DELIGNE GROUPOID REVISITED

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ABSTRACT. We show that for a differential graded Lie algebra $g$ whose components vanish in degrees below $-1$ the nerve of the Deligne 2-groupoid is homotopy equivalent to the simplicial set of $g$-valued differential forms introduced by V. Hinich [Hinich, 1997].

1. Introduction

The principal result of the present note compares two spaces (simplicial sets) naturally associated with a nilpotent differential graded Lie algebra (DGLA) subject to certain restrictions. Our interest in this problem has its origins in formal deformation theory of associative algebras and, more generally, algebroid stacks ([Bressler, Gorokhovsky, Nest & Tsygan, 2007]). The results of the present note are used in [Bressler, Gorokhovsky, Nest & Tsygan, 2015] to deduce a quasi-classical description of the deformation theory of a gerbe from the formality theorem of M. Kontsevich ([Kontsevich, 2003]).

To a nilpotent DGLA $h$ which satisfies the additional condition

$$h^i = 0 \text{ for } i < -1$$

P. Deligne [Deligne, 1994] and, independently, E. Getzler [Getzler, 2009] associated a (strict) 2-groupoid which we denote $MC^2(h)$ and refer to as the Deligne 2-groupoid.

Our principal result (Theorem 4.2) compares the simplicial nerve $\mathfrak{N}MC^2(h)$ of the 2-groupoid $MC^2(h)$, $h$ a nilpotent DGLA satisfying (1), to another simplicial set, denoted $\Sigma(h)$, introduced by V. Hinich [Hinich, 1997]:

1.1. Theorem. (Main theorem) Suppose that $h$ is a nilpotent DGLA such that $h^i = 0$ for $i < -1$. Then, the simplicial sets $\mathfrak{N}MC^2(h)$ and $\Sigma(h)$ are weakly homotopy equivalent.

In the case when the nilpotent DGLA $h$ satisfies $h^i = 0$ for $i < 0$ and, consequently, $MC^2(h)$ is an ordinary groupoid a homotopy equivalence between $\Sigma(h)$ and the nerve of $MC^2(h)$ was constructed by V. Hinich in [Hinich, 1997].

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Differential graded Lie algebras satisfying (1) arise in formal deformation theory of algebraic structures such as Lie algebras, commutative algebras, associative algebras to name a few. In what follows we shall concentrate on the latter example. Let $k$ denote an algebraically closed field of characteristic zero. For an associative algebra $A$ over $k$ the shifted Hochschild cochain complex $C^\bullet(A)[1]$ has a canonical structure of a DGLA under the Gerstenhaber bracket; we denote this DGLA by $g(A)$ for short. Suppose that $\mathfrak{m}$ is a nilpotent commutative $k$-algebra (without unit). Then, $g(A) \otimes_k \mathfrak{m}$ is a nilpotent DGLA which satisfies (1). Thus, the Deligne 2-groupoid $MC^2(g(A) \otimes_k \mathfrak{m})$ is defined. For an Artin $k$-algebra $R$ with maximal ideal $m_R$ the 2-groupoid $MC^2(g(A) \otimes_k m_R)$ is naturally equivalent to the 2-groupoid of $R$-deformations of the algebra $A$. In this sense the DGLA $g(A)$ controls the formal deformation theory of $A$.

The reason for considering the space $\Sigma(\mathfrak{h})$ is that it is defined not just for a DGLA (V. Hinich, [Hinich, 1997]), but, more generally, for any nilpotent $L_\infty$ algebra (E. Getzler, [Getzler, 2009]). Homotopy invariance properties of the functor $\Sigma$ (Proposition 3.9), the theory of J.W. Duskin ([Duskin, 2001/02]) and the theorem above yield the following result. If $\mathfrak{h}$ is a DGLA satisfying (1), $g$ is a $L_\infty$ algebra $L_\infty$-quasi-isomorphic to $\mathfrak{h}$ and $\mathfrak{m}$ is a nilpotent commutative $k$-algebra, then $\mathcal{M}MC^2(\mathfrak{h} \otimes_k \mathfrak{m})$ is homotopy equivalent to $\Sigma(\mathfrak{g} \otimes_k \mathfrak{m})$. Thus, the 2-groupoid $MC^2(\mathfrak{h} \otimes_k \mathfrak{m})$ can be reconstructed, up to equivalence, from the space $\Sigma(\mathfrak{g} \otimes_k \mathfrak{m})$. The situation envisaged above arises naturally. Any DGLA $\mathfrak{h}$ is $L_\infty$-quasi-isomorphic to an $L_\infty$ algebra with trivial univalent operation (the differential).

The paper is organized as follows. In Section 2 we review various constructions of nerves of 2-groupoids and their properties. In section 3 we recall the definitions of the functor $\Sigma$ (3.4) and of the Deligne 2-groupoid (3.10) and prove basic properties thereof. The proof of the main theorem (Theorem 4.2) given in Section 4 proceeds by exhibiting canonical weak homotopy equivalences from $\Sigma(\mathfrak{h})$ and $\mathcal{M}MC^2(\mathfrak{h})$ to a third naturally defined simplicial set.

2. The homotopy type of a strict 2-groupoid
2.1. Nerves of simplicial groupoids.

2.1.1. Simplicial groupoids. In what follows a simplicial category is a category enriched over the category of simplicial sets. A small simplicial category consists of a set of objects and a simplicial set of morphisms for each pair of objects.

A simplicial category $\mathcal{G}$ is a particular case of a simplicial object $[p] \mapsto \mathcal{G}_p$ in $\text{Cat}$ whose simplicial set of objects $[p] \mapsto N_0\mathcal{G}_p$ is constant.

A simplicial category is a simplicial groupoid if it is a groupoid in each (simplicial) degree.

2.1.2. The naïve nerve. Suppose that $\mathcal{G}$ is a simplicial category. Applying the nerve functor degree-wise we obtain the bi-simplicial set $N\mathcal{G} : ([p], [q]) \mapsto N_q\mathcal{G}_p$ whose diagonal we denote by $N\mathcal{G}$ and refer to as the naïve nerve of $\mathcal{G}$. 
2.1.3. The simplicial nerve. For a simplicial category $G$ the simplicial nerve, also known as the homotopy coherent nerve, $\mathcal{N}G$ is represented by the cosimplicial object in $[p] \mapsto \Delta^p_{\mathcal{N}} \in \text{Cat}_{\Delta}$, i.e.

$$\mathcal{N}G = \text{Hom}_{\text{Cat}_{\Delta}}(\Delta^p_{\mathcal{N}}, G).$$

Here, $\Delta^p_{\mathcal{N}}$ is the canonical free simplicial resolution of $[p]$ which admits the following explicit description ([Cordier, 1982]).

The set of objects of $\Delta^p_{\mathcal{N}}$ is $\{0, 1, \ldots, p\}$. For $0 \leq i \leq j \leq p$ the simplicial set of morphisms is given by $\text{Hom}_{\Delta^p_{\mathcal{N}}}(i, j) = N\mathcal{P}(i, j)$. The category $\mathcal{P}(i, j)$ is a sub-poset of $2^{\{0,\ldots,p\}}$ (with the induced partial ordering whereby viewed as a category) given by

$$\mathcal{P}(i, j) = \{I \subset \mathbb{Z} \mid (i, j \in I) \& (k \in I \implies i \leq k \leq j)\}.$$

The composition in $\Delta^p_{\mathcal{N}}$ is induced by functors

$$\mathcal{P}(i, j) \times \mathcal{P}(j, k) \to \mathcal{P}(i, k) : (I, J) \mapsto I \cup J.$$

In particular, $\Delta^0_{\mathcal{N}} = [0]$ and $\Delta^1_{\mathcal{N}} = [1]$

We refer the reader to [Hinich, 2007] for applications to deformation theory and to [Lurie, 2009] for the connection with higher category theory. The simplicial nerve of a simplicial groupoid is a Kan complex which reduces to the usual nerve for ordinary groupoids.

Since $\Delta^0_{\mathcal{N}} = [0]$ (respectively, $\Delta^1_{\mathcal{N}} = [1]$) it follows that $\mathcal{N}_0G$ (respectively, $\mathcal{N}_1G$) is the set of objects (respectively, the set of morphisms) of $G_0$.

2.1.4. Comparison of nerves. We refer the reader to [Hinich, 2007] for the definition of the canonical map of simplicial sets $\mathcal{N}G \to \mathcal{N}G$. In what follows we will make use of the following result of loc. cit.

2.2. Theorem. ([Hinich, 2007], Corollary 2.6.3) For any simplicial groupoid $G$ the canonical map $\mathcal{N}G \to \mathcal{N}G$ is a weak homotopy equivalence.

2.3. Strict 2-groupoids.

2.3.1. From strict 2-groupoids to simplicial groupoids. Suppose that $G$ is a strict 2-groupoid, i.e. a groupoid enriched over the category of groupoids. Thus, for every $g, g' \in G$, we have the groupoid $\text{Hom}_G(g, g')$ and the composition is strictly associative.

The nerve functor $[p] \mapsto N_p(\cdot) := \text{Hom}_{\text{Cat}}([p], \cdot)$ commutes with products. Let $G_p$ denote the category with the same objects as $G$ and with morphisms defined by $\text{Hom}_{G_p}(g, g') = N_p \text{Hom}_G(g, g')$; the composition of morphisms is induced by the composition in $G$. Note that the groupoid $G_0$ is obtained from $G$ by forgetting the 2-morphisms.

The assignment $[p] \mapsto G_p$ defines a simplicial object in groupoids with the constant simplicial set of objects, i.e. a simplicial groupoid which we denote by $\mathcal{G}$. 

2.4. Lemma. The simplicial nerve \( \mathfrak{N}\tilde{G} \) admits the following explicit description:

1. There is a canonical bijection between \( \mathfrak{N}_0\tilde{G} \) and the set of objects of \( \mathcal{G} \).

2. For \( n \geq 1 \) there is a canonical bijection between \( \mathfrak{N}_n\tilde{G} \) and the set of data of the form 

\[
((\mu_1)_0 \leq i \leq n, (g_{ij})_0 \leq i < j \leq n, (c_{ijk})_0 \leq i < j < k \leq n), \text{ where } (\mu_i) \text{ is an } (n+1)\text{-tuple of objects of } \mathcal{G},
\]

\( (g_{ij}) \) is a collection of 1-morphisms \( g_j : \mu_j \rightarrow \mu_i \) and \( (c_{ijk}) \) is a collection of 2-morphisms \( c_{ijk} : g_{ij}g_{jk} \rightarrow g_{ik} \) which satisfies

\[
c_{ijkl} = c_{ikl}c_{ijk} \quad (2)
\]

(in the set of 2-morphisms \( g_{ij}g_{jk}g_{kl} \rightarrow g_{il} \)).

For a morphism \( f : [m] \rightarrow [n] \) in \( \Delta \) the induced structure map \( f^* : \mathfrak{N}\tilde{G} \rightarrow \mathfrak{N}_{n}\tilde{G} \) is given (under the above bijection) by \( f^*(\mu_i, (g_{ij}), (c_{ijk})) = ((\nu_i), (h_{ij}), (d_{ijk})), \) where \( \nu_i = \mu_{f(i)}, h_{ij} = g_{f(i), f(j)} \), \( d_{ijk} = c_{f(i), f(j), f(k)} \) (cf. [Duskin, 2001/02]).

Proof. An \( n \)-simplex of \( \mathfrak{N}\tilde{G} \) is the following collection of data:

1. objects \( \mu_0, \ldots, \mu_n \) of \( \mathcal{G} \);

2. morphisms of simplicial sets \( N\mathcal{P}(i, j) \rightarrow N\text{Hom}_{\mathcal{G}}(\mu_i, \mu_j) \) intertwining the maps induced on the nerves by composition functors \( \mathcal{P}(i, j) \times \mathcal{P}(j, k) \rightarrow \mathcal{P}(i, k) \) and \( \text{Hom}_{\mathcal{G}}(\mu_i, \mu_j) \times \text{Hom}_{\mathcal{G}}(\mu_j, \mu_k) \rightarrow \text{Hom}_{\mathcal{G}}(\mu_i, \mu_k) \).

Since the nerve functor is fully faithful, the above data are equivalent to the following:

1. objects \( \mu_0, \ldots, \mu_n \) of \( \mathcal{G} \);

2. for any \( I \in N_0\mathcal{P}(i, j) \), a 1-morphism \( g_I : \mu_j \rightarrow \mu_i \) in \( \mathcal{G} \);

3. for any morphism \( J \rightarrow I \) in \( \mathcal{P}(i, j) \), a 2-morphism \( c_{IJ} : g_J \rightarrow g_I \), such that

\[
c_{IJ}c_{JK} = c_{IK} \quad (3)
\]

These data have to be compatible with the composition pairings \( \mathcal{P}(i, j) \times \mathcal{P}(j, k) \rightarrow \mathcal{P}(i, k) \) and \( \text{Hom}_{\mathcal{G}}(\mu_i, \mu_j) \times \text{Hom}_{\mathcal{G}}(\mu_j, \mu_k) \rightarrow \text{Hom}_{\mathcal{G}}(\mu_i, \mu_k) \).

Let \( g_{ij} : \mu_j \rightarrow \mu_i \) denote the morphism \( g_{i,j} \). By compatibility with compositions, if \( I = \{i, i_1, \ldots, i_k, j\} \) then \( g_I = g_{ii_1} \cdots g_{i_kj} \). Let \( c_{ijk} \) denote the two-morphism \( c_{\{i,j,k\}, \{i,k\}} : g_{ik} \rightarrow g_{ij}g_{jk} \). Now, by virtue of (3) and of compatibility with compositions, \( c_{ijk} \) satisfy the two-cocycle identity (3) and determine \( c_{IJ} \) for any \( I, J \). \(\blacksquare\)
In what follows, for a strict 2-groupoid \( \mathcal{G} \), we will denote by \( \mathcal{N} \mathcal{G} \) (respectively \( \mathcal{N} \mathcal{G} \)) the naïve (respectively simplicial) nerve of the associated simplicial groupoid \( \tilde{\mathcal{G}} \).

3. Homotopy types associated with \( L_\infty \)-algebras

3.1. \( L_\infty \)-algebras. We follow the notation of [Getzler, 2009] and refer the reader to loc. cit. for details.

Recall that an \( L_\infty \)-algebra is a graded vector space \( g \) equipped with operations

\[ \bigwedge^k g \rightarrow g[2-k] : x_1 \wedge \ldots \wedge x_k \mapsto [x_1, \ldots, x_k] \]

defined for \( k = 1, 2, \ldots \) which satisfy a sequence of Jacobi identities.

It follows from the Jacobi identities that the unary operation \([.] : g \rightarrow g[1]\) is a differential, which we will denote by \( \delta \).

An \( L_\infty \)-algebra is abelian if all operations with valency two and higher (i.e. all operations except for \( \delta \)) vanish. In other words, an abelian \( L_\infty \)-algebra is a complex. An \( L_\infty \)-algebra structure with vanishing operations of valency three and higher reduces to a structure of a DGLA.

The lower central series of an \( L_\infty \)-algebra \( g \) is the canonical decreasing filtration \( F^*g \) with \( F^0g = g \) for \( i \leq 1 \) and defined recursively for \( i \geq 1 \) by

\[ F^{i+1}g = \sum_{k=2}^{\infty} \sum_{i_1 + \ldots + i_k = i} [F^{i_1}g, \ldots, F^{i_k}g]. \]

An \( L_\infty \)-algebra is nilpotent if there exists an \( i \) such that \( F^ig = 0 \).

3.1.1. Maurer-Cartan elements. Suppose that \( g \) is a nilpotent \( L_\infty \)-algebra. For \( \mu \in g^1 \) let

\[ \mathcal{F}(\mu) = \delta \mu + \sum_{k=2}^{\infty} \frac{1}{k!} [\mu^{\wedge k}]. \] (4)

The element \( \mathcal{F}(\mu) \) of \( g^2 \) is called the curvature of \( \mu \). For any \( \mu \in g^1 \) the curvature \( \mathcal{F}(\mu) \) satisfies the Bianchi identity ([Getzler, 2009], Lemma 4.5)

\[ \delta \mathcal{F}(\mu) + \sum_{k=1}^{\infty} \frac{1}{k!} [\mu^{\wedge k}, \mathcal{F}(\mu)] = 0. \] (5)

An element \( \mu \in g^1 \) is called a Maurer-Cartan element (of \( g \)) if it satisfies the condition \( \mathcal{F}(\mu) = 0 \). The set of Maurer-Cartan elements of \( g \) will be denoted \( \text{MC}(g) \):

\[ \text{MC}(g) := \{ \mu \in g^1 \mid \mathcal{F}(\mu) = 0 \}. \]

The set \( \text{MC}(g) \) is pointed by the distinguished element \( 0 \in g^1 \).

Suppose that \( a \) is an abelian \( L_\infty \)-algebra. Then,

\[ \text{MC}(a) = Z^1(a) := \ker(\delta : a^1 \rightarrow a^2), \]

hence is equipped with a canonical structure of an abelian group.
3.1.2. Central Extensions. Suppose that \( g \) is an \( L_\infty \)-algebra and \( a \) is a subcomplex of \((g, \delta)\) such that \([a \wedge g^\otimes k] = 0\) for all \( k \geq 1\). In this case we will say that \( a \) is central in \( g \).

If \( a \) is central in \( g \), then there is a unique structure of an \( L_\infty \)-algebra on \( g/a \) such that the projection \( g \to g/a \) is a map of \( L_\infty \)-algebras. If \( g \) is nilpotent, then so is \( g/a \).

In what follows we assume that \( g \) is a nilpotent \( L_\infty \)-algebra and \( a \) is central in \( g \).

3.2. Lemma.

1. The addition operation on \( g^1 \) restricts to a free action of the abelian group \( MC(a) \) on the set \( MC(g) \).

2. The map \( MC(g) \to MC(g/a) \) is constant on the orbits of the action.

3. The induced map \( MC(g)/MC(a) \to MC(g/a) \) is injective.

Proof. Suppose that \( \alpha \in a^1 \) and \( \mu \in g^1 \). Since \( a \) is central in \( g \), \([(\alpha + \mu)^\otimes k] = [\mu^\otimes k] \) for \( k \geq 2 \) and \( F(\alpha + \mu) = \delta \alpha + F(\mu) \) (in the notation of (4)). Therefore, \( MC(a) + MC(g) = MC(g) \). In other words, the addition operation in \( g^1 \) restricts to an action of the abelian group \( MC(a) \) on the set \( MC(g) \) which is obviously free. Since the map \( MC(g) \to MC(g/a) \) is the restriction of the map \( g \to g/a \) constant on the orbits of the action, i.e. factors through \( MC(g)/MC(a) \), and the induced map \( MC(g)/MC(a) \to MC(g/a) \) is injective.

3.2.1. The Obstruction Map. The image of the map \( MC(g) \to MC(g/a) \) may be described in terms of the obstruction map (6) which we construct presently.

If \( \mu \in g^1 \) and \( \mu + a^1 \in MC(g/a) \), then \( F(\mu + a^1) = F(\mu) + \delta a^1 \subset a^2 \) and the Bianchi identity (5) reduces to \( \delta F(\mu + a^1) = 0 \), i.e. the assignment \( \mu + a^1 \mapsto F(\mu + a^1) \) gives rise to a well-defined map

\[
o_2: MC(g/a) \to H^2(a) \tag{6}\]

(abbreviation borrowed from [Goldman, Millson, 1988], 2.6).

3.3. Lemma. The sequence of pointed sets

\[
0 \to MC(g)/MC(a) \to MC(g/a) \xrightarrow{o_2} H^2(a) \tag{7}
\]

is exact.

Proof. If \( F(\mu + a^1) \subset \delta a^1 \), then there exists \( \alpha \in a^1 \) such that \( F(\mu + \alpha) = 0 \), i.e. \( \mu + a^1 \) is in the image of \( MC(g) \to MC(g/a) \).

3.4. The Functor \( \Sigma \). In what follows we denote by \( \Omega_n, n = 0, 1, 2, \ldots \) the commutative differential graded algebra over \( \mathbb{Q} \) with generators \( t_0, \ldots, t_n \) of degree zero and \( dt_0, \ldots, dt_n \) of degree one subject to the relations \( t_0 + \cdots + t_n = 1 \) and \( dt_0 + \cdots + dt_n = 0 \). The differential \( d: \Omega_n \to \Omega_n[1] \) is defined by \( t_i \mapsto dt_i \) and \( dt_i \mapsto 0 \). The assignment \([n] \mapsto \Omega_n \) extends in a natural way to a simplicial commutative differential graded algebra.
3.4.1. The simplicial set $\Sigma(g)$. For a nilpotent $L_\infty$-algebra $g$ and a non-negative integer $n$ let

$$\Sigma_n(g) = \text{MC}(g \otimes \Omega_n).$$

Equipped with structure maps induced by those of $\Omega_\bullet$, the assignment $n \mapsto \Sigma_n(g)$ defines a simplicial set denoted $\Sigma(g)$.

The simplicial set $\Sigma(g)$ was introduced by V. Hinich in [Hinich, 1997] for DGLA and used by E. Getzler in [Getzler, 2009] (where it is denoted $\text{MC}_\bullet(g)$) for general nilpotent $L_\infty$-algebras.

3.4.2. Abelian DGLA. If $a$ is an abelian $L_\infty$-algebra, then $\Sigma(a)$ is given by $\Sigma_n(a) = Z^1(\Omega_n \otimes a) = Z^0(\Omega_n \otimes a[1])$ and has a canonical structure of a simplicial abelian group. In particular, it is a Kan simplicial set.

Recall that the Dold-Kan correspondence associates to a complex of abelian groups $A$ a simplicial abelian group $K(A)$ defined by $K_n(A) = Z^0(C^\bullet([n]; A))$, the group of cocycles of (total) degree zero in the complex of simplicial cochains on the $n$-simplex with coefficients in $A$.

The integration map $\int : \Omega_n \otimes a \to C^\bullet([n]; a)$ induces a homotopy equivalence

$$\int : \Sigma(a) \to K(a[1]);$$

see [Getzler, 2009], Section 3. Thus, $\pi_i\Sigma(a) \cong H^{1-i}(a)$.

3.4.3. Central extensions. Suppose that $g$ is a nilpotent $L_\infty$-algebra and $a$ is a central subalgebra in $g$. Then, for $n = 0, 1, \ldots$, $\Omega_n \otimes a$ is central in $\Omega_n \otimes g$.

3.5. Lemma.

1. The addition operation on $(\Omega_n \otimes g)^1$ induces a principal action of the simplicial abelian group $\Sigma(a)$ on the simplicial set $\Sigma(g)$.

2. The map $\Sigma(g) \to \Sigma(g/a)$ factors through $\Sigma(g)/\Sigma(a)$.

3. The induced map $\Sigma(g)/\Sigma(a) \to \Sigma(g/a)$ is injective.

Proof. Follows from Lemma 3.2 and the naturality properties of the constructions in 3.1.2.

For $n = 0, 1, \ldots$ the map $([n] \to [0])^* : \mathbb{Q} \to \Omega_n$ is a quasi-isomorphism, with the quasi-inverse provided by the map induced by any morphism $[0] \to [n]$. Therefore, the map $a \to \Omega_n \otimes a$ is a quasi-isomorphism as well. The induced isomorphisms $H^2(a) \cong H^2(\Omega_n \otimes a)$ give rise to the isomorphism of the constant simplicial set $H^2(a)$ and $n \mapsto H^2(\Omega_n \otimes a)$.

The maps

$$o_{2,n} : \Sigma_n(g/a) = \text{MC}(\Omega_n \otimes g/a) \to H^2(\Omega_n \otimes a) \cong H^2(a)$$
assemble into the map of simplicial sets

\[ o_2: \Sigma(\mathfrak{g}/\mathfrak{a}) \to H^2(\mathfrak{a}). \]  

(9)

which factors as \( \Sigma(\mathfrak{g}/\mathfrak{a}) \to \pi_0\Sigma(\mathfrak{g}/\mathfrak{a}) \to H^2(\mathfrak{a}). \)

Let \( \Sigma(\mathfrak{g}/\mathfrak{a})_0 = o_2^{-1}(0) \). Thus, by (7), \( \Sigma(\mathfrak{g}/\mathfrak{a})_0 \) is a union of connected components of \( \Sigma(\mathfrak{g}/\mathfrak{a}) \) equal to the range of the map \( \Sigma(\mathfrak{g})/\Sigma(\mathfrak{a}) \to \Sigma(\mathfrak{g}/\mathfrak{a}) \).

It follows that the map \( \Sigma(\mathfrak{g}) \to \Sigma(\mathfrak{g}/\mathfrak{a})_0 \) is a principal fibration with group \( \Sigma(\mathfrak{a}) \), in particular, a Kan fibration ([May, 1967], Lemma 18.2).

3.6. Lemma. Suppose that \( \mathfrak{g} \) is a nilpotent \( L_\infty \)-algebra. Then, \( \Sigma(\mathfrak{g}) \) is a Kan simplicial set.

Proof. If \( \mathfrak{g} \) is an abelian \( L_\infty \)-algebra then \( \Sigma(\mathfrak{g}) \) is a simplicial group and therefore a Kan simplicial set.

Let \( F^\bullet \mathfrak{g} \) denote the lower central series. Assume that \( Gr^i_F \mathfrak{g} \neq 0 \) if and only if \( 0 \leq i \leq n \); that is, \( \mathfrak{g} \) is nilpotent of length \( n \). By induction assume that \( \Sigma(h) \) is a Kan simplicial set for any nilpotent \( L_\infty \)-algebra \( h \) of length at most \( n - 1 \).

Since \( \mathfrak{g} \) is nilpotent of length \( n \), it follows that \( F^n\mathfrak{g} = Gr^n\mathfrak{g} \) is central in \( \mathfrak{g} \) and \( \mathfrak{g}/F^n\mathfrak{g} \) is nilpotent of length \( n - 1 \). Therefore, \( \Sigma(\mathfrak{g}/F^n\mathfrak{g}) \) is a Kan simplicial set and so is \( \Sigma(\mathfrak{g}/F^n\mathfrak{g})_0 \). Since \( \Sigma(\mathfrak{g}) \to \Sigma(\mathfrak{g}/F^n\mathfrak{g})_0 \) is a Kan fibration it follows that \( \Sigma(\mathfrak{g}) \) is a Kan simplicial set as well.

3.7. Lemma. Suppose that \( \mathfrak{g} \) is a nilpotent \( L_\infty \)-algebra such that \( \mathfrak{g}^q = 0 \) for \( q \leq -k, k \) a positive integer. Then, for any connected component \( X \) of \( \Sigma(\mathfrak{g}) \), \( \pi_i(X) = 0 \) for \( i > k \).

Proof. Suppose that \( \mathfrak{g} \) is an abelian \( L_\infty \)-algebra. Then, \( \pi_i\Sigma(\mathfrak{g}) \cong H^{1-i}(\mathfrak{g}) \). For an \( L_\infty \)-algebra \( \mathfrak{g} \) which is not necessarily abelian the statement follows by induction on the nilpotency length, the abelian case establishing the base of the induction.

Let \( F^\bullet \mathfrak{g} \) denote the lower central series. Assume that \( Gr^i_F \mathfrak{g} \neq 0 \) if and only if \( 0 \leq i \leq n \); that is, \( \mathfrak{g} \) is nilpotent of length \( n \). By induction assume that the conclusion holds for all nilpotent \( L_\infty \)-algebras of length at most \( n - 1 \).

Since \( \mathfrak{g} \) is nilpotent of length \( n \), it follows that \( F^n\mathfrak{g} = Gr^n\mathfrak{g} \) is central in \( \mathfrak{g} \) and \( \mathfrak{g}/F^n\mathfrak{g} \) is nilpotent of length \( n - 1 \). Let \( X \subseteq \Sigma(\mathfrak{g}) \) be a connected component of \( \Sigma(\mathfrak{g}) \) and let \( Y \subseteq \Sigma(\mathfrak{g}/F^n\mathfrak{g}) \) be the image of \( X \) under the map induced by the quotient map \( \mathfrak{g} \to \mathfrak{g}/F^n\mathfrak{g} \). Then, \( X \to Y \) is a principal fibration with group the connected component of the identity in \( \Sigma(F^n\mathfrak{g}) \). The desired vanishing of higher homotopy groups of \( X \) follows from the induction hypotheses using the long exact sequence of homotopy groups.

3.7.1. Homotopy invariance.

3.8. Lemma. Suppose that \( f: \mathfrak{a} \to \mathfrak{b} \) is a quasi-isomorphism of abelian \( L_\infty \)-algebras. Then, the induced map \( \Sigma(f): \Sigma(\mathfrak{a}) \to \Sigma(\mathfrak{b}) \) is a weak homotopy equivalence.
Proof. Note that $\Sigma(f)$ is a morphism of simplicial abelian groups. It is sufficient to show that the maps $\pi_n \Sigma(f) : \pi_n \Sigma(a) \to \pi_n \Sigma(b)$ are isomorphisms for $n \geq 0$. To this end note that $\pi_n \Sigma(f)$ factors as the composition of isomorphisms

$$\pi_n \Sigma(a) \cong H^{1-n}(a) \xrightarrow{H^{1-n}(\Sigma(f))} H^{1-n}(b) \cong \pi_n \Sigma(b).$$

3.9. Proposition. ([Getzler, 2009], Proposition 4.9) Suppose that $f : g \to h$ is a quasi-isomorphism of $L_\infty$-algebras and $R$ is an Artin algebra with maximal ideal $m_R$. Then, the map $\Sigma(f \otimes \text{Id}) : \Sigma(g \otimes m_R) \to \Sigma(h \otimes m_R)$ is a weak homotopy equivalence.

Proof. We use induction on the nilpotency length of $m_R$, which is to say the largest integer $l$ such that $m^l_R \neq 0$.

If $m^2_R = 0$, then $f \otimes \text{Id} : g \otimes m_R \to h \otimes m_R$ is a quasi-isomorphism of abelian $L_\infty$-algebras and the claim follows from Lemma 3.8.

Suppose that $m^{l+1}_R = 0$. By the induction hypothesis

- the map $\Sigma(g \otimes m_{R}/m^l_R) \to \Sigma(h \otimes m_{R}/m^l_R)$ is a weak homotopy equivalence and
- the map $\pi_0 \Sigma(g \otimes m_{R}/m^l_R) \to \pi_0 \Sigma(h \otimes m_{R}/m^l_R)$ is a bijection.

The map $f \otimes \text{Id}_{m^l_R}$ is a quasi-isomorphism of abelian $L_\infty$-algebras, therefore the map $H^2(g \otimes m^l_R) \to H^2(h \otimes m^l_R)$ is an isomorphism. The commutativity of

$$\begin{array}{ccc}
\pi_0 \Sigma(g \otimes m_{R}/m^l_R) & \longrightarrow & \pi_0 \Sigma(h \otimes m_{R}/m^l_R) \\
\downarrow & & \downarrow \\
H^2(g \otimes m^l_R) & \longrightarrow & H^2(h \otimes m^l_R)
\end{array}$$

implies that the map

$$\pi_0 \Sigma(g \otimes m_{R}/m^l_R)_0 \to \pi_0 \Sigma(h \otimes m_{R}/m^l_R)_0$$

is a bijection. Therefore, the map

$$\Sigma(g \otimes m_{R}/m^l_R)_0 \to \Sigma(h \otimes m_{R}/m^l_R)_0$$

is a weak homotopy equivalence. The map $\Sigma(f)$ restricts to a map of principal fibrations

$$\begin{array}{ccc}
\Sigma(g \otimes m_{R}) & \longrightarrow & \Sigma(h \otimes m_{R}) \\
\downarrow & & \downarrow \\
\Sigma(g \otimes m_{R}/m^l_R)_0 & \longrightarrow & \Sigma(h \otimes m_{R}/m^l_R)_0
\end{array}$$

relative to the map of simplicial groups $\Sigma(g \otimes m^l_R) \to \Sigma(h \otimes m^l_R)$. The latter is a weak homotopy equivalence by Lemma 3.8. Therefore, so is the map $\Sigma(g \otimes m_{R}) \to \Sigma(h \otimes m_{R})$. \qed
3.10. Deligne groupoids.

3.10.1. Gauge transformations. Suppose that \( \mathfrak{h} \) is a nilpotent DGLA. Then, \( \mathfrak{h}^0 \) is a nilpotent Lie algebra. The unipotent group \( \exp \mathfrak{h}^0 \) acts on the space \( \mathfrak{h}^1 \) by affine transformations. The action of \( \exp X, X \in \mathfrak{h}^0 \), on \( \gamma \in \mathfrak{h}^1 \) is given by the formula

\[
(\exp X) \cdot \gamma = \gamma - \sum_{i=0}^{\infty} \frac{(\text{ad} X)^i}{(i + 1)!} (\delta X + [\gamma, X]).
\]

The effect of the above action on the curvature \( F(\gamma) = \delta \gamma + \frac{1}{2}[\gamma, \gamma] \) is given by

\[
F((\exp X) \cdot \gamma) = \exp(\text{ad} X)(F(\gamma)).
\]

3.10.2. The functor \( \text{MC}^1 \). Suppose that \( \mathfrak{h} \) is a nilpotent DGLA. It follows from (11) that gauge transformations (10) preserve the subset of Maurer-Cartan elements \( \text{MC}(\mathfrak{h}) \subseteq \mathfrak{h}^1 \).

We denote by \( \text{MC}^1(\mathfrak{h}) \) the Deligne groupoid (denoted \( C(\mathfrak{h}) \) in [Hinich, 1997]) defined as the groupoid associated with the action of the group \( \exp \mathfrak{h}^0 \) by gauge transformations on the set \( \text{MC}(\mathfrak{h}) \).

Thus, \( \text{MC}^1(\mathfrak{h}) \) is the category with the set of objects \( \text{MC}(\mathfrak{h}) \). For \( \gamma_1, \gamma_2 \in \text{MC}(\mathfrak{h}) \), \( \text{Hom}_{\text{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2) \) is the set of gauge transformations between \( \gamma_1, \gamma_2 \). The composition

\[
\text{Hom}_{\text{MC}^1(\mathfrak{h})}(\gamma_2, \gamma_3) \times \text{Hom}_{\text{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2) \to \text{Hom}_{\text{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_3)
\]

is given by the product in the group \( \exp(\mathfrak{h}^0) \).

3.10.3. The functor \( \text{MC}^2 \). For \( \mathfrak{h} \) as above satisfying the additional vanishing condition \( \mathfrak{h}^i = 0 \) for \( i < -1 \) we denote by \( \text{MC}^2(\mathfrak{h}) \) the Deligne 2-groupoid as defined by P. Deligne [Deligne, 1994] and independently by E. Getzler, [Getzler, 2009]. Below we review the construction of Deligne 2-groupoid of a nilpotent DGLA following [Getzler, 2009, Getzler, 2002] and references therein.

The objects and the 1-morphisms of \( \text{MC}^2(\mathfrak{h}) \) are those of \( \text{MC}^1(\mathfrak{h}) \). That is, for \( \gamma_1, \gamma_2 \in \text{MC}(\mathfrak{h}) \) the set \( \text{Hom}_{\text{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2) \) is the set of objects of the groupoid \( \text{Hom}_{\text{MC}^2(\mathfrak{h})}(\gamma_1, \gamma_2) \). The morphisms in \( \text{Hom}_{\text{MC}^2(\mathfrak{h})}(\gamma_1, \gamma_2) \) (i.e. the 2-morphisms of \( \text{MC}^2(\mathfrak{h}) \)) are defined as follows.

For \( \gamma \in \text{MC}(\mathfrak{h}) \) let \([\cdot, \cdot]_\gamma\) denote the Lie bracket on \( \mathfrak{h}^{-1} \) defined by

\[
[a, b]_\gamma = [a, \delta b + [\gamma, b]].
\]

Equipped with this bracket, \( \mathfrak{h}^{-1} \) becomes a nilpotent Lie algebra. We denote by \( \exp_\gamma \mathfrak{h}^{-1} \) the corresponding unipotent group, and by

\[
\exp_\gamma : \mathfrak{h}^{-1} \to \exp_\gamma \mathfrak{h}^{-1}
\]
the corresponding exponential map. If $\gamma_1$, $\gamma_2$ are two Maurer-Cartan elements, then the group $\exp_{\gamma_2} \mathfrak{h}^{-1}$ acts on $\text{Hom}_{MC^1(\mathfrak{h})}(\gamma_1, \gamma_2)$. For $\exp_{\gamma_2} t \in \exp_{\gamma_2} \mathfrak{h}^{-1}$ and $\text{Hom}_{MC^1(\mathfrak{h})}(\gamma_1, \gamma_2)$ the action is given by

$$(\exp_{\gamma_2} t) \cdot (\exp X) = \exp(\delta t + [\gamma_2, t]) \exp X \in \exp \mathfrak{h}^0.$$ 

By definition, $\text{Hom}_{MC^2(\mathfrak{h})}(\gamma_1, \gamma_2)$ is the groupoid associated with the above action.

The horizontal composition in $MC^2(\mathfrak{h})$, i.e. the map of groupoids

$$\otimes : \text{Hom}_{MC^2(\mathfrak{h})}(\exp X_{23}, \exp Y_{23}) \times \text{Hom}_{MC^2(\mathfrak{h})}(\exp X_{12}, \exp Y_{12}) \to \text{Hom}_{MC^2(\mathfrak{h})}(\exp X_{23} \exp X_{12}, \exp X_{23} \exp Y_{12}),$$

where $\gamma_i \in MC(\mathfrak{h})$, $\exp X_{ij}, \exp Y_{ij}, 1 \leq i, j \leq 3$ is defined by

$$\exp_{\gamma_3} t_{23} \otimes \exp_{\gamma_2} t_{12} = \exp_{\gamma_3} t_{23} \exp_{\gamma_3}(\exp(\text{ad} X_{23})(t_{12})), $$

where $\exp_{\gamma} t_{ij} \in \text{Hom}_{MC^2(\mathfrak{h})}(\exp X_{ij}, \exp Y_{ij})$.

3.11. Remark. There is a canonical map of 2-groupoids $MC^1(\mathfrak{h}) \to MC^2(\mathfrak{h})$ which induces a bijection $\pi_0(MC^1(\mathfrak{h})) \to \pi_0(MC^2(\mathfrak{h}))$ on sets of isomorphism classes of objects.


3.12.1. Abelian DGLA.

3.13. Lemma. Suppose that $\mathfrak{a}$ is an abelian DGLA satisfying $\mathfrak{a}^i = 0$ for $i < -1$. Then, the simplicial sets $\mathfrak{N}MC^2(\mathfrak{a})$ and $K(\mathfrak{a}[1])$ are isomorphic naturally in $\mathfrak{a}$.

Proof. The claim is an immediate consequence of the definitions and the explicit description of the nerve of $MC^2(\mathfrak{a})$ given in Lemma 2.4.

Combining Lemma 3.13 with the integration map (8) we obtain the map of simplicial abelian groups

$$\int : \Sigma(\mathfrak{a}) \to \mathfrak{N}MC^2(\mathfrak{a})$$

which is a weak homotopy equivalence.

3.13.1. Central extensions. Suppose that $\mathfrak{g}$ is a nilpotent DGLA satisfying $\mathfrak{g}^i = 0$ for $i < -1$ and $\mathfrak{a}$ is a central subalgebra in $\mathfrak{g}$. Note that $MC^2$ commutes with products, $\mathfrak{N}$ commutes with products and the addition map $+: \mathfrak{a} \times \mathfrak{g} \to \mathfrak{g}$ is a morphism of DGLAs. Thus, we obtain an action of the simplicial abelian group $\mathfrak{N}MC^2(\mathfrak{a})$ on the simplicial set $\mathfrak{N}MC^2(\mathfrak{g})$

$$\mathfrak{N}MC^2(+): \mathfrak{N}MC^2(\mathfrak{a}) \times \mathfrak{N}MC^2(\mathfrak{g}) \to \mathfrak{N}MC^2(\mathfrak{g}).$$

Note that the group structure on $\mathfrak{N}MC^2(\mathfrak{a})$ is obtained from the case $\mathfrak{a} = \mathfrak{g}$. Clearly, the action is free and the map $\mathfrak{N}MC^2(\mathfrak{g}) \to \mathfrak{N}MC^2(\mathfrak{g}/\mathfrak{a})$ factors through $\mathfrak{N}MC^2(\mathfrak{g})/\mathfrak{N}MC^2(\mathfrak{a})$.

3.13.2. The obstruction map.
3.14. Lemma. The obstruction map (6) factors as

$$\text{MC}(g/a) \rightarrow \pi_0 \text{MC}^2(g/a) \rightarrow H^2(a)$$

Proof. Suppose $\mu + a^1 \in \text{MC}(g/a)$. It follows from the formula (10) that

$$\exp(X + a^0) \cdot (\mu + a^1) = (\exp X) \cdot (\mu + a^1).$$

The formula (11) implies that

$$\mathcal{F}(\exp(X + a^0) \cdot (\mu + a^1)) = \mathcal{F}((\exp X) \cdot (\mu + a^1)) = \exp(ad X)(\mathcal{F}(\mu) + \delta a^1).$$

Since $\mathcal{F}(\mu) + \delta a^1 \subset a^2$, it follows that $\exp(ad X)(\mathcal{F}(\mu) + \delta a^1) = \mathcal{F}(\mu) + \delta a^1$ or, equivalently,

$$o_2(\exp(X + a^0) \cdot (\mu + a^1)) = o_2(\mu + a^1).$$

Recall (Lemma 2.4) that an $n$-simplex of $\mathfrak{N} \text{MC}^2(g/a)$, i.e. an element of $\mathfrak{N}_n \text{MC}^2(g/a)$ includes, among other things, a collection of $n + 1$ gauge-equivalent Maurer-Cartan elements of $g/a$. By Lemma 3.14 all of these Maurer-Cartan elements give rise to the same element of $H^2(a)$ under the map (6). Therefore, the assignment of this common value to an element of $\mathfrak{N}_n \text{MC}^2(g/a)$ give rise to a well-defined map

$$o_{2,n}: \mathfrak{N}_n \text{MC}^2(g/a) \rightarrow H^2(a)$$

for each $n = 0, 1, 2, \ldots$ such that the sequence of pointed sets

$$0 \rightarrow \mathfrak{N}_n \text{MC}^2(g)/\mathfrak{N}_n \text{MC}^2(a) \rightarrow \mathfrak{N}_n \text{MC}^2(g/a) \xrightarrow{o_{2,n}} H^2(a)$$

is exact. The maps (14) assemble into a map of simplicial sets

$$o_2: \mathfrak{N} \text{MC}^2(g/a) \xrightarrow{o_2} H^2(a),$$

where $H^2(a)$ is constant. Let $\mathfrak{N} \text{MC}^2(g/a)_0 = o_2^{-1}(0)$. The simplicial subset $\mathfrak{N} \text{MC}^2(g/a)_0$ is a union of connected components of $\mathfrak{N} \text{MC}^2(g/a)$ equal to the range of the map $\mathfrak{N} \text{MC}^2(g)/\mathfrak{N} \text{MC}^2(a) \rightarrow \mathfrak{N} \text{MC}^2(g/a)$.

It follows that $\mathfrak{N} \text{MC}^2(g) \rightarrow \mathfrak{N} \text{MC}^2(g/a)_0$ is a principal fibration with the group $\mathfrak{N} \text{MC}^2(a)$.

4. $\mathfrak{N} \text{MC}^2$ vs. $\Sigma$

In this section we show that for a DGLA $\mathfrak{h}$ satisfying $\mathfrak{h}^i = 0$ for $i < -1$ the simplicial sets $\mathfrak{N} \text{MC}^2(\mathfrak{h})$ and $\Sigma(\mathfrak{h})$ are isomorphic in the homotopy category of simplicial sets.
4.1. The main theorem. Let $\Sigma^2_n(h) = MC^2(\Omega_n \otimes h)$, where the latter is the simplicial groupoid associated with the strict 2-groupoid $MC^2(\Omega_n \otimes h)$ (see 2.3.1). Let $\Sigma^2(h) : [n] \mapsto \Sigma^2_n(h)$ denote the corresponding simplicial object in simplicial groupoids. Note that $\Sigma(h)$ is the simplicial set of objects of $\Sigma^2(h)$, hence there is a canonical map

$$\Sigma(h) \to \mathcal{N}\Sigma^2(h). \quad (15)$$

The map $Q \to \Omega_\bullet$ of simplicial DGA induces the map of simplicial objects in simplicial groupoids

$$MC^2(h) \to \Sigma^2(h). \quad (16)$$

Consider the diagram

$$\Sigma(h) \xrightarrow{(15)} \mathcal{N}\Sigma^2(h) \xleftarrow{\mathcal{N}((16))} \mathcal{N}MC^2(h). \quad (17)$$

4.2. Theorem. Suppose that $h$ is a nilpotent DGLA satisfying $h^i = 0$ for $i < -1$. Then, the morphisms (15) and $\mathcal{N}((16))$ are weak homotopy equivalences so that the diagram (17) represents an isomorphism $\Sigma(h) \cong \mathcal{N}MC^2(h)$ in the homotopy category of simplicial sets.

The rest of Section 4 is devoted to a proof of Theorem 4.2 which borrows techniques from the proof of Proposition 3.2.1 of [Hinich, 2004].

4.3. The map (15) is a weak homotopy equivalence. Let $\Sigma^1_n(h)$ denote the simplicial object in groupoids defined by $\Sigma^1_n(h) = MC^1(\Omega_n \otimes h)$. Note that $\Sigma(h)$ is the simplicial set of objects of $\Sigma^1(h)$ and hence there is a canonical map

$$\Sigma(h) \to \mathcal{N}\Sigma^1(h); \quad (18)$$

by Remark 3.11 there is a canonical map of simplicial objects in simplicial groupoids

$$\Sigma^1(h) \to \Sigma^2(h). \quad (19)$$

The map (15) is equal to the composition

$$\Sigma(h) \xrightarrow{(18)} \mathcal{N}\Sigma^1(h) \xrightarrow{\mathcal{N}((19))} \mathcal{N}\Sigma^2(h) \to \mathcal{N}\Sigma^2(h),$$

where the last map is the weak homotopy equivalence of Theorem 2.2.

4.4. Lemma. ([Hinich, 2004], Proposition 3.2.1) The map (18) is a weak homotopy equivalence.

Proof. Let $G_n(h) := \exp((\Omega_n \otimes h)^0)$. Then, $G(h) : [n] \mapsto G_n(h)$ is a simplicial group acting on $\Sigma(h)$, and $\Sigma(h)$ is the associated groupoid. Therefore,

$$N_q\Sigma(h) = \Sigma(h) \times G(h)^\times \quad \text{and} \quad N_q\Sigma(h),$$

and the map

$$\Sigma(h) \to N_q\Sigma(h)$$

is a weak homotopy equivalence because $G(h)$ is contractible.
4.5. **Proposition.** The map $\mathcal{N}((19))$ is a weak homotopy equivalence.

**Proof.** Let $\Gamma^1(\mathfrak{h})$ (respectively, $\Gamma^2(\mathfrak{h})$) denote the full subcategory of $\Sigma^1(\mathfrak{h})$ (respectively, of $\Sigma^2(\mathfrak{h})$) whose set of objects is $\text{MC}(\mathfrak{h})$ (a constant simplicial set). There is a commutative diagram

$$
\begin{array}{ccc}
\Gamma^1(\mathfrak{h}) & \longrightarrow & \Gamma^2(\mathfrak{h}) \\
\downarrow & & \downarrow \\
\Sigma^1(\mathfrak{h}) & \xrightarrow{(19)} & \Sigma^2(\mathfrak{h})
\end{array}
$$

The vertical arrows induce weak homotopy equivalences on respective nerves since, for each $n$ the functors $\Gamma^1(\mathfrak{h})_n \to \Sigma^1(\mathfrak{h})_n = \text{MC}^1(\Omega_n \otimes \mathfrak{h})$ and $\Gamma^2(\mathfrak{h})_n \to \Sigma^2(\mathfrak{h})_n = \text{MC}^2(\Omega_n \otimes \mathfrak{h})$ are equivalences by [Hinich, 2001], Proposition 8.2.5.

The map $\Gamma^1(\mathfrak{h}) \to \Gamma^2(\mathfrak{h})$ induces a bijection between sets of isomorphism classes of objects. For $\mu \in \text{MC}(\mathfrak{h})$, $\text{Hom}_{\text{MC}^2(\mathfrak{h})}(\mu, \mu)$ is naturally identified with the nerve of the groupoid associated to the action of the simplicial group $H(\mathfrak{h}, \mu): [n] \mapsto \exp((\Omega_n \otimes \mathfrak{h})_\mu)$ on the simplicial set $\text{Hom}_{\text{MC}^1(\mathfrak{h})}(\mu, \mu)$. Since the group $H(\mathfrak{h}, \mu)$ is contractible (it is isomorphic as a simplicial set to $[n] \mapsto \Omega^0_n \otimes \mathfrak{h}^{-1}$) the induced map $\text{Hom}_{\Gamma^1(\mathfrak{h})}(\mu, \mu) \to \text{Hom}_{\Gamma^2(\mathfrak{h})}(\mu, \mu)$ is an equivalence.

4.6. **The map $\mathfrak{N}((16)): \mathfrak{N} \text{MC}^2(\mathfrak{h}) \to \mathfrak{N} \Sigma^2(\mathfrak{h})$ is a weak homotopy equivalence.**

It suffices to show that the map

$$
\mathfrak{N} \text{MC}^2(\mathfrak{h}) \to \mathfrak{N} \text{MC}^2(\Omega_n \otimes \mathfrak{h})
$$

is a weak homotopy equivalence for all $n$. This follows from Proposition 4.7.

4.7. **Proposition.** Suppose that $\mathfrak{h}$ is a nilpotent DGLA concentrated in degrees greater than or equal to $-1$. The functor

$$
\text{MC}^2(\mathfrak{h}) \to \text{MC}^2(\Omega_n \otimes \mathfrak{h})
$$

is an equivalence.

**Proof.** The induced map $\pi_0((20))$ is a bijection by Remark 3.11 and (the proof of) [Hinich, 1997], Lemma 2.2.1. The result now follows from Lemma 4.8 below.

4.8. **Lemma.** Suppose $\mu \in \text{MC}(\mathfrak{h})$. The functor

$$
\text{Hom}_{\text{MC}^2(\mathfrak{h})}(\mu, \mu) \to \text{Hom}_{\text{MC}^2(\Omega_n \otimes \mathfrak{h})}(\mu, \mu)
$$

is an equivalence.
Proof. According to the description given in 3.10.3, for any nilpotent DGLA \((\mathfrak{g}, \delta)\) with \(\mathfrak{g}^i = 0\) for \(i < -1\) and \(\mu \in \text{MC}(\mathfrak{g})\) the groupoid \(\text{Hom}_{\text{MC}^2(\mathfrak{g})}(\mu, \mu)\) is isomorphic to the groupoid associated with the action of the group \(\exp_{\mu} \mathfrak{g}^{-1}\) on the set \(\exp(\ker(\delta^{-1})) \subset \exp(\mathfrak{g}^0)\) where \(\delta_{\mu} = \delta + [\mu, .]\).

Note that, for any \(X \in \ker(\delta_{\mu}^{-1})\), the automorphism group \(\text{Aut}(\exp(X))\) is isomorphic to (the additive group) \(\ker(\delta_{\mu}^{-1})\).

The map
\[
([n] \to [0])^* \otimes \text{Id}: (\mathfrak{h}, \delta) \to (\Omega_n \otimes \mathfrak{h}, d + \delta)
\] (22)
is a quasi-isomorphism of DGLA with the quasi-inverse given by the evaluation map \(\text{ev}_0 := ([0] \to [n])^* \otimes \text{Id}: \Omega_n \otimes \mathfrak{h} \to \mathfrak{h}\) (for any choice of a morphism \([0] \to [n]\)) which is a morphism of DGLA as well. The same maps are mutually quasi-inverse quasi-isomorphisms of DGLA

\[(\mathfrak{h}, \delta_{\mu}) \rightleftarrows (\Omega_n \otimes \mathfrak{h}, d + \delta_{\mu}).\]

Since (22) is a quasi-isomorphism and both DGLA are concentrated in degrees greater than or equal to \(-1\), the induced map \(\ker(\delta_{\mu}^{-1}) \to \ker((d + \delta_{\mu})^{-1})\) an isomorphism, hence so are the maps of automorphism groups.

Since the map (21) admits a left inverse (namely, ev_0) it remains to show that the induced map on sets of isomorphism classes is surjective. Note that, since ev_0 is a surjective quasi-isomorphism, the map \(d + \delta_{\mu}: \ker(\text{ev}_0)^{-1} \to \ker(\text{ev}_0)^0 \cap \ker((d + \delta_{\mu})^0)\) is an isomorphism.

Consider \(X \in (\Omega_n \otimes \mathfrak{g})^0\). Then, \(X = \text{ev}_0(X) + Y\) with \(Y \in \ker(\text{ev}_0)\), and \((d + \delta_{\mu})X = 0\) if and only if \(\delta_{\mu} \text{ev}_0(X) = 0\) and \((d + \delta_{\mu})Y = 0\).

Suppose \(X \in \ker((d + \delta_{\mu})^0)\). Then, \(\exp(X) = \exp(\text{ev}_0(X)) \cdot \exp(Z)\) where \(Z \in \ker(\text{ev}_0)^0 \cap \ker((d + \delta_{\mu})^0)\), and, therefore, \(Z = (d + \delta_{\mu})U\) for a uniquely determined \(U\). \(\blacksquare\)

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