

ACTIONS IN MODIFIED CATEGORIES OF INTEREST WITH APPLICATION TO CROSSED MODULES

Dedicated to Teimuraz Pirashvili on his 60th birthday

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ABSTRACT. The existence of the split extension classifier of a crossed module in the category of associative algebras is investigated. According to the equivalence of categories $\mathbf{XAss} \simeq \mathbf{Cat}^1\text{-Ass}$ we consider this problem in $\mathbf{Cat}^1\text{-Ass}$. This category is not a category of interest, it satisfies its all axioms except one. The action theory developed in the category of interest is adapted to the new type of category, which will be called modified category of interest. Applying the results obtained in this direction and the equivalence of categories we find a condition under which there exists the split extension classifier of a crossed module and give the corresponding construction.

1. Introduction

Categories of interest were introduced in order to study properties of different algebraic categories and different algebras simultaneously. The idea comes from P. G. Higgins [Higgins, 1956] and the definition is due to M. Barr and G. Orzech [Orzech, 1972]. The categories of groups, modules over a ring, vector spaces, associative algebras, associative commutative algebras, Lie algebras, Leibniz algebras, alternative algebras, Poisson algebras, left-right non-commutative Poisson algebras are categories of interest [Orzech, 1972, Casas, Datuashvili and Ladra, 2009*, Casas, Datuashvili and Ladra, 2014]. Note that the category of noncommutative Leibniz-Poisson algebras defined and studied in [Casas and Datuashvili, 2006] is not a category of interest. In [Montoli, 2010] there are

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given new examples of categories of interest, these are associative dialgebras and trialgebras, which were defined and studied in [Loday, 1995, Loday, 2001, Loday and Ronco, 2004]. The categories of crossed modules and precrossed modules in the category of groups, respectively, are equivalent to categories of interest as well, in the sense of [Casas, Datuashvili and Ladra, 2010, Casas, Datuashvili and Ladra, 2007].

In the procedure of investigation of representability of actions in the category of precrossed modules in the categories of Lie algebras [Casas, Datuashvili, Ladra and Uslu, 2012] and associative algebras, we recognized that the categories of cat^1 -Lie algebras and that of cat^1 -associative algebras (see Section 2 for definitions) satisfy all axioms of category of interest except one. Consequently, we plan to introduce and study a new type of category of interest; namely, a category which satisfies all axioms of a category of groups with operations stated in [Porter, 1987] except one, which is replaced by a new axiom; this category satisfies as well two additional axioms introduced in [Orzech, 1972] for categories of interest. The examples are mainly those categories which are equivalent to the categories of crossed modules and precrossed modules in the categories of Lie algebras, Leibniz algebras, associative and associative commutative algebras. Also, applying a result of [Borceux, Janelidze and Kelly, 2005*], the category of commutative Von Neumann regular rings is isomorphic to a category of commutative rings with a unary operation satisfying two axioms, which is a new type of category of interest as well. Therefore we decided to give a name to this sort of a category, which could be called “modified category of interest”.

In this work our main purpose is to unify the study of actions in certain algebraic categories by means of this new kind of category. We describe main notions, in particular, the notion of actor defined in [Casas, Datuashvili and Ladra, 2010] in categories of interest, or, equivalently, split extension classifier defined in the more general setting of semi-abelian categories [Borceux, Janelidze and Kelly, 2005], universal strict general actor, defined in [Casas, Datuashvili and Ladra, 2010]. At the same time we plan to study concrete examples of modified categories of interest and their equivalent ones. In this paper we find sufficient conditions for the existence of the split extension classifier of a crossed module in the category of associative algebras and give the corresponding construction. The analogous problem for (pre)crossed modules in the category of groups, crossed modules in the categories of Lie algebras and associative commutative algebras and in the category of Lie-Leibniz algebras were considered in [Norrie, 1990, Casas and Ladra, 1998, Arvasi and Ege, 2003, Casas, Datuashvili and Ladra, 2009, Casas, Datuashvili, Ladra and Uslu, 2012, Casas, Datuashvili and Ladra, 2013]. Note that the results and the constructions obtained in the cases of (pre)crossed modules in the categories of Lie and associative commutative algebras could be obtained in the way considered in this paper, i.e. by application of action theory in modified categories of interest. The analogous is true for the future investigations in the cases of precrossed modules in the category of associative algebras and (pre)crossed modules in the category of Leibniz algebras. This kind of results, like in the cases of associative algebras [Hochschild, 1947] and rings [Mac Lane, 1958], could be applied in the cohomology and obstruction theories of the corresponding

objects.

The notion of crossed module in the category of associative algebras was defined by Dedecker and Lue in [Dedecker and Lue, 1966]. It was used as a fundamental gadget to determine the coefficients in low-dimensional non-abelian cohomology [Lue, 1968]. On the other hand, it was used for the representation of Hochschild cohomology in [Baues and Minian, 2002]. Naturally, it will be important to investigate the existence of the split extension classifier of a crossed module in the category of associative algebras.

At this vein, for a given crossed module $\mathcal{A} : A_1 \xrightarrow{d} A_0$, we found a condition under which we construct an actor of the corresponding cat^1 -associative algebra $(A_1 \rtimes A_0, \omega_1, \omega_0)$ by using the general construction of universal strict general actor of $(A_1 \rtimes A_0, \omega_1, \omega_0)$. Then applying the equivalence of the categories $\mathbf{Cat}^1\text{-Ass} \simeq \mathbf{XAss}$ of cat^1 -associative algebras (cat^1 -algebras in what follows) and crossed modules, we carry the construction of an actor of $(A_1 \rtimes A_0, \omega_1, \omega_0)$ to the category of crossed modules, which is a split extension classifier of the crossed module $\mathcal{A} : A_1 \xrightarrow{d} A_0$ under the appropriate condition on it. Therefore we found a new example of a category and individual objects there with representable actions and described the representing objects. This problem is stated in [Borceux, Janelidze and Kelly, 2005*] (Problem 2).

The outline of the paper is as follows: in Section 2 we recall some well-known definitions of the category of crossed modules in the category of associative algebras and introduce some new notions, such as bimultipliers and crossed multipliers. In Section 3 we introduce a notion of modified category of interest and actions in this category. Then we introduce notions of general and strict general properties and universal strict general actors and construct a universal strict general actor of an object in a modified category of interest. We start Section 4 by constructing an object $(\mathfrak{A}(A), \bar{\omega}_0, \bar{\omega}_1)$ for a cat^1 -algebra (A, ω_0, ω_1) , and prove that if $\text{Ann}(A) = 0$ or $A^2 = A$, then the constructed object is an actor of (A, ω_0, ω_1) . We finish this section by examining a particular case; namely, an actor of a cat^1 -algebra $(A_1 \rtimes A_0, \omega_1, \omega_0)$, corresponding to a given crossed module $\mathcal{A} : A_1 \xrightarrow{d} A_0$. Finally, in Section 5, applying this result, we prove, that if $\text{Ann}(A_i) = 0$ or $A_i^2 = A_i, i = 0, 1$, for a crossed module $\mathcal{A} : A_1 \xrightarrow{d} A_0$, then there exists the split extension classifier of this crossed module and give the corresponding construction.

2. Crossed Modules in the Category of Associative Algebras

In this section we will give some well-known definitions and results about crossed modules in the category of associative algebras. We also define new notions such as bimultiplier and crossed bimultiplier of a crossed module and give some related results, which will be needed in the rest of the paper.

Let k be a fixed commutative ring with unit. All algebras in the rest of the paper will be associative algebras over k .

Let A, B be associative algebras. An action (i.e. a derived action) of B on A is a pair

of bilinear maps

$$B \times A \longrightarrow A, \quad A \times B \longrightarrow A$$

which we denote respectively as $(b, a) \mapsto b * a, (a, b) \mapsto a * b$, with conditions

$$\begin{aligned} (b_1 * b_2) * a &= b_1 * (b_2 * a) \\ a * (b_1 * b_2) &= (a * b_1) * b_2 \\ (b_1 * a) * b_2 &= b_1 * (a * b_2) \\ b * (a_1 * a_2) &= (b * a_1) * a_2 \\ (a_1 * a_2) * b &= a_1 * (a_2 * b) \\ a_1 * (b * a_2) &= (a_1 * b) * a_2 \end{aligned} \tag{2.1}$$

for all $a, a_1, a_2 \in A, b, b_1, b_2 \in B$.

We recall the construction of the algebra $Bim(A)$ of bimultipliers of an associative algebra A defined by G. Hochschild and by S. Mac Lane for rings (called bimultiplications in [Mac Lane, 1958] and multiplications in [Hochschild, 1947], from where the notion comes [Lavendhomme and Lucas, 1996]). An element of $Bim(A)$ is a pair $f = (f_l, f_r)$ of k -linear maps from A to A with

$$\begin{aligned} f_l(a * a') &= f_l(a) * a', \\ f_r(a * a') &= a * f_r(a'), \\ a * f_l(a') &= f_r(a) * a', \end{aligned}$$

for all $a, a' \in A$.

As it is well-known, $Bim(A)$ has an associative algebra structure, defined by componentwise scalar multiplication and addition, and composition as the multiplication. Note that since the bimultipliers f_l and f_r are written from the left side of an element, for the product of two bimultipliers we have $(f_l, f_r) * (g_l, g_r) = (f_l g_l, f_r g_r)$, where $(f_l g_l)(a) = f_l(g_l(a))$ and $(f_r g_r)(a) = g_r(f_r(a))$, for all $a \in A$.

The following definition of a crossed module is well-known from [Dedecker and Lue, 1966] and it is a special case of the definition of a crossed module in categories of interest [Casas, Datuashvili and Ladra, 2010] (cf. [Porter, 1987]).

2.1. DEFINITION. *A precrossed module $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$ in the category of associative algebras consists of an associative algebra homomorphism $d : A_1 \longrightarrow A_0$, called boundary map, together with an action of A_0 on A_1 , satisfying the conditions*

$$d(a_0 * a_1) = a_0 * d(a_1), \quad d(a_1 * a_0) = d(a_1) * a_0,$$

for all $a_1 \in A_1, a_0 \in A_0$. In addition, if

$$d(a'_1) * a_1 = a'_1 * a_1, \quad a_1 * d(a'_1) = a_1 * a'_1,$$

for all $a_1, a'_1 \in A_1$, then $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$ is called a crossed module.

Let $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$ and $\mathcal{A}' : (A'_1 \xrightarrow{d'} A'_0)$ be crossed modules. A homomorphism from \mathcal{A} to \mathcal{A}' is a pair (μ_1, μ_0) where $\mu_1 : A_1 \rightarrow A'_1$ and $\mu_0 : A_0 \rightarrow A'_0$ are associative algebra homomorphisms, such that $d'\mu_1 = \mu_0 d$ and

$$\mu_1(a_0 * a_1) = \mu_0(a_0) * \mu_1(a_1), \quad \mu_1(a_1 * a_0) = \mu_1(a_1) * \mu_0(a_0),$$

for all $a_1 \in A_1, a_0 \in A_0$. The corresponding category of crossed modules in the category of associative algebras will be denoted by **XAss**.

2.2. EXAMPLE. Let I be a two-sided ideal of an associative algebra A . Then, $inc. : I \rightarrow A$ is a crossed module with the action by conjugation (product) of A on I . Consequently, $id : A \rightarrow A$ and $inc. : 0 \rightarrow A$ are crossed modules.

2.3. EXAMPLE. Let C be a singular associative algebra (i.e. an associative algebra with trivial multiplication) with an action of the associative algebra A . Then, $0 : C \rightarrow A$ is a crossed module. Take $C = A$, then $0 : A \rightarrow A$ and $0 : A \rightarrow 0$ are crossed modules. Nevertheless, if C is non-singular, then $0 : C \rightarrow A$ is a precrossed module, in general.

2.4. DEFINITION. Let $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$ be a crossed module in the category of associative algebras. A bimultiplier of \mathcal{A} is a pair (α, β) of bimultipliers α and β of A_1, A_0 , respectively, such that $d\alpha_l = \beta_l d, d\alpha_r = \beta_r d$ and

$$a) \quad \alpha_l(a_0 * a_1) = \beta_l(a_0) * a_1,$$

$$b) \quad \alpha_l(a_1 * a_0) = \alpha_l(a_1) * a_0,$$

$$c) \quad a_0 * \alpha_l(a_1) = \beta_r(a_0) * a_1,$$

$$d) \quad \alpha_r(a_1 * a_0) = a_1 * \beta_r(a_0),$$

$$e) \quad \alpha_r(a_0 * a_1) = a_0 * \alpha_r(a_1),$$

$$f) \quad \alpha_r(a_1) * a_0 = a_1 * \beta_l(a_0),$$

for all $a_0 \in A_0, a_1 \in A_1$.

2.5. NOTATION. In the definition, instead of writing $d\alpha_l = \beta_l d, d\alpha_r = \beta_r d$, we may write these two equalities in one as $d\alpha_{l,r} = \beta_{l,r} d$. In the rest of the paper we will use this notation for shortness.

The set of all bimultipliers of a crossed module \mathcal{A} is denoted by $Bim(\mathcal{A})$. It can be easily checked that $Bim(\mathcal{A})$ is an associative algebra with usual scalar multiplication and addition, and the multiplication defined by

$$(\alpha, \beta) * (\alpha', \beta') = (\alpha * \alpha', \beta * \beta'),$$

for all $(\alpha, \beta), (\alpha', \beta') \in Bim(\mathcal{A})$, where $\alpha * \alpha' = (\alpha_l \alpha'_l, \alpha_r \alpha'_r)$ and $\beta * \beta' = (\beta_l \beta'_l, \beta_r \beta'_r)$.

2.6. DEFINITION. Let $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$ be a crossed module and $\partial_l : A_0 \rightarrow A_1$, $\partial_r : A_0 \rightarrow A_1$ be k -linear maps such that $\partial_r(a_0 * a'_0) = a_0 * \partial_r(a'_0)$, $\partial_l(a_0 * a'_0) = \partial_l(a_0) * a'_0$ and $a_0 * \partial_l(a'_0) = \partial_r(a_0) * a'_0$, for all $a_0, a'_0 \in A_0$. Then the pair $\partial := (\partial_l, \partial_r)$ will be called a crossed bimultiplier of \mathcal{A} .

The set of all crossed bimultipliers of a crossed module $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$ will be denoted by $\mathcal{BM}(\mathcal{A})$.

2.7. REMARK. Definitions 2.4 and 2.6 are deduced from the construction of bimultipliers of a semidirect product in Section 4. In Section 5 one can see that bimultipliers and crossed bimultipliers of a crossed module $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$ can be easily obtained from certain type of bimultipliers of the semidirect product $A_1 \rtimes A_0$ (Propositions 5.3 and 5.4). In the case of commutative associative algebras these definitions coincide with the ones given in [Arvasi and Ege, 2003].

2.8. PROPOSITION. Let $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$ be a crossed module. Then $\mathcal{BM}(\mathcal{A})$ is a non-empty set.

PROOF. Let a_1 be a fixed element of A_1 . Define $(\partial_{a_1})_l(a_0) = -a_1 * a_0$, $(\partial_{a_1})_r(a_0) = -a_0 * a_1$, for all $a_0 \in A_0$. Then by a direct calculation we find that $\partial_{a_1} := ((\partial_{a_1})_l, (\partial_{a_1})_r)$ is a crossed bimultiplier. ■

We can endow $\mathcal{BM}(\mathcal{A})$ with addition and scalar multiplication operations, which are defined in the usual ways. Multiplication operation is defined as follows:

$$\partial * \partial' = ((\partial * \partial')_l, (\partial * \partial')_r),$$

where

$$\begin{aligned} (\partial * \partial')_l &= \partial_l d \partial'_l, \\ (\partial * \partial')_r &= \partial'_r d \partial_r, \end{aligned}$$

for all $\partial, \partial' \in \mathcal{BM}(\mathcal{A})$. Note that, in an analogous way, as it is in the case of groups [Casas, Datuashvili and Ladra, 2009], it can be easily seen from the results of Section 4, that the product of two crossed bimultipliers corresponds to the composition of two bimultipliers of the semidirect product $A_1 \rtimes A_0$, where $\beta_{l,r} : A_0 \rightarrow A_0$ is zero (see Section 4, 4.1). The identity element and the opposite of an element in addition are also defined in the usual ways.

2.9. PROPOSITION. $\mathcal{BM}(\mathcal{A})$ endowed with the above defined operations is an associative algebra.

PROOF. We will only show that the product $\partial * \partial'$ is a crossed bimultiplier. Other conditions need straightforward verifications. Let $\partial, \partial' \in \mathcal{BM}(\mathcal{A})$. We have

$$\begin{aligned} (\partial * \partial')_l(a_0 * a'_0) &= \partial_l d \partial'_l(a_0 * a'_0) \\ &= \partial_l d (\partial'_l(a_0) * a'_0) \\ &= \partial_l (d \partial'_l(a_0) * a'_0) \\ &= \partial_l d \partial'_l(a_0) * a'_0, \end{aligned}$$

and similarly, we have $(\partial * \partial')_r(a_0 * a'_0) = a_0 * \partial_r d \partial'_r(a'_0)$, for all $a_0, a'_0 \in A_0$. On the other hand we have

$$\begin{aligned} a_0 * (\partial * \partial')_l(a'_0) &= a_0 * \partial_l(d \partial'_l(a'_0)) \\ &= \partial_r(a_0) * d \partial'_l(a'_0) \\ &= \partial_r(a_0) * \partial'_l(a'_0) \\ &= d \partial_r(a_0) * \partial'_l(a'_0) \\ &= \partial'_r d \partial_r(a_0) * a'_0, \end{aligned}$$

for all $a_0, a'_0 \in A_0$. So the operation is well-defined, as required. ■

2.10. DEFINITION. A crossed module $\mathcal{A}' : (A'_1 \xrightarrow{d'} A'_0)$ is a crossed submodule of a crossed module $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$, if A'_1, A'_0 are subalgebras of A_1, A_0 respectively, $d' = d|_{A'_1}$ and the action of A'_0 on A'_1 is induced by the action of A_0 on A_1 .

2.11. DEFINITION. A crossed submodule $\mathcal{A}' : (A'_1 \xrightarrow{d} A'_0)$ of a crossed module $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$ is an ideal if A'_1, A'_0 are ideals of A_1, A_0 , respectively; $a_0 * a'_1, a'_1 * a_0 \in A'_1$, for all $a_0 \in A_0, a'_1 \in A'_1$ and $a'_0 * a_1, a_1 * a'_0 \in A'_1$, for all $a'_0 \in A'_0, a_1 \in A_1$. This situation is denoted by $\mathcal{A}' \trianglelefteq \mathcal{A}$.

Let $\mathcal{A}' : (A'_1 \xrightarrow{d} A'_0)$ be an ideal of a crossed module $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$. Then the quotient crossed module \mathcal{A}/\mathcal{A}' is the crossed module $A_1/A'_1 \rightarrow A_0/A'_0$ with the induced boundary map and action.

2.12. DEFINITION. A *cat¹-associative algebra* (or, for shortness, *cat¹-algebra*) A is an associative algebra with two additional unary operations $\omega_0, \omega_1 : A \rightarrow A$ such that

$$\omega_0 \omega_1 = \omega_1, \omega_1 \omega_0 = \omega_0 \quad \text{and} \quad \ker \omega_0 * \ker \omega_1 = 0 = \ker \omega_1 * \ker \omega_0,$$

where ω_0 and ω_1 are associative algebra homomorphisms.

Obviously, a homomorphism between two *cat¹-algebras* is an associative algebra homomorphism, which preserves the unary operations. We will denote a *cat¹-algebra* by (A, ω_0, ω_1) and the corresponding category of such triples and homomorphisms between them by **Cat¹-Ass**. Note that *catⁿ-unitary associative algebras* are defined in [Ellis, 1988].

Note that from the conditions on unary operations it follows that $\omega_i \omega_i = \omega_i, i = 0, 1$.

Define a functor $P : \mathbf{Cat}^1\text{-Ass} \rightarrow \mathbf{XAss}$ as follows; for any object (A, ω_1, ω_0) in **Cat¹-Ass**, $P(A, \omega_0, \omega_1)$ is the crossed module $A_1 \xrightarrow{d} A_0$, where $A_1 = \ker \omega_0, A_0 = \text{Im } \omega_0$, the action is given by multiplication in A and $d = \omega_1|_{\ker \omega_0}$. Now we define a functor $S : \mathbf{XAss} \rightarrow \mathbf{Cat}^1\text{-Ass}$ as follows; for any crossed module $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$, $S(\mathcal{A}) := (A_1 \rtimes A_0, \omega_0, \omega_1)$, where

$$\omega_0(a_1, a_0) = (0, a_0), \quad \omega_1(a_1, a_0) = (0, d(a_1) + a_0),$$

for all $a_1 \in A_1, a_0 \in A_0$. Note that the semidirect product $A_1 \rtimes A_0$ is defined by the action of A_0 on A_1 , as it is in the corresponding crossed module. These two functors give rise to an equivalence of categories $\mathbf{XAss} \simeq \mathbf{Cat}^1\text{-Ass}$.

Precrossed modules in the category of associative algebras suggest the appropriate definition of precat¹-associative algebras, and we have an equivalence of categories $\mathbf{PreXAss} \simeq \mathbf{PreCat}^1\text{-Ass}$. Similarly, for Lie and Leibniz algebras. Some details can be found in [Ellis, 1988, Ellis, 1993].

3. Modified Categories of interest

We will have the main definitions and the statements given for categories of interest in [Casas, Datuashvili and Ladra, 2010, Datuashvili, 1995, Orzech, 1972] with certain modifications which we present as follows.

Let \mathbb{C} be a category of groups with a set of operations Ω and with a set of identities \mathbb{E} , such that \mathbb{E} includes the group identities and the following conditions hold. If Ω_i is the set of i -ary operations in Ω , then:

- (a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
- (b) the group operations (written additively: $0, -, +$) are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega'_2 = \Omega_2 \setminus \{+\}$, $\Omega'_1 = \Omega_1 \setminus \{-\}$. Assume that if $*$ $\in \Omega_2$, then Ω'_2 contains $*^\circ$ defined by $x *^\circ y = y * x$ and assume $\Omega_0 = \{0\}$;
- (c) for each $*$ $\in \Omega'_2$, \mathbb{E} includes the identity $x * (y + z) = x * y + x * z$;
- (d) for each $\omega \in \Omega'_1$ and $*$ $\in \Omega'_2$, \mathbb{E} includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and either the identity $\omega(x * y) = \omega(x) * \omega(y)$ or the identity $\omega(x) * y = \omega(x * y)$.

Denote by Ω'_{1S} the subset of those elements in Ω'_1 , which satisfy the identity $\omega(x) * y = \omega(x * y)$, and by Ω''_1 all other unary operations, i.e. those which satisfy the first identity from (d).

Let C be an object of \mathbb{C} and $x_1, x_2, x_3 \in C$:

Axiom 1. $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$, for each $*$ $\in \Omega'_2$.

Axiom 2. For each ordered pair $(*, \bar{*}) \in \Omega'_2 \times \Omega'_2$ there is a word W such that

$$(x_1 * x_2)\bar{*}x_3 = W(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

where each juxtaposition represents an operation in Ω'_2 .

We will denote the right side of Axiom 2 by $W(x_1, x_2, x_3; *, \bar{*})$.

3.1. DEFINITION. A category of groups with operations \mathbb{C} satisfying conditions (a) – (d), Axiom 1 and Axiom 2, will be called a modified category of interest.

According to this definition every category of interest is a modified category of interest as well.

3.2. REMARK. In the original definition of category of interest there are only the identities $\omega(x + y) = \omega(x) + \omega(y)$ and $\omega(x) * y = \omega(x * y)$, for all $\omega \in \Omega'_1$ and $* \in \Omega'_2$.

Denote by \mathbb{E}_G the subset of identities of \mathbb{E} which includes the group identities and the identities (c) and (d). We denote by \mathbb{C}_G the corresponding category of groups with operations. Thus we have $\mathbb{E}_G \hookrightarrow \mathbb{E}$, $\mathbb{C} = (\Omega, \mathbb{E})$, $\mathbb{C}_G = (\Omega, \mathbb{E}_G)$ and there is a full inclusion functor $\mathbb{C} \hookrightarrow \mathbb{C}_G$. We will call \mathbb{C}_G a general category of groups with operations of a modified category of interest \mathbb{C} .

3.3. EXAMPLE. The categories **Cat¹-Ass**, **Cat¹-Lie**, **Cat¹-Leibniz**, **PreCat¹-Ass**, **PreCat¹-Lie** and **PreCat¹-Leibniz** are modified categories of interest, which are not categories of interest. Also the category of commutative von Neumann regular rings is isomorphic to the category of commutative rings with a unary operation $(\)^*$ satisfying two axioms, defined in [Borceux, Janelidze and Kelly, 2005*], which is a modified category of interest.

3.4. DEFINITION. Let $C \in \mathbb{C}$. A subobject of C is called an ideal if it is the kernel of some morphism.

3.5. PROPOSITION. Let A be a subobject of B in \mathbb{C} . Then like in the case of categories of interest we have that A is an ideal of B if and only if the following conditions hold:

1. A is a normal subgroup of B ,
2. $a * b \in A$, for all $a \in A$, $b \in B$ and $* \in \Omega'_2$.

PROOF. Can be proved in a similar way as Theorem 1.7 in [Orzech, 1972]. ■

3.6. DEFINITION. Let $A, B \in \mathbb{C}$. An extension of B by A is a sequence

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0, \quad (3.1)$$

in which p is surjective and i is the kernel of p . We say that an extension is split if there is a morphism $s : B \longrightarrow E$ such that $ps = 1_B$.

3.7. DEFINITION. For $A, B \in \mathbb{C}$ we will say that we have a set of actions of B on A , whenever there is a map $f_* : A \times B \longrightarrow A$ for each $* \in \Omega_2$.

A split extension of B by A , induces an action of B on A corresponding to the operations in \mathbb{C} . For a given split extension (3.1), we have

$$b \cdot a = s(b) + a - s(b), \quad (3.2)$$

$$b * a = s(b) * a, \quad (3.3)$$

for all $b \in B$, $a \in A$ and $* \in \Omega_2'$. Actions defined by (3.2) and (3.3) will be called *derived actions of B on A* . We will often use the notation $b * a$, where we mean both the dot and the star actions.

3.8. DEFINITION. *Given an action of B on A , a semidirect product $A \rtimes B$ is a universal algebra, whose underlying set is $A \times B$ and the operations are defined by*

$$\begin{aligned} \omega(a, b) &= (\omega(a), \omega(b)), \\ (a', b') + (a, b) &= (a' + b' \cdot a, b' + b), \\ (a', b') * (a, b) &= (a' * a + a' * b + b' * a, b' * b), \end{aligned}$$

for all $a, a' \in A$, $b, b' \in B$, $* \in \Omega_2'$.

3.9. THEOREM. *An action of B on A is a derived action if and only if $A \rtimes B$ is an object of \mathbb{C} .*

PROOF. Let B have a derived action on A defined by the split extension (3.1). We will only show that $A \rtimes B$ satisfies the new condition. Other proofs can be found in [Orzech, 1972]. By the definition of semidirect product, we have:

$$\begin{aligned} \omega((a', b') * (a, b)) &= \omega(a' * a + a' * b + b' * a, b' * b) = \\ &(\omega(a') * \omega(a) + \omega(a') * \omega(b) + \omega(b') * \omega(a), \omega(b') * \omega(b)) = \\ &(\omega(a'), \omega(b')) * (\omega(a), \omega(b)) = \omega(a', b') * \omega(a, b), \end{aligned}$$

for all $(a, b), (a', b') \in A \rtimes B$, and $\omega \in \Omega_1''$. ■

3.10. PROPOSITION. *A set of actions of B on A in \mathbb{C}_G is a set of derived actions if and only if it satisfies the following conditions:*

1. $0 \cdot a = a$,
2. $b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2$,
3. $(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$,
4. $b * (a_1 + a_2) = b * a_1 + b * a_2$,
5. $(b_1 + b_2) * a = b_1 * a + b_2 * a$,
6. $(b_1 * b_2) \cdot (a_1 * a_2) = a_1 * a_2$,
7. $(b_1 * b_2) \cdot (a * b) = a * b$,
8. $a_1 * (b \cdot a_2) = a_1 * a_2$,
9. $b * (b_1 \cdot a) = b * a$,
10. $\omega(b \cdot a) = \omega(b) \cdot \omega(a)$,

11. $\omega(a * b) = \omega(a) * b = a * \omega(b)$, for any $\omega \in \Omega'_{1S}$ and $\omega(a * b) = \omega(a) * \omega(b)$, for any $\omega \in \Omega''_1$,

12. $x * y + z * t = z * t + x * y$,

for each $\omega \in \Omega'_1$, $* \in \Omega'_2$, $b, b_1, b_2 \in B$, $a, a_1, a_2 \in A$ and for $x, y, z, t \in A \cup B$ whenever each side of 12 makes sense.

PROOF. Follows from Definitions 3.7, 3.8 and Theorem 3.9. \blacksquare

3.11. REMARK. In the case of a category of interest, condition 11 is $\omega(a * b) = \omega(a) * b$ [Casas, Datuashvili and Ladra, 2010].

3.12. EXAMPLE. Let $(A, \omega_1^A, \omega_0^A), (B, \omega_1^B, \omega_0^B) \in \mathbf{Cat}^1\text{-Ass}$ and let $(B, \omega_1^B, \omega_0^B)$ have a derived action on $(A, \omega_1^A, \omega_0^A)$. By Definition 3.7 we have the identities (2.1) and also the identities

$$\begin{aligned} a * b = b * a = 0, & \text{ if } b \in \ker \omega_0^B, a \in \ker \omega_1^A \text{ or } b \in \ker \omega_1^B, a \in \ker \omega_0^A; \\ \omega_i^B(b) * \omega_i^A(a) &= \omega_i^A(b * a), \quad \omega_i^A(a) * \omega_i^B(b) = \omega_i^A(a * b), \\ \omega_j^B(b) * \omega_i^A(a) &= \omega_i^A(\omega_j^B(b) * a), \quad \omega_i^A(a) * \omega_j^B(b) = \omega_i^A(a * \omega_j^B(b)), \\ \omega_j^B(b) * \omega_i^A(a) &= \omega_j^A(b * \omega_i^A(a)), \quad \omega_i^A(a) * \omega_j^B(b) = \omega_j^A(\omega_i^A(a) * b), \end{aligned}$$

$i, j = 0, 1, i \neq j$, for any $a \in (A, \omega_1^A, \omega_0^A)$, $b \in (B, \omega_1^B, \omega_0^B)$.

3.13. DEFINITION. A precrossed module in a modified category of interest \mathbb{C} is a triple (C_1, C_0, ∂) , where $C_0, C_1 \in \mathbb{C}$, the object C_0 has a derived action on C_1 and $\partial : C_1 \rightarrow C_0$ is a morphism in \mathbb{C} with the conditions:

$$a) \partial(c_0 \cdot c_1) = c_0 + \partial(c_1) - c_0,$$

$$b) \partial(c_0 * c_1) = c_0 * \partial(c_1),$$

for all $c_0 \in C_0$, $c_1 \in C_1$, and $* \in \Omega'_2$.

In addition, if $\partial : C_1 \rightarrow C_0$ satisfies the conditions

$$c) \partial(c_1) \cdot c'_1 = c_1 + c'_1 - c_1,$$

$$d) \partial(c_1) * c'_1 = c_1 * c'_1,$$

for all $c_1, c'_1 \in C_1$, and $* \in \Omega'_2$, then the triple (C_1, C_0, ∂) is called a crossed module in \mathbb{C} .

3.14. DEFINITION. A morphism between two (pre)crossed modules $(C_1, C_0, \partial) \rightarrow (C'_1, C'_0, \partial')$ in \mathbb{C} is a pair of morphisms (μ_1, μ_0) in \mathbb{C} , $\mu_0 : C_0 \rightarrow C'_0$, $\mu_1 : C_1 \rightarrow C'_1$, such that

$$a) \mu_0 \partial(c_1) = \partial' \mu_1(c_1),$$

$$b) \mu_1(c_0 \cdot c_1) = \mu_0(c_0) \cdot \mu_1(c_1),$$

$$c) \mu_1(c_0 * c_1) = \mu_0(c_0) * \mu_1(c_1),$$

for all $c_0 \in C_0$, $c_1 \in C_1$ and $* \in \Omega'_2$.

3.15. DEFINITION. *Let $A \in \mathbb{C}$. The center of A is defined by*

$$Z(A) = \{z \in A \mid a + z = z + a, a + \omega(z) = \omega(z) + a, a * z = 0, a * \omega'(z) = 0, \text{ for all } a \in A, \omega \in \Omega_1, \omega' \in \Omega'_1 \text{ and } * \in \Omega_2'\}.$$

3.16. DEFINITION. *If A is an ideal of B , then*

$$Z(B, A) = \{b \in B \mid a + b = b + a, a + \omega(b) = \omega(b) + a, a * b = 0, a * \omega'(b) = 0, \text{ for all } a \in A, \omega \in \Omega_1, \omega' \in \Omega'_1 \text{ and } * \in \Omega_2'\}$$

is called the centralizer of A in B .

3.17. LEMMA. *In a modified category of interest, $Z(B, A)$ is an ideal of B .*

PROOF. Follows from Definition 3.16 ■

The definition of split extension classifier (object which represents actions), is formulated in [Borceux, Janelidze and Kelly, 2005] for semi-abelian categories in terms of categorical notions of internal object action and semidirect product. Categories of interest are semi-abelian categories. According to [Bourn and Janelidze, 1998] in this special case these notions coincide with the ones given in [Orzech, 1972]. Analogous situation we have in the cases of modified categories of interest and categories equivalent to them. Therefore the definition of a split extension classifier for modified categories of interest has the following form. Consider the category of all split extensions with fixed kernel A ; thus the objects are

$$0 \rightarrow A \rightarrow C \xrightarrow{s} C' \rightarrow 0$$

and the arrows are the triples of morphisms $(1_A, \gamma, \gamma')$ between the extensions, which commute with the section homomorphisms as well. By definition, an object $[A]$ is a split extension classifier for A if there exists a derived action of $[A]$ on A , such that the corresponding extension

$$0 \rightarrow A \rightarrow A \times [A] \xrightarrow{s} [A] \rightarrow 0$$

is a terminal object in the above defined category.

3.18. PROPOSITION. *Let \mathbb{C} be a modified category of interest and A be an object in \mathbb{C} . An object $B \in \mathbb{C}$ is a split extension classifier for A in the sense of [Borceux, Janelidze and Kelly, 2005] if and only if it satisfies the following condition: B has a derived action on A such that for all C in \mathbb{C} and a derived action of C on A there is a unique morphism $\varphi : C \rightarrow B$, with $c \cdot a = \varphi(c) \cdot a$, $c * a = \varphi(c) * a$, for all $* \in \Omega_2'$, $a \in A$ and $c \in C$.*

PROOF. Analogous to the one for categories of interest [Casas, Datuashvili and Ladra, 2010]. ■

The object B in \mathbb{C} satisfying the above stated condition will be called an *actor* of A and denoted by $\text{Act}(A)$. The corresponding universal acting object, which represents actions in the sense of [Borceux, Janelidze and Kelly, 2005, Borceux, Janelidze and Kelly, 2005*], in the categories equivalent to modified categories of interest will be called a split extension classifier and denoted by $[A]$, as it is generally in semi-abelian categories.

3.19. REMARK. As a consequence of this proposition, an actor of an object is unique up to isomorphism.

3.20. DEFINITION. *Let $A, B \in \mathbb{C}$. We will say that a set of actions of B on A is strict if for any two elements $b, b' \in B$, from the conditions $b \cdot a = b' \cdot a$, $\omega(b) \cdot a = \omega(b') \cdot a$, $b * a = b' * a$ and $\omega'(b) * a = \omega'(b') * a$, for all $a \in A$, $\omega \in \Omega_1'$, $\omega' \in \Omega_1''$ and $*$ $\in \Omega_2'$, it follows that $b = b'$.*

3.21. REMARK. In the case of a category of interest, the condition $\omega(b) * a = \omega(b') * a$ is always satisfied.

3.22. EXAMPLE. For any object $A \in \mathbb{C}$ we have an action of A on itself defined by $a \cdot a' = a + a' - a$, $a * a' = a * a'$, for all $a, a' \in A$, $*$ $\in \Omega_2'$, where $*$ on the left side denotes the action and on the right side the operation in A . This action is called an action by conjugation of A on itself.

3.23. EXAMPLE. Let $A \in \mathbb{C}$ and $Z(A) = 0$. Then the action by conjugation of A on itself is a strict action.

3.24. PROPOSITION. *Let $A, B \in \mathbb{C}$. A set of derived actions of B on A is strict if and only if in the corresponding split extension (3.1), we have $\text{Im}(s) \cap Z(E, A) = 0$.*

PROOF. Follows from Definitions 3.16 and 3.20. ■

3.25. EXAMPLE. Let $A \in \mathbb{C}$. If $\text{Act}(A)$ exists, then the derived action of $\text{Act}(A)$ on A is strict.

3.26. REMARK. Let $A \in \mathbb{C}$. If $\text{Act}(A)$ exists, then by Definition 3.18 and Example 3.22, there is a unique morphism $\beta : A \rightarrow \text{Act}(A)$ in \mathbb{C} , determined by $\beta(a) \cdot a' = a \cdot a'$, and $\beta(a) * a' = a * a'$, for all $a, a' \in A$. $\beta : A \rightarrow \text{Act}(A)$ is a crossed module and a terminal object in the category of crossed modules with the same domain A . Up to isomorphism, there is a unique crossed module with this property.

3.27. DEFINITION. *We will say that an object $G \in \mathbb{C}_G$ has a general actor property to the object $A \in \mathbb{C}$ (for shortness $GA(A)$ -property) if G has a set of actions on $A \in \mathbb{C}$, which is a set of derived actions in \mathbb{C}_G and for any object $C \in \mathbb{C}$ and a derived action of C on A in \mathbb{C} , there exists in \mathbb{C}_G a unique morphism $\varphi : C \rightarrow G$ such that $c \cdot a = \varphi(c) \cdot a$, for all $c \in C$, $a \in A$ and $*$ $\in \Omega_2'$.*

3.28. DEFINITION. We will say that an object $G \in \mathbb{C}_G$ has a strict general actor property to the object $A \in \mathbb{C}$ (for shortness $SGA(A)$ -property) if G has $GA(A)$ -property and the action of G on A is strict.

Below we formulate a condition on objects with $GA(A)$ -property, which will be used in the definition of universal strict general actor of an object in \mathbb{C} .

Condition 1. Let $A \in \mathbb{C}$ and $\{B_j\}_{j \in J}$ denote the set of all objects in \mathbb{C} which have derived actions on A . Let G be an object in \mathbb{C}_G with $GA(A)$ -property and $\varphi_j : B_j \rightarrow G$, $j \in J$, denote the corresponding unique morphism such that $b_j \dot{*} a = \varphi_j(b_j) \dot{*} a$, for all $b_j \in B_j$, $a \in A$, $* \in \Omega_2'$. The elements of G of the type $\varphi_i(b_i)$, $i \in J$ satisfy the following equality:

$$(\varphi_i(b_i) * \varphi_j(b_j))\bar{*}a = W(\varphi_i(b_i), \varphi_j(b_j), a; *, \bar{*})$$

for any $b_i \in B_i$, $b_j \in B_j$, $*, \bar{*} \in \Omega_2'$, $i, j \in J$ and $a \in A$.

3.29. DEFINITION. A universal strict general actor of an object A , denoted by $USGA(A)$, is an object in \mathbb{C}_G with $SGA(A)$ -property and with Condition 1, such that for any object G with $SGA(A)$ -property and with Condition 1 there exists a unique morphism $\eta : USGA(A) \rightarrow G$ in the category \mathbb{C}_G , with $\eta\psi_j = \varphi_j$, for any $j \in J$, where $\varphi_j : B_j \rightarrow G$ and $\psi_j : B_j \rightarrow USGA(A)$ denote the corresponding unique morphisms with the appropriate properties from the definition of general actor property. By Corollary 3.33 a universal strict general actor is the unique object (up to isomorphism) satisfying the corresponding properties.

We establish the following two statements without proofs, since they are the same as the ones of Proposition 3.8 and Theorem 3.9 in the case of categories of interest [Casas, Datuashvili and Ladra, 2010].

3.30. PROPOSITION. Let \mathbb{C} be a modified category of interest and $A \in \mathbb{C}$. If an actor $Act(A)$ exists, then the unique morphism $\eta : USGA(A) \rightarrow Act(A)$ is an isomorphism with $x \dot{*} a = \eta(x) \dot{*} (a)$, for all $x \in USGA(A)$, $a \in A$.

3.31. THEOREM. Let \mathbb{C} be a modified category of interest and $A \in \mathbb{C}$. A has an actor if and only if the semidirect product $A \rtimes USGA(A)$ is an object in \mathbb{C} . If it is the case, then $Act(A) \cong USGA(A)$.

Now we will show the existence of $USGA(A)$ for any object A in a modified category of interest \mathbb{C} . The construction given here is similar to that given for the case of categories of interest in [Casas, Datuashvili and Ladra, 2010] with some modifications.

Let $A \in \mathbb{C}$; consider all split extensions of A in \mathbb{C}

$$E_j : 0 \longrightarrow A \xrightarrow{i_j} C_j \xrightarrow{p_j} B_j \longrightarrow 0, \quad j \in \mathbb{J}.$$

When $B_j = B_k = B$, for $j \neq k$, in this case the corresponding extensions derive different actions of B on A . Let $\{b_j \cdot, b_j * \mid b_j \in B_j, * \in \Omega'_2\}$ be the set of functions defined by the action of B_j on A . For any element $b_j \in B_j$ denote $\mathbf{b}_j = \{b_j \cdot, b_j * , * \in \Omega'_2\}$. Let $\mathbb{B} = \{\mathbf{b}_j \mid b_j \in B_j, j \in \mathbb{J}\}$.

Thus each element $\mathbf{b}_j \in \mathbb{B}, j \in \mathbb{J}$ is a special type of a function $\mathbf{b}_j : \Omega_2 \rightarrow \text{Maps}(A \rightarrow A)$, defined by

$$\mathbf{b}_j(+) = b_j \cdot -, \mathbf{b}_j(*) = b_j * - : A \rightarrow A,$$

$* \in \Omega'_2$.

We define the $*$ operation, $\mathbf{b}_i * \mathbf{b}_k, * \in \Omega'_2$, for the elements of \mathbb{B} according to Axiom 2.

$$((\mathbf{b}_i * \mathbf{b}_k)(\bar{*}))(a) = W(b_i, b_k, a; *, \bar{*}), \text{ and } ((\mathbf{b}_i * \mathbf{b}_k)(+))(a) = a.$$

We define the operation of addition by

$$((\mathbf{b}_i + \mathbf{b}_k)(+))(a) = b_i \cdot (b_k \cdot a),$$

$$((\mathbf{b}_i + \mathbf{b}_k)(*))(a) = b_i * a + b_k * a.$$

For a unary operation $\omega \in \Omega'_1$ we define

$$(\omega(\mathbf{b}_k)(+))(a) = \omega(b_k) \cdot (a), (\omega(\mathbf{b}_k)(*))(a) = \omega(b_k) * (a),$$

$\omega(b * b') = \omega(b) * \omega(b')$ if $\omega \in \Omega'_{1S}$ and $\omega(b * b') = \omega(b) * b'$ if $\omega \in \Omega'_{1S}$ (in the case of categories of interest, this condition is $\omega(b * b') = \omega(b) * b'$)

$$\omega(b_1 + \dots + b_n) = \omega(b_1) + \dots + \omega(b_n),$$

$$((-\mathbf{b}_k)(+))(a) = (-b_k) \cdot a, (-b) \cdot (a) = a$$

$$((-\mathbf{b}_k)(*))(a) = -(b_k * a), ((-b)(*))(a) = -((b(*))(a)),$$

$$-(b_1 + \dots + b_n) = -b_n - \dots - b_1,$$

where b, b', b_1, \dots, b_n are certain combinations of $*$ operations on the elements of \mathbb{B} , i.e. the elements of the type $\mathbf{b}_{i_1} *_{i_1} \dots *_{i_{n-1}} \mathbf{b}_{i_n}, n > 1$.

Denote by $\mathfrak{B}'(A)$ the set of all functions ($\Omega_2 \rightarrow \text{Maps}(A \rightarrow A)$) obtained by performing all kind of operations defined above on the elements of \mathbb{B} and on the new obtained elements as the results of operations. In what follows we will write for simplicity $b * a$ or $b \cdot a$ instead of $(b(*))(a)$ or $(b(+))(a)$, respectively, where $b \in \mathfrak{B}'(A), a \in A$. Note that it may happen that $b \dot{*} a = b' \dot{*} a$, for any $a \in A$ and any $* \in \Omega'_2$, but we do not have the equality $\omega(b) \dot{*} a = \omega(b') \dot{*} a$, for any $a \in A$, any $* \in \Omega'_2$ and any unary operation ω (i.e. for any ω which is a finite combination of elements of Ω'_1). We define the following relation: we will write $b \sim b'$, for $b, b' \in \mathfrak{B}'(A)$, if and only if $b \dot{*} a = b' \dot{*} a$ and $\omega(b) \cdot a = \omega(b') \cdot a, \omega'(b) * a = \omega'(b') * a$, for any $a \in A, \omega \in \Omega'_1, \omega' \in \Omega'_1$ and $* \in \Omega'_2$. Let R be a congruence relation generated by \sim . We define $\mathfrak{B}(A) = \mathfrak{B}'(A)/R$. The operations defined on $\mathfrak{B}'(A)$ define the corresponding operations on $\mathfrak{B}(A)$. For simplicity we will denote the elements of $\mathfrak{B}(A)$ by the same letters b, b' etc. instead of the classes clb, clb' etc.

3.32. THEOREM. Let $A \in \mathbb{C}$, then we have:

1. $\mathfrak{B}(A)$ is an object of \mathbb{C}_G ;

2. The set of actions of $\mathfrak{B}(A)$ on A , defined as in [Casas, Datuashvili and Ladra, 2010] for categories of interest, is a set of strict derived actions in \mathbb{C}_G ;
3. $\mathfrak{B}(A)$ is a universal strict general actor for A .

PROOF. The proofs of the statements 1., 2., and 3. are similar to the ones of Proposition 4.1, Proposition 4.2 and Theorem 4.3, respectively, for categories of interest [Casas, Datuashvili and Ladra, 2010]. ■

3.33. COROLLARY. Let \mathbb{C} be a modified category of interest. For any universal strict general actor C of an object A , we have an isomorphism $\theta : C \approx \mathfrak{B}(A)$ with $\theta(c) \dot{*} a = c \dot{*} a$, for all $c \in C, a \in A$. The unique morphism η given in Definition 3.29 satisfies the analogous condition $\eta(x) \dot{*} a = x \dot{*} a$, for all $x \in \text{USGA}(A), a \in A$.

PROOF. See the proof of Corollary 4.4 in [Casas, Datuashvili and Ladra, 2010]. ■

Define a map $d : A \rightarrow \mathfrak{B}(A)$ by $d(a) = \mathbf{a}$, where $\mathbf{a} = \{a \cdot, a * : * \in \Omega'_2\}$. By definition we have $d(a) \cdot a' = a + a' - a$, and $d(a) * a' = a * a'$, for all $a, a' \in A, * \in \Omega'_2$. By direct checking we have also $\ker d = Z(A)$.

3.34. PROPOSITION. For any object A in \mathbb{C} we have:

1. $d : A \rightarrow \mathfrak{B}(A)$ is a crossed module in \mathbb{C}_G .
2. For any crossed module $\gamma : A \rightarrow B$ in \mathbb{C} there exists a unique crossed module morphism $A : (A, B, \gamma) \rightarrow (A, \mathfrak{B}(A), d)$ in \mathbb{C}_G under the object A .
3. $d : A \rightarrow \mathfrak{B}(A)$ is the unique crossed module (up to isomorphism) with the property that the operator object on the right side is a universal strict general actor of A .

PROOF. 1. In the proof we will only show that the new equalities, which occur in the modified category of interest case, are satisfied. The others can be found in [Casas, Datuashvili and Ladra, 2010].

First we will prove that d is a homomorphism in \mathbb{C}_G . For this we will show that $d(\omega(a)) = \omega(d(a))$, for any $\omega \in \Omega'_1$ and $a \in A$. Therefore, we need to prove for all $a, a' \in A, \omega, \omega' \in \Omega'_1, * \in \Omega'_2$, that

$$\begin{aligned} d(\omega(a)) \cdot a' &= \omega(d(a)) \cdot a', \\ \omega'(d(\omega(a))) \cdot a' &= \omega'(\omega(d(a))) \cdot a', \\ d(\omega(a)) * a' &= \omega(d(a)) * a', \\ \omega'(d(\omega(a))) * a' &= \omega'(\omega(d(a))) * a', \text{ for all } a, a' \in A, \omega' \in \Omega'_1, * \in \Omega'_2 \end{aligned}$$

We will prove the fourth equality.

Since

$$\begin{aligned} d(\omega(a)) * a' &= \omega(a) * a', \\ \omega(d(a)) * a' &= \omega(\mathbf{a}) * a' = \omega(a) * a', \end{aligned}$$

we have

$$\omega'(d(\omega(a))) * a' = \omega'(\omega(a)) * a' = \omega'(\omega(d(a))) * a',$$

as required. On the other hand, for $\omega = -$ we have

$$d(-a) = -d(a)$$

and

$$d(a_1 + a_2) = d(a_1) + d(a_2).$$

The next equality that we must prove is $d(a_1 * a_2) = d(a_1) * d(a_2)$. For this we need to show the following four equalities:

$$\begin{aligned} d(a_1 * a_2) \cdot a &= (d(a_1) * d(a_2)) \cdot a, \\ \omega(d(a_1 * a_2)) \cdot a &= \omega(d(a_1) * d(a_2)) \cdot a, \\ d(a_1 * a_2) \bar{*} a &= (d(a_1) * d(a_2)) \bar{*} a, \\ \omega(d(a_1 * a_2)) \bar{*} a &= \omega(d(a_1) * d(a_2)) \bar{*} a. \end{aligned}$$

for all $a \in A, \omega \in \Omega'_1, \bar{*} \in \Omega'_2$.

We will only show the fourth equality. For this, we have

$$\begin{aligned} \omega(d(a_1 * a_2)) \bar{*} a &= d(\omega(a_1 * a_2)) \bar{*} a \\ &= (\omega(a_1) * \omega(a_2)) \bar{*} a \\ &= W(\omega(a_1), \omega(a_2); a; *, \bar{*}) \\ &= W(d\omega(a_1), d\omega(a_2); a; *, \bar{*}) \\ &= (d\omega(a_1) * d\omega(a_2)) \bar{*} a \\ &= (\omega d(a_1) * \omega d(a_2)) \bar{*} a \\ &= \omega(d(a_1) * d(a_2)) \bar{*} a \end{aligned}$$

as required.

Now we will show that d is a crossed module. We have to check conditions $a)-d)$ from the definition of a crossed module given in Definition 3.13. We will check condition $b)$. Other conditions are checked similarly, see also [Casas, Datuashvili and Ladra, 2010]. The condition $b)$ states

$$d(b * a) = b * d(a), \text{ for any } b \in \mathfrak{B}(A), a \in A, * \in \Omega'_2.$$

Thus we have to show

$d(b * a) \dot{\bar{*}} a' = (b * d(a)) \dot{\bar{*}} a', \omega(d(b * a)) \dot{\bar{*}} a' = \omega(b * da) \dot{\bar{*}} a'$, for the dot action for all $\omega \in \Omega'_1$, and for any $*, \bar{*}$ actions for all $\omega \in \Omega''_1$. We only show the second equality of the second one, others are proved analogously. Consider the case $b = \mathbf{b}_i, i \in \mathbb{J}$. Recall that by

definition $\omega(\mathbf{b}_i) * a = \omega(b_i) * a$. We have

$$\begin{aligned}
\omega(d(\mathbf{b}_i * a)) \bar{*} a' &= d(\omega(\mathbf{b}_i * a)) \bar{*} a' \\
&= d(\omega(b_i * a)) \bar{*} a' \\
&= d(\omega(b_i) * \omega(a)) \bar{*} a' \\
&= (\omega(b_i) * \omega(a)) \bar{*} a' \\
&= W(\omega(b_i), \omega(a), a'; *, \bar{*}) \\
&= W(\omega(\mathbf{b}_i), \omega(a), a'; *, \bar{*}) \\
&= W(\omega(\mathbf{b}_i), d(\omega(a)), a'; *, \bar{*}) \\
&= (\omega(\mathbf{b}_i) * d(\omega(a))) \bar{*} a' \\
&= (\omega(\mathbf{b}_i) * \omega(d(a))) \bar{*} a' \\
&= \omega(\mathbf{b}_i * d(a)) \bar{*} a'.
\end{aligned}$$

The case $b = \mathbf{b}_{i_1} * \mathbf{b}_{i_2} \cdots * \mathbf{b}_{i_n}$, or b is the sum of elements of the type $\mathbf{b}_{i_1} * \mathbf{b}_{i_2} \cdots * \mathbf{b}_{i_n}$ is proved in a similar way as in the case of categories of interest [Casas, Datuashvili and Ladra, 2010], i.e. by application of the result in the case $b = \mathbf{b}_i$ and Axiom 2.

The proofs of 2. and 3. are the same as the ones of Proposition 4.6 for categories of interest [Casas, Datuashvili and Ladra, 2010]. \blacksquare

4. Actor of an object in $\mathbf{Cat}^1\text{-Ass}$

In this section we will construct an object $(\mathfrak{A}(A), \overline{\omega_0}, \overline{\omega_1})$ for an object (A, w_0, w_1) in $\mathbf{Cat}^1\text{-Ass}$ and prove that under certain conditions it is an actor of (A, w_0, w_1) . The construction is deduced from the general construction of a universal strict general actor in modified categories of interest given in Section 3 and its interpretation for the case $\mathbb{C} = \mathbf{Cat}^1\text{-Ass}$, i.e. from the construction of $(\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)})$, for $(A, w_0, w_1) \in \mathbf{Cat}^1\text{-Ass}$.

Consider the triples $(f_{l,r}, f_{l,r}^0, f_{l,r}^1)$, which consist of bimultipliers of A such that

- M1. $f_{l,r}^i \omega_i = \omega_i f_{l,r}$, for $i = 0, 1$,
- M2. $f_{l,r}^j \omega_i = \omega_i f_{l,r}^j$, for $i = 0, 1, j = 0, 1$, and $\omega_i f_{l,r}^j = \omega_j f_{l,r} \omega_i$, for $i = 0, 1, j = 0, 1, i \neq j$,
- M3. $f_{l,r}(x) = f_{l,r}^1(x)$, for all $x \in \ker \omega_0$,
- M4. $f_{l,r}(x) = f_{l,r}^0(x)$, for all $x \in \ker \omega_1$.

Denote the set of all this kind of triples by $\mathfrak{A}(A)$. This set is not empty, we will show now that the elements of $(\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)})$ are elements of $\mathfrak{A}(A)$. We have to show that the elements of $(\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)})$ satisfy conditions M1-M4. The proofs of M1 and M2 are obvious. We will demonstrate M3; M4 is proved in an analogous way. First we will show that if $(B_i, w_0^{B_i}, w_1^{B_i})$ has a derived action on (A, w_0^A, w_1^A) in $\mathbf{Cat}^1\text{-Ass}$, then for any $b_i \in B_i$ the triple $(b_{i(l,r)} *, \omega_0^{B_i}(b_{i(l,r)})*, \omega_1^{B_i}(b_{i(l,r)}*))$ (i.e. $cl(b_{i(l,r)} *, \omega_0^{B_i}(b_{i(l,r)})*, \omega_1^{B_i}(b_{i(l,r)}*))$) is an element of $\mathfrak{A}(A)$. We will show this fact for the left multipliers and omit the corresponding index l . Let $a \in \ker \omega_0^A$, then since $(b_i - w_1^{B_i}(b_i)) \in \ker \omega_1^{B_i}$, we have

$(b_i - w_1^{B_i}(b_i)) * a = 0$ (see Example 3.12 for the derived action conditions in **Cat¹-Ass**), from which follows that condition M3 is satisfied. Now let $b = b_i * b_j$, where $b_i \in B_i, b_j \in B_j$, and B_i and B_j have derived actions on A . Then by definition of action of $(\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)})$ on A we have

$(b_i * b_j) * a - \omega_1^{\mathfrak{B}(A)}(b_i * b_j) * a = (b_i * b_j) * a - (\omega_1^{B_i}(b_i) * \omega_1^{B_j}(b_j)) * a = b_i * (b_j * a) - \omega_1^{B_i}(b_i) * (\omega_1^{B_j}(b_j) * a) = b_i * (\omega_1^{B_j}(b_j) * a) - \omega_1^{B_i}(b_i) * (\omega_1^{B_j}(b_j) * a) = 0$, since $\omega_1^{B_j}(b_j) * a \in \ker \omega_0^A$. The case where b is any type of element in $(\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)})$ is proved by induction on the length of b , applying the distributive property of the action and the fact that ω_0 and ω_1 are homomorphisms for the addition. The fact that $a * b = a * \omega_1^{\mathfrak{B}(A)}(b)$ is proved in an analogous way.

An easy checking shows that if $(f_{l,r}, f_{l,r}^0, f_{l,r}^1)$ satisfies conditions M1-M4, then the triples $(f_{l,r}^0, f_{l,r}^0, f_{l,r}^0)$ and $(f_{l,r}^1, f_{l,r}^1, f_{l,r}^1)$ also satisfy the same conditions. We will demonstrate this fact for the triple $(f_{l,r}^0, f_{l,r}^0, f_{l,r}^0)$; the proof for the case $(f_{l,r}^1, f_{l,r}^1, f_{l,r}^1)$ is analogous.

Checking of condition M1. We have to show that

$$f_{l,r}^0 \omega_0 = \omega_0 f_{l,r}^0 \text{ and } f_{l,r}^0 \omega_1 = \omega_1 f_{l,r}^0.$$

The first equality follows from M2, for the triple $(f_{l,r}, f_{l,r}^0, f_{l,r}^1)$, for $i = j = 0$, and the second one follows again from M2, for $i = 1$ and $j = 0$.

Checking of condition M2, the case $i=j=0$. We have to prove

$$f_{l,r}^0 \omega_0 = \omega_0 f_{l,r}^0.$$

This equality was proved above as condition M1. The case $i = 0, j = 1$.

We have to prove the following equalities

$$f_{l,r}^0 \omega_0 = \omega_0 f_{l,r}^0 = \omega_1 f_{l,r}^0 \omega_0.$$

The first equality has been already proved in the previous case. For the second equality we apply the first equality of this case, the fact that $\omega_1 \omega_0 = \omega_0$ and obtain $\omega_0 f_{l,r}^0 = \omega_1 \omega_0 f_{l,r}^0 = \omega_1 f_{l,r}^0 \omega_0$.

The case $i = 1, j = 0$. We have to prove

$$f_{l,r}^0 \omega_1 = \omega_1 f_{l,r}^0 = \omega_0 f_{l,r}^0 \omega_1.$$

The first equality follows from M2 for the triple $(f_{l,r}, f_{l,r}^0, f_{l,r}^1)$, for $i = 1, j = 0$. For the second equality we apply the fact that $\omega_1 \omega_1 = \omega_1$, the first equality of this case and obtain $\omega_1 f_{l,r}^0 = \omega_0 \omega_1 f_{l,r}^0 = \omega_0 f_{l,r}^0 \omega_1$.

The case $i = 1, j = 1$. We have to prove

$$f_{l,r}^0 \omega_1 = \omega_1 f_{l,r}^0.$$

This equality was proved already in the previous case.

It is obvious that conditions M3 and M4 are satisfied for the triple $(f_{l,r}^0, f_{l,r}^0, f_{l,r}^0)$.

$\mathfrak{A}(A)$ has an associative algebra structure with componentwise addition, scalar multiplication and multiplication of the corresponding bimultipliers. The zero element is the triple $(0, 0, 0)$, where 0 is the zero map. $(\mathfrak{A}(A), \overline{\omega}_0, \overline{\omega}_1)$ is a cat^1 -algebra with unary operations, $\overline{\omega}_0, \overline{\omega}_1 : \mathfrak{A}(A) \rightarrow \mathfrak{A}(A)$ defined by $\overline{\omega}_0(f, f^0, f^1) = (f^0, f^0, f^0)$ and $\overline{\omega}_1(f, f^0, f^1) = (f^1, f^1, f^1)$, respectively. Define an action of $\mathfrak{A}(A)$ on A by the maps $\mathfrak{A}(A) \times A \rightarrow A$, $((f, f^0, f^1), a) = f_l(a)$, and $A \times \mathfrak{A}(A) \rightarrow A$, $(a, (f, f^0, f^1)) = f_r(a)$, for all $a \in A$ and $(f, f^0, f^1) \in \mathfrak{A}(A)$. We have an injective homomorphism

$$\psi : (\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)}) \hookrightarrow (\mathfrak{A}(A), \overline{\omega}_0, \overline{\omega}_1).$$

It is worth to recall that we write left or right bimultipliers as maps on the left side of an element (i.e. $f_l(a)$ and $f_r(a)$, see Section 2). Therefore for the product of two right bimultipliers $*b$ and $*b'$, from the triples in $\mathfrak{A}(A)$, which is a composition of bimultipliers by definition of product in $\mathfrak{A}(A)$ we have $(*b)(*b')(a) = (a * b') * b$, since $(*b)(*b')$ denotes the composition and we apply first $*b'$ and then $*b$.

Condition 2. Let A be an associative algebra such that $\text{Ann}(A) = 0$ or $A^2 = A$.

Let $A_0, A_1 \in \mathbf{Ass}$ and A_0 has an action on A_1 . It is easy to see that if $A_i, i = 0, 1$, satisfy Condition 2, then the semidirect product $A_1 \rtimes A_0$ also satisfies this condition. We will show first that if $\text{Ann}(A_i) = 0$, then $\text{Ann}(A_1 \rtimes A_0) = 0$. Suppose $(a'_1, a'_0) * (a_1, a_0) = 0$, for any $a_i \in A_i, i = 0, 1$. By the definition of multiplication in $A_1 \rtimes A_0$, it follows that $a'_0 * a_0 = 0$ and $a'_1 * a_1 + a'_1 * a_0 + a'_0 * a_1 = 0$, for any $a_i \in A_i, i = 0, 1$. From the first equality it follows that $a'_0 = 0$. Taking in the second equality $a_0 = 0$ we obtain that $a'_1 = 0$, which proves that $\text{Ann}(A_1 \rtimes A_0) = 0$. Now suppose that $A_i^2 = A_i, i = 0, 1$; we have to show that $(A_1 \rtimes A_0)^2 = A_1 \rtimes A_0$. Suppose $(a_1, a_0) \in A_1 \rtimes A_0$, where $a_0 = a_0^{11} * a_0^{12} + \dots + a_0^{i1} * a_0^{i2}$ and $a_1 = a_1^{11} * a_1^{12} + \dots + a_1^{j1} * a_1^{j2}$. We have the following equalities

$$(a_1, a_0) = (0, a_0^{11} * a_0^{12} + \dots + a_0^{i1} * a_0^{i2}) + (a_1^{11} * a_1^{12} + \dots + a_1^{j1} * a_1^{j2}, 0) = (0, a_0^{11}) * (0, a_0^{12}) + \dots + (0, a_0^{i1}) * (0, a_0^{i2}) + (a_1^{11}, 0) * (a_1^{12}, 0) + \dots + (a_1^{j1}, 0) * (a_1^{j2}, 0),$$

which proves that $(A_1 \rtimes A_0)^2 = A_1 \rtimes A_0$.

4.1. PROPOSITION. *If A satisfies Condition 2, then there is an isomorphism*

$$(\mathfrak{A}(A), \overline{\omega}_0, \overline{\omega}_1) \approx (\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)}).$$

PROOF. It is a well-known fact that under Condition 2 the action of $\text{Bim}(A)$ on A is a derived action in the category of associative algebras [Lavendhomme and Lucas, 1996, Casas, Datuashvili and Ladra, 2010, Borceux, Janelidze and Kelly, 2005*]. In an analogous way, from the definition of action of $\mathfrak{A}(A)$ on A , one can easily see that this action is a derived action in \mathbf{Ass} and in $\mathbf{Cat}^1\text{-Ass}$ as well. Therefore, since $(\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)})$ has general actor property, by definition (see Definition 3.27) there exists a unique morphism $\varphi : \mathfrak{A}(A) \rightarrow \mathfrak{B}(A)$ in $\mathbf{Cat}^1\text{-Ass}_G$ such that $\varphi((f, f^0, f^1)) * a = (f, f^0, f^1) * a$ and

$a*\varphi((f, f^0, f^1)) = a*(f, f^0, f^1)$, for all $a \in A$ and $(f, f^0, f^1) \in \mathfrak{A}(A)$. We have shown above that there exists an injective homomorphism $\psi : (\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)}) \hookrightarrow (\mathfrak{A}(A), \overline{w_0}, \overline{w_1})$. For any $b \in (\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)})$ and $a \in A$ we have $\psi(b)*a = b*a$ and $a*\psi(b) = a*b$. By the construction of $(\mathfrak{A}(A), \overline{w_0}, \overline{w_1})$ its action on (A, ω_0, ω_1) is strict. By Theorem 3.32.2. the action of $(\mathfrak{B}(A), w_0^{\mathfrak{B}(A)}, w_1^{\mathfrak{B}(A)})$ on (A, ω_0, ω_1) is also strict. From these facts it follows that $\varphi\psi = 1$ and $\psi\varphi = 1$. Therefore we find that φ is an isomorphism. ■

4.2. COROLLARY. *If A satisfies Condition 2, then $(\mathfrak{A}(A), \overline{w_0}, \overline{w_1})$ is an actor of (A, ω_0, ω_1) .*

PROOF. Since $(\mathfrak{A}(A), \overline{w_0}, \overline{w_1}) \in \mathbf{Cat}^1\text{-Ass}$ and its action on A is a derived action in this category, the semidirect product $(A, \omega_0, \omega_1) \rtimes (\mathfrak{A}(A), \overline{w_0}, \overline{w_1}) \in \mathbf{Cat}^1\text{-Ass}$. Now the result follows from Proposition 4.1 and Theorems 3.31 and 3.32.3. ■

4.1 Actor of a cat^1 -algebra corresponding to a given crossed module

Let A_1, A_0 be associative algebras with a derived action of A_0 on A_1 . Let $f = (f_l, f_r) \in \text{Bim}(A_1 \rtimes A_0)$. Then $f_l : A_1 \rtimes A_0 \rightarrow A_1 \rtimes A_0$ can be represented by four k -linear maps

$$\alpha_l : A_1 \rightarrow A_1, \quad \gamma_l : A_1 \rightarrow A_0, \quad \beta_l : A_0 \rightarrow A_0, \quad \partial_l : A_0 \rightarrow A_1$$

such that

$$f_l(a_1, a_0) = (\alpha_l(a_1) + \partial_l(a_0), \beta_l(a_0) + \gamma_l(a_1))$$

for all $a_1 \in A_1, a_0 \in A_0$. Also, $f_r : A_1 \rtimes A_0 \rightarrow A_1 \rtimes A_0$ can be represented by four k -linear maps

$$\alpha_r : A_1 \rightarrow A_1, \quad \gamma_r : A_1 \rightarrow A_0, \quad \beta_r : A_0 \rightarrow A_0, \quad \partial_r : A_0 \rightarrow A_1$$

such that

$$f_r(a_1, a_0) = (\alpha_r(a_1) + \partial_r(a_0), \beta_r(a_0) + \gamma_r(a_1))$$

for all $a_1 \in A_1, a_0 \in A_0$. Let $f = (f_l, f_r) \in \text{Bim}(A_1 \rtimes A_0)$ and suppose f satisfies the condition M1, for $i = 0$ and certain $f^0 \in \text{Bim}(A_1 \rtimes A_0)$. Then we find that $\gamma_r = \gamma_l = 0$ and so we can represent f by the triple $(\alpha, \partial, \beta)$. One can easily see that $\alpha \in \text{Bim}(A_1)$, $\beta \in \text{Bim}(A_0)$ and we have the following conditions

A1. $\alpha_l(a_0 * a_1) = \partial_l(a_0) * a_1 + \beta_l(a_0) * a_1,$

A2. $\alpha_l(a_1 * a_0) = \alpha_l(a_1) * a_0,$

A3. $a_0 * \alpha_l(a_1) = \partial_r(a_0) * a_1 + \beta_r(a_0) * a_1,$

A4. $\alpha_r(a_1 * a_0) = a_1 * \partial_r(a_0) + a_1 * \beta_r(a_0),$

A5. $\alpha_r(a_0 * a_1) = a_0 * \alpha_r(a_1),$

A6. $\alpha_r(a_1) * a_0 = a_1 * \partial_l(a_0) + a_1 * \beta_l(a_0),$

A7. $\partial_r(a_0 * a'_0) = a_0 * \partial_r(a'_0),$

A8. $\partial_l(a_0 * a'_0) = \partial_l(a_0) * a'_0,$

A9. $a_0 * \partial_l(a'_0) = \partial_r(a_0) * a'_0,$

for all $a_1 \in A_1, a_0, a'_0 \in A_0.$

We have an analogous result for the representations of bimultipliers f^0 and f^1 from any triple $(f, f^0, f^1) \in \mathfrak{A}(A_1 \rtimes A_0),$ since we have showed that (f^0, f^0, f^0) and (f^1, f^1, f^1) are elements in $\mathfrak{A}(A_1 \rtimes A_0),$ and therefore f^0 and f^1 satisfy condition M1 for $i = 0.$

Note that in the case where the semidirect product $A_1 \rtimes A_0$ corresponds to a certain crossed module $A_1 \xrightarrow{d} A_0,$ Conditions A7-A9 are crossed bimultiplier conditions of the crossed module (Section 2, Definition 2.6).

4.3. NOTATION. In the rest of the paper, \mathcal{A} will denote a crossed module $\mathcal{A} : (A_1 \xrightarrow{d} A_0),$ $S(\mathcal{A})$ will denote the corresponding cat¹-algebra $(A_1 \rtimes A_0, \omega_1, \omega_0).$

4.4. PROPOSITION. *Let $f := (\alpha, \partial, \beta), f^0 := (\alpha^0, \partial^0, \beta^0), f^1 := (\alpha^1, \partial^1, \beta^1)$ be bimultipliers of $A_1 \rtimes A_0.$ Then $(f, f^0, f^1) \in \mathfrak{A}(A_1 \rtimes A_0)$ if and only if (f, f^0, f^1) satisfies the following conditions.*

1. $\beta_{l,r}(a_0) = \beta_{l,r}^0(a_0),$
2. $\partial_{l,r}^i(a_0) = 0,$ for $i = 0, 1,$
3. $\beta_{l,r}^1(a_0) = d\partial_{l,r}(a_0) + \beta_{l,r}(a_0),$
4. $\beta_{l,r}^1 d(a_1) = d\alpha_{l,r}(a_1),$
5. $\beta_{l,r}^i d(a_1) = d\alpha_{l,r}^i(a_1),$ for $i = 0, 1,$
6. $\alpha_{l,r}(a_1) = \alpha_{l,r}^1(a_1),$
7. $\alpha_{l,r}(a_1) = \alpha_{l,r}^0(a_1) + \partial_{l,r} d(a_1),$

for all $a_0 \in A_0, a_1 \in A_1.$

PROOF. Let $(f, f^0, f^1) \in \mathfrak{A}(A_1 \rtimes A_0).$ All the properties 1-7 follow from the conditions M1-M4 and the presentations of the bimultipliers $f_{l,r}, f_{l,r}^0,$ and $f_{l,r}^1$ in terms of the corresponding linear maps $\alpha, \beta, \partial,$ given in this section. We will demonstrate property 6.

We have

$$f_{l,r}(a_1, a_0) = (\alpha_{l,r}(a_1) + \partial_{l,r}(a_0), \beta_{l,r}(a_0)).$$

For any $a_1 \in A_1$ we obtain

$$f_{l,r}(a_1, 0) = (\alpha_{l,r}(a_1), 0).$$

Analogously,

$$f_{l,r}^1(a_1, 0) = (\alpha_{l,r}^1(a_1), 0).$$

Now the result follows from M3.

The converse statement of the proposition is proved by direct checking and is left to the reader. ■

4.5. COROLLARY. $\ker \bar{\omega}_0 = \{(f, 0, f^1) \in \mathfrak{A}(A_1 \times A_0)\}$, and any element $(f, 0, f^1) \in \ker \bar{\omega}_0$ can be represented by $((\alpha, \partial, 0), (0, 0, 0), (\alpha, 0, \beta^1))$.

PROOF. Follows from the definition of the map $\bar{\omega}_0$ and Proposition 4.4. ■

4.6. PROPOSITION. For any $((\alpha, \partial, 0), (0, 0, 0), (\alpha, 0, \beta^1)) \in \ker \bar{\omega}_0$ we have:

1. (α, β^1) is a bimultiplier of \mathcal{A} ,
2. $\alpha_{l,r}(a_1) = \partial_{l,r}d(a_1)$,
3. $\beta_{l,r}^1(a_0) = d\partial_{l,r}(a_0)$,

for all $a_0 \in A_0, a_1 \in A_1$.

PROOF. 1. By Proposition 4.4, condition 6, we have $\alpha = \alpha^1$ and by A1-A6 we obtain that (α^1, β^1) satisfies conditions of Definition 2.4.

Property 2. Follows from Proposition 4.4, condition 7

Property 3. Follows from Proposition 4.4, conditions 1, 2 and 3 and from the fact that $\beta_{l,r}^0 = 0$ ■

5. Split extension classifier of a crossed module

According to the definition of action in semi-abelian categories [Borceux, Janelidze and Kelly, 2005*], it is natural to define an action (i.e. a derived action) in \mathbf{XAss} in an analogous way as it is defined in a modified category of interest, thus as an action derived from a split extension in this category. In this section we will construct a crossed module Δ for a given crossed module $\mathcal{A} : A_1 \xrightarrow{d} A_0$ and prove, that if A_0 and A_1 satisfy Condition 2, then Δ is isomorphic to $P(\text{Act}(S(\mathcal{A})))$, which means that Δ is the split extension classifier of \mathcal{A} .

5.1. LEMMA. *The bilinear maps*

$$\begin{aligned} \text{Bim}(\mathcal{A}) \times \mathcal{BM}(\mathcal{A}) &\longrightarrow \mathcal{BM}(\mathcal{A}), \\ \mathcal{BM}(\mathcal{A}) \times \text{Bim}(\mathcal{A}) &\longrightarrow \mathcal{BM}(\mathcal{A}), \end{aligned}$$

given by

$$\begin{aligned} ((\alpha', \beta'), \partial) &\longmapsto \bar{\partial}, \\ (\partial, (\alpha', \beta')) &\longmapsto \bar{\bar{\partial}}, \end{aligned}$$

define a derived action of $\text{Bim}(\mathcal{A})$ on $\mathcal{BM}(\mathcal{A})$ in the category of associative algebras, where

$$\bar{\partial}_l = \alpha'_l \partial_l, \bar{\partial}_r = \partial_r \beta'_r$$

and

$$\bar{\bar{\partial}}_l = \partial_l \beta'_l, \bar{\bar{\partial}}_r = \alpha'_r \partial_r,$$

for all $(\alpha', \beta') \in \text{Bim}(\mathcal{A}), \partial \in \text{Bim}(A_0, A_1)$.

PROOF. The proof is a direct consequence of the definitions. ■

Define a map $\Delta : \mathcal{BM}(\mathcal{A}) \rightarrow \text{Bim}(\mathcal{A})$ by $\Delta(\partial) = (\alpha, \beta)$, for all $\partial \in \mathcal{BM}(\mathcal{A})$, where $\alpha = \partial d, \beta = d\partial$.

5.2. PROPOSITION. $\Delta : \mathcal{BM}(\mathcal{A}) \rightarrow \text{Bim}(\mathcal{A})$ is a crossed module with the action defined in Lemma 5.1.

PROOF. Direct checking by using the definitions. ■

5.3. PROPOSITION. $\ker \bar{\omega}_0 \cong \mathcal{BM}(\mathcal{A})$.

PROOF. By Corollary 4.5 and Proposition 4.6 any element of $\ker \bar{\omega}_0$ has the form $((\partial d, \partial, 0), (0, 0, 0), (\partial d, 0, d\partial))$, where $\partial \in \mathcal{BM}(\mathcal{A})$, from which follows the result. ■

5.4. PROPOSITION. $\text{Im} \bar{\omega}_0 \cong \text{Bim}(\mathcal{A})$.

PROOF. We have

$$\text{Im} \bar{\omega}_0 = \{(f^0, f^0, f^0) : (f, f^0, f^1) \in \mathfrak{A}(A_1 \rtimes A_0)\}.$$

By Proposition 4.4.2 we have $\partial_{l,r}^0 = 0$. Therefore we can represent an element of $\text{Im} \bar{\omega}_0$ by $((\alpha^0, 0, \beta^0), (\alpha^0, 0, \beta^0), (\alpha^0, 0, \beta^0))$, where $\alpha^0 \in \text{Bim}(A_1), \beta^0 \in \text{Bim}(A_0)$. From properties A1-A6 it follows that (α^0, β^0) is a bimultiplier of the crossed module $\mathcal{A} : (A_1 \xrightarrow{d} A_0)$. Conversely, any bimultiplier $(\alpha, \beta) \in \text{Bim}(\mathcal{A})$ can be considered as an element of $\text{Im} \bar{\omega}_0$. Namely, define $f^0 := (\alpha, 0, \beta)$. Since $f^0 \in \text{Bim}(A_1 \rtimes A_0)$, by Proposition 4.4 we have $(f^0, f^0, f^0) \in \mathfrak{A}(A_1 \rtimes A_0)$ and $\bar{\omega}_0(f^0, f^0, f^0) = (f^0, f^0, f^0)$, as required. ■

5.5. PROPOSITION. $\Delta \cong P(\mathfrak{A}(S(\mathcal{A})))$.

PROOF. Follows from Propositions 5.2, 5.3 and 5.4, since Δ is isomorphic to the restriction of $\bar{\omega}_1$. ■

5.6. THEOREM. If A_0 and A_1 satisfy Condition 2, then the crossed module $\Delta : \mathcal{BM}(\mathcal{A}) \rightarrow \text{Bim}(\mathcal{A})$ defined in Proposition 5.2 is the split extension classifier of the crossed module \mathcal{A} .

PROOF. As we have proved in Section 4, if A_0 and A_1 satisfy Condition 2, then the semidirect product $A_1 \rtimes A_0$ also satisfies this condition. Now the result follows from Corollary 4.2, Proposition 5.5 and the fact that P and S define an equivalence of the corresponding categories. ■

5.7. EXAMPLE. Let A_1 be an associative algebra which satisfies Condition 2, then $\mathcal{A} : (A_1 \xrightarrow{id} A_1)$ is a crossed module and the split extension classifier of \mathcal{A} is the crossed module $\Delta : \mathcal{BM}(\mathcal{A}) \rightarrow \text{Bim}(\mathcal{A})$, where $\mathcal{BM}(\mathcal{A}) \cong \text{Bim}(A_1)$, $\text{Bim}(\mathcal{A}) = \{(\alpha, \alpha) \mid \alpha \in \text{Bim}(A_1)\} \cong \text{Bim}(A_1)$ and Δ is defined by $\Delta(\partial) = (\partial, \partial)$. So $[A] \cong (\text{Bim}(A_1), \text{Bim}(A_1), \text{id})$.

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