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HOMOTOPY UNITAL A_∞-MORPHISMS WITH SEVERAL ENTRIES

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Abstract. We show that morphisms from n homotopy unital A_∞-algebras to a single one are maps over an operad module with $n+1$ commuting actions of the operad A_∞^{hu}, whose algebras are homotopy unital A_∞-algebras. The operad A_∞ and modules over it have two useful gradings related by isomorphisms which change the degree. The composition of A_∞^{hu}-morphisms with several entries is presented as a convolution of a coalgebra-like and an algebra-like structures.

The present work is a sequel to [Lyu15]. We use freely notations and notions from the previous article. There polymodule cooperads were defined and an example of A_∞-polymodule cooperad F was given. Here we describe three more examples of polymodule cooperads: an A_∞-polymodule cooperad F, an A_∞^{hu}-polymodule cooperad F^{hu} and $\mathrm{A}_\infty^{\mathsf{hu}}$-polymodule cooperad F^{hu}. Here A_∞ (resp. $\mathrm{A}_\infty^{\mathsf{hu}}$) is an operad of conventional (resp. homotopy unital) A_∞-algebras, and A_∞, A_∞^{hu} are their signless versions. Also F (resp. F^{hu}) is a signless version of F (resp. F^{hu}). We develop the idea of "isomorphism" of operads and polymodule cooperads changing degrees. Operads A_∞ and A_∞, A_∞^{hu} and $\mathrm{A}_\infty^{\mathsf{hu}}$, and polymodule cooperads F and F, F^{hu} and F^{hu} are "isomorphic" in this sense.

Both categories of **dg**-operads and of polymodule **dg**-cooperads have a model structure. It is known that the **dg**-operad A_∞ (resp. $\mathrm{A}_\infty^{\mathsf{hu}}$) is a cofibrant resolution of the **dg**-operad *As* (resp. *As1*) of non-unital (resp. unital) associative **dg**-algebras. We show that the polymodule cooperad F (resp. F^{hu}) is a cofibrant resolution of the polymodule cooperad responsible for morphisms and composition in the multicategory of non-unital (resp. unital) associative **dg**-algebras. Polymodule cooperads F, F (resp. F^{hu}, F^{hu}) are means to represent morphisms and their composition in multicategories of conventional (resp. homotopy unital) A_∞-algebras or A_∞-algebras. The composition is recovered via convolution of polymodule cooperad and a lax $\mathcal{C}at$-multifunctor $\mathcal{H}om$ built from **dg**-modules.

Verification that changing degrees does not lead out of polymodule cooperads is straightforward but lengthy.

Contents

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We assume that all modules are graded modules over a commutative ring. A lot of signs disappear due to chiral system of notations, see Section 1.1: we use right operators, homogeneous elements of the right homomorphism object in closed symmetric monoidal category of graded modules. We discuss model structures on categories related to complexes (Hinich‘s theorem) in Section 1.2. The notion of morphism of **dg**-operads of certain (non-zero) degree is recalled in Section 1.6. (Homotopy unital) A_∞-algebras and the related operads A_∞, A_∞, $\mathrm{A}_\infty^{\mathsf{hu}}$ and A_∞^{hu} are recalled in Examples 1.5, 1.10, 1.11.

The categorical basement to constructions in this article is the notion of $\mathcal{C}at$-multicategories and (co)lax $\mathcal{C}at$-multifunctors, considered in detail in [Lyu15]. We study the category ${}_n\mathrm{Op}_1$ of $n \wedge 1$-operad modules. We prove that the category ${}_n\mathrm{Op}_1$ with quasi-isomorphisms as weak equivalences and degreewise surjections as fibrations is a model category (Proposition 2.2). Starting with a symmetric **dg**-multicategory C we construct in Section 3.1 a lax $\mathcal{C}at$-multifunctor *hom*, which to a sequence $(A_i)_{i\in I}$, B of objects of C and a vector $(n^i)_{i\in I} \in \mathbb{N}^I$ assigns a complex $hom((A_i)_{i\in I}; B)((n^i)_{i\in I}) = \mathsf{C}\big(({}^{n^i}A_i)_{i\in I}; B\big)$ equipped with compositions coming from C. We compute some signs for *hom* arising from shifts [1] in Lemma 3.2 and Corollary 3.3.

The $n\wedge 1$-operad module *hom* is revisited in Section 3.4. Polymodule homomorphisms of certain (non-zero) degree are introduced in Definition 3.6. They are motivated by relationship between *hom* for shifted and non-shifted complexes (Example 3.7). We describe the polymodules $FAs1_n$ responsible for homomorphisms $f : A_1 \otimes \cdots \otimes A_n \to B$ of unital associative **dg**-algebras in Section 3.8 after dealing with non-unital case and polymodules FAs_n in Example 3.5. We gradually begin to construct cofibrant **dg**-resolution of $FAs1_n$. The first step is made in [Lyu15, Proposition 3.4], where we construct differential graded A_∞-polymodules F_n, signless shifted version of A_∞-polymodules F_n from Proposition 3.9. We prove that the $n\wedge 1$-operad module $(\mathrm{A}_\infty, \mathrm{F}_n)$ is a cofibrant replacement of (As, FAs_n) and even homotopy isomorphic to it in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$ (Theorem 3.13). At last, in Theorem 3.19 it is shown that the $n \wedge 1$-operad **dg**-module $(\mathrm{A}_\infty^{\mathsf{hu}}, \mathrm{F}_n^{\mathsf{hu}})$ is a cofibrant replacement of $(As1, FAs1_n)$ and even homotopy isomorphic to it in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$.

In Section 4 we equip some collections of $n \wedge 1$-operad modules, $n \geqslant 0$, with comultiplication turning them into polymodule cooperads. This comultiplication encodes composition in a certain multicategory. In Section 4.1 we equip $(\mathrm{A}_\infty, \mathrm{F}_\bullet)$ with comultiplication Δ^{G} targeted at $\circledast_{\mathsf{G}}$. The formulas for $(\mathrm{A}_\infty, \mathrm{F}_\bullet, \Delta^{\mathsf{G}})$ resemble those for $(A_\infty, F_\bullet, \mathit{\Delta}^{\mathsf{G}})$, but contain numerous signs. Theorem B.1 allows to conclude that unitality and associativity of the latter implies unitality and associativity of the former. Thus, $(\mathrm{A}_\infty, \mathrm{F}_\bullet, \Delta^{\mathsf{G}})$ is a graded polymodule cooperad (Proposition 4.3). It is shown in Proposition 4.2 that $\Delta^{\mathsf{G}}(t)$ depends rather on planar tree t, than on the ordering of the set of internal vertices of t. When comultiplication Δ^{M} is targeted at $\circledast_{\mathsf{M}}$ instead of $\circledast_{\mathsf{G}}$, it makes $(\mathrm{A}_\infty, \mathrm{F}_\bullet)$ into a **dg**-polymodule cooperad (Theorem 4.4). We extend comultiplications $\mathit{\Delta}^{\mathsf{M}}$, Δ^{M}

to $(A_\infty^{\mathsf{hu}}, F^{\mathsf{hu}}, \Delta^{\mathsf{M}})$, $(\mathrm{A}_\infty^{\mathsf{hu}}, \mathrm{F}^{\mathsf{hu}}, \Delta^{\mathsf{M}})$ in Proposition 4.6 making them into **dg**-polymodule cooperads.

Appendix B is devoted to proof of Theorem B.1 which allows to transfer graded polymodule cooperad structures along polymodule isomorphisms of non-zero degree. The supplied proof is straightforward although lengthy and consists in checking that several signs coincide. Besides, I would not call this verification a simple exercise.

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1. Preliminaries

Here we describe notations, recall some notions and results needed in the following parts of the article.

1.1. Notations and conventions. We denote by $\mathbb{N}$ the set of non-negative integers $\mathbb{Z}_{\geqslant 0}$. By norm on $\mathbb{N}^n$ we mean the function $|\cdot| : \mathbb{N}^n \to \mathbb{N}$, $j \mapsto |j| = \sum_{i=1}^n j^i$.

Let $\mathcal{V} = (\mathcal{V}, \otimes, \mathbb{1})$ be a complete and cocomplete closed symmetric monoidal category with the right inner hom $\underline{\mathcal{V}}(X, Y)$. Mostly we shall be interested in the category of (differential) graded $\Bbbk$-modules $\mathcal{V} = \mathbf{gr}$ (resp. $\mathcal{V} = \mathbf{dg}$). When a $\Bbbk$-linear map f is applied to an element x, the result is typically written as $x.f = xf$. Thus we work with right homomorphism objects which we denote $\underline{\mathbf{gr}}(X, Y)$ (resp. $\underline{\mathbf{dg}}(X, Y)$) for (differential) graded $\Bbbk$-modules X, Y. The tensor product of two maps of graded $\Bbbk$-modules f, g of certain degree is defined so that for elements x, y of arbitrary degree

$$(x \otimes y).(f \otimes g) = (-1)^{\deg y \cdot \deg f} x.f \otimes y.g.$$

In other words, we follow the Koszul rule imposed by the symmetry. Composition of homogeneous $\Bbbk$-linear maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ is usually denoted $f \cdot g = fg : X \to Z$. For other types of maps composition is often written as $g \circ f = gf$.

We consider the category of totally ordered finite sets and their non-decreasing maps. An arbitrary totally ordered finite set is isomorphic to a unique set $\mathbf{n} = \{1 < 2 < \cdots < n\}$ via a unique isomorphism, $n \geqslant 0$. Functions of totally ordered finite set that we use in this article are assumed to *depend only on the isomorphism class of the set*. Thus, it suffices to define them only for skeletal totally ordered finite sets $\mathbf{n}$. The full subcategory of such sets and their non-decreasing maps is denoted $\mathcal{O}_{\mathsf{sk}}$.

Whenever $I \in \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}$, there is another totally ordered set $[I] = \{0\} \sqcup I$ containing I, where element 0 is the smallest one. Thus, $[n] = [\mathbf{n}] = \{0 < 1 < 2 < \cdots < n\}$.

Let $(I, \leqslant)$, $(X_i, \leqslant)$, $i \in I$, be partially ordered sets. When $\bigsqcup_{i \in I} X_i$ is equipped with the lexicographic order it is denoted $\trianglelefteq_{i \in I} X_i$. Thus $(i, x) < (j, y)$ iff $i < j$ or ($i = j$ and $x < y \in X_i$).

The list $A, \ldots, A$ consisting of n copies of the same object A is denoted ${}^n A$.

For any graded $\Bbbk$-module M denote by $sM = M[1]$ the same module with the grading shifted by 1: $M[1]^k = M^{k+1}$. Denote by $\sigma : M \to M[1]$, $M^k \ni x \mapsto x \in M[1]^{k-1}$ the "identity map" of degree $\deg \sigma = -1$.

1.2. Model category structures. The following theorem is proved by Hinich in [Hin97, Section 2.2], except that he relates a category with the category of complexes $\mathbf{dg}$, not with its power $\mathbf{dg}^S$. A generalization is given in [Lyu12, Theorem 1.2]. It has the same formulation as below, however, $\mathbf{dg}$ means there the category of differential graded modules over a graded commutative ring.

1.3. Theorem. [Hin97, Lyu12] *Suppose that S is a set, a category $\mathcal{C}$ is complete and cocomplete and $F : \mathbf{dg}^S \rightleftarrows \mathcal{C} : U$ is an adjunction. Assume that U preserves filtering colimits. For any $x \in S$, $p \in \mathbb{Z}$ consider the object $\mathbb{K}[-p]_x$ of $\mathbf{dg}^S$, $\mathbb{K}[-p]_x(x) = \bigl(0 \to \Bbbk \xrightarrow{1} \Bbbk \to 0\bigr)$ (concentrated in degrees p and $p+1$), $\mathbb{K}[-p]_x(y) = 0$ for $y \ne x$. Assume that the chain map $U(\mathrm{in}_2) : UA \to U(F(\mathbb{K}[-p]_x) \sqcup A)$ is a quasi-isomorphism for all objects A of $\mathcal{C}$ and all $x \in S$, $p \in \mathbb{Z}$. Equip $\mathcal{C}$ with the classes of weak equivalences (resp. fibrations) consisting of morphisms f of $\mathcal{C}$ such that Uf is a quasi-isomorphism (resp. an epimorphism). Then the category $\mathcal{C}$ is a model category.*

We shall recall also several constructions used in the proof of this theorem. They describe cofibrations and trivial cofibrations in $\mathcal{C}$. Assume that $M \in \operatorname{Ob} \mathbf{dg}^S$, $A \in \operatorname{Ob} \mathcal{C}$, $\alpha : M \to UA \in \mathbf{dg}^S$. Denote by $C = \operatorname{Cone} \alpha = (M[1] \oplus UA, d_{\mathrm{Cone}}) \in \operatorname{Ob} \mathbf{dg}^S$ the cone taken pointwise, that is, for any $x \in S$ the complex $C(x) = \operatorname{Cone}\bigl(\alpha(x) : M(x) \to (UA)(x)\bigr)$ is the usual cone. Denote by $\bar\imath : UA \to C$ the obvious embedding. Let $\varepsilon : FU(A) \to A$ be the adjunction counit. Following Hinich [Hin97, Section 2.2.2] define an object $A\langle M, \alpha\rangle \in \operatorname{Ob} \mathcal{C}$ as the pushout

$$\begin{array}{ccc} FU(A) & \xrightarrow{\varepsilon} & A \\ {\scriptstyle F\bar\imath}\downarrow & & \downarrow{\scriptstyle \bar\jmath} \\ FC & \xrightarrow{g} & A\langle M, \alpha\rangle \end{array} \tag{1.1}$$

If $\alpha = 0$, then $A\langle M, 0\rangle \simeq F(M[1]) \sqcup A$ and $\bar\jmath = \mathrm{in}_2$ is the canonical embedding. We say that M consists of free $\Bbbk$-modules if for any $x \in S$, $p \in \mathbb{Z}$ the $\Bbbk$-module $M(x)^p$ is free.

The proof contains the following important statements. If M consists of free $\Bbbk$-modules and $d_M = 0$, then $\bar\jmath : A \to A\langle M, \alpha\rangle$ is a cofibration. It might be called an *elementary standard cofibration*. If

$$A \to A_1 \to A_2 \to \cdots$$

is a sequence of elementary standard cofibrations, B is a colimit of this diagram, then the "infinite composition" map $A \to B$ is a cofibration called a *standard cofibration* [Hin97, Section 2.2.3].

Assume that $N \in \operatorname{Ob} \mathbf{dg}^S$ consists of free $\Bbbk$-modules, $d_N = 0$ and $M = \operatorname{Cone}\bigl(1_{N[-1]}\bigr) = (N \oplus N[-1], d_{\mathrm{Cone}})$. Then for any morphism $\alpha : M \to UA \in \mathbf{dg}^S$ the morphism $\bar\jmath$:

$A \to A\langle M, \alpha\rangle$ is a trivial cofibration in $\mathcal{C}$ and a standard cofibration, composition of two elementary standard cofibrations. It is called a *standard trivial cofibration.* Any (trivial) cofibration is a retract of a standard (trivial) cofibration [Hin97, Remark 2.2.5].

When $F : \mathbf{dg}^S \to \mathcal{C}$ is the functor of constructing a free **dg**-algebra of some kind, the maps $\bar{\jmath}$ are interpreted as "adding variables to kill cycles".

The category Op of operads admits an adjunction $F : \mathbf{dg}^{\mathbb{N}} \rightleftarrows \mathrm{Op} : U$. Applying Theorem 1.3 to this category one gets [Lyu11, Proposition 1.8]

1.4. Proposition. *Define weak equivalences (resp. fibrations) in* Op *as morphisms f of* Op *such that Uf is a quasi-isomorphism (resp. an epimorphism). These classes make* Op *into a model category.*

This statement was proven previously in [Hin97], [Spi01, Remark 2] and follows from [Mur11, Theorem 1.1].

1.5. Example. Using Stasheff's associahedra one proves that there is a cofibrant replacement $\mathrm{A}_\infty \to As$ where the graded operad A_∞ is freely generated by n-ary operations m_n of degree $2-n$ for $n \geqslant 2$. The differential is found as

$$m_n\partial = -\sum_{j+p+q=n}^{1<p<n} (-)^{jp+q}(1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}) \cdot m_{j+1+q}. \tag{1.2}$$

Basis $(m(t))$ of $\mathrm{A}_\infty = T\big(\Bbbk\{m_n \mid n \geqslant 2\}\big)$ over $\Bbbk$ is indexed by isomorphism classes of planar rooted trees t without unary vertices (those with one incoming edge). The tree $|$ without internal vertices (the root and the leaf) corresponds to the unit from $\mathrm{A}_\infty(1)$.

Algebras over the **dg**-operad A_∞ are precisely A_∞-algebras, that is, complexes $A \in \mathbf{dg}$ with the differential m_1 and operations $m_n : A^{\otimes n} \to A$, $\deg m_n = 2-n$, for $n \geqslant 2$ such that

$$\sum_{j+p+q=n} (-1)^{jp+q}(1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}) \cdot m_{j+1+q} = 0.$$

Furthermore, the chain map $\mathrm{A}_\infty(n) \to As(n)$ is homotopy invertible for each $n \geqslant 1$. One way to prove it is implied by a remark of Markl [Mar96, Example 4.8]. Another proof uses the operad of Stasheff associahedra [Sta63] and the configuration space of $(n+1)$-tuples of points on a circle considered by Seidel in his book [Sei08]. Details can be found in [BLM08, Proposition 1.19].

1.6. Morphisms of operads. Besides usual homomorphisms of **dg**-operads, which are chain maps of degree 0, we consider also maps that change the degree.

1.7. Definition. *A* **dg***-operad homomorphism $t : \mathcal{O} \to \mathcal{P}$ of degree $\bar{t} = r \in \mathbb{Z}$ is a collection of homogeneous $\Bbbk$-linear maps $t(n) : \mathcal{O}(n) \to \mathcal{P}(n)$, $n \geqslant 0$, of degree $g(n) = (1-n)r$ such that*

- $1_{\mathcal{O}}.t(1) = 1_{\mathcal{P}}$;

- *for all $k, n_1, \dots, n_k \in \mathbb{N}$ the following square commutes up to a certain sign:*

$$\begin{array}{ccc} \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \otimes \mathcal{O}(k) & \xrightarrow{\mu} & \mathcal{O}(n_1 + \cdots + n_k) \\ {\scriptstyle t(n_1)\otimes\cdots\otimes t(n_k)\otimes t(k)} \downarrow & (-1)^c & \downarrow {\scriptstyle t(n_1+\cdots+n_k)} \\ \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \otimes \mathcal{P}(k) & \xrightarrow{\mu} & \mathcal{P}(n_1 + \cdots + n_k) \end{array} \tag{1.3}$$

where the tensor product of homogeneous right maps $t(\_)$ is that of the **dg**-*category* $\underline{\mathbf{dg}}$ *and the sign is determined by*

$$c = r\sum_{i=1}^{k}(i-1)(1-n_i) + \frac{r(r-1)}{2}\sum_{1\leqslant i<j\leqslant k}(1-n_i)(1-n_j) + \frac{r(r-1)}{2}(1-k)\sum_{i=1}^{k}(1-n_i); \tag{1.4}$$

- *for all $n \in \mathbb{Z}$*

$$d \cdot t(n) = (-1)^{g(n)} t(n) \cdot d : \mathcal{O}(n) \to \mathcal{P}(n).$$

Notice that the only functions $g : \mathbb{N} \to \mathbb{Z}$ that satisfy the equations

$$g(1) = 0, \qquad g(n_1) + \cdots + g(n_k) + g(k) = g(n_1 + \cdots + n_k)$$

are functions $g(n) = (1 - n)r$ for some $r \in \mathbb{Z}$.

1.8. Example. Let X, Y be objects of **dg** (complexes of $\Bbbk$-modules). Define a collection $\mathcal{H}om(X,Y)$ as $\mathcal{H}om(X,Y)(n) = \underline{\mathbf{dg}}(X^{\otimes n}, Y)$. Substitution composition $\mathcal{H}om(X,Y) \odot \mathcal{H}om(Y,Z) \to \mathcal{H}om(X,Z)$ and obvious units $\mathbb{1} \to \mathcal{H}om(X,X)$ make the category of complexes enriched in the monoidal category $(\mathbf{dg}^{\mathbb{N}}, \odot)$. In particular, $\mathcal{E}nd\, X = \mathcal{H}om(X,X)$ are algebras in $\mathbf{dg}^{\mathbb{N}}$ (**dg**-operads).

Let (X, d_X) be a complex of $\Bbbk$-modules and let $(X[1], d_{X[1]} = -\sigma^{-1} \cdot d_X \cdot \sigma)$ be its suspension. There is a **dg**-operad morphism

$$\Sigma = \mathcal{H}om(\sigma; \sigma^{-1}) = \mathcal{H}om(\sigma; 1) \cdot \mathcal{H}om(1; \sigma^{-1}) : \mathcal{E}nd(X[1]) \to \mathcal{E}nd\, X$$

of degree 1. That is, the mapping $f \mapsto (-1)^{nf} \sigma^{\otimes n} \cdot f \cdot \sigma^{-1}$,

$$\Sigma(n) = \underline{\mathbf{dg}}(\sigma^{\otimes n}; 1) \cdot \underline{\mathbf{dg}}(1; \sigma^{-1}) : \underline{\mathbf{dg}}(X[1]^{\otimes n}, X[1]) \to \underline{\mathbf{dg}}(X^{\otimes n}, X),$$

has degree $1 - n$. The sign $(-1)^c$, $c = \sum_{i=1}^{k}(i-1)(1-n_i)$, pops out in the following procedure. Write down the tensor product corresponding to the left vertical arrow of (1.3) for $t = \Sigma$:

$$(\sigma^{\otimes n_1} \otimes \sigma^{-1}) \otimes (\sigma^{\otimes n_2} \otimes \sigma^{-1}) \otimes \cdots \otimes (\sigma^{\otimes n_k} \otimes \sigma^{-1}) \otimes (\sigma^{\otimes k} \otimes \sigma^{-1});$$

move factors σ^{-1} using the Koszul rule to their respective opponents, factors σ of $\sigma^{\otimes k}$, in order to cancel them and to obtain finally $\sigma^{\otimes(n_1+\cdots+n_k)} \otimes \sigma^{-1}$.

Maps $\Sigma(n)$ commute with the differential in the graded sense because their factors $\sigma^{\pm 1}$ do.

1.9. Remark. Summands $r(r-1)/2\sum_{1\leqslant i<j\leqslant k}(1-n_i)(1-n_j)+(1-k)r(r-1)/2\sum_{i=1}^k(1-n_i)$ of c make sure that the composition of two morphisms of operads $t:\mathcal{O}\to\mathcal{P}$ and $u:\mathcal{P}\to\mathcal{Q}$ of degree $\bar{t}$ and $\bar{u}$ respectively be an operad morphism of degree $\bar{t}+\bar{u}$. Furthermore, if all homogeneous maps $t(n):\mathcal{O}(n)\to\mathcal{P}(n)$ for $t:\mathcal{O}\to\mathcal{P}$ are invertible, than there is an inverse morphism of operads $t^{-1}:\mathcal{P}\to\mathcal{O}$ of degree $-\bar{t}$ with $t^{-1}(n)=t(n)^{-1}$.

Let $\mathcal{O}$ be a **dg**-operad, $\mathcal{P}$ be a graded operad and $t:\mathcal{O}\to\mathcal{P}$ be an invertible graded operad homomorphism of degree r ($1_{\mathcal{O}}.t(1)=1_{\mathcal{P}}$ and (1.3) holds). Then $\mathcal{P}$ has a unique differential d which turns it into a **dg**-operad and $t:\mathcal{O}\to\mathcal{P}$ into a **dg**-operad homomorphism of degree r.

1.10. Example. The **dg**-operad A_∞ has a version denoted A_∞ in [Lyu15]. This is a **dg**-operad freely generated as a graded operad by n-ary operations b_n of degree 1 for $n\geqslant 2$. The differential is defined as

$$b_n\partial=-\sum_{j+p+q=n}^{1<p<n}(1^{\otimes j}\otimes b_p\otimes 1^{\otimes q})\cdot b_{j+1+q}.$$

Comparing the differentials we find that these two operads are isomorphic via an isomorphism of degree 1

$$\Sigma:A_\infty\to\mathrm{A}_\infty,\qquad b_i\mapsto m_i.$$

In fact, due to (1.3)

$$\begin{aligned}&[(1^{\otimes j}\otimes b_p\otimes 1^{\otimes q})b_{j+1+q}].\Sigma(j+p+q)\\&=(-1)^{j(1-p)+1-p}[(1^{\otimes j}\otimes b_p\otimes 1^{\otimes q}).(\Sigma(1)^{\otimes j}\otimes\Sigma(p)\otimes\Sigma(1)^{\otimes q})]\cdot[b_{j+1+q}.\Sigma(j+1+q)]\\&=(-1)^{(j+1)(1-p)}[(1^{\otimes j}\otimes m_p\otimes 1^{\otimes q})m_{j+1+q}].\end{aligned}$$

Therefore,

$$\begin{aligned}m_n\partial&=(b_n.\Sigma(n))\partial=(-1)^{1-n}(b_n\partial).\Sigma(n)\\&=(-1)^n\sum_{j+p+q=n}^{1<p<n}[(1^{\otimes j}\otimes b_p\otimes 1^{\otimes q})b_{j+1+q}].\Sigma(n)\\&=(-1)^n\sum_{j+p+q=n}^{1<p<n}(-1)^{(j+1)(1-p)}(1^{\otimes j}\otimes m_p\otimes 1^{\otimes q})m_{j+1+q}\\&=-\sum_{j+p+q=n}^{1<p<n}(-1)^{jp+q}(1^{\otimes j}\otimes m_p\otimes 1^{\otimes q})\cdot m_{j+1+q}\end{aligned}$$

which coincides with (1.2). This fixes the differential on A_∞ since m_n generate the graded operad. An easy lemma shows that it suffices to verify graded commutation of the differential and any operad homomorphism on generators. In particular, $\Sigma:A_\infty\to\mathrm{A}_\infty$ commutes with ∂ in the graded sense.

Knowing that A_∞ is homotopy isomorphic to its cohomology As, we conclude that A_∞ is homotopy isomorphic to its cohomology as well. There is an isomorphism of degree 1 between graded operads $\Sigma : H^\bullet(A_\infty) \to As$. Hence, $H^\bullet(A_\infty(n)) = \Bbbk[1-n]$ for $n \geqslant 1$.

For any algebra $A \in \mathbf{dg}$ over the $\mathbf{dg}$-operad A_∞ the $\mathbf{dg}$-module $A[1]$ becomes an algebra over the $\mathbf{dg}$-operad A_∞ so that the square of operad homomorphisms commutes:

$$\begin{array}{ccc} A_\infty & \longrightarrow & \mathcal{E}nd\, A[1] \\ {\scriptstyle\Sigma}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{H}om(\sigma;\sigma^{-1})} \\ \mathrm{A}_\infty & \longrightarrow & \mathcal{E}nd\, A \end{array}, \qquad (-1)^n \sigma^{\otimes n} \cdot b_n \cdot \sigma^{-1} = m_n : A^{\otimes n} \to A, \quad n \geqslant 1. \tag{1.5}$$

Verification is straightforward.

1.11. Example. Approaching homotopy unital A_∞-algebras we start with strictly unital ones. They are governed by the operad $\mathrm{A}_\infty^{\mathsf{su}}$ generated over A_∞ by a nullary degree 0 cycle $\mathbf{1}^{\mathsf{su}}$ subject to the following relations:

$$(1 \otimes \mathbf{1}^{\mathsf{su}})m_2 = 1, \quad (\mathbf{1}^{\mathsf{su}} \otimes 1)m_2 = 1, \quad (1^{\otimes a} \otimes \mathbf{1}^{\mathsf{su}} \otimes 1^{\otimes c})m_{a+1+c} = 0 \ \text{ if } \ a + c > 1.$$

There is a standard trivial cofibration and a homotopy isomorphism $\mathrm{A}_\infty^{\mathsf{su}} \rightarrowtail\!\sim\!\to \mathrm{A}_\infty^{\mathsf{su}}\langle \mathbf{1}^{\mathsf{su}} - \mathsf{i}, \mathsf{j}\rangle = \mathrm{A}_\infty^{\mathsf{su}}\langle \mathsf{i}, \mathsf{j}\rangle$, where i, j are two nullary operations, $\deg \mathsf{i} = 0$, $\deg \mathsf{j} = -1$, with $\mathsf{i}\partial = 0$, $\mathsf{j}\partial = \mathbf{1}^{\mathsf{su}} - \mathsf{i}$.

A cofibrant replacement $\mathrm{A}_\infty^{\mathsf{hu}} \to As1$ is constructed as a $\mathbf{dg}$-suboperad of $\mathrm{A}_\infty^{\mathsf{su}}\langle \mathsf{i}, \mathsf{j}\rangle$ generated as a graded operad by i and n-ary operations of degree $4 - n - 2k$

$$m_{n_1;n_2;\dots;n_k} = (1^{\otimes n_1} \otimes \mathsf{j} \otimes 1^{\otimes n_2} \otimes \mathsf{j} \otimes \dots \otimes 1^{\otimes n_{k-1}} \otimes \mathsf{j} \otimes 1^{\otimes n_k})m_{n+k-1},$$

where $n = \sum_{q=1}^k n_q$, $k \geqslant 1$, $n_q \geqslant 0$, $n + k \geqslant 3$. Notice that the graded operad $\mathrm{A}_\infty^{\mathsf{hu}}$ is free. See [Lyu11, Section 1.11] for the proofs.

One can perform all the above steps also for the operad A_∞:

1) Adding to A_∞ a nullary degree -1 cycle $\mathbf{1}^{\mathsf{su}}$ subject to the relations:

$$(1 \otimes \mathbf{1}^{\mathsf{su}})b_2 = 1, \quad (\mathbf{1}^{\mathsf{su}} \otimes 1)b_2 = -1, \quad (1^{\otimes a} \otimes \mathbf{1}^{\mathsf{su}} \otimes 1^{\otimes c})b_{a+1+c} = 0 \ \text{ if } \ a + c > 1. \tag{1.6}$$

The resulting operad is denoted A_∞^{su}.

2) Adding to A_∞^{su} two nullary operations $\mathbf{i}$, $\mathbf{j}$, $\deg \mathbf{i} = -1$, $\deg \mathbf{j} = -2$, with $\mathbf{i}\partial = 0$, $\mathbf{j}\partial = \mathbf{i} - \mathbf{1}^{\mathsf{su}}$. The standard trivial cofibration $A_\infty^{\mathsf{su}} \rightarrowtail\!\sim\!\to A_\infty^{\mathsf{su}}\langle \mathbf{i}, \mathbf{j}\rangle$ is a homotopy isomorphism.

3) A_∞^{hu} is a $\mathbf{dg}$-suboperad of $A_\infty^{\mathsf{su}}\langle \mathbf{i}, \mathbf{j}\rangle$ generated as a graded operad by $\mathbf{i}$ and n-ary operations of degree $3 - 2k$

$$b_{n_1;n_2;\dots;n_k} = (1^{\otimes n_1} \otimes \mathbf{j} \otimes 1^{\otimes n_2} \otimes \mathbf{j} \otimes \dots \otimes 1^{\otimes n_{k-1}} \otimes \mathbf{j} \otimes 1^{\otimes n_k})b_{n+k-1},$$

where $n = \sum_{q=1}^k n_q$, $k \geqslant 1$, $n_q \geqslant 0$, $n + k \geqslant 3$.

The obtained operads are related to the previous ones by invertible homomorphisms of degree 1, extending $\Sigma : b_n \mapsto m_n$,

$$\Sigma : A^{\mathsf{su}}_\infty \to \mathrm{A}^{\mathsf{su}}_\infty,\ \mathbf{1}^{\mathsf{su}} \mapsto 1^{\mathsf{su}}; \quad \Sigma : A^{\mathsf{su}}_\infty\langle \mathbf{i}, \mathbf{j}\rangle \to \mathrm{A}^{\mathsf{su}}_\infty\langle \mathsf{i}, \mathsf{j}\rangle,\ \mathbf{i} \mapsto \mathsf{i},\ \mathbf{j} \mapsto \mathsf{j}; \quad \Sigma : A^{\mathsf{hu}}_\infty \to \mathrm{A}^{\mathsf{hu}}_\infty.$$

The latter is a restriction of the previous map. For algebras A over operads $\mathrm{A}^{\mathsf{su}}_\infty$, $\mathrm{A}^{\mathsf{su}}_\infty\langle \mathsf{i}, \mathsf{j}\rangle$, $\mathrm{A}^{\mathsf{hu}}_\infty$ the complex $A[1]$ obtains a structure of an algebra over the operad A^{su}_∞, $A^{\mathsf{su}}_\infty\langle \mathbf{i}, \mathbf{j}\rangle$ or A^{hu}_∞ due to a property similar to (1.5), in particular,

$$\mathbf{1}^{\mathsf{su}}\sigma^{-1} = 1^{\mathsf{su}}, \qquad \mathbf{i}\sigma^{-1} = \mathsf{i}, \qquad \mathbf{j}\sigma^{-1} = \mathsf{j} : \Bbbk \to A.$$

2. Model structure of the category of operad polymodules

2.1. The operad $T\mathcal{N} \sqcup \mathcal{O}$. As explained in [Lyu11, Proposition 1.8] the operad $T\mathcal{N} \sqcup \mathcal{O}$ is a direct sum over ordered rooted trees t with inputs whose vertices are coloured with $\mathcal{N}$ and $\mathcal{O}$ that are terminal the following sense. Two conditions hold:

$\lightning$) t contains no edge whose both ends are coloured with $\mathcal{O}$;

$\lightning\lightning$) insertion of a unary vertex coloured with $\mathcal{O}$ into an arbitrary edge of t breaks condition $\lightning$).

By the way, these conditions imply that t contains no edge whose both ends are coloured with $\mathcal{N}$. The second condition is related to inserting a unit $\eta : \Bbbk \to T\mathcal{N} \sqcup \mathcal{O}$ identified with the unit $\eta : \Bbbk \to \mathcal{O}$ in an arbitrary summand of $T\mathcal{N} \sqcup \mathcal{O}$ represented by a coloured tree. The first condition means that one can not apply binary composition in $\mathcal{O}$ inside an element of

$$C(\mathcal{N}, \mathcal{O}; t) \simeq \otimes^{v\in(\mathrm{v}(t),\leqslant)} c(v)(|v|) \subset C(\mathcal{N} \oplus \mathcal{O}; t) \subset T(\mathcal{N} \oplus \mathcal{O})$$

represented by t. Here $\leqslant$ is an admissible order on $\mathrm{v}(t)$, the set of internal vertices of an ordered rooted trees t with inputs $\operatorname{Inp} t$, see Section 1.5 of [Lyu11]. The set $\mathrm{v}(t)$ is coloured by the function $c : \mathrm{v}(t) \to \{\mathcal{N}, \mathcal{O}\}$, which singles out an individual summand of $(\mathcal{N} \oplus \mathcal{O})^{\otimes \mathrm{v}(t)}$.

Any sequence of contractions of edges whose ends are coloured with $\mathcal{O}$ and insertions of unary vertices coloured with $\mathcal{O}$ applied to given tree t' may lead to no more than one terminal tree t. The mapping $t' \mapsto t$ is well defined. The summand $C(\mathcal{N}, \mathcal{O}; t')$ of $T(\mathcal{N}\oplus\mathcal{O})$ corresponding to t' is mapped by binary compositions in $\mathcal{B}$ and insertions of the unit of $\mathcal{B}$ to the summand $C(\mathcal{N}, \mathcal{O}; t)$. Associativity and unitality of $\mathcal{O}$ implies that this map is unique.

The requirement of $\mathcal{O} \to C = T\mathcal{N} \sqcup \mathcal{O}$ being a morphism of operads (see [Lyu15, Corollary A.4]) reduces to agreeing with binary compositions and units. Therefore, in the case of operads equation (A.10) of [Lyu15] is equivalent to a family of similar diagrams with $\alpha : T\mathcal{O} \to \mathcal{O}$ replaced with the unit $1_{\mathcal{O}} : \Bbbk \to \mathcal{O}$ and with the binary compositions $\mathcal{O} \odot \mathcal{O} \supset \Bbbk\otimes\cdots\otimes\Bbbk\otimes\mathcal{O}\otimes\Bbbk\otimes\cdots\otimes\Bbbk\otimes\mathcal{O} \xrightarrow{\mu} \mathcal{O}$. That is, the kernel $\mathcal{I}$ of $T(\mathcal{N}\oplus\mathcal{O}) \to T\mathcal{N}\sqcup\mathcal{O}$ is generated as an ideal by relations coming from binary composition in $\mathcal{O}$ (corresponding to contraction of an $\mathcal{O}$-coloured edge) and from inserting a unit of $\mathcal{O}$ (corresponding to insertion of $\mathcal{O}$-coloured unary vertex into an edge). Together with the above this implies that an element of $C(\mathcal{N},\mathcal{O};t')$ is equivalent to a unique element of $C(\mathcal{N},\mathcal{O};t)$ modulo $\mathcal{I}$. In fact, for elements $x'\in C(\mathcal{N},\mathcal{O};t')$ and $x''\in C(\mathcal{N},\mathcal{O};t'')$ related by an elementary relation as above this holds true since there is either a path $(t',t'',\dots,t)$ or a path $(t'',t',\dots,t)$ consisting of contractions or unary insertions. We conclude that [Lyu11, eq. (1.2)]

$$T\mathcal{N}\sqcup\mathcal{O}=\coprod_{t\in\mathcal{L}}C(\mathcal{N},\mathcal{O};t),$$

where $\mathcal{L}$ is the list of terminal trees.

2.2. Proposition. *Let $\mathcal{V}=\mathbf{dg}$. Then the category ${}_n\mathrm{Op}_1$ of $n\wedge 1$-operad modules with quasi-isomorphisms as weak equivalences and degreewise surjections as fibrations is a model category.*

Proof. Let us apply Theorem 1.3 to the adjunction $F:\mathbf{dg}^S\leftrightarrows {}_n\mathrm{Op}_1:U$, where $S=n\mathbb{N}\sqcup\mathbb{N}^n\sqcup\mathbb{N}$. By [Lyu15, Corollary 2.20] the category ${}_n\mathrm{Op}_1$ is complete and cocomplete.

Let $\mathcal{N}\in\operatorname{Ob}\mathbf{dg}^{\mathbb{N}}$ or $\mathcal{N}\in\operatorname{Ob}\mathbf{dg}^{\mathbb{N}^n}$ be a complex. For instance, $\mathcal{N}=\mathbb{K}_x[-p]$ for $p\in\mathbb{Z}$, $x\in\mathbb{N}$ or $x\in\mathbb{N}^n$. Let $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ be an $n\wedge 1$-operad module. Denote by $\tilde{\mathcal{N}}$ the object $(0,\dots,0;0;\mathcal{N})$ or $(0,\dots,0;\mathcal{N};0)$ or $(0,\dots,0,\mathcal{N},0,\dots,0;0;0)$ ($\mathcal{N}$ on i^{th} place) of $\mathbf{dg}^S$. We shall prove case by case that if $\mathcal{N}$ is contractible then so is $U(F\tilde{\mathcal{N}}\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B}))$.

By [Lyu15, Corollary A.4] the operad module in question $F\tilde{\mathcal{N}}\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ is the quotient of $F(\tilde{\mathcal{N}}\oplus(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B}))$ by the smallest ideal $\mathcal{I}$ generated by relations which tell that $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})\to F(\tilde{\mathcal{N}}\oplus(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B}))/\mathcal{I}$ is a morphism of $n\wedge 1$-operad modules. Notice by the way that

$$\begin{aligned}
F(0,\dots,0;0;\mathcal{N})\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})&=(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{Q};T\mathcal{N}\sqcup\mathcal{B}),\\
\mathcal{Q}&=\mathcal{P}\odot^0_{\mathcal{B}}(T\mathcal{N}\sqcup\mathcal{B}),\\
F(0,\dots,0;\mathcal{N};0)\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})&=(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{R};\mathcal{B}),\\
\mathcal{R}&=\mathcal{P}\oplus\odot_{\geqslant 0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{N};\mathcal{B}),\\
F(0,\dots,0,\mathcal{N},0,\dots,0;0;0)\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})&=(\mathcal{A}_1,\dots,T\mathcal{N}\sqcup\mathcal{A}_i,\dots,\mathcal{A}_n;\mathcal{S};\mathcal{B}),\\
\mathcal{S}&=(T\mathcal{N}\sqcup\mathcal{A}_i)\odot^i_{\mathcal{A}_i}\mathcal{P}.
\end{aligned}$$

More important is the presentation of operad polymodules as direct sums over some kind of trees. This presentation we use for the proof. Similarly to the operad case considered in Section 2.1 we find that

$$\mathcal{Q}=\textstyle\coprod_{\tau\in\mathcal{L}_{\mathcal{Q}}}\mathcal{Q}(\mathcal{P},\mathcal{N},\mathcal{B};\tau),\qquad\text{where}\qquad\mathcal{Q}(\mathcal{P},\mathcal{N},\mathcal{B};\tau)=\operatorname*{colim}_{\leqslant\in\mathcal{G}_{\mathcal{Q}}(\tau)}\mathcal{Q}(\mathcal{P},\mathcal{N},\mathcal{B};\tau)(\leqslant).$$

Here objects of the groupoid $\mathcal{G}_{\mathcal{Q}}(\tau)$ are admissible (compatible with the natural partial order $\preccurlyeq$ with the root vertex as the biggest element) total orderings $\leqslant$ of the set of vertices

$\overline{\mathrm{v}}(\tau)$. By definition between any two objects of $\mathcal{G}(\tau)$ there is precisely one morphism. Thus $\mathcal{G}(\tau)$ is contractible (equivalent to the terminal category with one object and one morphism). Therefore the colimit is isomorphic to any of

$$\mathcal{Q}(\mathcal{P},\mathcal{N},\mathcal{B};\tau)(\leqslant) = \otimes^{v\in(\overline{\mathrm{v}}(\tau),\leqslant)} c(v)(|v|).$$

Here $\tau = (\tau, c, |\cdot|)$ is a planar rooted tree τ with inputs $\operatorname{Inp}\tau$, a colouring $c : \overline{\mathrm{v}}(\tau) \to \{\mathcal{P},\mathcal{N},\mathcal{B}\}$ such that $c(\operatorname{Inp}\tau) \subset \{\mathcal{P}\}$, $c(\mathrm{v}(\tau)) \subset \{\mathcal{N},\mathcal{B}\}$ and an arbitrary function $|\cdot| : \operatorname{Inp}(\tau) \to \mathbb{N}^n$, which complements the valency $|\cdot| : \mathrm{v}(\tau) \to \mathbb{N}$. The set $\mathcal{L}_{\mathcal{Q}}$ of terminal trees consists of τ such that

$*$) τ contains no edge whose both ends are coloured with $\mathcal{B}$;

$**$) τ contains no vertex coloured with $\mathcal{B}$ whose all entering edges have other ends coloured with $\mathcal{P}$;

$***$) insertion of a unary vertex coloured with $\mathcal{B}$ into an arbitrary edge of τ breaks condition $*$) or $**$).

Respectively,

$$\mathcal{S} = \textstyle\coprod_{\tau\in\mathcal{L}_{\mathcal{S}}} \mathcal{S}(\mathcal{N},\mathcal{A}_i,\mathcal{P};\tau), \qquad \text{where} \qquad \mathcal{S}(\mathcal{N},\mathcal{A}_i,\mathcal{P};\tau) = \operatorname*{colim}_{\leqslant\in\mathcal{G}_{\mathcal{S}}(\tau)} \mathcal{S}(\mathcal{N},\mathcal{A}_i,\mathcal{P};\tau)(\leqslant).$$

The colimit over the contractible groupoid $\mathcal{G}_{\mathcal{S}}(\tau)$ is isomorphic to expression under colimit in any vertex $\leqslant$. Assuming that $\ell \in \mathbb{N}^n$, $\ell^i = |\operatorname{Inp}\tau|$, we have

$$\mathcal{S}(\mathcal{N},\mathcal{A}_i,\mathcal{P};\tau)(\leqslant)(\ell) = \big[\otimes^{v\in(\mathrm{v}(\tau)-\{\mathrm{rv}\},\leqslant)} c(v)(|v|)\big] \otimes \mathcal{P}(\ell,\ell^i \mapsto q).$$

Here $\tau = (\tau, c)$ is an ordered rooted tree τ with inputs $\operatorname{Inp}\tau$ and a colouring $c : \mathrm{v}(\tau) - \{\mathrm{rv}\} \to \{\mathcal{N},\mathcal{A}_i\}$. The set $\mathcal{L}_{\mathcal{S}}$ of terminal trees consists of τ such that

$*$) τ contains no edge whose both ends are coloured with $\mathcal{B}$;

$**$) τ contains no edge adjacent to the root vertex whose one end is coloured with $\mathcal{B}$;

$***$) insertion of a unary vertex coloured with $\mathcal{B}$ into an arbitrary edge of τ breaks condition $*$) or $**$).

Let us prove existence of contracting homotopy similarly to the case of operads [Lyu11, Proposition 1.8]. Let $\mathcal{N} \in \operatorname{Ob}\mathbf{dg}^{\mathbb{N}}$ or $\mathcal{N} \in \operatorname{Ob}\mathbf{dg}^{\mathbb{N}^n}$ be contractible and let $h : \mathcal{N} \to \mathcal{N}$ be a contracting homotopy, $\deg h = -1$, $dh + hd = 1_{\mathcal{N}}$. Let us show that the operad module morphism $\alpha = \mathrm{in}_2 : \mathcal{M} = (\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B}) \to F\tilde{\mathcal{N}} \sqcup (\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ is homotopy invertible. Consider the operad module morphism $\beta : F\tilde{\mathcal{N}} \sqcup (\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B}) \to (\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ which restricts to $\beta\big|_{(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})} = \mathrm{id}$ and $\beta\big|_{F\tilde{\mathcal{N}}}$, adjunct to $0 : \tilde{\mathcal{N}} \to U(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$. Then $\alpha\cdot\beta = \mathrm{id}$ and $g = \beta\cdot\alpha$ is homotopic to $f = \mathrm{id}_{F\tilde{\mathcal{N}}\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})}$ in the $\mathbf{dg}$-category $\mathbf{dg}^S$, as we show next. The homotopy h is extended by 0 to the map $h' = h \oplus 0 : \tilde{\mathcal{N}} \oplus U(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to \tilde{\mathcal{N}} \oplus U(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$, which satisfies $dh' + h'd = f| - g| : \tilde{\mathcal{N}} \oplus U(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to \tilde{\mathcal{N}} \oplus U(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$. In all three cases the endomorphisms f, g of $F\tilde{\mathcal{N}} \sqcup (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$ lift to endomorphisms of $F(\tilde{\mathcal{N}} \oplus (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}))$ obtained by applying $f\big|_{\tilde{\mathcal{N}}} = 1 : \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}$, $f\big|_{\mathcal{M}} = 1 : \mathcal{M} \to \mathcal{M}$, $g\big|_{\tilde{\mathcal{N}}} = 0 : \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}$, $g\big|_{\mathcal{M}} = 1 : \mathcal{M} \to \mathcal{M}$ to each $\otimes$-factor corresponding to a vertex of the tree. For an arbitrary pair of trees (t, τ) choose admissible orderings $(\leqslant, \leqslant)$. Then the summands of $F\tilde{\mathcal{N}} \sqcup (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$ are preserved by f and g and the restriction to the summand is $f \otimes \dots \otimes f$ and $g \otimes \dots \otimes g$ respectively. Define a $\Bbbk$-endomorphism $\hat{h} = \sum_{v \in (\mathrm{v}(t), \leqslant)} f \otimes \dots \otimes f \otimes h' \otimes g \otimes \dots \otimes g$ of degree -1, where h' is applied on place indexed by v. Then

$$d\hat{h} + \hat{h}d = \sum_{v \in (\mathrm{v}(t), \leqslant)} f \otimes \dots \otimes f \otimes (f - g) \otimes g \otimes \dots \otimes g = f \otimes \dots \otimes f - g \otimes \dots \otimes g = f - g.$$

Therefore, f and g are homotopic to each other and α is homotopy invertible. ∎

Proposition A.6 of [Lyu15] gives a recipe of computing colimits in the category ${}_n\mathrm{Op}_1$ of $n \wedge 1$-operad modules in two steps. First of all compute colimits $\mathcal{B}_i$ on each of $n+1$ operadic places $i \in [n]$. Take induced module over $(\mathcal{B}_1, \dots, \mathcal{B}_n; \mathcal{B}_0)$ on each node of the diagram and consider the obtained diagram in the category $(\mathcal{B}_1, \dots, \mathcal{B}_n)$-mod-$\mathcal{B}_0$. Secondly find the colimit of the latter diagram, by finding its colimit in $\mathcal{V}^{\mathbb{N}^n}$, then generating by it a free $(\mathcal{B}_1, \dots, \mathcal{B}_n; \mathcal{B}_0)$-module F, dividing it precisely by such relations that canonical mapping from any module to the quotient of F were a morphism of $(\mathcal{B}_1, \dots, \mathcal{B}_n; \mathcal{B}_0)$-modules.

3. Morphisms with several entries

Here we give support to the observation that morphisms with n entries of algebras over operads form an $n \wedge 1$-operad module. In particular, we find this module for A_∞-algebras.

3.1. Features of the lax $\mathcal{C}at$-multifunctor *hom*. An example of symmetric **dg**-multicategory C comes from $\mathsf{C}_\Bbbk$ – the closed symmetric multicategory of complexes of $\Bbbk$-modules and their chain maps [BLM08, Example 3.18]. It is representable by the symmetric Monoidal category **dg** of complexes and chain maps [BLM08, Example 3.27]. We take for C the associated enriched symmetric multicategory $\underline{\mathsf{C}}_\Bbbk$, which is a $\mathsf{C}_\Bbbk$-multicategory, or equivalently, a **dg**-multicategory. The composition in $\underline{\mathsf{C}}_\Bbbk$ has a natural meaning: this is a composition (of tensor products) of homogeneous maps, taking into account the Koszul rule. We use $\underline{\mathsf{C}}_\Bbbk$ to define the lax $\mathcal{C}at$-multifunctor *hom*.

Let us discuss the relationship between the suspension and the composition $\mu^T_{\underline{\mathsf{C}}_\Bbbk}$ for a functor $T : [l] \to \mathcal{S}_{\mathsf{sk}}$, where $\mathcal{S}_{\mathsf{sk}}$ is the full subcategory of $\mathcal{S}et$ formed by $\mathbf{n}$, $n \geqslant 0$.

Let $g : U \to W$, $f_i : X_i \to Y_i$, $1 \leqslant i \leqslant k$ be homogeneous maps of certain degrees. For any $1 \leqslant j \leqslant k$ the maps

$$\underline{\mathsf{C}}_\Bbbk\big((1)_{i<j}, f_j, (1)_{i>j}; 1\big) : \underline{\mathsf{C}}_\Bbbk\big((X_i)_{i<j}, (Y_i)_{i \geqslant j}; U\big) \to \underline{\mathsf{C}}_\Bbbk\big((X_i)_{i \leqslant j}, (Y_i)_{i>j}; U\big)$$

are defined as the precomposition with f_j, $h \mapsto (-1)^{h \cdot f_j}(1^{j-1} \times f_j \times 1^{k-j}) \cdot h$. The map

$$\underline{\mathsf{C}}_{\Bbbk}\big((1)_{i=1}^k; g\big) : \underline{\mathsf{C}}_{\Bbbk}\big((X_i)_{i=1}^k; U\big) \to \underline{\mathsf{C}}_{\Bbbk}\big((X_i)_{i=1}^k; W\big)$$

is defined as the postcomposition with g, $h \mapsto h \cdot g$. By convention, the map

$$\underline{\mathsf{C}}_{\Bbbk}\big(f_1, f_2, \dots, f_k; g\big) : \underline{\mathsf{C}}_{\Bbbk}\big((Y_i)_{i=1}^k; U\big) \to \underline{\mathsf{C}}_{\Bbbk}\big((X_i)_{i=1}^k; W\big)$$

is the composition (in this order)

$$\underline{\mathsf{C}}_{\Bbbk}\big(f_1, 1, \dots, 1; 1\big) \cdot \underline{\mathsf{C}}_{\Bbbk}\big(1, f_2, \dots, 1; 1\big) \cdot \dots \cdot \underline{\mathsf{C}}_{\Bbbk}\big(1, \dots, 1, f_k; 1\big) \cdot \underline{\mathsf{C}}_{\Bbbk}\big(1, \dots, 1, 1; g\big).$$

Factors of this product commute up to the sign depending on parity of the product of degrees of factors.

3.2. Lemma. *For arbitrary complexes $A_e \in \operatorname{Ob} \underline{\mathsf{C}}_{\Bbbk}$, $e \in E(T)$, and strongly ordered tree $(T, \leqslant)$ the following square commutes up to the sign $(-1)^{c(T)}$*

$$\begin{array}{ccc}
\bigotimes^{v \in \mathrm{v}(T)} \underline{\mathsf{C}}_{\Bbbk}\big((sA_e)_{e \in \mathrm{in}(v)}; sA_{\mathrm{ou}(v)}\big) & \xrightarrow{\mu^T_{\underline{\mathsf{C}}_{\Bbbk}}} & \underline{\mathsf{C}}_{\Bbbk}\big((sA_a)_{a \in \operatorname{Inp} T}; sA_{\mathit{root\ edge}(T)}\big) \\
\Big\downarrow {\scriptstyle \otimes^{v \in \mathrm{v}(T)} \underline{\mathsf{C}}_{\Bbbk}(^{\mathrm{in}(v)}\sigma; \sigma^{-1})} & (-1)^{c(T)} & \Big\downarrow {\scriptstyle \underline{\mathsf{C}}_{\Bbbk}(^{\operatorname{Inp} T}\sigma; \sigma^{-1})} \\
\bigotimes_{v \in \mathrm{v}(T)} \underline{\mathsf{C}}_{\Bbbk}\big((A_e)_{e \in \mathrm{in}(v)}; A_{\mathrm{ou}(v)}\big) & \xrightarrow[\mu^T_{\underline{\mathsf{C}}_{\Bbbk}}]{} & \underline{\mathsf{C}}_{\Bbbk}\big((A_a)_{a \in \operatorname{Inp} T}; A_{\mathit{root\ edge}(T)}\big)
\end{array}$$

where

$$c(T, \leqslant) = \sum_{v \in (\mathrm{v}(T), \leqslant)} \Big(1 - \#(v) - \sum_{v < x < Pv}^{x \in \mathrm{v}(T)} |x|\Big) + |\{(v, x) \in \mathrm{v}(T)^2 \mid v < x,\ Px < Pv\}|$$
$$+ |\{(v, x) \in \mathrm{v}(T)^2 \mid v < x < Pv = Px,\ x \lessdot v\}|$$
$$+ |\{(v, x) \in \operatorname{Inpv}(T)^2 \mid v < x,\ Px < Pv\}|,$$

the function $\# : \mathrm{v}(T) \to \mathbb{N}$, $v \mapsto \#(v)$ is determined by its restrictions $\# : \operatorname{inV}(Pv) \xrightarrow{\cong} \mathbf{z}$, $z = |Pv|$, the unique order-preserving maps.

Proof. The sign coincides with the sign of permutation of the expression $\otimes^{v \in \mathrm{v}(T)}(\sigma^{\otimes |v|} \otimes \sigma^{-1})$, followed by cancellation of matching σ^{-1} and σ, resulting in $(-1)^{c(T)} \sigma^{\otimes \operatorname{Inp} T} \otimes \sigma^{-1}$. The summands $\#(v) - 1$ come from cancelling σ^{-1} corresponding to v against the $\#(v)$-th factor of $\sigma^{\otimes |v|}$. The summand $|x|$ comes from moving this σ^{-1} past factor indexed by x. The last summand is of similar nature. ■

Let $g : U \to W$, $f_i : X_i \to Y_i$, $1 \leqslant i \leqslant k$ be homogeneous maps of certain degrees. Then

$$\mathit{hom}\big((f_i)_{i\in I}; g\big) : \mathit{hom}\big((Y_i)_{i\in I}; U\big) \to \mathit{hom}\big((X_i)_{i\in I}; W\big)$$

denotes the collection of homogeneous maps

$$\mathit{hom}\big((f_i)_{i\in I}; g\big) = \underline{\mathsf{C}}_{\Bbbk}\big(({}^{n^i} f_i)_{i\in I}; g\big) : \underline{\mathsf{C}}_{\Bbbk}\big(({}^{n^i} Y_i)_{i\in I}; U\big) \to \underline{\mathsf{C}}_{\Bbbk}\big(({}^{n^i} X_i)_{i\in I}; W\big).$$

3.3. Corollary. *For each t-tree τ the following square commutes up to the sign* $(-1)^{c(\tilde\tau)}$

$$\begin{array}{ccc}
\displaystyle\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} \mathit{hom}\big(sA_{\mathrm{in}(v)}; sA_{\mathrm{ou}(v)}\big)\big((|\tau(e)^{-1}(p)|)_{e\in\mathrm{in}(v)}\big) & \xrightarrow{\mathrm{comp}_\tau} & \mathit{hom}\big(sA_{\mathrm{Inp}\,t}; sA_{\mathit{root\ edge}(t)}\big)\big((|\tau(a)|)_{a\in\mathrm{Inpv}\,t}\big) \\
\Big\downarrow {\scriptstyle \otimes^{v\in\mathrm{v}(t)}\otimes^{p\in\tau(v)}\mathit{hom}({}^{\mathrm{in}(v)}\sigma;\sigma^{-1})((|\tau(e)^{-1}(p)|)_{e\in\mathrm{in}(v)})} & (-1)^{c(\tilde\tau)} & \Big\downarrow {\scriptstyle \mathit{hom}({}^{\mathrm{Inp}\,t}\sigma;\sigma^{-1})((|\tau(a)|)_{a\in\mathrm{Inpv}\,t})} \\
\displaystyle\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} \mathit{hom}\big(A_{\mathrm{in}(v)}; A_{\mathrm{ou}(v)}\big)\big((|\tau(e)^{-1}(p)|)_{e\in\mathrm{in}(v)}\big) & \xrightarrow{\mathrm{comp}_\tau} & \mathit{hom}\big(A_{\mathrm{Inp}\,t}; A_{\mathit{root\ edge}(t)}\big)\big((|\tau(a)|)_{a\in\mathrm{Inpv}\,t}\big)
\end{array}$$

where $\tilde\tau \in \mathrm{tr}$ *corresponds to* $\tau : t \to \mathcal{O}_{\mathsf{sk}}$,

$$\begin{aligned}
c(\tilde\tau) = \sum_{v\in\mathrm{v}(t)} \sum_{p\in\tau(v)} \Big(1 - \#(v,p) - \sum_{(v,p)<(x,y)<(Pv,\tau(\mathrm{ou}(v)).p)}^{(x,y)\in\mathrm{v}(\tilde\tau)} |(x,y)|\Big) \\
+ |\{(v,p,x,y) \in \mathrm{v}(\tilde\tau)^2 \mid (v,p) < (x,y),\ (Px,\tau(\mathrm{ou}(x)).y) < (Pv,\tau(\mathrm{ou}(v)).p)\}| \\
+ |\{(v,p,x,y) \in \mathrm{v}(\tilde\tau)^2 \mid (v,p) < (x,y) < P(v,p) = P(x,y),\ (x,y) \lessdot (v,p)\}| \\
+ |\{(v,p,x,y) \in \mathrm{Inpv}(\tilde\tau)^2 \mid (v,p) < (x,y),\ (Px,\tau(\mathrm{ou}(x)).y) < (Pv,\tau(\mathrm{ou}(v)).p)\}|.
\end{aligned}$$

Recall that the orders $<$, $\lessdot$ in $\mathrm{v}(\tilde\tau)$ and $<$ in $\mathrm{Inpv}(\tilde\tau)$ are lexicographical. Thus $(v,p) < (x,y)$ in $\mathrm{Inpv}(\tilde\tau)$ iff either $v \lhd x$ or $v = x$ and $p < y$ in $\tau(v)$.

3.4. Main source of $n \wedge 1$-operad modules. From now on we assume tacitly that $\mathcal{V} = \mathbf{dg}$. When the differential is not concerned we may use $\mathcal{V} = \mathbf{gr}$.

3.5. Example. Let As denote $\mathcal{V}$-operad with $As(0) = \emptyset$, the initial object, and $As(n) = \mathbb{1}$, the unit object of $\mathcal{V}$, for $n > 0$. Let us describe an $n \wedge 1$-operad As-module FAs_n with $FAs_n(j^1, \dots, j^n) = \mathbb{1}$ for all non–vanishing $(j^1, \dots, j^n) \in \mathbb{N}^n$, while $FAs_n(0, \dots, 0) = \emptyset$. In particular, $FAs_0 = \emptyset$. The actions for FAs_n are given by multiplication for $\mathbb{1}$. Associate *hom* with the symmetric $\mathcal{V}$-multicategory $\underline{\mathcal{V}}$, represented by the symmetric monoidal $\mathcal{V}$-category $\underline{\mathcal{V}}$. A morphism of $n \wedge 1$-operad modules

$$(As, \dots, As; FAs_n; As) \to (\mathcal{E}nd\, A_1, \dots, \mathcal{E}nd\, A_n; \mathit{hom}(A_1, \dots, A_n; B); \mathcal{E}nd\, B)$$

amounts to a family of morphisms $f_i : A_i \to B$ of associative algebras without units, $i \in \mathbf{n}$, such that the following diagrams commute for all $1 \leqslant i < j \leqslant n$:

$$\begin{array}{ccccc} A_i \otimes A_j & \xrightarrow[\cong]{c} & A_j \otimes A_i & \xrightarrow{f_j \otimes f_i} & B \otimes B \\ {\scriptstyle f_i \otimes f_j}\downarrow & & & & \downarrow{\scriptstyle m_B} \\ B \otimes B & & \xrightarrow{\quad m_B \quad} & & B \end{array} \tag{3.1}$$

In fact, morphisms $f_i = f_{(e_i)}$ are particular cases of the action map

$$f_{(e_i)} : \mathbb{1} = FAs_n(e_i) \to \underline{\mathcal{V}}(A_i; B),$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^n$ has 1 on i-th place. The equations hold for all $1 \leqslant i < j \leqslant n$:

$$\begin{array}{ccc} FAs_n(e_i) \otimes FAs_n(e_j) \otimes As(2) & \xrightarrow{\text{mult}} & FAs_n(e_i + e_j) \\ \downarrow & = & \downarrow \\ \mathcal{H}(e_i) \otimes \mathcal{H}(e_j) \otimes (\mathcal{E}nd\, B)(2) & \xrightarrow{\mu_{\text{id}}} & \mathcal{H}(e_i + e_j) \end{array}$$

$$\begin{array}{ccc} FAs_n(e_j) \otimes FAs_n(e_i) \otimes As(2) & \xrightarrow{\text{mult}} & FAs_n(e_i + e_j) \\ \downarrow & = & \downarrow \\ \mathcal{H}(e_j) \otimes \mathcal{H}(e_i) \otimes (\mathcal{E}nd\, B)(2) & \xrightarrow{\mu_{(12)}} & \mathcal{H}(e_i + e_j) \end{array}$$

Here the compositions μ_{id} and $\mu_{(12)}$ in $\underline{\mathcal{V}}$ correspond to the two maps

$$\begin{aligned} \text{id} &: \mathbf{0} \sqcup \dots \sqcup \mathbf{0} \sqcup \mathbf{1} \sqcup \mathbf{0} \sqcup \dots \sqcup \mathbf{0} \sqcup \mathbf{1} \sqcup \mathbf{0} \sqcup \dots \sqcup \mathbf{0} = \mathbf{2} \to \mathbf{2}, \\ (12) &: \mathbf{0} \sqcup \dots \sqcup \mathbf{0} \sqcup \mathbf{1} \sqcup \mathbf{0} \sqcup \dots \sqcup \mathbf{0} \sqcup \mathbf{1} \sqcup \mathbf{0} \sqcup \dots \sqcup \mathbf{0} = \mathbf{2} \to \mathbf{2}. \end{aligned}$$

The equations are more explicit in the form

$$\begin{array}{ccc} \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\qquad \text{mult} \qquad} & \mathbb{1} \\ {\scriptstyle f_{(e_i)} \otimes f_{(e_j)} \otimes m_B}\downarrow & = & \downarrow{\scriptstyle f_{(e_i+e_j)}} \\ \underline{\mathcal{V}}(A_i; B) \otimes \underline{\mathcal{V}}(A_j; B) \otimes \underline{\mathcal{V}}(B, B; B) & \xrightarrow{\mu_{\text{id}}} & \underline{\mathcal{V}}(A_i, A_j; B) \end{array}$$

$$\begin{array}{ccc} \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\qquad \text{mult} \qquad} & \mathbb{1} \\ {\scriptstyle f_{(e_j)} \otimes f_{(e_i)} \otimes m_B}\downarrow & = & \downarrow{\scriptstyle f_{(e_i+e_j)}} \\ \underline{\mathcal{V}}(A_j; B) \otimes \underline{\mathcal{V}}(A_i; B) \otimes \underline{\mathcal{V}}(B, B; B) & \xrightarrow{\mu_{(12)}} & \underline{\mathcal{V}}(A_i, A_j; B) \end{array}$$

Abusing the notation the same equations can be written as

$$(f_{(e_i)} \otimes f_{(e_j)})m_B = f_{(e_i+e_j)} = c(f_{(e_j)} \otimes f_{(e_i)})m_B : A_i \otimes A_j \to B,$$

which coincides with condition (3.1).

For non-vanishing $j = (j^1, \dots, j^n) \in \mathbb{N}^n$ the map $FAs1_n(j) = \mathbb{1} \to \underline{\mathcal{V}}(({}^{j^i}A_i)_{i=1}^n; B)$ is given by the morphism $(\otimes_{i=1}^n f_i^{\otimes j^i}) \cdot m_B^{\|j\|} : \otimes_{i=1}^n A_i^{\otimes j^i} \to B \in \mathcal{V}$. For $\mathcal{V} = (\mathcal{S}\mathrm{et}, \times)$ it is $\prod_{i=1}^n \prod_{k^i=1}^{j^i} a_i^{k^i} \mapsto f_1(a_1^1, \dots, a_1^{j^1}) \cdot \dots \cdot f_n(a_n^1, \dots, a_n^{j^n})$.

Notice that if A_i, $i \in \mathbf{n}$, B are unital algebras, a collection of unital morphisms $f_i : A_i \to B$ that satisfy equation (3.1) is the same as a single unital morphism $f : A_1 \otimes \dots \otimes A_n \to B$. In fact, such f determines $f_i(x) = f(1 \otimes \dots \otimes 1 \otimes x \otimes 1 \otimes \dots \otimes 1)$ and can be recovered from the whole collection of f_i's.

3.6. Definition. *Let $\mathcal{V} = \mathbf{dg}$. An $n \wedge 1$-operad module homomorphism*

$$(f_1, \dots, f_n; h; g) : (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to (\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{Q}; \mathcal{D}),$$

of degree $r \in \mathbb{Z}$ is a family of $\mathbf{dg}$-operad homomorphisms $g : \mathcal{B} \to \mathcal{D}$, $f_i : \mathcal{A}_i \to \mathcal{C}_i$, $0 \leqslant i \leqslant n$, of degree r and a collection of homogeneous $\mathbb{k}$-linear maps $h(j) : \mathcal{P}(j) \to \mathcal{Q}(j)$, $j \in \mathbb{N}^n$, of degree $r(1 - \|j\|)$ such that

- *for all $l \in \mathbb{N}$, $(k_q \in \mathbb{N}^n \mid 1 \leqslant q \leqslant l)$, the following square commutes up to the sign*

$$\begin{array}{ccc} \Big(\bigotimes_{q=1}^{l} \mathcal{P}(k_q)\Big) \otimes \mathcal{B}(l) & \xrightarrow{\rho} & \mathcal{P}\Big(\sum_{q=1}^{l} k_q\Big) \\ {\scriptstyle (\otimes_{q=1}^l h(k_q)) \otimes g(l)} \Big\downarrow & (-1)^{c_\rho} & \Big\downarrow {\scriptstyle h(\sum_{q=1}^l k_q)} \\ \Big(\bigotimes_{q=1}^{l} \mathcal{Q}(k_q)\Big) \otimes \mathcal{D}(l) & \xrightarrow{\rho} & \mathcal{Q}\Big(\sum_{q=1}^{l} k_q\Big) \end{array} \tag{3.2}$$

$$c_\rho = r \sum_{q=1}^{l} (q-1)(1 - \|k_q\|) + r \sum_{1 \leqslant b < a \leqslant l}^{1 \leqslant c < d \leqslant n} k_a^c k_b^d$$
$$+ \frac{r(r-1)}{2} \Big\{ (1 - l) \sum_{q=1}^{l} (1 - \|k_q\|) + \sum_{1 \leqslant q < s \leqslant l} (1 - \|k_q\|)(1 - \|k_s\|) \Big\}; \tag{3.3}$$

- *for all $k \in \mathbb{N}^n$, $(j_p^i \in \mathbb{N} \mid 1 \leqslant i \leqslant n,\ 0 \leqslant p \leqslant k^i)$, the following square commutes up to the sign*

$$\begin{array}{ccc} \Big[\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p^i)\Big] \otimes \mathcal{P}\big((k^i)_{i=1}^n\big) & \xrightarrow{\lambda} & \mathcal{P}\Big(\Big(\sum_{p=1}^{k^i} j_p^i\Big)_{i=1}^n\Big) \\ {\scriptstyle [\otimes_{i=1}^n \otimes_{p=1}^{k^i} f_i(j_p^i)] \otimes h(k)} \Big\downarrow & (-1)^{c_\lambda} & \Big\downarrow {\scriptstyle h((\sum_{p=1}^{k^i} j_p^i)_{i=1}^n)} \\ \Big[\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k^i} \mathcal{C}_i(j_p^i)\Big] \otimes \mathcal{Q}\big((k^i)_{i=1}^n\big) & \xrightarrow{\lambda} & \mathcal{Q}\Big(\Big(\sum_{p=1}^{k^i} j_p^i\Big)_{i=1}^n\Big) \end{array} \tag{3.4}$$

$$c_\lambda = r\sum_{i=1}^{n}\sum_{p=1}^{k^i}(1-j^i_p)\Bigl(p-1+\sum_{q=1}^{i-1}k^q\Bigr) + \frac{r(r-1)}{2}\Bigl\{(1-\|k\|)\sum_{i=1}^{n}\sum_{p=1}^{k^i}(1-j^i_p) + \sum_{1\leqslant i<l\leqslant n}\Bigl[\sum_{p=1}^{k^i}(1-j^i_p)\Bigr]\Bigl[\sum_{q=1}^{k^l}(1-j^l_q)\Bigr] + \sum_{i=1}^{n}\sum_{1\leqslant p<q\leqslant k^i}(1-j^i_p)(1-j^i_q)\Bigr\};$$

- *for all* $j \in \mathbb{N}^n$
$$d\cdot h(j) = (-1)^{r(1-\|j\|)}h(j)\cdot d : \mathcal{P}(j) \to \mathcal{Q}(j).$$

The second (shuffle) part of c_ρ, c_λ proportional to $r(r-1)/2$ makes sure that the composition of morphisms of degrees r and r' be a morphism of degree $r+r'$. The first part coincides with $rc(\tilde\tau_\rho)$, $rc(\tilde\tau_\lambda)$, see (2.8) and (2.9) of [Lyu15].

The last condition using λ can be replaced with n conditions using λ^i, $1\leqslant i\leqslant n$:

$$\begin{array}{ccc}
\Bigl[\bigotimes_{p=1}^{k^i}\mathcal{A}_i(j_p)\Bigr]\otimes\mathcal{P}(k) & \xrightarrow{\lambda^i} & \mathcal{P}\Bigl(k, k^i\mapsto\sum_{p=1}^{k^i}j_p\Bigr) \\
{\scriptstyle [\otimes_{p=1}^{k^i}f_i(j_p)]\otimes h(k)}\downarrow & (-1)^{c_{\lambda^i}} & \downarrow{\scriptstyle h(k,k^i\mapsto\sum_{p=1}^{k^i}j_p)} \\
\Bigl[\bigotimes_{p=1}^{k^i}\mathcal{C}_i(j_p)\Bigr]\otimes\mathcal{Q}(k) & \xrightarrow{\lambda^i} & \mathcal{Q}\Bigl(k, k^i\mapsto\sum_{p=1}^{k^i}j_p\Bigr)
\end{array}$$

$$c_{\lambda^i} = r\sum_{p=1}^{k^i}(1-j_p)\Bigl(p-1+\sum_{q=1}^{i-1}k^q\Bigr)+\frac{r(r-1)}{2}\Bigl\{(1-\|k\|)\sum_{p=1}^{k^i}(1-j_p)+\sum_{1\leqslant p<q\leqslant k^i}(1-j_p)(1-j_q)\Bigr\}. \tag{3.5}$$

3.7. Example. For all complexes A_1, …, A_n, B the collection

$$\varSigma = ({}^n\,hom(\sigma;\sigma^{-1}); hom({}^n\sigma;\sigma^{-1}); hom(\sigma;\sigma^{-1})) : \mathcal{H}om(sA_1,\dots,sA_n;sB)\to\mathcal{H}om(A_1,\dots,A_n;B)$$

is an $n\wedge 1$-operad morphism of degree 1. In fact, equations for $\varSigma$ involving λ and ρ are particular cases of Corollary 3.3.

3.8. A_∞-morphisms with several entries. Let us describe an $n\wedge1$-operad $As1$-module $FAs1_n$ with $FAs1_n(j^1,\dots,j^n)=\Bbbk$ for all $(j^1,\dots,j^n)\in\mathbb{N}^n$. In particular, $FAs1_0=\Bbbk$. The actions for $FAs1_n$ are given by multiplication in $\Bbbk$. Associate *hom* with the symmetric **dg**-multicategory $\underline{\mathsf{C}}_\Bbbk$. A morphism of $n\wedge1$-operad modules

$$(As1,\dots,As1;FAs1_n;As1)\to(\mathcal{E}nd\,A_1,\dots,\mathcal{E}nd\,A_n;hom(A_1,\dots,A_n;B);\mathcal{E}nd\,B)$$

amounts to a family of unital morphisms $f_i : A_i \to B$ of associative unital differential graded $\Bbbk$-algebras, $i \in \mathbf{n}$, such that diagrams (3.1) commute for all $1 \leqslant i < j \leqslant n$. These data are in bijection with unital homomorphisms $f : A_1 \otimes \cdots \otimes A_n \to B$, where $A_1, \dots, A_n$, B are unital associative **dg**-algebras.

In fact, each complex $A_1, \dots, A_n, B$ acquires a unital associative **dg**-algebra structure through morphisms $As1 \to \mathcal{E}nd\, A_i$, $As1 \to \mathcal{E}nd\, B$. Particular cases of actions

$$\lambda_{e_i} : \mathcal{A}_i(0) \otimes \mathcal{P}(e_i) \to \mathcal{P}(0),$$
$$\rho_\varnothing : \mathcal{B}(0) = \Bbbk \otimes \mathcal{B}(0) \to \mathcal{P}(0),$$

for the module $(As1, \dots, As1; FAs1_n; As1)$ take unity to unity:

$$\lambda_{e_i} : As1(0) \otimes FAs1_n(e_i) \ni 1 \otimes 1 \mapsto 1 \in FAs1_n(0),$$
$$\rho_\varnothing : As1(0) \ni 1 \mapsto 1 \in FAs1_n(0).$$

Commutative diagram (2.2) of [Lyu11] with $\mathcal{H}om(A_1, \dots, A_n; B)(0)$ in place of $\mathcal{H}om(A; B)(0)$ shows that $1 \in FAs1_n(0)$ is represented by 1_B. Since the representation agrees with λ_{e_i} the equation $1_{A_i}.f_i = 1_B$ holds, thus, f_i is unital.

3.9. Proposition. *There is an $n \wedge 1$-operad module $(\mathrm{A}_\infty, \mathrm{F}_n)$ freely generated as graded module by elements $\mathsf{f}_{j^1,\dots,j^n} \in \mathrm{F}_n(j^1, \dots, j^n)$, $(j^1, \dots, j^n) \in \mathbb{N}^n - 0$, of degree $1 - j^1 - \cdots - j^n = 1 - \|j\|$. The differential for it is given by*

$$\mathsf{f}_\ell \partial = \sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>1} (-1)^{(1-x)(\ell^1+\cdots+\ell^{q-1}+r)+1-\|\ell\|} \lambda^q_{(^r1,x,^t1)}(^r1, m_x, {}^t1; \mathsf{f}_{\ell-(x-1)e_q})$$
$$+ \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\cdots+j_k=\ell}}^{k>1} (-1)^{k + \sum_{1 \leqslant b < a \leqslant k}^{1 \leqslant c < d \leqslant n} j_a^c j_b^d + \sum_{p=1}^k (p-1)(\|j_p\|-1)} \rho_{(j_p^i)}((\mathsf{f}_{j_p})_{p=1}^k; m_k). \quad (3.6)$$

There is an invertible morphism of degree 1 between these $n \wedge 1$-operad modules

$$(\varSigma, \varSigma) : (A_\infty, F_n) \to (\mathrm{A}_\infty, \mathrm{F}_n), \qquad b_i \mapsto m_i, \quad f_j \mapsto \mathsf{f}_j. \quad (3.7)$$

F_n-maps are A_∞-algebra morphisms $A_1, \dots, A_n \to B$ (for algebras written with operations m_n). The two notions of A_∞-morphisms agree in the sense that the square of $n \wedge 1$-operad module maps

$$\begin{array}{ccc}
(^nA_\infty; F_n; A_\infty) & \longrightarrow & ((\mathcal{E}nd\, A_i[1])_{i=1}^n; hom((A_i[1])_{i=1}^n; B[1]); \mathcal{E}nd\, B[1]) \\
\Big\downarrow {\scriptstyle (^n\varSigma;\varSigma;\varSigma)} & & \Big\downarrow {\scriptstyle (^n hom(\sigma;\sigma^{-1}); hom(^n\sigma;\sigma^{-1}); hom(\sigma;\sigma^{-1}))} \\
(^n\mathrm{A}_\infty; \mathrm{F}_n; \mathrm{A}_\infty) & \longrightarrow & ((\mathcal{E}nd\, A_i)_{i=1}^n; hom((A_i)_{i=1}^n; B); \mathcal{E}nd\, B)
\end{array} \quad (3.8)$$

commutes.

Proof. The existence of F_n is proven in [Lyu15]. This implies the existence of F_n as the following lemma shows.

3.10. Lemma. *Let* $(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$ *be a* $\mathbf{dg}$*-*$n \wedge 1$*-operad module,* $(\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{Q}; \mathcal{D})$ *be a graded* $n \wedge 1$*-operad module and*

$$(f_1, \dots, f_n; h; g) : (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to (\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{Q}; \mathcal{D}),$$

be an invertible graded $n \wedge 1$*-operad module homomorphism of degree* r *(equations* (3.2) *and* (3.4) *hold). Then* $\mathcal{C}_1, \dots, \mathcal{C}_n$*,* $\mathcal{D}$ *are* $\mathbf{dg}$*-operads (see Remark 1.9) and* $\mathcal{P}$ *has a unique differential* d *which turns it into a* $\mathbf{dg}$*-*$n \wedge 1$*-operad module and* $(f_1, \dots, f_n; h; g)$ *into a* $\mathbf{dg}$*-*$n \wedge 1$*-operad module isomorphism of degree* r*.*

Proof. The differential is given by a unique expression

$$d = \bigl(\mathcal{Q}(j) \xrightarrow{h(j)^{-1}} \mathcal{P}(j) \xrightarrow{(-1)^{r(1-\|j\|)}d} \mathcal{P}(j) \xrightarrow{h(j)} \mathcal{Q}(j)\bigr).$$

Clearly, $\deg d = 1$ and $d^2 = 0$. Verification that ρ and λ for $\mathcal{Q}$ are chain maps is straightforward. ∎

Let us compute the value of the differential on generators f_ℓ:

$$\begin{aligned}
\mathsf{f}_\ell \partial &= (f_\ell.\varSigma(\ell))\partial = (-1)^{1-\|\ell\|}(f_\ell.\partial)\varSigma(\ell) \\
&= (-1)^{1-\|\ell\|} \sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>1} \lambda^q_{({}^r1,x,{}^t1)}({}^r1, b_x, {}^t1; f_{\ell-(x-1)e_q}).\varSigma(\ell) \\
&\qquad + (-1)^{\|\ell\|} \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} \rho_{(j^i_p)}((f_{j_p})^k_{p=1}; b_k).\varSigma(\ell) \\
&= \sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>1} (-1)^{c(\tilde{\tau}_{\lambda^q})+1-\|\ell\|} \lambda^q_{({}^r1,x,{}^t1)}({}^r1, m_x, {}^t1; \mathsf{f}_{\ell-(x-1)e_q}) \\
&\qquad + \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} (-1)^{k+c(\tilde{\tau}_\rho)} \rho_{(j^i_p)}((\mathsf{f}_{j_p})^k_{p=1}; m_k),
\end{aligned}$$

which coincides with (3.6), if one plugs in expressions $c(\tilde{\tau}_{\lambda^q}) = c_{\lambda^q}$ from (3.5) and $c(\tilde{\tau}_\rho) = c_\rho$ from (3.3) for $r = 1$.

The image of $\mathsf{f}_\ell \partial$ in $\mathit{hom}((A_i)_i; B)$ is

$$\sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>1} (-1)^{(1-x)(\ell^1+\dots+\ell^{q-1}+r)+1-\|\ell\|} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} A_i \xrightarrow{1^{\otimes(q-1)}\otimes(1^{\otimes r}\otimes m_x\otimes 1^{\otimes t})\otimes 1^{\otimes(n-q)}}$$
$$T^{\ell^1}A_1 \otimes \dots \otimes T^{\ell^{q-1}}A_{q-1} \otimes T^{r+1+t}A_q \otimes T^{\ell^{q+1}}A_{q+1} \otimes \dots \otimes T^{\ell^n}A_n \xrightarrow{\mathsf{f}_{\ell-(x-1)e_q}} B\big]$$
$$+ \sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>1} (-1)^{k+\sum_{1\leqslant b<a\leqslant k}^{1\leqslant c<d\leqslant n} j_a^c j_b^d + \sum_{p=1}^{k}(p-1)(\|j_p\|-1)} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} A_i \xrightarrow{\otimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \otimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j_p^i} A_i$$
$$\xrightarrow{\varkappa^{-1}} \otimes^{p\in\mathbf{k}} \otimes^{i\in\mathbf{n}} T^{j_p^i} A_i \xrightarrow{\otimes^{p\in\mathbf{k}}\mathsf{f}_{j_p}} \otimes^{p\in\mathbf{k}} B \xrightarrow{m_k} B\big].$$

F_n-algebra maps consist of A_∞-algebras $A_1, \dots, A_n$, B, and a collection $(\mathsf{f}_j)_{j\in\mathbb{N}^n-0}$ that satisfies the following equation for all $\ell \in \mathbb{N}^n - 0$:

$$\mathsf{f}_\ell m_1 + (-1)^{\|\ell\|} \Big[\sum_{q=1}^{n} \sum_{r+1+t=\ell^q} 1^{\otimes(q-1)} \otimes (1^{\otimes r} \otimes m_1 \otimes 1^{\otimes t}) \otimes 1^{\otimes(n-q)} \Big] \mathsf{f}_\ell = \mathsf{f}_\ell \partial.$$

In expanded form the equation says:

$$\sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>0} (-1)^{(1-x)(\ell^1+\dots+\ell^{q-1}+r)-\|\ell\|} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} A_i \xrightarrow{1^{\otimes(q-1)}\otimes(1^{\otimes r}\otimes m_x\otimes 1^{\otimes t})\otimes 1^{\otimes(n-q)}}$$
$$T^{\ell^1}A_1 \otimes \dots \otimes T^{\ell^{q-1}}A_{q-1} \otimes T^{r+1+t}A_q \otimes T^{\ell^{q+1}}A_{q+1} \otimes \dots \otimes T^{\ell^n}A_n \xrightarrow{\mathsf{f}_{\ell-(x-1)e_q}} B\big]$$
$$= \sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>0} (-1)^{k+\sum_{1\leqslant b<a\leqslant k}^{1\leqslant c<d\leqslant n} j_a^c j_b^d + \sum_{p=1}^{k}(p-1)(\|j_p\|-1)} \big[\otimes^{i\in\mathbf{n}} T^{\ell^i} A_i \xrightarrow{\otimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \otimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j_p^i} A_i$$
$$\xrightarrow{\varkappa^{-1}} \otimes^{p\in\mathbf{k}} \otimes^{i\in\mathbf{n}} T^{j_p^i} A_i \xrightarrow{\otimes^{p\in\mathbf{k}}\mathsf{f}_{j_p}} \otimes^{p\in\mathbf{k}} B \xrightarrow{m_k} B\big]. \quad (3.9)$$

This is actually the definition of an A_∞-algebra morphism $A_1, \dots, A_n \to B$ for algebras written with operations m_n, adopted in the current article.

Relationship between f_j and f_j in $\mathit{hom}((A_i)_{i=1}^n; B)(j)$,

$$\begin{array}{ccc} T^{j^1}A_1 \otimes \dots \otimes T^{j^n}A_n & \xrightarrow{\mathsf{f}_j} & B \\ \sigma^{\otimes j^1}\otimes\dots\otimes\sigma^{\otimes j^n} \downarrow & & \downarrow \sigma \\ T^{j^1}sA_1 \otimes \dots \otimes T^{j^n}sA_n & \xrightarrow{f_j} & sB \end{array}$$

shows that diagram (3.8) commutes on generators. Therefore, it is commutative. ■

Reducing the data used in [Lyu15, Definition 2.12] we call an *n-dimensional right operad module* the pair $(\mathcal{P};\mathcal{B})$ consisting of a $\mathbf{dg}$-operad $\mathcal{B}$ and an object $\mathcal{P} \in \mathbf{dg}^{\mathbb{N}^n}$, equipped with a unital associative action

$$\rho : \Big(\bigotimes_{q=1}^{l} \mathcal{P}(k_q)\Big) \otimes \mathcal{B}(l) \longrightarrow \mathcal{P}\Big(\sum_{q=1}^{l} k_q\Big) \in \mathbf{dg}\,.$$

3.11. Definition. *An n-dimensional right operad module homomorphism $(h;g) : (\mathcal{P};\mathcal{B}) \to (\mathcal{Q};\mathcal{D})$ of degree $(p;0)$, $p \in \mathbb{Z}^n$, is a $\mathbf{dg}$-operad homomorphism $g : \mathcal{B} \to \mathcal{D}$ of degree 0 and a collection of homogeneous $\Bbbk$-linear maps $h(j) : \mathcal{P}(j) \to \mathcal{Q}(j)$, $j \in \mathbb{N}^n$, of degree $(p|j) = \sum_{i=1}^{n} p^i j^i$ such that*

- *for all $l \in \mathbb{N}$, $(k_q \in \mathbb{N}^n \mid 1 \leqslant q \leqslant l)$, the following square commutes up to the sign*

$$\begin{array}{ccc} \Big(\bigotimes_{q=1}^{l} \mathcal{P}(k_q)\Big) \otimes \mathcal{B}(l) & \xrightarrow{\quad\rho\quad} & \mathcal{P}\Big(\sum_{q=1}^{l} k_q\Big) \\ {\scriptstyle (\otimes_{q=1}^{l} h(k_q))\otimes g(l)}\downarrow & (-1)^{c(k_1,\dots,k_l)} & \downarrow{\scriptstyle h(\sum_{q=1}^{l} k_q)} \\ \Big(\bigotimes_{q=1}^{l} \mathcal{Q}(k_q)\Big) \otimes \mathcal{D}(l) & \xrightarrow{\quad\rho\quad} & \mathcal{Q}\Big(\sum_{q=1}^{l} k_q\Big) \end{array} \tag{3.10}$$

$$c(k_1,\dots,k_l) = \sum_{1 \leqslant t < q \leqslant l} \chi(k_t,k_q), \tag{3.11}$$

where $\chi : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{Z}/2$ is an arbitrary bilinear form (it is specified by a matrix $\chi \in \mathrm{Mat}(n,\mathbb{Z}/2)$);

- *for all $j \in \mathbb{N}^n$*

$$d \cdot h(j) = (-1)^{(p|j)} h(j) \cdot d : \mathcal{P}(j) \to \mathcal{Q}(j). \tag{3.12}$$

3.12. Lemma. *Let $(\mathcal{P};\mathcal{B})$ be an n-dimensional right $\mathbf{dg}$-operad module, let $g : \mathcal{B} \to \mathcal{D}$ be a $\mathbf{dg}$-operad isomorphism of degree 0. Let $h(j) : \mathcal{P}(j) \to \mathcal{Q}(j)$, $j \in \mathbb{N}^n$, be a collection of invertible homogeneous $\Bbbk$-linear maps of degree $(p|j)$ for some $p \in \mathbb{Z}^n$. Let $\chi : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{Z}/2$ be a bilinear form. Then $\mathcal{Q}$ admits a unique structure of an n-dimensional right $\mathcal{D}$-module such that $(h;g) : (\mathcal{P};\mathcal{B}) \to (\mathcal{Q};\mathcal{D})$ is a homomorphism of degree $(p;0)$ with respect to χ.*

Proof. The value of the differential in $\mathcal{Q}$ is fixed by (3.12). The unique candidate ρ for action of $\mathcal{D}$ on $\mathcal{Q}$ is found from diagram (3.10). This ρ is a chain map, as follows from a cubical diagram consisting of two faces (3.10) joined by differentials. Opposite faces of the cube commute up to the same sign, since $(p|\sum_{q=1}^{l} k_q) = \sum_{q=1}^{l}(p|k_q)$. Therefore, the both squares expressing commutation of ρ with the differential commute simultaneously.

Associativity of the action of $\mathcal{D}$ on $\mathcal{Q}$ is expressed by the pentagon

$$\begin{array}{ccccc}
\bigotimes_{q=1}^{l}\Big(\bigotimes_{t=1}^{n_q}\mathcal{Q}({}_tk_q)\otimes\mathcal{D}(n_q)\Big)\otimes\mathcal{D}(l) & \xrightarrow{\otimes_{q=1}^{l}\rho\otimes 1} & \Big(\bigotimes_{q=1}^{l}\mathcal{Q}\Big(\sum_{t=1}^{n_q}{}_tk_q\Big)\Big)\otimes\mathcal{D}(l) & \xrightarrow{\rho} & \mathcal{Q}\Big(\sum_{q=1}^{l}\sum_{t=1}^{n_q}{}_tk_q\Big) \\
\wr\downarrow & & & \nearrow\rho & \\
\Big(\bigotimes_{q=1}^{l}\bigotimes_{t=1}^{n_q}\mathcal{Q}({}_tk_q)\Big)\otimes\Big(\bigotimes_{q=1}^{l}\mathcal{D}(n_q)\Big)\otimes\mathcal{D}(l) & \xrightarrow{1\otimes\mu_{\mathcal{D}}} & \Big(\bigotimes_{q=1}^{l}\bigotimes_{t=1}^{n_q}\mathcal{Q}({}_tk_q)\Big)\otimes\mathcal{D}\Big(\sum_{q=1}^{l}n_q\Big) & &
\end{array}$$

lying at the bottom of a rectangular prism, whose top face is the pentagon, expressing associativity of the action of $\mathcal{B}$ on $\mathcal{P}$. Vertical maps are tensor products of h and g. The walls commute up to sign. The product of these signs is $+1$, since

$$c\Big(\big(({}_tk_q)_{t=1}^{n_q}\big)_{q=1}^{l}\Big)=c\bigg(\Big(\sum_{t=1}^{n_q}{}_tk_q\Big)_{q=1}^{l}\bigg)+\sum_{q=1}^{l}c\big(({}_tk_q)_{t=1}^{n_q}\big)$$

due to definition (3.11) of c and bilinearity of χ.

Unitality of the action of $\mathcal{D}$ on $\mathcal{Q}$ follows from that for $\mathcal{B}$ and $\mathcal{P}$, since $c(k)=0$, $k\in\mathbb{N}^n$. ■

Cofibrant replacement of an $n\wedge 1$-operad module $(\mathcal{O},\mathcal{P})\overset{\text{def}}{=}(\mathcal{O},\dots,\mathcal{O};\mathcal{P};\mathcal{O})$ is a trivial fibration $(\mathcal{A},\mathcal{F})\to(\mathcal{O},\mathcal{P})$ such that the only map from the initial $n\wedge 1$-operad module $(\mathbb{1},0)\to(\mathcal{A},\mathcal{F})$ is a cofibration in ${}_n\mathrm{Op}_1$.

3.13. Theorem. *The $n\wedge 1$-operad module $(\mathrm{A}_\infty,\mathrm{F}_n)$ is a cofibrant replacement of (As,FAs_n). Moreover, $(\mathrm{A}_\infty,\mathrm{F}_n)\to(As,FAs_n)$ is a homotopy isomorphism in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$.*

Proof. Generate a free $n\wedge 1$-operad As-module $\overline{\mathrm{F}}_n$ by elements $\mathsf{f}_{j^1,\dots,j^n}\in\overline{\mathrm{F}}_n(j^1,\dots,j^n)$, $(j^1,\dots,j^n)\in\mathbb{N}^n-0$, of degree $1-j^1-\dots-j^n$. Actually, $(As,\overline{\mathrm{F}}_n)$ is the coequalizer in ${}_n\mathrm{Op}_1$ of the pair of morphisms of collections

$$0,\mathrm{in}:(\Bbbk\{(m_2\otimes 1)m_2-(1\otimes m_2)m_2,m_n\mid n\geqslant 3\},0)\rightrightarrows(\mathrm{A}_\infty,\mathrm{F}_n),$$

the second arrow is just the embedding. Therefore, the differential in $\overline{\mathrm{F}}_n$ reduces to

$$\mathsf{f}_\ell\partial=\sum_{q=1}^{n}\sum_{r+2+t=\ell^q}(-1)^{1+t+\ell^{q+1}+\dots+\ell^n}\lambda^q({}^r1,m,{}^t1;\mathsf{f}_{\ell-e_q})-\sum_{q+r=\ell}^{q,r\in\mathbb{N}^n-0}(-1)^{\|r\|+\sum_{c>d}q^cr^d}\rho(\mathsf{f}_q,\mathsf{f}_r;m),$$

$m=m_2$, and the equation $\partial^2=0$ follows. Notice that the quadratic part of the differential

$$\mathsf{f}_\ell\bar\partial=\sum_{q+r=\ell}^{q,r\in\mathbb{N}^n-0}(-1)^{1-\|r\|+\sum_{c>d}q^cr^d}\rho(\mathsf{f}_q,\mathsf{f}_r;m)\tag{3.13}$$

is a differential itself, $\bar\partial^2=0$.

The $n \wedge 1$-operad module morphism in question decomposes as

$$(\mathrm{A}_\infty, \mathrm{F}_n) \xrightarrow{htis} (As, \overline{\mathrm{F}}_n) \xrightarrow{(1,p)} (As, FAs_n).$$

The first epimorphism is a homotopy isomorphism, since $\mathrm{A}_\infty \to As$ is. Let us describe the second epimorphism and prove for it the same property, that is, the $n \wedge 1$-operad As-module epimorphism $p : \overline{\mathrm{F}}_n \to FAs_n$ is a homotopy isomorphism. We prove more: the zero degree cycle $p : \overline{\mathrm{F}}_n \to FAs_n$ is homotopy invertible in the **dg**-category n-As-mod.

Any left n-operad As-module $\mathcal{P}$ decomposes into a direct sum of submodules. Any subset $S \subset \mathbf{n}$ with the induced total ordering is viewed as the isomorphic ordinal with $|S|$ elements. For any $k \in \mathbb{N}^n$ denote by $\operatorname{supp} k = \{i \in \mathbf{n} \mid k^i \neq 0\}$ its support. Consider the n-operad As-submodule

$$\mathcal{P}^S(k) = \begin{cases} \mathcal{P}(k) & \text{if } \operatorname{supp} k = S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{P} = \oplus_{S \subset \mathbf{n}} \mathcal{P}^S$. Since $As(0) = 0$, the $\mathbb{Z}^n$-graded collection $\mathcal{P}^S$ is a left n-operad As-module. This structure boils down to a $\mathbb{Z}^S$-graded collection $\mathcal{P}^S$, which is a left S-operad As-module (that is, a $|S|$-operad As-module). A left n-operad As-module $\mathcal{P}$ is freely generated iff left S-operad As-modules $\mathcal{P}^S$ are freely generated for all $S \subset \mathbf{n}$.

Let $e_S \in \mathbb{N}^n$ have the coordinates $e_S^i = \chi(i \in S) \in \{0, 1\}$, $e_i \overset{\text{def}}{=} e_{\{i\}}$. For $j \in \mathbb{N}^n$, $j \neq 0$, consider the basic element $u_j = 1 \in \Bbbk = FAs_n(j)$. For $S \neq \varnothing$ the element $u_{e_S} = 1 \in \Bbbk = FAs_n(e_S)$ freely generates the left S-operad As-module FAs_n^S, while $FAs_n^\varnothing = 0$. Namely, for any $j \in \mathbb{N}^n$, $j \neq 0$, with support $S = \operatorname{supp} j$ we have $u_j = \lambda((m^{(j^i)})_{i \in S}; u_{e_S})$.

The left n-operad As-module $\overline{\mathrm{F}}_n$ is also freely generated. Its basis is given by elements $\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)})$, where $k > 0$ and $j_t \in \mathbb{N}^n - 0$ for all t.

The $n \wedge 1$-operad As-module map p is specified on the generators as follows:

$$\mathsf{f}_j.p = \begin{cases} u_j, & \text{if } \|j\| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

On the basis of the left n-operad As-module $\overline{\mathrm{F}}_n$ the map p is computed as

$$\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).p = \begin{cases} u_j, & \text{if } \ \|j_1\| = \dots = \|j_k\| = 1, \quad j = \sum_r j_r, \\ 0, & \text{otherwise.} \end{cases}$$

In order to prove that p is a chain map it suffices to prove that $\mathsf{f}_{2e_a}.\partial p = 0$, $1 \leqslant a \leqslant n$, and $\mathsf{f}_{e_a+e_b}.\partial p = 0$ for all $1 \leqslant a < b \leqslant n$. These equations are verified straightforwardly:

$$\mathsf{f}_{2e_a}.\partial = -\lambda(m; \mathsf{f}_{e_a}) + \rho(\mathsf{f}_{e_a}, \mathsf{f}_{e_a}; m) \overset{p}{\longmapsto} -\lambda(m; u_{e_a}) + u_{2e_a} = 0,$$

$$\mathsf{f}_{e_a+e_b}.\partial = \rho(\mathsf{f}_{e_a}, \mathsf{f}_{e_b}; m) - \rho(\mathsf{f}_{e_b}, \mathsf{f}_{e_a}; m) \overset{p}{\longmapsto} u_{e_a+e_b} - u_{e_b+e_a} = 0.$$

A zero degree cycle $\beta : FAs_n \to \overline{\mathrm{F}}_n$ in n-As-mod is given on generators u_j of free $\Bbbk$-modules $FAs_n(j)$ by the formula

$$u_j.\beta = \rho\big((\lambda(m^{(j^i)}; \mathsf{f}_{e_i}))_{i\in\operatorname{supp} j}; m^{(|\operatorname{supp} j|)}\big).$$

The composition

$$\begin{array}{ccccc} FAs_n & \xrightarrow{\quad\beta\quad} & \overline{\mathrm{F}}_n & \xrightarrow{\quad p \quad} & FAs_n \\ u_{e_S} & \longmapsto \rho((\mathsf{f}_{e_i})_{i\in S}; m^{(|S|)}) = & (\otimes_{i\in S}\mathsf{f}_{e_i})m^{(S)} & \longmapsto (\otimes_{i\in S}u_{e_i})m^{(S)} = & u_{e_S} \end{array}$$

is the identity map. Let us prove that $p\beta$ is homotopy invertible. These two statements would imply that p is homotopy invertible in n-As-mod and β is its homotopy inverse.

Let $\overline{\mathrm{F}}_n^{(q)}$ be a n-As-submodule generated by $\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)})$, $k \leqslant q$, $\overline{\mathrm{F}}_n^{(0)} = 0$. This filtration induces the graded n-As-module with the components $\overline{\mathrm{F}}_n^{\{k\}} = \overline{\mathrm{F}}_n^{(k)}/\overline{\mathrm{F}}_n^{(k-1)}$. Since the differential in As vanishes, the differential $\partial : \overline{\mathrm{F}}_n^{(q)} \to \overline{\mathrm{F}}_n^{(q+1)}$ is a left n-operad As-module map. We look for a left n-operad As-module map $h : \overline{\mathrm{F}}_n \to \overline{\mathrm{F}}_n$ of degree -1 such that $\overline{\mathrm{F}}_n^{(q)}.h \subset \overline{\mathrm{F}}_n^{(q-1)}$. Consider the zero degree cycle

$$N = 1 - p\beta + h\partial + \partial h \overline{\mathrm{F}}_n \to \overline{\mathrm{F}}_n.$$

It satisfies $\overline{\mathrm{F}}_n^{(q)}.N \subset \overline{\mathrm{F}}_n^{(q)}$. We are going to choose h in such a way that N be locally nilpotent. Thus, $1 - N$ is invertible with the (well–defined) inverse $\sum_{a=0}^\infty N^a$. Therefore,

$$p\beta = 1 - N + h\partial + \partial h : \overline{\mathrm{F}}_n \to \overline{\mathrm{F}}_n$$

is homotopy invertible.

Since $\rho\big(\overline{\mathrm{F}}_n^{(q_1)}(j_1)\otimes\cdots\otimes\overline{\mathrm{F}}_n^{(q_k)}(j_k)\otimes As(k)\big) \subset \overline{\mathrm{F}}_n^{(q_1+\cdots+q_k)}(j_1+\cdots+j_k)$ there is an induced map between quotients:

$$\bar\rho : \overline{\mathrm{F}}_n^{\{q_1\}}(j_1) \otimes \cdots \otimes \overline{\mathrm{F}}_n^{\{q_k\}}(j_k) \otimes As(k) \to \overline{\mathrm{F}}_n^{\{q_1+\cdots+q_k\}}(j_1 + \cdots + j_k)$$

The actions $\bar\rho$ assemble to an action of As on the sum $\overline{\mathrm{F}}_n^{\{\}} = \coprod_{q=0}^\infty \overline{\mathrm{F}}_n^{\{q\}}$. The quadratic differential $\bar\partial : \overline{\mathrm{F}}_n^{\{q\}}(j) \to \overline{\mathrm{F}}_n^{\{q+1\}}(j)$ from (3.13) induces a differential $\bar\partial$ in $\overline{\mathrm{F}}_n^{\{\}}$, thereby making it into a differential graded $n\wedge 1$-As-module. As a left n-As-module it is generated by its n-dimensional right As-**dg**-submodule $\bar{\mathsf{f}}_n^{\{\}}$:

$$\bar{\mathsf{f}}_n^{\{\}} = \coprod_{k=0}^\infty \bar{\mathsf{f}}_n^{\{k\}}, \qquad \bar{\mathsf{f}}_n^{\{k\}}(j) = \Bbbk\big\{\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}) \mid j_1+\cdots+j_k = j,\ \forall\, q \leqslant k\ j_q \in \mathbb{N}^n - 0\big\}.$$

The matrix coefficients of $\bar\partial : \bar{\mathsf{f}}_n^{\{k\}}(j) \to \bar{\mathsf{f}}_n^{\{k+1\}}(j)$ are integers and we shall find $h : \bar{\mathsf{f}}_n^{\{k\}}(j) \to \bar{\mathsf{f}}_n^{\{k-1\}}(j)$ with the same property. Thus, instead of working over a general ring $\Bbbk$ we can assume that $\Bbbk = \mathbb{Z}$, and we do it till the end of the proof. Any such map h extends to a morphism of left n-As-modules in a unique way.

The operator induced by N in the graded n-As-module $\overline{\mathrm{F}}_n^{\{\}}$ is denoted $\bar{N} : \overline{\mathrm{F}}_n^{\{\}} \to \overline{\mathrm{F}}_n^{\{\}}$. It can be described via a simplified formula

$$\bar{N} = 1 - \overline{p\beta} + h\bar{\partial} + \bar{\partial}h : \overline{\mathrm{F}}_n^{\{k\}} \to \overline{\mathrm{F}}_n^{\{k\}}, \tag{3.14}$$

where $\bar{\partial}$ is given by (3.13), $h : \overline{\mathrm{F}}_n^{\{p\}} \to \overline{\mathrm{F}}_n^{\{p-1\}}$, and $\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).\overline{p\beta}$ vanishes unless $\|j_1\| = \dots = \|j_k\| = 1$ and $\operatorname{supp} j_q$ are all distinct for $1 \leqslant q \leqslant k$. When $(j_1, \dots, j_k)$ is a permutation of $(e_{a_1}, \dots, e_{a_k})$ with $1 \leqslant a_1 < \dots < a_k \leqslant n$, then

$$\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).\overline{p\beta} = \rho(\mathsf{f}_{e_{a_1}}, \dots, \mathsf{f}_{e_{a_k}}; m^{(k)}).$$

Otherwise, $\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).\overline{p\beta}$ vanishes. The operator N is locally nilpotent iff $\bar{N}$ is. We shall achieve $\bar{N} = 0$.

Let us define a family of graded abelian groups $\tilde{f}_n(j)$, $j \in \mathbb{N}^n$,

$$\tilde{f}_n(j)^k = \mathbb{Z}\big\{x(j_1, \dots, j_k) \mid j_q \in \mathbb{N}^n - 0,\ j_1 + \dots + j_k = j\big\}.$$

The family $\tilde{f}_n$ has an obvious structure of a graded n-dimensional right As-module, namely,

$$\tilde{\rho}\big(x\big(({}_tj_1)_{t=1}^{n_1}\big), \dots, x\big(({}_tj_k)_{t=1}^{n_k}\big); m^{(k)}\big) = x\big(({}_tj_1)_{t=1}^{n_1}\big), \dots, ({}_tj_k)_{t=1}^{n_k}\big).$$

This structure is completely fixed by the requirement

$$\tilde{\rho}\big(x(j_1), \dots, x(j_k); m^{(k)}\big) = x(j_1, \dots, j_k) \tag{3.15}$$

Consider the bilinear form $\chi : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{Z}$, $\chi(t,p) = \sum_{c \leqslant d} t^c p^d$, and define the corresponding c by (3.11). There are invertible mappings $\psi(j) : \bar{\mathsf{f}}_n^{\{\}}(j) \to \tilde{f}_n(j)$ of degree $\|j\| = ((1,1,\dots,1)|j)$ such that

- $(\mathsf{f}_j).\psi(j) = x(j)$;
- the right As-module structure obtained from (ψ, id_{As}) and the bilinear form χ as in Lemma 3.12 satisfies condition (3.15).

Existence and uniqueness of ψ is shown in the following computation in square (3.10):

$$\begin{array}{ccccc}
\mathsf{f}_{j_1} \otimes \dots \otimes \mathsf{f}_{j_k} \otimes m^{(k)} & \xmapsto{\bar{\rho}} & \rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}) & \xmapsto{\psi(j_1 + \dots + j_k)} & \rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).\psi \\
\Big\downarrow{\scriptstyle \psi(j_1) \otimes \dots \otimes \psi(j_k) \otimes 1} & & & & \Big\downarrow{\scriptstyle (-1)^{\sum_{q<r}^{c \leqslant d} j_q^c j_r^d}} \\
(-1)^{\sum_{q<r} \|j_q\|(1 - \|j_r\|)} x(j_1) \otimes \dots \otimes x(j_k) \otimes m^{(k)} & \xmapsto{\tilde{\rho}} & & & (-1)^{\sum_{q=1}^k (k-q)\|j_q\| - \sum_{q<r} \|j_q\| \cdot \|j_r\|} x(j_1, \dots, j_k)
\end{array}$$

wherefore

$$\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).\psi = (-1)^{\sum_{q=1}^k (k-q)\|j_q\| - \sum_{q<r}^{c>d} j_q^c j_r^d} x(j_1, \dots, j_k)$$

and $\deg\psi(j) = \|j\|$ as claimed. We conclude that for this ψ and χ the induced (by Lemma 3.12) right action of As on $\tilde{f}_n$ is the natural one.

Let us compute the differential $\tilde{\partial}$ in $\tilde{f}_n$. For $\ell = j_1 + \cdots + j_k$ the expression

$$\begin{aligned}
&(-1)^{\|\ell\|}\rho(\mathsf{f}_{j_1},\dots,\mathsf{f}_{j_k};m^{(k)}).\bar{\partial}\psi \\
&= \sum_{q=1}^{k}\sum_{t+p=j_q}^{t,p\neq 0}(-1)^{\sum_{r=q+1}^{k}(1-\|j_r\|)+\sum_{c>d}t^cp^d+\|p\|+1+\|\ell\|}\rho(\mathsf{f}_{j_1},\dots,\mathsf{f}_{j_{q-1}},\mathsf{f}_t,\mathsf{f}_p,\mathsf{f}_{j_{q+1}},\dots,\mathsf{f}_{j_k};m^{(k+1)}).\psi \\
&= \sum_{q=1}^{k}\sum_{t+p=j_q}^{t,p\neq 0}(-1)^{\sum_{r=q+1}^{k}(1-\|j_r\|)+\sum_{c>d}t^cp^d+\|p\|+1+\|\ell\|+\sum_{r=1}^{q-1}(k+1-r)\|j_r\|+(k+1-q)\|t\|+(k-q)\|p\|} \\
&\qquad \times(-1)^{\sum_{r=q+1}^{k}(k-r)\|j_r\|-\sum_{u<r}^{c>d}j_u^cj_r^d+\sum_{c>d}t^cp^d}x(j_1,\dots,j_{q-1},t,p,j_{q+1},\dots,j_k)
\end{aligned}$$

has to coincide with

$$\rho(\mathsf{f}_{j_1},\dots,\mathsf{f}_{j_k};m^{(k)}).\psi\tilde{\partial} = (-1)^{\sum_{q=1}^{k}(k-q)\|j_q\|-\sum_{q<r}^{c>d}j_q^cj_r^d}x(j_1,\dots,j_k).\tilde{\partial}.$$

This gives the differential $\tilde{\partial}$:

$$x(j_1,\dots,j_k).\tilde{\partial} = \sum_{q=1}^{k}(-1)^{k+1-q}\sum_{t+p=j_q}^{t,p\neq 0}x(j_1,\dots,j_{q-1},t,p,j_{q+1},\dots,j_k). \tag{3.16}$$

Note that the differential $\bar{\partial}:\bar{\mathsf{f}}_n^{\{k\}}(\ell)\to\bar{\mathsf{f}}_n^{\{k+1\}}(\ell)$ makes

$$0\longrightarrow\bar{\mathsf{f}}_n^{\{1\}}(\ell)\xrightarrow{\bar{\partial}}\dots\xrightarrow{\bar{\partial}}\bar{\mathsf{f}}_n^{\{k\}}(\ell)\xrightarrow{\bar{\partial}}\dots\xrightarrow{\bar{\partial}}\bar{\mathsf{f}}_n^{\{\|\ell\|\}}(\ell)\longrightarrow 0 \tag{3.17}$$

into a bounded complex of abelian groups. The term $\bar{\mathsf{f}}_n^{\{k\}}(\ell)$ is placed in degree $k-\|\ell\|$.

Consider the operad morphism $As\to\mathbb{Z}$, $m^{(k)}\mapsto 0$ for $k\geqslant 2$, where $\mathbb{Z}$ is the unit operad, $\mathbb{Z}(1)=\mathbb{Z}$, $\mathbb{Z}(n)=0$ for $n\neq 1$. We may view $\coprod_{\ell\in\mathbb{N}^n-0}\bar{\mathsf{f}}_n^{\{k\}}(\ell)$ as a left n-operad $\mathbb{Z}$-module, quotient of $\bar{\mathsf{f}}_n^{\{k\}}$ by the submodule spanned by images of all left actions of elements $m^{(k)}$ for $k\geqslant 2$. Applying the same quotient procedure to FAs_n we get

$$\overline{FAs}_n(\ell) = \begin{cases}\mathbb{Z}=\mathbb{Z}u(\ell)=\mathbb{Z}u(e_{\operatorname{supp}\ell}), & \text{if } \|\ell\|=|\operatorname{supp}\ell|,\\ 0, & \text{if } \|\ell\|>|\operatorname{supp}\ell|.\end{cases}$$

We are going to prove that complex (3.17) is homotopy isomorphic via $\bar{p}$ and $\bar{\beta}$ to its cohomology $\overline{FAs}_n(\ell)$. If $\ell=e_S$ for some $S\subset\mathbf{n}$, then the cohomology is concentrated in degree 0 and equals $\overline{FAs}_n(e_S)=\mathbb{Z}=\mathbb{Z}u(e_S)$. If $\|\ell\|>|\operatorname{supp}\ell|$, then the cohomology vanishes.

We construct mappings of abelian groups $h:\bar{\mathsf{f}}_n^{\{p\}}(\ell)\to\bar{\mathsf{f}}_n^{\{p-1\}}(\ell)$ such that $\bar{N}:\bar{\mathsf{f}}_n^{\{k\}}(\ell)\to\bar{\mathsf{f}}_n^{\{k\}}(\ell)$ given by (3.14) vanishes. These h induce the left n-As-module morphism $h:\overline{\mathrm{F}}_n\to\overline{\mathrm{F}}_n$ compatibly with the generator–to–generator mapping $\bar{\mathsf{f}}_n^{\{k\}}\to\overline{\mathrm{F}}_n^{(k)}$.

Thus, vanishing of $\bar{N} : \bar{\mathsf{f}}_n^{\{k\}}(\ell) \to \bar{\mathsf{f}}_n^{\{k\}}(\ell)$ implies vanishing of $\bar{N} : \overline{\mathrm{F}}_n^{\{\}} \to \overline{\mathrm{F}}_n^{\{\}}$ and local nilpotency of $N : \overline{\mathrm{F}}_n \to \overline{\mathrm{F}}_n$.

We have reduced the proposition to proving that the chain maps $\bar{p}$, $\bar{\beta}$ in

$$\begin{array}{ccccccccccccc}
0 \longrightarrow \bar{\mathsf{f}}_n^{\{1\}}(\ell) & \xrightarrow{\bar{\partial}} & \dots & \xrightarrow{\bar{\partial}} & \bar{\mathsf{f}}_n^{\{k\}}(\ell) & \xrightarrow{\bar{\partial}} & \dots & \xrightarrow{\bar{\partial}} & \bar{\mathsf{f}}_n^{\{\|\ell\|-1\}}(\ell) & \xrightarrow{\bar{\partial}} & \bar{\mathsf{f}}_n^{\{\|\ell\|\}}(\ell) & \longrightarrow & 0 \\
 & & & & & & & & & & \bar{\beta}\uparrow \ \downarrow \bar{p} & & \\
 & & & & & & & & 0 & \longrightarrow & \overline{FAs}_n(\ell) & \longrightarrow & 0
\end{array} \tag{3.18}$$

are homotopy inverse to each other for any $\ell \in \mathbb{N}^n - 0$. We add formally the case of $\ell = 0$ by defining the top and the bottom rows as complexes $\bar{\mathsf{f}}_n^{\{0\}}(0) = \mathbb{Z}$ and $\overline{FAs}_n(0) = \mathbb{Z}$ concentrated in degree 0. Here $\bar{p}$, $\bar{\beta}$ are defined as the identity maps.

Chain maps $\bar{p}$, $\bar{\beta}$ give rise to other chain maps $\tilde{p}$, $\tilde{\beta}$ in the commutative diagram

$$\begin{array}{ccccccccccc}
0 \to \bar{\mathsf{f}}_n^{\{1\}}(\ell) & \xrightarrow{\bar{\partial}} & \dots & \xrightarrow{\bar{\partial}} & \bar{\mathsf{f}}_n^{\{k\}}(\ell) & \xrightarrow{\bar{\partial}} & \dots & \xrightarrow{\bar{\partial}} & \bar{\mathsf{f}}_n^{\{\|\ell\|\}}(\ell) \to 0 & & \\
 & & & & & & & & & \overset{\bar{p}}{\searrow}\ \underset{\bar{\beta}}{\nwarrow} & \\
\downarrow \psi & & & & \downarrow \psi & & & & \downarrow \psi & & 0 \to \overline{FAs}_n(\ell) \to 0 \\
0 \to \tilde{f}_n(\ell)^1 & \xrightarrow{\tilde{\partial}} & \dots & \xrightarrow{\tilde{\partial}} & \tilde{f}_n(\ell)^k & \xrightarrow{\tilde{\partial}} & \dots & \xrightarrow{\tilde{\partial}} & \tilde{f}_n(\ell)^{\|\ell\|} \to 0 & & \| \\
 & & & & & & & & & \overset{\tilde{p}}{\searrow}\ \underset{\tilde{\beta}}{\nwarrow} & \\
 & & & & & & & & & & 0 \to \overline{FAs}_n(\ell) \to 0
\end{array}$$

In fact, the maps $\tilde{p}$, $\tilde{\beta}$ have to be defined if $l^i \in \{0, 1\}$ for all $1 \leqslant i \leqslant n$. For $k = \|\ell\|$ we find

$$x(e_{a_1}, \dots, e_{a_k}).\tilde{p} = \operatorname{sign}(a_1, \dots, a_k) u(\ell) \overset{\text{def}}{=} (-1)^{\sum_{q<r} \chi(a_q > a_r)} u(\ell), \tag{3.19}$$

where $\chi(b > c)$ is 1 or 0 depending on the case whether the inequality holds or not. The exponent is the number of inversions in the sequence $(a_1, \dots, a_k)$. If $\ell = e_{\mathbf{n}} = (1, 1, \dots, 1)$, then $k = n$ and the sign is just the sign of the permutation $(a_1, \dots, a_k)$. The map $\tilde{\beta} = \bar{\beta}\psi : \overline{FAs}_n(\ell) \to \tilde{f}_n(\ell)^{\|\ell\|}$ satisfies

$$u(\ell).\tilde{\beta} = x(e_{c_1}, \dots, e_{c_k}), \tag{3.20}$$

where $\{c_1 < c_2 < \dots < c_k\} = \operatorname{supp} \ell$. In particular, if $\ell = e_{\mathbf{n}}$, then $k = n$ and $u(e_{\mathbf{n}})\tilde{\beta} = x(e_1, \dots, e_n)$.

Consider the augmented coalgebra $C_n = \mathbb{Z}\{\mathbb{N}^n\}$ with the comultiplication $j.\Delta = \sum_{q+r=j} q \otimes r$ where $j, q, r \in \mathbb{N}^n$. Generators $j \in \mathbb{N}^n$ of the free abelian group C_n are denoted also $x(j)$. The augmentation is $\eta : \mathbb{Z} \to C_n$, $1 \mapsto x(0)$. The counit is $\varepsilon : C_n \to \mathbb{Z}$, $x(j) \mapsto \delta_{j0}$. The reduced comultiplication is defined as

$$\bar{\Delta} = \Delta - \eta \otimes \mathrm{id} - \mathrm{id} \otimes \eta + \varepsilon\eta \otimes \eta, \qquad \vec{0}.\bar{\Delta} = 0, \quad j.\bar{\Delta} = \sum_{q+r=j}^{q,r \neq 0} q \otimes r \quad \text{for } j \neq 0.$$

The abelian subgroup $\bar{C}_n = \operatorname{Ker} \varepsilon = \mathbb{Z}\{\mathbb{N}^n - 0\} \subset C_n$ equipped with the comultiplication $\bar{\Delta}$ is a coassociative coalgebra, which is not counital. The complex $\tilde{f}_n$ is nothing else but the cohomology complex $K'(\bar{C}_n)$ of the coassociative coalgebra $\bar{C}_n$, which is the upper row of the diagram

$$\begin{array}{ccccccccccccccc} 0\to\mathbb{Z} & \xrightarrow{0} & \bar{C}_n & \xrightarrow{\bar{\Delta}} & \bar{C}_n^{\otimes 2} & \xrightarrow{\tilde{\partial}} & \dots & \xrightarrow{\tilde{\partial}} & \bar{C}_n^{\otimes k} & \xrightarrow{\tilde{\partial}} & \dots & \xrightarrow{\tilde{\partial}} & \bar{C}_n^{\otimes n} & \xrightarrow{\tilde{\partial}} & \dots \\ \| & & \tilde{\beta}\uparrow\downarrow\tilde{p} & & \tilde{\beta}\uparrow\downarrow\tilde{p} & & & & \tilde{\beta}\uparrow\downarrow\tilde{p} & & & & \tilde{\beta}\uparrow\downarrow\tilde{p} & & \\ 0\to\mathbb{Z} & \xrightarrow{0} & \wedge^1_{\mathbb{Z}}(\mathbb{Z}^n) & \xrightarrow{0} & \wedge^2_{\mathbb{Z}}(\mathbb{Z}^n) & \xrightarrow{0} & \dots & \xrightarrow{0} & \wedge^k_{\mathbb{Z}}(\mathbb{Z}^n) & \xrightarrow{0} & \dots & \xrightarrow{0} & \wedge^n_{\mathbb{Z}}(\mathbb{Z}^n) & \xrightarrow{0} & 0 \end{array} \tag{3.21}$$

We identify $x(j_1,\dots,j_k)\in\tilde{f}_n$ with $j_1\otimes\dots\otimes j_k\in\bar{C}_n^{\otimes k}$. The exterior algebra $\wedge_{\mathbb{Z}}(\mathbb{Z}^n)=T_{\mathbb{Z}}(\mathbb{Z}^n)/(x\otimes x\mid x\in\mathbb{Z}^n)$ has the basis $(e_{\{c_1<c_2<\dots<c_k\}}=e_{c_1}\wedge e_{c_2}\wedge\dots\wedge e_{c_k})$, where $1\leqslant c_1<c_2<\dots<c_k\leqslant n$. The mappings in this diagram are

$$(j_1\otimes\dots\otimes j_k).\tilde{\partial}=\sum_{q=1}^{k}(-1)^{k-q+1}j_1\otimes\dots\otimes j_{q-1}\otimes j_q.\bar{\Delta}\otimes j_{q+1}\otimes\dots\otimes j_k,$$

$$x(j_1,\dots,j_k).\tilde{p}=0 \text{ unless } \Big\|\sum_{q=1}^{k}j_q\Big\|=k=\Big|\operatorname{supp}\sum_{q=1}^{k}j_q\Big|,\quad j_q\in\mathbb{N}^n-0,$$

$$x(e_{a_1},\dots,e_{a_k}).\tilde{p}=(-1)^{\sum_{q<r}\chi(a_q>a_r)}e_{\{a_1,\dots,a_k\}},\quad a_1,\dots,a_k-\text{distinct},$$

$$e_{\{c_1<c_2<\dots<c_k\}}.\tilde{\beta}=x(e_{c_1},\dots,e_{c_k}),$$

which coincides with (3.16), (3.19) and (3.20).

It remains to prove that the maps $\tilde{p}$, $\tilde{\beta}$ are homotopy inverse to each other. Clearly, $\tilde{\beta}\tilde{p}=1$.

3.14. Lemma. *For $n=1$ the maps $\tilde{p}$, $\tilde{\beta}$ are homotopy inverse to each other.*

Proof. For $n=1$ the chain maps in question become

$$\begin{array}{ccccccccccc} 0\to\mathbb{Z} & \xrightarrow{0} & \bar{C}_1 & \xrightarrow{\bar{\Delta}} & \bar{C}_1^{\otimes 2} & \xrightarrow{\tilde{\partial}} & \dots & \xrightarrow{\tilde{\partial}} & \bar{C}_1^{\otimes k} & \xrightarrow{\tilde{\partial}} & \dots \\ \| & & \tilde{\beta}\uparrow\downarrow\tilde{p} & & \uparrow\downarrow & & & & \uparrow\downarrow & & \\ 0\to\mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

$$x(j).\tilde{p}=\delta_{j1},\qquad 1.\tilde{\beta}=x(1).$$

Define a map of graded abelian groups $h:K'(\bar{C}_1)\to K'(\bar{C}_1)$ of degree -1 by the formula

$$x(j_1,\dots,j_{k-1},j_k).h=\begin{cases}x(j_1,\dots,j_{k-2},j_{k-1}+1), & \text{if } k>1,\ j_k=1,\\ 0, & \text{otherwise.}\end{cases}$$

We claim that the chain map $E=\tilde{p}\tilde{\beta}-h\tilde{\partial}-\tilde{\partial}h:K'(\bar{C}_1)\to K'(\bar{C}_1)$ is the identity map. In fact, $x(1).E=x(1)$, and for $j\geqslant 2$ we have

$$x(j).E=\sum_{q+r=j}^{q,r>0}x(q,r).h=x(j-1+1)=x(j).$$

For $k > 1$ and $j_k \geqslant 2$ we find

$$x(j_1, \dots, j_k).E = \sum_{q+r=j_k} x(j_1, \dots, j_{k-1}, q, r).h = x(j_1, \dots, j_{k-1}, j_k).$$

It remains to consider for $k > 1$ the value

$$\begin{aligned} &x(j_1, \dots, j_{k-1}, 1).E = -x(j_1, \dots, j_{k-2}, j_{k-1}+1).\tilde\partial - x(j_1, \dots, j_{k-1}, 1).\tilde\partial h \\ = &\sum_{q+r=j_{k-1}+1} x(j_1, \dots, j_{k-1}, q, r) - \sum_{q+t=j_{k-1}} x(j_1, \dots, j_{k-2}, q, t, 1).h = x(j_1, \dots, j_{k-2}, j_{k-1}, 1). \end{aligned}$$

Hence, $E = \mathrm{id}$, and $\tilde p \tilde\beta$ is homotopic to the identity map. ■

3.14.1. Homology of augmented algebras. Let $\mathcal{C}$ be a symmetric monoidal category with the tensor product $\otimes$ and the unit object $\mathbb{1}$. Assume that $A = (A, \mu : A \otimes A \to A, \eta : \mathbb{1} \to A, \varepsilon : A \to \mathbb{1})$ is an augmented unital associative algebra in $\mathcal{C}$. There is an associated simplicial object $S(A)$:

$$\cdots A\otimes A\otimes A \;\begin{array}{c} \xrightarrow{d_3=1\otimes1\otimes\varepsilon} \\ \xleftarrow{s_2=1\otimes1\otimes\eta} \\ \xrightarrow{d_2=1\otimes\mu} \\ \xleftarrow{s_1=1\otimes\eta\otimes1} \\ \xrightarrow{d_1=\mu\otimes1} \\ \xleftarrow{s_0=\eta\otimes1\otimes1} \\ \xrightarrow[d_0=\varepsilon\otimes1\otimes1]{} \end{array}\; A\otimes A \;\begin{array}{c} \xrightarrow{d_2=1\otimes\varepsilon} \\ \xleftarrow{s_1=1\otimes\eta} \\ \xrightarrow{d_1=\mu} \\ \xleftarrow{s_0=\eta\otimes1} \\ \xrightarrow[d_0=\varepsilon\otimes1]{} \end{array}\; A \;\begin{array}{c} \xrightarrow{d_1=\varepsilon} \\ \xleftarrow{s_0=\eta} \\ \xrightarrow[d_0=\varepsilon]{} \end{array}\; \mathbb{1},$$

where d_i and s_i are face maps and degeneracy maps respectively. When B is another augmented algebra in $\mathcal{C}$, the Cartesian product $S(A) \times S(B)$ of simplicial objects [Mac63, Section VIII.8] is naturally isomorphic to the simplicial object $S(A \otimes B)$.

Assume also that $\mathcal{C}$ is abelian and the tensor product $\otimes$ is bilinear. A complex $K(A) \overset{\text{def}}{=} K(S(A))$ is associated with the simplicial object $S(A)$. It has the differential $\partial = \sum_{i=0}^q (-1)^i d_i : A^{\otimes q} \to A^{\otimes q-1}$. Homology of $K(A)$ gives the torsion objects $\operatorname{Tor}^A_\bullet(\mathbb{1}, \mathbb{1})$, where the left and the right A-module $\mathbb{1}$ obtains its structure via $\varepsilon : A \to \mathbb{1}$. Given two augmented algebras A and B in $\mathcal{C}$ we can form a bisimplicial object in $\mathcal{C}$, whose terms are $A^{\otimes p} \otimes B^{\otimes q}$. By Eilenberg–Zilber theorem [Wei94, Theorem 8.5.1] the complexes $K(A \otimes B) = K(S(A \otimes B)) \simeq K(S(A) \times S(B))$ and $K(A) \otimes K(B)$ are quasi-isomorphic.

Consider associative algebras $\bar A = (\bar A, \mu : \bar A \otimes \bar A \to \bar A)$ in $\mathcal{C}$, which are not required to have a unit. Such an algebra gives rise to a unital one $A = \mathbb{1} \oplus \bar A$ for which $\eta = \mathrm{in}_{\mathbb{1}} : \mathbb{1} \to A$ is the unit and $\varepsilon = \mathrm{pr}_{\mathbb{1}} : A \to \mathbb{1}$ is an augmentation. Introduce another monoidal product in $\mathcal{C}$ (not bilinear) via the formula

$$\bar A \circledast \bar B = \bar A \oplus \bar B \oplus (\bar A \otimes \bar B).$$

There is an obvious isomorphism

$$\mathbb{1} \oplus (\bar A \circledast \bar B) = (\mathbb{1} \oplus \bar A) \otimes (\mathbb{1} \oplus \bar B).$$

If $\bar{A}$, $\bar{B}$ are associative algebras in $(\mathcal{C}, \otimes)$, then $\bar{A} \circledast \bar{B}$ obtains an associative algebra structure in $(\mathcal{C}, \otimes)$ via this isomorphism, namely, $\mathbb{1} \oplus (\bar{A} \circledast \bar{B}) = A \otimes B$.

There is a normalised chain complex $K_N(\bar{A}) = K_N(S(A))$ of the simplicial complex $S(A)$:

$$\dots \longrightarrow \bar{A}^{\otimes 3} \xrightarrow{-\mu\otimes 1 + 1\otimes\mu} \bar{A}^{\otimes 2} \xrightarrow{-\mu} \bar{A} \xrightarrow{0} \mathbb{1} \to 0,$$

with the differential $\partial = \sum_{i=0}^{q-2} (-1)^{i+1} 1^{\otimes i} \otimes \mu \otimes 1^{\otimes q-i-2} : \bar{A}^{\otimes q} \to \bar{A}^{\otimes q-1}$, where $\mathbb{1}$ is placed in degree 0. By a generalization of normalization theorem of Eilenberg and Mac Lane [Mac63, Theorem VIII.6.1] the natural projection $K(A) \to K_N(\bar{A})$ is a homotopy isomorphism. As a corollary, we get the following

3.15. Proposition. *For associative algebras $\bar{A}$, $\bar{B}$ in $(\mathcal{C}, \otimes)$ there is a natural quasi-isomorphism*

$$K_N(\bar{A} \circledast \bar{B}) \rightleftarrows K_N(\bar{A}) \otimes K_N(\bar{B}).$$

3.15.1. Conclusion of the proof of Theorem 3.13. Let us take for $(\mathcal{C}, \otimes)$ the category $(\mathrm{Ab}^{\mathrm{op}}, \otimes_{\mathbb{Z}}^{\mathrm{op}})$ opposite to the category of abelian groups with the opposite tensor product. Clearly, the category $\mathrm{Ab}^{\mathrm{op}}$ is abelian. An associative algebra in this monoidal category is a coassociative coalgebra over $\mathbb{Z}$ in the ordinary sense. In particular, such is $\bar{C}_n = \mathbb{Z}\{\mathbb{N}^n - 0\}$. Adding a unit to it in $\mathrm{Ab}^{\mathrm{op}}$ gives $C_n = \mathbb{Z}\{\mathbb{N}^n\}$. Since $C_n \otimes_{\mathbb{Z}} C_m \simeq C_{n+m}$, we conclude that $\bar{C}_n \circledast \bar{C}_m \simeq \bar{C}_{n+m}$. The homological complex $K_N(\bar{C}_n)$ in $\mathrm{Ab}^{\mathrm{op}}$ and the top line of (3.21), the cohomological complex $K'(\bar{C}_n)$ in Ab are identified: the n-th abelian groups and the differentials between them coincide. Thus, the results of the previous section apply to $K'(\bar{C}_n)$.

We claim that the cohomology of $K'(\bar{C}_n)$ is isomorphic to $\wedge_{\mathbb{Z}}(\mathbb{Z}^n)$. In fact, using induction we deduce from Lemma 3.14 and Proposition 3.15 the quasi-isomorphism

$$K'(\bar{C}_{n+1}) \underset{qis}{\longrightarrow} K'(\bar{C}_n) \otimes K'(\bar{C}_1) \xrightarrow{q\otimes\tilde{p}} \wedge_{\mathbb{Z}} (\mathbb{Z}^n) \otimes \wedge_{\mathbb{Z}}(\mathbb{Z}) \simeq \wedge_{\mathbb{Z}}(\mathbb{Z}^{n+1}).$$

Here q, $\tilde{p}$ are quasi-isomorphisms. So is their tensor product $q \otimes \tilde{p}$, since complexes $K'(\bar{C}_n)$, $\wedge_{\mathbb{Z}}(\mathbb{Z}^n)$ consist of free abelian groups, $\wedge_{\mathbb{Z}}(\mathbb{Z}^n)$ are bounded and $K'(\bar{C}_n)$ are direct sums of bounded complexes.

Both rows of diagram (3.21) have the same homology, which coincides with the bottom row and consists of finitely generated free abelian groups. Since $H(\tilde{\beta})H(\tilde{p}) = 1$, the matrices of $H(\tilde{\beta})$ and $H(\tilde{p})$ are invertible. Thus, $\tilde{\beta}$ and $\tilde{p}$ induce isomorphisms in homology. They are quasi-isomorphisms of complexes consisting of free abelian groups. Therefore, their cones are acyclic complexes consisting of free abelian groups. They split into short exact sequences whose terms are also free abelian groups (as subgroups of such). Hence, these short exact sequences split and the cones are contractible. Thus, $\tilde{p}$ and $\tilde{\beta}$ are homotopy isomorphisms. Clearly, they are homotopy inverse to each other. This implies the same conclusion for $\bar{p}$ and $\bar{\beta}$ and for p and β. ■

3.16. Corollary. [to Proposition 3.9, Theorem 3.13] *The polymodule F_n is homotopy isomorphic to its cohomology and $H^\bullet(F_n(j)) = \Bbbk[1 - \|j\|]$ for $j \in \mathbb{N}^n - 0$.*

This is due to existence of a degree 1 isomorphism $\Sigma : H^\bullet(F_n) \to FAs_n$.

3.17. Homotopy unital A_∞-morphisms. Consider the free $n \wedge 1$-$\mathrm{A}^{\mathsf{su}}_\infty$-module

$$\tilde{\mathrm{F}}_n = \bigodot_{i=1}^n \mathrm{A}^{\mathsf{su}}_\infty \odot^i_{\mathrm{A}_\infty} \mathrm{F}_n \odot^0_{\mathrm{A}_\infty} \mathrm{A}^{\mathsf{su}}_\infty = \odot_{\geqslant 0}({}^n\mathrm{A}^{\mathsf{su}}_\infty; \Bbbk\{\mathsf{f}_j \mid j \in \mathbb{N}^n - 0\}; \mathrm{A}^{\mathsf{su}}_\infty).$$

In particular, $\tilde{\mathrm{F}}_0 = \mathrm{A}^{\mathsf{su}}_\infty(0) = \Bbbk 1^{\mathsf{su}}$ by [Lyu15, Lemma A.9]. The graded ideal generated by the following system of relations in it

$$\rho_\varnothing(1^{\mathsf{su}}) = \lambda^i_{e_i}(1^{\mathsf{su}}; \mathsf{f}_{e_i}), \ \forall\, i, \quad \lambda^i_\ell({}^a1, 1^{\mathsf{su}}, {}^b1; \mathsf{f}_\ell) = 0 \ \text{ if } \ a + 1 + b = \ell^i, \ \|\ell\| > 1,$$

is stable under the differential, as one easily verifies. Therefore the quotient $\mathrm{F}^{\mathsf{su}}_n$ of $\tilde{\mathrm{F}}_n$ by these relations is an $n \wedge 1$-$\mathrm{A}^{\mathsf{su}}_\infty$-module. We still have $\mathrm{F}^{\mathsf{su}}_0 = \mathrm{A}^{\mathsf{su}}_\infty(0) = \Bbbk 1^{\mathsf{su}}$. Note that $\mathrm{F}^{\mathsf{su}}_n$-algebra maps coincide with *strictly unital* A_∞*-algebra morphisms*, which are by [BLM08, Definition 9.2] A_∞-morphisms $\mathsf{f} : (A_1, \dots, A_n) \to B$ between strictly unital A_∞-algebras such that all components of f vanish if any of its entries is $1^{\mathsf{su}}_{A_i}$, except $1^{\mathsf{su}}_{A_i}\mathsf{f}_{e_i} = 1^{\mathsf{su}}_B$.

The rows of the following diagram in $\mathbf{dg}^{\mathbb{N} \sqcup \mathbb{N}^n}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathrm{A}_\infty, \mathrm{F}_n) & \longrightarrow & (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n) & \longrightarrow & (\Bbbk 1^{\mathsf{su}}, \Bbbk 1^{\mathsf{su}}\rho_\varnothing) & \longrightarrow & 0 \\ & & \scriptstyle htis \big\downarrow & & \scriptstyle htis \big\downarrow p' & & \big\| & & \\ 0 & \longrightarrow & (As, FAs_n) & \longrightarrow & (As1, FAs1_n) & \longrightarrow & (\Bbbk 1^{\mathsf{su}}, \Bbbk 1^{\mathsf{su}}\rho_\varnothing) & \longrightarrow & 0 \end{array} \tag{3.22}$$

are exact sequences, split in the obvious way. Therefore, the middle vertical arrow p' is a homotopy isomorphism.

Consider the embedding of free graded operads $\mathrm{A}^{\mathsf{su}}_\infty \to \mathrm{A}^{\mathsf{su}}_\infty\langle \mathsf{i}, \mathsf{j}\rangle$, where i, j are two nullary operations, $\deg \mathsf{i} = 0$, $\deg \mathsf{j} = -1$. Assuming $\mathsf{i}\partial = 0$, $\mathsf{j}\partial = 1^{\mathsf{su}} - \mathsf{i}$, we make the second operad differential graded and the embedding becomes a chain map. It is proven in [Lyu11] (end of proof of Proposition 1.8) that this embedding is a homotopy isomorphism. Or, the reader can simplify the lines of the proof given below and adopt it to the case of $\mathrm{A}^{\mathsf{su}}_\infty \to \mathrm{A}^{\mathsf{su}}_\infty\langle \mathsf{i}, \mathsf{j}\rangle$.

3.18. Proposition. *The embedding $\iota : (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n) \to (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle \mathsf{i}, \mathsf{j}\rangle$ is a homotopy isomorphism.*

Proof. An arbitrary chain $n \wedge 1$-module map $\phi : (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle \mathsf{i}, \mathsf{j}\rangle \to (\mathcal{A}, \mathcal{P})$ is fixed by specifying a chain $n \wedge 1$-module map $(\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n) \to (\mathcal{A}, \mathcal{P})$ and the image $\phi(\mathsf{j}) \in \mathcal{A}(0)^{-1}$. In particular, there is a unique chain $n \wedge 1$-module map

$$\pi : (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle \mathsf{i}, \mathsf{j}\rangle \to (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n), \qquad \mathsf{i} \mapsto 1^{\mathsf{su}}, \qquad \mathsf{j} \mapsto 0,$$

whose restriction to $(\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)$ is identity. Let us prove that π is homotopy inverse to ι.

The restrictions of the above chain maps $\iota' : \Bbbk\mathsf{1}^{\mathsf{su}} \hookrightarrow \Bbbk\{\mathsf{1}^{\mathsf{su}}, \mathsf{i}, \mathsf{j}\}$ and $\pi' : \Bbbk\{\mathsf{1}^{\mathsf{su}}, \mathsf{i}, \mathsf{j}\} \to \Bbbk\mathsf{1}^{\mathsf{su}}$, $\mathsf{1}^{\mathsf{su}} \mapsto \mathsf{1}^{\mathsf{su}}$, $\mathsf{i} \mapsto \mathsf{1}^{\mathsf{su}}$, $\mathsf{j} \mapsto 0$, are homotopy isomorphisms: the homotopy $h : \Bbbk\{\mathsf{1}^{\mathsf{su}}, \mathsf{i}, \mathsf{j}\} \to \Bbbk\{\mathsf{1}^{\mathsf{su}}, \mathsf{i}, \mathsf{j}\}$, $\mathsf{1}^{\mathsf{su}}.h = 0$, $\mathsf{i}.h = \mathsf{j}$, $\mathsf{j}.h = 0$, satisfies $\partial h + h\partial = \pi'\iota' - 1$. We know from Proposition A.1 that the $n \wedge 1$-module $(\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle \mathsf{i}, \mathsf{j}\rangle$ coincides with $(\mathrm{A}^{\mathsf{su}}_\infty\langle \mathsf{i}, \mathsf{j}\rangle, \mathcal{P} = \bigodot_{k=0}^n \mathrm{A}^{\mathsf{su}}_\infty\langle \mathsf{i}, \mathsf{j}\rangle \odot^k_{\mathrm{A}^{\mathsf{su}}_\infty} \mathrm{F}^{\mathsf{su}}_n)$. The component $\mathcal{P}(l)$ is obtained from components $\mathrm{F}^{\mathsf{su}}_n(r)$ with $r^q \geqslant l^q$ for all $q \in \mathbf{n}$ by plugging the unused $r^q - l^q$ entries with i and j in all possible ways determined by injections $\psi^q : \mathbf{l}^q \hookrightarrow \mathbf{r}^q$:

$$\mathcal{P}(l) = \coprod_{(\psi^q:\mathbf{l}^q\hookrightarrow\mathbf{r}^q)_{q=1}^n} \Big(\bigotimes_{q\in\mathbf{n}} \bigotimes_{\mathbf{r}^q - \operatorname{Im}\psi^q} \Bbbk\{\mathsf{i}, \mathsf{j}\} \Big) \otimes \mathrm{F}^{\mathsf{su}}_n(r).$$

There is a split surjection

$$\lambda : \mathcal{Q}(l) = \coprod_{(\psi^q:\mathbf{l}^q\hookrightarrow\mathbf{r}^q)_{q=1}^n} \Big(\bigotimes_{q\in\mathbf{n}} \bigotimes_{\mathbf{r}^q - \operatorname{Im}\psi^q} (\Bbbk\{\mathsf{i}, \mathsf{j}\} \oplus \Bbbk\mathsf{1}^{\mathsf{su}}) \Big) \otimes \mathrm{F}^{\mathsf{su}}_n(r) \longrightarrow \mathcal{P}(l),$$

obtained by acting with all $\mathsf{1}^{\mathsf{su}}$ on $\mathrm{F}^{\mathsf{su}}_n$ on the left via λ. This reduces the quantities r^q by the number of factors $\mathsf{1}^{\mathsf{su}}$.

Denote $f = \pi'\iota'$, $g = \mathrm{id} : \Bbbk\{\mathsf{1}^{\mathsf{su}}, \mathsf{i}, \mathsf{j}\} \to \Bbbk\{\mathsf{1}^{\mathsf{su}}, \mathsf{i}, \mathsf{j}\}$. Equip the set $S = \sqcup_{q\in\mathbf{n}}(\mathbf{r}^q - \operatorname{Im}\psi^q)$ with the lexicographic order, $q < y \in \mathbf{n}$ implies $(q,c) < (y,z)$. The maps ∂ and $\pi\iota$ satisfy

$$\begin{array}{ccc}
\mathcal{Q}(l) & \xrightarrow[\widehat{\partial}]{\oplus_{(\psi^q)_q}(1\otimes\partial + \sum_{(y,z)\in S}(\otimes^{(q,c)<(y,z)}1)\otimes\partial\otimes(\otimes^{(q,c)>(y,z)}1)\otimes 1)} & \mathcal{Q}(l) \\
\lambda\downarrow & = & \downarrow\lambda \\
\mathcal{P}(l) & \xrightarrow{\partial} & \mathcal{P}(l)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{Q}(l) & \xrightarrow[\widehat{\pi\iota}]{\oplus_{(\psi^q)_q}(\otimes^{(q,c)\in S}f)\otimes 1} & \mathcal{Q}(l) \\
\lambda\downarrow & = & \downarrow\lambda \\
\mathcal{P}(l) & \xrightarrow{\pi\iota} & \mathcal{P}(l)
\end{array}$$

Since $\mathsf{1}^{\mathsf{su}}.f = \mathsf{1}^{\mathsf{su}}.g = \mathsf{1}^{\mathsf{su}}$, $\mathsf{1}^{\mathsf{su}}.h = 0$, there is a unique map $H : \mathcal{P}(l) \to \mathcal{P}(l)$ of degree -1 such that

$$\begin{array}{ccc}
\mathcal{Q}(l) & \xrightarrow[\widehat{H}]{\oplus_{(\psi^q)_q}\sum_{(y,z)\in S}(\otimes^{(q,c)<(y,z)}f)\otimes h\otimes(\otimes^{(q,c)>(y,z)}g)\otimes 1} & \mathcal{Q}(l) \\
\lambda\downarrow & = & \downarrow\lambda \\
\mathcal{P}(l) & \xrightarrow{H} & \mathcal{P}(l)
\end{array}$$

In order to find the commutator $\partial H + H\partial$ we can compute

$$\begin{aligned}
\widehat{\partial}\widehat{H} + \widehat{H}\widehat{\partial} &= \coprod_{(\psi^q:\mathbf{l}^q\hookrightarrow\mathbf{r}^q)_{q=1}^n} \sum_{(y,z)\in S} \Big(\bigotimes_{(q,c)<(y,z)} f\Big) \otimes (f-g) \otimes \Big(\bigotimes_{(q,c)>(y,z)} g\Big) \otimes 1 \\
&= \coprod_{(\psi^q:\mathbf{l}^q\hookrightarrow\mathbf{r}^q)_{q=1}^n} \Big(\bigotimes_{(q,c)\in S} f - \bigotimes_{(q,c)\in S} g\Big) \otimes 1 = \widehat{\pi\iota} - 1.
\end{aligned}$$

Therefore, $\partial H + H\partial = \pi\iota - 1$. ∎

The projection p' decomposes into a standard trivial cofibration and an epimorphism p''

$$p' = \bigl((\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n) \overset{htis}{\rightarrowtail\!\!\!\sim\!\!\to} (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle \mathsf{i}, \mathsf{j}\rangle \xrightarrow{p''} (As1, FAs1_n)\bigr),$$

where $p''(1^{\mathsf{su}}) = 1^{\mathsf{su}}$, $p''(\mathsf{i}) = 1^{\mathsf{su}}$, $p''(m_2) = m_2$, $p''(\mathsf{f}_{e_i}) = 1 \in FAs1_n(e_i)$, and other generators go to 0. As a corollary $p''(1^{\mathsf{su}}\rho_\varnothing) = 1^{\mathsf{su}}\rho_\varnothing$. Hence, the projection p'' is a homotopy isomorphism as well.

Generators f_ℓ of the $n \wedge 1$-operad module F_n are interpreted as maps $\mathsf{f}_\ell : \otimes^{k\in\mathbf{n}} T^{\ell^k} A_k \to B$ of degree $\deg \mathsf{f}_\ell = 1 - \|\ell\|$. A cofibrant replacement $(\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n) \to (As1, FAs1_n)$ is constructed as a $\mathbf{gr}$-submodule of $(\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle \mathsf{i}, \mathsf{j}\rangle$ generated in operadic part by i and g-ary operations of degree $4 - g - 2k$

$$m_{g_1;g_2;\dots;g_k} = (1^{\otimes g_1} \otimes \mathsf{j} \otimes 1^{\otimes g_2} \otimes \mathsf{j} \otimes \cdots \otimes 1^{\otimes g_{k-1}} \otimes \mathsf{j} \otimes 1^{\otimes g_k}) m_{g+k-1},$$

where $g = \sum_{q=1}^k g_q$, $k \geqslant 1$, $g_q \geqslant 0$, $g + k \geqslant 3$ and in module part by the nullary elements $\mathsf{v}_k = \lambda^k_{e_k}(\mathsf{j}; \mathsf{f}_{e_k}) - \mathsf{j}\rho_\varnothing = \mathsf{j}\mathsf{f}_{e_k} - \mathsf{j}\rho_\varnothing$, $k \in \mathbf{n}$, $\deg \mathsf{v}_k = -1$, and by elements

$$\begin{aligned} \mathsf{f}_{(\ell^k_1;\ell^k_2;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}} &= \lambda_{\hat\ell}\bigl(({}^{\ell^k_1}1, \mathsf{j}, {}^{\ell^k_2}1, \mathsf{j}, \dots, {}^{\ell^k_{t^k-1}}1, \mathsf{j}, {}^{\ell^k_{t^k}}1)_{k\in\mathbf{n}}; \mathsf{f}_{\hat\ell}\bigr) \\ &= \bigl[\otimes^{k\in\mathbf{n}} T^{\ell^k} A_k \xrightarrow{\otimes^{k\in\mathbf{n}}(1^{\otimes \ell^k_1} \otimes \mathsf{j} \otimes 1^{\otimes \ell^k_2} \otimes \mathsf{j} \otimes \cdots \otimes 1^{\otimes \ell^k_{t^k-1}} \otimes \mathsf{j} \otimes 1^{\otimes \ell^k_{t^k}})} \otimes^{k\in\mathbf{n}} T^{\hat\ell^k} A_k \xrightarrow{\mathsf{f}_{\hat\ell}} B\bigr] \end{aligned} \tag{3.23}$$

of arity $\ell = \bigl(\sum_{p=1}^{t^k} \ell^k_p\bigr)_{k\in\mathbf{n}}$, where the intermediate arity is $\hat\ell = \bigl(t_k - 1 + \sum_{p=1}^{t^k} \ell^k_p\bigr)_{k\in\mathbf{n}} = \bigl(-1 + \sum_{p=1}^{t^k} (\ell^k_p + 1)\bigr)_{k\in\mathbf{n}}$, and of degree $\deg \mathsf{f}_{(\ell^k_1;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}} = 1 + 2n - \sum_{k=1}^n \sum_{p=1}^{t^k} (\ell^k_p + 2)$. We assume that $t_k \geqslant 1$ for all $k \in \mathbf{n}$ and either $\|\hat\ell\| = \sum_{k=1}^n (t^k - 1) + \sum_{k=1}^n \sum_{p=1}^{t^k} \ell^k_p \geqslant 2$, or all $t_k = 1$ and there is $m \in \mathbf{n}$ such that $\ell^k_1 = \delta^k_m$. The last condition eliminates from the list the summands $\mathsf{f}_{0,\dots,0,(0;0),0,\dots,0} = \mathsf{j}\mathsf{f}_{e_k}$ of v_k. Setting $\mathsf{i}\partial = 0$, $\mathsf{j}\partial = 1^{\mathsf{su}} - \mathsf{i}$, we turn $(\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle \mathsf{i}, \mathsf{j}\rangle$ into a $\mathbf{dg}$-module and $(\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n)$ into its $\mathbf{dg}$-submodule. Note that $\mathsf{v}_k\partial = \mathsf{i}\rho_\varnothing - \mathsf{i}\mathsf{f}_{e_k}$.

Let us prove that the graded $n \wedge 1$-module $(\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n)$ is free over $(\Bbbk, 0)$. The graded $n \wedge 1$-module $(\mathrm{A}_\infty, \mathrm{F}_n)\langle \mathsf{j}\rangle$ can be presented as

$$\begin{aligned} &\bigl(\mathrm{A}_\infty, \odot_{\geqslant 0}({}^{[n]}\mathrm{A}_\infty; \Bbbk\{\mathsf{f}_\ell \mid \ell \in \mathbb{N}^n - 0\})\bigr)\langle \mathsf{j}\rangle \\ &\quad \simeq \bigl(\mathrm{A}_\infty\langle \mathsf{j}\rangle, \odot_{\geqslant 0}({}^n\Bbbk\langle m_{n_1;\dots;n_k} \mid k + \textstyle\sum_{q=1}^k n_q \geqslant 3\rangle; \Bbbk\{\mathsf{f}_{(\ell^k_1;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}} \mid \|\hat\ell\| \geqslant 1\}; \mathrm{A}_\infty\langle \mathsf{j}\rangle)\bigr). \end{aligned} \tag{3.24}$$

The free graded $n \wedge 1$-operad module generated by $m_{n_1;\dots;n_k}$ and $\mathsf{f}_{(\ell^k_1;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}}$ has the form

$$\begin{aligned} K &= F\bigl(\Bbbk\{m_{n_1;\dots;n_k} \mid k + \textstyle\sum_{q=1}^k n_q \geqslant 3\}, \Bbbk\{\mathsf{f}_{(\ell^k_1;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}} \mid \|\hat\ell\| \geqslant 1\}\bigr) \\ &= \bigl(\Bbbk\langle m_{n_1;\dots;n_k} \mid k + \textstyle\sum_{q=1}^k n_q \geqslant 3\rangle, \\ &\qquad \odot_{\geqslant 0}\,({}^{[n]}\Bbbk\langle m_{n_1;\dots;n_k} \mid k + \textstyle\sum_{q=1}^k n_q \geqslant 3\rangle; \Bbbk\{\mathsf{f}_{(\ell^k_1;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}} \mid \|\hat\ell\| \geqslant 1\})\bigr). \end{aligned}$$

It is a direct summand of (3.24), so we have a split exact sequence in $\mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$

$$0 \longrightarrow K \underset{\pi}{\overset{\alpha}{\rightleftarrows}} (\mathrm{A}_\infty, \mathrm{F}_n)\langle \mathsf{j}\rangle \underset{\omega}{\overset{\varkappa}{\rightleftarrows}} (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) \longrightarrow 0,$$

where ω takes $\mathsf{j}\rho_\varnothing$ to the nullary generator $\mathsf{j}\rho_\varnothing$. Consider also the graded $n \wedge 1$-module

$$L = F\bigl(\Bbbk\{m_{n_1;\dots;n_k} \mid k + \textstyle\sum_{q=1}^k n_q \geqslant 3\}, \Bbbk\{\mathsf{v}_k, \mathsf{f}_{e_k}, \mathsf{f}_{(\ell_1^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}} \mid \|\hat{\ell}\| \geqslant 2\}\bigr).$$

Notice that the map $L \to K$, $\mathsf{v}_k \mapsto \mathsf{j}\mathsf{f}_{e_k}$, which maps other generators identically, identifies the $n \wedge 1$-modules L and K.

Consider the graded module morphism

$$\beta : L = \bigl(\Bbbk\langle m_{n_1;\dots;n_k} \mid k + \textstyle\sum_{q=1}^k n_q \geqslant 3\rangle, \overline{L}\bigr) \to (\mathrm{A}_\infty, \mathrm{F}_n)\langle \mathsf{j}\rangle = \bigl(\mathrm{A}_\infty\langle \mathsf{j}\rangle, \overline{(\mathrm{A}_\infty, \mathrm{F}_n)\langle \mathsf{j}\rangle}\bigr),$$
$$\mathsf{v}_k \mapsto \mathsf{j}\mathsf{f}_{e_k} - \mathsf{j}\rho_\varnothing,$$

which maps other generators identically. The morphism β extends to basic elements so that each factor $\mathsf{j}\rho_\varnothing$ arising from a vertex of type v_k gives its j to subsequent $m_{;;;}$ adding another semicolon to its indexing sequence. This follows by associativity of ρ. The basic elements v_k are mapped by $\beta\varkappa$ to $-\mathsf{j}\rho_\varnothing$. For any other basic element $b(t) \in L$ we have $b(t).\beta\varkappa = 0$.

The map $\beta - \beta\varkappa\omega \in \mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$ factors through α as the following diagram shows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \underset{\pi}{\overset{\alpha}{\rightleftarrows}} & (\mathrm{A}_\infty, \mathrm{F}_n)\langle \mathsf{j}\rangle & \underset{\omega}{\overset{\varkappa}{\rightleftarrows}} & (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) & \longrightarrow & 0 \\ & & {\scriptstyle\exists!\gamma}\uparrow & \nearrow_{\beta-\beta\varkappa\omega} & & & & & \\ & & L & & & & & & \end{array}$$

The unique map $\gamma = (\beta - \beta\varkappa\omega)\pi = \beta\pi : L \to K \in \mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$, such that $\beta - \beta\varkappa\omega = \gamma\alpha$, has a triangular matrix. In fact, L and K have an $\mathbb{N}$-grading, $L^q = \oplus\Bbbk b(t)$, $K^q = \oplus\Bbbk b(t)$, where the summation is over forests t with q vertices labelled by one of v_k (resp. one of $\mathsf{j}\mathsf{f}_{e_k}$). The map γ takes the filtration $L_q = L^0 \oplus \dots \oplus L^q$ to the filtration $K_q = K^0 \oplus \dots \oplus K^q$. The diagonal entries $\gamma^{qq} : L^q \to K^q$ are identity maps. Thus, the matrix of γ equals $1 - N$, where N is locally nilpotent, and γ is invertible. We obtained a split exact sequence

$$0 \longrightarrow L \xrightarrow{\beta-\beta\varkappa\omega} (\mathrm{A}_\infty, F_1)\langle \mathsf{j}\rangle \underset{\omega}{\overset{\varkappa}{\rightleftarrows}} (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) \longrightarrow 0. \tag{3.25}$$

Let us decompose the first two terms into direct sums

$$\overline{L} = \Bbbk\{\mathsf{v}_s \mid s \in \mathbf{n}\} \oplus \bigl(\overline{L} \ominus \Bbbk\{\mathsf{v}_s \mid s \in \mathbf{n}\}\bigr),$$
$$\overline{(\mathrm{A}_\infty, \mathrm{F}_n)\langle \mathsf{j}\rangle} = \Bbbk\{\mathsf{j}\rho_\varnothing, \mathsf{j}\mathsf{f}_{e_s} \mid s \in \mathbf{n}\} \oplus \bigl(\overline{(\mathrm{A}_\infty, \mathrm{F}_n)\langle \mathsf{j}\rangle} \ominus \Bbbk\{\mathsf{j}\rho_\varnothing, \mathsf{j}\mathsf{f}_{e_s} \mid s \in \mathbf{n}\}\bigr),$$

where the complements are spanned by all basic elements except those listed in the first summands. The maps β and $\beta - \beta\varkappa\omega$ preserve this decomposition. Their restriction to the second summand coincide and this is an isomorphism due to exactness of (3.25). If we drop out complements, this split exact sequence takes the form

$$0 \longrightarrow \Bbbk\{\mathsf{v}_s \mid s \in \mathbf{n}\} \xrightarrow{\beta-\beta\varkappa\omega} \Bbbk\{\mathsf{j}\rho_\varnothing, \mathsf{jf}_{e_s} \mid s \in \mathbf{n}\} \underset{\omega}{\overset{\varkappa}{\rightleftarrows}} \Bbbk\mathsf{j}\rho_\varnothing \longrightarrow 0,$$

where $\mathsf{v}_s.(\beta - \beta\varkappa\omega) = \mathsf{jf}_{e_s}$. Let us replace it with another split exact sequence

$$0 \longrightarrow \Bbbk\{\mathsf{v}_s \mid s \in \mathbf{n}\} \underset{\tau}{\overset{\beta}{\rightleftarrows}} \Bbbk\{\mathsf{j}\rho_\varnothing, \mathsf{jf}_{e_s} \mid s \in \mathbf{n}\} \underset{\omega}{\overset{\theta}{\rightleftarrows}} \Bbbk\mathsf{j}\rho_\varnothing \longrightarrow 0,$$

where $\mathsf{jf}_{e_s}.\theta = \mathsf{j}\rho_\varnothing.\theta = \mathsf{j}\rho_\varnothing$ and $\mathsf{j}\rho_\varnothing.\tau = 0$, $\mathsf{jf}_{e_s}.\tau = \mathsf{v}_s$. Restoring back the dropped isomorphism of second summands we obtain from the above the split exact sequence

$$0 \longrightarrow L \xrightarrow{\beta} (\mathrm{A}_\infty, F_1)\langle\mathsf{j}\rangle \underset{\omega}{\overset{\theta}{\rightleftarrows}} (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) \longrightarrow 0,$$

such that θ vanishes on the complement.

Adding freely i we deduce the split exact sequence in $\mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$

$$0 \to L\langle\mathsf{i}\rangle \to (\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{j}, \mathsf{i}\rangle \to (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) \to 0. \tag{3.26}$$

The image of the embedding is precisely $(\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n)$, thus the latter graded $n \wedge 1$-module is free. In the particular case of $n = 0$ the module part is generated by the empty set of generators. Therefore, $\mathrm{F}^{\mathsf{hu}}_0 = \mathrm{A}^{\mathsf{hu}}_\infty(0)$ by [Lyu15, Lemma A.9].

Furthermore, from the top row of diagram (3.22) we deduce a splittable exact sequence in $\mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$

$$0 \to (\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{i}\rangle \to (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle\mathsf{i}\rangle \to (\Bbbk 1^{\mathsf{su}}, \Bbbk 1^{\mathsf{su}}\rho_\varnothing) \to 0.$$

We may choose the splitting of this exact sequence as indicated below:

$$0 \to (\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{i}\rangle \to (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle\mathsf{i}\rangle \to (\Bbbk\{1^{\mathsf{su}} - \mathsf{i}\}, \Bbbk\{1^{\mathsf{su}}\rho_\varnothing - \mathsf{i}\rho_\varnothing\}) \to 0.$$

Adding freely j we get the split exact sequence

$$0 \to (\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{i}, \mathsf{j}\rangle \to (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle\mathsf{i}, \mathsf{j}\rangle \to (\Bbbk\{1^{\mathsf{su}} - \mathsf{i}\}, \Bbbk\{(1^{\mathsf{su}} - \mathsf{i})\rho_\varnothing\}) \to 0. \tag{3.27}$$

Combining (3.26) with (3.27) we get a split exact sequence

$$0 \to (\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n) \xrightarrow{i'} (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle\mathsf{i}, \mathsf{j}\rangle \to (\Bbbk\{1^{\mathsf{su}} - \mathsf{i}, \mathsf{j}\}, \Bbbk\{(1^{\mathsf{su}} - \mathsf{i})\rho_\varnothing, \mathsf{j}\rho_\varnothing\}) \to 0. \tag{3.28}$$

The differential in $(\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n)$ is computed through that of $(\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle\mathsf{i}, \mathsf{j}\rangle$. Actually, (3.28) is a split exact sequence in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$, where the third term obtains the differential $\mathsf{j}.\partial = 1^{\mathsf{su}} - \mathsf{i}$, $\mathsf{j}\rho_\varnothing.\partial = 1^{\mathsf{su}}\rho_\varnothing - \mathsf{i}\rho_\varnothing$. The third term is contractible, which shows that the inclusion i' is a homotopy isomorphism in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$. Hence, the epimorphism $p = i' \cdot p'' : (\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n) \to (\mathit{As1}, \mathit{FAs1}_n)$ is a homotopy isomorphism as well.

In order to prove that $(\mathbb{1},0) \to (\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n)$ is a standard cofibration we present it as a colimit of sequence of elementary cofibrations

$$(\mathbb{1},0) \to \mathcal{D}_0 = F(\Bbbk\{\mathsf{i}, m_2\}, \Bbbk\{\mathsf{f}_{e_s} \mid s \in \mathbf{n}\}) \to \mathcal{D}_1 \to \mathcal{D}_2 \to \dots,$$

where for $r > 0$

$$\mathcal{D}_r = F\big(\Bbbk\{\mathsf{i}, m_{n_1;\dots;n_k} \mid \deg m_{n_1;\dots;n_k} \geqslant -r\}, \Bbbk\{\mathsf{v}_s, \mathsf{f}_{(\ell^k_1;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}} \mid s \in \mathbf{n},\ \deg \mathsf{f}_{(\ell^k_1;\dots;\ell^k_{t^k})} \geqslant -r\}\big).$$

Summing up, we have

3.19. Theorem. *The $n \wedge 1$-operad* $\mathbf{dg}$*-module* $(\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n)$ *is a cofibrant replacement of* $(As1, FAs1_n)$*. Moreover,* $(\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n) \to (As1, FAs1_n)$ *is a homotopy isomorphism in* $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$.

Algebra maps over $(\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n)$ are identified with homotopy unital A_∞-morphisms, which we define in the spirit of Fukaya's approach:

3.20. Definition. *A* homotopy unital structure *of an* A_∞*-morphism* $\mathsf{f} : A_1, \dots, A_n \to B$ *is an* A_∞*-morphism* $\mathsf{f}^+ : (A^+_k)_{k\in\mathbf{n}} = (A_k \oplus \Bbbk 1^{\mathsf{su}}_{A_k} \oplus \Bbbk \mathsf{j}^{A_k})_{k\in\mathbf{n}} \to B \oplus \Bbbk 1^{\mathsf{su}}_B \oplus \Bbbk \mathsf{j}^B = B^+$ *between given homotopy unital* A_∞*-algebra structures such that:*

(1) f^+ *is a strictly unital: for all* $1 \leqslant k \leqslant n$

$$1^{\mathsf{su}}_{A_k}\mathsf{f}^+_{e_k} = 1^{\mathsf{su}}_B, \quad \big[1^{\otimes(k-1)} \otimes (1^{\otimes a} \otimes 1^{\mathsf{su}}_{A_k} \otimes 1^{\otimes b}) \otimes 1^{\otimes(n-k)}\big]\mathsf{f}^+_\ell = 0 \quad \text{if} \quad a+1+b = \ell^k,\ \|\ell\| > 1.$$

(2) the element $\mathsf{v}^B_k = \mathsf{j}^{A_k}\mathsf{f}^+_{e_k} - \mathsf{j}^B$ *is contained in* B*;*
(3) the restriction of f^+ *to* $A_1, \dots, A_n$ *gives* f*;*
(4) $\big[\otimes^{k\in\mathbf{n}}(A_k \oplus \Bbbk \mathsf{j}^{A_k})^{\otimes \ell^k}\big]\mathsf{f}^+_\ell \subset B$*, for each* $\ell \in \mathbb{N}^n$*,* $\|\ell\| > 1$*.*

Homotopy unital structure of an A_∞-morphism f means a *choice* of such f^+. There is another notion of unitality which is a *property* of an A_∞-morphism:

3.21. Definition. [See [BLM08, Proposition 9.13]] *An* A_∞*-morphism* $\mathsf{f} : A_1, \dots, A_n \to B$ *between unital* A_∞*-algebras is* unital *if the cycles* $\mathsf{i}^{A_k}\mathsf{f}_{e_k}$ *and* i^B *differ by a boundary for all* $1 \leqslant k \leqslant n$*.*

For a homotopy unital A_∞-morphism $\mathsf{f} : A_1, \dots, A_n \to B$ the equation holds $\mathsf{v}^B_k m_1 = \mathsf{v}_k\partial = \mathsf{i}\rho_\varnothing - \mathsf{i}\mathsf{f}_{e_k} = \mathsf{i}^B - \mathsf{i}^{A_k}\mathsf{f}_{e_k}$. Thus an A_∞-morphism with a homotopy unital structure is unital.

3.22. Conjecture. Unitality of an A_∞-morphism is equivalent to homotopy unitality: any unital A_∞-morphism admits a homotopy unital structure.

All reasoning of this section can be applied to F_n in place of F_n. A nullary degree -1 cycle $\mathbf{1}^{\mathsf{su}}$ subject to relations (1.6) is added to A_∞. The resulting operad is denoted A^{su}_∞. We consider the A^{su}_∞-module

$$\tilde{F}_n = \bigodot_{i=1}^n A^{\mathsf{su}}_\infty \odot^i_{A_\infty} F_n \odot^0_{A_\infty} A^{\mathsf{su}}_\infty = \odot_{\geqslant 0}({}^n A^{\mathsf{su}}_\infty; \Bbbk\{f_j \mid j \in \mathbb{N}^n - 0\}; A^{\mathsf{su}}_\infty).$$

It is divided by the graded ideal generated by the following system of relations

$$\rho_\varnothing(\mathbf{1}^{\mathsf{su}}) = \lambda^i_{e_i}(\mathbf{1}^{\mathsf{su}}; f_{e_i}),\ \forall\, i, \quad \lambda^i_\ell({}^a1, \mathbf{1}^{\mathsf{su}}, {}^b1; f_\ell) = 0 \ \text{ if } \ a+1+b = \ell^i,\ \|\ell\| > 1.$$

The quotient is denoted F^{su}_n. Similarly to the above we add two nullary operations $\mathbf{i}$, $\mathbf{j}$ to A^{su}_∞ with $\deg \mathbf{i} = -1$, $\deg \mathbf{j} = -2$, $\mathbf{i}\partial = 0$, $\mathbf{j}\partial = \mathbf{i} - \mathbf{1}^{\mathsf{su}}$. The obtained $A^{\mathsf{su}}_\infty\langle \mathbf{i}, \mathbf{j}\rangle$-module $F^{\mathsf{su}}_n\langle \mathbf{i}, \mathbf{j}\rangle$ contains an A^{hu}_∞-submodule F^{hu}_n spanned by the nullary elements $\mathbf{v}_k = \lambda^k_{e_k}(\mathbf{j}; f_{e_k}) - \mathbf{j}\rho_\varnothing = \mathbf{j}f_{e_k} - \mathbf{j}\rho_\varnothing$, $k \in \mathbf{n}$, $\deg \mathbf{v}_k = -2$, and by elements $f_{(\ell^k_1;\ell^k_2;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}}$ similar to (3.23). There are invertible operad module homomorphisms Σ of degree 1 sending $f_j \mapsto \mathsf{f}_j$, $\mathbf{1}^{\mathsf{su}} \mapsto 1^{\mathsf{su}}$, $\mathbf{i} \mapsto \mathsf{i}$, $\mathbf{j} \mapsto \mathsf{j}$, $\mathbf{v}_k \mapsto \mathsf{v}_k$:

$$\Sigma : (A^{\mathsf{su}}_\infty, F^{\mathsf{su}}_n) \to (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n), \qquad \Sigma : (A^{\mathsf{su}}_\infty, F^{\mathsf{su}}_n)\langle \mathbf{i}, \mathbf{j}\rangle \to (\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)\langle \mathsf{i}, \mathsf{j}\rangle,$$
$$\Sigma : (A^{\mathsf{hu}}_\infty, F^{\mathsf{hu}}_n) \to (\mathrm{A}^{\mathsf{hu}}_\infty, \mathrm{F}^{\mathsf{hu}}_n).$$

4. Composition of morphisms with several arguments

4.1. Non-shifted A_∞-morphisms. Below we shall prove that the convolution H of A_∞-polymodule cooperad F and the lax $\mathcal{C}at$-multifunctor $\mathcal{H}om$ built from $\underline{\mathsf{C}}_\Bbbk$ gives a multicategory of A_∞-algebras and A_∞-morphisms. Its objects are A_∞-algebras and morphisms $(A_i)_{i\in I} \to B \in \mathsf{H}$ are morphisms of $n \wedge 1$-operad modules

$$({}^I\mathrm{A}_\infty; \mathrm{F}_n; \mathrm{A}_\infty) \to ((\mathcal{E}nd\, A_i)_{i\in I}; \mathit{hom}((A_i)_{i\in I}; B); \mathcal{E}nd\, B),$$

which are precisely A_∞-morphisms with several arguments. Their composition is the composition in H.

Let us denote by a_∞ the multiquiver of A_∞-algebras and their morphisms, that is, $\mathsf{a}_\infty = \mathsf{H}$ for $\mathcal{A} = \mathrm{A}_\infty$. Due to reasoning after equation (8.20.2) in [BLM08] the map

$$Ts : \mathsf{a}_\infty((A_i)_{i\in\mathbf{n}}; B) \to \mathbf{dgac}(\otimes^{i\in\mathbf{n}} TsA_i, TsB), \quad \mathsf{f} \mapsto f,$$

is a bijection.

Diagram (3.8) implies commutativity of

$$\begin{array}{ccc} \mathrm{F}_{|v|} & \xrightarrow{g_v} & \mathit{hom}((sA_e)_{e\in\mathrm{in}(v)}; sA_{\mathrm{ou}(v)}) \\ \Sigma\big\downarrow & & \big\downarrow \mathit{hom}((\sigma)_{e\in\mathrm{in}(v)};\sigma^{-1}) \\ \mathrm{F}_{|v|} & \xrightarrow{\mathsf{g}_v} & \mathit{hom}((A_e)_{e\in\mathrm{in}(v)}; A_{\mathrm{ou}(v)}) \end{array}$$

Here $\mathit{hom}((\sigma)_{e\in\mathrm{in}(v)};\sigma^{-1}) = \mathit{hom}((\sigma)_{e\in\mathrm{in}(v)};1) \cdot \mathit{hom}((1)_{e\in\mathrm{in}(v)};\sigma^{-1})$ is the product of right operators.

Put differently, for each $k \in \mathbb{N}^{\mathrm{in}(v)} - 0$ there is a bijection

$$\mathbf{gr}\Big(\bigotimes^{e\in\mathrm{in}(v)} T^{k^e} sA_e, sA_{\mathrm{ou}(v)}\Big) \xrightarrow{\cong} \underline{\mathbf{gr}}\Big(\bigotimes^{e\in\mathrm{in}(v)} T^{k^e} A_e, A_{\mathrm{ou}(v)}\Big)^{1-\|k\|},\ g^v_k \mapsto \mathsf{g}^v_k = \Big(\bigotimes^{e\in\mathrm{in}(v)} \sigma^{\otimes k^e}\Big)\cdot g^v_k \cdot \sigma^{-1}.$$

Using it we write equation (4.15) of [Lyu15] composed with projection pr_j as

$$\prod_{v\in\mathrm{v}(t)} \prod_{k\in\mathbb{N}^{\mathrm{in}(v)}-0} \underline{\mathbf{gr}}\big(\bigotimes_{e\in\mathrm{in}(v)} T^{k^e}A_e, A_{\mathrm{ou}(v)}\big)^{1-\|k\|} \to \underline{\mathbf{gr}}\big(\bigotimes_{a\in\mathrm{Inp}\,t} T^{j^a}A_a, A_{\text{root edge}}\big)^{1-\|j\|},$$
$$\big(\mathsf{g}^v_k\big)^{v\in\mathrm{v}(t)}_{k\in\mathbb{N}^{\mathrm{in}(v)}-0} \mapsto \mathrm{comp}$$
$$\Big(\bigotimes^{a\in\mathrm{Inp}\,t}\sigma^{\otimes j^a}\Big)\Big\langle \sum^{\text{surjective } t\text{-tree } \tau}_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=j^a} \bigotimes^{v\in\mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \Big[\Big(\bigotimes^{e\in\mathrm{in}(v)}\sigma^{\otimes|\tau(e)^{-1}(p)|}\Big)^{-1}\mathsf{g}^v_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}}\sigma\Big]\Big\rangle\sigma^{-1}. \tag{4.1}$$

Let us compute this expression. Assume that $t \ne |$ and take the smallest vertex 1 of $(\mathrm{v}(t), \leqslant)$. Consider subtree t' of t with $V(t') = V(t) - \mathrm{inV}(1)$, $E(t') = E(t) - \mathrm{in}(1)$, $\mathrm{v}(t') = \mathrm{v}(t) - \{1\}$. Thus, $\mathrm{Inpv}(t') = \{1\} \sqcup \mathrm{Inpv}(t) - \mathrm{inV}(1)$. Define t'-tree $\tau' = \tau|_{t'}$ for a t-tree τ. Decompose correspondingly the set of inputs of t into three subsets,

$$\begin{aligned}(\mathrm{Inp}\,t, \triangleleft) &= \mathrm{Inp}^-\,t \,\underline{\sqcup}_{\triangleleft}\, \mathrm{Inp}^0\,t \,\underline{\sqcup}_{\triangleleft}\, \mathrm{Inp}^+\,t, \text{ where}\\ \mathrm{Inp}^0\,t &= \mathrm{in}(1),\\ \mathrm{Inp}^-\,t &= \{a\in\mathrm{Inp}\,t \mid a\triangleleft 1,\ a\notin\mathrm{in}(1)\},\\ \mathrm{Inp}^+\,t &= \{a\in\mathrm{Inp}\,t \mid a\triangleright 1\}.\end{aligned}$$

Respectively the set of inputs of $\tilde\tau$ is decomposed as $(\mathrm{Inp}\,\tilde\tau, <) = \mathrm{Inp}^-\,\tilde\tau \,\underline{\sqcup}_<\, \mathrm{Inp}^0\,\tilde\tau \,\underline{\sqcup}_<\, \mathrm{Inp}^+\,\tilde\tau$, where $\mathrm{Inp}^\diamondsuit\,\tilde\tau = \{(a,x) \mid a\in\mathrm{Inp}^\diamondsuit\,t,\ x\in\tau(\mathrm{tail}(a))\}$ for $\diamondsuit\in\{-,0,+\}$.

A summand of the considered expression is written as

$$\begin{aligned}&\mathrm{comp}\,\sigma^{\otimes\,\mathrm{Inp}\,\tilde\tau}\big(\otimes^{p\in\tau(1)} g_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(1)}}\big)\big(\otimes^{v\in\mathrm{v}(t')}\ \otimes^{p\in\tau(v)}\ g_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}}\big)\sigma^{-1}\\ &= (-1)^{\mathrm{sign}(\phi_0)}\,\mathrm{comp}\big\langle\sigma^{\otimes\,\mathrm{Inp}^-\,\tilde\tau}\otimes\big[\otimes^{p\in\tau(1)}\big(\otimes^{e\in\mathrm{in}(1)}\sigma^{\otimes\tau(e)^{-1}(p)} g_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(1)}}\big)\big]\otimes\sigma^{\otimes\,\mathrm{Inp}^+\,\tilde\tau}\big\rangle\\ &\qquad\big(\otimes^{v\in\mathrm{v}(t')}\ \otimes^{p\in\tau(v)}\ g_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}}\big)\sigma^{-1}.\end{aligned}$$

Here the identity map

$$\phi_0 : (\mathrm{Inp}^0\,\tilde\tau, <) = \underline{\bigsqcup}^{<}_{e\in\mathrm{in}(1)} \tau(\mathrm{tail}(e)) = \underline{\bigsqcup}^{<}_{e\in\mathrm{in}(1)} \underline{\bigsqcup}^{<}_{p\in\tau(1)} \tau(e)^{-1}(p) \xrightarrow{\cong} \underline{\bigsqcup}^{<}_{p\in\tau(1)} \underline{\bigsqcup}^{<}_{e\in\mathrm{in}(1)} \tau(e)^{-1}(p)$$

is viewed as an order changing permutation. The sign of this permutation is

$$\mathrm{sign}(\phi_0) = \sum_{q<p\in\tau(1)} \sum_{e<g\in\mathrm{in}(1)} |\tau(e)^{-1}(p)|\cdot|\tau(g)^{-1}(q)|.$$

Thus the above expression is

$$\begin{aligned}&(-1)^{\mathrm{sign}(\phi_0)+\sum_{q<p\in\tau(1)}(-1+\sum_{e\in\mathrm{in}(1)}|\tau(e)^{-1}(p)|)+|\mathrm{Inp}^-\,\tilde\tau|\sum_{p\in\tau(1)}(-1+\sum_{e\in\mathrm{in}(1)}|\tau(e)^{-1}(p)|)}\\ &\qquad\mathrm{comp}\big(\otimes^{p\in\tau(1)}\mathsf{g}_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(1)}}\big)\sigma^{\otimes\,\mathrm{Inp}\,\tilde\tau'}\big(\otimes^{v\in\mathrm{v}(t')}\ \otimes^{p\in\tau(v)}\ g_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}}\big)\sigma^{-1}.\end{aligned}$$

Let us denote the obtained sign function by

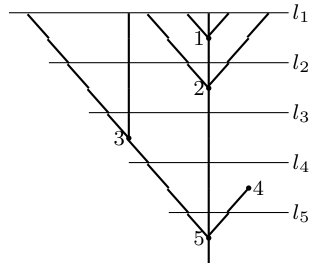
$$\operatorname{sg}'(\tau) = \operatorname{sign}(\phi_0) + \sum_{p\in\tau(1)} (p-1)\Big(\sum_{e\in\operatorname{in}(1)} |\tau(e)^{-1}(p)| - 1\Big) + |\operatorname{Inp}^-\tilde{\tau}|\Big(\sum_{a\in\operatorname{inV}(1)} j^a - |\tau(1)|\Big).$$

The right part of the obtained expression is similar to the initial expression with t', τ',

$$j'^a = \begin{cases} j^a, & \text{for } a \notin \operatorname{Inpv}^0 t, \\ |\tau(1)|, & \text{for } a = 1, \end{cases}$$

in place of t, τ, j. This allows to conclude by induction.

Let us draw an ordered tree t as a tree with height: all internal vertices are placed at points of the plane with different height (=ordinate) so that the order $\leqslant$ of vertices agrees with reversed order of heights, see example.



Naturally, the planar tree has to become a plane tree, which means that the order $\lhd$ and the orientation of the plane agree. Furthermore, all input vertices have to be placed on one horizontal line l_1, whose height exceeds all heights of internal vertices. For technical purposes we draw also a horizontal line l_i between vertices $i-1$ and i. The subset of $E(t)$ consisting of edges that intersect l_i is denoted $\operatorname{Inp}_i t$. This subset is naturally $\lhd$-ordered by intersection points of edges with l_i, e.g. $(\operatorname{Inp}_1 t, \lhd) = (\operatorname{Inp} t, \lhd)$. The set of tails of edges from $\operatorname{Inp}_i t$ is denoted $\operatorname{Inpv}_i t$. The bijection $(\operatorname{Inpv}_i t, \lhd) \cong (\operatorname{Inp}_i t, \lhd)$ preserves the order. Decompose correspondingly the set of inputs of t into three subsets,

$$\begin{aligned} (\operatorname{Inpv}_i t, \lhd) &= \operatorname{Inpv}_i^- t \,\underline{\lhd}\, \operatorname{Inpv}_i^0 t \,\underline{\lhd}\, \operatorname{Inpv}_i^+ t, \text{ where} \\ \operatorname{Inpv}_i^0 t &= \operatorname{inV}(i), \\ \operatorname{Inpv}_i^- t &= \{u \in \operatorname{Inpv}_i t \mid u \lhd i,\ u \notin \operatorname{inV}(i)\}, \\ \operatorname{Inpv}_i^+ t &= \{u \in \operatorname{Inpv}_i t \mid u \rhd i\}. \end{aligned}$$

We conclude that map (4.1) takes $\big(\mathsf{g}_k^v\big)_{k\in\mathbb{N}^{\operatorname{in}(v)}-0}^{v\in\operatorname{v}(t)}$ to

$$\operatorname{comp} \sum_{\forall a\in\operatorname{Inpv} t\, |\tau(a)|=j^a}^{\text{surjective } t\text{-tree } \tau} (-1)^{\operatorname{sg}(\tau)} \otimes^{v\in\operatorname{v}(t)} \otimes^{p\in\tau(v)} \mathsf{g}_{|\tau(e)^{-1}(p)|_{e\in\operatorname{in}(v)}},$$

where

$$\operatorname{sg}(\tau) = \sum_{v\in \mathrm{v}(t)} \sum_{q<p\in\tau(v)} \sum_{e<g\in \mathrm{in}(v)} |\tau(e)^{-1}(p)|\cdot|\tau(g)^{-1}(q)|$$
$$+\sum_{v\in\mathrm{v}(t)}\sum_{p\in\tau(v)}(p-1)\big(\sum_{e\in\mathrm{in}(v)}|\tau(e)^{-1}(p)|-1\big)+\sum_{v\in\mathrm{v}(t)}\big(\sum_{u\in\mathrm{Inpv}_v^- t}|\tau(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)}|\tau(a)|-|\tau(v)|\big).$$

Correspondingly, we define a degree 0 map $\Delta^{\mathsf{G}}(t)(j) : \mathrm{F}_{\mathrm{Inp}\,t}(j) \to \circledast_{\mathsf{G}}(t)(\mathrm{F}_{|v|})_{v\in\mathrm{v}(t)}(j)$,

$$\Delta^{\mathsf{G}}(t)(\mathsf{f}_j) = \sum_{\forall a\in \mathrm{Inpv}\,t\,|\tau(a)|=j^a}^{\text{surjective } t\text{-tree } \tau} (-1)^{\operatorname{sg}(\tau)} \otimes^{v\in\mathrm{v}(t)} \otimes^{p\in\tau(v)} \mathsf{f}_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}}. \tag{4.2}$$

or, equivalently,

$$\Delta^{\mathsf{G}}(t)(j)\cdot \mathrm{pr}_\tau = (-1)^{\operatorname{sg}(\tau)}\Big\langle \mathrm{F}_{\mathrm{Inp}\,t}(j) \xrightarrow{\varSigma(j)^{-1}} F_{\mathrm{Inp}\,t}(j) \xrightarrow{\varDelta^{\mathsf{G}}(t)(j)\cdot\mathrm{pr}_\tau} \tag{4.3}$$
$$\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} F_{|v|}\Big(\big(|\tau(e)^{-1}(p)|\big)_{e\in\mathrm{in}(v)}\Big) \xrightarrow{\otimes^{v\in\mathrm{v}(t)}\otimes^{p\in\tau(v)}\varSigma} \bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} \mathrm{F}_{|v|}\Big(\big(|\tau(e)^{-1}(p)|\big)_{e\in\mathrm{in}(v)}\Big)\Big\rangle.$$

According to Theorem B.1 the map Δ is a morphism of $\mathrm{Inp}(t)\wedge 1$-A_∞-modules. We may compute it recursively:

$$\Delta^{\mathsf{G}}(t)(\mathsf{f}_j) = \sum_{(\tau(e):\mathbf{j}^e\to\tau(1))_{e\in\mathrm{in}(1)}} (-1)^{\operatorname{sg}'(\tau)}\big(\otimes^{p\in\tau(1)}\mathsf{f}_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(1)}}\big)\otimes\Delta^{\mathsf{G}}(t')(\mathsf{f}_{j'}), \tag{4.4}$$

the summation is taken over all families of isotonic maps $\tau(e)$, $e\in\mathrm{in}(1)$, with varying target $\tau(1)$ such that $\cup_e \operatorname{Im}\tau(e) = \tau(1)$.

By the way we get a recursive formula for comultiplication in F_n:

$$\varDelta^{\mathsf{G}}(t)(f_j) = \sum_{(\tau(e):\mathbf{j}^e\to\tau(1))_{e\in\mathrm{in}(1)}} \big(\otimes^{p\in\tau(1)} f_{|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(1)}}\big)\otimes\varDelta^{\mathsf{G}}(t')(f_{j'}).$$

The following statement elucidates in what sense $\Delta^{\mathsf{G}}(t)$ depends on a planar tree t rather than on ordered tree.

4.2. Proposition. *Let $(\mathrm{v}(t),\leqslant)$ and $(\mathrm{v}(t),\leqslant')$ be two orders admissible for the same planar tree t. Let $\sigma : (\mathrm{v}(t),\leqslant)\to(\mathrm{v}(t),\leqslant')$ be the isotonic bijection. Denoting the corresponding symmetry map also by σ we have*

$$\sigma.\Delta^{\mathsf{G}}(t,\leqslant) = \Delta^{\mathsf{G}}(t,\leqslant').$$

PROOF. It suffices to consider elementary transposition $\sigma = (i-1\, i)$ and two orders $\leqslant$, $\leqslant'$, which differ only by interchanging two vertices $i-1 \leqslant i$, not related by $\preccurlyeq$. Denote by t' the planar tree t with the ordering $\leqslant'$ of internal vertices, $i-1 \geqslant' i$.

Decompose $\operatorname{Inpv}_{i-1} t = \operatorname{Inpv}_{i-1} t'$ as

$$\operatorname{Inpv}^-_{i-1} t \boxed{<} \operatorname{inV}(i-1) \boxed{<} A \boxed{<} \operatorname{inV}(i) \boxed{<} \operatorname{Inpv}^+_i t.$$

Notice that $\operatorname{Inpv}^-_{i-1} t' = \operatorname{Inpv}^-_{i-1} t$ and $\operatorname{Inpv}^-_i t = \{i-1\} \boxed{<} A$, $\operatorname{Inpv}^-_i t' = \operatorname{inV}(i-1) \boxed{<} A$. We claim that

$$\Bigl(|\tau(i-1)| - \sum_{a\in \operatorname{inV}(i-1)} |\tau(a)|\Bigr) \cdot \Bigl(|\tau(i)| - \sum_{b\in \operatorname{inV}(i)} |\tau(b)|\Bigr)$$
$$+ \Bigl(\sum_{u\in \operatorname{Inpv}^-_i t} |\tau(u)|\Bigr) \cdot \Bigl(\sum_{b\in \operatorname{inV}(i)} |\tau(b)| - |\tau(i)|\Bigr) = \Bigl(\sum_{u\in \operatorname{Inpv}^-_i t'} |\tau(u)|\Bigr) \cdot \Bigl(\sum_{b\in \operatorname{inV}(i)} |\tau(b)| - |\tau(i)|\Bigr).$$

In fact,

$$\sum_{u\in \operatorname{Inpv}^-_i t'} |\tau(u)| - \sum_{u\in \operatorname{Inpv}^-_i t} |\tau(u)| = \sum_{u\in \operatorname{inV}(i-1)} |\tau(u)| - |\tau(i-1)|.$$

This implies the required equation between signs. ∎

Applying Theorem B.1 to degree 1 homomorphism $(\varSigma, \varSigma) : (A_\infty, F_n) \to (\mathrm{A}_\infty, \mathrm{F}_n)$, see (3.7), we deduce via (B.2)

4.3. PROPOSITION. *$(\mathrm{A}_\infty, \mathrm{F}_n)$ is a graded polymodule cooperad with the comultiplication given by (4.3).*

4.4. THEOREM. *Define comultiplication $\Delta^{\mathsf{M}}(t)$ for the A_∞-polymodule $\mathrm{F}_\bullet$ by*

$$\Delta^{\mathsf{M}}(t)(j) = \bigl[\mathrm{F}_{\operatorname{Inp} t}(j) \xrightarrow{\Delta^{\mathsf{G}}(t)} \circledast_{\mathsf{G}} (t)(\mathrm{F}_{|v|})_{v\in \mathrm{v}(t)}(j) \xrightarrow{\pi} \circledast_{\mathsf{M}} (t)(\mathrm{F}_{|v|})_{v\in \mathrm{v}(t)}(j)\bigr].$$

*On generators it is given by (4.2). For the tree $t = |$ define $\Delta^{\mathsf{M}}(|)(j) : \mathrm{F}_1(j) \to \mathrm{A}_\infty(j)$, $\mathsf{f}_j \mapsto \delta_{j1}$, $j \geqslant 1$. Then all $\Delta^{\mathsf{M}}(t)(j)$ are chain maps, thus, $(\mathrm{A}_\infty, \mathrm{F}_\bullet, \Delta^{\mathsf{M}})$ is a **dg**-polymodule cooperad.*

PROOF. It suffices to prove that $\mathsf{f}_j.\Delta^{\mathsf{M}}(t)\partial = \mathsf{f}_j.\partial\Delta^{\mathsf{M}}(t)$, $j \in \mathbb{N}^{\operatorname{Inp} t} - 0$, for the tree $t = |$ and all trees t with two internal vertices. The proof follows the lines of [Lyu15, Proposition 4.13] with extra care for signs. For the tree $t = |$ and a positive integer j we have

$$\mathsf{f}_j.\partial\Delta^{\mathsf{M}}(|) = \sum_{r+n+t=j}^{n>1} (-1)^{(1-n)r+1-j} (1^{\otimes r} \otimes m_n \otimes 1^{\otimes t}) \mathsf{f}_{r+1+t}.\Delta^{\mathsf{M}}(|)$$
$$+ \sum_{i_1+\dots+i_l=j}^{l>1} (-1)^{l+\sum_{p=1}^{l}(p-1)(i_p-1)} (\mathsf{f}_{i_1} \otimes \mathsf{f}_{i_2} \otimes \dots \otimes \mathsf{f}_{i_l}) m_l.\Delta^{\mathsf{M}}(|)$$
$$= (-1)^{1-j} m_j \chi(j>1) + (-1)^j m_j \chi(j>1) = 0 = \delta_{j1}.\partial = \mathsf{f}_j.\varDelta^{\mathsf{M}}(|)\partial.$$

Let us consider the tree t and the t-tree τ from [Lyu15, (4.17)] with adjacent notation. In particular,

$$j = (j^1, \dots, j^{n+q-1}) = (u^1, \dots, u^{c-1}, i^1, \dots, i^q, u^{c+1}, \dots, u^n) = (u^-, i, u^+).$$

We have

$$\mathsf{f}_j.\Delta^{\mathsf{M}}(t) = \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} r_p = i}^{r_1,\dots,r_{u^c}\in\mathbb{N}^q-0} (-1)^{\mathrm{sg}(\tau)} \Bigl(\bigotimes_{p=1}^{u^c} \mathsf{f}_{r_p}\Bigr) \otimes \mathsf{f}_u,$$

$$\mathrm{sg}(\tau) = \sum_{1\leqslant z<p\leqslant u^c} \sum_{1\leqslant e<g\leqslant q} r_p^e r_z^g + \sum_{p=1}^{u^c}(p-1)\Bigl(\sum_{e=1}^{q} r_p^e - 1\Bigr) + \Bigl(\sum_{g=1}^{q} i^g - u^c\Bigr)\sum_{v=1}^{c-1} u^v. \tag{4.5}$$

We find

$$\mathsf{f}_j.\Delta^{\mathsf{M}}(t)\partial = \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} r_p=i}^{r_1,\dots,r_{u^c}\in\mathbb{N}^q-0} \sum_{h=1}^{n} \sum_{a+w+z=u^h}^{w>1} (-1)^{\mathrm{sg}(\tau)+(1-w)(u^1+\dots+u^{h-1}+a)+1-\|u\|} \tag{4.6}$$
$$\bigl(\otimes_{p=1}^{u^c}\mathsf{f}_{r_p}\bigr) \otimes \lambda^h({}^a1, m_w, {}^z1; \mathsf{f}_{u-(w-1)e_h})$$
$$+ \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} r_p=i}^{r_1,\dots,r_{u^c}\in\mathbb{N}^q-0} \sum_{w=2}^{\infty} \sum_{u_1+\dots+u_w=u}^{u_1,\dots,u_w\in\mathbb{N}^n-0} (-1)^{\mathrm{sg}(\tau)+w+\sum_{1\leqslant b<a\leqslant w}^{1\leqslant e<d\leqslant n} u_a^e u_b^d + \sum_{q=1}^{w}(q-1)(\|u_q\|-1)} \tag{4.7}$$
$$\bigl(\otimes_{p=1}^{u^c}\mathsf{f}_{r_p}\bigr) \otimes \rho((\mathsf{f}_{u_v})_{v=1}^{w}; m_w).$$
$$+ \sum_{u^c=0}^{\infty} \sum_{\sum_{p=1}^{u^c} r_p=i}^{r_1,\dots,r_{u^c}\in\mathbb{N}^q-0} \sum_{h=1}^{u_c}\sum_{g=1}^{q} \sum_{a+x+z=r_h^g}^{x>1} (-1)^{\mathrm{sg}(\tau)+1-\|u\|+u^c-h-\sum_{v=h+1}^{u^c}\|r_v\|+(1-x)(a+\sum_{y=1}^{g-1} r_h^y)} \tag{4.8}$$
$$(-1)^{1-\|r_h\|}\bigl(\otimes_{p=1}^{h-1}\mathsf{f}_{r_p}\bigr) \otimes \lambda^g({}^a1, m_x, {}^z1; \mathsf{f}_{r_h-(x-1)e_g}) \otimes \bigl(\otimes_{p=h+1}^{u^c}\mathsf{f}_{r_p}\bigr) \otimes \mathsf{f}_u$$
$$+ \sum_{k^c=0}^{\infty} \sum_{\sum_{p=1}^{k^c} s_p=i}^{s_1,\dots,s_{k^c}\in\mathbb{N}^q-0} \sum_{a=0}^{k_c-1}\sum_{w=2}^{\infty} \sum_{l_1+\dots+l_w=s_{a+1}}^{l_1,\dots,l_w\in\mathbb{N}^q-0} (-1)^{\mathrm{sg}(\tilde\tau)-\|k\|+k^c-a-\sum_{v=a+2}^{k^c}\|s_v\|+w+\sum_{1\leqslant b<g\leqslant w}^{1\leqslant e<d\leqslant q} l_g^e l_b^d}$$
$$(-1)^{\sum_{\nu=1}^{w}(\nu-1)(\|l_\nu\|-1)}\bigl(\otimes_{p=1}^{a}\mathsf{f}_{s_p}\bigr) \otimes \rho((\mathsf{f}_{l_\nu})_{\nu=1}^{w}; m_w) \otimes \bigl(\otimes_{p=a+2}^{k^c}\mathsf{f}_{s_p}\bigr) \otimes \mathsf{f}_k, \tag{4.9}$$

where $\tilde\tau$ means t-tree τ determined by (k, s) in place of (u, r). Also by (3.6)

$$\mathsf{f}_j.\partial\Delta^{\mathsf{M}}(t) = \sum_{h=1}^{c-1} \sum_{a+w+z=u^h}^{w>1} (-1)^{(1-w)(a+\sum_{v=1}^{h-1}u^v)+1-\|j\|}\lambda^h({}^a1, m_w, {}^z1; \mathsf{f}_{j-((w-1)e_h,0,0)}.\Delta^{\mathsf{M}}(t))$$
$$+ \sum_{g=1}^{q} \sum_{a+x+z=i^g}^{x>1} (-1)^{(1-x)(a+\sum_{v=1}^{c-1}u^v+\sum_{y=1}^{g-1}i^y)+1-\|j\|}\lambda^g({}^a1, m_x, {}^z1; \mathsf{f}_{j-(0,(x-1)e_g,0)}.\Delta^{\mathsf{M}}(t))$$
$$+ \sum_{h=c+1}^{n} \sum_{a+w+z=u^h}^{w>1} (-1)^{(1-w)(a+\sum_{v=1}^{h-1,v\neq c}u^v+\sum_{g=1}^{q}i^g)+1-\|j\|}\lambda^h({}^a1, m_w, {}^z1; \mathsf{f}_{j-(0,0,(w-1)e_h)}.\Delta^{\mathsf{M}}(t))$$

$$+\sum_{w=2}^{\infty}\sum_{j_1+\dots+j_w=j}^{j_1,\dots,j_w\in\mathbb{N}^{\operatorname{Inp}t}-0}(-1)^{w+\sum_{1\leqslant b<a\leqslant w}^{1\leqslant e<d\leqslant n+q-1}j_a^ej_b^d+\sum_{p=1}^{w}(p-1)(\|j_p\|-1)}\rho((\mathsf{f}_{j_v}.\Delta^{\mathsf{M}}(t))_{v=1}^{w};m_w)$$

$$=\sum_{h=1}^{c-1}\sum_{a+w+z=u^h}^{w>1}\sum_{u^c=0}^{\infty}\sum_{\sum_{p=1}^{u^c}r_p=i}^{r_1,\dots,r_{u^c}\in\mathbb{N}^q-0}(-1)^{(1-w)(a+\sum_{v=1}^{h-1}u^v)+1-\|j\|+\operatorname{sg}(\tau|u^h\mapsto u^h-w+1)} \tag{4.10}$$
$$(-1)^{-w(u^c-\sum_{p=1}^{u^c}\|r_p\|)}\big(\otimes_{p=1}^{u^c}\mathsf{f}_{r_p}\big)\otimes\lambda^h\big({}^a1,m_w,{}^z1;\mathsf{f}_{u-(w-1)e_h}\big)$$

$$+\sum_{g=1}^{q}\sum_{b+x+l=i^g}^{x>1}\sum_{u^c=0}^{\infty}\sum_{\sum_{p=1}^{u^c}s_p=i-(x-1)e_g}^{s_1,\dots,s_{u^c}\in\mathbb{N}^q-0}(-1)^{(1-x)(b+\sum_{v=1}^{c-1}u^v+\sum_{y=1}^{g-1}i^y)+1-\|j\|+\operatorname{sg}(\tau|r\mapsto s)} \tag{4.11}$$
$$\lambda^g\big({}^b1,m_x,{}^l1;(\otimes_{p=1}^{u^c}\mathsf{f}_{s_p})\otimes\mathsf{f}_u\big)$$

$$+\sum_{h=c+1}^{n}\sum_{a+w+z=u^h}^{w>1}\sum_{u^c=0}^{\infty}\sum_{\sum_{p=1}^{u^c}r_p=i}^{r_1,\dots,r_{u^c}\in\mathbb{N}^q-0}(-1)^{(1-w)(a+\sum_{v=1}^{h-1,v\neq c}u^v+\sum_{g=1}^{q}i^g)+1-\|j\|} \tag{4.12}$$
$$(-1)^{\operatorname{sg}(\tau|u^h\mapsto u^h-w+1)-w(u^c-\sum_{p=1}^{u^c}\|r_p\|)}\big(\otimes_{p=1}^{u^c}\mathsf{f}_{r_p}\big)\otimes\lambda^h\big({}^a1,m_w,{}^z1;\mathsf{f}_{u-(w-1)e_h}\big)$$

$$+\sum_{w=2}^{\infty}\sum_{j_1+\dots+j_w=j}^{j_1,\dots,j_w\in\mathbb{N}^{\operatorname{Inp}t}-0}(-1)^{w+\sum_{1\leqslant b<a\leqslant w}^{1\leqslant e<d\leqslant n+q-1}j_a^ej_b^d+\sum_{p=1}^{w}(p-1)(\|j_p\|-1)+\sum_{v=1}^{w}\operatorname{sg}(\tau_v)}$$
$$\rho\Bigg(\Bigg(\sum_{u_v^c\in\mathbb{N}}\sum_{\sum_{p=u_1^c+\dots+u_{v-1}^c+1}^{u_1^c+\dots+u_{v-1}^c+u_v^c}r_p=i_v}^{\forall p\,r_p\in\mathbb{N}^q-0}\Big(\bigotimes_{p=1+\sum_{\alpha=1}^{v-1}u_\alpha^c}^{\sum_{\alpha=1}^{v}u_\alpha^c}\mathsf{f}_{r_p}\Big)\otimes\mathsf{f}_{u_v}\Bigg)_{v=1}^{w};m_w\Bigg). \tag{4.13}$$

In the last sum we denote $\sum_{v=1}^{w}u_v^c$ by u^c, thus, $\sum_{v=1}^{w}u_v=u$.

Due to [Lyu15, (2.13)] expression (4.13) can be transformed using

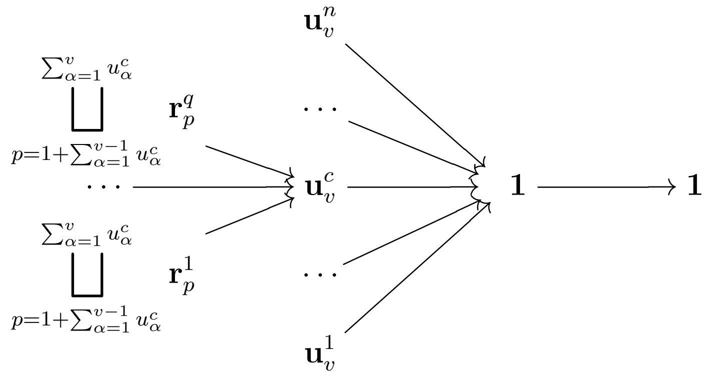
$$\rho\Bigg(\Bigg(\sum_{u_v^c\in\mathbb{N}}\sum_{\sum_{p=u_1^c+\dots+u_{v-1}^c+1}^{u_1^c+\dots+u_{v-1}^c+u_v^c}r_p=i_v}^{\forall p\,r_p\in\mathbb{N}^q-0}\Big(\bigotimes_{p=1+\sum_{\alpha=1}^{v-1}u_\alpha^c}^{\sum_{\alpha=1}^{v}u_\alpha^c}\mathsf{f}_{r_p}\Big)\otimes\mathsf{f}_{u_v}\Bigg)_{v=1}^{w};m_w\Bigg)$$
$$=(-1)^{\sum_{1\leqslant v<y\leqslant w}(1-\|u_v\|)(u_y^c-\sum_{p=1+\sum_{\alpha=1}^{v-1}u_\alpha^c}^{\sum_{\alpha=1}^{v}u_\alpha^c}\|r_p\|)}\big(\otimes_{p=1}^{u^c}\mathsf{f}_{r_p}\big)\otimes\rho((\mathsf{f}_{u_v})_{v=1}^{w};m_w).$$

Let us verify that (4.7) equals (4.13). In fact, due to [Lyu15, (2.13)] all summands of two sums are pairwise equal. It remains to check that signs are equal as well:

$$\operatorname{sg}(\tau)+\sum_{1\leqslant b<a\leqslant w}^{1\leqslant e<d\leqslant n}u_a^eu_b^d+\sum_{q=1}^{w}(q-1)(\|u_q\|-1)\equiv\sum_{1\leqslant b<a\leqslant w}^{1\leqslant e<d\leqslant n+q-1}j_a^ej_b^d+\sum_{p=1}^{w}(p-1)(\|j_p\|-1)$$
$$+\sum_{v=1}^{w}\operatorname{sg}(\tau_v)+\sum_{1\leqslant v<y\leqslant w}(1-\|u_v\|)(u_y^c-\sum_{p=1+\sum_{\alpha=1}^{v-1}u_\alpha^c}^{\sum_{\alpha=1}^{v}u_\alpha^c}\|r_p\|)\pmod 2.$$

Here τ_v, $1 \leqslant v \leqslant w$, is the t-tree

$\tau_v =$



However this identity is precisely the claim of Lemma B.2 for our tree t with two internal vertices. It can be checked also directly as an exercise.

Let us show that sum (4.8) equals sum (4.11). In fact, due to [Lyu15, (2.14)] expression (4.11) can be transformed using

$$\lambda^g\big({}^b1, m_x, {}^l1; (\otimes_{p=1}^{u^c}\mathsf{f}_{s_p}) \otimes \mathsf{f}_u\big) \\ = (-1)^{-x(h-1-\sum_{p=1}^{h-1}\|s_p\|)}(\otimes_{p=1}^{h-1}\mathsf{f}_{s_p}) \otimes \lambda^g\big({}^a1, m_x, {}^z1; \mathsf{f}_{s_h}\big) \otimes (\otimes_{p=h+1}^{u^c}\mathsf{f}_{s_p}) \otimes \mathsf{f}_u,$$

where $h \in \mathbb{N}$ is found from the inequalities

$$1 \leqslant h \leqslant u^c, \quad a \overset{\text{def}}{=} b - \sum_{p=1}^{h-1} s_p^g \geqslant 0, \quad z \overset{\text{def}}{=} \sum_{p=1}^{h} s_p^g - b - 1 \geqslant 0.$$

We identify $s_p = r_p$ for $p \neq h$ and $s_h = r_h - (x-1)e_g$. Thus terms of the two sums are pairwise equal and it remains to check that their signs are pairwise equal as well:

$$\operatorname{sg}(\tau) + 1 - \|u\| + u^c - h - \sum_{v=h+1}^{u^c}\|r_v\| + (1-x)(a + \sum_{y=1}^{g-1} r_h^y) - \|r_h\| \\ \equiv (1-x)(b + \sum_{v=1}^{c-1} u^v + \sum_{y=1}^{g-1} i^y) - \|j\| + \operatorname{sg}(\tau \mid r_h^g \mapsto r_h^g - x + 1) - x(h - 1 - \sum_{p=1}^{h-1}\|s_p\|) \pmod 2.$$

This equation can be transformed into equivalent one:

$$\operatorname{sg}(\tau) \equiv \operatorname{sg}(\tau \mid r_h^g \mapsto r_h^g - x + 1) \\ + (1-x)(-\sum_{y=1}^{g-1} r_h^y + \sum_{p=1}^{h-1} r_p^g + \sum_{v=1}^{c-1} u^v + \sum_{y=1}^{g-1} i^y + h - 1 - \sum_{p=1}^{h-1}\|r_p\|) \pmod 2. \quad (4.14)$$

In order to prove it we assign the following values to variables j_q, $1 \leqslant q \leqslant u^c$:

$$j_q = \begin{cases} 0, & \text{if } 1 + \sum_{p=1}^{h-1} r_p^g \leqslant q \leqslant x - 1 + \sum_{p=1}^{h-1} r_p^g, \\ 1, & \text{otherwise.} \end{cases}$$

Apply Lemma B.3 to our t-tree τ and notice that τ' is obtained from τ replacing r_h^g with $r_h^g - x + 1$. The identity proven in Lemma B.3 is precisely identity (4.14) that we had to establish.

The subsum of (4.6) containing terms with $h < c$ is equal to sum (4.10). In fact, terms of these sums are pairwise identified. It remains to show that the signs are equal:

$$\begin{aligned}&\mathrm{sg}(\tau) + (1-w)(a + \sum_{l=1}^{h-1} u^l) - \|u\| \\ &\equiv (1-w)(a + \sum_{v=1}^{h-1} u^v) - \|j\| + \mathrm{sg}(\tau \mid u^h \mapsto u^h - w + 1) - w(u^c - \sum_{p=1}^{u^c} \|r_p\|) \pmod 2.\end{aligned}$$

Setting

$$j_q = \begin{cases} 0, & \text{if } 1 \leqslant q \leqslant w-1, \\ 1, & \text{if } w \leqslant q \leqslant u^h, \end{cases} \tag{4.15}$$

we see that the above identity is precisely the statement of Lemma B.3 applied to our t-tree τ and described $(j_q)_q$.

Similarly, the subsum of (4.6) containing terms with $h > c$ is equal to sum (4.12). Again we have to show that signs coincide:

$$\begin{aligned}&\mathrm{sg}(\tau) + (1-w)(u^1 + \dots + u^{h-1} + a) - \|u\| \\ &\equiv (1-w)(a+ \sum_{v=1}^{h-1, v\neq c} u^v + \sum_{g=1}^{q} i^g) - \|j\| + \mathrm{sg}(\tau \mid u^h \mapsto u^h - w + 1) - w(u^c - \sum_{p=1}^{u^c} \|r_p\|) \pmod 2,\end{aligned}$$

which actually simplifies to

$$\mathrm{sg}(\tau) \equiv \mathrm{sg}(\tau \mid u^h \mapsto u^h - w + 1) \pmod 2.$$

Using j_q from (4.15) we deduce the required identity from Lemma B.3.

It remains to prove that subsum of (4.6) containing terms with $h = c$ plus sum (4.9) gives 0. Consider the element

$$\begin{aligned}&x = (\otimes_{p=1}^{u^c} \mathsf{f}_{r_p}) \otimes (1^{\otimes a} \otimes m_w \otimes 1^{\otimes z}) \otimes \mathsf{f}_{u-(w-1)e_c} \in \\ &\big(\otimes_{p=1}^{u^c} \mathrm{F}_q(r_q)\big) \otimes (\mathrm{A}_\infty(1)^{\otimes a} \otimes \mathrm{A}_\infty(w) \otimes \mathrm{A}_\infty(1)^{\otimes z}) \otimes \mathrm{F}_n(u-(w-1)e_c) \subset \circledast_{\mathsf{G}}(t^*)(\mathrm{A}_\infty, \mathrm{F}_{|v|})_{v \in \mathrm{v}(t)},\end{aligned}$$

where $a + w + z = u^c$ and t^* is obtained from t by adding one unary vertex on the internal edge. By definition, elements

$$x.(1 \otimes \lambda^c) = (\otimes_{p=1}^{u^c} \mathsf{f}_{r_p}) \otimes \lambda^c({}^a 1, m_w, {}^z 1; \mathsf{f}_{u-(w-1)e_h}) \quad \text{and}$$

$$x.(\rho \otimes 1) = (-1)^{w \sum_{p=a+w+1}^{u^c} (\|r_p\| - 1)} \Big(\bigotimes_{p=1}^{a} \mathsf{f}_{r_p}\Big) \otimes \rho((\mathsf{f}_{r_p})_{p=a+1}^{a+w}; m_w) \otimes \Big(\bigotimes_{p=a+w+1}^{u^c} \mathsf{f}_{r_p}\Big) \otimes \mathsf{f}_{u-(w-1)e_h}$$

of $\circledast_{\mathsf{G}}(t)(\mathrm{F}_{|v|})_{v\in\mathrm{v}(t)}$ are identified in $\circledast_{\mathsf{M}}(t)(\mathrm{F}_{|v|})_{v\in\mathrm{v}(t)}$. Consequently, terms of (4.6) with $h=c$ and terms of sum (4.9) are pairwise equal in $\circledast_{\mathsf{M}}(t)(\mathrm{F}_{|v|})_{v\in\mathrm{v}(t)}$ provided we identify: $u^v=k^v$ for $v\in\mathbf{n}-\{c\}$, $u^c=k^c+w-1$, $s_p=r_p$ for $1\leqslant p\leqslant a$, $l_\nu=r_{\nu+a}$ for $1\leqslant\nu\leqslant w$, $s_p=r_{p+w-1}$ for $a+2\leqslant p\leqslant k^c$, and $s_{a+1}=\sum_{\nu=1}^w l_w=\sum_{\nu=1}^w r_{a+\nu}$. We have to check that the terms occur with opposite signs:

$$\mathrm{sg}(\tau)+(1-w)(a+\sum_{v=1}^{c-1}u^v)+1-\|u\|+w\sum_{p=a+w+1}^{u^c}(\|r_p\|-1)+1$$
$$\equiv \mathrm{sg}(\tilde\tau)+1-\|k\|+\sum_{v=a+2}^{k^c}(1-\|s_v\|)+w+\sum_{1\leqslant b<g\leqslant w}^{1\leqslant e<d\leqslant q}l_g^e l_b^d+\sum_{\nu=1}^{w}(\nu-1)(\|l_\nu\|-1) \pmod 2.$$

Plug in explicit value of $\mathrm{sg}(\tau)$ from (4.5):

$$\sum_{1\leqslant z<p\leqslant u^c}^{1\leqslant e<g\leqslant q}r_p^e r_z^g+\sum_{p=1}^{u^c}(p-1)(\|r_p\|-1)+(\|i\|-u^c)\sum_{v=1}^{c-1}u^v+(1-w)(a+\sum_{v=1}^{c-1}u^v)$$
$$+(w-1)\sum_{p=a+w+1}^{u^c}(\|r_p\|-1)\equiv\sum_{1\leqslant z<p\leqslant u^c}^{1\leqslant e<g\leqslant q}s_p^e s_z^g+\sum_{p=1}^{k^c}(p-1)(\|s_p\|-1)$$
$$+(\|i\|-k^c)\sum_{v=1}^{c-1}k^v+\sum_{1\leqslant b<g\leqslant w}^{1\leqslant e<d\leqslant q}l_g^e l_b^d+\sum_{\nu=1}^{w}(\nu-1)(\|l_\nu\|-1) \pmod 2.$$

Terms quadratic in r, s and l cancel each other. More cancellations occur leading to the obvious identity:

$$\sum_{p=a+1}^{a+w}(p-1)(\|r_p\|-1)+(1-w)a\equiv a\big(\sum_{\nu=1}^{w}\|r_{a+\nu}\|-1\big)+\sum_{\nu=1}^{w}(\nu-1)(\|r_{a+\nu}\|-1) \pmod 2.$$

Thus, $\mathsf{f}_j.\Delta^{\mathsf{M}}(t)\partial=\mathsf{f}_j.\partial\Delta^{\mathsf{M}}(t)$. ■

4.5. Comultiplication for homotopy unital case. Let multiquiver $\mathsf{a}_\infty^{\mathsf{hu}}=\mathsf{H}$ be convolution of $\mathrm{F}^{\mathsf{hu}}:\mathsf{F}\to\mathsf{M}$ and $\mathcal{H}om:\mathsf{B}\to\mathsf{M}$ coming from $\underline{\mathsf{C}}_{\Bbbk}$. Objects of $\mathsf{a}_\infty^{\mathsf{hu}}$ are homotopy unital A_∞-algebras and morphisms are homotopy unital A_∞-morphisms.

There is a multiquiver map $\text{-}^+:\mathsf{a}_\infty^{\mathsf{hu}}\to\mathsf{a}_\infty$, $(A,\mathsf{i},m_1,m_{n_1;n_2;\dots;n_k}\mid k+\sum_{q=1}^k n_q\geqslant3)\to(A^+,m_n^+\mid n\geqslant1)$, where $A^+=A\oplus\Bbbk1^{\mathsf{su}}\oplus\Bbbk\mathsf{j}$ is strictly unital with the strict unit 1^{su}, $m_n^+\big|_{A^{\otimes n}}=m_n$, $\mathsf{j}m_1^+=1^{\mathsf{su}}-\mathsf{i}$ and

$$(1^{\otimes n_1}\otimes\mathsf{j}\otimes1^{\otimes n_2}\otimes\mathsf{j}\otimes\dots\otimes1^{\otimes n_{k-1}}\otimes\mathsf{j}\otimes1^{\otimes n_k})m_{n+k-1}^+=m_{n_1;n_2;\dots;n_k}:A^{\otimes n+k-1}\to A$$

for $k\geqslant1$, $n_q\geqslant0$, $n=\sum_{q=1}^k n_q$, $n+k\geqslant3$. On morphisms with n arguments we have

$$\mathsf{f}=(\mathsf{v}_k,\mathsf{f}_{(\ell_1^k;\ell_2^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}})\mapsto\mathsf{f}^+=(\mathsf{f}_j^+\mid j\in\mathbb{N}^n-0),$$

where $\mathsf{j}\mathsf{f}^+_{e_k} = \mathsf{v}_k + \mathsf{j}\rho_\varnothing$ and for $\|\hat{\ell}\| \geqslant 2$

$$\bigl[\otimes^{k\in\mathbf{n}} T^{\ell^k} A_k \xrightarrow{\otimes^{k\in\mathbf{n}}(1^{\otimes \ell^k_1}\otimes\mathsf{j}\otimes 1^{\otimes\ell^k_2}\otimes\mathsf{j}\otimes\cdots\otimes 1^{\otimes \ell^k_{t^k-1}}\otimes\mathsf{j}\otimes 1^{\otimes\ell^k_{t^k}})} \otimes^{k\in\mathbf{n}} T^{\hat{\ell}^k} A_k^+ \xrightarrow{\mathsf{f}^+_{\hat{\ell}}} B\bigr]$$
$$= \lambda_{\hat{\ell}}\bigl(({}^{\ell^k_1}1, \mathsf{j}, {}^{\ell^k_2}1, \mathsf{j}, \dots, {}^{\ell^k_{t^k-1}}1, \mathsf{j}, {}^{\ell^k_{t^k}}1)_{k\in\mathbf{n}}; \mathsf{f}^+_{\hat{\ell}}\bigr) = \mathsf{f}_{(\ell^k_1;\ell^k_2;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}}.$$

This multiquiver map is injective on morphisms and the conditions of Definition 3.20 describe its image. The image is closed under composition in a_∞, hence, it is a submulticategory. In this way $\mathsf{a}^{\mathsf{hu}}_\infty$ becomes a multicategory and $\text{-}^+ : \mathsf{a}^{\mathsf{hu}}_\infty \to \mathsf{a}_\infty$ becomes a multifunctor. Composing it with the multifunctor $Ts : \mathsf{a}_\infty \to \mathbf{dgac}$ we get again a full and faithful embedding $Ts(\text{-})^+ : \mathsf{a}^{\mathsf{hu}}_\infty \to \mathbf{dgac}$. Its image is described by conditions parallel to that of Definition 3.20:

(1) f^+ is a strictly unital;

(2) $\widehat{\mathsf{f}^+}(1\otimes\cdots\otimes 1\otimes(A_k + \mathsf{j}^{A_k})\otimes 1\otimes\cdots\otimes 1) \subset B + \mathsf{j}^B$;

(3) $\widehat{\mathsf{f}^+}(\otimes^{k\in\mathbf{n}} TA_k) \subset TB$;

(4) $\widehat{\mathsf{f}^+}(\otimes^{k\in\mathbf{n}} T^{\ell^k}(A_k \oplus \Bbbk\mathsf{j}^{A_k})) \subset B \oplus T^{>1}(B\oplus\Bbbk\mathsf{j}^B)$ for each $\ell\in\mathbb{N}^n$, $\|\ell\| > 1$.

One checks directly that the set of such coalgebra morphisms is closed under composition.

4.5.1. Comultiplication for unital case. Comultiplication (4.2) extends in a unique way to $(\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}}_n)$, which differs from $(\mathrm{A}_\infty, \mathrm{F}_n)$ by a direct summand $(\Bbbk 1^{\mathsf{su}}, \Bbbk 1^{\mathsf{su}}\rho_\varnothing)$, see (3.22). In fact, for a tree t the equation

$$\rho_\varnothing = \bigl[\mathrm{A}^{\mathsf{su}}_\infty(0) \xrightarrow[\cong]{\rho_\varnothing} \mathrm{F}^{\mathsf{su}}_{\operatorname{Inp} t}(0) \xrightarrow{\Delta(t)} \circledast_{\mathsf{M}}(t)(\mathrm{F}^{\mathsf{su}}_{|v|})_{v\in\mathrm{v}(t)}(0)\bigr] \tag{4.16}$$

is one of those saying that $\Delta(t)$ agree with ρ (see (3.2) with $l = 0$). So we set $\Delta(t)\bigl(\rho_\varnothing(1^{\mathsf{su}})\bigr) = \rho_\varnothing(1^{\mathsf{su}})$. For non-empty $\mathrm{v}(t)$ the image of (4.16) is contained in the image of the summand $\mathrm{F}^{\mathsf{su}}_{|\,\mathrm{rv}\,|}(0)$ of $\circledast_{\mathsf{G}}(t)(\mathrm{F}^{\mathsf{su}}_{|v|})_{v\in\mathrm{v}(t)}(0)$ indexed by the t-tree τ_0 with $\tau_0(v) = \varnothing$ for all v except the root vertex, while $\tau_0(\mathrm{rv}) = \mathbf{1}$. For the tree $t = |$ (4.16) is the right action in the regular bimodule $\mathrm{A}^{\mathsf{su}}_\infty$:

$$\mathrm{id} = \rho_\varnothing = \bigl[\mathrm{A}^{\mathsf{su}}_\infty(0) \xrightarrow[\sim]{\rho_\varnothing} \mathrm{F}^{\mathsf{su}}_1(0) \xrightarrow{\Delta(|)} \mathrm{A}^{\mathsf{su}}_\infty(0)\bigr].$$

We can be more precise in this case: $\Delta(|)\bigl(\rho_\varnothing(1^{\mathsf{su}})\bigr) = \rho_\varnothing(1^{\mathsf{su}}) = 1^{\mathsf{su}}$.

So extended comultiplication obviously agrees with the left action λ (see (3.4) with $k = 0$). It agrees also with the right action ρ, see (3.2) for $l > 0$ with $J = \{q\in\mathbf{l} \mid k_q = 0\}$. We may take elements $1^{\mathsf{su}}\rho_\varnothing$ in each place $\mathcal{P}(0) = \mathrm{F}^{\mathsf{su}}_n(0)$ for $q\in J$. Then $1^{\mathsf{su}}\rho_\varnothing$ will appear also in $\mathcal{Q}(0) = \circledast_{\mathsf{M}}(t)(\mathrm{F}^{\mathsf{su}}_{|v|})_{v\in\mathrm{v}(t)}(0)$ for the same q. Using associativity of ρ we can absorb those 1^{su} into an element of $\mathrm{A}^{\mathsf{su}}_\infty$ and get rid of 1^{su}'s completely. The equation is reduced to the case of $(\mathrm{A}_\infty, \mathrm{F}_n)$, which is already verified. Coassociativity of extended comultiplication is obvious.

Let us extend comultiplication further to

$$(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle \mathsf{i}, \mathsf{j}\rangle \simeq \big(\mathrm{A}_\infty^{\mathsf{su}}\langle \mathsf{i}, \mathsf{j}\rangle, \bigcirc_{k=0}^{n} \mathrm{A}_\infty^{\mathsf{su}}\langle \mathsf{i}, \mathsf{j}\rangle \odot^{k}_{\mathrm{A}_\infty^{\mathsf{su}}} \mathrm{F}_n^{\mathsf{su}}\big)$$

using Proposition A.1. Let $n = |\operatorname{Inp} t|$. The comultiplication is the lower diagonal in

$$\begin{array}{ccc}
\circledast_{\mathsf{M}}(t)(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_{|v|}^{\mathsf{su}})_{v\in \mathrm{v}(t)} & \hookrightarrow & \circledast_{\mathsf{M}}(t)(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_{|v|}^{\mathsf{su}})_{v\in \mathrm{v}(t)}\langle \mathsf{i}, \mathsf{j}\rangle \\
\uparrow{\scriptstyle \Delta^{\mathsf{M}}(t)} & & \downarrow \wr \\
(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}}) & & \big(\mathrm{A}_\infty^{\mathsf{su}}\langle \mathsf{i}, \mathsf{j}\rangle, \bigcirc_{k=0}^{n} \mathrm{A}_\infty^{\mathsf{su}}\langle \mathsf{i}, \mathsf{j}\rangle \odot^{k}_{\mathrm{A}_\infty^{\mathsf{su}}} \circledast_{\mathsf{M}}(t)(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_{|v|}^{\mathsf{su}})_{v\in \mathrm{v}(t)}\big) \\
\downarrow & \nearrow{\scriptstyle \Delta^{\mathsf{M}}(t)\langle \mathsf{i}, \mathsf{j}\rangle} & \downarrow \\
(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle \mathsf{i}, \mathsf{j}\rangle & & \circledast_{\mathsf{M}}(t)(\mathrm{A}_\infty^{\mathsf{su}}\langle \mathsf{i}, \mathsf{j}\rangle, \bigcirc_{k\in \mathrm{in}(v)\cup \mathrm{ou}(v)} \mathrm{A}_\infty^{\mathsf{su}}\langle \mathsf{i}, \mathsf{j}\rangle \odot^{k}_{\mathrm{A}_\infty^{\mathsf{su}}} \mathrm{F}_{|v|}^{\mathsf{su}})_{v\in \mathrm{v}(t)} \\
 & \searrow{\scriptstyle \Delta^{\mathsf{M}}(t)} & \downarrow \wr \\
 & & \circledast_{\mathsf{M}}(t)(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_{|v|}^{\mathsf{su}})\langle \mathsf{i}, \mathsf{j}\rangle_{v\in \mathrm{v}(t)}
\end{array}$$

Proof of coassociativity is contained in diagram (4.17). The operad module $(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})$ is short-handed to $\mathrm{F}_n^{\mathsf{su}}$. Similarly $\mathrm{F}_n^{\mathsf{su}}\langle \mathsf{i}, \mathsf{j}\rangle$ stands for $(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle \mathsf{i}, \mathsf{j}\rangle$. Being diagram (4.2) of [Lyu15] the top square commutes. The middle square parallel to the top face is obtained by adding freely operations i and j. Hence it also commutes. The vertical faces commute as well, therefore, the bottom quadrangle is commutative.

Thus, a **dg**-polymodule cooperad $(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}^{\mathsf{su}})\langle \mathsf{i}, \mathsf{j}\rangle$ is constructed. Quite similarly, there is a **dg**-polymodule cooperad $(A_\infty^{\mathsf{su}}, F^{\mathsf{su}})\langle \mathbf{i}, \mathbf{j}\rangle$ isomorphic to it via a degree 1 isomorphism.

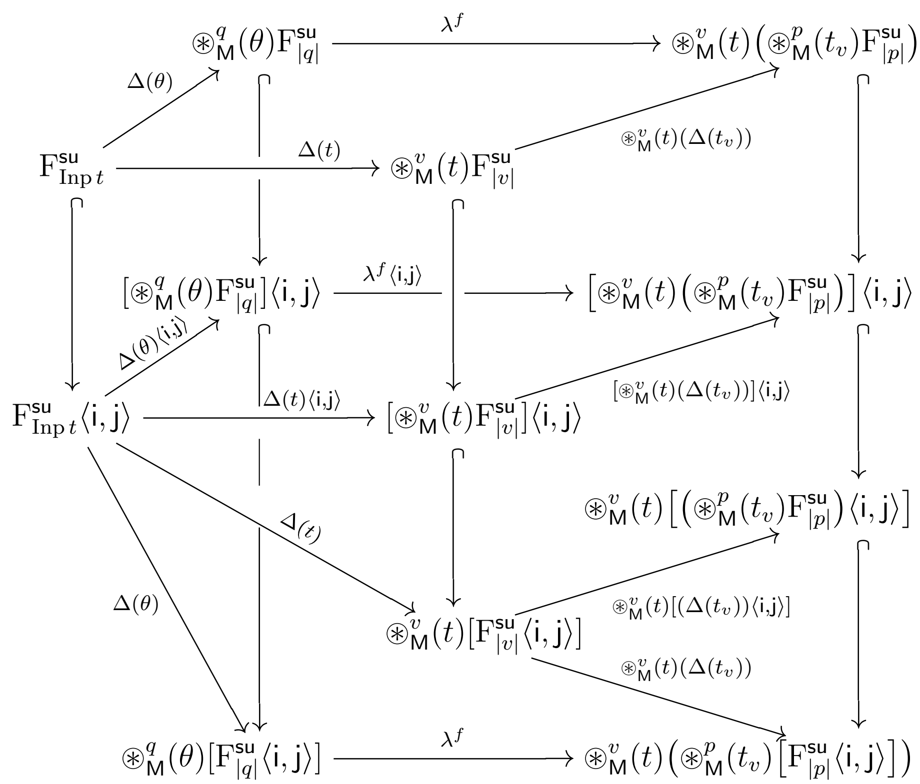
4.6. Proposition. *The collections of operad submodules* $(A_\infty^{\mathsf{hu}}, F^{\mathsf{hu}}) \subset (A_\infty^{\mathsf{su}}, F^{\mathsf{su}})\langle \mathbf{i}, \mathbf{j}\rangle$, $(\mathrm{A}_\infty^{\mathsf{hu}}, \mathrm{F}^{\mathsf{hu}}) \subset (\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}^{\mathsf{su}})\langle \mathsf{i}, \mathsf{j}\rangle$ *are* **dg**-*polymodule subcooperads.*

Proof. Assume that $k \in \operatorname{Inp} t$ for a tree t. Let us compute $\Delta^{\mathsf{M}}(t)(f_{e_k})$. Notice that there exists the only t-tree τ such that $\tilde{\tau}$ is surjective and $|\tau(a)| = \delta_k^a$ for all $a \in \operatorname{Inpv} t$. In fact, let $\operatorname{tail}(k) = v_0 \xrightarrow{k} v_1 \to v_2 \to \dots \to v_m = \operatorname{rv}(t)$ be the oriented path from the tail v_0 of chosen $k \in \operatorname{Inp} t$ to the root vertex. The tree τ is given by the formula

$$\tau(v) = \begin{cases} \mathbf{1}, & \text{if } v = v_j \text{ for some } j,\ 0 \leqslant j \leqslant m, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Denoting $e_p^S \in \mathbb{N}^S$ a basis vector for $p \in S$ we find

$$\Delta^{\mathsf{M}}(t)(f_{e_k^{\operatorname{Inp} t}}) = f_{e_{v_0}^{\operatorname{inV}(v_1)}} \otimes f_{e_{v_1}^{\operatorname{inV}(v_2)}} \otimes f_{e_{v_2}^{\operatorname{inV}(v_3)}} \otimes \dots \otimes f_{e_{v_{m-2}}^{\operatorname{inV}(v_{m-1})}} \otimes f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}}$$



(4.17)

(factors 1 assigned to vertices outside the path are omitted). Now we compute

$$
\begin{aligned}
\Delta^{\mathsf{M}}(t)(\mathbf{v}_k^{\operatorname{Inp} t}) &= \Delta^{\mathsf{M}}(t)(\lambda^k_{e_k}(\mathbf{j}; f_{e_k^{\operatorname{Inp} t}}) - \mathbf{j}\rho_\varnothing) = \lambda^k_{e_k}(\mathbf{j}; \Delta^{\mathsf{M}}(t)(f_{e_k^{\operatorname{Inp} t}})) - \mathbf{j}\rho_\varnothing \\
&= \mathbf{j} f_{e_{v_0}^{\operatorname{inV}(v_1)}} \otimes f_{e_{v_1}^{\operatorname{inV}(v_2)}} \otimes f_{e_{v_2}^{\operatorname{inV}(v_3)}} \otimes \dots \otimes f_{e_{v_{m-2}}^{\operatorname{inV}(v_{m-1})}} \otimes f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}} \\
&\quad - \mathbf{j}\rho_\varnothing \otimes f_{e_{v_1}^{\operatorname{inV}(v_2)}} \otimes f_{e_{v_2}^{\operatorname{inV}(v_3)}} \otimes \dots \otimes f_{e_{v_{m-2}}^{\operatorname{inV}(v_{m-1})}} \otimes f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}} \\
&\quad + \mathbf{j} f_{e_{v_1}^{\operatorname{inV}(v_2)}} \otimes f_{e_{v_2}^{\operatorname{inV}(v_3)}} \otimes \dots \otimes f_{e_{v_{m-2}}^{\operatorname{inV}(v_{m-1})}} \otimes f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}} \\
&\quad - \mathbf{j}\rho_\varnothing \otimes f_{e_{v_2}^{\operatorname{inV}(v_3)}} \otimes \dots \otimes f_{e_{v_{m-2}}^{\operatorname{inV}(v_{m-1})}} \otimes f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}} \\
&\quad + \mathbf{j} f_{e_{v_2}^{\operatorname{inV}(v_3)}} \otimes \dots \otimes f_{e_{v_{m-2}}^{\operatorname{inV}(v_{m-1})}} \otimes f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}} \\
&\quad \dots \\
&\quad - \mathbf{j}\rho_\varnothing \otimes f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}} \\
&\quad + \mathbf{j} f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}} \\
&\quad - \mathbf{j}\rho_\varnothing \\
&= \mathbf{v}_{v_0}^{\operatorname{inV}(v_1)} \otimes f_{e_{v_1}^{\operatorname{inV}(v_2)}} \otimes f_{e_{v_2}^{\operatorname{inV}(v_3)}} \otimes \dots \otimes f_{e_{v_{m-2}}^{\operatorname{inV}(v_{m-1})}} \otimes f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}} \\
&\quad + \mathbf{v}_{v_1}^{\operatorname{inV}(v_2)} \otimes f_{e_{v_2}^{\operatorname{inV}(v_3)}} \otimes \dots \otimes f_{e_{v_{m-2}}^{\operatorname{inV}(v_{m-1})}} \otimes f_{e_{v_{m-1}}^{\operatorname{inV}(v_m)}}
\end{aligned}
$$

$$+\mathbf{v}_{v_2}^{\mathrm{inV}(v_3)} \otimes \cdots \otimes f_{e_{v_{m-2}}^{\mathrm{inV}(v_{m-1})}} \otimes f_{e_{v_{m-1}}^{\mathrm{inV}(v_m)}}$$
$$\dots$$
$$+\mathbf{v}_{v_{m-2}}^{\mathrm{inV}(v_{m-1})} \otimes f_{e_{v_{m-1}}^{\mathrm{inV}(v_m)}}$$
$$+\mathbf{v}_{v_{m-1}}^{\mathrm{inV}(v_m)}.$$

On other generators we transform

$$\varDelta^{\mathsf{M}}(t)(f_{(\ell_1^k;\ell_2^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}}) = \lambda_{\hat{\ell}}\big(({}^{\ell_1^k}1,\mathbf{j},{}^{\ell_2^k}1,\mathbf{j},\dots,{}^{\ell_{t^k-1}^k}1,\mathbf{j},{}^{\ell_{t^k}^k}1)_{k\in\mathbf{n}};\varDelta^{\mathsf{M}}(t)(f_{\hat{\ell}}^+)\big)$$

as follows. First factors f_{i_q} are replaced with generators $f_{a_1;a_2;\dots;a_p}$ accordingly with the set of $\mathbf{j}$'s appearing among the arguments of f_{i_q}. The only exception is the case of $\mathbf{j}f_{e_k}$ which is replaced with $\mathbf{v}_k+\mathbf{j}\rho_\varnothing$. In obtained summands all instances of $\mathbf{j}\rho_\varnothing$ are moved to the right as arguments $\mathbf{j}$ of f_p indexed by a τ-parent vertex to the considered one due to defining the tensor product as a colimit, and this procedure goes on until no $\mathbf{j}\rho_\varnothing$ are left. Notice that the separate term $\mathbf{j}\rho_\varnothing$ can not appear elsewhere but in the expression $\varDelta^{\mathsf{M}}(t)(\mathbf{j}f_{e_k^{\mathrm{Inp}\,t}})$, which is not considered by itself, but only as a summand of $\mathbf{v}_k^{\mathrm{Inp}\,t}$.

The case of $(\mathrm{A}_\infty^{\mathsf{hu}},\mathrm{F}^{\mathsf{hu}},\Delta^{\mathsf{M}})$ is completely analogous, although with additional signs. ■

A. Induced operad modules

A.1. Proposition. *Let $\mathcal{V}=\mathbf{dg}$ and $A=(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{A}_0)$ be an $n\wedge 1$-operad module. Let $\beta_i:\mathcal{M}_i\to\mathcal{A}_i\in\mathbf{dg}^{\mathbb{N}}$ be chain maps for $i\in[n]$. Denote $M=(\mathcal{M}_1,\dots,\mathcal{M}_n;0;\mathcal{M}_0)$ and $\beta=(\beta_1,\dots,\beta_n;0;\beta_0):(\mathcal{M}_1,\dots,\mathcal{M}_n;0;\mathcal{M}_0)\to(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{A}_0)$. Then $A\langle M,\beta\rangle$ defined in diagram (1.1) is isomorphic to $((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle)_{i\in[n]};\bigodot_{i=0}^n\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle\odot^i_{\mathcal{A}_i}\mathcal{P})$.*

Proof. Denote $\mathcal{R}=\bigodot_{i=0}^n\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle\odot^i_{\mathcal{A}_i}\mathcal{P}$ and $\mathcal{C}_i=T(\mathcal{M}_i[1])$. As we know from [Lyu11, Section 1.10] in $\mathscr{G}={}_n\mathrm{Op}_1^{\mathbf{gr}}$ the graded module underlying $A\langle M,\beta\rangle$ is isomorphic to $((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))\sqcup((\mathcal{A}_i)_{i\in[n]};\mathcal{P})$. Clearly, this coproduct is also a colimit of the following diagram (a pushout)

$$\begin{array}{ccc} & ((\mathcal{A}_i)_{i\in[n]};\mathcal{A}_0(0)) \xrightarrow{((1);!)} & ((\mathcal{A}_i)_{i\in[n]};\mathcal{P}) \\ & \downarrow \mathrm{in}_2 & \downarrow \mathrm{in}_2 \\ ((\mathcal{C}_i\sqcup\mathcal{A}_i)_{i\in[n]};(\mathcal{C}_0\sqcup\mathcal{A}_0)(0)) = & ((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))\sqcup((\mathcal{A}_i)_{i\in[n]};\mathcal{A}_0(0)) \xrightarrow{1\sqcup((1);!)} & A\langle M,\beta\rangle \end{array}$$

Here the equation is due to [Lyu15, Corollary A.10]. Denote $\mathcal{B}_i=\mathcal{C}_i\sqcup\mathcal{A}_i\simeq\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle$ in $\mathrm{Op}^{\mathbf{gr}}$. Applying [Lyu15, Corollary A.7] to the canonical embeddings $\mathrm{in}_2:\mathcal{A}_i\to\mathcal{B}_i$ we deduce an isomorphism $\psi_3:A\langle M,\beta\rangle\to((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle)_{i\in[n]};\bigodot_{i=0}^n\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle\odot^i_{\mathcal{A}_i}\mathcal{P})$ in ${}_n\mathrm{Op}_1^{\mathbf{gr}}$.

In order to show that the isomorphism is actually in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$ we consider diagram on the following page, where $\phi_k=\psi_k^{-1}$, $1\leqslant k\leqslant 3$. Notice that the isomorphism ψ_2 is nothing

$$
\begin{array}{ccccc}
((T\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant 0}((T\mathcal{A}_i)_{i\in[n]};\mathcal{P})) & \xrightarrow{((\alpha_i);\odot_{\geqslant 0}((\alpha_i);1))} & ((\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P})) & \xrightarrow{((1)_{i\in[n]};\alpha)} & ((\mathcal{A}_i)_{i\in[n]};\mathcal{P}) \\
\Big\downarrow{\scriptstyle ((T\bar{\imath}_i);\odot_{\geqslant 0}((T\bar{\imath}_i);1))} & \boxed{1} & \Big\downarrow{\scriptstyle ((\bar{\jmath}_i)_{i\in[n]};\odot_{\geqslant 0}((\bar{\jmath}_i)_{i\in[n]};1))} & \boxed{2} & \Big\downarrow{\scriptstyle ((\bar{\jmath}_i)_{i\in[n]};\eta)} \\
((TC_i)_{i\in[n]};\odot_{\geqslant 0}((TC_i)_{i\in[n]};\mathcal{P})) & \xrightarrow{((g_i);\odot_{\geqslant 0}((g_i);1))} & ((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle);\odot_{\geqslant 0}((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle);\mathcal{P})) & \xrightarrow{((1)_{i\in[n]};z)} & ((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle);\mathcal{R}) \\
{\scriptstyle\phi_1}\Big\downarrow\wr\Big\uparrow{\scriptstyle\psi_1} & \boxed{3} & {\scriptstyle\phi_2}\Big\downarrow\wr\Big\uparrow{\scriptstyle\psi_2} & \boxed{4} & {\scriptstyle\phi_3}\Big\downarrow\wr\Big\uparrow{\scriptstyle\psi_3} \\
\begin{array}{l}((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))\sqcup \\ ((T\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant 0}((T\mathcal{A}_i)_{i\in[n]};\mathcal{P}))\end{array} & \xrightarrow{1\sqcup((\alpha_i);\odot_{\geqslant 0}((\alpha_i);1))} & \begin{array}{l}((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))\sqcup \\ ((\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P}))\end{array} & \xrightarrow{1\sqcup((1);\alpha)} & \begin{array}{l}((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0)) \\ \sqcup((\mathcal{A}_i)_{i\in[n]};\mathcal{P})\end{array}
\end{array}
$$

else but the isomorphism ψ_3, written for the operad module $((\mathcal{A}_i)_{i\in[n]};\mathcal{P})$ instead of $\mathcal{P}$, taking into account that

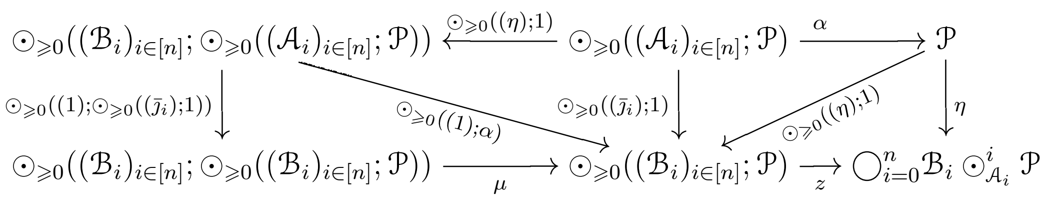
$$\bigodot_{i=0}^{n}\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle\odot^i_{\mathcal{A}_i}\odot_{\geqslant0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P}))\simeq\odot_{\geqslant0}((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle)_{i\in[n]};\mathcal{P}).$$

The isomorphism ψ_1 follows from the fact that $\bar{F}:{}_n\mathrm{Op}_1^{\mathbf{gr}}\to\mathbf{gr}^{n\mathbb{N}\sqcup\mathbb{N}^n\sqcup\mathbb{N}}$ preserves colimits. The map $z:\odot_{\geqslant0}((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle)_{i\in[n]};\mathcal{P})\to\bigodot_{i=0}^{n}\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle\odot^i_{\mathcal{A}_i}\mathcal{P}$ is the canonical projection.

We claim that the diagram commutes. Its two top squares are in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$, while the bottom vertical isomorphisms are constructed only in $\mathscr{G}={}_n\mathrm{Op}_1^{\mathbf{gr}}$. Thus, squares $\boxed{3}$ and $\boxed{4}$ make sense in $\mathscr{G}$. First of all, square $\boxed{1}$ commutes due to definition (1.1) of $\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle$. Commutativity of square $\boxed{2}$ follows from the equation

$$\begin{array}{ccc}\odot_{\geqslant0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P}) & \xrightarrow{\quad\alpha\quad} & \mathcal{P}\\ {\scriptstyle\odot_{\geqslant0}((\jmath_i)_{i\in[n]};1)}\downarrow & = & \downarrow{\scriptstyle\eta}\\ \odot_{\geqslant0}((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle)_{i\in[n]};\mathcal{P}) & \xrightarrow{z} & \bigodot_{i=0}^{n}\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle\odot^i_{\mathcal{A}_i}\mathcal{P}\end{array}\tag{A.1}$$

proven directly from the definition of the induced operad module. Namely paths in the following diagram



satisfy the relations

$$\begin{aligned}\alpha\cdot\eta=\alpha\cdot\odot_{\geqslant0}((\eta);1)\cdot z&=\odot_{\geqslant0}((\eta);1)\cdot\odot_{\geqslant0}((1);\alpha)\cdot z\\&=\odot_{\geqslant0}((\eta);1)\cdot\odot_{\geqslant0}((1);\odot_{\geqslant0}((\jmath_i);1))\cdot\mu\cdot z=\odot_{\geqslant0}((\jmath_i);1)\cdot z.\end{aligned}$$

Commutativity of squares $\boxed{3}$ and $\boxed{4}$ with the vertical arrows ψ_k is proven separately on each of summands of the source of the square. On $((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))$ commutativity holds due to [Lyu15, Lemma A.9]. On $((\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P}))$ verification reduces to diagram (A.1).

One easily finds out that all three columns of diagram on the preceding page compose to in_2:

$$((T\bar{\imath}_i);\odot_{\geqslant0}((T\bar{\imath}_i);1))\cdot\phi_1=\mathrm{in}_2,\ ((\bar{\jmath}_i)_{i\in[n]};\odot_{\geqslant0}((\bar{\jmath}_i)_{i\in[n]};1))\cdot\phi_2=\mathrm{in}_2,\ ((\bar{\jmath}_i)_{i\in[n]};\eta)\cdot\phi_3=\mathrm{in}_2\,.$$

Therefore, the exterior of this diagram drawn with isomorphisms ϕ_k is a pushout square. Hence, the pasting $\boxed{1\cup2}$ of squares $\boxed{1}$ and $\boxed{2}$ (a diagram in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$) is a pushout square in ${}_n\mathrm{Op}_1^{\mathbf{gr}}$. However, a cone for a diagram $\mathcal{D}\to{}_n\mathrm{Op}_1^{\mathbf{dg}}$ is a colimiting cone iff its image under the forgetful functor ${}_n\mathrm{Op}_1^{\mathbf{dg}}\to{}_n\mathrm{Op}_1^{\mathbf{gr}}$ is a colimiting cone for the composition $\mathcal{D}\to{}_n\mathrm{Op}_1^{\mathbf{dg}}\to{}_n\mathrm{Op}_1^{\mathbf{gr}}$. Thus, the pasting $\boxed{1\cup2}$ is a pushout square in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$, and the colimit $A\langle M,\beta\rangle$ is isomorphic to $((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle)_{i\in[n]};\bigodot_{i=0}^{n}\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle\odot^i_{\mathcal{A}_i}\mathcal{P})$ in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$. ∎

B. Isomorphisms of degree 1 of polymodule cooperads

Introduce a $\Bbbk$-linear involution

$$\Xi = \sum_{\forall a\in \operatorname{Inpv} t\,|\tau(a)|=j^a}^{t-\text{tree } \tau} (-1)^{\operatorname{sg}(\tau)} \operatorname{pr}_\tau \cdot \operatorname{in}_\tau : \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in \mathrm{v}(t)}(j) \to \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in \mathrm{v}(t)}(j).$$

B.1. Theorem. *Let $({}^n g; h_n; g) : ({}^n\mathcal{A}; G_n; \mathcal{A}) \to ({}^n\mathcal{B}; H_n; \mathcal{B})$, $n \geqslant 0$, be a family of invertible $n \wedge 1$-operad module homomorphisms of degree 1. If the family $(\mathcal{A}, G_\bullet, \Delta^{\mathsf{G}})$ defines a graded polymodule cooperad $\mathsf{F} \to (\mathsf{M}, \circledast_{\mathsf{G}})$, then $(\mathcal{B}, H_\bullet, \Delta^{\mathsf{G}})$ defines a graded polymodule cooperad such that the following diagram commutes*

$$\begin{array}{ccccc} G_{\operatorname{Inp} t} & \xrightarrow{\Delta^{\mathsf{G}}(t)} & \circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in \mathrm{v}(t)} & \xrightarrow{\Xi} & \circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in \mathrm{v}(t)} \\ {\scriptstyle h}\downarrow & & & & \downarrow{\scriptstyle \circledast_{\mathsf{G}}(t)(h)} \\ H_{\operatorname{Inp} t} & & \xrightarrow{\qquad\Delta^{\mathsf{G}}(t)\qquad} & & \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in \mathrm{v}(t)} \end{array}$$

In particular, Δ^{G} is multiplicative, see equation (4.2) of [Lyu15].

Proof. Given a colax $\mathcal{C}at$-multifunctor $(\mathcal{A}, G_\bullet, \Delta^{\mathsf{G}}) : \mathsf{F} \to (\mathsf{M}, \circledast_{\mathsf{G}})$ let us construct comultiplication Δ^{G} for $H_\bullet$. The map

$$\Delta^{\mathsf{G}}(t) : H_{\operatorname{Inp} t}(j) \to \coprod_{\forall a\in \operatorname{Inpv} t\,|\tau(a)|=j^a}^{t-\text{tree } \tau} \bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} H_{|v|}\Big(\big(|\tau(e)^{-1}(p)|\big)_{e\in \operatorname{in}(v)}\Big) \tag{B.1}$$

is determined by

$$\Delta^{\mathsf{G}}(t)(j) \cdot \operatorname{pr}_\tau = (-1)^{\operatorname{sg}(\tau)} \Big\langle H_{\operatorname{Inp} t}(j) \xrightarrow{h^{-1}} G_{\operatorname{Inp} t}(j) \xrightarrow{\Delta^{\mathsf{G}}(t)(j)\cdot \operatorname{pr}_\tau} \tag{B.2}$$
$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} G_{|v|}\Big(\big(|\tau(e)^{-1}(p)|\big)_{e\in \operatorname{in}(v)}\Big) \xrightarrow{\otimes^{v\in \mathrm{v}(t)}\otimes^{p\in\tau(v)} h} \bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} H_{|v|}\Big(\big(|\tau(e)^{-1}(p)|\big)_{e\in \operatorname{in}(v)}\Big)\Big\rangle.$$

These are maps of degree 0 since

$$\begin{aligned} \deg \otimes^{v\in \mathrm{v}(t)} \otimes^{p\in\tau(v)} h &= \sum_{v\in \mathrm{v}(t)} \sum_{p\in\tau(v)} \big(1 - \sum_{e\in \operatorname{in}(v)} |\tau(e)^{-1}(p)|\big) \\ &= \sum_{v\in \mathrm{v}(t)} |\tau(v)| - \sum_{v\in \mathrm{v}(t)} \sum_{u\in \operatorname{inV}(v)} |\tau(u)| = 1 - \sum_{u\in \operatorname{Inpv}(t)} |\tau(u)| = 1 - \|j\| = -\deg h^{-1}. \end{aligned} \tag{B.3}$$

First claim: (B.1) is a homomorphism of right $\mathcal{B}$-modules. In detail, for all trees $t \in \operatorname{tr}(n)$, all partitions $j = j_1 + \cdots + j_m \in \mathbb{N}^n$ we have to prove that the square

$$\begin{array}{ccc} \big(\otimes_{r=1}^m H_n(j_r)\big) \otimes \mathcal{B}(m) & \xrightarrow{\qquad\rho\qquad} & H_n(j) \\ {\scriptstyle (\otimes_{r=1}^m \Delta^{\mathsf{G}}(t)(j_r))\otimes 1}\downarrow & & \downarrow{\scriptstyle \Delta^{\mathsf{G}}(t)(j)} \\ \big(\otimes_{r=1}^m \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in \mathrm{v}(t)}(j_r)\big) \otimes \mathcal{B}(m) & \xrightarrow{\rho} & \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in \mathrm{v}(t)}(j) \end{array} \tag{B.4}$$

commutes departing from a similar strictly commutative square for G. Passing to summands of the product we perform the following computations.

First, we prove the above square for $t = |$, $n = 1$, $j = j_1 + \dots + j_m \in \mathbb{N}$. Since $\circledast_{\mathsf{G}}(|)(j) = \delta_{j1}\Bbbk$, it suffices to consider $j = 1$. By definition

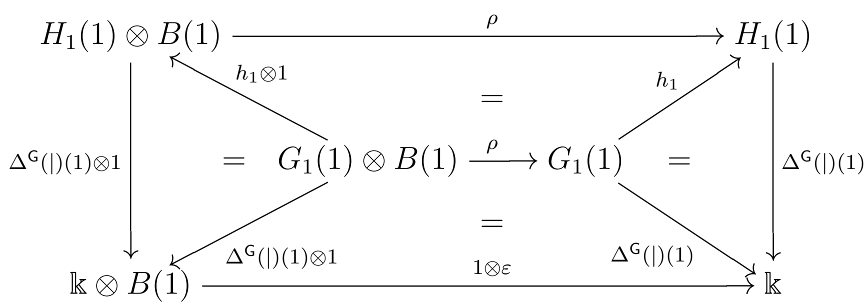
$$\Delta^{\mathsf{G}}(|)(1) = \big\langle G_1(1) \xrightarrow{h_1} H_1(1) \xrightarrow{\Delta^{\mathsf{G}}(|)(1)} \Bbbk \big\rangle.$$

If $m > 1$, we have

$$\Big\langle \big(\otimes_{r=1}^m H_1(j_r)\big) \otimes \mathcal{B}(m) \xrightarrow{\rho} H_1(1) \xrightarrow{\Delta^{\mathsf{G}}(|)(1)} \Bbbk \Big\rangle = 0$$

due to similar equation for G.

For $m = 1$ the required equation is given by the exterior of commutative diagram



Now we provide a statement useful for the case of $t \neq |$.

B.2. Lemma. *Assume that $t \neq |$. For all $m \in \mathbb{N}$, all sequences $(\tau_r)_{r=1}^m$ of t-trees*

$$\sum_{r=1}^m \operatorname{sg}(\tau_r) + \sum_{1\leqslant r<s\leqslant m}^{v>w,\, v,w\in \mathrm{v}(t)} \Big(|\tau_r(v)| - \sum_{u\in \mathrm{inV}(v)} |\tau_r(u)|\Big)\Big(|\tau_s(w)| - \sum_{x\in \mathrm{inV}(w)} |\tau_s(x)|\Big) + c_\rho^{root}$$
$$\equiv \operatorname{sg}(\tau) + c_\rho \pmod 2,$$

where $\tau = \tau_1 + \dots + \tau_m$ is the t-tree such that $\tau(v) = \trianglelefteq_{r=1}^m \tau_r(v)$ for all $v \in \overline{\mathrm{v}}(t) - \{\mathrm{rv}\}$. Maps $\tau(u \to v)$ are $\trianglelefteq_{r=1}^m \tau_r(u \to v)$ if v is not the root vertex. In detail, we claim that

$$\sum_{r=1}^m \sum_{v\in \mathrm{v}(t)} \sum_{q<p\in\tau_r(v)} \sum_{e\lessdot g\in \mathrm{in}(v)} |\tau_r(e)^{-1}(p)| \cdot |\tau_r(g)^{-1}(q)|$$
$$+ \sum_{r=1}^m \sum_{v\in \mathrm{v}(t)} \sum_{p\in\tau_r(v)} (p-1)\big(-1 + \sum_{e\in \mathrm{in}(v)} |\tau_r(e)^{-1}(p)|\big)$$
$$+ \sum_{r=1}^m \sum_{v\in \mathrm{v}(t)} \big(\sum_{u\in \mathrm{Inpv}_v^- t} |\tau_r(u)|\big) \cdot \big(-|\tau_r(v)| + \sum_{a\in \mathrm{inV}(v)} |\tau_r(a)|\big)$$

$$+\sum_{1\leqslant r<s\leqslant m}^{v>w,\,v,w\in\mathrm{v}(t)}\Big(|\tau_r(v)|-\sum_{u\in\mathrm{inV}(v)}|\tau_r(u)|\Big)\Big(|\tau_s(w)|-\sum_{x\in\mathrm{inV}(w)}|\tau_s(x)|\Big)$$

$$+\sum_{r=1}^{m}(r-1)\big(1-\sum_{x\in\mathrm{inV}(\mathrm{rv})}|\tau_r(x)|\big)+\sum_{1\leqslant r<s\leqslant m}^{x\lessdot y,\,x,y\in\mathrm{inV}(\mathrm{rv})}|\tau_r(y)|\cdot|\tau_s(x)|$$

$$\equiv\sum_{v\in\mathrm{v}(t)}\sum_{q<p\in\tau(v)}\sum_{e\lessdot g\in\mathrm{in}(v)}|\tau(e)^{-1}(p)|\cdot|\tau(g)^{-1}(q)|$$

$$+\sum_{v\in\mathrm{v}(t)}\sum_{p\in\tau(v)}(p-1)\big(\sum_{e\in\mathrm{in}(v)}|\tau(e)^{-1}(p)|-1\big)+\sum_{v\in\mathrm{v}(t)}\big(\sum_{u\in\mathrm{Inpv}_v^-t}|\tau(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)}|\tau(a)|-|\tau(v)|\big)$$

$$+\sum_{r=1}^{m}(r-1)\big(1-\sum_{a\in\mathrm{Inpv}\,t}|\tau_r(a)|\big)+\sum_{1\leqslant r<s\leqslant m}^{c\lhd d,\,c,d\in\mathrm{Inpv}\,t}|\tau_r(d)|\cdot|\tau_s(c)|\pmod 2.$$

Proof. The detailed form of the equation is obtained using the formula $\deg\otimes^{p\in\tau_r(v)}h=|\tau_r(v)|-\sum_{u\in\mathrm{inV}(v)}|\tau_r(u)|$, which is a part of (B.3). The equation is obvious for $m=0$ and $m=1$. Let us prove it by induction on m. Assume that it holds for $m=k-1$. Let us deduce it for $m=k$. The difference of two expressions, one with $\tau=\tau_1+\dots+\tau_{k-1}+\tau_k$ and another with $\tau'=\tau_1+\dots+\tau_{k-1}$ is the following:

$$\sum_{v\in\mathrm{v}(t)}\sum_{q<p\in\tau_k(v)}\sum_{e\lessdot g\in\mathrm{in}(v)}|\tau_k(e)^{-1}(p)|\cdot|\tau_k(g)^{-1}(q)|+\sum_{v\in\mathrm{v}(t)}\sum_{p\in\tau_k(v)}(p-1)\big(\sum_{e\in\mathrm{in}(v)}|\tau_k(e)^{-1}(p)|-1\big)$$

$$+\sum_{v\in\mathrm{v}(t)}\big(\sum_{u\in\mathrm{Inpv}_v^-t}|\tau_k(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)}|\tau_k(a)|-|\tau_k(v)|\big)$$

$$+\sum_{1\leqslant r<k}^{v>w,\,v,w\in\mathrm{v}(t)}\Big(|\tau_r(v)|-\sum_{u\in\mathrm{inV}(v)}|\tau_r(u)|\Big)\Big(|\tau_k(w)|-\sum_{x\in\mathrm{inV}(w)}|\tau_k(x)|\Big)$$

$$+(k-1)\big(1-\sum_{x\in\mathrm{inV}(\mathrm{rv})}|\tau_k(x)|\big)+\sum_{1\leqslant r<k}^{x\lessdot y,\,x,y\in\mathrm{inV}(\mathrm{rv})}|\tau_r(y)|\cdot|\tau_k(x)|$$

$$\equiv\sum_{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\sum_{e\lessdot g\in\mathrm{in}(v)}\sum_{1\leqslant r<k}\sum_{q\in\tau_r(v)}\sum_{p\in\tau_k(v)}|\tau_k(e)^{-1}(p)|\cdot|\tau_r(g)^{-1}(q)|$$

$$+\sum_{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\sum_{e\lessdot g\in\mathrm{in}(v)}\sum_{q<p\in\tau_k(v)}|\tau_k(e)^{-1}(p)|\cdot|\tau_k(g)^{-1}(q)|$$

$$+\sum_{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\sum_{p\in\tau_k(v)}\big(p-1+\sum_{1\leqslant r<k}|\tau_r(v)|\big)\big(\sum_{e\in\mathrm{in}(v)}|\tau(e)^{-1}(p)|-1\big)$$

$$+\sum_{v\in\mathrm{v}(t)}\big(\sum_{u\in\mathrm{Inpv}_v^-t}|\tau_k(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)}|\tau_k(a)|-|\tau_k(v)|\big)$$

$$+\sum_{v\in\mathrm{v}(t)}\big(\sum_{u\in\mathrm{Inpv}_v^-t}\sum_{1\leqslant r<k}|\tau_r(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)}|\tau_k(a)|-|\tau_k(v)|\big)$$

$$+\sum_{v\in \mathrm{v}(t)}\Bigl(\sum_{u\in \mathrm{Inpv}_v^- t}|\tau_k(u)|\Bigr)\cdot\sum_{1\leqslant r<k}\Bigl(\sum_{a\in \mathrm{inV}(v)}|\tau_r(a)|-|\tau_r(v)|\Bigr)$$
$$+(k-1)\bigl(1-\sum_{a\in \mathrm{Inpv}\,t}|\tau_k(a)|\bigr)+\sum_{1\leqslant r<k}^{c\lhd d,\,c,d\in \mathrm{Inpv}\,t}|\tau_r(d)|\cdot|\tau_k(c)| \pmod 2.$$

So it remains to prove the above induction step. Denote for $v\in\overline{\mathrm{v}}(t)$

$$n(v)=|\tau_k(v)|,\qquad z(v)=\sum_{1\leqslant r<k}|\tau_r(v)|.$$

After certain cancellations the above can be written as

$$\sum_{w<v\neq \mathrm{rv}}^{v,w\in \mathrm{v}(t)} z(v)\Bigl(n(w)-\sum_{x\in \mathrm{inV}(w)}n(x)\Bigr)+(k-1)\sum_{w\neq \mathrm{rv}}^{w\in \mathrm{v}(t)}\Bigl(n(w)-\sum_{x\in \mathrm{inV}(w)}n(x)\Bigr)$$
$$-\sum_{v>w}^{v,w\in \mathrm{v}(t)}\Bigl(\sum_{u\in \mathrm{inV}(v)}z(u)\Bigr)\Bigl(n(w)-\sum_{x\in \mathrm{inV}(w)}n(x)\Bigr)+(k-1)\bigl(1-\sum_{x\in \mathrm{inV(rv)}}n(x)\bigr)+\sum_{x\lessdot y}^{x,y\in \mathrm{inV(rv)}}z(y)n(x)$$
$$\equiv\sum_{v\in \mathrm{v}(t)-\{\mathrm{rv}\}}\ \sum_{x\lessdot y\in \mathrm{inV}(v)}z(y)n(x)-\sum_{v\in \mathrm{v}(t)-\{\mathrm{rv}\}}z(v)\Bigl(n(v)-\sum_{x\in \mathrm{inV}(v)}n(x)\Bigr)$$
$$+\sum_{v\in \mathrm{v}(t)}\Bigl(\sum_{u\in \mathrm{Inpv}_v^- t}z(u)\Bigr)\Bigl(\sum_{a\in \mathrm{inV}(v)}n(a)-n(v)\Bigr)+\sum_{v\in \mathrm{v}(t)}\Bigl(\sum_{a\in \mathrm{inV}(v)}z(a)-z(v)\Bigr)\sum_{u\in \mathrm{Inpv}_v^- t}n(u)$$
$$+(k-1)\bigl(1-\sum_{a\in \mathrm{Inpv}\,t}n(a)\bigr)+\sum_{c\lhd d}^{c,d\in \mathrm{Inpv}\,t}z(d)\cdot n(c) \pmod 2. \tag{B.5}$$

Summands proportional to $(k-1)$ cancel each other since

$$\sum_{w\in \mathrm{v}(t)-\{\mathrm{rv}\}}\bigl(n(w)-\sum_{x\in \mathrm{inV}(w)}n(x)\bigr)=\sum_{x\in \mathrm{inV(rv)}}n(x)-\sum_{a\in \mathrm{Inpv}\,t}n(a).$$

Equivalently, for all $p,q\in\overline{\mathrm{v}}(t)$ the coefficient at $z(p)n(q)$ has to vanish:

$$\chi(\mathrm{rv}>p>q\in \mathrm{v}(t))-\chi(Pq<p<\mathrm{rv})+\chi(Pq<Pp\leqslant \mathrm{rv})-\chi(\mathrm{rv}\geqslant Pp>q\in \mathrm{v}(t))$$
$$\equiv\chi(Pp=Pq)\chi(q\lessdot p)+\chi(p=Pq<\mathrm{rv})-\chi(\mathrm{rv}>p=q\in \mathrm{v}(t))+\chi(p\in \mathrm{Inpv}_{Pq}^- t)$$
$$-\chi(p\in \mathrm{Inpv}_q^- t)+\chi(q\in \mathrm{Inpv}_{Pp}^- t)-\chi(q\in \mathrm{Inpv}_p^- t)+\chi(q\lhd p\in \mathrm{Inpv}\,t) \pmod 2.$$

By convention $\mathrm{Inpv}_x^- t=\varnothing$ for $x\in V(t)-\mathrm{v}(t)$.

First of all we prove this identity in several particular cases, when at least one of p, q is an input vertex. The equation holds true for $t=|$, let us proceed for $t\neq|$.

1. Assume that $p, q \in \operatorname{Inpv} t$. The equation simplifies to

$$-\chi(Pq < Pp) + \chi(Pp = Pq)\chi(q \lessdot p) + \chi(p \in \operatorname{Inpv}^-_{Pq} t) + \chi(q \in \operatorname{Inpv}^-_{Pp} t) + \chi(q \lhd p) \equiv 0 \pmod 2.$$

   In all 9 cases, $p \lhd q$, $p = q$, $p \rhd q$ and $Pp < Pq$, $Pp = Pq$, $Pp > Pq$, the equation holds.

2. Assume that $p \in \operatorname{Inpv} t$, $q \in \mathrm{v}(t)$. The equation simplifies to

$$-\chi(Pq < Pp) + \chi(Pp > q) + \chi(Pp = Pq)\chi(q \lessdot p) + \chi(p \in \operatorname{Inpv}^-_{Pq} t) + \chi(p \in \operatorname{Inpv}^-_{q} t) + \chi(q \in \operatorname{Inpv}^-_{Pp} t) \equiv 0 \pmod 2.$$

   If $q \geqslant Pp$ the equation holds, assume that $q < Pp$. The equation holds in all 6 cases, $Pp < Pq$, $Pp = Pq$, $Pp > Pq$, and $p \lhd q$, $p \rhd q$.

3. Assume that $p \in \mathrm{v}(t)$, $q \in \operatorname{Inpv} t$. The equation reduces to

$$-\chi(Pq < p < \mathrm{rv}) + \chi(Pq < Pp \leqslant \mathrm{rv}) - \chi(Pp = Pq)\chi(q \lessdot p) - \chi(p = Pq < \mathrm{rv}) - \chi(p \in \operatorname{Inpv}^-_{Pq} t) - \chi(q \in \operatorname{Inpv}^-_{Pp} t) + \chi(q \in \operatorname{Inpv}^-_{p} t) \equiv 0 \pmod 2.$$

   The equation holds if $p > Pq$ or $p = Pq$, assume that $p < Pq$. The equation holds in all 6 cases, $Pp < Pq$, $Pp = Pq$, $Pp > Pq$, and $p \lhd q$, $p \rhd q$.

We conclude that validity of the identity for t is equivalent to its validity for the subtree $\bar t \subset t$, $\mathrm{v}(\bar t) = \mathrm{v}(t)$, $\operatorname{Inp} \bar t = \varnothing$. Considering the subtree $\bar t$ we single out the smallest vertex $1 \in \mathrm{v}(\bar t)$ and convert it to an input vertex. The expressions computed for the obtained tree t' and the previous tree $\bar t$ are equal. However, the number of internal vertices of t' is smaller than that of $\bar t$. This allows to conclude the proof by induction. ∎

Using the above lemma we verify equation (B.4)

$$\Big\langle \Big(\bigotimes_{r=1}^{m} H_n(j_r)\Big) \otimes \mathcal{B}(m) \xrightarrow{(\otimes_{r=1}^m \Delta^{\mathsf{G}}(t)(j_r))\otimes 1} \Big(\bigotimes_{r=1}^{m} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in \mathrm{v}(t)}(j_r)\Big) \otimes \mathcal{B}(m) \xrightarrow{\rho} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(j) \Big\rangle$$

$$= \sum_{\forall a \in \operatorname{Inpv} t\, |\tau_r(a)| = j_r^a}^{t-\text{trees } (\tau_r)_{r=1}^m} \Big\langle \Big(\bigotimes_{r=1}^{m} H_n(j_r)\Big) \otimes \mathcal{B}(m) \xrightarrow{(\otimes_{r=1}^m (\Delta^{\mathsf{G}}(t)(j_r)\operatorname{pr}_{\tau_r}))\otimes 1}$$

$$\Big(\bigotimes_{r=1}^{m} \bigotimes^{v\in\mathrm{v}(t)} \bigotimes^{p\in\tau_r(v)} H_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in \mathrm{in}(v)}\big)\Big) \otimes \mathcal{B}(m) \xrightarrow{\varkappa\otimes 1}$$

$$\Big(\bigotimes^{v\in\mathrm{v}(t)} \bigotimes_{r=1}^{m} \bigotimes^{p\in\tau_r(v)} H_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in \mathrm{in}(v)}\big)\Big) \otimes \mathcal{B}(m) \xrightarrow{\cong}$$

$$\Big(\bigotimes^{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\bigotimes_{r=1}^{m}\bigotimes^{p\in\tau_r(v)} H_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes$$

$$\otimes\Big(\bigotimes_{r=1}^{m} H_{|\,\mathrm{rv}\,|}\big(|\tau_r(x)|_{x\in\mathrm{inV}(\mathrm{rv})}\big)\Big)\otimes\mathcal{B}(m)\xrightarrow{1\otimes\rho}$$

$$\Big(\bigotimes^{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\bigotimes^{p\in\tau(v)} H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes H_{|\,\mathrm{rv}\,|}\big(|\tau(x)|_{x\in\mathrm{inV}(\mathrm{rv})}\big)\xrightarrow{\cong}$$

$$\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\xrightarrow{\mathrm{in}_\tau}\circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(j)\Big\rangle$$

$$=\sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau_r(a)|=j_r^a}^{t-\text{trees }(\tau_r)_{r=1}^m}(-1)^{\sum_{r=1}^m\mathrm{sg}(\tau_r)-\sum_{1\leqslant r<s\leqslant m}(1-\|j_r\|)(1-\|j_s\|)}\Big\langle\Big(\bigotimes_{r=1}^{m}H_n(j_r)\Big)\otimes\mathcal{B}(m)$$

$$\xrightarrow{(\otimes_{r=1}^m(h^{-1}\Delta^{\mathsf{G}}(t)\,\mathrm{pr}_{\tau_r}))\otimes 1}\Big(\bigotimes_{r=1}^{m}\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau_r(v)} G_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes\mathcal{B}(m)$$

$$\xrightarrow{(\otimes_{r=1}^m\otimes^{v\in\mathrm{v}(t)}\otimes^{p\in\tau_r(v)}h)\otimes(g^{-1}g)}\Big(\bigotimes_{r=1}^{m}\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau_r(v)} H_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes\mathcal{B}(m)\xrightarrow{\varkappa\otimes 1}\text{as above}\Big\rangle$$

$$=\sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau_r(a)|=j_r^a}^{t-\text{trees }(\tau_r)_{r=1}^m}(-1)^{\sum_{r=1}^m\mathrm{sg}(\tau_r)+\sum_{1\leqslant r<s\leqslant m}^{v>w,\,v,w\in\mathrm{v}(t)}(|\tau_r(v)|-\sum_{u\in\mathrm{inV}(v)}|\tau_r(u)|)(|\tau_s(w)|-\sum_{x\in\mathrm{inV}(w)}|\tau_s(x)|)}$$

$$(-1)^{\sum_{1\leqslant r<s\leqslant m}(1-\|j_r\|)(1-\|j_s\|)}\Big\langle\Big(\bigotimes_{r=1}^{m}H_n(j_r)\Big)\otimes\mathcal{B}(m)\xrightarrow{1\otimes g^{-1}}\Big(\bigotimes_{r=1}^{m}H_n(j_r)\Big)\otimes\mathcal{A}(m)\xrightarrow{(\otimes_{r=1}^m(h^{-1}))\otimes 1}$$

$$\Big(\bigotimes_{r=1}^{m}G_n(j_r)\Big)\otimes\mathcal{A}(m)\xrightarrow{(\otimes_{r=1}^m(\Delta^{\mathsf{G}}(t)\,\mathrm{pr}_{\tau_r}))\otimes 1}\Big(\bigotimes_{r=1}^{m}\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau_r(v)} G_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes\mathcal{A}(m)$$

$$\xrightarrow{\varkappa\otimes 1}\Big(\bigotimes^{v\in\mathrm{v}(t)}\bigotimes_{r=1}^{m}\bigotimes^{p\in\tau_r(v)} G_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes\mathcal{A}(m)\xrightarrow{\cong}$$

$$\Big(\bigotimes^{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\bigotimes_{r=1}^{m}\bigotimes^{p\in\tau_r(v)} G_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes$$

$$\otimes\Big(\bigotimes_{r=1}^{m} G_{|\,\mathrm{rv}\,|}\big(|\tau_r(x)|_{x\in\mathrm{inV}(\mathrm{rv})}\big)\Big)\otimes\mathcal{A}(m)\xrightarrow{(\otimes^v\otimes_{r=1}^m\otimes^{p\in\tau_r(v)}h)\otimes(\otimes_{r=1}^m h)\otimes g}$$

$$\Big(\bigotimes^{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\bigotimes_{r=1}^{m}\bigotimes^{p\in\tau_r(v)} H_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes$$

$$\otimes\Big(\bigotimes_{r=1}^{m} H_{|\,\mathrm{rv}\,|}\big(|\tau_r(x)|_{x\in\mathrm{inV}(\mathrm{rv})}\big)\Big)\otimes\mathcal{B}(m)\xrightarrow{1\otimes\rho}$$

$$\Big(\bigotimes^{v\in\mathrm{v}(t)-\{\mathrm{rv}\}} \bigotimes^{p\in\tau(v)} H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes H_{|\,\mathrm{rv}\,|}\big(|\tau(x)|_{x\in\mathrm{inV}(\mathrm{rv})}\big)\xrightarrow{\cong}$$
$$\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\xrightarrow{\mathrm{in}_\tau} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(j)\Big\rangle$$
$$=\sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau_r(a)|=j_r^a}^{t-\mathrm{trees}\,(\tau_r)_{r=1}^m}(-1)^{\sum_{r=1}^m \mathrm{sg}(\tau_r)+\sum_{1\leqslant r<s\leqslant m}^{v>w,\,v,w\in\mathrm{v}(t)}(|\tau_r(v)|-\sum_{u\in\mathrm{inV}(v)}|\tau_r(u)|)(|\tau_s(w)|-\sum_{x\in\mathrm{inV}(w)}|\tau_s(x)|)+c_\rho^{\mathrm{root}}}$$
$$\Big\langle\Big(\bigotimes_{r=1}^m H_n(j_r)\Big)\otimes\mathcal{B}(m)\xrightarrow{(1\otimes g^{-1})\cdot[(\otimes_{r=1}^m h)^{-1}\otimes 1]}\Big(\bigotimes_{r=1}^m G_n(j_r)\Big)\otimes\mathcal{A}(m)\xrightarrow{(\otimes_{r=1}^m(\Delta^{\mathsf{G}}(t)\,\mathrm{pr}_{\tau_r}))\otimes 1}$$
$$\Big(\bigotimes_{r=1}^m\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau_r(v)} G_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes\mathcal{A}(m)\xrightarrow{\varkappa\otimes 1}$$
$$\Big(\bigotimes^{v\in\mathrm{v}(t)}\bigotimes_{r=1}^m\bigotimes^{p\in\tau_r(v)} G_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes\mathcal{A}(m)\xrightarrow{\cong}$$
$$\Big(\bigotimes^{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\bigotimes_{r=1}^m\bigotimes^{p\in\tau_r(v)} G_{|v|}\big(|\tau_r(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes$$
$$\otimes\Big(\bigotimes_{r=1}^m G_{|\,\mathrm{rv}\,|}\big(|\tau_r(x)|_{x\in\mathrm{inV}(\mathrm{rv})}\big)\Big)\otimes\mathcal{A}(m)\xrightarrow{1\otimes\rho}$$
$$\Big(\bigotimes^{v\in\mathrm{v}(t)-\{\mathrm{rv}\}}\bigotimes^{p\in\tau(v)} G_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big)\otimes G_{|\,\mathrm{rv}\,|}\big(|\tau(x)|_{x\in\mathrm{inV}(\mathrm{rv})}\big)\xrightarrow{\cong}$$
$$\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} G_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\xrightarrow{\mathrm{in}_\tau}\circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(j)\xrightarrow{\circledast_{\mathsf{G}}(t)(h)}\circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(j)\Big\rangle.$$

Using Lemma B.2 and [Lyu15, (2.13)] we shorten the previous composition to

$$(-1)^{c_\rho}\Big\langle\Big(\bigotimes_{r=1}^m H_n(j_r)\Big)\otimes\mathcal{B}(m)\xrightarrow{(1\otimes g^{-1})\cdot[(\otimes_{r=1}^m h)^{-1}\otimes 1]}\Big(\bigotimes_{r=1}^m G_n(j_r)\Big)\otimes\mathcal{A}(m)$$
$$\xrightarrow{(\otimes_{r=1}^m\Delta^{\mathsf{G}}(t))\otimes 1}\Big(\bigotimes_{r=1}^m\circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(j_r)\Big)\otimes\mathcal{A}(m)\xrightarrow{\rho}$$
$$\circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(j)\xrightarrow{\Xi}\circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(j)\xrightarrow{\circledast_{\mathsf{G}}(t)(h)}\circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(j)\Big\rangle$$
$$=(-1)^{c_\rho}\Big\langle\Big(\bigotimes_{r=1}^m H_n(j_r)\Big)\otimes\mathcal{B}(m)\xrightarrow{(1\otimes g^{-1})\cdot[(\otimes_{r=1}^m h)^{-1}\otimes 1]}\Big(\bigotimes_{r=1}^m G_n(j_r)\Big)\otimes\mathcal{A}(m)\xrightarrow{\rho}G_n(j)$$
$$\xrightarrow{\Delta^{\mathsf{G}}(t)(j)}\circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(j)\xrightarrow{\Xi}\circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(j)\xrightarrow{\circledast_{\mathsf{G}}(t)(h)}\circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(j)\Big\rangle$$
$$=(-1)^{c_\rho}\Big\langle\Big(\bigotimes_{r=1}^m H_n(j_r)\Big)\otimes\mathcal{B}(m)\xrightarrow{(1\otimes g^{-1})\cdot[(\otimes_{r=1}^m h)^{-1}\otimes 1]}\Big(\bigotimes_{r=1}^m G_n(j_r)\Big)\otimes\mathcal{A}(m)$$

$$\xrightarrow{\rho} G_n(j) \xrightarrow{h} H_n(j) \xrightarrow{\Delta^{\mathsf{G}}(t)(j)} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(j)\Big\rangle$$

$$= \Big\langle \Big(\bigotimes_{r=1}^m H_n(j_r)\Big) \otimes \mathcal{B}(m) \xrightarrow{(1\otimes g^{-1})\cdot[(\otimes_{r=1}^m h)^{-1}\otimes 1]} \Big(\bigotimes_{r=1}^m G_n(j_r)\Big) \otimes \mathcal{A}(m)$$

$$\xrightarrow{(\otimes_{r=1}^m h)\otimes g} \Big(\bigotimes_{r=1}^m H_n(j_r)\Big) \otimes \mathcal{B}(m) \xrightarrow{\rho} H_n(j) \xrightarrow{\Delta^{\mathsf{G}}(t)(j)} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(j)\Big\rangle$$

$$= \Big\langle \Big(\bigotimes_{r=1}^m H_n(j_r)\Big) \otimes \mathcal{B}(m) \xrightarrow{\rho} H_n(j) \xrightarrow{\Delta^{\mathsf{G}}(t)(j)} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(j)\Big\rangle.$$

The first claim is proven.

Second claim: (B.2) is a homomorphism of left $\mathcal{B}$-modules with respect to i-th action. In detail, for all trees $t \in \mathrm{tr}(n)$, all elements $k \in \mathbb{N}^n$, all partitions $j = j_1 + \dots + j_{k^i} \in \mathbb{N}$, $1 \leqslant i \leqslant n$, we have to prove that the square

$$\begin{array}{ccc}
\big(\otimes_{q=1}^{k^i}\mathcal{B}(j_q)\big) \otimes H_n(k) & \xrightarrow{\quad\lambda^i\quad} & H_n(k; k^i \mapsto j) \\
\downarrow{\scriptstyle 1\otimes\Delta^{\mathsf{G}}(t)} & & \downarrow{\scriptstyle \Delta^{\mathsf{G}}(t)} \\
\big(\otimes_{q=1}^{k^i}\mathcal{B}(j_q)\big) \otimes \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k) & \xrightarrow{\lambda^i} & \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k; k^i \mapsto j)
\end{array}$$

commutes departing from a similar commutative square for G.

First we obtain a useful identity.

B.3. Lemma. *Let t-tree τ satisfy $|\tau(\mathrm{tail}(l))| = k^l$ for $l \in \mathrm{Inp}(t) \cong \mathbf{n}$. Let $i \in \mathrm{Inp}\, t$. For any vector $(j_q)_{q=1}^{k^i} \in \mathbb{N}^{k^i}$ define the t-tree τ' by*

$$\tau'(v) = \begin{cases} \tau(v), & \text{for } v \neq \mathrm{tail}(i), \\ \trianglelefteq_{p\in\tau(\mathrm{head}(i))} \trianglelefteq_{q\in\tau(i)^{-1}(p)} \mathbf{j}_q, & \text{for } v = \mathrm{tail}(i). \end{cases}$$

The map $\tau'(i) : \tau'(\mathrm{tail}(i)) \to \tau'(\mathrm{head}(i)) = \tau(\mathrm{head}(i))$ is the projection to the first index. Then the equation holds

$$\mathrm{sg}(\tau) + \sum_{q=1}^{k^i}(1 - j_q) \cdot \sum_{v<\mathrm{head}(i)}^{v\in\mathrm{v}(t)} \Big(|\tau(v)| - \sum_{u\in\mathrm{inV}(v)} |\tau(u)|\Big)$$
$$+ \sum_{p<r}^{p,r\in\tau(\mathrm{head}(i))} \Big(|\tau(i)^{-1}(r)| - \sum_{q\in\tau(i)^{-1}(r)} j_q\Big)\Big(1 - \sum_{e\in\mathrm{in}(\mathrm{head}(i))} |\tau(e)^{-1}(p)|\Big) + \sum_{p\in\tau(\mathrm{head}(i))} c_{\lambda^i}$$
$$\equiv \mathrm{sg}(\tau') + c_{\lambda^i} \pmod 2.$$

In detail this equation is

$$\sum_{v\in\mathrm{v}(t)}\sum_{q<p\in\tau(v)}\sum_{e\lessdot g\in\mathrm{in}(v)}|\tau(e)^{-1}(p)|\cdot|\tau(g)^{-1}(q)|$$
$$+\sum_{v\in\mathrm{v}(t)}\sum_{p\in\tau(v)}(p-1)\big(\sum_{e\in\mathrm{in}(v)}|\tau(e)^{-1}(p)|-1\big)+\sum_{v\in\mathrm{v}(t)}\big(\sum_{u\in\mathrm{Inpv}_v^- t}|\tau(u)|\big)\big(\sum_{a\in\mathrm{inV}(v)}|\tau(a)|-|\tau(v)|\big)$$
$$+\sum_{q=1}^{k^i}(1-j_q)\cdot\sum_{v<\mathrm{head}(i)}^{v\in\mathrm{v}(t)}\Big(|\tau(v)|-\sum_{u\in\mathrm{inV}(v)}|\tau(u)|\Big)$$
$$+\sum_{p<r\in\tau(\mathrm{head}(i))}\big(|\tau(i)^{-1}(r)|-\sum_{q\in\tau(i)^{-1}(r)}j_q\big)\big(1-\sum_{e\in\mathrm{in}(\mathrm{head}(i))}|\tau(e)^{-1}(p)|\big)$$
$$+\sum_{p\in\tau(\mathrm{head}(i))}\sum_{q\in\tau(i)^{-1}(p)}(1-j_q)\Big(q-\min(\tau(i)^{-1}(p))+\sum_{e\in\mathrm{in}(\mathrm{head}(i))}^{e\lessdot i}|\tau(e)^{-1}(p)|\Big)$$
$$\equiv\sum_{v\in\mathrm{v}(t)}\sum_{q<p\in\tau'(v)}\sum_{e\lessdot g\in\mathrm{in}(v)}|\tau'(e)^{-1}(p)|\cdot|\tau'(g)^{-1}(q)|$$
$$+\sum_{v\in\mathrm{v}(t)}\sum_{p\in\tau'(v)}(p-1)\big(-1+\sum_{e\in\mathrm{in}(v)}|\tau'(e)^{-1}(p)|\big)$$
$$+\sum_{v\in\mathrm{v}(t)}\big(\sum_{u\in\mathrm{Inpv}_v^- t}|\tau'(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)}|\tau'(a)|-|\tau'(v)|\big)+\sum_{q=1}^{k^i}(1-j_q)\Big(q-1+\sum_{l=1}^{i-1}k^l\Big)\pmod 2.$$

Proof. The case of $t=|$ being easy, assume that $t\neq|$. For the input edge $i\in\mathrm{Inp}(t)\cong\mathbf{n}$ there are vertices $\mathrm{head}(i)\in\mathrm{v}(t)$ and $\mathrm{tail}(i)\in\mathrm{Inpv}\,t$.

Substituting the definition of τ' into the equation to prove we get after cancellation

$$\sum_{q<p}^{q,p\in\tau(\mathrm{head}(i))}\sum_{g\gtrdot i}^{g\in\mathrm{in}(\mathrm{head}(i))}|\tau(i)^{-1}(p)|\cdot|\tau(g)^{-1}(q)|+\sum_{q<p}^{q,p\in\tau(\mathrm{head}(i))}\sum_{e\lessdot i}^{e\in\mathrm{in}(\mathrm{head}(i))}|\tau(e)^{-1}(p)|\cdot|\tau(i)^{-1}(q)|$$
$$+\sum_{p\in\tau(\mathrm{head}(i))}(p-1)|\tau(i)^{-1}(p)|+|\tau(\mathrm{tail}(i))|\sum_{v\in\mathrm{v}(t)}^{\mathrm{Inp}_v^- t\ni i}\Big(-|\tau(v)|+\sum_{a\in\mathrm{inV}(v)}|\tau(a)|\Big)$$
$$+\Big(\sum_{u\in\mathrm{Inpv}_{\mathrm{head}(i)}^- t}|\tau(u)|\Big)|\tau(\mathrm{tail}(i))|+\sum_{r=1}^{k^i}(1-j_r)\cdot\sum_{v<\mathrm{head}(i)}^{v\in\mathrm{v}(t)}\Big(|\tau(v)|-\sum_{u\in\mathrm{inV}(v)}|\tau(u)|\Big)$$
$$+\sum_{p<q\in\tau(\mathrm{head}(i))}\big(|\tau(i)^{-1}(q)|-\sum_{r\in\tau(i)^{-1}(q)}j_r\big)\big(1-\sum_{e\in\mathrm{in}(\mathrm{head}(i))}|\tau(e)^{-1}(p)|\big)$$
$$+\sum_{p\in\tau(\mathrm{head}(i))}\sum_{r\in\tau(i)^{-1}(p)}(1-j_r)\Big(r-\min(\tau(i)^{-1}(p))+\sum_{e\lessdot i}^{e\in\mathrm{in}(\mathrm{head}(i))}|\tau(e)^{-1}(p)|\Big)$$

$$\equiv \sum_{q<p}^{q,p\in\tau(\mathrm{head}(i))} \sum_{g>i}^{g\in\mathrm{in}(\mathrm{head}(i))} \sum_{r\in\tau(i)^{-1}(p)} j_r \cdot |\tau(g)^{-1}(q)| + \sum_{q<p}^{q,p\in\tau(\mathrm{head}(i))} \sum_{e<i}^{e\in\mathrm{in}(\mathrm{head}(i))} |\tau(e)^{-1}(p)| \cdot \sum_{r\in\tau(i)^{-1}(q)} j_r + \sum_{p\in\tau(\mathrm{head}(i))} (p-1) \sum_{r\in\tau(i)^{-1}(p)} j_r$$
$$+ \Big(\sum_{r\in\tau(\mathrm{tail}(i))} j_r\Big) \sum_{v\in\mathrm{v}(t)}^{\mathrm{Inp}_v^- t\ni i} \Big(-|\tau(v)| + \sum_{a\in\mathrm{inV}(v)} |\tau(a)|\Big)$$
$$+ \Big(\sum_{u\in\mathrm{Inpv}^-_{\mathrm{head}(i)} t} |\tau(u)|\Big) \sum_{r\in\tau(\mathrm{tail}(i))} j_r + \sum_{r=1}^{k^i}(1-j_r)\Big(r-1+\sum_{l=1}^{i-1} k^l\Big) \pmod 2. \quad \text{(B.6)}$$

Notice that j_r's enter this equation linearly. Let us prove that coefficients at j_r in both parts are equal:

$$\sum_{v<\mathrm{head}(i)}^{v\in\mathrm{v}(t)} \Big(-|\tau(v)| + \sum_{u\in\mathrm{inV}(v)} |\tau(u)|\Big) + \sum_{p<\tau(i)(r)}^{p\in\tau(\mathrm{head}(i))} \Big(-1 + \sum_{e\in\mathrm{in}(\mathrm{head}(i))} |\tau(e)^{-1}(p)|\Big)$$
$$- r + \min(\tau(i)^{-1}(\tau(i)(r))) - \sum_{e<i}^{e\in\mathrm{in}(\mathrm{head}(i))} |\tau(e)^{-1}(\tau(i)(r))|$$
$$\equiv \sum_{q<\tau(i)(r)}^{q\in\tau(\mathrm{head}(i))} \sum_{g>i}^{g\in\mathrm{in}(\mathrm{head}(i))} |\tau(g)^{-1}(q)| + \sum_{p>\tau(i)(r)}^{p\in\tau(\mathrm{head}(i))} \sum_{e<i}^{e\in\mathrm{in}(\mathrm{head}(i))} |\tau(e)^{-1}(p)| + \tau(i)(r) - 1$$
$$+ \sum_{v\in\mathrm{v}(t)}^{\mathrm{Inp}_v^- t\ni i} \Big(-|\tau(v)| + \sum_{a\in\mathrm{inV}(v)} |\tau(a)|\Big) + \sum_{u\in\mathrm{Inpv}^-_{\mathrm{head}(i)} t} |\tau(u)| - r + 1 - \sum_{l=1}^{i-1} k^l \pmod 2.$$

Using the presentation $\min(\tau(i)^{-1}(\tau(i)(r))) = 1 + \sum_{p<\tau(i)(r)}^{p\in\tau(\mathrm{head}(i))} |\tau(i)^{-1}(p)|$ we reduce the equation to

$$\sum_{v<\mathrm{head}(i)}^{v\in\mathrm{v}(t)} \Big(-|\tau(v)| + \sum_{u\in\mathrm{inV}(v)} |\tau(u)|\Big) + \sum_{e<i}^{e\in\mathrm{in}(\mathrm{head}(i))} |\tau(\mathrm{tail}(e))|$$
$$\equiv \sum_{v\in\mathrm{v}(t)}^{\mathrm{Inp}_v^- t\ni i} \Big(-|\tau(v)| + \sum_{a\in\mathrm{inV}(v)} |\tau(a)|\Big) + \sum_{u\in\mathrm{Inpv}^-_{\mathrm{head}(i)} t} |\tau(u)| - \sum_{l=1}^{i-1} k^l \pmod 2. \quad \text{(B.7)}$$

Notice that the difference of left and right hand sides is a linear form of variables $|\tau(u)|$, $u \in \overline{\mathrm{v}}(t)$. For a condition P denote by $\chi(P)$ the indicator function, $\chi(P) = 1$ if P holds, $\chi(P) = 0$ if it does not. Assume that $t \neq |$. The coefficient at $|\tau(u)|$, $u \in \overline{\mathrm{v}}(t)$, in difference (B.7) is

$$\begin{aligned} c^i_t(u) &= -\chi(u \in \mathrm{v}(t),\, u < \mathrm{head}(i)) + \chi(Pu < \mathrm{head}(i)) + \chi(Pu = \mathrm{head}(i),\, \mathrm{ou}(u) \lessdot i) \\ &+ \chi(u \in \mathrm{v}(t),\, \mathrm{Inp}^-_u\, t \ni i) - \chi(\mathrm{Inp}^-_{Pu}\, t \ni i) - \chi(\mathrm{Inpv}^-_{\mathrm{head}(i)}\, t) + \chi(u \in \mathrm{Inpv}\, t,\, u \lhd \mathrm{tail}(i)) \\ &= -\chi(u \in \mathrm{v}(t),\, u < \mathrm{head}(i)) + \chi(Pu < \mathrm{head}(i)) + \chi(Pu = \mathrm{head}(i),\, \mathrm{ou}(u) \lessdot i) \\ &+ \chi(u \in \mathrm{v}(t),\, u < \mathrm{head}(i),\, u \rhd \mathrm{head}(i)) - \chi(Pu < \mathrm{head}(i),\, Pu \rhd \mathrm{head}(i)) \\ &- \chi(u \lhd \mathrm{head}(i) < Pu,\, (u \in \mathrm{v}(t),\, u < \mathrm{head}(i)) \text{ or } u \in \mathrm{Inpv}\, t) + \chi(u \in \mathrm{Inpv}\, t,\, u \lhd \mathrm{tail}(i)). \end{aligned}$$

Our goal is to prove that $c^i_t(u) \equiv 0 \pmod 2$. We shall do it by induction on the number of internal vertices. The claim is obvious for corollas t. An ordered tree t has the smallest internal vertex 1 with respect to $<$. If $1 = \mathrm{head}(i)$, then

$$c^i_t(u) = \chi(Pu = 1,\, \mathrm{ou}(u) \lessdot i) - \chi(u \lhd 1 \neq Pu,\, u \in \mathrm{Inpv}\, t) + \chi(u \in \mathrm{Inpv}\, t,\, u \lhd \mathrm{tail}(i)).$$

Clearly, $c^i_t(u)$ vanishes unless $u \in \mathrm{Inpv}\, t$. It vanishes modulo 2 in all three cases: $Pu \lhd 1$, $Pu = 1$ and $Pu \rhd 1$.

Assume that $1 \neq \mathrm{head}(i)$. As earlier, consider subtree t' of t with $V(t') = V(t) - \mathrm{inV}(1)$, $E(t') = E(t) - \mathrm{in}(1)$, $\mathrm{v}(t') = \mathrm{v}(t) - \{1\}$, $\mathrm{Inpv}(t') = \{1\} \sqcup \mathrm{Inpv}(t) - \mathrm{inV}(1)$. We have $i \in \mathrm{Inp}\, t'$ so the expression $c^i_{t'}(u)$ makes sense for $u \in \overline{\mathrm{v}}(t')$. By induction hypothesis we may assume that $c^i_{t'}(u) \equiv 0 \pmod 2$. Let us deduce that $c^i_t(u) \equiv 0 \pmod 2$ case by case.

1. For $u \in \mathrm{v}(t) - \{1\} = \mathrm{v}(t')$ or for $u \in \mathrm{Inpv}(t) - \mathrm{inV}(1)$ we have $c^i_t(u) = c^i_{t'}(u) \equiv 0 \pmod 2$.

2. For $u \in \mathrm{Inpv}(t)$, $Pu = 1$ we have

$$c^i_t(u) = 1 - \chi(1 \rhd \mathrm{head}(i)) + \chi(u \lhd \mathrm{tail}(i)) \equiv 1 - \chi(1 \rhd \mathrm{head}(i)) - \chi(1 \lhd \mathrm{head}(i)) = 0.$$

3. For $u = 1 \neq \mathrm{head}(i)$ we have

$$\begin{aligned} c^i_t(1) &= -1 + \chi(P1 < \mathrm{head}(i)) + \chi(P1 = \mathrm{head}(i),\, \mathrm{ou}(1) \lessdot i) + \chi(1 \rhd \mathrm{head}(i)) \\ &\qquad - \chi(P1 < \mathrm{head}(i),\, P1 \rhd \mathrm{head}(i)) - \chi(1 \lhd \mathrm{head}(i) < P1), \\ c^i_{t'}(1) &= \chi(P1 < \mathrm{head}(i)) + \chi(P1 = \mathrm{head}(i),\, \mathrm{ou}(1) \lessdot i) \\ &\qquad - \chi(P1 < \mathrm{head}(i),\, P1 \rhd \mathrm{head}(i)) - \chi(1 \lhd \mathrm{head}(i) < P1) + \chi(1 \lhd \mathrm{tail}(i)). \end{aligned}$$

Hence,

$$c^i_t(1) \equiv c^i_t(1) - c^i_{t'}(1) = -1 + \chi(1 \rhd \mathrm{head}(i)) - \chi(1 \lhd \mathrm{tail}(i)) \equiv 0 \pmod 2.$$

Identity (B.7) is proven.

Therefore equation (B.6) is equivalent to any of its particular cases with fixed j_r's. When we put $j_r = 1$ for all $1 \leqslant r \leqslant k^i$, equation (B.6) becomes obvious. Therefore its validity is proven for arbitrary j_r as well. ∎

Let us prove that $\Delta^{\mathsf{G}}(t)(k)$ for H is a homomorphism of left $\mathcal{B}$-modules with respect to i-th action. Note that, in particular, $|\tau(\mathrm{tail}(i))| = k^i$.

$$\Big\langle\Big(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Big)\otimes H_n(k)\xrightarrow{1\otimes\Delta^{\mathsf{G}}(t)}\Big(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Big)\otimes\circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k)$$

$$\xrightarrow{\lambda^i}\circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)\Big\rangle$$

$$=\sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=k^a}^{t-\text{tree }\tau}\Big\langle\Big(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Big)\otimes H_n(k)\xrightarrow{1\otimes\Delta^{\mathsf{G}}(t)}\Big(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Big)\otimes\circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k)$$

$$\xrightarrow{1\otimes\mathrm{pr}_\tau}\Big(\bigotimes^{p\in\tau(\mathrm{head}(i))}\bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{B}(j_q)\Big)\otimes\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)}H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\xrightarrow[\cong]{\zeta}$$

$$\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)}\Big[\Big(\text{if }v=\mathrm{head}(i)\text{ then }\bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{B}(j_q)\otimes\Big)H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big]\xrightarrow{\otimes^v\otimes^p[1\text{ or }\lambda^i]}$$

$$\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)}H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)};\text{if }v=\mathrm{head}(i)\text{ then }|\tau(i)^{-1}(p)|\mapsto\textstyle\sum_{q\in\tau(i)^{-1}(p)}j_q\big)$$

$$\xrightarrow{\mathrm{in}_{\tau'}}\circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)\Big\rangle$$

$$=\sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=k^a}^{t-\text{tree }\tau}\Big\langle\Big(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Big)\otimes H_n(k)\xrightarrow{(1\otimes h^{-1})(\otimes_{q=1}^{k^i}g\otimes1)^{-1}}\Big(\bigotimes_{q=1}^{k^i}\mathcal{A}(j_q)\Big)\otimes G_n(k)$$

$$\xrightarrow{(-1)^{\mathrm{sg}(\tau)}\otimes\Delta^{\mathsf{G}}(t)\,\mathrm{pr}_\tau}\Big(\bigotimes^{p\in\tau(\mathrm{head}(i))}\bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{A}(j_q)\Big)\otimes\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)}G_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)$$

$$\xrightarrow{(\otimes\otimes g)\otimes(\otimes\otimes h)}\Big(\bigotimes^{p\in\tau(\mathrm{head}(i))}\bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{B}(j_q)\Big)\otimes\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)}H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\xrightarrow[\cong]{\zeta}\text{ as above}\Big\rangle$$

$$=\sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=k^a}^{t-\text{tree }\tau}(-1)^{\mathrm{sg}(\tau)+\sum_{q=1}^{k^i}(1-j_q)\cdot\sum_{v<\mathrm{head}(i)}^{v\in\mathrm{v}(t)}(|\tau(v)|-\sum_{u\in\mathrm{inV}(v)}|\tau(u)|)}$$

$$(-1)^{\sum_{p<r}^{p,r\in\tau(\mathrm{head}(i))}(|\tau(i)^{-1}(r)|-\sum_{q\in\tau(i)^{-1}(r)}j_q)(1-\sum_{e\in\mathrm{in}(\mathrm{head}(i))}|\tau(e)^{-1}(p)|)}$$

$$\Big\langle\Big(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Big)\otimes H_n(k)\xrightarrow{((\otimes_{q=1}^{k^i}g)\otimes h)^{-1}}\Big(\bigotimes_{q=1}^{k^i}\mathcal{A}(j_q)\Big)\otimes G_n(k)$$

$$\xrightarrow{1\otimes(\Delta^{\mathsf{G}}(t)\,\mathrm{pr}_\tau)}\Big(\bigotimes^{p\in\tau(\mathrm{head}(i))}\bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{A}(j_q)\Big)\otimes\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)}G_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)$$

$$\xrightarrow[\cong]{\zeta}\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)}\Big[\Big(\text{if }v=\mathrm{head}(i)\text{ then }\bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{A}(j_q)\otimes\Big)G_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\big)\Big]$$

$$\xrightarrow{\otimes^v\otimes^p[h\text{ or }((\otimes^{q\in\tau(i)^{-1}(p)}g)\otimes h)\lambda^i]}$$

$$\bigotimes^{v\in\mathrm{v}(t)} \bigotimes^{p\in\tau(v)} H_{|v|}\bigl(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}; \text{if } v=\mathrm{head}(i) \text{ then } |\tau(i)^{-1}(p)| \mapsto \textstyle\sum_{q\in\tau(i)^{-1}(p)} j_q\bigr)$$

$$\xrightarrow{\mathrm{in}_{\tau'}} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)\Bigr\rangle$$

$$= \sum_{\forall a\in\mathrm{Inpv}\,t\,|\tau(a)|=k^a}^{t-\mathrm{tree}\ \tau} (-1)^{\mathrm{sg}(\tau)+\sum_{q=1}^{k^i}(1-j_q)\cdot\sum_{v<\mathrm{head}(i)}^{v\in\mathrm{v}(t)}(|\tau(v)|-\sum_{u\in\mathrm{inV}(v)}|\tau(u)|)}$$

$$(-1)^{\sum_{p<r}^{p,r\in\tau(\mathrm{head}(i))}(|\tau(i)^{-1}(r)|-\sum_{q\in\tau(i)^{-1}(r)} j_q)(1-\sum_{e\in\mathrm{in}(\mathrm{head}(i))}|\tau(e)^{-1}(p)|)+\sum_{p\in\tau(\mathrm{head}(i))} c_{\lambda^i}}$$

$$\Bigl\langle\Bigl(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Bigr)\otimes H_n(k) \xrightarrow{((\otimes_{q=1}^{k^i} g)\otimes h)^{-1}} \Bigl(\bigotimes_{q=1}^{k^i}\mathcal{A}(j_q)\Bigr)\otimes G_n(k)$$

$$\xrightarrow{1\otimes(\Delta^{\mathsf{G}}(t)\,\mathrm{pr}_\tau)} \Bigl(\bigotimes^{p\in\tau(\mathrm{head}(i))}\bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{A}(j_q)\Bigr)\otimes \bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} G_{|v|}\bigl(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\bigr)$$

$$\xrightarrow[\cong]{\zeta} \bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)}\Bigl[\Bigl(\text{if } v=\mathrm{head}(i) \text{ then } \bigotimes^{q\in\tau(i)^{-1}(p)}\mathcal{A}(j_q)\otimes\Bigr)G_{|v|}\bigl(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}\bigr)\Bigr] \xrightarrow{\otimes^v\otimes^p[1 \text{ or } \lambda^i]}$$

$$\bigotimes^{v\in\mathrm{v}(t)}\bigotimes^{p\in\tau(v)} G_{|v|}\bigl(|\tau(e)^{-1}(p)|_{e\in\mathrm{in}(v)}; \text{if } v=\mathrm{head}(i) \text{ then } |\tau(i)^{-1}(p)|\mapsto \textstyle\sum_{q\in\tau(i)^{-1}(p)} j_q\bigr)$$

$$\xrightarrow{\mathrm{in}_{\tau'}} \circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j) \xrightarrow{\circledast_{\mathsf{G}}(t)(h)} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)\Bigr\rangle$$

$$= (-1)^{c_{\lambda^i}}\Bigl\langle\Bigl(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Bigr)\otimes H_n(k) \xrightarrow{((\otimes_{q=1}^{k^i} g)\otimes h)^{-1}} \Bigl(\bigotimes_{q=1}^{k^i}\mathcal{A}(j_q)\Bigr)\otimes G_n(k) \xrightarrow{1\otimes\Delta^{\mathsf{G}}(t)}$$

$$\Bigl(\bigotimes_{q=1}^{k^i}\mathcal{A}(j_q)\Bigr)\otimes\circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(k) \xrightarrow{\lambda^i} \circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)$$

$$\xrightarrow{\circledast_{\mathsf{G}}(t)(h)} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j) \xrightarrow{\Xi} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)\Bigr\rangle$$

$$= (-1)^{c_{\lambda^i}}\Bigl\langle\Bigl(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Bigr)\otimes H_n(k) \xrightarrow{((\otimes_{q=1}^{k^i} g)\otimes h)^{-1}} \Bigl(\bigotimes_{q=1}^{k^i}\mathcal{A}(j_q)\Bigr)\otimes G_n(k)$$

$$\xrightarrow{\lambda^i} G_n(k;k^i\mapsto j) \xrightarrow{\Delta^{\mathsf{G}}(t)} \circledast_{\mathsf{G}}(t)(G_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)$$

$$\xrightarrow{\circledast_{\mathsf{G}}(t)(h)} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j) \xrightarrow{\Xi} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)\Bigr\rangle$$

$$= (-1)^{c_{\lambda^i}}\Bigl\langle\Bigl(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Bigr)\otimes H_n(k) \xrightarrow{((\otimes_{q=1}^{k^i} g)\otimes h)^{-1}} \Bigl(\bigotimes_{q=1}^{k^i}\mathcal{A}(j_q)\Bigr)\otimes G_n(k)$$

$$\xrightarrow{\lambda^i} G_n(k;k^i\mapsto j) \xrightarrow{h} H_n(k;k^i\mapsto j) \xrightarrow{\Delta^{\mathsf{G}}(t)} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)\Bigr\rangle$$

$$= \Bigl\langle\Bigl(\bigotimes_{q=1}^{k^i}\mathcal{B}(j_q)\Bigr)\otimes H_n(k) \xrightarrow{\lambda^i} H_n(k;k^i\mapsto j) \xrightarrow{\Delta^{\mathsf{G}}(t)} \circledast_{\mathsf{G}}(t)(H_{|v|})_{v\in\mathrm{v}(t)}(k;k^i\mapsto j)\Bigr\rangle.$$

Here Lemma B.3 is applied.

Third claim: normalization equation (4.1) of [Lyu15] holds for (H,Δ^{G}), that is,

$$\big(H_n \xrightarrow{\Delta^{\mathsf{G}}(\tau[n])} \circledast_{\mathsf{G}}(\tau[n])(H_n) \xrightarrow{\cong} H_n\big) = 1.$$

In fact, for any $\tau[n]$-tree τ we have $\operatorname{sg}(\tau) = 0$ and the above identity follows from the commutative diagram

$$\begin{array}{ccccc} G_n & \xrightarrow{\Delta^{\mathsf{G}}(\tau[n])} & \circledast_{\mathsf{G}}(\tau[n])(G_n) & \xrightarrow{\cong} & G_n \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \circledast_{\mathsf{G}}(\tau[n])(h)} & & \downarrow{\scriptstyle h} \\ H_n & \xrightarrow{\Delta^{\mathsf{G}}(\tau[n])} & \circledast_{\mathsf{G}}(\tau[n])(H_n) & \xrightarrow{\cong} & H_n \end{array}$$

whose first row composes to the identity morphism.

Fourth claim: multiplicativity equation (4.2) of [Lyu15] holds for (H,Δ^{G}).

B.4. Lemma. *For all $t \in \operatorname{tr}$, all t-trees τ, any fixed vertex $x \in \mathrm{v}(t)$, all sequences $(t_v) \in \prod_{v\in\mathrm{v}(t)} \operatorname{tr}|v|$ such that t_v is the corolla $\tau[|v|]$ for $v \neq x$, all collections of t_v-trees τ_v^p, $v \in \mathrm{v}(t)$, $p \in \tau(v)$, such that for all $u \in \operatorname{Inpv}(t_v) \cong \operatorname{inV}(v)$ the bijection $\tau_v^p(u) \cong \tau(u \to v)^{-1}(p)$ holds, we have*

$$\begin{aligned} &\operatorname{sg}(T) - \operatorname{sg}(\tau) - \sum_{p\in\tau(x)} \operatorname{sg}(\tau_x^p) \\ &\quad\equiv \sum_{i<j\in\tau(x)} \sum_{q>p\in\mathrm{v}(t_x)} \Big(|\tau_x^i(q)| - \sum_{b\in\operatorname{inV}(q)} |\tau_x^i(b)|\Big)\Big(|\tau_x^j(p)| - \sum_{c\in\operatorname{inV}(p)} |\tau_x^j(c)|\Big) \pmod 2, \end{aligned}$$

where $I_t(t_v \mid v \in \mathrm{v}(t))$-tree T is constructed below (4.13) of [Lyu15].

Proof. Denote $r = t_x$ and use the simplified notation for r-trees $\tau^p = \tau_x^p$, $p \in \tau(x)$. Note that t, τ determine completely τ_v^p for $v \neq x$. In fact, $\tau_v^p(u) \cong \tau(u \to v)^{-1}(p)$ for $u \in \operatorname{Inpv}(t_v) \cong \operatorname{inV}(v)$, for $v \neq x$ as well as for $v = x$. We have

$$\mathrm{v}(I_t(t_v \mid v \in \mathrm{v}(t))) = (\mathrm{v}(t) - \{x\}) \sqcup \mathrm{v}(r).$$

Furthermore, for $v \in \overline{\mathrm{v}}(t) - \{x\}$

$$T(v) = \biguplus^{<}_{p\in\tau(v)} \tau_v^p(\operatorname{rv}(t_v)) = \biguplus^{<}_{p\in\tau(v)} \mathbf{1} \cong \tau(v).$$

For the same reasons $T(\operatorname{rv}(r)) \cong \tau(x)$. For $u \in \overline{\mathrm{v}}(r)$ we have $T(u) = \biguplus^{<}_{p\in\tau(x)} \tau^p(u)$. For an edge $(e : u \to v) \in \bar{E}(t)$ with $v \neq x$ we have

$$\begin{aligned} T(e) = \Big(T(u) = \tau(u) = \biguplus^{<}_{p\in\tau(v)} \tau(e)^{-1}(p) = \biguplus^{<}_{p\in\tau(v)} \tau_v^p(u) \xrightarrow[\sqcup\tau(e)]{\sqcup\tau_v^p(e)} \biguplus^{<}_{p\in\tau(v)} \tau_v^p(v) \\ = \tau(v) = T(v)\Big) = \tau(e). \end{aligned}$$

This holds also when $u = x$ if we substitute $T(\mathrm{rv}(r))$ instead of $T(x)$. An edge $(e : u \to v) \in \bar{E}(r)$ leads to

$$T(e) = \big(T(u) = \bigsqcup^{<}_{p\in\tau(x)} \tau^p(u) \xrightarrow{\sqcup\tau^p(e)} \bigsqcup^{<}_{p\in\tau(x)} \tau^p(v) = T(v)\big).$$

This completes the description of T.

The equation to prove is

$$\begin{aligned}
&\sum_{v\in\mathrm{v}(t)-\{x\}} \sum_{q<p\in\tau(v)} \sum_{e\lessdot g\in\mathrm{in}(v)} |\tau(e)^{-1}(p)|\cdot|\tau(g)^{-1}(q)| + \sum_{v\in\mathrm{v}(r)} \sum_{q<p\in\tau(x)} \sum_{w\lessdot y\in\mathrm{in}(v)} |\tau^p(w)|\cdot|\tau^q(y)| \\
&\quad + \sum_{v\in\mathrm{v}(r)} \sum_{j\in\tau(x)} \sum_{q<p\in\tau^j(v)} \sum_{e\lessdot g\in\mathrm{in}(v)} |\tau^j(e)^{-1}(p)|\cdot|\tau^j(g)^{-1}(q)| \\
&\quad + \sum_{v\in\mathrm{v}(t)-\{x\}} \sum_{p\in\tau(v)} (p-1)\big(\sum_{e\in\mathrm{in}(v)} |\tau(e)^{-1}(p)| - 1\big) \\
&\quad + \sum_{v\in\mathrm{v}(r)} \sum_{j\in\tau(x)} \sum_{p\in\tau^j(v)} \Big(p - 1 + \sum_{i<j}^{i\in\tau(x)} |\tau^i(v)|\Big)\Big(\sum_{e\in\mathrm{in}(v)} |\tau^j(e)^{-1}(p)| - 1\Big) \\
&\quad + \sum_{v\in\mathrm{v}(t)-\{x\}} \big(\sum_{u\in\mathrm{Inpv}_v^- t} |\tau(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)} |\tau(a)| - |\tau(v)|\big) \\
&\quad + \sum_{v\in\mathrm{v}(r)} \big(\sum_{u\in\mathrm{Inpv}_x^- t} |\tau(u)| + \sum_{u\in\mathrm{Inpv}_v^- r} \sum_{i\in\tau(x)} |\tau^i(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)} \sum_{j\in\tau(x)} |\tau^j(a)| - \sum_{j\in\tau(x)} |\tau^j(v)|\big) \\
&- \sum_{v\in\mathrm{v}(t)} \sum_{q<p\in\tau(v)} \sum_{e\lessdot g\in\mathrm{in}(v)} |\tau(e)^{-1}(p)|\cdot|\tau(g)^{-1}(q)| - \sum_{v\in\mathrm{v}(t)} \sum_{p\in\tau(v)} (p-1)\big(\sum_{e\in\mathrm{in}(v)} |\tau(e)^{-1}(p)| - 1\big) \\
&\quad - \sum_{v\in\mathrm{v}(t)} \big(\sum_{u\in\mathrm{Inpv}_v^- t} |\tau(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)} |\tau(a)| - |\tau(v)|\big) \\
&\quad - \sum_{j\in\tau(x)} \sum_{v\in\mathrm{v}(r)} \sum_{q<p\in\tau^j(v)} \sum_{e\lessdot g\in\mathrm{in}(v)} |\tau^j(e)^{-1}(p)|\cdot|\tau^j(g)^{-1}(q)| \\
&\quad - \sum_{j\in\tau(x)} \sum_{v\in\mathrm{v}(r)} \sum_{p\in\tau^j(v)} (p-1)\big(\sum_{e\in\mathrm{in}(v)} |\tau^j(e)^{-1}(p)| - 1\big) \\
&\quad - \sum_{j\in\tau(x)} \sum_{v\in\mathrm{v}(r)} \big(\sum_{u\in\mathrm{Inpv}_v^- r} |\tau^j(u)|\big)\cdot\big(\sum_{a\in\mathrm{inV}(v)} |\tau^j(a)| - |\tau^j(v)|\big) \\
&\equiv \sum_{i<j\in\tau(x)} \sum_{q>p\in\mathrm{v}(r)} \big(|\tau^i(q)| - \sum_{b\in\mathrm{inV}(q)} |\tau^i(b)|\big)\big(|\tau^j(p)| - \sum_{c\in\mathrm{inV}(p)} |\tau^j(c)|\big) \pmod 2.
\end{aligned}$$

Notice that two subsums proportional to $\sum_{u\in\mathrm{Inpv}_x^- t} |\tau(u)|$ cancel each other. In fact,

$$\sum_{v\in\mathrm{v}(r)} \big(\sum_{a\in\mathrm{inV}(v)} \sum_{j\in\tau(x)} |\tau^j(a)| - \sum_{j\in\tau(x)} |\tau^j(v)|\big) = \sum_{j\in\tau(x)} \sum_{v\in\mathrm{v}(r)} \big(\sum_{a\in\mathrm{inV}(v)} |\tau^j(a)| - |\tau^j(v)|\big)$$

$$= \sum_{j\in\tau(x)} \Big(\sum_{a\in\operatorname{Inpv}(r)} |\tau^j(a)| - 1 \Big) = \sum_{a\in\operatorname{inV}(x)} \sum_{j\in\tau(x)} |\tau(a\to x)^{-1}(j)| - |\tau(x)|$$

$$= \sum_{a\in\operatorname{inV}(x)} |\tau(a)| - |\tau(x)|.$$

Thus, after cancellations the considered equation becomes

$$\sum_{v\in\mathrm{v}(r)} \sum_{q<p\in\tau(x)} \sum_{w\lessdot y\in\operatorname{in}(v)} |\tau^p(w)|\cdot|\tau^q(y)| + \sum_{v\in\mathrm{v}(r)} \sum_{j\in\tau(x)} \Big(\sum_{i<j}^{i\in\tau(x)} |\tau^i(v)| \Big) \Big(\sum_{a\in\operatorname{inV}(v)} |\tau^j(a)| - |\tau^j(v)| \Big)$$
$$+ \sum_{v\in\mathrm{v}(r)} \Big(\sum_{u\in\operatorname{Inpv}_v^- r} \sum_{i\in\tau(x)} |\tau^i(u)| \Big) \cdot \Big(\sum_{a\in\operatorname{inV}(v)} \sum_{j\in\tau(x)} |\tau^j(a)| - \sum_{j\in\tau(x)} |\tau^j(v)| \Big)$$
$$- \sum_{q<p\in\tau(x)} \sum_{w\lhd y\in\operatorname{Inpv} r} |\tau^p(w)|\cdot|\tau^q(y)| - \sum_{p\in\tau(x)} (p-1) \Big(\sum_{a\in\operatorname{Inpv} r} |\tau^p(a)| - 1 \Big)$$
$$- \sum_{j\in\tau(x)} \sum_{v\in\mathrm{v}(r)} \Big(\sum_{u\in\operatorname{Inpv}_v^- r} |\tau^j(u)| \Big) \cdot \Big(\sum_{a\in\operatorname{inV}(v)} |\tau^j(a)| - |\tau^j(v)| \Big)$$
$$\equiv \sum_{i<j\in\tau(x)} \sum_{q>p\in\mathrm{v}(r)} \Big(|\tau^i(q)| - \sum_{b\in\operatorname{inV}(q)} |\tau^i(b)| \Big) \Big(|\tau^j(p)| - \sum_{c\in\operatorname{inV}(p)} |\tau^j(c)| \Big) \pmod 2.$$

Note that this expression does not depend on t. Explicit dependence on τ is only through the totally ordered set $\tau(x)$. Thus the both sides depend on a finite family of r-trees τ^j, $j\in\tau(x)$. When $\tau(x)=\varnothing$ the both sides vanish identically. Let us prove the equation by induction on $|\tau(x)|$. Assume that it holds true for $|\tau(x)|=k-1$. The difference of the equation containing $\tau^1, \dots, \tau^{k-1}, \tau^k$ and that containing $\tau^1, \dots, \tau^{k-1}$ depends on two kinds of variables:

$$n(v) = |\tau^k(v)|, \qquad z(v) = \sum_{1\leqslant j<k} |\tau^j(v)|, \qquad v\in\mathrm{v}(r).$$

The difference takes the form

$$\sum_{v\in\mathrm{v}(r)} \sum_{w\lessdot y\in\operatorname{in}(v)} n(w)\cdot z(y) + \sum_{v\in\mathrm{v}(r)} z(v) \Big(\sum_{a\in\operatorname{inV}(v)} n(a) - n(v) \Big)$$
$$+ \sum_{v\in\mathrm{v}(r)} \Big(\sum_{u\in\operatorname{Inpv}_v^- r} z(u) \Big) \cdot \Big(\sum_{a\in\operatorname{inV}(v)} n(a) - n(v) \Big) + \sum_{v\in\mathrm{v}(r)} \Big(\sum_{u\in\operatorname{Inpv}_v^- r} n(u) \Big) \cdot \Big(\sum_{a\in\operatorname{inV}(v)} z(a) - z(v) \Big)$$
$$- \sum_{w\lhd y\in\operatorname{Inpv} r} n(w)\cdot z(y) - (k-1) \Big(\sum_{a\in\operatorname{Inpv} r} n(a) - 1 \Big)$$
$$\equiv \sum_{q>p\in\mathrm{v}(r)} \Big(z(q) - \sum_{b\in\operatorname{inV}(q)} z(b) \Big) \Big(n(p) - \sum_{c\in\operatorname{inV}(p)} n(c) \Big) \pmod 2.$$

Notice that the value at root vertex of r is fixed:

$$n(\mathrm{rv}) = 1, \qquad z(\mathrm{rv}) = k-1.$$

Using this we may rewrite the equation once more. However we have already seen the result: up to change of notations this is nothing else, but equation (B.5), which is already proven. ■

B.5. Corollary. *For all $t \in \operatorname{tr}$, all t-trees τ, all sequences $(t_v) \in \prod_{v\in\mathrm{v}(t)} \operatorname{tr}|v|$, all collections of t_v-trees τ_v^p, $v \in \mathrm{v}(t)$, $p \in \tau(v)$, such that for all $u \in \operatorname{Inpv}(t_v) \cong \operatorname{inV}(v)$ the bijection $\tau_v^p(u) \cong \tau(u \to v)^{-1}(p)$ holds, we have*

$$\begin{aligned}\operatorname{sg}(T) - \operatorname{sg}(\tau) - \sum_{v\in\mathrm{v}(t)} \sum_{p\in\tau(v)} \operatorname{sg}(\tau_v^p) \\ \equiv \sum_{v\in\mathrm{v}(t)} \sum_{p<r\in\tau(v)} \sum_{q>s\in\mathrm{v}(t_v)} \big(|\tau_v^p(q)| - \sum_{b\in\operatorname{inV}(q)} |\tau_v^p(b)|\big)\big(|\tau_v^r(s)| - \sum_{c\in\operatorname{inV}(s)} |\tau_v^r(c)|\big) \pmod 2,\end{aligned}$$

where $I_t(t_v \mid v \in \mathrm{v}(t))$-tree T is constructed below (4.13) of [Lyu15].

Proof. Enumerate internal vertices of t as $\{v_1, \dots, v_k\} = \mathrm{v}(t)$. Consider the sequence of trees constructed in the proof of [Lyu15, Proposition 4.3]: $t^0 = t$, $t^1 = I(t^0; (t_v^1)_{v\in\mathrm{v}(t^0)}) = I(t; t_{v_1}, \text{corollas})$, $t^2 = I(t^1; (t_v^2)_{v\in\mathrm{v}(t^1)}) = I(t^1; t_{v_2}, \text{corollas})$, $\dots$, $t^k = I(t^{k-1}; (t_v^k)_{v\in\mathrm{v}(t^{k-1})}) = I(t^{k-1}; t_{v_k}, \text{corollas}) = I(t; (t_{v_i})_{i=1}^k)$. Accompany them with t_v^1-trees

$$({}^1\tau_v^p)_{v\in\mathrm{v}(t^0)}^{p\in\tau(v)} = \big((\tau_{v_1}^p)^{p\in\tau(v_1)}, \text{corollas}\big)$$

agreeing with τ, t_v^2-trees $({}^2\tau_v^p)_{v\in\mathrm{v}(t^1)}^{p\in{}^1\tau(v)} = \big((\tau_{v_2}^p)^{p\in\tau(v_2)}, \text{corollas}\big)$ agreeing with ${}^1\tau$ constructed from $({}^1\tau_v^p)$ as in Remark 4.11 of [Lyu15], $\dots$, t_v^k-trees $({}^k\tau_v^p)_{v\in\mathrm{v}(t^{k-1})}^{p\in{}^{k-1}\tau(v)} = \big((\tau_{v_k}^p)^{p\in\tau(v_k)}, \text{corollas}\big)$ agreeing with ${}^{k-1}\tau$ constructed from $({}^{k-1}\tau_v^p)$. We construct also ${}^k\tau$ from $({}^k\tau_v^p)$. Furthermore, ${}^1\tilde\tau = I\big(\tilde\tau; ({}^1\tilde\tau_v^p)_{v\in\mathrm{v}(t)}^{p\in\tau(v)}\big)$, ${}^2\tilde\tau = I\big({}^1\tilde\tau; ({}^2\tilde\tau_v^p)_{v\in\mathrm{v}(t^1)}^{p\in{}^1\tau(v)}\big)$, $\dots$, ${}^k\tilde\tau = I\big({}^{k-1}\tilde\tau; ({}^k\tilde\tau_v^p)_{v\in\mathrm{v}(t^{k-1})}^{p\in{}^{k-1}\tau(v)}\big) = I\big(\tilde\tau; (\tilde\tau_v^p)_{v\in\mathrm{v}(t)}^{p\in\tau(v)}\big) = \tilde T$. The last but one equation follows from the observation that almost all ${}^i\tilde\tau_v^p$ are corollas, except the case of $v = v_i$. The isomorphism ${}^k\tilde\tau \cong \tilde T$ is over $\theta = I(t; (t_v))$. Notice that total orderings on fibres of maps ${}^k\tilde\tau \to \theta$ and $\tilde T \to \theta$ agree. In fact, elements $(u_0, p_0, u_1, p_1, \dots, u_k, p_k) < (u_0, p_0', u_1, p_1', \dots, u_k, p_k')$ of $\overline{\mathrm{v}}({}^k\tilde\tau)$ over $(u_0, u_1, \dots, u_k) \in \overline{\mathrm{v}}(t_k)$ are related by this inequality iff $p_0 < p_0'$ or $(p_0 = p_0', p_1 < p_1')$ $\dots$ or $(p_0 = p_0', p_1 = p_1', \dots, p_k < p_k')$. However, only two possibilities among them may occur: $p_0 < p_0' \in \tau(u_0)$ and $(p_0 = p_0', p_1 = p_1', \dots, p_i < p_i')$, where $u_0 = v_i$. These coincide with two possibilities for inequality between two points of a fibre of $\tilde T \to \theta$. Hence, by [Lyu15, Remark 4.11] ${}^k\tau = T$ as a θ-tree.

The claimed identity is the sum of k identities proven in Lemma B.4 for ${}^1\tau$, ${}^2\tau$, $\dots$, ${}^k\tau$. ■

B.6. Remark. The scheme for deducing Corollary B.5 from Lemma B.4 is unveiled by [Lyu15, Proposition 4.3].

Now using Corollary B.5 we prove the fourth claim. Fix t, $(t_v)_{v\in\mathrm{v}(t)}$ and denote $I_t(t_v \mid v \in \mathrm{v}(t))$ by θ. Recall that θ-trees T are in bijection with admissible pairs $\big(\tau, (\tau_v^p)_{v\in\mathrm{v}(t)}^{p\in\tau(v)}\big)$.

Choose one of them. For $j \in \mathbb{N}^{\operatorname{Inp} t}$ we have

$$\Big\langle H_{\operatorname{Inp} t}(j) \xrightarrow{\Delta^{\mathsf{G}}(t)} \coprod_{\forall a\in \operatorname{Inpv} t\, |\tau(a)|=j^a}^{t\text{-tree } \tau} \bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in \operatorname{in}(v)}\big) \xrightarrow{\coprod\otimes\otimes\Delta^{\mathsf{G}}(t_v)}$$

$$\coprod_{\forall a\in \operatorname{Inpv} t\, |\tau(a)|=j^a}^{t\text{-tree } \tau} \bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \coprod_{\substack{\forall c\in \operatorname{Inpv} t_v=\operatorname{inV}(v)\\ \tau_v^p(c)=\tau(c\to v)^{-1}(p)}}^{t_v\text{-trees } \tau_v^p} \bigotimes^{u\in\mathrm{v}(t_v)} \bigotimes^{q\in\tau_v^p(u)} H_{|u|}\big(|\tau_v^p(y)^{-1}(q)|_{y\in \operatorname{in}(u)}\big)$$

$$\xrightarrow{\cong} \coprod_{\forall a\in \operatorname{Inpv} \theta\, |T(a)|=j^a}^{\theta\text{-tree } T} \bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{u\in\mathrm{v}(t_v)} \bigotimes^{p\in\tau(v)} \bigotimes^{q\in\tau_v^p(u)} H_{|u|}\big(|\tau_v^p(y)^{-1}(q)|_{y\in \operatorname{in}(u)}\big)$$

$$\xrightarrow{\operatorname{pr}_T} \bigotimes^{w\in\mathrm{v}(\theta)} \bigotimes^{r\in T(w)} H_{|w|}\big(|T(x)^{-1}(r)|_{x\in\operatorname{in}(w)}\big)\Big\rangle$$

$$= (-1)^{\operatorname{sg}(\tau)}\Big\langle H_{\operatorname{Inp} t}(j) \xrightarrow{h^{-1}} G_{\operatorname{Inp} t}(j) \xrightarrow{\Delta^{\mathsf{G}}(t)\operatorname{pr}_\tau} \bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} G_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in \operatorname{in}(v)}\big) \xrightarrow{\otimes\otimes h}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} H_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in \operatorname{in}(v)}\big) \xrightarrow{\otimes\otimes(\Delta^{\mathsf{G}}(t_v)\operatorname{pr}_{\tau_v^p})}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \bigotimes^{u\in\mathrm{v}(t_v)} \bigotimes^{q\in\tau_v^p(u)} H_{|u|}\big(|\tau_v^p(y)^{-1}(q)|_{y\in \operatorname{in}(u)}\big) \xrightarrow{\cong}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{u\in\mathrm{v}(t_v)} \bigotimes^{p\in\tau(v)} \bigotimes^{q\in\tau_v^p(u)} H_{|u|}\big(|\tau_v^p(y)^{-1}(q)|_{y\in \operatorname{in}(u)}\big) = \bigotimes^{w\in\mathrm{v}(\theta)} \bigotimes^{r\in T(w)} H_{|w|}\big(|T(x)^{-1}(r)|_{x\in\operatorname{in}(w)}\big)\Big\rangle$$

$$= (-1)^{\operatorname{sg}(\tau)+\sum_{v\in\mathrm{v}(t)}\sum_{p\in\tau(v)}\operatorname{sg}(\tau_v^p)}\Big\langle H_{\operatorname{Inp} t}(j) \xrightarrow{h^{-1}} G_{\operatorname{Inp} t}(j) \xrightarrow{\Delta^{\mathsf{G}}(t)\operatorname{pr}_\tau}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} G_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in \operatorname{in}(v)}\big) \xrightarrow{\otimes\otimes(\Delta^{\mathsf{G}}(t_v)\operatorname{pr}_{\tau_v^p})}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \bigotimes^{u\in\mathrm{v}(t_v)} \bigotimes^{q\in\tau_v^p(u)} G_{|u|}\big(|\tau_v^p(y)^{-1}(q)|_{y\in \operatorname{in}(u)}\big) \xrightarrow{\otimes\otimes\otimes\otimes h}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \bigotimes^{u\in\mathrm{v}(t_v)} \bigotimes^{q\in\tau_v^p(u)} H_{|u|}\big(|\tau_v^p(y)^{-1}(q)|_{y\in \operatorname{in}(u)}\big) \xrightarrow{\cong}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{u\in\mathrm{v}(t_v)} \bigotimes^{p\in\tau(v)} \bigotimes^{q\in\tau_v^p(u)} H_{|u|}\big(|\tau_v^p(y)^{-1}(q)|_{y\in \operatorname{in}(u)}\big) = \bigotimes^{w\in\mathrm{v}(\theta)} \bigotimes^{r\in T(w)} H_{|w|}\big(|T(x)^{-1}(r)|_{x\in\operatorname{in}(w)}\big)\Big\rangle$$

$$= (-1)^{\operatorname{sg}(\tau)+\sum_{v\in\mathrm{v}(t)}\sum_{p\in\tau(v)}\operatorname{sg}(\tau_v^p)}$$

$$(-1)^{\sum_{v\in\mathrm{v}(t)}\sum_{p<r\in\tau(v)}\sum_{q>s\in\mathrm{v}(t)}\big(|\tau_v^p(q)|-\sum_{b\in\operatorname{inV}(q)}|\tau_v^p(b)|\big)\big(|\tau_v^r(s)|-\sum_{c\in\operatorname{inV}(s)}|\tau_v^r(c)|\big)}\Big\langle H_{\operatorname{Inp} t}(j) \xrightarrow{h^{-1}}$$

$$G_{\operatorname{Inp} t}(j) \xrightarrow{\Delta^{\mathsf{G}}(t)\operatorname{pr}_\tau} \bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} G_{|v|}\big(|\tau(e)^{-1}(p)|_{e\in \operatorname{in}(v)}\big) \xrightarrow{\otimes\otimes(\Delta^{\mathsf{G}}(t_v)\operatorname{pr}_{\tau_v^p})}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{p\in\tau(v)} \bigotimes^{u\in\mathrm{v}(t_v)} \bigotimes^{q\in\tau_v^p(u)} G_{|u|}\big(|\tau_v^p(y)^{-1}(q)|_{y\in \operatorname{in}(u)}\big) \xrightarrow{\cong}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{u\in \mathrm{v}(t_v)} \bigotimes^{p\in \tau(v)} \bigotimes^{q\in \tau^p_v(u)} G_{|u|}\big(|\tau^p_v(y)^{-1}(q)|_{y\in \mathrm{in}(u)}\big) \xrightarrow{\otimes\otimes\otimes\otimes h}$$

$$\bigotimes^{v\in \mathrm{v}(t)} \bigotimes^{u\in \mathrm{v}(t_v)} \bigotimes^{p\in \tau(v)} \bigotimes^{q\in \tau^p_v(u)} H_{|u|}\big(|\tau^p_v(y)^{-1}(q)|_{y\in \mathrm{in}(u)}\big) = \bigotimes^{w\in \mathrm{v}(\theta)} \bigotimes^{r\in T(w)} H_{|w|}\big(|T(x)^{-1}(r)|_{x\in \mathrm{in}(w)}\big)\Big\rangle$$

$$= (-1)^{\mathrm{sg}(T)}\Big\langle H_{\mathrm{Inp}\,t}(j) \xrightarrow{h^{-1}} G_{\mathrm{Inp}\,t}(j) \xrightarrow{\Delta^{\mathsf{G}}(\theta)\,\mathrm{pr}_T} \bigotimes^{w\in \mathrm{v}(\theta)} \bigotimes^{r\in T(w)} G_{|w|}\big(|T(x)^{-1}(r)|_{x\in \mathrm{in}(w)}\big)$$

$$\xrightarrow{\otimes\otimes h} \bigotimes^{w\in \mathrm{v}(\theta)} \bigotimes^{r\in T(w)} H_{|w|}\big(|T(x)^{-1}(r)|_{x\in \mathrm{in}(w)}\big)\Big\rangle$$

$$= \Big\langle H_{\mathrm{Inp}\,t}(j) \xrightarrow{\Delta^{\mathsf{G}}(\theta)} \coprod_{\forall a\in \mathrm{Inpv}\,\theta\, |T(a)|=j^a}^{\theta\text{-tree } T} \bigotimes^{w\in \mathrm{v}(\theta)} \bigotimes^{r\in T(w)} H_{|w|}\big(|T(x)^{-1}(r)|_{x\in \mathrm{in}(w)}\big)$$

$$\xrightarrow{\mathrm{pr}_T} \bigotimes^{w\in \mathrm{v}(\theta)} \bigotimes^{r\in T(w)} H_{|w|}\big(|T(x)^{-1}(r)|_{x\in \mathrm{in}(w)}\big)\Big\rangle.$$

The theorem is proven. ■

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