CATEGORIES ENRICHED OVER A QUANTALOID: ALGEBRAS

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Abstract. Given a small quantaloid \( \mathcal{Q} \) with a set of objects \( \mathcal{Q}_0 \), it is proved that complete skeletal \( \mathcal{Q} \)-categories, completely distributive skeletal \( \mathcal{Q} \)-categories, and \( \mathcal{Q} \)-powersets of \( \mathcal{Q} \)-typed sets are all monadic over the slice category of \( \text{Set} \) over \( \mathcal{Q}_0 \).

1. Introduction

A quantaloid \([\text{Ros}1996]\) is a category enriched over the symmetric monoidal closed category \( \text{Sup} \) consisting of complete lattices and suprema-preserving functions. Since a quantaloid \( \mathcal{Q} \) is a bicategory \([\text{Ben}1967]\) (a 2-category indeed), following \([\text{BC1982, BCSW1983, Str1981, Wal1981}]\), a theory of categories enriched over \( \mathcal{Q} \) (or \( \mathcal{Q} \)-categories for short) has been developed, see e.g. \([\text{Stu2005, Stu2006, Stu2007}]\).

Given a small quantaloid \( \mathcal{Q} \), with \( \mathcal{Q}_0 \) its set of objects, objects in the slice category \( \text{Set} \downarrow \mathcal{Q}_0 \) are called \( \mathcal{Q} \)-typed sets. Then \( \mathcal{Q} \)-categories can be treated as structured \( \mathcal{Q} \)-typed sets. In this paper, we emphasize this aspect of \( \mathcal{Q} \)-categories. That is to say, we treat the theory of \( \mathcal{Q} \)-categories as one on the topos \( \text{Set} \downarrow \mathcal{Q}_0 \). It should be stressed that this theory is not developed within the topos \( \text{Set} \downarrow \mathcal{Q}_0 \), but rather, it depends heavily on the structure of \( \mathcal{Q} \) which is formed outside of that topos. The role of \( \mathcal{Q} \) is something like a “dynamic table of truth values” (c.f. \([\text{Stu2007}]\)). The purpose of this paper is to show that some interesting classes of \( \mathcal{Q} \)-categories are exactly the Eilenberg-Moore algebras corresponding to certain monads on the topos \( \text{Set} \downarrow \mathcal{Q}_0 \). These results show that the relationship between \( \mathcal{Q} \)-categories and \( \mathcal{Q} \)-typed sets are analogous to that between preordered sets and sets, exemplifying a benefit of treating \( \mathcal{Q} \)-categories as structured \( \mathcal{Q} \)-typed sets (instead of structured sets).

First, both the category \( \mathcal{Q} \text{-Sup} \) consisting of complete skeletal \( \mathcal{Q} \)-categories and cocontinuous \( \mathcal{Q} \)-functors and the category \( \mathcal{Q} \text{-CD} \) consisting of completely distributive skeletal \( \mathcal{Q} \)-categories and bicontinuous \( \mathcal{Q} \)-functors are monadic over \( \text{Set} \downarrow \mathcal{Q}_0 \). These conclusions extend the classical results that both the category \( \text{Sup} \) of complete lattices and join-preserving maps, and the category \( \text{CD} \) of completely distributive lattices and complete lattice homomorphisms, are monadic over \( \text{Set} \).

Second, the correspondence that sends each object \( A \) in \( \text{Set} \downarrow \mathcal{Q}_0 \) to its \( \mathcal{Q} \)-powerset...
\(|\mathcal{P}A|\) (defined below) yields a monadic functor \((\text{Set} \downarrow \mathcal{Q}_0)^{\text{op}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0\). We hasten to remark that the monadicity of \((\text{Set} \downarrow \mathcal{Q}_0)^{\text{op}}\) over \(\text{Set} \downarrow \mathcal{Q}_0\) is a special case of a general result in topos theory [MM1992] that states that for each topos \(\mathcal{E}\), the opposite category \(\mathcal{E}^{\text{op}}\) is monadic over \(\mathcal{E}\). The point of the result presented here is that for each non-empty set \(X\), there exist many monadic functors from \((\text{Set} \downarrow X)^{\text{op}}\) to \(\text{Set} \downarrow X\).

The contents are arranged as follows. In Section 2 we recall some basic concepts and results about \(\mathcal{Q}\)-categories and fix notations for later use. Section 3 proves that both \(\mathcal{Q}\text{-Sup}\) and \(\mathcal{Q}\text{-CD}\) are monadic over \(\text{Set} \downarrow \mathcal{Q}_0\). Section 4 proves the monadicity of the functor \((\text{Set} \downarrow \mathcal{Q}_0)^{\text{op}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0\) that sends each object \(A\) in \(\text{Set} \downarrow \mathcal{Q}_0\) to its \(\mathcal{Q}\)-powerset.

2. Categories enriched over a quantaloid

We refer to [Stu2005, Stu2006] for an overview of the theory of quantaloid-enriched categories. In this preliminary section, we recall some basic concepts and fix some notations for later use. It should be noted that the theory of quantaloid-enriched categories is a special case of that of \(\mathcal{W}\)-categories; and that some of the results in this section are also known to be valid for \(\mathcal{W}\)-categories, for example, the construction of \(\mathcal{P}A\) and the Yoneda lemma. The reader is referred to [BC1982, BCSW1983, Str1981, Str1983, Wal1981, Wal1982] for more on these categories.

\(\mathcal{Q}\)-categories, \(\mathcal{Q}\)-functors, and \(\mathcal{Q}\)-distributors. A quantaloid \(\mathcal{Q}\) is a category such that \(\mathcal{Q}(X,Y)\) is a complete lattice for any objects \(X, Y\) in \(\mathcal{Q}\) and that the composition \(\circ\) of arrows preserves suprema in both variables, i.e.

\[
g \circ \bigvee_i f_i = \bigvee_i g \circ f_i \quad \text{and} \quad \bigvee_i g_i \circ f = \bigvee_i g_i \circ f
\]

whenever the operations are defined. The identity arrow on an object \(X\) is written \(1_X\). The top and bottom elements in \(\mathcal{Q}(X,Y)\) are denoted by \(\top_{X,Y}\) and \(\bot_{X,Y}\) respectively. The identity \(1_X\) is required to be different from the bottom element \(\bot_{X,X}\) for all objects \(X\) in \(\mathcal{Q}\). However, for different objects \(X\) and \(Y\), it may happen that \(\top_{X,Y} = \bot_{X,Y}\). The class of objects in \(\mathcal{Q}\) is denoted by \(\mathcal{Q}_0\) as usual.

For any arrow \(f : X \rightarrow Y\) and any object \(Z\) in a quantaloid \(\mathcal{Q}\), both of the maps

\[
- \circ f : \mathcal{Q}(Y,Z) \rightarrow \mathcal{Q}(X,Z), \quad f \circ - : \mathcal{Q}(Z,X) \rightarrow \mathcal{Q}(Z,Y)
\]

have respective right adjoints

\[
- \Rightarrow f : \mathcal{Q}(X,Z) \rightarrow \mathcal{Q}(Y,Z), \quad f \Rightarrow - : \mathcal{Q}(Z,Y) \rightarrow \mathcal{Q}(Z,X).
\]

The operators \(\Rightarrow\) and \(\Rightarrow\) are called the right and left implication respectively.

In this paper, \(\mathcal{Q}\) is assumed to be a small quantaloid. This means that \(\mathcal{Q}_0\) is a set.

A \(\mathcal{Q}\)-typed set \(A\) is a pair \((A_0, t)\) with \(A_0\) being a set and \(t\) a function \(A_0 \rightarrow \mathcal{Q}_0\). The function \(t\) is called the type function of \(A\) with the value \(tx\) the type of \(x\). Type functions
of \(\mathcal{Q}\)-typed sets are all denoted by “\(t\)”, as usual. A type-preserving map \(F : A \rightarrow B\) between \(\mathcal{Q}\)-typed sets is a function \(F : A_0 \rightarrow B_0\) such that \(t(Fx) = tx\) for all \(x \in A_0\). The category of \(\mathcal{Q}\)-typed sets and type-preserving maps is exactly the slice category \(\text{Set} \downarrow \mathcal{Q}_0\).

For each \(X \in \mathcal{Q}_0\), we write \(\ast_X\) for the \(\mathcal{Q}\)-typed set with exactly one element \(\ast\) that is of type \(X\).

For a \(\mathcal{Q}\)-typed set \(A = (A_0, t)\), the underlying set \(A_0\) is often written \(A\) for simplicity if no confusion would arise.

A \(\mathcal{Q}\)-matrix \(\phi : A \rightarrow B\) between \(\mathcal{Q}\)-typed sets is a function that assigns to each pair \((x, y) \in A \times B\) an arrow \(\phi(x, y) \in \mathcal{Q}(tx, ty)\). In particular, if \(A\) (resp. \(B\)) is of the form \(\ast_X\), then we write \(\phi(x)\) for \(\phi(\ast, x)\) (resp. \(\phi(x, \ast)\)).

\(\mathcal{Q}\)-typed sets and \(\mathcal{Q}\)-matrices constitute a quantaloid \(\mathcal{Q}\text{-Mat}\) in which

- The composition \(\psi \circ \phi : A \rightarrow C\) of \(\phi : A \rightarrow B\) and \(\psi : B \rightarrow C\) is given by
  \[(\psi \circ \phi)(x, z) = \bigvee_{y \in B} \psi(y, z) \circ \phi(x, y)\].

- The identity \(\mathcal{Q}\)-matrix \(\text{id}_A : A \rightarrow A\) on a \(\mathcal{Q}\)-typed set \(A\) is given by
  \[\text{id}_A(x, y) = \begin{cases} 1_{tx}, & x = y; \\ \bot_{tx, ty}, & \text{otherwise.} \end{cases}\]

- The local order is defined pointwise, that is,
  \[\phi_1 \leq \phi_2 : A \rightarrow B\] if and only if \(\phi_1(x, y) \leq \phi_2(x, y)\) for all \((x, y) \in A \times B\).

- For any \(\mathcal{Q}\)-matrices \(\phi : A \rightarrow B\), \(\psi : B \rightarrow C\) and \(\lambda : A \rightarrow C\), \(\lambda \not\leq \phi : B \rightarrow C\) and \(\psi \not\leq \lambda : A \rightarrow B\) are respectively given by
  \[(\lambda \not\leq \phi)(y, z) = \bigwedge_{x \in A} \lambda(x, z) \not\leq \phi(x, y), \quad (\psi \not\leq \lambda)(x, y) = \bigwedge_{z \in C_0} \psi(y, z) \not\leq \lambda(x, z)\].

A \(\mathcal{Q}\)-category \(\mathcal{A}\) is a monad in the 2-category \(\mathcal{Q}\text{-Mat}\). Explicitly, a \(\mathcal{Q}\)-category is a pair \((\mathcal{A}, A)\) where \(A\) is a \(\mathcal{Q}\)-typed set and \(A : A \rightarrow A\) is a \(\mathcal{Q}\)-matrix such that \(\text{id}_A \leq A\) and \(A \circ A \leq A\).

In the following we write \(\mathcal{A}\) for a \(\mathcal{Q}\)-category, \(|\mathcal{A}|\) for its underlying \(\mathcal{Q}\)-typed set and \(\mathcal{A}_0\) for the underlying set of \(|\mathcal{A}|\).

A \(\mathcal{Q}\)-functor \(F : \mathcal{A} \rightarrow \mathcal{B}\) between \(\mathcal{Q}\)-categories is a type-preserving map \(F : |\mathcal{A}| \rightarrow |\mathcal{B}|\) such that \(\mathcal{A}(x, y) \leq \mathcal{B}(Fx, Fy)\) for all objects \(x, y\) in \(\mathcal{A}\). The category of \(\mathcal{Q}\)-categories and \(\mathcal{Q}\)-functors is denoted by \(\mathcal{Q}\text{-Cat}\).

The correspondence \(\mathcal{A} \mapsto |\mathcal{A}|\) defines a (forgetful) functor \(|-| : \mathcal{Q}\text{-Cat} \rightarrow \text{Set} \downarrow \mathcal{Q}_0\). Conversely, each \(\mathcal{Q}\)-typed set \(A\) together with the identity \(\mathcal{Q}\)-matrix on \(A\) is a \(\mathcal{Q}\)-category.

Such \(\mathcal{Q}\)-categories are said to be discrete. In this paper, we do not distinguish \(\mathcal{Q}\)-typed sets and discrete \(\mathcal{Q}\)-categories.
For a \( Q \)-functor \( F : \mathbf{A} \to \mathbf{B} \) between \( Q \)-categories, we write \( F \) instead of \(|F|\) for the underlying type-preserving map \(|A| \to |B|\).

The underlying order of a \( Q \)-category \( \mathbf{A} \) [Stu2005] refers to the preorder on the set of objects in \( \mathbf{A} \) defined by
\[
x \leq y \iff tx = ty \text{ and } 1_{tx} \leq A(x, y).
\]

It is trivial that \( Q \)-functors preserve underlying orders of \( Q \)-categories. Two objects \( x, y \) of \( \mathbf{A} \) are isomorphic, in symbols \( x \cong y \), if \( x \leq y \) and \( y \leq x \). A \( Q \)-category \( \mathbf{A} \) is skeletal if its underlying order is antisymmetric.

The underlying order of a \( Q \)-category \( \mathbf{B} \) induces a preorder on the set of all \( Q \)-functors from a \( Q \)-category \( \mathbf{A} \) to \( \mathbf{B} \):
\[
F \leq G \iff \forall x \in A, Fx \leq Gx.
\]

Thus, \( Q \)-\( \text{Cat} \) is indeed a locally ordered category. Two \( Q \)-functors \( F, G : \mathbf{A} \to \mathbf{B} \) are isomorphic, in symbols \( F \cong G \), if \( F \leq G \) and \( G \leq F \).

A pair of \( Q \)-functors \( F : \mathbf{A} \to \mathbf{B} \) and \( G : \mathbf{B} \to \mathbf{A} \) is said to form an adjunction, written \( \mathbf{F} \dashv \mathbf{G} : \mathbf{A} \to \mathbf{B} \), if \( 1_{\mathbf{A}} \leq G \circ F \) and \( F \circ G \leq 1_{\mathbf{B}} \). In this case, \( F \) is called a left adjoint of \( \mathbf{G} \) and \( \mathbf{G} \) a right adjoint of \( \mathbf{F} \).

A \( Q \)-distributor \( \phi : \mathbf{A} \to \mathbf{B} \) between \( Q \)-categories is a \( Q \)-matrix \( \phi : |A| \to |B| \) that is compatible with the structures on \( \mathbf{A} \) and \( \mathbf{B} \) in the sense that
\[
\mathbf{B}(y, y') \circ \phi(x, y) \leq \phi(x, y') \quad \text{and} \quad \phi(x, y) \circ \mathbf{A}(x', x) \leq \phi(x', y)
\]
for any objects \( x, x' \) in \( \mathbf{A} \) and \( y, y' \) in \( \mathbf{B} \); or equivalently, \( \phi \circ A = \phi = B \circ \phi \) in \( Q \)-\( \text{Mat} \).

\( Q \)-categories and \( Q \)-distributors constitute a quantaloid \( Q \)-\( \text{Dist} \) in which compositions, the left and right implications are calculated as in \( Q \)-\( \text{Mat} \).

Following [Lack2010], for a 2-category \( \mathbf{C} \), we denote by \( \mathbf{C}^{\text{op}} \) \( (\mathbf{C}^\text{co}, \text{resp.}) \) the 2-category obtained by reversing the 1-arrows (the 2-arrows, resp.) in \( \mathbf{C} \). For each quantaloid \( Q \), \( Q^{\text{op}} \) is also a quantaloid, but \( Q^\text{co} \) is not in general. Given a \( Q \)-category \( \mathbf{A} \), there is a corresponding \( Q \)-\text{op}-category \( \mathbf{A}^\text{op} \) with the same underlying \( Q \)-typed set as that of \( \mathbf{A} \) and with \( \mathbf{A}^\text{op}(x, y) = \mathbf{A}(y, x) \).

For each \( Q \)-distributor \( \phi : \mathbf{A} \to \mathbf{B} \), the assignment \( \phi^\text{op}(x, y) = \phi(x, y) \) defines a \( Q \)-\text{op}-distributor \( \mathbf{B}^\text{op} \to \mathbf{A}^\text{op} \). If \( F : \mathbf{A} \to \mathbf{B} \) is a \( Q \)-functor, then
\[
F^\text{op} : \mathbf{A}^\text{op} \to \mathbf{B}^\text{op}, \quad x \mapsto Fx
\]
is a \( Q \)-\text{op}-functor. Furthermore, \( F \leq G \) in \( Q \)-\( \text{Cat} \) if and only if \( G^\text{op} \leq F^\text{op} \) in \( Q \)-\( \text{Cat}^\text{op} \).

Therefore, \( (Q \- \text{Cat})^\text{co} \) is isomorphic to \( Q \)-\( \text{Cat}^\text{op} \) [Stu2005].

\footnote{We would like to point out that the terminologies adopted here are not exactly the same as in our main references, [Stu2005, Stu2006], on quantaloid-enriched categories. Our \( Q \)-categories and \( Q \)-distributors are exactly the \( Q \)-\text{op}-categories and \( Q \)-\text{op}-distributors in the sense of Stubbe. The difference arises in the interpretations of \( \mathbf{A}(x, y) \) for a \( Q \)-category \( \mathbf{A} \): it is interpreted as the hom-arrow from \( y \) to \( x \) in [Stu2005, Stu2006], but from \( x \) to \( y \) here. Note that this difference also leads to the swap of presheaves and co-presheaves.}
The graph and cograph of a \( Q \)-functor \( F : \mathcal{A} \to \mathcal{B} \) refer to the \( Q \)-distributors \( F_\sharp = \mathcal{B}(F-, -) : \mathcal{A} \leftrightarrow \mathcal{B} \) and \( F_\sharp^* = \mathcal{B}(-, F-) : \mathcal{B} \leftrightarrow \mathcal{A} \) respectively. \( F_\sharp \) is a left adjoint of \( F_\sharp^* \) in \( Q \text{-Dist} \), i.e., \( \mathfrak{A} \leq F_\sharp^* \circ F_\sharp \) and \( F_\sharp \circ F_\sharp^* \leq \mathcal{B} \).

The following proposition is a special case of an observation in [BCSW1983] about modules (= distributors) between \( W \)-categories. We record it here because of its usefulness.

**2.1. Proposition.** Let \( F : \mathcal{A} \to \mathcal{B} \) be a \( Q \)-functor.

(1) \( F \) is fully faithful in the sense that \( \mathfrak{A}(x, y) = \mathfrak{B}(Fx, Fy) \) for all \( x, y \in \mathfrak{A} \) if and only if \( F_\sharp^* \circ F_\sharp = \mathfrak{A} \).

(2) If \( F \) is essentially surjective in the sense that there is some \( x \in \mathfrak{A} \) such that \( Fx \cong y \) in \( \mathfrak{B} \) for all \( y \in \mathfrak{B} \), then \( F_\sharp \circ F_\sharp^* = \mathfrak{B} \).

A presheaf [Stu2005] on a \( Q \)-category \( \mathfrak{A} \) is a \( Q \)-distributor of the form \( \phi : \mathfrak{A} \leftrightarrow \ast \mathfrak{X} \). All presheaves on \( \mathfrak{A} \) constitute a skeletal \( Q \)-category \( \mathcal{P}\mathfrak{A} \) with

\[
\text{t}\phi = X \text{ and } \mathcal{P}\mathfrak{A}(\phi, \phi') = \phi' \swarrow \phi
\]

for any \( \phi : \mathfrak{A} \leftrightarrow \ast \mathfrak{X} \) and \( \phi' : \mathfrak{A} \leftrightarrow \ast \mathfrak{Y} \).

Dually, a co-presheaf on \( \mathfrak{A} \) is a \( Q \)-distributor of the form \( \psi : \ast \mathfrak{X} \leftrightarrow \mathfrak{A} \). All co-presheaves on \( \mathfrak{A} \) constitute a skeletal \( Q \)-category \( \mathcal{P}^\dagger \mathfrak{A} \) with

\[
\text{t}\psi = X \text{ and } \mathcal{P}^\dagger \mathfrak{A}(\psi, \psi') = \psi' \searrow \psi
\]

for any \( \psi : \ast \mathfrak{X} \leftrightarrow \mathfrak{A} \) and \( \psi' : \ast \mathfrak{Y} \leftrightarrow \mathfrak{A} \).

It should be stressed that the underlying order of \( \mathcal{P}\mathfrak{A} \) coincides with the local order in \( Q \text{-Dist} \) while the underlying order of \( \mathcal{P}^\dagger \mathfrak{A} \) is the reverse local order in \( Q \text{-Dist} \).

The correspondences

\[
x \mapsto \mathfrak{A}(-, x) : \mathfrak{A} \leftrightarrow \ast \text{tx}
\]

and

\[
x \mapsto \mathfrak{A}(x, -) : \ast \text{tx} \leftrightarrow \mathfrak{A}
\]

define two \( Q \)-functors

\[
\mathcal{Y}_\mathfrak{A} : \mathfrak{A} \to \mathcal{P}\mathfrak{A}
\]

and

\[
\mathcal{Y}^\dagger_\mathfrak{A} : \mathfrak{A} \to \mathcal{P}^\dagger \mathfrak{A}
\]

which are called respectively the Yoneda and the co-Yoneda embedding due to the following:
2.2. Lemma. (Yoneda lemma, [Stu2005]) \( \mathcal{P}_A(\mathcal{Y}_A(x), \phi) = \phi(x) \) and \( \mathcal{P}^!_A(\psi, \mathcal{Y}^!_A(x)) = \psi(x) \) for any \( x \in A \), \( \phi \in \mathcal{P}_A \), and \( \psi \in \mathcal{P}^!_A \).

The correspondence \( A \mapsto \mathcal{P}_A \) gives a contravariant functor
\[
\mathcal{P} : (\mathcal{Q}\text{-Cat})^{\text{op}} \longrightarrow \mathcal{Q}\text{-Cat}
\]
that sends a \( \mathcal{Q} \)-functor \( F : A \longrightarrow B \) to
\[
\mathcal{P}F : \mathcal{P}B \longrightarrow \mathcal{P}A, \quad \mathcal{P}F(\psi) = \psi \circ F_\circ.
\]
Dually, the correspondence \( A \mapsto \mathcal{P}^!_A \) gives a contravariant functor
\[
\mathcal{P}^! : \mathcal{Q}\text{-Cat} \longrightarrow (\mathcal{Q}\text{-Cat})^{\text{op}}
\]
that sends a \( \mathcal{Q} \)-functor \( F : A \longrightarrow B \) to
\[
\mathcal{P}^!F : \mathcal{P}^!B \longrightarrow \mathcal{P}^!A, \quad \mathcal{P}^!F(\psi) = F_\circ \circ \psi.
\]

2.3. Theorem. [Hoh2014, Stu2005] The functor \( \mathcal{P}^! : \mathcal{Q}\text{-Cat} \longrightarrow (\mathcal{Q}\text{-Cat})^{\text{op}} \) is left adjoint to \( \mathcal{P} : (\mathcal{Q}\text{-Cat})^{\text{op}} \longrightarrow \mathcal{Q}\text{-Cat} \).

2.4. Proposition. [Stu2013] Let \( F : A \longrightarrow B \) be a \( \mathcal{Q} \)-functor.

1. The \( \mathcal{Q} \)-functor \( \mathcal{P}F \) has a left adjoint \( \exists_F : \mathcal{P}A \longrightarrow \mathcal{P}B \) and a right adjoint \( \forall_F : \mathcal{P}A \longrightarrow \mathcal{P}B \) given by \( \exists_F(\phi) = \phi \circ F_\circ \) and \( \forall_F(\phi) = \phi \circ F_\circ \), respectively.

2. The \( \mathcal{Q} \)-functor \( \mathcal{P}^!F : \mathcal{P}^!B \longrightarrow \mathcal{P}^!A \) has a left adjoint \( \forall^!_F : \mathcal{P}^!A \longrightarrow \mathcal{P}^!B \) and a right adjoint \( \exists^!_F : \mathcal{P}^!A \longrightarrow \mathcal{P}^!B \) given by \( \forall^!_F(\psi) = F_\circ \circ \psi \) and \( \exists^!_F(\psi) = F_\circ \circ \psi \), respectively.

Different notations have been used for the \( \mathcal{Q} \)-functors \( \exists_F, \forall_F, \exists^!_F \) and \( \forall^!_F \) in [SZ2013a, Stu2013]. The notations adopted here originate from topos theory [MM1992].

2.5. Proposition. Given a pair of \( \mathcal{Q} \)-functors \( F : A \longrightarrow B \) and \( G : B \longrightarrow A \), the following are equivalent:

1. \( F \circ G : A \longrightarrow B \).
2. \( \exists_F \circ \exists_G : \mathcal{P}A \longrightarrow \mathcal{P}B \).
3. \( \mathcal{P}F \circ \mathcal{P}G : \mathcal{P}B \longrightarrow \mathcal{P}A \).
4. \( \exists^!_F \circ \exists^!_G : \mathcal{P}^!A \longrightarrow \mathcal{P}^!B \).
5. \( \mathcal{P}^!F \circ \mathcal{P}^!G : \mathcal{P}^!B \longrightarrow \mathcal{P}^!A \).

Proof. We prove the equivalence of (1) and (2) for example.

1. \( \Rightarrow \) (2) This follows from the fact that a 2-functor preserves adjunctions [Lack2010].
2. \( \Rightarrow \) (1) For any object \( x \) in \( A \),
\[
\mathcal{Y}_A(x) \leq \exists_G \circ \exists_F(\mathcal{Y}_A(x)) = \mathcal{Y}_A(x) \circ (G \circ F)^2 = \mathcal{Y}_A(GFx)
\]
showing that \( x \leq GFx \). Thus \( 1_A \leq G \circ F \). Similarly it can be verified that \( F \circ G \leq 1_B \).

Hence \( F \circ G : A \longrightarrow B \). \( \blacksquare \)
It is clear that the assignments \( F \mapsto \exists_F \) and \( F \mapsto \exists^\dagger_F \) give rise to two functors:

\[
P_\exists : \text{Q-Cat} \to \text{Q-Cat} \quad \text{and} \quad P^\dagger_\exists : \text{Q-Cat} \to \text{Q-Cat}.
\]

Both \( P_\exists \) and \( P^\dagger_\exists \) preserve the local order in \( \text{Q-Cat} \), hence both of them are 2-functorial \( \text{Q-Cat} \to \text{Q-Cat} \). Both of the contravariant functors \( P \) and \( P^\dagger \) reverse the local order, so, both of them are 2-functorial from \( \text{Q-Cat}^{\text{coop}} \) to \( \text{Q-Cat} \).

For any \( \text{Q-functor} \) \( F \), it follows from Proposition 2.4 that \( P^\dagger_\exists F \dashv P_\exists F \). Thus, \( P_\exists (P^\dagger_\exists F) \dashv P_\exists (P^\dagger F) \). By Proposition 2.5(2) one has \( P_\exists (P^\dagger F) \dashv P_\exists (P^\dagger_\exists F) \). This proves the following:

2.6. Corollary. \([\text{Hoh2014, Stu2013]}\) \( P \circ P^\dagger = P_\exists \circ P^\dagger_\exists \).

\[
\begin{array}{c|c}
\text{Q-Cat} & P^\dagger_\exists \\
\hline
P_\exists & P \to \text{Q-Cat} \\
\end{array}
\]

The following conclusion is a direct consequence of Proposition 2.1, it will be useful in the last section.

2.7. Proposition. Let \( F : \mathbb{A} \to \mathbb{B} \) be a \( \text{Q-functor} \).

(1) \( F \) is fully faithful if and only if \( P F \circ \exists_F = 1_{P \mathbb{A}} \) if and only if \( P^\dagger F \circ \exists^\dagger_F = 1_{P^\dagger \mathbb{A}} \).

(2) If \( F \) is essentially surjective, then \( \exists_F \circ P F = 1_{P \mathbb{B}} \) and \( \exists^\dagger_F \circ P^\dagger F = 1_{P^\dagger \mathbb{B}} \).

Complete and completely distributive \( \text{Q-categories} \). Let \( \mathbb{A} \) be a \( \text{Q-category} \) and \( \phi : \mathbb{A} \to \ast_X \) a presheaf on \( \mathbb{A} \). A supremum of \( \phi \) is an object \( \text{sup} \phi \) in \( \mathbb{A} \) of type \( X \) such that for any \( x \) in \( \mathbb{A} \),

\[
\mathbb{A}(\text{sup} \phi, x) = P\mathbb{A}(\phi, Y_\mathbb{A}(x));
\]

or equivalently, \( \mathbb{A}(\text{sup} \phi, -) = \mathbb{A} \vee \phi \). It is clear that the supremum of a presheaf \( \mathbb{A} \to \ast_X \), if exists, is unique up to isomorphism. Dually, the infimum of a co-presheaf \( \psi : \ast_X \to \mathbb{A} \) is an object \( \text{inf} \psi \) in \( \mathbb{A} \) of type \( X \) such that for any \( x \) in \( \mathbb{A} \),

\[
\mathbb{A}(x, \text{inf} \psi) = P^\dagger\mathbb{A}(Y^\dagger_\mathbb{A}(x), \psi);
\]

or equivalently, \( \mathbb{A}(-, \text{inf} \psi) = \psi \downarrow \mathbb{A} \).

2.8. Definition. \([\text{Stu2005]}\) A \( \text{Q-category} \) \( \mathbb{A} \) is cocomplete if every presheaf on \( \mathbb{A} \) has a supremum; \( \mathbb{A} \) is complete if every co-presheaf on \( \mathbb{A} \) has an infimum.

It is known that (i) \( \mathbb{A} \) is cocomplete if and only if the Yoneda embedding \( Y_\mathbb{A} : \mathbb{A} \to P\mathbb{A} \) has a left adjoint \( \text{sup}_\mathbb{A} : P\mathbb{A} \to \mathbb{A} \); (ii) \( \mathbb{A} \) is complete if the co-Yoneda embedding \( Y^\dagger_\mathbb{A} : \mathbb{A} \to P^\dagger\mathbb{A} \) has a right adjoint \( \text{inf}_\mathbb{A} : P^\dagger\mathbb{A} \to \mathbb{A} \); and (iii) \( \mathbb{A} \) is complete if and only if it is cocomplete.
2.9. Example. [Stu2005] Let $A$ be a $Q$-category. Then both $PA$ and $P^A$ are complete, hence cocomplete. Explicitly, for any $\Phi \in P(PA)$ and $\Psi \in P(PA)$,

$$\sup \Phi = \Phi \circ (YA)_2, \quad \inf \Psi = \Psi \setminus (YA)_2;$$

for any $\Phi \in P(P^A)$ and $\Psi \in P(P^A)$,

$$\sup \Phi = (YA)^\sharp \neg \Phi, \quad \inf \Psi = (YA)^\sharp \circ \Psi.$$

In particular, $\sup_{PA} = PY_A$ and $\inf_{P^A} = P(Y_A)^\dagger$.

A $Q$-functor $F : A \longrightarrow B$ is said to be cocontinuous if it preserves suprema in the sense that $F(\sup_A \phi)$ is a supremum of $\exists_F(\phi)$ whenever $\sup_A \phi$ exists. Dually, $F : A \longrightarrow B$ is continuous if it preserves infima in the sense that $F(\inf_A \phi)$ is an infimum of $\exists_F^1(\psi)$ whenever $\inf_A \psi$ exists. $F : A \longrightarrow B$ is bicontinuous if it is both cocontinuous and continuous.

It is known [Stu2005] that a $Q$-functor $F : A \longrightarrow B$ between complete $Q$-categories is a left adjoint (resp. right adjoint) if and only if $F$ is cocontinuous (resp. continuous). In particular, for each $Q$-functor $F : A \longrightarrow B$, $PF : PB \longrightarrow PA$ is bicontinuous; $\exists_F : PA \longrightarrow PB$ is cocontinuous; and $\forall_F : PA \longrightarrow PB$ is continuous.

2.10. Definition. [Stu2007] A $Q$-category $A$ is completely distributive if it is cocomplete and the left adjoint $\sup_A : PA \longrightarrow A$ of the Yoneda embedding $YA : A \longrightarrow PA$ has a left adjoint $\downarrow_A : A \longrightarrow PA$.

Note that completely distributive $Q$-categories are said to be \textit{totally continuous} in [Stu2007]. Here we call them completely distributive following the practice in lattice theory, e.g. [Joh1982, Ran1952, Wood2004].

2.11. Example. [Stu2007] For a $Q$-category $A$, it follows from Example 2.9 that $\sup_{PA} = PY_A$. Thus, $\sup_{PA}$ is a right adjoint by Proposition 2.4. This shows that $PA$ is completely distributive.

2.12. Proposition. Let $A, B$ be skeletal $Q$-categories, $F : A \longrightarrow B$ a left and right adjoint $Q$-functor.

(1) If $F$ is an epimorphism in $Q\text{-}\mathbf{Cat}$ and $A$ is completely distributive, then so is $B$.

(2) If $F$ is a monomorphism in $Q\text{-}\mathbf{Cat}$ and $B$ is completely distributive, then so is $A$.

Proof. (1) Suppose that $H \dashv F \dashv G$. Then $F \circ G \circ F = F$, hence $F \circ G = 1_B$ since $F$ is an epimorphism. It follows that for any $y \in B$,

$$(PG \circ YA \circ G)(y) = PG(YA(Gy)) = A(G-, Gy) = B(F \circ G-, y) = B(-, y) = Y_B(y),$$

showing that $PG \circ YA \circ G = Y_B$.

By assumption, the Yoneda embedding $YA$ has a left adjoint $\sup_A$ that also has a left adjoint $\downarrow_A$. By virtue of Proposition 2.5 it holds that $PH \dashv PF \dashv PG$, hence

$$(PH \circ \downarrow_A \circ H) \dashv (F \circ \sup_A \circ PF) \dashv PG \circ YA \circ G = Y_B.$$
Therefore, $\mathbb{B}$ is completely distributive with $\sup_B = F \circ \sup_A \circ \mathcal{P}F$.

(2) Suppose that $H \dashv F \dashv G$. Then $F \circ H \circ F = F \circ G \circ F = F$, and thus $H \circ F = G \circ F = 1_A$ since $F$ is a monomorphism. Hence, for each $x$ in $A$,

$$(\mathcal{P}F \circ \mathcal{Y}_B \circ F)(x) = \mathcal{P}F(\mathcal{Y}_B(Fx)) = \mathbb{B}(F-,Fx) = A(H \circ F-,x) = A(-,x) = Y_A(x).$$

That means $\mathcal{P}F \circ \mathcal{Y}_B \circ F = Y_A$. Since $\exists F \dashv \mathcal{P}F$, $\sup_B \dashv \mathcal{Y}_B$, and $H \dashv F$, it follows that $H \circ \sup_B \circ \exists F$ is a left adjoint of $Y_A = \mathcal{P}F \circ \mathcal{Y}_B \circ F$. Therefore, $A$ is complete with $\sup_A = H \circ \sup_B \circ \exists F$. Since $F$ is cocontinuous (being a left adjoint), we have that $F \circ \sup_A = \sup_B \circ \exists F$. Hence

$$\sup_A = G \circ F \circ \sup_A = G \circ \sup_B \circ \exists F.$$ 

This shows that $\sup_A : \mathcal{P}A \longrightarrow A$ is a composite of right adjoints ($\exists F$ is a right adjoint by Proposition 2.5, $\sup_B$ is a right adjoint by complete distributivity of $\mathbb{B}$), hence it is itself a right adjoint. The conclusion thus follows.

Now we form the following categories:

- **$Q$-Sup**, the category of skeletal cocomplete $Q$-categories and cocontinuous $Q$-functors.
- **$Q$-Inf**, the category of skeletal complete $Q$-categories and continuous $Q$-functors.
- **$Q$-CD**, the category of skeletal completely distributive $Q$-categories and bicontinuous $Q$-functors.

The categories $Q$-Sup and $Q$-Inf are dually isomorphic. For each cocontinuous $Q$-functor $F : A \longrightarrow \mathbb{B}$ between complete $Q$-categories, let $F^+$ denote its right adjoint. Dually, for each continuous $Q$-functor $G : \mathbb{B} \longrightarrow A$ between complete $Q$-categories, let $G^+$ denote its left adjoint. Then we obtain a pair of functors

$$Q-\text{Inf} \xleftarrow{(\cdot)^{op}} \xrightarrow{(\cdot)^{-}} Q-\text{Sup}^{op}$$

that are inverse to each other.

The questions. The categories $Q$-Sup, $Q$-CD are respectively the $Q$-analogue of the category $\text{Sup}$ of complete lattices and join-preserving maps, and the category $\text{CD}$ of completely distributive lattices and complete lattice homomorphisms. Since both $\text{Sup}$ and $\text{CD}$ are monadic over $\text{Set}$ [Joh1982], our first question is whether the categories $Q$-Sup and $Q$-CD are monadic over $\text{Set} \downarrow Q_0$?

The forgetful functor $|-| : Q\text{-Cat} \longrightarrow \text{Set} \downarrow Q_0$ has a left adjoint $I : \text{Set} \downarrow Q_0 \longrightarrow Q\text{-Cat}$, given by identifying $Q$-typed sets with discrete $Q$-categories. Consider the adjunction $|\mathcal{P}| \dashv |\mathcal{P}|^\text{op}$ obtained by composing the following

$$\text{Set} \downarrow Q_0 \xleftarrow{\text{I}} Q\text{-Cat} \xrightarrow{\mathcal{P}} Q\text{-Cat}^{\text{op}} \xrightarrow{\mathcal{P}^{\text{op}}} \text{Set} \downarrow Q_0^{\text{op}}.$$
It is clear that $|P^\dagger| \dashv |P|$ is the $Q$-version of the adjunction $P^{\text{op}} \dashv P : \text{Set} \to \text{Set}^{\text{op}}$. It is known that $P : \text{Set}^{\text{op}} \to \text{Set}$ is monadic and the corresponding algebras are the complete atomic Boolean algebras [Joh1982]. So, it is natural to ask what are the algebras of the monad determined by the adjunction $|P^\dagger| \dashv |P|$? This is the second question we’ll consider in this paper.

Before proceeding, we list below some known facts about $Q$-$\text{Cat}$, $Q$-$\text{Sup}$, and $Q$-$\text{CD}$.

(a) The monad corresponding to the adjunction $\mathcal{I} \dashv |\cdot| : \text{Set} \downarrow Q_0 \to Q$-$\text{Cat}$ is the identity monad on $\text{Set} \downarrow Q_0$, hence its Eilenberg-Moore category is $\text{Set} \downarrow Q_0$. Therefore, the forgetful functor $|\cdot| : Q$-$\text{Cat} \to \text{Set} \downarrow Q_0$ is not monadic. This is an extension of the well-known fact that the forgetful functor from preordered sets to sets is not monadic.

(b) The composite of the adjunctions

$$Q$-$\text{Cat} \xrightarrow{\mathcal{E}} Q$-$\text{Inf} \xrightarrow{(-)^\dagger} Q$-$\text{Sup}^{\text{op}} \xrightarrow{\mathcal{P}^{\text{op}}_3} Q$-$\text{Cat}^{\text{op}}$$

is exactly the adjunction $P^\dagger \dashv P : Q$-$\text{Cat} \to Q$-$\text{Cat}^{\text{op}}$ in Theorem 2.3. It is proved in [Stu2013] that the algebras of the monad corresponding to the adjunction $P^\dagger \dashv P$ are the completely distributive $Q$-categories with bicontinuous $Q$-functors as morphisms. Hence, the category $Q$-$\text{CD}$ is monadic over $Q$-$\text{Cat}$.

(c) Restricting the codomain of the 2-functor $\mathcal{P}_3 : Q$-$\text{Cat} \to Q$-$\text{Cat}$ to $Q$-$\text{Sup}$ gives a left adjoint, also written $\mathcal{P}_3$, to the forgetful functor $\mathcal{E} : Q$-$\text{Sup} \to Q$-$\text{Cat}$. It is proved in [Stu2013] that the forgetful functor $Q$-$\text{Sup} \to Q$-$\text{Cat}$ is lax-idempotent monadic (see Theorem 3.16 below).

### 3. $Q$-$\text{Sup}$ and $Q$-$\text{CD}$ are monadic over $\text{Set} \downarrow Q_0$

The aim of this section is to show that both $Q$-$\text{Sup}$ and $Q$-$\text{CD}$ are strictly monadic over $\text{Set} \downarrow Q_0$. Recall that a right adjoint functor $G : \mathcal{D} \to \mathcal{C}$ is monadic (resp. strictly monadic) [Mac1998, MM1992] if the comparison functor $K : \mathcal{D} \to \mathcal{C}^\mathcal{T}$ is an equivalence (resp. isomorphism) of categories, where $\mathcal{T}$ is the corresponding monad and $\mathcal{C}^\mathcal{T}$ is the Eilenberg-Moore category of $\mathcal{T}$-algebras and homomorphisms. A category $\mathcal{D}$ is (strictly) monadic over a category $\mathcal{C}$ if there exists a (strictly) monadic functor $G : \mathcal{D} \to \mathcal{C}$.

For an object $x$ in a $Q$-category $\mathcal{A}$ and an arrow $f : tx \to Y$ in $Q$, the tensor of $f$ and $x$, denoted by $f \otimes x$, is an object in $\mathcal{A}$ of type $Y$ such that $\mathcal{A}(f \otimes x, -) = \mathcal{A}(x, -) \vee f$. Dually, for an arrow $g : Y \to tx$, the cotensor of $g$ and $x$, denote by $g \mapsto x$, is an object in $\mathcal{A}$ of type $Y$ such that $\mathcal{A}(-, g \mapsto x) = g \sqcap -$. A $Q$-category $\mathcal{A}$ is tensored if the tensor $f \otimes x$ exists for all objects $x$ in $\mathcal{A}$ and all arrows $f$ in $Q$ with codomain $tx$ [Stu2006]. The dual notion is cotensored.

It is easy to see that the tensor $f \otimes x$ is the supremum of the presheaf $f \circ \mathcal{A}(-, x)$; the cotensor $g \mapsto x$ is the infimum of the co-presheaf $\mathcal{A}(x, -) \circ g$. So, a complete $Q$-category is both tensored and cotensored.
For each $\mathcal{Q}$-category $\mathbb{A}$ and each object $X$ in $\mathcal{Q}$, write $\mathbb{A}_X$ for the preordered set consisting of objects of type $X$ in $\mathbb{A}$ together with the underlying order. It is known that if $\mathbb{A}$ is a complete $\mathcal{Q}$-category then $\mathbb{A}_X$ is a complete preordered set for each $X$ in $\mathcal{Q}$. The following proposition was observed in [LZ09] for quantale-enriched categories and in [Shen2014] for the general setting.

3.1. Proposition. Let $\mathbb{A}$ and $\mathbb{B}$ be $\mathcal{Q}$-categories, and $F : \mathbb{A} \rightarrow \mathbb{B}$ be a type-preserving map. If both $\mathbb{A}$ and $\mathbb{B}$ are tensored, then $F : \mathbb{A} \rightarrow \mathbb{B}$ is a $\mathcal{Q}$-functor if and only if

1. For any object $x$ in $\mathbb{A}$ and arrow $f : tx \rightarrow Y$, $f \otimes Fx \leq F(f \otimes x)$;
2. For any object $X$ in $\mathcal{Q}$, $F : \mathbb{A}_X \rightarrow \mathbb{B}_X$ is order-preserving.

Dually, if both $\mathbb{A}$ and $\mathbb{B}$ are cotensored, then $F : \mathbb{A} \rightarrow \mathbb{B}$ is a $\mathcal{Q}$-functor if and only if

1’ For any object $x$ in $\mathbb{A}$ and arrow $g : Y \rightarrow tx$, $F(g \mapsto x) \leq g \mapsto Fx$;
2’ For any object $X$ in $\mathcal{Q}$, $F : \mathbb{A}_X \rightarrow \mathbb{B}_X$ is order-preserving.

3.2. Proposition. [Stu2006] Let $\mathbb{A}$ and $\mathbb{B}$ be $\mathcal{Q}$-categories, $F : \mathbb{A} \rightarrow \mathbb{B}$ a type-preserving map. If $\mathbb{A}$ is tensored, then $F : \mathbb{A} \rightarrow \mathbb{B}$ is a left adjoint $\mathcal{Q}$-functor if and only if

1. $F$ preserves tensors in the sense that $F(f \otimes x) = f \otimes Fx$ for all objects $x$ in $\mathbb{A}$ and all arrows $f : tx \rightarrow Y$;
2. For all objects $X$ in $\mathcal{Q}$, $F : \mathbb{A}_X \rightarrow \mathbb{B}_X$ is a left adjoint.

Dually, if $\mathbb{A}$ is cotensored, then $F : \mathbb{A} \rightarrow \mathbb{B}$ is a right adjoint $\mathcal{Q}$-functor if and only if

1’ $F$ preserves cotensors in the sense that $F(g \mapsto x) = g \mapsto Fx$ for all objects $x$ in $\mathbb{A}$ and all arrows $g : Y \rightarrow tx$;
2’ For all objects $X$ in $\mathcal{Q}$, $F : \mathbb{A}_X \rightarrow \mathbb{B}_X$ is a right adjoint.

3.3. Definition. [SZ2013a] A closure operator on a $\mathcal{Q}$-category $\mathbb{A}$ is a $\mathcal{Q}$-functor $c : \mathbb{A} \rightarrow \mathbb{A}$ such that $1_{\mathbb{A}} \leq c$ and $c^2 \leq c$.

3.4. Lemma. If $c : \mathbb{A} \rightarrow \mathbb{A}$ is a closure operator on a skeletal $\mathcal{Q}$-category $\mathbb{A}$, then $c(\mathbb{A}) = \{ x \in \mathbb{A} \mid c(x) = x \}$ and $c : \mathbb{A} \rightarrow c(\mathbb{A})$ is left adjoint to the inclusion $i : c(\mathbb{A}) \hookrightarrow \mathbb{A}$.

Proof. Since $\mathbb{A}$ is skeletal and $c^2(x)$ is isomorphic to $c(x)$ for each $x$ in $\mathbb{A}$, it follows immediately that $c(\mathbb{A}) = \{ x \in \mathbb{A} \mid c(x) = x \}$.

Since $c \circ i(y) = c(y) = y$ for any $y$ in $c(\mathbb{A})$ and $i \circ c(x) = c(x) \geq x$ for any $x$ in $\mathbb{A}$, it follows that $c \circ i = 1_{c(\mathbb{A})}$ and $i \circ c \geq 1_{\mathbb{A}}$. Hence, $c$ is left adjoint to $i$. 


3.5. Definition. A congruence on a complete skeletal $Q$-category $\mathbb{A}$ is an equivalence relation $R$ on the underlying set $\mathbb{A}_0$ subject to the following conditions:

(i) $(x, y) \in R$ implies $tx = ty$, that is, equivalent elements have the same type.

(ii) For each object $X$ in $Q$, the subset $R \cap (\mathbb{A}_X \times \mathbb{A}_X)$ is closed w.r.t. joins in $\mathbb{A}_X \times \mathbb{A}_X$.

(iii) If $(x, y) \in R$, then $(f \otimes x, f \otimes y) \in R$ for all $f : tx \to ty$.

A congruence $R$ is complete if it satisfies moreover:

(iv) For each object $X$ in $Q$, the subset $R \cap (\mathbb{A}_X \times \mathbb{A}_X)$ is closed w.r.t. meets in $\mathbb{A}_X \times \mathbb{A}_X$.

(v) If $(x, y) \in R$, then $(g \mapsto x, g \mapsto y) \in R$ for all $g : Y \to tx$ in $Q$.

For a congruence $R$ on a complete skeletal $Q$-category $\mathbb{A}$, define a map $c : \mathbb{A}_0 \to \mathbb{A}_0$ by putting $c(x)$ to be the greatest element in the equivalence class of $x$ (which is a subset of the complete lattice $\mathbb{A}_{tx}$). Then $c$ is clearly type-preserving.

3.6. Lemma. If $R$ is a congruence on a complete skeletal $Q$-category $\mathbb{A}$, then $c : \mathbb{A} \to \mathbb{A}$ is a closure operator. Furthermore, if $R$ is complete then $c : \mathbb{A} \to \mathbb{A}$ is also a right adjoint.

Proof. It is easy to check that $c$ has the following properties:

(a) $c : \mathbb{A}_X \to \mathbb{A}_X$ preserves order for each object $X$ in $Q$.

(b) For each $X$ in $\mathbb{A}$, $x \leq c(x) = c^2(x)$.

(c) For any object $x$ in $\mathbb{A}$ and any $f : tx \to Y$ in $Q$, $f \otimes c(x) \leq c(f \otimes x)$.

Properties (a) and (c) ensure that $c : \mathbb{A} \to \mathbb{A}$ is a $Q$-functor by virtue of Proposition 3.1, hence a closure operator by (b).

It remains to show that $c : \mathbb{A} \to \mathbb{A}$ is a right adjoint if $R$ is a complete congruence. We apply Proposition 3.2 to accomplish this.

Since $c : \mathbb{A} \to \mathbb{A}$ is a $Q$-functor, one has that $g \mapsto c(x) \geq g \mapsto x$ for all $x$ and $g : Y \to tx$ by Proposition 3.1. Meanwhile, condition (v) ensures that $g \mapsto c(x) \leq c(g \mapsto x)$.

Therefore, $g \mapsto c(x) = c(g \mapsto x)$. This proves that $c$ preserves cotensors.

Let $\{x_i\}$ be a family of elements in $\mathbb{A}_X$. On one hand, since $(x_i, c(x_i)) \in R$ for any $x_i$ and $R$ is closed w.r.t. meets, it follows that $(\bigwedge x_i, \bigwedge c(x_i)) \in R$. Thus, $c(\bigwedge x_i) \geq \bigwedge c(x_i)$.

On the other hand, since $c : \mathbb{A}_X \to \mathbb{A}_X$ preserves order, it is clear that $c(\bigwedge x_i) \leq \bigwedge c(x_i)$.

Therefore, $c : \mathbb{A}_X \to \mathbb{A}_X$ is meet-preserving, hence a right adjoint since $\mathbb{A}_X$ is a complete lattice.

3.7. Lemma. Let $\mathbb{A}$ be a skeletal complete $Q$-category, $c : \mathbb{A} \to \mathbb{A}$ a closure operator. Then $c(\mathbb{A})$, as a subcategory of $\mathbb{A}$, is complete.

Proof. Let $\iota$ be the embedding $c(\mathbb{A}) \subseteq \mathbb{A}$. It is easy to check that $\mathcal{P}i \circ Y_A \circ \iota = Y_{c(A)}$.

Since $\iota \dashv i$ (Lemma 3.4), $\sup_A \dashv Y_A$ ($\mathbb{A}$ is cocomplete) and $\mathcal{P}c \dashv \mathcal{P}i$ (Proposition 2.5), then

$$c \circ \sup_A \circ \mathcal{P}c \dashv \mathcal{P}i \circ Y_A \circ \iota = Y_{c(A)}$$

showing that the Yoneda embedding $Y_{c(A)}$ has a left adjoint, hence $c(\mathbb{A})$ is cocomplete, hence complete.
3.8. Theorem. The forgetful functor $|-| : \mathcal{Q-\text{Sup}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0$ is strictly monadic.

Proof. Since both of the forgetful functors $\mathcal{Q-\text{Sup}} \rightarrow \mathcal{Q-\text{Cat}}$ and $\mathcal{Q-\text{Cat}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0$ are right adjoints, it follows that the forgetful functor $|-| : \mathcal{Q-\text{Sup}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0$, being a composite of right adjoints, is itself a right adjoint. Thus, by virtue of Beck's theorem (Theorem 1 on page 151 in [Mac1998]), it suffices to show that $|-| : \mathcal{Q-\text{CD}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0$ creates split coequalizers.

Given a pair of cocontinuous $\mathcal{Q}$-functors $F,G : \mathcal{A} \rightarrow \mathcal{B}$ between complete skeletal $\mathcal{Q}$-categories, a split coequalizer of $F,G : |\mathcal{A}| \rightarrow |\mathcal{B}|$ in $\text{Set} \downarrow \mathcal{Q}_0$ is, by definition, a type-preserving map $H : |\mathcal{B}| \rightarrow \mathcal{C}$ with type-preserving maps $\mathcal{C} \xrightarrow{K} |\mathcal{B}| \xrightarrow{L} |\mathcal{A}|$ such that

$$H \circ F = H \circ G, \quad F \circ L = 1, \quad G \circ K = 1, \quad G \circ L = K \circ H.$$ 

Define a relation $R$ on $\mathcal{B}_0$ by

$$R = \{(y_1, y_2) \in \mathcal{B}_0 \times \mathcal{B}_0 \mid H(y_1) = H(y_2)\}.$$

**Claim 1:** For any $y_1, y_2 \in \mathcal{B}_0$, $(y_1, y_2) \in R$ if and only if there is a pair $(x_1, x_2) \in \mathcal{A}_0 \times \mathcal{A}_0$ such that $G(x_1) = G(x_2)$ and $y_1 = F(x_1), y_2 = F(x_2)$.

Sufficiency is easy. For necessity, let $x_1 = L(y_1)$ and $x_2 = L(y_2)$. Then

$$G(x_1) = G \circ L(y_1) = K \circ H(y_1) = K \circ H(y_2) = G \circ L(y_2) = G(x_2),$$

and

$$F(x_1) = F \circ L(y_1) = y_1, \quad F(x_2) = F \circ L(y_2) = y_2.$$

**Claim 2:** The relation $R$ is a congruence on $\mathcal{B}$.

This follows from Claim 1 and the fact that both $F$ and $G$ preserve tensors and joins (with respect to the underlying orders).

Thus, $R$ determines a closure operator $c : \mathcal{B} \rightarrow \mathcal{B}$ by Lemma 3.6. It follows from Lemma 3.7 that $c(\mathcal{B})$ is complete. Since the underlying $\mathcal{Q}$-typed set of $c(\mathcal{B})$ is essentially the $\mathcal{Q}$-typed set $\mathcal{C}$, hence $\mathcal{C}$ can be made into a complete $\mathcal{Q}$-category $\mathcal{C}$ (which is isomorphic to $c(\mathcal{B})$) such that $H : \mathcal{B} \rightarrow \mathcal{C}$ is a cocontinuous $\mathcal{Q}$-functor. This proves that the forgetful functor $|-| : \mathcal{Q-\text{CD}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0$ creates split coequalizers.

Since $(\mathcal{Q-\text{Inf}})^{\text{co}}$ is isomorphic to $\mathcal{Q}^{\text{op}}\text{-Sup}$ as 2-categories, applying the above theorem to $\mathcal{Q}^{\text{op}}$ yields:

3.9. Theorem. The forgetful functor $\mathcal{Q-\text{Inf}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0$ is strictly monadic.

Our next task is to show that the forgetful functor $|-| : \mathcal{Q-\text{CD}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0$ is strictly monadic. We show that it is a right adjoint first. Given a continuous $\mathcal{Q}$-functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between complete $\mathcal{Q}$-categories, it follows from Proposition 2.4 and 2.5 that $\mathcal{P}_{\mathcal{A}} F : \mathcal{P}\mathcal{A} \rightarrow \mathcal{P}\mathcal{B}$ is bicontinuous. Therefore, by restricting the domain and the codomain of the functor $\mathcal{P}_{\mathcal{A}} : \mathcal{Q-\text{Cat}} \rightarrow \mathcal{Q-\text{Sup}}$ one obtains a functor $\mathcal{P}_{\mathcal{A}}^{\text{inf}} : \mathcal{Q-\text{Inf}} \rightarrow \mathcal{Q-\text{CD}}$ that is left adjoint to the forgetful functor $\mathcal{E}^{\text{inf}} : \mathcal{Q-\text{CD}} \rightarrow \mathcal{Q-\text{Inf}}$. Then the forgetful functor $|-| : \mathcal{Q-\text{CD}} \rightarrow \text{Set} \downarrow \mathcal{Q}_0$, as a composite of right adjoints, is a right adjoint.
3.10. Lemma. Let $\mathbb{A}$ be a skeletal completely distributive $\mathcal{Q}$-category; $c : \mathbb{A} \to \mathbb{A}$ be a right adjoint and a closure operator. Then $c(\mathbb{A})$ is completely distributive.

Proof. This follows from Proposition 2.12(1) and the fact that $c : \mathbb{A} \to c(\mathbb{A})$ is both a left and a right adjoint.

3.11. Theorem. The forgetful functor $|\cdot| : \mathcal{Q}\text{-CD} \to \mathcal{Set} \downarrow \mathcal{Q}_0$ is strictly monadic.

Proof. It suffices to check that the forgetful functor $|\cdot| : \mathcal{Q}\text{-CD} \to \mathcal{Set} \downarrow \mathcal{Q}_0$ creates split coequalizers. We only include here a sketch of the proof since it is similar to that of Theorem 3.8.

Suppose $F,G : \mathbb{A} \to \mathbb{B}$ are bicontinuous $\mathcal{Q}$-functors between completely distributive skeletal $\mathcal{Q}$-categories and $H : |\mathbb{B}| \to C$ is a split coequalizer of $F,G : |\mathbb{A}| \to |\mathbb{B}|$ in $\mathcal{Set} \downarrow \mathcal{Q}_0$. By definition there exist type-preserving maps $C \xrightarrow{K} |\mathbb{B}| \xleftarrow{L} |\mathbb{A}|$ such that $H \circ F = H \circ G$, $F \circ L = 1$, $H \circ K = 1$, $G \circ L = K \circ H$.

Define a relation $R$ on $\mathbb{B}_0$ by $R = \{(y_1, y_2) \in \mathbb{B}_0 \times \mathbb{B}_0 \mid H(y_1) = H(y_2)\}$. Then $R$ is a complete congruence on $\mathbb{B}$. This follows easily from Claim 1 in Theorem 3.8 and the fact that both $F$ and $G$ preserve tensors, cotensors, joins and meets (with respect to the underlying orders). By virtue of Lemma 3.6, the relation $R$ determines a $\mathcal{Q}$-functor $c : \mathbb{B} \to \mathbb{B}$ which is both a closure operator and a right adjoint. Then $c(\mathbb{B})$ is completely distributive by Lemma 3.10. Since the underlying $\mathcal{Q}$-typed set of $c(\mathbb{B})$ is isomorphic to $C$, it follows that $C$ can be made into a completely distributive $\mathcal{Q}$-category $\mathbb{C}$ (isomorphic to $c(\mathbb{B})$) such that $H : \mathbb{B} \to \mathbb{C}$ is a bicontinuous $\mathcal{Q}$-functor. This proves that the forgetful functor $|\cdot| : \mathcal{Q}\text{-CD} \to \mathcal{Set} \downarrow \mathcal{Q}_0$ creates split coequalizers.

In the remainder of this section, we show that the forgetful functor $\mathcal{Q}\text{-CD} \to \mathcal{Q}\text{-Inf}$ is monadic. But, we do not know whether so is the forgetful functor $\mathcal{Q}\text{-CD} \to \mathcal{Q}\text{-Sup}$.

Consider the adjunction $\mathcal{P}_3 \dashv \mathcal{E} : \mathcal{Q}\text{-Cat} \to \mathcal{Q}\text{-Sup}$. The corresponding monad is given by

$\mathcal{P}_3 = \{\mathcal{P}_3 : \mathcal{Q}\text{-Cat} \to \mathcal{Q}\text{-Cat}, \ Y : 1 \Rightarrow \mathcal{P}_3, \sup : \mathcal{P}_3^2 \Rightarrow \mathcal{P}_3\}$.

The monad $\mathcal{P}_3$ is an example of monads that are of Kock-Zöberlein type. The following proposition, extracted from [Kock1995, Zob1976], is taken from [Hof2013].

3.12. Proposition. Let $T = (T, e, m)$ be a monad on a locally ordered category $\mathcal{C}$ with $T$ a 2-functor. Then the following are equivalent:

1. $Te_X \leq e_{TX}$ for all objects $X$.
2. $Te_X \dashv m_X$ for all objects $X$.
3. $m_X \dashv e_{TX}$ for all objects $X$.
4. For any object $X$ and morphism $h : TX \to X$, the pair $(X, h)$ is a $T$-algebra if and only if $h \circ e_X = 1_X$. In this case, $h \dashv e_X$. 


A monad on a locally-ordered category is said to be of Kock-Zöberlein type, if it satisfies one (hence all) of the equivalent conditions in Proposition 3.12. This kind of monads are examples of lax-idempotent 2-monads on 2-categories introduced by G.M. Kelly and S. Lack [KL1997], so, we'll call them lax-idempotent in this paper.

A 2-functor $T : C \to D$ between locally-ordered categories is lax-idempotent monadic if it is monadic and the corresponding monad is lax-idempotent.

3.13. PROPOSITION. [Stu2013] The monad $P_\exists = (P_\exists, Y, \sup)$ is lax-idempotent.

PROOF. The conclusion was proved in [Stu2013]. Here we repeat the proof for later use. For any $Q$-category $A$, since $\sup_P A = PY_A$ (Example 2.9) and $P_\exists Y_A \dashv P Y_A$ (Proposition 2.4), it follows that $P_\exists Y_A \dashv \sup_P A$. Hence $P_\exists Y_A = P_\exists Y_A \circ \sup_P A \leq Y_P A$, completing the proof.

3.14. COROLLARY. [Stu2013] For a $Q$-category $A$, the following are equivalent:

(1) $A$ is complete.

(2) The Yoneda embedding $Y_A : A \to P A$ has a left inverse $P A \to A$.

3.15. COROLLARY. Given a $Q$-category $A$ and a $Q$-functor $F : P A \to A$, $(A, F)$ is a $P_\exists$-algebra if and only if $A$ is a skeletal complete $Q$-category and $F = \sup_A$.

It follows from Corollary 3.15 that the category of $P_\exists$-algebras is equivalent to the category of skeletal complete $Q$-categories and cocontinuous $Q$-functors.

3.16. THEOREM. [Stu2013] The forgetful functor $Q$-$\text{Sup} \to Q$-$\text{Cat}$ is lax-idempotent monadic.

A 2-functor $T : C \to D$ between locally-ordered categories is colax-idempotent monadic if $T^\co : C^\co \to D^\co$ is lax-idempotent monadic. Since the 2-category $(Q$-$\text{Cat})^\co$ is isomorphic to $Q^{\text{op}}$-$\text{Cat}$, and $(Q$-$\text{Inf})^\co$ to $Q^{\text{op}}$-$\text{Sup}$, applying the above theorem to $Q^{\text{op}}$ we obtain:

3.17. COROLLARY. The forgetful functor $Q$-$\text{Inf} \to Q$-$\text{Cat}$ is colax-idempotent monadic.

Now we come to the last conclusion in this section.

3.18. PROPOSITION. The forgetful functor $Q$-$\text{CD} \to Q$-$\text{Inf}$ is lax-idempotent monadic.

PROOF. Consider the monad $P^{\inf}_\exists$ generated by the adjunction $P^{\inf}_\exists \dashv E^{\inf}$ (see the paragraph following Theorem 3.9). By the same argument for $P_\exists$ one deduces that the monad $P^{\inf}_\exists$ is lax-idempotent. So, it remains to check that the forgetful functor $E^{\inf} : Q$-$\text{CD} \to Q$-$\text{Inf}$ is monadic.

Let $A$ be a complete skeletal $Q$-category and $F : P A \to A$ a continuous $Q$-functor. If $(A, F)$ is a $P^{\inf}_\exists$-algebra, then $F$ is a left inverse of the Yoneda embedding $Y_A$ by Proposition 3.12(4), hence $A$ is complete and $F = \sup_A$ by corollaries 3.14 and 3.15. Thus, $\sup_A$ is a right adjoint, showing that $A$ is completely distributive. Therefore, the correspondence $(A, F) \mapsto A$ defines a functor $Q$-$\text{Inf} P^{\inf}_\exists \to Q$-$\text{CD}$ that is inverse to the comparison functor.
\[ \mathcal{Q}\text{-CD} \longrightarrow \mathcal{Q}\text{-Inf}^{\mathcal{P}_{\inf}^\mathcal{Q}}, \quad \mathcal{A} \mapsto (\mathcal{A}, \sup_{\mathcal{A}}). \]

The conclusion thus follows.

The monadicity of the forgetful functor \( \mathcal{Q}\text{-CD} \longrightarrow \text{Set} \Downarrow \mathcal{Q}_0 \) does not follow from that of the forgetful functors \( \mathcal{Q}\text{-Inf} \longrightarrow \mathcal{Q}\text{-Cat} \) and \( \mathcal{Q}\text{-CD} \longrightarrow \mathcal{Q}\text{-Inf} \), since the composite of monadic functors need not be monadic, see [Bor1994], page 214.

4. \( \mathcal{Q}\)-powersets as algebras

It is well-known (e.g. [Joh1982, MM1992]) that the contravariant powerset functor \( \mathcal{P} : \text{Set}^{\text{op}} \longrightarrow \text{Set} \) is monadic with a left adjoint given by \( \mathcal{P}^{\text{op}} : \text{Set} \longrightarrow \text{Set}^{\text{op}} \); and that the algebras corresponding to the monad generated by the adjunction \( \mathcal{P}^{\text{op}} \dashv \mathcal{P} \) are powersets (or equivalently, complete atomic Boolean algebras). In this section, we establish a \( \mathcal{Q}\)-version of this conclusion. That is, if we denote by \( \mathcal{P}^{\dagger} \dashv \mathcal{P} \) the adjunction obtained by composing the following

\[
\begin{array}{ccc}
\text{Set} \Downarrow \mathcal{Q}_0 & \xrightarrow{\mathcal{Q}} & \text{Q-Cat} \\
\mathcal{I} & \mapsto & \mathcal{Q}\text{-Cat}^{\text{op}} \\
\mathcal{Q}\text{-Cat}^{\text{op}} & \xrightarrow{\mathcal{P}\text{op}} & (\text{Set} \Downarrow \mathcal{Q}_0)^{\text{op}},
\end{array}
\]

then the functor \( \mathcal{P} : (\text{Set} \Downarrow \mathcal{Q}_0)^{\text{op}} \longrightarrow \text{Set} \Downarrow \mathcal{Q}_0 \) is monadic and the corresponding Eilenberg-Moore algebras are exactly the \( \mathcal{Q}\)-powersets of \( \mathcal{Q}\)-typed sets.

The monadicity of the powerset functor \( \mathcal{P} : \text{Set}^{\text{op}} \longrightarrow \text{Set} \) is a special case of a general result in topos theory [MM1992] that states that for each topos \( \mathcal{E} \), the opposite category \( \mathcal{E}^{\text{op}} \) is monadic over \( \mathcal{E} \). In particular, for each set \( \mathcal{X} \), \( (\text{Set} \Downarrow \mathcal{X})^{\text{op}} \) is monadic over \( \text{Set} \Downarrow \mathcal{X} \) with the (internal) powerset functor being a monadic one. We’d like to remark that the conclusion presented here shows that, for each non-empty set \( \mathcal{X} \), there exist many monadic functors from \( (\text{Set} \Downarrow \mathcal{X})^{\text{op}} \) to \( \text{Set} \Downarrow \mathcal{X} \).

Before proceeding, we spell out some facts of the adjunction \( \mathcal{P}^{\dagger} \dashv \mathcal{P} \).

First, the functor \( \mathcal{P} \) sends each \( \mathcal{Q}\)-typed set \( \mathcal{A} \) to the underlying \( \mathcal{Q}\)-typed set \( \mathcal{P}\mathcal{A} \), where \( \mathcal{A} \) is regarded as a discrete \( \mathcal{Q}\)-category. The \( \mathcal{Q}\)-typed set \( \mathcal{P}\mathcal{A} \) is called the \( \mathcal{Q}\)-powerset of \( \mathcal{A} \).

Second, for each type-preserving map \( \mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B} \) between \( \mathcal{Q}\)-typed sets, \( \mathcal{P}\mathcal{F} \) is the underlying type-preserving map of \( \mathcal{P}\mathcal{F} : \mathcal{P}\mathcal{B} \longrightarrow \mathcal{P}\mathcal{A} \). Similarly, \( \mathcal{P}^{\dagger}\mathcal{F} \) is the underlying type-preserving map of \( \mathcal{P}^{\dagger}\mathcal{F} : \mathcal{P}^{\dagger}\mathcal{B} \longrightarrow \mathcal{P}^{\dagger}\mathcal{A} \). So, for a type-preserving map \( \mathcal{F} \) between \( \mathcal{Q}\)-typed sets, we simply write \( \mathcal{P}\mathcal{F} \) (\( \mathcal{P}^{\dagger}\mathcal{F} \), resp.) for \( \mathcal{P}\mathcal{F} \) (\( \mathcal{P}^{\dagger}\mathcal{F} \), resp.) if no confusion would arise.

Third, the unit and counit of the adjunction \( \mathcal{P}^{\dagger} \dashv \mathcal{P} \) are respectively given by

\[
\epsilon_A = Y_{\mathcal{P}^{\dagger}\mathcal{A}} \circ Y_A : \mathcal{A} \longrightarrow |\mathcal{P}^{\dagger}\mathcal{A}| \longrightarrow |\mathcal{P}\mathcal{P}^{\dagger}\mathcal{A}|,
\]

and

\[
\gamma_A = Y_{\mathcal{P}\mathcal{A}} \circ Y_A : \mathcal{A} \longrightarrow |\mathcal{P}\mathcal{A}| \longrightarrow |\mathcal{P}^{\dagger}\mathcal{P}\mathcal{A}|.
\]
for any $Q$-typed set $A$.

The following lemma is a counterpart of the Beck-Chevalley condition in [MM1992], Theorem 2, page 206.

4.1. Lemma. Let

$$
\begin{array}{ccc}
A & \xrightarrow{F} & C \\
H \downarrow & & \downarrow K \\
B & \xrightarrow{G} & D
\end{array}
$$

be a pullback square in $\text{Set} \downarrow Q_0$. Then the square of $Q$-distributors between discrete $Q$-categories

$$
\begin{array}{ccc}
A & \xrightarrow{F_2} & C \\
\Phi_{H^2} \downarrow & & \downarrow \Phi_{K^2} \\
B & \xrightarrow{G_2} & D
\end{array}
$$

commutes; or equivalently, the square of $Q$-functors

$$
\begin{array}{ccc}
\mathcal{P}C & \xrightarrow{\mathcal{P}F} & \mathcal{P}A \\
\exists_{\Phi} \downarrow & & \downarrow \exists_{\Phi} \\
\mathcal{P}D & \xrightarrow{\mathcal{P}G} & \mathcal{P}B
\end{array}
$$

commutes.

Proof. By hypothesis, we can assume that the underlying set of $A$ is

$$\{(y, z) \in B \times C \mid G_y = K_z\},$$

the type function is given by $t[(y, z)] = ty = tz$ for all $(y, z) \in A$, and that both $H$ and $F$ are projections. For all $b \in B$ and $c \in C$,

$$(F_2 \circ H^2)(b, c) = \bigvee_{(y, z) \in A} \operatorname{id}_C(z, c) \circ \operatorname{id}_B(b, y) = \begin{cases} 1_{tb}, & Gb = Kc; \\ \bot_{tb,tc}, & \text{otherwise.} \end{cases}$$

It follows that

$$K^2 \circ G_2(b, c) = \bigvee_{d \in D} \operatorname{id}_D(d, Kc) \circ \operatorname{id}_D(Gb, d) = \operatorname{id}_D(Gb, Kc) = F_2 \circ H^2(b, c).$$
That is, the second square commutes.

It remains to check that the second square commutes if and only if so does the third one. Since $\exists_\phi \circ \mathcal{P} F(\phi) = \phi \circ F \circ H^2$ and $\mathcal{P} G \circ \exists_K(\phi) = \phi \circ K^2 \circ G_2$, all $\phi \in \mathcal{P} C$, the commutativity of the third square follows trivially from that of the second one. Conversely, if the third square commutes, then for all $c \in C$,

$$F_2 \circ H^2(-,c) = \text{id}_{C}(-,c) \circ F_2 \circ H^2 = \exists_H \circ \mathcal{P} F(\text{id}_{C}(-,c))$$

and

$$K^2 \circ G_2(-,c) = \text{id}_{C}(-,c) \circ K^2 \circ G_2 = \mathcal{P} G \circ \exists_K(\text{id}_{C}(-,c)),$$

hence, the second one commutes. □

4.2. Theorem. The functor $|\mathcal{P}| : (\text{Set } \downarrow \mathcal{Q}_0)^{\text{op}} \rightarrow \text{Set } \downarrow \mathcal{Q}_0$ is monadic.

Proof. Since $\text{Set } \downarrow \mathcal{Q}_0$ is a complete category, we apply Corollary 3 on page 180 in [MM1992] to prove the conclusion. That is, we show that $|\mathcal{P}| : (\text{Set } \downarrow \mathcal{Q}_0)^{\text{op}} \rightarrow \text{Set } \downarrow \mathcal{Q}_0$ reflects isomorphisms and preserves coequalizers of reflexive pairs.

Since $|\mathcal{P}| : (\text{Set } \downarrow \mathcal{Q}_0)^{\text{op}} \rightarrow \text{Set } \downarrow \mathcal{Q}_0$ is faithful, it reflects both monomorphisms and epimorphisms. Since the slice category $\text{Set } \downarrow \mathcal{Q}_0$ is a topos, an arrow in $\text{Set } \downarrow \mathcal{Q}_0$ is an isomorphism if and only if it is both a monomorphism and an epimorphism. Consequently, $|\mathcal{P}| : (\text{Set } \downarrow \mathcal{Q}_0)^{\text{op}} \rightarrow \text{Set } \downarrow \mathcal{Q}_0$ reflects isomorphisms.

It remains to check that $|\mathcal{P}| : (\text{Set } \downarrow \mathcal{Q}_0)^{\text{op}} \rightarrow \text{Set } \downarrow \mathcal{Q}_0$ preserves coequalizers of reflexive pairs. Recall that a pair of arrows $r, s : X \rightarrow Y$ in a category is reflexive if there exists an arrow $i : Y \rightarrow X$ such that $r \circ i = 1_Y = s \circ i$. So, a reflexive pair in $(\text{Set } \downarrow \mathcal{Q}_0)^{\text{op}}$ is a pair of arrows $F, G : A \rightarrow B$ in $\text{Set } \downarrow \mathcal{Q}_0$ together with an arrow $K : B \rightarrow A$ such that $K \circ F = K \circ G = 1_A$. We must show that if $H : C \rightarrow A$ is an equalizer of $F$ and $G$ in $\text{Set } \downarrow \mathcal{Q}_0$ then $|\mathcal{P}| H = \mathcal{P} H : |\mathcal{P} C| \rightarrow |\mathcal{P} A|$ is a coequalizer of $\mathcal{P} F$ and $\mathcal{P} G$. That is, for each $L : |\mathcal{P} A| \rightarrow D$ in $\text{Set } \downarrow \mathcal{Q}_0$ with $L \circ \mathcal{P} F = L \circ \mathcal{P} G$, there exists a unique $L : |\mathcal{P} C| \rightarrow D$ such that $L \circ \mathcal{P} H = L$.

Uniqueness. It is obvious that, as an equalizer, $H : C \rightarrow A$ is a monomorphism in $\text{Set } \downarrow \mathcal{Q}_0$. Hence $H$ is a fully faithful $\mathcal{Q}$-functor between discrete $\mathcal{Q}$-categories $C$ and $A$. Thus, $\bar{L} \circ \mathcal{P} H \circ \exists_{\mathcal{H}} = L \circ \exists_{\mathcal{H}}$ by Proposition 2.7(1).

Existence. It suffices to verify that $L \circ \exists_{\mathcal{H}} \circ \mathcal{P} H = L$. First, we check that the square

$$
\begin{array}{ccc}
C & \xrightarrow{H} & A \\
\downarrow{H} & & \downarrow{F} \\
A & \xrightarrow{G} & B \\
\end{array}
$$

is a pullback in $\text{Set } \downarrow \mathcal{Q}_0$. Given a pair of arrows $F', G' : D \rightarrow A$ with $F \circ F' = G \circ G'$, since $K \circ F = K \circ G = 1_A$, we have that

$$F' = K \circ F \circ F' = K \circ G \circ G' = G'.$$
Because $H : C \to A$ is an equalizer of $F$ and $G$, there is a unique $U : D \to A$ such that $F' = H \circ U = G'$. This proves that the square is a pullback. Then it follows from Lemma 4.1 that $\mathcal{P}G \circ \exists_F = \exists_H \circ \mathcal{P}H$. Finally, since $K \circ F = 1_A$, it follows that $F : A \to B$ is a fully faithful $Q$-functor if we treat $A$ and $B$ as discrete $Q$-categories. Thus, $\mathcal{P}F \circ \exists_F = 1_{\mathcal{P}A}$ by Proposition 2.7(1). Therefore,

$$L = L \circ \mathcal{P}F \circ \exists_F = L \circ \mathcal{P}G \circ \exists_F = L \circ \exists_H \circ \mathcal{P}H.$$  

The proof is thus completed. 

4.3. Remark. In general, the monadic functor $|\mathcal{P}| : (\text{Set} \downarrow Q_0)^{\text{op}} \to \text{Set} \downarrow Q_0$ is different from the “internal” powerset functor for the topos $\text{Set} \downarrow Q_0$. It is easily verified that $|\mathcal{P}|$ coincides with the internal powerset functor if $Q$ is given by

$$Q(X,Y) = \begin{cases} 2 = \{0,1\} & \text{if } X = Y; \\ 1 = \{0\} & \text{otherwise}. \end{cases}$$

Furthermore, if there exist different objects $X,Y$ in $Q$ with $Q(X,Y)$ containing at least two elements, then the functor $|\mathcal{P}|$ cannot be isomorphic to the internal powerset functor $[-, B] : (\text{Set} \downarrow Q_0)^{\text{op}} \to \text{Set} \downarrow Q_0$ for any $B$ in $\text{Set} \downarrow Q_0$. To see this, we first note that for each $A$ in $\text{Set} \downarrow Q_0$ and $Z$ in $Q_0$, an element in $[A,B]$ with type $Z$ is exactly a function $A_Z \to B_Z$, where $A_Z$ is the set of elements in $A$ with type $Z$, and likewise for $B_Z$. Now, let $C$ be a $Q$-typed set consisting of only one element with type $X$. Then there is exactly one element in $[C,B]$ that is of type $Y$ (namely, the unique map from the empty set to $B_Y$), but there are at least two elements in $|\mathcal{P}|C$ that are of type $Y$. Therefore, $[-, B]$ and $|\mathcal{P}|$ cannot be isomorphic.

In the following we describe the Eilenberg-Moore algebras of the monad generated by the adjunction $|\mathcal{P}| \dashv |\mathcal{P}|$. The corresponding monad is given by

$$|\mathcal{B}| = \{ |\mathcal{B}| : \text{Set} \downarrow Q_0 \to \text{Set} \downarrow Q_0, \epsilon : 1 \Rightarrow |\mathcal{B}|, \delta : |\mathcal{B}|^2 \Rightarrow |\mathcal{B}| \}$$

where

- $|\mathcal{B}|F = |\mathcal{P}||\mathcal{P}^\dagger F| : |\mathcal{P}||\mathcal{P}^\dagger A| \to |\mathcal{P}||\mathcal{P}^\dagger B|$ for any type-preserving map $F : A \to B$,
- $\epsilon_A = \mathcal{E}_{\mathcal{P}^\dagger A} \circ \mathcal{Y}_A^\dagger : A \to |\mathcal{P}^\dagger A| \to |\mathcal{P}||\mathcal{P}^\dagger A|$ for any $Q$-typed set $A$,
- $\delta_A = \mathcal{P}\gamma_{|\mathcal{P}^\dagger A|} : |\mathcal{B}|^2 A \to |\mathcal{B}|A$ for any $Q$-typed set $A$.

For each $Q$-typed set $B$, $(|\mathcal{P}B|, \mathcal{P}\gamma_B)$ is a $|\mathcal{B}|$-algebra. The following theorem says that all $|\mathcal{B}|$-algebras are of this form.

4.4. Theorem. Every $|\mathcal{B}|$-algebra is of the form $(|\mathcal{P}B|, \mathcal{P}\gamma_B)$ for some $Q$-typed set $B$. 
**Proof.** Suppose that \((A, F)\) is a \(|B|\)-algebra. That is, \(A\) is a \(Q\)-typed set, \(F : |B|A \rightarrow A\) is a type-preserving map such that \(F \circ \epsilon_A = 1_A\) and \(F \circ \delta_A = F \circ |B|F\). We show that there is some \(Q\)-typed set \(B\) such that \((A, F)\) is isomorphic to \(\langle |PB|, \mathcal{P} \gamma_B \rangle\).

Consider the pullback

\[
\begin{array}{ccc}
B & \xrightarrow{i} & |\mathcal{P}^t A| \\
\downarrow{i'} & & \downarrow{\mathcal{P}^t F} \\
|\mathcal{P}^t A| & \xrightarrow{\gamma|\mathcal{P}^t A|} & |\mathcal{P}^t |B|A| \\
\end{array}
\]

in \(\text{Set} \downarrow \mathbb{Q}_0\). We claim that \(B\) satisfies the requirement. The proof is divided into three steps.

**Step 1.** \(i = i'\). This follows easily from the triangular identity \(\mathcal{P}^t \epsilon_A \circ \gamma|\mathcal{P}^t A| = 1|\mathcal{P}^t A|\) and the equality \(\mathcal{P}^t \epsilon_A \circ \mathcal{P}^t F = \mathcal{P}^t (F \circ \epsilon_A) = 1|\mathcal{P}^t A|\). Consequently, \(i\) is an equalizer of \(\mathcal{P}^t F\) and \(\gamma|\mathcal{P}^t A|\).

**Step 2.** \(K_A = \mathcal{P} i \circ \epsilon_A : (A, F) \rightarrow \langle |PB|, \mathcal{P} \gamma_B \rangle\) is a homomorphism between \(|B|\)-algebras, i.e., \(K_A \circ F = \mathcal{P} \gamma_B \circ |B|K_A\). To see this, we calculate:

\[
K_A \circ F = \mathcal{P} i \circ \epsilon_A \circ F
= \mathcal{P} i \circ |B|F \circ \epsilon_{|B|A} \quad \text{(naturality of} \ \epsilon) \\
= \mathcal{P} (\mathcal{P}^t F \circ i) \circ \epsilon_{|B|A} \quad \text{((} \mathcal{P}^t F : |\mathcal{P}^t A| \rightarrow |\mathcal{P}^t (|B|A)|) \\
= \mathcal{P} (\gamma|\mathcal{P}^t A| \circ i) \circ \epsilon_{|B|A} \quad \text{(} i \text{ equalizes} \ \mathcal{P}^t F \text{ and} \ \gamma|\mathcal{P}^t A|) \\
= \mathcal{P} i \circ \delta_A \circ \epsilon_{|B|A} \\
= \mathcal{P} i \quad \text{((} \delta_A \circ \epsilon_{|B|A} = 1_{|B|A} \text{) } \text{and} \ F \circ \epsilon_A = 1_A) \\
= \mathcal{P} i \circ |B|F \circ |B|\epsilon_A \quad \text{(} i \text{ equalizes} \ \mathcal{P}^t F \text{ and} \ \gamma|\mathcal{P}^t A|) \\
= \mathcal{P} (\mathcal{P}^t (\mathcal{P} i \circ \gamma_B) \circ |B|\epsilon_A) \quad \text{(naturality of} \ \gamma) \\
= \mathcal{P} \gamma_B \circ |B| (\mathcal{P} i \circ |B|\epsilon_A) \\
= \mathcal{P} \gamma_B \circ |B|K_A.
\]

**Step 3.** \(K_A : (A, F) \rightarrow \langle |PB|, \mathcal{P} \gamma_B \rangle\) is an isomorphism between \(|B|\)-algebras. It suffices to check that \(K_A : A \rightarrow |PB|\) is an isomorphism in \(\text{Set} \downarrow \mathbb{Q}_0\).

Let \(L_A = F \circ \exists_i\). On the one hand, it follows from the calculations in Step 2 that

\[
K_A \circ L_A = \mathcal{P} i \circ \epsilon_A \circ F \circ \exists_i = \mathcal{P} i \circ \exists_i = 1_{|PB|},
\]

where the last equality holds due to Proposition 2.7(1).

On the other hand, by virtue of Lemma 4.1 and the definition of \(\delta_A\) one has that

\[
\exists_i \circ \mathcal{P} i = \mathcal{P} \gamma|\mathcal{P}^t A| \circ \exists \mathcal{P}^t F = \delta_A \circ \exists \mathcal{P}^t F.
\]
Since $\mathcal{P}^\dagger F : |\mathcal{P}^\dagger A| \to |\mathcal{P}^\dagger (|B|A)|$ is fully faithful $(\mathcal{P}^\dagger \epsilon_A \circ \mathcal{P}^\dagger F = 1_{|B|A})$, it holds that

$$|B|F \circ \exists_{\mathcal{P}^\dagger F} = \mathcal{P}(\mathcal{P}^\dagger F) \circ \exists_{\mathcal{P}^\dagger F} = 1_{|B|A}$$

by Proposition 2.7(1). Consequently,

$$L_A \circ K_A = F \circ \exists_i \circ \mathcal{P}_i \circ \epsilon_A$$

$$= F \circ \delta_A \circ \exists_{\mathcal{P}^\dagger F} \circ \epsilon_A$$

$$= F \circ |B|F \circ \exists_{\mathcal{P}^\dagger F} \circ \epsilon_A \quad ((A, F) \text{ is a } |B| \text{-algebra})$$

$$= F \circ \epsilon_A \quad ((|B|F \circ \exists_{\mathcal{P}^\dagger F} = 1_{|B|A})$$

$$= 1_A.$$

Therefore, $K_A : A \to |\mathcal{P}B|$ is an isomorphism between $\mathcal{Q}$-typed sets.

**4.5. Remark.** From the point of view of fuzzy sets [Zad1965], a $\mathcal{Q}$-typed set is nothing but a “fuzzy set valued in $\mathcal{Q}_0$”. Viewed in this perspective, the functor

$$|\mathcal{P}| : (\text{Set} \downarrow \mathcal{Q}_0)^{\text{op}} \to \text{Set} \downarrow \mathcal{Q}_0$$

is a fuzzy counterpart of the contravariant powerset functor $\mathcal{P} : \text{Set}^{\text{op}} \to \text{Set}$; the above theorem can be thought of as a fuzzy version of the Stone duality between sets and complete atomic Boolean algebras. Thus, it is not surprising that the functor $|\mathcal{P}|$ has applications in the theory of fuzzy sets. Interested readers are referred to [Hoh2014, SZ2013b, Stu2014] for more discussions on related topics.

Finally, consider the adjunction $|-|^{\text{op}} \circ \mathcal{P}^\dagger \dashv \mathcal{P} \circ I^{\text{op}}$ obtained by composing the following adjunctions

$$\mathcal{Q} \text{-Cat} \xleftarrow{p^\dagger} \mathcal{Q} \text{-Cat}^{\text{op}} \xrightarrow{\tau |^{\text{op}}} (\text{Set} \downarrow \mathcal{Q}_0)^{\text{op}}.$$  

Let $\tau$ be the counit of the adjunction $|-|^{\text{op}} \circ \mathcal{P}^\dagger \dashv \mathcal{P} \circ I^{\text{op}}$ and let $\mathcal{B}$ be the monad on $\mathcal{Q} \text{-Cat}$ corresponding to this adjunction. Then the following theorem says that the Eilenberg-Moore algebras of $\mathcal{B}$ are also the $\mathcal{Q}$-powersets of $\mathcal{Q}$-typed sets.

**4.6. Theorem.** If $(A, F)$ is a $\mathcal{B}$-algebra, then there exists a $\mathcal{Q}$-typed set $B$ such that $(A, F)$ is isomorphic to $(\mathcal{P}B, \mathcal{P} \tau_B)$.

**Proof.** Consider the pullback

$$\begin{array}{ccc}
B & \xrightarrow{i} & |\mathcal{P}^\dagger A| \\
\downarrow i' & & \downarrow |\mathcal{P}^\dagger F| \\
|\mathcal{P}^\dagger A| & \xrightarrow{\tau |^{\text{op}}_{|\mathcal{P}^\dagger A|}} & |\mathcal{P}^\dagger (BA)|
\end{array}$$

in $\text{Set} \downarrow \mathcal{Q}_0$, where $B = \mathcal{P} \circ I^{\text{op}} \circ |-|^{\text{op}} \circ \mathcal{P}^\dagger$. Then $B$ satisfies the requirement. The proof is similar to that of Theorem 4.4 and is thus omitted here.
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CATEGORIES ENRICHED OVER A QUANTALOID: ALGEBRAS


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