to vertical isomorphism, the existence of the cells (18) does not depend on the choice of the precompanions structure. Furthermore, when we have a collection of these cells that satisfy the conditions of the definition above, conjugation by the isomorphisms of Lemma 3.20 will give another set of cells that satisfy the conditions.

Recall that an icon $\psi: F \Rightarrow G$ between W -homomorphisms is a W -icon if and only if for each $w \in W$ and choice of pseudo-inverses $(Fw)^*$ and $(Gw)^*$ of Fw and Gw respectively there is a 2-cell $\alpha_w^*: (Fw)^* \Rightarrow (Gw)^*$ such that the pastings of the diagrams

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccccc}
 FA & \xrightarrow{Fw} & FB & \xrightarrow{(Fw)^*} & FA \\
 \parallel & & \parallel & & \parallel \\
 GA & \xrightarrow{Gw} & GB & \xrightarrow{(Gw)^*} & GA \\
 \parallel & & \parallel & & \parallel \\
 \eta_{Fw} & & \eta_{Gw}^{-1} & &
 \end{array} \\
 \hline
 \end{array} & \text{and} & \begin{array}{c}
 \begin{array}{ccccc}
 FB & \xrightarrow{(Fw)^*} & FA & \xrightarrow{Fw} & FB \\
 \parallel & & \parallel & & \parallel \\
 GB & \xrightarrow{(Gw)^*} & GA & \xrightarrow{Gw} & GB \\
 \parallel & & \parallel & & \parallel \\
 \varepsilon_{Fw}^{-1} & & \varepsilon_{Gw} & &
 \end{array} \\
 \hline
 \end{array}
 \end{array}$$

are identity 2-cells. It is now straightforward to prove the following result.

4.13. LEMMA. *An icon $\psi: F \Rightarrow G$ between W -homomorphisms is a W -icon if and only if the corresponding vertical transformation $\mathbf{Dbl}(\psi): \mathbf{Dbl}(F) \Rightarrow \mathbf{Dbl}(G)$ is a W -transformation.*

Note that the horizontal composition of a W -transformation with a pseudo-functor is again a W -transformation. We conclude that, since every weakly globular double category is vertically equivalent to one of the form $\mathbf{Dbl}(\mathcal{D})$ for some bicategory \mathcal{D} , the vertical universal property of $\mathbf{Dbl}(\mathbf{C}(W^{-1}))$ is as stated in the following theorem.

4.14. THEOREM. *For any category \mathbf{C} with a collection W of arrows satisfying the conditions to form a bicategory of fractions, composition with the functor $\mathbf{Dbl}(J_W): \mathbf{Dbl}\mathbf{C} \rightarrow \mathbf{Dbl}(\mathbf{C}(W^{-1}))$ induces an equivalence of categories*

$$\mathbf{WGDbl}_{\text{ps},v}(\mathbf{Dbl}(\mathbf{C}(W^{-1})), \mathbb{X}) \simeq \mathbf{WGDbl}_{\text{ps},v,W}(\mathbf{Dbl}(\mathbf{C}), \mathbb{X})$$

where $\mathbf{WGDbl}_{\text{ps},v,W}(\mathbf{Dbl}(\mathbf{C}), \mathbb{X})$ is the category of W -pseudo-functors and vertical W -transformations.

We would like to have a horizontal universal property for $\mathbf{Dbl}(\mathbf{C}(W^{-1}))$ as well, but there is no obvious one at this point. Furthermore, the description of the arrows to be inverted in $\mathbf{Dbl}(\mathbf{C})$ is rather cumbersome. So in the next section we will introduce a new candidate for the weakly globular double category of fractions. In order to get a better handle on the horizontal arrows that will be inverted and to keep the complexity of the construction down, we will just consider the case of a class of horizontal arrows in a weakly globular double category of the form $H\mathbf{C}$ where \mathbf{C} is a category. The resulting double category $\mathbf{C}\{W\}$ will be vertically equivalent to the image $\mathbf{Dbl}(\mathbf{C}(W^{-1}))$ of the bicategory of fractions and will hence inherit its vertical universal property from the one given above. However, the horizontal universal property will be a completely new one.

5. The construction of $\mathbf{C}\{W\}$

In this section we introduce a weakly globular double category $\mathbf{C}\{W\}$ that is vertically equivalent to $\mathbf{Dbl}(\mathbf{C}(W^{-1}))$, and hence shares its vertical universal property. Furthermore, we will show that it has an additional horizontal universal property as a double category with companions and conjoints. The two properties together determine $\mathbf{C}\{W\}$ up to vertical and horizontal equivalence of weakly globular double categories.

Furthermore, one draw-back of both the category and bicategory of fractions constructions is that the resulting category or bicategory does not necessarily have small hom-sets. The same thing is true for $\mathbf{Dbl}(\mathbf{C}(W^{-1}))$, but not for $\mathbf{C}\{W\}$: it has small homsets in both directions, both for arrows and double cells.

The reason we can achieve this is as follows. For categories and bicategories of fractions we kept the same objects, but added new arrows and 2-cells, in order to obtain

(pseudo) inverses for the arrows in the class W . However, for the weakly globular double category of fractions, we are not looking for the structure of a pseudo inverse, but rather for the structure of a precompanion, since we saw in Propositions 3.21 and 3.22 that precompanions in weakly globular double categories correspond to equivalences in bicategories and we established the universal property of $\mathbf{DbI}(\mathbf{C}(W^{-1}))$ (Theorem 4.14, with Definitions 4.9 and 4.11) in terms of precompanions. Intuitively we can think of precompanions as having a horizontal pseudo inverse up to a vertical isomorphism. So we can add new objects with our new arrows, as long as the new objects are vertically isomorphic to the old ones. The construction of $\mathbf{DbI}(\mathbf{C}(W^{-1}))$ also added new objects, but there the idea was that the new objects were dictated by the new paths. In this section we want to give a free construction that starts by adding new objects and lets the arrows and double cells follow from what is needed to obtain the precompanion structure. We will add an additional object for each arrow w in W , with a horizontal arrow to the domain of the original arrow. The idea is that this additional arrow will be l_w , so that we need to give $w \circ l_w$ the structure of a companion. We will not take everything as free as possible, because the vertical universal property is in terms of pseudo-functors, and this gives us a bit of flexibility to make additional assumptions. For instance, we will make l_w a horizontal isomorphism. This tightening in the horizontal direction is what will give us eventually a horizontal universal property for the resulting weakly globular double category.

The idea of the construction comes from a kind of symmetrization of the original construction of a (bi)category of (right) fractions. An arrow in a (bi)category of fractions is given as a span $A \xleftarrow{w} A' \xrightarrow{f} B$ where w is in the class of arrows to be inverted. Sometimes these arrows are also called generalized arrows. We could think of all objects that are connected to each other by arrows that are to be inverted as different representations of the same object in a quotient of the category. The Gabriel-Zisman conditions assure us that an arrow in the quotient category can be given by taking a different representation followed by an arrow out of the new representation in the original category. The (bi)category of fractions construction does not go as far as to take the quotient, but rather keeps track of how the new representation is related to the domain of the generalized arrow. For the weakly globular double category of fractions, we will allow for this type of change of representation in both the domain and the codomain of an arrow. Furthermore, rather than using generalized arrows, it turns out that we can do this by adding new objects. The new objects will correspond to the arrows in W , so they will be objects with a second representation (which may be thought of as some type of refinement). Horizontal arrows will then be given by arrows between the refinements. In the traditional constructions of (bi)categories of fractions one always needs to consider an equivalence relation for the cells at the highest level and make two of those cells equal if they agree on a further refinement. In the case of weakly globular double categories we can use the vertical structure to deal with this as we will show below.

5.1. THE CONSTRUCTION. We define the weakly globular double category $\mathbf{C}\{W\}$ as follows:

- *Objects* are arrows in W , $(w) = (A \xrightarrow{w} B)$.
- A *vertical arrow* $(u_1, C, u_2): (A_1 \xrightarrow{w_1} B) \dashrightarrow (A_2 \xrightarrow{w_2} B)$ is an equivalence class of commutative diagrams

$$\begin{array}{ccc} A_1 & \xrightarrow{w_1} & B \\ u_1 \uparrow & & \parallel \\ C & & \\ u_2 \downarrow & & \\ A_2 & \xrightarrow{w_2} & B \end{array}$$

with $w_1 u_1 = w_2 u_2$ in W . Two such diagrams, (u_1, C, u_2) and (v_1, D, v_2) , are equivalent (i.e., represent the same vertical arrow) when there are arrows $C \xleftarrow{r_1} E \xrightarrow{r_2} D$ such that

$$\begin{array}{ccccc} & & A_1 & & \\ & u_1 \nearrow & & \nwarrow v_1 & \\ C & \xleftarrow{r_1} & E & \xrightarrow{r_2} & D \\ & u_2 \searrow & & \swarrow v_2 & \\ & & A_2 & & \end{array}$$

commutes and $w_1 u_1 r_1 = w_1 v_1 r_2 = w_2 v_2 r_2 = w_2 u_2 r_1$ is in W .

- A *horizontal arrow* $(A \xrightarrow{w} B) \xrightarrow{f} (A' \xrightarrow{w'} B')$ is given by an arrow $A \xrightarrow{f} A'$ in \mathbf{C} . We will usually draw this as $(B \xleftarrow{w} A) \xrightarrow{f} (A' \xrightarrow{w'} B')$.
- A *double cell*

$$\begin{array}{ccc} (w_1) & \xrightarrow{f_1} & (w'_1) \\ [u_1, C, u_2] \downarrow & (\varphi) & \downarrow [u'_1, C', u'_2] \\ (w_2) & \xrightarrow{f_2} & (w'_2). \end{array}$$

is an equivalence class of commutative diagrams of the form

$$\begin{array}{ccccccc} B & \xleftarrow{w_1} & A_1 & \xrightarrow{f_1} & A'_1 & \xrightarrow{w'_1} & B' \\ \parallel & & \uparrow u_1 & & \uparrow u'_1 & & \parallel \\ & & C & \xrightarrow{\varphi} & C' & & \\ & & \downarrow u_2 & & \downarrow u'_2 & & \\ B & \xleftarrow{w_2} & A_2 & \xrightarrow{f_2} & A'_2 & \xrightarrow{w'_2} & B' \end{array} \tag{19}$$

The diagram (19) is equivalent to the diagram

$$\begin{array}{ccccccc}
 B & \xleftarrow{w_1} & A_1 & \xrightarrow{f_1} & A'_1 & \xrightarrow{w'_1} & B' \\
 \parallel & & \uparrow v_1 & & \uparrow v'_1 & & \parallel \\
 & & D & \xrightarrow{\psi} & D' & & \\
 & & \downarrow v_2 & & \downarrow v'_2 & & \\
 B & \xleftarrow{w_2} & A_2 & \xrightarrow{f_2} & A'_2 & \xrightarrow{w'_2} & B'
 \end{array}$$

if and only if there are arrows r, s, r', s' and χ as in

$$\begin{array}{ccc}
 C & \xrightarrow{\varphi} & C' \\
 r \uparrow & & \uparrow r' \\
 E & \xrightarrow{\chi} & E' \\
 s \downarrow & & \downarrow s' \\
 D & \xrightarrow{\psi} & D'
 \end{array}$$

such that $w_1 u_1 r \in W$, $w'_1 u'_1 r' \in W$, and making the following diagram commute:

$$\begin{array}{ccccc}
 \xleftarrow{w_1} & & \xrightarrow{f_1} & & \xrightarrow{w'_1} \\
 \parallel & \swarrow u_1 & \nearrow v_1 & \xrightarrow{\psi} & \nearrow v'_1 & \parallel \\
 & \nearrow r & \xrightarrow{s} & \xrightarrow{\chi} & \nearrow s' & \\
 & \searrow \varphi & \nearrow v_2 & \xrightarrow{r'} & \searrow v'_2 & \\
 \xleftarrow{w_2} & & \xrightarrow{f_2} & & \xrightarrow{w'_2}
 \end{array} \tag{20}$$

Note that a double cell may not have a representative for each combination of representatives of each vertical arrows, however, there is always a representative with the given representative of the codomain vertical arrow as in the following lemma.

5.2. LEMMA. *Given a representative of a double cell,*

$$\begin{array}{ccccccc}
 B & \xleftarrow{w_1} & A_1 & \xrightarrow{f_1} & A'_1 & \xrightarrow{w'_1} & B' \\
 \parallel & & \uparrow u_1 & & \uparrow u'_1 & & \parallel \\
 & & C & \xrightarrow{\varphi} & C' & & \\
 & & \downarrow u_2 & & \downarrow u'_2 & & \\
 B & \xleftarrow{w_2} & A_2 & \xrightarrow{f_2} & A'_2 & \xrightarrow{w'_2} & B',
 \end{array} \tag{21}$$

and another representative (v_1, D, v_2) of the codomain vertical arrow, then there are arrows $r: E \rightarrow C$ and $\psi: E \rightarrow D$ such that

$$\begin{array}{ccccccc}
 B & \xleftarrow{w_1} & A_1 & \xrightarrow{f_1} & A'_1 & \xrightarrow{w'_1} & B' \\
 \parallel & & \uparrow u_1 r & & \uparrow v_1 & & \parallel \\
 & & E & \xrightarrow{\psi} & D & & \\
 & & \downarrow u_2 r & & \downarrow v_2 & & \\
 B & \xleftarrow{w_2} & A_2 & \xrightarrow{f_2} & A'_2 & \xrightarrow{w'_2} & B'
 \end{array} \tag{22}$$

and (21) represent the same double cell.

PROOF. Since

$$\begin{array}{ccc}
 A'_1 \xrightarrow{w'_1} B' & & A'_1 \xrightarrow{w'_1} B' \\
 \uparrow u'_1 & & \uparrow v_1 \\
 C' & \text{and} & D \\
 \downarrow u'_2 & & \downarrow v_2 \\
 A'_2 \xrightarrow{w'_2} B' & & A'_2 \xrightarrow{w'_2} B'
 \end{array}$$

represent the same vertical arrow, there are arrows r_1 and r_2 such that $w'_1 u'_1 r_1, w'_1 v_1 r_2 \in W$ and

$$\begin{array}{ccccc}
 & & A'_1 & & \\
 & u'_1 \nearrow & & \nwarrow v_1 & \\
 C' & \xleftarrow{r_1} & F & \xrightarrow{r_2} & D \\
 & u'_2 \searrow & & \swarrow v_2 & \\
 & & A'_2 & &
 \end{array}$$

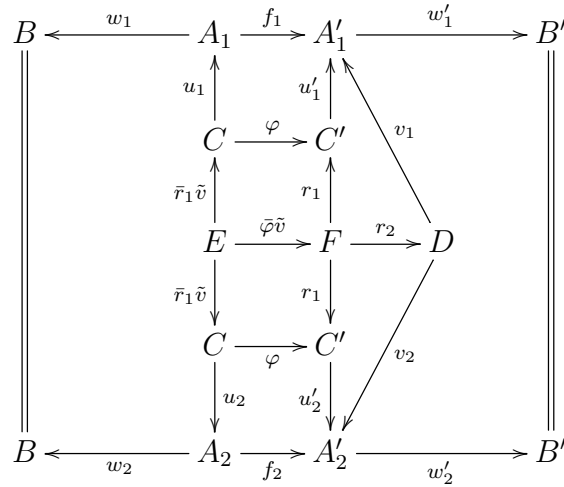
commutes. By condition **CF2** there are arrows $\bar{r}_1 \in W$ and $\bar{\varphi}$ that make the following square commute,

$$\begin{array}{ccc}
 \bar{F} & \xrightarrow{\bar{\varphi}} & F \\
 \bar{r}_1 \downarrow & & \downarrow w'_2 v_2 r_1 \\
 C & \xrightarrow{\varphi} C' & \xrightarrow{w'_2 v_2} B'
 \end{array}$$

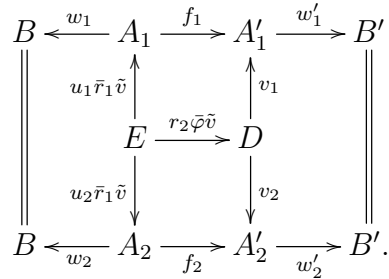
By condition **CF3** there exists an arrow $(\tilde{v}: E \rightarrow \bar{F}) \in W$ such that

$$\begin{array}{ccc}
 E & \xrightarrow{\bar{\varphi} \tilde{v}} & F \\
 \bar{r}_1 \tilde{v} \downarrow & & \downarrow r_1 \\
 C & \xrightarrow{\varphi} & C'
 \end{array}$$

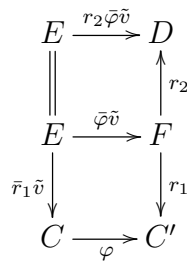
commutes. We can use all this to construct the following commutative diagram



So we obtain a double cell



This cell is equivalent to (21) since we have a commutative diagram



that fits in a diagram like (20). By taking $r = \bar{r}_1 \tilde{v}$ and $\psi = r_2 \bar{\varphi} \tilde{v}$ we see that we have indeed a cell as in (22). ■

5.3. EXAMPLE. In the example where $\mathbf{C} = \mathbf{Atlases}$ and W the class of atlas refinements, the situation is as follows. The objects of $\mathbf{C}\{W\}$ are atlases with a refinement. There is a vertical arrow between any two such objects if there is a common refinement, and a horizontal arrow corresponds to an atlas map between the refinements. Double cells correspond to common refinements where the two atlas maps corresponding to the horizontal arrows agree.

5.4. SOME BASIC PROPERTIES. We will now study some of the basic properties of $\mathbf{C}\{W\}$ and in particular, we will show that $\mathbf{C}\{W\}$ is a weakly globular double category.

The following lemma is equivalent to stating that the codomain map $d_1: \mathbf{C}\{W\}_1 \rightarrow \mathbf{C}\{W\}_0$, sending squares and horizontal arrows to their horizontal codomains is an isofibration.

5.5. LEMMA. *For every pair of a horizontal arrow $(w_2) \xrightarrow{g} (w'_2)$ and a vertical arrow $(w'_1) \xrightarrow{[v_1, D, v_2]} (w'_2)$ in $\mathbf{C}\{W\}$, there is a double cell*

$$\begin{array}{ccc} (w_1) & \xrightarrow{f} & (w'_1) \\ [u_1, C, u_2] \downarrow \bullet & (\varphi) & \downarrow \bullet [v_1, D, v_2] \\ (w_2) & \xrightarrow{g} & (w'_2) . \end{array}$$

PROOF. Let $A_2 \xrightarrow{w_2} B$ and $A'_i \xrightarrow{w'_i} B'$, for $i = 1, 2$ in W . By condition **CF2** there exists a commutative square

$$\begin{array}{ccc} E & \xrightarrow{\varphi_E} & D \\ \bar{v}_2 \downarrow & & \downarrow w'_2 v_2 \\ A_2 & \xrightarrow{w'_2 g} & B' \end{array}$$

in \mathbf{C} . Since $w'_2 \in W$, there is an arrow $(C \xrightarrow{\tilde{w}'_2} E) \in W$ such that

$$\begin{array}{ccc} C & \xrightarrow{\varphi_E \tilde{w}'_2} & D \\ \bar{v}_2 \tilde{w}'_2 \downarrow & & \downarrow v_2 \\ A_2 & \xrightarrow{g} & A'_2 \end{array}$$

also commutes (by condition **CF3**). So let $\varphi_C = \varphi_E \tilde{w}'_2 u_2 = \bar{v}_2 \tilde{w}'_2$. Note that $u_2 \in W$, and the diagram

$$\begin{array}{ccccc} (B \xleftarrow{w_2 u_2} C) & \xrightarrow{v_1 \varphi_C} & (A'_1 \xrightarrow{w'_1} B') & & \\ \parallel & & \uparrow v_1 & & \parallel \\ & & C \xrightarrow{\varphi_C} D & & \\ & & \downarrow u_2 \quad \downarrow v_2 & & \\ (B \xleftarrow{w_2} A_2) & \xrightarrow{g} & (A'_2 \xrightarrow{w'_2} B') & & \end{array}$$

commutes in \mathbf{C} and satisfies all the conditions to represent a double cell in $\mathbf{C}\{W\}$. ■

The following lemma implies that the vertical arrow category is a posetal groupoid.

5.6. LEMMA. *There is a vertical arrow $(A \xrightarrow{w} B) \dashrightarrow (A' \xrightarrow{w'} B')$ if and only if $B = B'$ and furthermore, this arrow is unique.*

PROOF. It is obvious that the existence of such a vertical arrow implies that $B = B'$.

Now suppose that we have objects $(A \xrightarrow{w} B)$ and $(A' \xrightarrow{w'} B)$ in $\mathbf{C}\{W\}$. Since W satisfies condition **CF2** above, there is a commutative square

$$\begin{array}{ccc} C & \xrightarrow{u} & A \\ u' \downarrow & & \downarrow w \\ A' & \xrightarrow{w'} & B \end{array}$$

in \mathbf{C} with $u' \in W$ and consequently, $wu = w'u' \in W$. So

$$\begin{array}{ccc} A & \xrightarrow{w} & B \\ u \uparrow & & \parallel \\ C & & \\ u' \downarrow & & \\ A' & \xrightarrow{w'} & B \end{array}$$

represents a vertical arrow as required.

To show that this arrow is unique, suppose that we have two representatives

$$\begin{array}{ccc} A & \xrightarrow{w} & B \\ u \uparrow & & \parallel \\ C & & \\ u' \downarrow & & \\ A' & \xrightarrow{w'} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{w} & B \\ v \uparrow & & \parallel \\ D & & \\ v' \downarrow & & \\ A' & \xrightarrow{w'} & B. \end{array}$$

By conditions **CF1** and **CF2**, there are arrows $C \xleftarrow{r} E \xrightarrow{s} D$ such that $wur = wvs \in W$. Now it follows that $w'v's = wvs = wur = w'u'r$ and $w' \in W$. By condition **CF3**, there is an arrow $E' \xrightarrow{\tilde{w}} E$ such that $v's\tilde{w} = u'r\tilde{w}$. So the following diagram commutes:

$$\begin{array}{ccccc} & & A & & \\ & u \nearrow & & \nwarrow v & \\ C & \xleftarrow{r\tilde{w}} & E' & \xrightarrow{s\tilde{w}} & D \\ & \searrow u' & & \swarrow v' & \\ & & A' & & \end{array}$$

and hence $[u, C, u'] = [v, D, v']$. ■

Note that this implies that π_0 of the vertical arrow category of $\mathbf{C}\{W\}$ is isomorphic to the set of objects of \mathbf{C} .

The category of double cells in $\mathbf{C}\{W\}$ with vertical composition (and horizontal arrows as objects) is also posetal, as stated in the following lemma.

5.7. LEMMA. *For any pairs of horizontal arrows and vertical arrows fitting together as in*

$$\begin{array}{ccc} (w_1) & \xrightarrow{f_1} & (w'_1) \\ \downarrow [u_1, C, u_2] & & \downarrow [v_1, D, v_2] \\ (w_2) & \xrightarrow{f_2} & (w'_2) \end{array}$$

there is at most one double cell in $\mathbf{C}\{W\}$ that fills this.

PROOF. Suppose that both (φ) and (ψ) fit in this frame. By Lemma 5.2 we may assume that φ and ψ are of the following form:

$$\begin{array}{ccc} \left\| \begin{array}{ccc} \xleftarrow{w_1} & \xrightarrow{f_1} & \xrightarrow{w'_1} \\ \uparrow u'_1 & \xrightarrow{\varphi} & \uparrow v_1 \\ \downarrow u'_2 & & \downarrow v_2 \\ \xleftarrow{w_2} & \xrightarrow{f_2} & \xrightarrow{w'_2} \end{array} \right\| & \text{and} & \left\| \begin{array}{ccc} \xleftarrow{w_1} & \xrightarrow{f_1} & \xrightarrow{w'_1} \\ \uparrow u''_1 & \xrightarrow{\psi} & \uparrow v_1 \\ \downarrow u''_2 & & \downarrow v_2 \\ \xleftarrow{w_2} & \xrightarrow{f_2} & \xrightarrow{w'_2} \end{array} \right\| \end{array}$$

Since $[u'_1, u'_2] = [u_1, u_2] = [u''_1, u''_2]$, there are arrows r' and r'' such that $w_1 u'_1 r', w_1 u''_1 r'' \in W$ and the following diagram commutes

$$\begin{array}{ccc} & u'_1 & & u''_1 & \\ & \nearrow & & \nearrow & \\ & r' & & r'' & \\ & \nwarrow & & \nwarrow & \\ & u'_2 & & u''_2 & \end{array}$$

Note that $w'_1 v_1 \varphi r' = w'_1 f_1 u'_1 r' = w'_1 f_1 u''_1 r'' = w'_1 v_1 \psi r''$. Since $w'_1 v_1 \in W$, we can apply condition **CF3** to obtain an arrow \tilde{w} such that $\varphi r' \tilde{w} = \psi r'' \tilde{w}$. Finally, we see that

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \leftarrow w_1 & \xrightarrow{f_1} & w'_1 \\
 \uparrow u'_1 & \xrightarrow{\varphi} & \uparrow v_1 \\
 \leftarrow w_2 & \xrightarrow{f_2} & w'_2
 \end{array} & \sim & \begin{array}{ccc}
 \leftarrow w_1 & \xrightarrow{f_1} & w'_1 \\
 \uparrow u'_1 r' \tilde{w} & \xrightarrow{\varphi r' \tilde{w}} & \uparrow v_1 \\
 \leftarrow w_2 & \xrightarrow{f_2} & w'_2
 \end{array} \\
 \\
 \begin{array}{ccc}
 \leftarrow w_1 & \xrightarrow{f_1} & w'_1 \\
 \uparrow u''_1 r'' \tilde{w} & \xrightarrow{\psi r'' \tilde{w}} & \uparrow v_1 \\
 \leftarrow w_2 & \xrightarrow{f_2} & w'_2
 \end{array} & \sim & \begin{array}{ccc}
 \leftarrow w_1 & \xrightarrow{f_1} & w'_1 \\
 \uparrow u''_1 & \xrightarrow{\psi} & \uparrow v_1 \\
 \leftarrow w_2 & \xrightarrow{f_2} & w'_2
 \end{array}
 \end{array} =$$

So (φ) and (ψ) represent the same double cell. ■

5.8. **REMARK.** Since the vertical arrow category of $\mathbf{C}\{W\}$ is posetal and groupoidal, this lemma implies that the vertical category $(\mathbf{C}\{W\})_1$ of horizontal arrows and double cells is also posetal and groupoidal.

5.9. **THEOREM.** *The double category $\mathbf{C}\{W\}$ is weakly globular.*

PROOF. By Lemma 5.6 the vertical arrow category of $\mathbf{C}\{W\}$ is groupoidal and posetal as required. Furthermore, the codomain $d_1: (\mathbf{C}\{W\})_1 \rightarrow (\mathbf{C}\{W\})_0$ is an isofibration. For any pair of a horizontal arrow $(w_2) \xrightarrow{g} (w'_2)$ and a vertical arrow $(w'_1) \xrightarrow{[v_1, D, v_2]} (w'_2)$, there is a double cell

$$\begin{array}{ccc}
 (w_1) & \xrightarrow{f} & (w'_1) \\
 \downarrow [u_1, C, u_2] & (\varphi) & \downarrow [v_1, D, v_2] \\
 (w_2) & \xrightarrow{g} & (w'_2)
 \end{array}$$

by Lemma 5.5. And this double cell is vertically invertible since the vertical category $(\mathbf{C}\{W\})_1$ is posetal groupoidal by Lemma 5.7. ■

5.10. **COMPOSITION.** *Composition of vertical arrows* is defined using condition **CF2**. For

a pair of composable vertical arrows,

$$\begin{array}{ccc}
 A_1 & \xrightarrow{w_1} & B \\
 u_1 \uparrow & & \parallel \\
 C & & \\
 u_2 \downarrow & & \\
 A_2 & \xrightarrow{w_2} & B \\
 v_1 \uparrow & & \parallel \\
 D & & \\
 v_2 \downarrow & & \\
 A_3 & \xrightarrow{w_3} & B,
 \end{array}$$

there is a commutative square in \mathbf{C} of the form

$$\begin{array}{ccc}
 E & \xrightarrow{\bar{v}_1} & C \\
 \bar{u}_2 \downarrow & & \downarrow w_2 u_2 \\
 D & \xrightarrow{w_2 v_1} & A_2
 \end{array}$$

with $u_2 \bar{v}_1 = v_1 \bar{u}_2 \in W$ (by Condition **CF2**). So $w_1 u_1 \bar{v}_1 = w_2 u_2 \bar{v}_1 = w_2 v_1 \bar{u}_2 = w_3 v_2 \bar{u}_2$, and a representative for the vertical composition is given by

$$\begin{array}{ccc}
 A_1 & \xrightarrow{w_1} & B \\
 u_1 \bar{v}_1 \uparrow & & \parallel \\
 D & & \\
 v_2 \bar{u}_2 \downarrow & & \\
 A_3 & \xrightarrow{w_3} & B
 \end{array}$$

Note that this is well-defined on equivalence classes, doesn't depend on the choice of the square in condition **CF2**, and is associative and unital, because the vertical arrow category is posetal.

Analogously, to define *vertical composition of double cells*,

$$\begin{array}{ccccc}
 (B \xleftarrow{w_1} A_1) & \xrightarrow{f_1} & (A'_1 \xrightarrow{w'_1} B') & & \\
 \parallel & & \parallel & & \\
 u_1 \uparrow & & u'_1 \uparrow & & \\
 C & (\varphi) & C' & & \\
 u_2 \downarrow & & u'_2 \downarrow & & \\
 (B \xleftarrow{w_2} A_2) & \xrightarrow{f_2} & (A'_2 \xrightarrow{w'_2} B') & & \\
 \parallel & & \parallel & & \\
 v_1 \uparrow & & v'_1 \uparrow & & \\
 D & (\psi) & D' & & \\
 v_2 \downarrow & & v'_2 \downarrow & & \\
 (B \xleftarrow{w_3} A_3) & \xrightarrow{f_3} & (A'_3 \xrightarrow{w'_3} B) & & \\
 \parallel & & \parallel & &
 \end{array}$$

we only need to give a representative to fit the square with the composed boundary arrows. Without loss of generality we may assume that the double cells we are composing are such that $\varphi: C \rightarrow C'$ and $\psi: D \rightarrow D'$ fit in the diagrams given. So suppose that we have commutative squares

$$\begin{array}{ccc} E & \xrightarrow{\bar{v}_1} & C \\ \bar{u}_2 \downarrow & & \downarrow u_2 \\ D & \xrightarrow{v_1} & A_2 \end{array} \quad \begin{array}{ccc} E' & \xrightarrow{\bar{v}'_1} & C \\ \bar{u}'_2 \downarrow & & \downarrow u'_2 \\ D' & \xrightarrow{v'_1} & A'_2, \end{array}$$

giving the following composition of the vertical arrows involved:

$$\begin{array}{ccccc} (B \xleftarrow{w_1} A_1) & \xrightarrow{f_1} & (A'_1 \xrightarrow{w'_1} B') & & \\ \parallel & & & & \parallel \\ & u_1 \bar{v}_1 \uparrow & u'_1 \bar{v}'_1 \uparrow & & \\ & E & E' & & \\ & v_2 \bar{u}_2 \downarrow & v'_2 \bar{u}'_2 \downarrow & & \\ (B \xleftarrow{w_3} A_3) & \xrightarrow{f_3} & (A'_3 \xrightarrow{w'_3} B') & & \end{array}$$

$(\psi) \cdot (\varphi)$

We will now construct $(\xi) = (\psi) \cdot (\varphi)$. Note that it is sufficient to find an arrow $\bar{E} \xrightarrow{r} E$ in W with an arrow $\xi: \bar{E} \rightarrow E'$ such that $\bar{v}'_1 \xi = \varphi \bar{v}_1 r$ and $\bar{u}'_2 \xi = \psi \bar{u}_2 r$.

Apply condition **CF2** to obtain a commutative square

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\bar{\varphi}} & E' \\ \bar{v}'_1 \downarrow & & \downarrow \bar{v}'_1 \\ E & \xrightarrow{\bar{v}_1} & C \xrightarrow{\varphi} C'. \end{array}$$

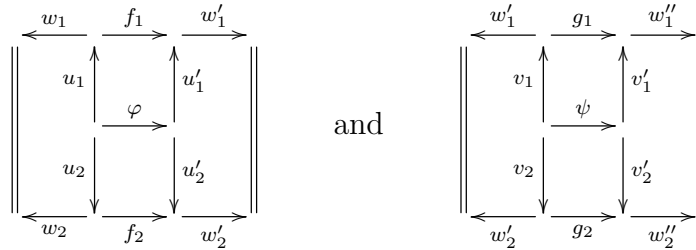
in \mathbf{C} with $\bar{v}'_1 \in W$. Then

$$\begin{aligned} w'_2 v'_1 \bar{u}'_2 \bar{\varphi} &= w'_2 u'_2 \bar{v}'_1 \bar{\varphi} \\ &= w'_2 u'_2 \varphi \bar{v}_1 \bar{v}'_1 \\ &= w'_2 f_2 u_2 \bar{v}_1 \bar{v}'_1 \\ &= w'_2 f_2 v_1 \bar{u}_2 \bar{v}'_1 \\ &= w'_2 v'_1 \psi \bar{u}_2 \bar{v}'_1 \end{aligned}$$

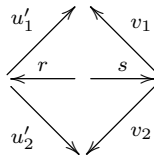
Since $w'_2 v'_1 \in W$, there is an arrow $s: \bar{E} \rightarrow \tilde{E}$ such that $\bar{u}'_2 \bar{\varphi} s = \psi \bar{u}_2 \bar{v}'_1 s$ (by condition **CF3**). So let $r = \bar{v}'_1 s$ and $\xi = \bar{\varphi} s$.

Composition of horizontal arrows in $\mathbf{C}\{W\}$ is defined directly in terms of composition of arrows in \mathbf{C} : the horizontal composition of $(B \xleftarrow{w} A) \xrightarrow{f} (A' \xrightarrow{w'} B')$ and $(B' \xleftarrow{w'} A') \xrightarrow{f'} (A'' \xrightarrow{w''} B'')$ is $(B \xleftarrow{w} A) \xrightarrow{f'f} (A' \xrightarrow{w''} B'')$.

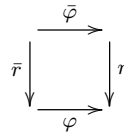
Horizontal composition of double cells is defined as follows. Let



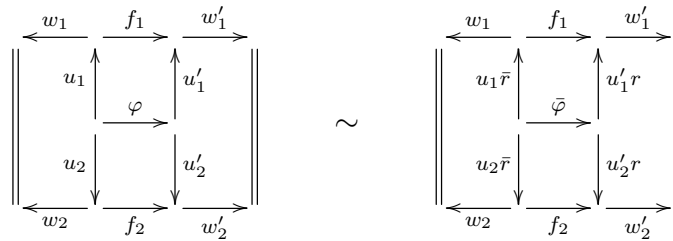
be two horizontally composable double cells. This means that $[u'_1, u'_2] = [v_1, v_2]$, so there exist arrows r and s such that $w'_1 u'_1 r \in W$ and the following diagram commutes,



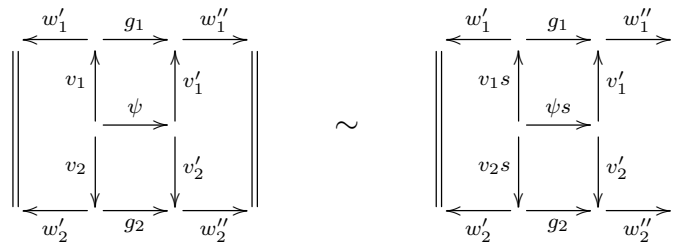
By conditions **CF2** and **CF3** there is a commutative square of the form



with $\bar{r} \in W$. Now it is not difficult to check that



and



Since $u'_1 r = v_1 s$ and $u'_2 r = v_2 s$, the horizontal composition is given by

$$\begin{array}{ccccc}
 & \xleftarrow{w_1} & \xrightarrow{g_1 f_1} & \xrightarrow{w'_1} & \\
 & \uparrow u_1 \bar{r} & \xrightarrow{\psi s \bar{\varphi}} & \uparrow v'_1 & \\
 & \downarrow u_2 \bar{r} & & \downarrow v'_2 & \\
 & \xleftarrow{w_2} & \xrightarrow{g_2 f_2} & \xrightarrow{w'_2} & .
 \end{array}$$

Middle four interchange holds because the double cells are posetal in the vertical direction.

5.11. **REMARK.** For any objects $w_1, w_2 \in \mathbf{C}\{W\}_{00}$, the hom-set $\mathbf{C}\{W\}_h(w_1, w_2)$ is small since it is isomorphic to $\mathbf{C}(d_0(w_1), d_0(w_2))$ and $\mathbf{C}\{W\}_v(A, B)$ is small since it contains at most one element. Furthermore, for any given pair of horizontal or vertical arrows, there is a set of double cells with those arrows as domain and codomain, since given any frame of horizontal and vertical arrows, there is at most one double cell that fills it. This shows that although $\mathbf{C}(W^{-1})$ is generally not a bicategory with small hom-sets (and consequently, its weakly globular double category of marked paths does not have small horizontal hom-sets either), the double category $\mathbf{C}\{W\}$ does. This is one of the advantages of $\mathbf{C}\{W\}$ over $\mathbf{C}(W^{-1})$.

5.12. **THE FUNDAMENTAL BICATEGORY OF $\mathbf{C}\{W\}$.** In this section we establish the relationship between the weakly globular double category $\mathbf{C}\{W\}$ and the bicategory of fractions $\mathbf{C}(W^{-1})$. We will construct a biequivalence of bicategories

$$\omega: \mathbf{Bic}(\mathbf{C}\{W\}) \rightarrow \mathbf{C}(W^{-1}),$$

with pseudo-inverse $\alpha: \mathbf{C}(W^{-1}) \rightarrow \mathbf{Bic}(\mathbf{C}\{W\})$.

The objects of the associated bicategory $\mathbf{Bic}(\mathbf{C}\{W\})$ are the connected components of the vertical arrow category of $\mathbf{C}\{W\}$. By Lemma 5.6 the map

$$\omega_0: \mathbf{Bic}(\mathbf{C}\{W\})_0 \rightarrow \mathbf{C}(W^{-1})_0 = \mathbf{C}_0,$$

sending the connected component of $(A \xrightarrow{w} B)$ to B is an isomorphism.

An arrow in $\mathbf{Bic}(\mathbf{C}\{W\})$ corresponds to a horizontal arrow in $\mathbf{C}\{W\}$, so it is given by a diagram $B \xleftarrow{w} A \xrightarrow{f} A' \xrightarrow{w'} B'$ in \mathbf{C} with $w, w' \in W$. We define

$$\omega_1(B \xleftarrow{w} A \xrightarrow{f} A' \xrightarrow{w'} B') = (B \xleftarrow{w} A \xrightarrow{w' f} B').$$

A 2-cell φ from $B \xleftarrow{w_1} A_1 \xrightarrow{f_1} A'_1 \xrightarrow{w'_1} B'$ to $B \xleftarrow{w_2} A_2 \xrightarrow{f_2} A'_2 \xrightarrow{w'_2} B'$ in $\mathbf{Bic}(\mathbf{C}\{W\})$

corresponds to an equivalence class of commutative diagrams of the form

$$\begin{array}{ccccc}
 B & \xleftarrow{w_1} & A_1 & \xrightarrow{f_1} & A'_1 & \xrightarrow{w'_1} & B' \\
 \parallel & & \uparrow u_1 & & \uparrow v_1 & & \parallel \\
 & & C & \xrightarrow{\varphi} & C' & & \\
 & & \downarrow u_2 & & \downarrow v_2 & & \\
 B & \xleftarrow{w_2} & A_2 & \xrightarrow{f_2} & A'_2 & \xrightarrow{w'_2} & B'
 \end{array}$$

This means that $w'_1 f_1 u_1 = w'_1 v_1 \varphi = w'_2 v_2 \varphi = w'_2 f_2 u_2$ and $w_1 u_1 = w_2 u_2$, so we define

$$\omega_2(\varphi) = \left(\begin{array}{ccccc} & & A_1 & & \\ & w_1 & \uparrow u_1 & & w'_1 f_1 \\ B & & C & & B' \\ & w_2 & \downarrow u_2 & & w'_2 f_2 \\ & & A_2 & & \end{array} \right)$$

The fact that this is well defined on equivalence classes of double cells follows from the following lemma about the 2-cells in a bicategory of fractions of a category.

5.13. LEMMA. *For any category \mathbf{C} with a class of arrows W satisfying the conditions **CF1**, **CF2**, and **CF3** above, the bicategory of fractions $\mathbf{C}(W^{-1})$ has at most one 2-cell between any two arrows.*

PROOF. The arrows in this bicategory are spans $A \xleftarrow{w} S \xrightarrow{f} B$ with $w \in W$. A 2-cell from $A \xleftarrow{w_1} S_1 \xrightarrow{f_1} B$ to $A \xleftarrow{w_2} S_2 \xrightarrow{f_2} B$ is represented by a commutative diagram of the form

$$\begin{array}{ccccc}
 & & S_1 & & \\
 & w_1 & \uparrow & f_1 & \\
 A & & T & & B \\
 & w_2 & \downarrow & f_2 & \\
 & & S_2 & &
 \end{array}$$

Two such diagrams,

$$\begin{array}{ccccc}
 & & S_1 & & \\
 & w_1 & \uparrow & f_1 & \\
 A & & T & & B \\
 & w_2 & \downarrow & f_2 & \\
 & & S_2 & &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccccc}
 & & S_1 & & \\
 & w_1 & \uparrow u'_1 & f_1 & \\
 A & & T' & & B \\
 & w_2 & \downarrow u'_2 & f_2 & \\
 & & S_2 & &
 \end{array}
 \tag{23}$$

represent the same 2-cell when there exist arrows t and t' such that the diagram

$$\begin{array}{ccccc}
 & & S_1 & & \\
 & u_1 \nearrow & & \nwarrow u'_1 & \\
 T & \xleftarrow{t} & \bar{T} & \xrightarrow{t'} & T' \\
 & u_2 \searrow & & \swarrow u'_2 & \\
 & & S_2 & &
 \end{array} \tag{24}$$

commutes and $w_1 u_1 t \in W$.

Note that for any two 2-cells as in (23), we can find t and t' as in (24) by first using condition **CF2** to find a commutative square

$$\begin{array}{ccc}
 & r \rightarrow & \\
 r' \downarrow & \square & \downarrow w_1 u_1 \\
 & w_1 u'_1 \rightarrow & .
 \end{array}$$

We have then also that $w_2 u_2 r = w_1 u_1 r = w_1 u'_1 r' = w_2 u'_2 r'$. By condition **CF3** applied to both w_1 and w_2 we find that there is an arrow $\tilde{w} \in W$ such that $u_1 r \tilde{w} = u'_1 r' \tilde{w}$ and $u_2 r \tilde{w} = u'_2 r' \tilde{w}$. So $t = r \tilde{w}$ and $t' = r' \tilde{w}$ fit in the diagram (24). ■

5.14. THEOREM. *The bicategories, $\mathbf{Bic}(\mathbf{C}\{W\})$ and $\mathbf{C}(W^{-1})$ are equivalent in the 2-category $\mathbf{Bicat}_{\text{icon}}$.*

PROOF. We first show that the functions ω_0 , ω_1 and ω_2 defined above give a homomorphism of bicategories: $\omega: \mathbf{Bic}(\mathbf{C}\{W\}) \rightarrow \mathbf{C}(W^{-1})$. To prove this we only need to give the comparison 2-cells for units and composition. (They will automatically be coherent since the hom-categories in $\mathbf{C}(W^{-1})$ are posetal.)

We will represent each connected component of $(\mathbf{C}\{W\})_0$ by the identity arrow in it. So an identity arrow in $\mathbf{Bic}(\mathbf{C}\{W\})$ has the form $B \xleftarrow{1_B} B \xrightarrow{1_B} B \xrightarrow{1_B} B$, and

$$\omega_1 \left(B \xleftarrow{1_B} B \xrightarrow{1_B} B \xrightarrow{1_B} B \right) = \left(B \xleftarrow{1_B} B \xrightarrow{1_B} B \right).$$

So ω_1 preserves identities strictly.

For any two composable arrows

$$B_1 \xleftarrow{w_1} A_1 \xrightarrow{f} A_2 \xrightarrow{w_2} B_2 \quad \text{and} \quad B_2 \xleftarrow{w_3} A_3 \xrightarrow{g} A_4 \xrightarrow{w_4} B_3 \tag{25}$$

in $\mathbf{Bic}(\mathbf{C}\{W\})$, the composition is found by first constructing the following double cell in $\mathbf{C}\{W\}$,

$$\begin{array}{ccccc}
 (B_1 \xleftarrow{w_1 \bar{w}_3} C) & \xrightarrow{\bar{w}_2 \bar{f}} & (A_3 \xrightarrow{w_3} B_2) & & \\
 \parallel & & \parallel & & \parallel \\
 & C \xrightarrow{\bar{f}} D & & & \\
 & \bar{w}_3 \downarrow & \uparrow \bar{w}_2 & & \\
 (B_1 \xleftarrow{w_1} A_1) & \xrightarrow{f} & (A_2 \xrightarrow{w_2} B_2) & & \\
 & & \downarrow \bar{w}_3 & &
 \end{array}$$

(using chosen squares in \mathbf{C} to obtain $\bar{w}_2, \bar{w}_3, \bar{w}_3$ and \bar{f}) and then composing the domain horizontal arrow with $(B_2 \xleftarrow{w_3} A_3) \xrightarrow{g} (A_4 \xrightarrow{w_4} B_3)$ to get

$$(B_1 \xleftarrow{w_1 \bar{w}_3} C) \xrightarrow{g \bar{w}_2 \bar{f}} (A_4 \xrightarrow{w_4} B_3).$$

So ω_1 applied to the composite gives

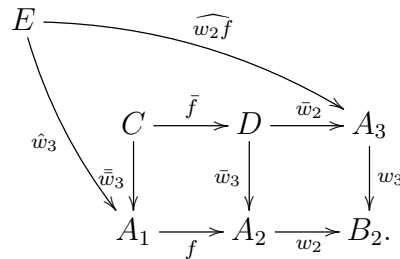
$$B_1 \xleftarrow{w_1 \bar{w}_3} C \xrightarrow{w_4 g \bar{w}_2 \bar{f}} B_3. \tag{26}$$

On the other hand, the composite of the images of the arrows in (25) under ω is

$$B_1 \xleftarrow{w_1 \hat{w}_3} E \xrightarrow{w_4 g \widehat{w_2 f}} B_3, \tag{27}$$

where \hat{w}_3 and $\widehat{w_2 f}$ are obtained using chosen squares.

To construct a 2-cell from (26) to (27) we consider the following diagram containing the three chosen squares used above:

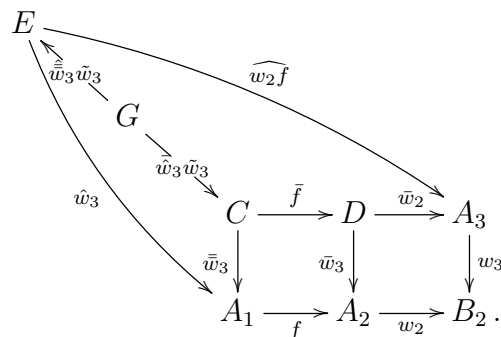


We want to construct a span $E \leftarrow G \rightarrow C$ that will make this diagram commutative.

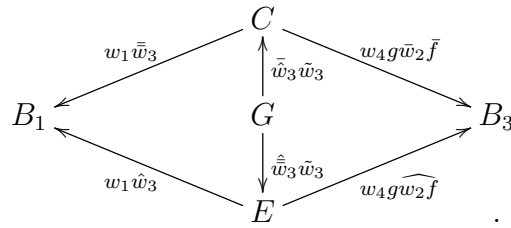
Using Condition **CF2** we find a commutative square

$$\begin{array}{ccc} F & \xrightarrow{\bar{w}_3} & C \\ \bar{w}_3 \downarrow & & \downarrow \bar{w}_3 \\ E & \xrightarrow{\hat{w}_3} & A_1 \end{array}$$

and using Condition **CF3** we find an arrow $G \xrightarrow{\bar{w}_3} F$ in W such that all parts of the following diagram commute:



So the required comparison 2-cell is given by



We will now construct a pseudo-inverse $\alpha: \mathbf{C}(W^{-1}) \rightarrow \mathbf{Bic}(\mathbf{C}\{W\})$ for ω . On objects, $\alpha(A) = (1_A)$. On arrows,

$$\alpha(A \xleftarrow{w} C \xrightarrow{f} B) = (A \xleftarrow{w} C \xrightarrow{f} B \xrightarrow{1_B} B),$$

and on 2-cells,

$$\alpha \left(\begin{array}{ccc} & w & \\ & \swarrow & \uparrow v \\ & & f \\ & \nwarrow & \downarrow \\ w' & & f' \end{array} \right) = \left\| \begin{array}{ccc} & w & f \\ & \swarrow & \uparrow v \\ & & f v \\ & \nwarrow & \downarrow v' \\ w' & & f' \end{array} \right\|.$$

We will show that α is a homomorphism of bicategories. First, note that α preserves units strictly: $\alpha(\xleftarrow{1_A} \xrightarrow{1_A}) = (\xleftarrow{1_A} \xrightarrow{1_A} \xrightarrow{1_A})$. Composition is also preserved strictly as long as we choose the same chosen squares. We will now show this. So let $A \xleftarrow{w_1} S \xrightarrow{f_1} B$ and $B \xleftarrow{w_2} T \xrightarrow{f_2} C$ be a composable pair of arrows in $\mathbf{C}(W^{-1})$. Let

$$\begin{array}{ccc} U & \xrightarrow{\bar{f}_1} & T \\ \bar{w}_2 \downarrow & & \downarrow w_2 \\ S & \xrightarrow{f_1} & B \end{array} \tag{28}$$

be a chosen square. Then the composition in $\mathbf{C}(W^{-1})$ is $A \xleftarrow{w_1 \bar{w}_2} U \xrightarrow{f_2 \bar{f}_1} C$, and

$$\alpha(A \xleftarrow{w_1 \bar{w}_2} U \xrightarrow{f_2 \bar{f}_1} C) = (A \xleftarrow{w_1 \bar{w}_2} U \xrightarrow{f_2 \bar{f}_1} C \xrightarrow{1_C} C).$$

To calculate the composition in $\mathbf{Bic}(\mathbf{C}\{W\})$ of

$$\alpha(A \xleftarrow{w_1} S \xrightarrow{f_1} B) = (A \xleftarrow{w_1} S \xrightarrow{f_1} B \xrightarrow{1_B} B)$$

and

$$\alpha(B \xleftarrow{w_2} T \xrightarrow{f_2} C) = (B \xleftarrow{w_2} T \xrightarrow{f_2} C \xrightarrow{1_C} C),$$

we note that we can use the same chosen square (28) to construct a pair of horizontal arrows in $\mathbf{C}\{W\}$ that is composable there: there is a double cell

$$\begin{array}{ccc} (\overleftarrow{w_1 \bar{w}_2}) & \xrightarrow{\bar{f}_1} & (\overrightarrow{w_2}) \\ \parallel & & \parallel \\ \parallel & \xrightarrow{\bar{f}_1} & \parallel \\ \parallel & \downarrow \bar{w}_2 & \downarrow w_2 \\ (\overleftarrow{w_1}) & \xrightarrow{f_1} & (\overline{\quad}) \end{array}$$

Composing the top row of this double cell with $(\overleftarrow{w_2}) \xrightarrow{f_2} (\overline{\quad})$ gives us

$$(\overleftarrow{w_1 \bar{w}_2}) \xrightarrow{f_2 \bar{f}_1} (\overline{\quad}).$$

Now it remains for us to calculate the composites $\alpha \circ \omega$ and $\omega \circ \alpha$. First, a straightforward calculation shows that $\omega \alpha = \text{Id}_{\mathbf{C}(W^{-1})}$. The other composition requires a bit more attention.

On objects, $\alpha \omega(\overline{(\overrightarrow{w})}) = \overline{(\overrightarrow{1_B})}$ for $A \xrightarrow{w} B$, and $\overline{(\overrightarrow{w})} = \overline{(\overrightarrow{1_B})}$ in $\mathbf{Bic}(\mathbf{C}\{W\})$. So $\alpha \omega$ is the identity on objects.

On arrows, $\alpha \omega(\overleftarrow{w_1} \xrightarrow{f} \overrightarrow{w_2}) = (\overleftarrow{w_1} \xrightarrow{w_2 f} \overline{\quad})$. And on 2-cells,

$$\begin{aligned} \alpha \omega \left(\begin{array}{ccc} \overleftarrow{w_1} & \xrightarrow{f} & \overrightarrow{w_2} \\ \parallel & & \parallel \\ \parallel & \begin{array}{c} \uparrow u_1 \\ \xrightarrow{h} \\ \downarrow v_1 \end{array} & \parallel \\ \parallel & \begin{array}{c} \uparrow u_2 \\ \downarrow v_2 \end{array} & \parallel \\ \parallel & & \parallel \\ \overleftarrow{w'_1} & \xrightarrow{f'} & \overrightarrow{w'_2} \end{array} \right) = \alpha \left(\begin{array}{ccc} w_1 & \begin{array}{c} \uparrow u_1 \\ \downarrow v_1 \end{array} & w_2 f \\ \swarrow & & \searrow \\ w'_1 & \downarrow v_1 & w'_2 f' \end{array} \right) \\ = \begin{array}{ccc} \overleftarrow{w_1} & \xrightarrow{w_2 f} & \overline{\quad} \\ \parallel & \begin{array}{c} \uparrow u_1 \\ \xrightarrow{w_2 f u_1} \\ \downarrow v_1 \end{array} & \parallel \\ \parallel & & \parallel \\ \overleftarrow{w'_1} & \xrightarrow{w'_2 f'} & \overline{\quad} \end{array} \end{aligned}$$

So we see that $\alpha \circ \omega$ is not the identity, but there is an invertible icon $\zeta : \alpha \omega \Rightarrow \text{Id}_{\mathbf{Bic}(\mathbf{C}(W)^{-1})}$ such that the component of ζ at $\overleftarrow{w_1} \xrightarrow{f} \overrightarrow{w_2}$ is

$$\begin{array}{ccc} \overleftarrow{w_1} & \xrightarrow{w_2 f} & \overline{\quad} \\ \parallel & & \parallel \\ \parallel & \begin{array}{c} \uparrow w_2 \\ \xrightarrow{f} \end{array} & \parallel \\ \parallel & & \parallel \\ \overleftarrow{w_1} & \xrightarrow{f} & \overrightarrow{w_2} \end{array}$$

The fact that the naturality squares of 2-cells commute follows from the fact that this bicategory is locally posetal.

We conclude that ω and α form a biequivalence of bicategories $\mathbf{Bic}(\mathbf{C}\{W\}) \simeq \mathbf{C}(W^{-1})$. ■

5.15. COROLLARY. $\mathbf{C}\{W\}$ and $\mathbf{DbI}(\mathbf{C}(W^{-1}))$ are equivalent in the 2-category $\mathbf{WGDbI}_{\text{ps},v}$.

PROOF. Apply \mathbf{DbI} to the equivalence of Theorem 5.14 and then compose the resulting vertical equivalence of weakly globular double categories with the equivalence of $\mathbf{C}\{W\}$ with $\mathbf{DbIBic}(\mathbf{C}\{W\})$. ■

6. The Universal Property of $\mathbf{C}\{W\}$

The goal of this section is to describe the vertical and horizontal universal properties of the weakly globular double category $\mathbf{C}\{W\}$ we constructed in the previous section. First we derive the universal property of $\mathbf{C}\{W\}$ inherited from $\mathbf{C}(W^{-1})$ by Corollary 5.15. This is a property in terms of pseudo-functors and vertical transformations. The second universal property we give for $\mathbf{C}\{W\}$ is in terms of strict double functors and horizontal transformations. The two properties together determine $\mathbf{C}\{W\}$ up to horizontal and vertical equivalence.

6.1. THE VERTICAL UNIVERSAL PROPERTY OF $\mathbf{C}\{W\}$. Recall that Theorem 4.14 states that composition with $\mathbf{DbI}(J_W): \mathbf{DbI}(\mathbf{C}) \rightarrow \mathbf{DbI}(\mathbf{C}(W^{-1}))$ induces an equivalence of categories,

$$\mathbf{WGDbI}_{\text{ps},v}(\mathbf{DbI}(\mathbf{C}(W^{-1})), \mathbb{X}) \simeq \mathbf{WGDbI}_{\text{ps},v,W}(\mathbf{DbI}(\mathbf{C}), \mathbb{X})$$

for any weakly globular double category \mathbb{X} .

To obtain a vertical universal property for $\mathbf{C}\{W\}$, note that $\mathbf{C}\{W\}$ is vertically equivalent to $\mathbf{DbI}(\mathbf{C}(W^{-1}))$. We compose $\mathbf{DbI}(J_W)$ with the equivalence $\mathbf{DbI}(\mathbf{C}(W^{-1})) \xrightarrow{\sim} \mathbf{C}\{W\}$ to get a pseudo-functor $\tilde{J}_W: \mathbf{DbI}(\mathbf{C}) \rightarrow \mathbf{C}\{W\}$ with the following universal property:

6.2. THEOREM. *Composition with the pseudo-functor $\tilde{J}_W: \mathbf{DbI}(\mathbf{C}) \rightarrow \mathbf{C}\{W\}$, gives rise to an equivalence of categories,*

$$\mathbf{WGDbI}_{\text{ps},v}(\mathbf{C}\{W\}, \mathbb{D}) \simeq \mathbf{WGDbI}_{\text{ps},v,W}(\mathbf{DbI}(\mathbf{C}), \mathbb{D}).$$

Recall that $\mathbf{WGDbI}_{v,\text{ps},W}(\mathbf{DbI}(\mathbf{C}), \mathbb{D})$ is the category of W -pseudo-functors (which send W -doubly marked paths to precompanions), and vertical W -transformations as described in Definition 4.11.

Since furthermore, $H(\mathbf{C})$ is equivalent to $\mathbf{DbI}(\mathbf{C})$, we may compose \tilde{J}_W with this equivalence and obtain $\mathcal{J}_W: H(\mathbf{C}) \rightarrow \mathbf{C}\{W\}$. This functor has the universal property expressed in the following corollary.

6.3. COROLLARY. *Composition with $\mathcal{J}_W: H\mathbf{C} \rightarrow \mathbf{C}\{W\}$ gives rise to an equivalence of categories,*

$$\mathbf{WGDbI}_{\text{ps},v}(\mathbf{C}\{W\}, \mathbb{D}) \simeq \mathbf{WGDbI}_{\text{ps},v,W}(H\mathbf{C}, \mathbb{D}).$$

6.4. REMARKS.

1. In this corollary, $\mathbf{WGDbI}_{\text{ps},v,W}(HC, \mathbb{D})$ has as objects pseudo-functors that send the elements of W to precompanions. Note that this vertical universal property is easier to express than the one in terms of $\mathbf{DbI}(\mathbf{C})$, as given in Theorem 6.2.
2. This characterization of $\mathbf{C}\{W\}$ determines this weakly globular double category up to a vertical equivalence.
3. It is straightforward to check that the composition $\mathcal{J}_W: HC \rightarrow \mathbf{C}\{W\}$ used in the last corollary is the obvious inclusion functor, sending an object of C to the object in $\mathbf{C}\{W\}$ represented by its identity arrow. It is clear that this is a strict functor of weakly globular double categories.

In the rest of this section we will discuss the universal property of the functor $\mathcal{J}_W: HC \rightarrow \mathbf{C}\{W\}$ in terms of 2-categories of strict functors and horizontal transformations. This will then characterize $\mathbf{C}\{W\}$ up to a horizontal equivalence. This characterization will be in terms of companions rather than precompanions.

6.5. COMPANIONS AND CONJOINTS IN $\mathbf{C}\{W\}$. As we saw in an Section 3.4, companions in weakly globular double categories are closely related to quasi units in bicategories. Requiring in the universal property that the elements of W themselves become companions is too strong, but requiring that they become companions after composition with a horizontal isomorphism turns out to be just right for a universal property with respect to strict functors and horizontal transformations.

For this part we will make the added assumption on the class W that it satisfies the 3 for 2 property: if f, g, h are arrows in \mathbf{C} such that $gf = h$ and two of f, g and h are in W , so is the third.

The following lemma characterizes the horizontal and vertical arrows in $\mathbf{C}\{W\}$ that have companions or conjoinants.

6.6. LEMMA.

1. A vertical arrow in $\mathbf{C}\{W\}$ has a horizontal companion if and only if it has a representative of the form

$$\begin{array}{ccc}
 (A_1 \xrightarrow{wu} B) & & (29) \\
 \parallel & & \parallel \\
 C & & \\
 u \downarrow & & \\
 (A_2 \xrightarrow{w} B) & &
 \end{array}$$

2. A vertical arrow in $\mathbf{C}\{W\}$ has a horizontal conjoint if and only if it has a representative of the form

$$\begin{array}{ccc} (A_1 \xrightarrow{w} B) & & \\ u \uparrow & & \parallel \\ A_2 & & \\ \parallel & & \parallel \\ (A_2 \xrightarrow{wu} B) & & \end{array} \tag{30}$$

3. A horizontal arrow in $\mathbf{C}\{W\}$ has a vertical companion and conjoint if and only if it is of the form

$$(B \xleftarrow{wu} A_2) \xrightarrow{u} (A_1 \xrightarrow{w} B).$$

PROOF. To prove the first part, the companion for (29) is the horizontal arrow

$$(B \xleftarrow{wu} A_1) \xrightarrow{u} (A_2 \xrightarrow{w} B)$$

with binding cells

$$\begin{array}{ccc} (B \xleftarrow{wu} A_1) \xlongequal{\quad} (A_1 \xrightarrow{wu} B) & & \\ \parallel & \parallel & \parallel \\ \psi_{u,w} = & A_1 \xlongequal{\quad} A_1 & \\ \parallel & \parallel & \parallel \\ (B \xleftarrow{wu} A_1) \xrightarrow{u} (A_2 \xrightarrow{w} B) & & \\ & u \downarrow & \\ & & \\ (B \xleftarrow{wu} A_1) \xrightarrow{u} (A_2 \xrightarrow{w} B) & & \\ \parallel & \parallel & \parallel \\ \chi_{u,w} = & A_1 \xrightarrow{u} A_2 & \\ \parallel & \parallel & \parallel \\ (B \xleftarrow{w} A_2) \xlongequal{\quad} (A_2 \xrightarrow{w} B) & & \\ & u \downarrow & \\ & & \end{array} \tag{31}$$

We leave it up to the reader to check that these cells satisfy the equations (11).

Conversely, suppose that a vertical arrow

$$\begin{array}{ccc} A_1 \xrightarrow{w_1} B & & \\ u_1 \uparrow & & \parallel \\ C & & \\ u_2 \downarrow & & \parallel \\ A_2 \xrightarrow{w_2} B & & \end{array} \tag{32}$$

has a companion $(B \xleftarrow{w_1} A_1) \xrightarrow{f} (A_2 \xrightarrow{w_2} B)$. Then there is a binding cell

$$\begin{array}{ccccc}
 (B \xleftarrow{w_1} A_1) & \xrightarrow{f} & (A_2 \xrightarrow{w_2} B) & & \\
 \parallel & & \parallel & & \parallel \\
 & u_1 \uparrow & & (\chi) & \\
 & C & & A_2 & \\
 & u_2 \downarrow & & & \\
 (B \xleftarrow{w_2} A_2) & \xlongequal{\quad} & (A_2 \xrightarrow{w_2} B) & &
 \end{array}$$

So there is an arrow $(r: D \rightarrow C) \in W$ and an arrow $\chi: D \rightarrow A_2$ such that the following diagram commutes,

$$\begin{array}{ccccc}
 (B \xleftarrow{w_1} A_1) & \xrightarrow{f} & (A_2 \xrightarrow{w_2} B) & & \\
 \parallel & & \parallel & & \parallel \\
 & u_1 \uparrow & & & \\
 & C & & & \\
 & r \uparrow & & \chi & \\
 & D & \xrightarrow{\quad} & A_2 & \\
 & r \downarrow & & & \\
 & C & & & \\
 & u_2 \downarrow & & & \\
 (B \xleftarrow{w_2} A_2) & \xlongequal{\quad} & (A_2 \xrightarrow{w_2} B) & & .
 \end{array}$$

So $fu_1r = \chi = u_2r$. It follows that (32) is equivalent to

$$\begin{array}{ccc}
 A_1 & \xrightarrow{w_1} & B \\
 \parallel & & \parallel \\
 A_1 & & \\
 f \downarrow & & \\
 A_2 & \xrightarrow{w_2} & B
 \end{array}$$

This completes the proof of the first part.

The second part follows from the first part, together with Remark 3.3: a vertical arrow in $\mathbf{C}\{W\}$ has a horizontal conjoint if and only if its inverse has this horizontal arrow as companion.

The last part follows from the proofs of the previous two parts. ■

It is easy to see that the vertical companions and conjoints generate the category of vertical arrows:

$$\begin{array}{ccc}
 B & \xleftarrow{w_1} & A_1 \\
 \parallel & & \uparrow u_1 \\
 & & C \\
 & & \downarrow u_2 \\
 B & \xleftarrow{w_2} & A_2
 \end{array}$$

is the vertical composition of

$$\begin{array}{ccc}
 B \xleftarrow{w_1} A_1 & & B \xleftarrow{w_2 u_2} C \\
 \parallel & \uparrow u_1 & \parallel \\
 B \xleftarrow{w_1 u_1} C & & B \xleftarrow{w_2} A_2 \\
 \parallel & \parallel & \\
 B & & B
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 B \xleftarrow{w_2 u_2} C & & B \xleftarrow{w_2} A_2 \\
 \parallel & \parallel & \parallel \\
 B \xleftarrow{w_1 u_1} C & & B \xleftarrow{w_1} A_1 \\
 \parallel & \downarrow u_2 & \parallel \\
 B & & B
 \end{array}$$

which are composable since $w_1 u_1 = w_2 u_2$.

The following lemma shows that the companion binding cells and their vertical inverses (the conjoint binding cells) generate all the cells of $\mathbf{C}\{W\}$.

6.7. LEMMA. *Each double cell in $\mathbf{C}\{W\}$ can be written as a pasting diagram of companion binding cells as in (31) and their vertical inverses.*

PROOF. Let

$$\begin{array}{ccc}
 (B \xleftarrow{w_1} A_1) \xrightarrow{f_1} (A'_1 \xrightarrow{w'_1} B') & & (33) \\
 \parallel & \uparrow u_1 \quad \xi \quad \uparrow u'_1 & \parallel \\
 C & & C' \\
 \downarrow u_2 & & \downarrow u'_2 \\
 (B \xleftarrow{w_2} A_2) \xrightarrow{f_2} (A'_2 \xrightarrow{w'_2} B') & &
 \end{array}$$

be a representative of a double cell in $\mathbf{C}\{W\}$. This cell can be written as a pasting of the following array of double cells by first composing the double cells in each row horizontally and then composing the resulting double cells vertically.

$$\begin{array}{ccc}
 (B \xleftarrow{w_1} A_1) \xrightarrow{=} (A_1 \xrightarrow{w_1} B) & & (B \xleftarrow{w_1} A_1) \xrightarrow{f_1} (A'_1 \xrightarrow{w'_1} B') \\
 \parallel & \uparrow u_1 & \parallel \\
 C \xrightarrow{u_1} A_1 & & A_1 \xrightarrow{f_1} A'_1 \\
 \parallel & \parallel & \parallel \\
 (B \xleftarrow{w_1 u_1} C) \xrightarrow{u_1} (A_1 \xrightarrow{w_1} B) & & (B \xleftarrow{w_1} A_1) \xrightarrow{f_1} (A'_1 \xrightarrow{w'_1} B')
 \end{array}$$

$$\begin{array}{ccc}
 (B \xleftarrow{w_1 u_1} C) \xrightarrow{\xi} (C' \xrightarrow{w'_1 u'_1} B') & & (B' \xleftarrow{w'_1 u'_1} C') \xrightarrow{u'_1} (A'_1 \xrightarrow{w'_1} B') \\
 \parallel & \parallel & \parallel \\
 C \xrightarrow{\xi} C' & & C' \xrightarrow{=} C' \\
 \parallel & \parallel & \parallel \\
 (B \xleftarrow{w_1 u_1} C) \xrightarrow{\xi} (C' \xrightarrow{w'_1 u'_1} B') & & (B' \xleftarrow{w'_1 u'_1} C') \xrightarrow{=} (C' \xrightarrow{w'_1 u'_1} B')
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 (B \xleftarrow{w_1 u_1} C) & \xrightarrow{\xi} & (C' \xrightarrow{w'_2 u'_2} B') \\
 \parallel & & \parallel \\
 & C \xrightarrow{\xi} & C' \\
 & \parallel & \parallel \\
 (B \xleftarrow{w_2 u_2} C) & \xrightarrow{\xi} & (C' \xrightarrow{w'_2 u'_2} B')
 \end{array} & &
 \begin{array}{ccc}
 (B' \xleftarrow{w'_2 u'_2} C') & \xrightarrow{=} & (C' \xrightarrow{w'_2 u'_2} B') \\
 \parallel & & \parallel \\
 & C' & \xrightarrow{=} & C' \\
 & \parallel & & \parallel \\
 & & \downarrow u'_2 & \\
 (B' \xleftarrow{w'_2 u'_2} C') & \xrightarrow{u'_2} & (A'_2 \xrightarrow{w'_2} B')
 \end{array} \\
 \\
 \begin{array}{ccc}
 (B \xleftarrow{w_2 u_2} C) & \xrightarrow{u_2} & (A_2 \xrightarrow{w_2} B) \\
 \parallel & & \parallel \\
 & C \xrightarrow{u_2} & A_2 \\
 & \parallel & \parallel \\
 & \downarrow u_2 & \\
 (B \xleftarrow{w_2} A_2) & \xrightarrow{=} & (A_2 \xrightarrow{w_2} B)
 \end{array} & &
 \begin{array}{ccc}
 (B \xleftarrow{w_2} A_2) & \xrightarrow{f_2} & (A'_2 \xrightarrow{w'_2} B') \\
 \parallel & & \parallel \\
 & A_2 \xrightarrow{f_2} & A'_2 \\
 & \parallel & \parallel \\
 (B \xleftarrow{w_2} A_2) & \xrightarrow{f_2} & (A'_2 \xrightarrow{w'_2} B')
 \end{array}
 \end{array}$$

Each one of these cells is either a horizontal or vertical identity cell or a companion binding cell or the inverse of a companion binding cell. ■

6.8. *W-FRIENDLY FUNCTORS AND TRANSFORMATIONS.* As we have seen in Lemma 6.6 and Lemma 6.7, the strict inclusion functor $\mathcal{J}_W : HC \rightarrow \mathbf{C}\{W\}$ adds companions and conjoints to HC . We want to make precise in what sense it does so freely. The goal of this section is to introduce a notion of W -friendly double functor out of HC and a notion of W -friendly horizontal transformation between these W -friendly double functors, such that \mathcal{J}_W the universal W -friendly double functor out of HC in the sense that we have an equivalence of categories,

$$\text{WGDbI}_{\text{st},h}(\mathbf{C}\{W\}, \mathbb{D}) \simeq \text{WGDbI}_{\text{st},h,W}(HC, \mathbb{D}),$$

where $\text{WGDbI}_{\text{st},h,W}(HC, \mathbb{D})$ is the category of W -friendly functors and W -friendly horizontal transformations.

The first question is for which arrows it adds the companions and conjoints. It is tempting to think that it does this for the horizontal arrows in the image of W . However, those would be the arrows of the form

$$(1_A) \xrightarrow{w} (1_B)$$

and the arrows that obtain companions and conjoints are of the form

$$(uw) \xrightarrow{w} (u).$$

In particular, there are companions and conjoints for arrows of the form

$$(w) \xrightarrow{w} (1_B), \tag{34}$$

but not for

$$(1_A) \xrightarrow{w} (1_B)$$

unless $A = B$ and $w = 1_B$. The arrow in (34) can be factored as

$$(B \xleftarrow{w} A) \xrightarrow{1_A} (A \xrightarrow{1_A} A) \xrightarrow{w} (B \xrightarrow{1_B} B)$$

We will denote the first arrow in this factorization by $\varphi_w: (w) \rightarrow (1_A)$. Note that this is a horizontal isomorphism in $\mathbf{C}\{W\}$. Our first step will be to show that the elements of the family $\{\varphi_w\}$ for $w \in W$ form the components of an invertible natural transformation. To express this property we need the following comma category derived from W .

6.9. DEFINITION. Let \mathbf{C} be a category with a class W of arrows. We define ∇W to be the category with arrows in W as objects and an arrow from $A \xrightarrow{w} B$ to $A' \xrightarrow{w'} B$ is given by a commutative triangle of arrows in \mathbf{C} :

$$\begin{array}{ccc} A & \xrightarrow{v} & A' \\ & \searrow w & \swarrow w' \\ & & B \end{array}$$

We denote this arrow by $(v, w'): w \rightarrow w'$. There is a functor $D_0: \nabla W \rightarrow \mathbf{C}$ defined by $D_0(w) = d_0(w)$ (the domain) on objects and $D_0(v, w') = v$.

6.10. REMARK. When W satisfies the 3 for 2 property, the commutativity of the triangle in Definition 6.9 implies that $v \in W$. Furthermore, by condition **CF1**, W forms a subcategory of \mathbf{C} that contains all objects of \mathbf{C} . So we can view ∇W as the comma category $(1_W \downarrow L_W)$, where $L_W: d(C_0) \rightarrow W$ is the inclusion functor of the discrete category on the objects of \mathbf{C} into the category W . In this notation, D_0 is just the first projection functor $(1_W \downarrow L_W) \rightarrow W$ followed by the inclusion into \mathbf{C} .

Let $\Phi: \nabla W \rightarrow \text{Comp}(\mathbf{C}\{W\})$ be the functor that sends an arrow

$$\begin{array}{ccc} A & \xrightarrow{u} & A' \\ & \searrow wu & \swarrow w \\ & & B \end{array}$$

to the horizontal companion arrow $(B \xleftarrow{wu} A) \xrightarrow{u} (A' \xrightarrow{w} B)$ with the vertical companion and binding cells in $\mathbf{C}\{W\}$ as indicated in the proof of Lemma 6.6. Now we can view φ as a natural transformation in the following diagram

$$\begin{array}{ccc} \nabla W & \xrightarrow{\Phi} & \text{Comp}(\mathbf{C}\{W\}) \\ D_0 \downarrow & \cong & \downarrow h_- \\ \mathbf{C} & \xrightarrow{h_{\mathcal{J}_W}} & h\mathbf{C}\{W\}. \end{array}$$

This is almost enough to describe \mathcal{J}_W as a W -friendly functor. The problem is that the information given thus far is not enough to describe W -friendly transformations between W -friendly functors. Such transformations need to have components that interact well with the companion binding cells. In order to ensure this, Φ needs to be viewed as a double functor $V\nabla W \rightarrow \text{Comp}(\mathbf{C}\{W\})$. This leads us to the following definition.

6.11. DEFINITION. A W -friendly structure (Γ, γ) for a functor $G: HC \rightarrow \mathbb{D}$ consists of a functor $\Gamma: V\nabla W \rightarrow \text{Comp}(\mathbb{D})$ together with an invertible natural transformation γ ,

$$\begin{array}{ccc} \nabla W & \xrightarrow{v\Gamma} & v\text{Comp}(\mathbb{D}) \\ D_0 \downarrow & \xrightarrow{\gamma} & \downarrow h_- \\ \mathbf{C} & \xrightarrow{hG} & h\mathbb{D}, \end{array}$$

where h_- is the identity on objects and takes the horizontal component of each arrow (companion pair). We will also refer to G (and to the pair (G, Γ)) as a W -friendly functor.

6.12. PROPOSITION. There is a canonical W -friendly structure

$$(\Phi: V\nabla W \rightarrow \text{Comp}(\mathbf{C}\{W\}), \varphi)$$

for the functor $\mathcal{J}_{\mathbf{C}}: HC \rightarrow \mathbf{C}\{W\}$.

PROOF. On objects, $\Phi(w) = (w)$. On horizontal arrows, Φ is the identity. On vertical arrows, $\Phi((u, w)) = (h_{(u,w)}, v_{(u,w)}, \psi_{(u,w)}, \chi_{(u,w)})$, where

$$\begin{aligned} h_{(u,w)} &= (\leftarrow wu) \xrightarrow{u} (\rightarrow w) \\ v_{(u,w)} &= \begin{array}{ccc} (\rightarrow wu) & & \\ \parallel & & \parallel \\ u \downarrow & & \\ (\rightarrow w) & & \end{array} \end{aligned}$$

and the binding cells are

$$\begin{array}{ccc} \begin{array}{ccc} (\leftarrow wu) & \equiv & (\rightarrow wu) \\ \parallel & & \parallel \\ \parallel & & \parallel \\ (\leftarrow wu) & \xrightarrow{u} & (\rightarrow w) \end{array} & \text{and} & \begin{array}{ccc} (\leftarrow wu) & \xrightarrow{u} & (\rightarrow w) \\ \parallel & & \parallel \\ \parallel & & \parallel \\ (\leftarrow w) & \equiv & (\rightarrow w) \end{array} \end{array}$$

Furthermore, the invertible natural transformation φ in

$$\begin{array}{ccc} \nabla W & \xrightarrow{v\Phi} & v\text{Comp}(\mathbf{C}\{W\}) \\ D_0 \downarrow & \xrightarrow{\varphi} & \downarrow h_- \\ \mathbf{C} & \xrightarrow{hF} & h\mathbf{C}\{W\} \end{array}$$

has components φ_w ,

$$(\leftarrow w) \equiv (==).$$

■

Let $L: \mathbf{C}\{W\} \rightarrow \mathbb{D}$ be a functor of double categories. Composition with $\mathcal{J}_{\mathbf{C}}: HC \rightarrow \mathbf{C}\{W\}$ gives rise to a functor $L\mathcal{J}_{\mathbf{C}}: HC \rightarrow \mathbb{D}$ with the W -friendly structure $(\text{Comp}(L) \circ \Phi, \lambda)$. Here, $\text{Comp}(L) \circ \Phi: V\nabla W \rightarrow \text{Comp}(\mathbb{D})$ sends w to $L(w)$ and a vertical arrow $(u, v): vu \xrightarrow{u} v$ to

$$\left(L \left(\begin{array}{c} \leftarrow vu \quad \xrightarrow{u} \quad \xrightarrow{v} \end{array} \right), L \left(\begin{array}{ccc} & \xleftarrow{vu} & \\ \parallel & & \parallel \\ & \xrightarrow{v} & \\ & \downarrow u & \end{array} \right), L \left(\begin{array}{ccc} \xleftarrow{vu} & \xleftarrow{vu} & \xrightarrow{vu} \\ \parallel & \parallel & \parallel \\ \xrightarrow{vu} & \xrightarrow{u} & \xrightarrow{v} \\ \parallel & \parallel & \parallel \\ \xleftarrow{vu} & \xrightarrow{u} & \xrightarrow{v} \end{array} \right), \\ L \left(\begin{array}{ccc} \xleftarrow{vu} & \xrightarrow{u} & \xrightarrow{v} \\ \parallel & \parallel & \parallel \\ \xrightarrow{vu} & \xrightarrow{u} & \xrightarrow{v} \\ \parallel & \parallel & \parallel \\ \xleftarrow{vu} & \xrightarrow{u} & \xrightarrow{v} \end{array} \right) \right).$$

Furthermore, the natural transformation λ has components $\lambda_w = L \left(\begin{array}{c} \xleftarrow{w} \\ \parallel \\ \xleftarrow{w} \end{array} \right)$.

The appropriate horizontal transformations between W -friendly functors are described in the following definition.

6.13. DEFINITION. For W -friendly functors

$$(G, \Gamma, \gamma), (L, \Lambda, \lambda): HC \rightrightarrows \mathbb{D}, V\nabla W \rightrightarrows \text{Comp}(\mathbb{D}),$$

a W -friendly horizontal transformation is a pair $(a: G \rightrightarrows L, \alpha: \Gamma \rightrightarrows \Lambda)$ of horizontal transformations such that the following square commutes:

$$\begin{array}{ccc} h_- \circ v\Gamma & \xrightarrow{h_- \circ v\alpha} & h_- \circ v\Lambda \\ \gamma \downarrow & & \downarrow v\lambda \\ hG \circ D_0 & \xrightarrow{h\alpha \circ D_0} & hL \circ D_0. \end{array} \tag{35}$$

6.14. THE HORIZONTAL UNIVERSAL PROPERTY OF $\mathbf{C}\{W\}$. We can now state and prove the universal property of $\mathcal{J}_W: HC \rightarrow \mathbf{C}\{W\}$ in terms of W -friendly double functors and W -friendly horizontal transformations.

6.15. THEOREM. Composition with $\mathcal{J}_W: HC \rightarrow \mathbf{C}\{W\}$ induces an equivalence of categories

$$\text{WGDbI}_{\text{st},h}(\mathbf{C}\{W\}, \mathbb{D}) \simeq \text{WGDbI}_{\text{st},h,W}(HC, \mathbb{D}),$$

where $\text{WGDbI}_{\text{st},h,W}(HC, \mathbb{D})$ is the category of W -friendly functors and W -friendly horizontal transformations.

PROOF. Composition with \mathcal{J}_W sends a functor $L: \mathbf{C}\{W\} \rightarrow \mathbb{D}$ of weakly globular double categories to the triple $(L\mathcal{J}_W, L \circ \Phi, L\varphi)$, i.e., the functor $L\mathcal{J}_W$ with W -friendly structure $(L\Phi, L\varphi)$, as described above.

To show that composition with \mathcal{J}_W is essentially surjective on objects, we show how to lift a functor $G: HC \rightarrow \mathbb{D}$ with a W -friendly structure (Γ, γ) to a functor $\tilde{G}: \mathbf{C}\{W\} \rightarrow \mathbb{D}$.

Define \tilde{G} on objects by $\tilde{G}(A \xrightarrow{w} B) = \Gamma(w)$. On vertical arrows, \tilde{G} is defined by

$$\tilde{G} \left(\begin{array}{ccc} & (\xrightarrow{w_1}) & \\ u_1 \uparrow & & \parallel \\ & (\xrightarrow{w_2}) & \\ & u_2 \downarrow & \end{array} \right) = v_{\Gamma(u_2, w_2)} \cdot (v_{\Gamma(u_1, w_1)})^{-1}$$

(where the notation is as in the proof of Proposition 6.12) and on horizontal arrows, \tilde{G} is defined by $\tilde{G} \left((\xleftarrow{w}) \xrightarrow{f} (\xrightarrow{v}) \right) = \gamma_v^{-1} G(f) \gamma_w$. To define \tilde{G} on double cells, we use the factorization of a generic double cell in $\mathbf{C}\{W\}$ given in the proof of Lemma 6.7. The result of applying \tilde{G} to (33) is given in Figure 1. It is not hard to see that there is an invertible horizontal transformation $\bar{\gamma}: \tilde{G} \circ \mathcal{J}_W \Rightarrow G$ with components $\bar{\gamma}_A = \gamma_{1_A}$. Furthermore, it is straightforward to check that $\tilde{G}\Phi = \Gamma$, and $(\bar{\gamma}, \text{id})$ is an invertible W -friendly transformation from $(\tilde{G}\mathcal{J}_W, \tilde{G}\Phi, \tilde{G}\varphi)$ to (G, Γ, γ) .

We now want to show that composition with \mathcal{J}_W is fully faithful on arrows, i.e., horizontal transformations. So we will show that horizontal transformations

$$b: G \Rightarrow L: \mathbf{C}\{W\} \rightrightarrows \mathbb{D}$$

are in one-to-one correspondence with W -friendly transformations

$$(b\mathcal{J}_W, \beta): (G\mathcal{J}_W, G\Phi, G\varphi) \rightrightarrows (L\mathcal{J}_W, L\Phi, L\varphi)$$

where $\beta: G\Phi \Rightarrow L\Phi$ is defined by $\beta_w = b_w$. The condition that the square in (35) commutes for $(b\mathcal{J}_W, \beta)$ is equivalent to the following square of horizontal arrows commuting for each $w \in W$:

$$\begin{array}{ccc} G(w) & \xrightarrow{b_w} & L(w) \\ G(\xleftarrow{v} \Rightarrow \Rightarrow) \downarrow & & \downarrow G(\xleftarrow{v} \Rightarrow \Rightarrow) \\ G(\text{id}_A) & \xrightarrow{b_{\text{id}_A}} & L(\text{id}_A) \end{array}$$

and the commutativity of this diagram follows immediately from the vertical naturality of b .

Conversely, given a W -friendly horizontal transformation $(a: G\mathcal{J}_W \Rightarrow L\mathcal{J}_W, \alpha: G\Phi \Rightarrow$

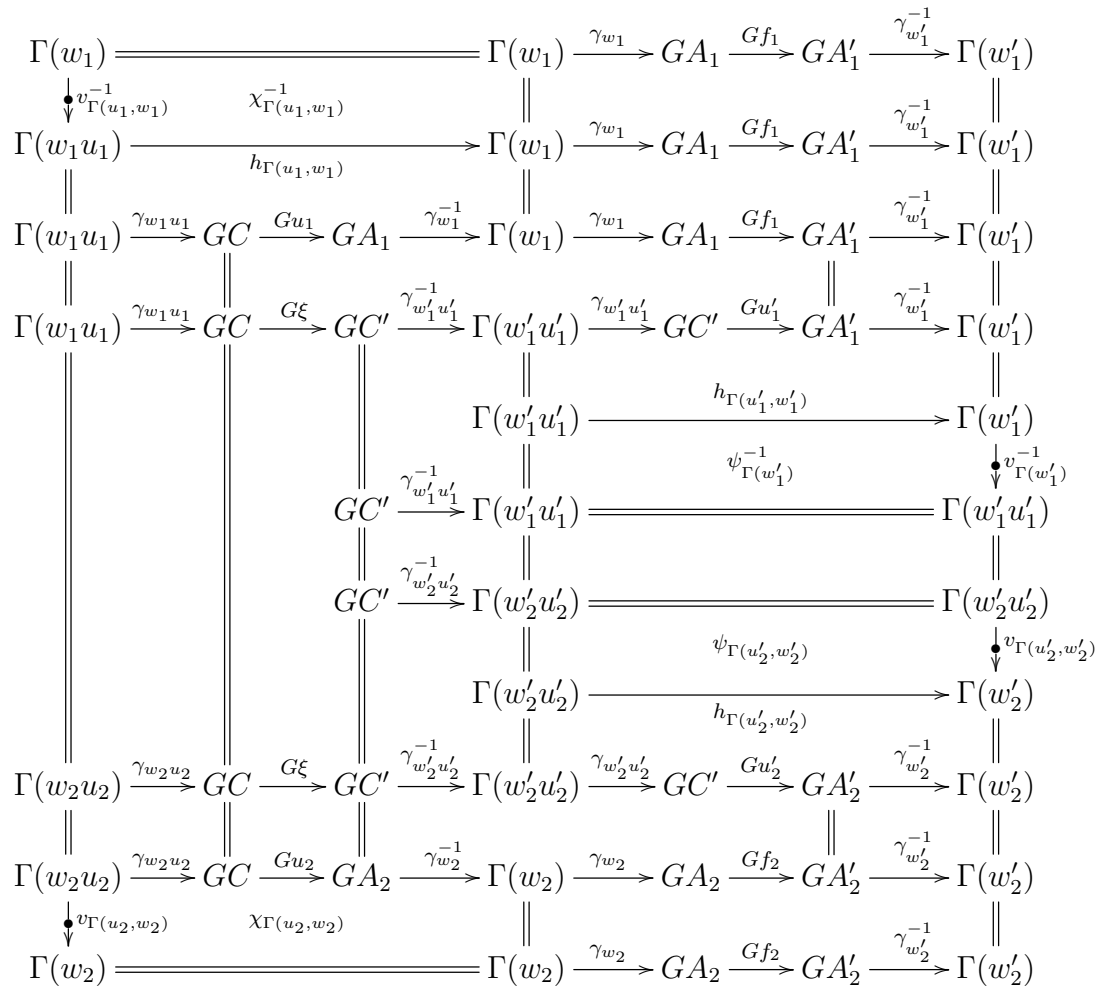


Figure 1: The image of a generic double cell in $\mathbf{C}\{W\}$ under \tilde{G} .

$L\Phi$), we get $\tilde{a}: G \rightrightarrows L$ with components $\tilde{a}_w = \alpha_w$ and

$$\tilde{a} \left(\begin{array}{c} \left(\begin{array}{c} \xrightarrow{wu} \\ \parallel \\ \xrightarrow{u} \\ \parallel \\ \xrightarrow{w} \end{array} \right) \end{array} \right) = \begin{array}{c} G(wu) \xrightarrow{h_{\alpha(wu)}} L(wu) \xlongequal{\quad} L(wu) \\ \parallel \qquad \qquad \qquad \parallel \quad L(\psi_{\Phi(u,v)}) \quad \downarrow L(v_{\Phi(u,w)}) \\ G(wu) \xlongequal{\quad} G(wu) \xrightarrow{h_{\alpha(wu)}} L(wu) \xrightarrow{L(h_{\Phi(u,w)})} L(w) \\ \parallel \qquad \qquad \qquad \parallel \\ G(wu) \xlongequal{\quad} G(wu) \xrightarrow{G(h_{\Phi(u,w)})} G(w) \xrightarrow{h_{\alpha(w)}} L(w) \\ \downarrow G(v_{\Phi(u,v)}) \quad \downarrow G(v_{\Phi(u,w)}) \quad \downarrow G(\chi_{\Phi(u,w)}) \quad \parallel \\ G(w) \xlongequal{\quad} G(w) \xlongequal{\quad} G(w) \xrightarrow{h_{\alpha(w)}} L(w) \end{array}$$

Note that it is sufficient to define \tilde{a} on these particular vertical arrows, since all others are generated by these and their inverses. ■

6.16. REMARKS.

1. One might be tempted to think that if we take $W = \mathbf{C}_1$, the class of all arrows in \mathbf{C} , then every horizontal arrow in $\mathbf{C}\{W\}$ will have a companion and a conjoint. Unfortunately that is not the case: only arrows of the form $(\xleftarrow{uv}) \xrightarrow{u} (\xrightarrow{v})$ get a companion and a conjoint. However, if we restrict ourselves to such horizontal arrows (for any chosen class W), take all vertical arrows and the full sub double category of double cells, we do get a double category in which every horizontal arrow has a companion and a conjoint. That double category is the transpose of what Shulman [31] has called a fibrant double category and what was called a gregarious double category in [7].
2. The vertical and horizontal property together determine $\mathbf{C}\{W\}$ up to both horizontal and vertical equivalence of weakly globular double categories, but note that for the vertical equivalence the arrows are pseudo-functors, where for the horizontal equivalence the arrows are strict functors.
3. The construction of $\mathbf{C}\{W\}$ given in Section 5.1 can be extended to 2-categories, where the universal properties of $\mathcal{C}\{W\}$ can be formulated using the horizontal double category HC . It can even be extended to arbitrary weakly globular double categories. We will present these results in a sequel to this paper, as they involve a lot of technical details.

7. Conclusions

In this paper we studied what the horizontal and vertical universal properties of a weakly globular double category of fractions should be and we have seen how this determines this weakly globular double category both up to horizontal and vertical equivalence. Both companions and precompanions play an important role in the description of its universal properties. We have seen that the notion of precompanion generalizes both the notions of companion and of horizontal isomorphism in a double category.

For double categories with strict functors, the hom-object $\mathbf{DbICat}(\mathbb{C}, \mathbb{D})$ can be given the structure of a double category where the horizontal arrows are horizontal transformations and the vertical arrows are vertical transformations, and modifications are given as families of double cells that are functorial and natural in appropriate ways (the precise definition can be found in [10]). This leads to the question whether the universal property of $\mathbf{C}\{W\}$ as a coinverter can be expressed in terms of double categories rather than 2-categories. This is indeed possible, but requires a further study of the modifications for weakly globular double categories. We will discuss this in a separate paper.

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