ON THE 3-REPRESENTATIONS OF GROUPS AND THE 2-CATEGORICAL TRACES

WEI WANG

Abstract. To 2-categorify the theory of group representations, we introduce the notions of the 3-representation of a group in a strict 3-category and the strict 2-categorical action of a group on a strict 2-category. We also 2-categorify the concept of the trace by introducing the 2-categorical trace of a 1-endomorphism in a strict 3-category. For a 3-representation \( \rho \) of a group \( G \) and an element \( f \) of \( G \), the 2-categorical trace \( \text{Tr}_2 \rho_f \) is a category. Moreover, the centralizer of \( f \) in \( G \) acts categorically on this 2-categorical trace. We construct the induced strict 2-categorical action of a finite group, and show that the 2-categorical trace \( \text{Tr}_2 \) takes an induced strict 2-categorical action into an induced categorical action of the initia groupoid. As a corollary, we get the 3-character formula of the induced strict 2-categorical action.

Contents

1 Introduction 1999
2 The 3-representations of groups 2002
3 The 2-categorical traces of 3-representations 2016
4 The induced strict 2-categorical action on the induced 2-category 2026
5 The 3-character of the induced strict 2-categorical action 2029
6 The categorical action of the centralizer of \( f \) on \( \text{Tr}_2 \rho_f \) 2034

1. Introduction

The notion of a group acting on a category goes back to Grothendieck’s Tohoku paper [15]. Recently Ganter, Kapranov [13] and Bartlett [5] categorified the concept of the trace of a linear transformation by introducing the notion of the category trace. This is a set associated to any endofunctor on a small category, and is a vector space in the linear case. Moreover, a functor commuting with the endofunctor defines a linear transformation on this vector space, whose ordinary trace defines a joint trace. This allowed these authors to define 2-characters. When a group acts on a \( k \)-linear category, the joint trace of a commuting pair of group elements is the 2-character of the categorical 

Supported by National Nature Science Foundation in China (No. 11171298; No. 11571305)
Received by the editors 2015-02-14 and, in revised form, 2015-12-23.
Transmitted by Larry Breen. Published on 2015-12-31.
2010 Mathematics Subject Classification: 18D05; 18D99; 20J99; 20C99.
Key words and phrases: the 3-representation of a group in a 3-category; the 2-categorical trace; the 3-cocycle condition; the induced strict 2-categorical actions; the 3-character; 2-categorification.
© Wei Wang, 2015. Permission to copy for private use granted.

1999
action. This is an analogue of the character of the representation of a group on a vector space and is a 2-class function. In general, an \( n \)-class function is a function defined on \( n \)-tuples of commuting elements of a group and invariant under simultaneous conjugation. Such functions already appear in equivariant Morava \( E \)-theory \([16]\). The theory of 2-representations was developed further in \([6]\) \([10]\) \([11]\) \([12]\) \([14]\) \([23]\) \([26]\) etc..

During the past two decades an active direction of research has been the categorification of some algebraic, geometric or analytic concepts. For example, 2-vector spaces, 2-bundles (gerbes), 2-connections and 2-curvatures. All involve 2-categorical constructions and have various applications, such as a geometric definition of elliptic cohomology \([1]\), 2-gauge theory \([3]\) \([4]\) and the 2-dimensional Langlands correspondence \([17]\) \([22]\). It is believed that higher categorification is necessary for many geometric and physical applications. 3-categorical constructions already appear in the theory of 2-gerbes (3-bundles) \([7]\) \([8]\) and in 3-gauge theory \([21]\) \([24]\) \([27]\), which involves more general Gray-categories. The purpose of this paper is to 2-categorify the theory of group representations and characters by introducing the notions of the 3-representation of a group in a 3-category, the strict 2-categorical action of a group on a 2-category and the 2-categorical trace. The problem of investigating representations of groups in higher categories has already been mentioned in \([13]\).

A geometric motivation for considering higher representations of groups is as follows. Suppose that \( G \) is a Lie group and that \( H \) is a Lie subgroup. Let \( V \) be a finite dimensional representation of \( H \). We can construct a homogeneous vector bundle \( G \times_H V \) over the homogeneous space \( G/H \) as \( G \times V \) modulo the equivalent relation

\[
(g, v) \sim (gh, h^{-1}.v) \quad \text{for} \quad g \in G, h \in H, v \in V.
\]

The space of sections of this bundle is exactly the space \( \text{Ind}_H^G V \) of the induced representation. When \( V \) is a 2- or 3-representation of \( H \), a similar construction will give us a homogeneous 2- or 3-bundle over the homogeneous space \( G/H \). This will provide us good examples of higher bundles in higher differential geometry and higher gauge theory. But for a higher representation \( \pi \) of the Lie group \( H \), the functors \( \pi(h) \) usually depend on \( h \in H \) “discontinuously”. Thus it is not easy to describe the space of “sections” of the resulting higher bundles. However, when \( G \) and \( H \) are finite, \( G/H \) is discrete, and so we have a clear picture. This is why we only consider 3-representations of a finite group in this paper.

For simplicity, we only consider strict 2- and 3-categories. A 3-representation of a group \( G \) in a 3-category is given by a 1-isomorphism for each element of \( G \), a 2-isomorphism for each pair of elements of \( G \), and a 3-isomorphism for each triple of elements of \( G \). These 3-isomorphisms must satisfy the 3-cocycle condition. This condition has a simple geometric interpretation: the composition of 3-isomorphisms corresponding to 5 tetrahedrons in the boundary of a 4-simplex is equal to the identity 3-arrow. Given a 2-category \( \mathcal{V} \), a strict 2-categorical action of \( G \) on \( \mathcal{V} \) is given by an endofunctor of \( \mathcal{V} \) for each element of \( G \), a pseudonatural transformation between functors for each pair of elements of \( G \), and a modification for each triple of elements of \( G \). Details are given in Section 2.3-2.4.
Recall that given a 2-representation $\varrho$ of a finite group $G$ in a 2-category $V$ and an element $f$ of $G$, we have a 1-isomorphism $\varrho_f : x \to x$, where $x$ is an object of $V$ that $G$ acts on. In [5] [13], the authors introduced the notion of the categorical trace $\text{Tr} \varrho_f$. This is the set of 2-arrows in $V$, whose 1-source is the unit arrow $1_x$ and whose 1-target is $\varrho_f$. The centralizer of $f$ in $G$ acts on this set naturally.

In our case, given a 3-representation $\rho$ of $G$ in a 3-category $C$ and an element $f$ of $G$, we have a 1-isomorphism $\rho_f : x \to x$ in $C$. The 2-categorical trace $\text{Tr}_2 \rho_f$ is a category. Its objects are 2-arrows with 1-source the unit arrow $1_x$ and 1-target $\rho_f$, and its morphisms are 3-isomorphisms between such 2-isomorphisms:

Moreover, the centralizer of $f$ in $G$, denoted by $C_G(f)$, acts categorically on the 2-categorical trace $\text{Tr}_2 \rho_f$ in the following sense. We can define an invertible functor $\psi_g$ acting on $\text{Tr}_2 \rho_f$ for each $g \in C_G(f)$, and for any $h, g \in C_G(f)$, define a natural isomorphism $\Gamma_{h,g} : \psi_h \circ \psi_g \Rightarrow \psi_{hg}$ between such functors on the category $\text{Tr}_2 \rho_f$. This construction is given in Section 3. To prove the action to be categorical, we have to show the associativity in the definition of categorical action, i.e.,

$$\Gamma_{k,hg} \# (\psi_k \circ \Gamma_{h,g}) = \Gamma_{kh,g} \# (\Gamma_k \circ \psi_g) : \psi_k \circ \psi_h \circ \psi_g \Rightarrow \psi_{khg},$$

(1)

for any $k, h, g \in C_G(f)$, where $\#$ is the composition of natural transformations between functors on the category $\text{Tr}_2 \rho_f$. This is the most difficult and technical part of this paper. By applying the 3-cocycle identity (15) repeatedly, we prove in Section 6 that

$$\{\psi_g, \Gamma_{h,g}\}_{g,h \in C_G(f)}$$

is a categorical action of the centralizer $C_G(f)$ on the category $\text{Tr}_2 \rho_f$.

An easy and interesting example of 3-representations is the 1-dimensional one, which is given by a 3-cocycle on a finite group $G$. A 3-cocycle is a function $c : G \times G \times G \to k^\ast$ such that

$$c(g_3, g_2, g_1)c(g_4, g_3g_2, g_1)c(g_4, g_3, g_2) = c(g_4, g_3, g_2g_1)c(g_4g_3, g_2, g_1)$$

(2)

for any $g_4, \ldots, g_1 \in G$. Here $k$ is a field of characteristic 0. Such a 3-cocycle gives us a strict action of $G$ on a 2-category with only one object, one 1-arrow and the set of 2-arrows isomorphic to $k^\ast$. For an element $f$ of $G$, its 2-categorical trace $\text{Tr}_2 \rho_f$ is a category.
with only one object and the set of 1-arrows isomorphic to $k^*$. For any $h$ and $g$ in the centralizer $C_G(f)$, we can construct an element $\Gamma_{h,g}$ from the 3-cocycle $c$ in (2) such that $\Gamma_{\ast,\ast}$ is a 2-cocycle on the centralizer. This can be proved quite easily and elementarily by using the condition (2) for 3-cocycles repeatedly in Section 6.1. This corresponds step by step to the proof of the general case carried out in Section 6.4. It can be viewed as a simple model of the proof of (1). The difficulty in the general case is that we have to handle diagrams, while in the 1-dimensional case we only need to handle element of the field $k$.

Suppose that $C$ is a $k$-linear 3-category. Then $T_{2}\rho_f$ is also a $k$-linear category. If $k, g$ and $f$ are pairwise commutative, then $\psi_k$ and $\psi_g$ are $k$-linear endofunctors acting on $T_{2}\rho_f$. We define the 3-character of a 3-representation $\rho$ to be

$$\chi_{\rho}(f, g, k) := \text{the joint trace of functors } \psi_k \text{ and } \psi_g \text{ on } T_{2}\rho_f.$$ 

It is the trace of the linear transformation induced by the functor $\psi_k$ on the $k$-vector space $\text{Tr}\psi_g$.

Suppose that a subgroup $H$ of a finite group $G$ acts strictly 2-categorically on a 2-category $V$. In Section 4, we define the induced 2-category $\text{Ind}^G_H(V)$ and strict 2-categorical action of $G$ on it. In Section 5, we calculate the 2-categorical trace of the induced strict 2-categorical action as

$$T_{2}(\text{Ind}^G_H(\rho)) = \text{Ind}^{\Lambda(G)}_{\Lambda(H)}T_{2}(\rho),$$

where $\Lambda(H)$ and $\Lambda(G)$ are initia groupoids associated to groups $H$ and $G$, respectively. As a corollary, we derive the 3-character of the induced strict 2-categorical action, which coincides with the formula in [16] for $n$-characters when $n = 3$. These results are the generalization of induced categorical action and the 2-character formula in [13].

It would be interesting to investigate the $m$-representation of a group in an $m$-category, the $m$-cocycle condition and $(m - 1)$-categorical trace for a positive integer $m > 3$.

I would like to thank the anonymous referee for his/her many inspiring and valuable suggestions.

2. The 3-representations of groups

2.1. Strict 2-categories. A 2-category is a category enriched over the category of all small categories. In particular, a strict 2-category $C$ consists of collections $C_0$ of objects, $C_1$ of arrows and $C_2$ of 2-arrows, together with

- functions $s_n, t_n : C_i \to C_n$ for all $0 \leq n < i \leq 2$, called $n$-source and $n$-target,
- functions $#_n : C_{n+1} \times C_{n+1} \to C_{n+1}$ for all $n = 0, 1$, called vertical composition,
- a function $#_0 : C_2 \times C_2 \to C_2$, called the horizontal composition,
- a function $1_* : C_i \to C_{i+1}$ for $i = 0, 1$, called the identity.

For a 1-arrow $x \xrightarrow{A} y$, its 0-source and 0-target are $x$ and $y$, respectively. For
a 2-arrow $x \xrightarrow{A} y$ in $C_2$, its 1-source and 1-target are $x \xrightarrow{A} y$ and $x \xrightarrow{B} y$, respectively, while its 0-source and 0-target are $x$ and $y$, respectively.

Two 1-arrows $A$ and $A'$ are called 0-composable if the 0-target of $A$ coincides with the 0-source of $A'$. In this case, their vertical composition is $A#_0A'$: $x \xrightarrow{A} y \xrightarrow{A'} z$. Two 2-arrows $\phi$ and $\psi$ are called 1-composable if the 1-target of $\phi$ coincide with the 1-source of $\psi$. In this case, their vertical composition $\phi#_1\psi$ is

\[
\begin{array}{c}
\phi \\
\bigcirc \\
\psi \\
\end{array}
\]

where $A = s_1(\phi)$, $B = t_1(\phi) = s_1(\psi)$, $C = t_1(\psi)$, $x = s_0(\phi) = s_0(\psi)$, $y = t_0(\phi) = t_0(\psi)$. In general, two arrows are composable if the target matching condition is satisfied.

Two 2-arrows $\phi$ and $\psi$ are called horizontally composable (0-composable) if the 0-target of $\phi$ coincides with the 0-source of $\psi$. In this case, their horizontal composition $\phi#_0\psi$ is

\[
\begin{array}{c}
A \\
\bigcirc \\
B \\
\end{array}
\]

In particular, when $\phi = 1_A$ we call $1_A#_0\psi$ whiskering from left by 1-arrow $A$, and denote it by

\[
A#_0\psi: \quad x \xrightarrow{A} y \xrightarrow{\psi} z,
\]

Similarly, we define whiskering from right by a 1-arrow.

The identities satisfy

\[
1_x#_0 A = A = A#_0 1_y, \quad \text{for any } 1\text{-arrow } A: x \rightarrow y; \quad 1_A#_1 \phi = \phi = \phi#_1 1_B, \quad \text{for any } 2\text{-arrow } \phi: A \rightarrow B.
\] (4)

The composition $\#_p$ satisfies the associativity

\[
(\phi#_p\psi)#_p\omega = \phi#_p(\psi#_p\omega),
\] (5)

if the corresponding arrows are $p$-composable, for $p = 0$ or 1.

The horizontal composition satisfies the interchange law:

\[
(A#_0\psi)#_1(\phi#_0 D) = \phi#_0\psi = (\phi#_0 B)#_1(C#_0\psi).
\] (6)
Namely,

\[
\begin{array}{c}
A \xrightarrow{\phi} B \\
\downarrow \gamma \\
C \xrightarrow{\psi} D
\end{array}
\quad \quad \quad \quad \quad \quad 
\begin{array}{c}
A \xrightarrow{\phi} B \\
\downarrow \gamma \\
C \xrightarrow{\psi} D
\end{array}
\]

the vertical composition of left two 2-arrows coincides with the vertical composition of right two 2-arrows. They are both equal to the horizontal composition \(\phi \#_0 \psi\). The interchange law allows us to change the order of compositions of 2-arrows, up to whiskerings. This is essentially the paste theorem for 2-categories (cf. §2.13 in [18]).

The interchange law (6) is a special case of the following more general compatibility condition for different compositions. If \((\beta, \beta'), (\gamma, \gamma') \in C_k \times C_k\) are \(p\)-composable and \((\beta, \gamma), (\beta', \gamma') \in C_k \times C_k\) are \(q\)-composable, \(p, q = 0, 1\), then we have

\[
(\beta \#_p \beta') \#_q (\gamma \#_p \gamma') = (\beta \#_q \gamma) \#_p (\beta' \#_q \gamma'). \tag{7}
\]

The left-hand side of the interchange law (6) is exactly the compatibility condition (7) with \(p = 0, q = 1, \beta = 1_A, \beta' = \psi, \gamma = \phi, \gamma' = 1_B\), by using the property (4) of identities. (4) (5) and (7) are the main axioms that a strict 2-category satisfies.

A 1-arrow \(A : x \to y\) is called invertible or a 1-isomorphism, if there exists another 1-arrow \(B : y \to x\) such that \(1_x = A \#_0 B\) and \(B \#_0 A = 1_y\). A strict 2-category in which every 1-arrow is invertible is called a strict 2-groupoid. A 2-arrow \(\varphi : A \Rightarrow B\) is called invertible or a 2-isomorphism if there exists another 2-arrow \(\psi : B \Rightarrow A\) such that \(\psi \#_1 \varphi = 1_B\) and \(\varphi \#_1 \psi = 1_A\). \(\psi\) is uniquely determined and called the inverse of \(\varphi\).

Let \(\mathcal{S}\) and \(\mathcal{T}\) be two strict 2-categories. A (strict) 2-functor \(F : \mathcal{S} \to \mathcal{T}\) is an assignment of a 2-arrow

\[
\begin{array}{c}
F(X) \\
\downarrow F(\varphi) \\
F(Y)
\end{array}
\]

to each 2-arrow \(x \xrightarrow{\varphi} y\) such that \(F\) preserves compositions \#_p and identities. More explicitly, we have

- \(F(\varphi \#_1 \psi) = F(\varphi) \#_1 F(\psi)\) and \(F(1_f) = 1_{F(f)}\) for all composable 2-arrows \(\varphi\) and \(\psi\) and any 0- or 1-arrow \(f\);
- \(F(g) \#_0 F(f) = F(g \#_0 f)\) for all composable 1-arrows \(g\) and \(f\), and \(F(\varphi) \#_0 F(\psi) = F(\varphi \#_0 \psi)\) for all horizontally composable 2-arrows \(\varphi\) and \(\psi\).
Let $F_1$ and $F_2$ be two 2-functors from $S$ to $T$. A \textit{pseudonatural transformation} $\rho : F_1 \to F_2$ is an assignment of a 1-arrow $\rho(X)$ in $T$ to each object $X$ in $S$ and a 2-isomorphism $\rho(f)$ in $T$ to each 1-arrow $f : X \to Y$ in $S$ such that they satisfy two axioms

- The composition of 1-arrows in $S$:

- The compatibility with 2-arrows:

for any 2-arrow $\varphi : f \Rightarrow g$.

Let $F_1, F_2 : S \to T$ be two strict 2-functors and let $\rho_1, \rho_2 : F_1 \to F_2$ be pseudonatural transformations. A \textit{modification} $\Phi : \rho_1 \Longrightarrow \rho_2$ is an assignment of a 2-arrow
2006

WEI WANG

in $\mathcal{T}$ to any object $X$ in $\mathcal{S}$, which satisfies

\[
\begin{array}{ccc}
F_1(X) & \xrightarrow{F_1(f)} & F_1(Y) \\
\rho_2(X) & \xleftarrow{\Phi(X)} & \rho_1(X) \\
F_2(X) & \xrightarrow{F_2(f)} & F_2(Y) \\
\rho_1(Y) & \xrightarrow{\rho_2(Y)} & \rho_1(Y)
\end{array}
\]

\[
\begin{array}{ccc}
F_1(X) & \xrightarrow{F_1(f)} & F_1(Y) \\
\rho_2(X) & \xleftarrow{\Phi(Y)} & \rho_2(X) \\
F_2(X) & \xrightarrow{F_2(f)} & F_2(Y) \\
\rho_1(Y) & \xrightarrow{\rho_2(Y)} & \rho_1(Y)
\end{array}
\]

2.2. STRICT 3-CATEGORIES. A 3-category is a category enriched over the category of all small strict 2-categories. In particular, a strict 3-category $\mathcal{C}$ consists of collections $\mathcal{C}_0$ of objects, $\mathcal{C}_1$ of 1-arrows, $\mathcal{C}_2$ of 2-arrows, and $\mathcal{C}_3$ of 3-arrows, together with

- functions $s_n, t_n : \mathcal{C}_i \to \mathcal{C}_n$ for all $0 \leq n < i \leq 3$, called $n$-source and $n$-target,
- functions $\#_n : \mathcal{C}_{n+1} \times \mathcal{C}_{n+1} \to \mathcal{C}_{n+1}$ for all $n = 0, 1, 2$, called vertical composition,
- a function $\#_p : \mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i$, $p + 2 \leq i$, called the horizontal composition,
- a function $1_* : \mathcal{C}_i \to \mathcal{C}_{i+1}$ for $i = 0, 1$, called identity.

For a 3-arrow $\varphi : x \xrightarrow{\gamma} y$, its 2-source and 2-target are $\gamma$ and $\gamma'$ respectively.

The 3-arrows $\varphi$ and $\varphi' : x \xrightarrow{\gamma'} y$ are 2-composable, and their composition $\varphi \#_2 \varphi'$ is

\[
\begin{array}{ccc}
x & \xrightarrow{\gamma} & y \\
\varphi & \xrightarrow{\varphi'} & y
\end{array}
\]

In a strict 3-category, 0-, 1- and 2-arrows behave as in a 2-category. We call two 3-arrows $\varphi$ and $\psi$ horizontally $p$-composable if the $p$-target of $\varphi$ coincides with the $p$-source of $\psi$, $p = 0, 1$, and denote their horizontal composition as $\varphi \#_p \psi$.

For a 2-arrow $\delta$, 3-arrows $1_\delta$ and $\varphi$ are horizontally 1-composable if the 1-target of $\delta$ coincides with the 1-source of $\varphi$. In this case,

\[
\delta \#_1 \varphi := 1_\delta \#_1 \varphi := x \xrightarrow{\gamma} y,
\]

is called whiskering from above by a 2-arrow $\delta$. It is similar to define whiskering from
There is also whiskering from left (or right) by a 1-arrow $A\#_0\varphi := 1_A\#_0\varphi$ (or $\varphi\#_0B$):

\begin{align*}
  z & \xrightarrow{A} x \\
  & \xrightarrow{f} y \\
  & \xleftarrow{g'}
\end{align*}

\begin{align*}
  & \downarrow f \\
  & \downarrow g \\
  & \uparrow \gamma \\
  & \uparrow \gamma'
\end{align*}

\begin{align*}
  & \xrightarrow{\eta} \downarrow \eta \\
  & \xleftarrow{f'} \uparrow f' \\
  & \xrightarrow{\gamma} \downarrow \gamma
\end{align*}

The properties of identities, the associativity and the compatibility condition for different compositions, similar to (4) (5) and (7) for a strict 2-category, also hold in a strict 3-category. See page 8 of [19] for an explicit definition of a strict $m$-category.

A strict 3-functor (or a functor) is a map preserving compositions and identities.

2.3. Remark. In a strict 3-category, the interchange law (6) for the horizontal composition of 2-arrows is also satisfied. But in general, a 3-category does not satisfy the interchange law. Gray-categories are the greatest possible semi-strictification of 3-categories, and appear naturally in 3-gauge theory [27]. The 3-representation in a Gray-category is more natural, but is much more complicated. So we restrict to the 3-representation in strict 3-categories in this paper.

In a strict 3-category $C$, a 1-arrow $B : x \to y$ is called a 1-isomorphism if there exists 1-arrow $C : y \to x$ such that there exist 2-isomorphisms $u : 1_y \Rightarrow C\#_k B$ and $v : 1_x \Rightarrow B\#_k C$. We call $C$ a quasi-inverse to $B$, and vice versa. However, when $k = 2$ or 3, we call a $k$-arrow a $k$-isomorphism if it is strictly invertible.

2.4. The 3-representations of a group in a strict 3-category. Let $C$ be a strict 3-category and let $G$ be a group. $G$ can be viewed as a strict 3-category with only one object $\bullet$, $G$ as the set of 1-arrows $g : \bullet \to \bullet$, the set of 2-arrows consisting of the identities of 1-arrows, and the set of 3-arrows consisting of the identities of 2-arrows. A 3-representation of a group $G$ in $C$ is a weak functor $\rho$ from $G$ to $C$ in the following sense. We have

(1) an object $x$ of $C$;
(2) for each $g \in G$, a 1-isomorphism $\rho_g : x \to x$;
(3) for each $h, g \in G$, a 2-isomorphism $\phi_{h,g} : \rho_h\rho_g \Rightarrow \rho_{hg}$ (here and in the following
we write $\rho_h \#_0 \rho_g$ as $\rho_h \rho_g$ for simplicity), corresponding to the 2-cell

(4) for each $g_3, g_2, g_1 \in G$, a 3-isomorphism, called the assiciator,

$$\Phi_{g_3,g_2,g_1} : (\rho_{g_3} \#_0 \phi_{g_2,g_1}) \#_1 \phi_{g_3,g_2,g_1} \Longrightarrow (\phi_{g_3,g_2} \#_0 \rho_{g_1}) \#_1 \phi_{g_3,g_2,g_1},$$

(12) corresponding to the 3-cell

It can be viewed as exchanging the diagonals of the quadrilateral:

(5) a 2-isomorphism $\phi_1 : \rho_1 \Longrightarrow 1_x;$

(13)
such that the following conditions are satisfied:

- \( \phi_{1,g} = \phi_1 \#_0 \rho_g, \phi_{g,1} = \rho_g \#_0 \phi_1 \).
- the 3-cocycle condition that for any \( g_4, \ldots, g_1 \in G \), we have
  \[
  \{[\rho_{g_4} \#_0 \Phi_{g_3,g_2,g_1}] \#_1 \phi_{g_4,g_3g_2g_1}] \#_2 \{[\rho_{g_4} \#_0 \phi_{g_3,g_2} \#_0 \rho_{g_1}] \#_1 \Phi_{g_4,g_3g_2,g_1}] \#_2 \{[\Phi_{g_4,g_3,g_2} \#_0 \phi_{g_1}] \#_1 \phi_{g_4,g_3g_2,g_1}] \#_2 \{[\phi_{g_4,g_3} \#_0 (\rho_{g_2} \rho_{g_1})] \#_1 \Phi_{g_4,g_3g_2,g_1}] \} \}.
  \]  

Equivalently, the composition of the 3-isomorphisms represented by 5 tetrahedrons above in the boundary of a 4-simplex is the identity. This comes from the fact that the boundary of the corresponding 4-simplex in the 3-category \( G \) is the identity 3-arrow.

2.5. Remark. (1) For simplicity, we assume in this paper that \( \rho_1 = 1_x \) and that \( \phi_1 \) is the identity.

(2) The 3-cocycle \( \{\Phi_{g_3,g_2,g_1}\} \) defines an element of the 3-dimensional non-abelian cohomology. A first attempt at an explicit description of the 3-dimensional non-abelian cohomology of a group goes back to Dedecker [9]. See section 4 of [7] for 3-dimensional non-abelian Čech cocycles, which can be used to construct a 2-gerbe.

2.6. The 3-cocycle condition. We will give a clear geometric description of the 3-cocycle condition (15) in terms of 5 tetrahedrons in the boundary of a 4-simplex above. This is equivalent to triviality of the 3-holonomy. See section 5 C of [27] for the 3-holonomy in the lattice 3-gauge theory (the cubical case), where 3-gauge theory from the point of view of Gray-categories is investigated.

In the left-hand side of the 3-cocycle condition (15), the first 3-isomorphism is

\[
A_1 = [\rho_{g_4} \#_0 \Phi_{g_3,g_2,g_1}] \#_1 \phi_{g_4,g_3g_2g_1}.
\]  

Here \( \Phi_{g_3,g_2,g_1} \) is a 3-isomorphism whiskered from left by the 1-isomorphism \( \rho_{g_4} \), and \( \rho_{g_4} \#_0 \Phi_{g_3,g_2,g_1} \) is whiskered from below by the 2-isomorphism \( \phi_{g_4,g_3g_2g_1} \). \( A_1 \) corresponds
to the 3-cell

\begin{align*}
\text{The 3-arrow } A_1
\end{align*}

whose 2-source and 2-target are the 2-isomorphisms

\begin{align*}
s_2(A_1) &= [(\rho_{g_4}\rho_{g_3})\#_0\phi_{g_3,g_1}]\#_1[(\rho_{g_4}\#_0\phi_{g_3,g_2})\rho_{g_1} : \rho_{g_4}\rho_{g_3}\rho_{g_2}\rho_{g_1} \rightarrow \rho_{g_4}\rho_{g_3}\rho_{g_2}]; \tag{17} \\
t_2(A_1) &= [\rho_{g_4}\#_0\phi_{g_3,g_2}\#_0\rho_{g_1}]\#_1[(\rho_{g_4}\#_0\phi_{g_3,g_2})\rho_{g_1} : \rho_{g_4}\rho_{g_3}\rho_{g_2}\rho_{g_1} \rightarrow \rho_{g_4}\rho_{g_3}\rho_{g_2}.
\end{align*}

corresponding to 2-cells

\begin{align*}
\text{The 2-arrow } s_2(A_1) \\
\text{The 2-arrow } t_2(A_1)
\end{align*}

respectively, where \( \rho_a := \rho_{g_3g_2} ; \rho_b := \rho_{g_3g_2g_1} \). It is fundamental in this paper to write down the \( p \)-arrow corresponding to \( p \)-cells as whiskered vertical compositions. For example, \( s_2(A_1) \) in (17) is the composition of the following three whiskered 2-isomorphisms.
The second 3-isomorphism in the left-hand side of the 3-cocycle condition (15) is
\[ A_2 = [\rho_{g_4} \#_0 \phi_{g_3,g_2} \#_0 \rho_{g_1}] \#_1 \Phi_{g_4,g_3,g_2,g_1}, \]
corresponding to the 3-cell

The 3-arrow \( A_2 \)

(here \( \rho_a := \rho_{g_4 g_3 g_2}, \rho_b := \rho_{g_3 g_2} \)) with 2-source \( s_2(A_2) = t_2(A_1) \) in (17) and 2-target
\[ t_2(A_2) = [\rho_{g_4} \#_0 \phi_{g_3,g_2} \#_0 \rho_{g_1}] \#_1 [\phi_{g_4,g_3,g_2} \#_0 \rho_{g_1}] \#_1 \phi_{g_4,g_3,g_2,g_1} \]
(18)
corresponding to 2-cells

And the third 3-isomorphism in the left-hand side of the 3-cocycle condition (15) is
\[ A_3 = [\Phi_{g_4,g_3,g_2} \#_0 \rho_{g_1}] \#_1 \phi_{g_4,g_3,g_2,g_1}, \]
corresponding to the 3-cell

\[ \begin{array}{c}
\bullet \\
\hspace{1cm} \rho g_4 \\
\hspace{2.5cm} \rho a \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]

The 3-arrow \( A_3 \)

\begin{align*}
\text{(here } \rho_a := \rho_{g_4 g_3 g_2}, \; \phi_b := \phi_{g_4 g_3 g_2 g_1} \text{)} & \text{ with 2-source } s_2(A_3) = t_2(A_2) \text{ in (18) and 2-target } \\
t_2(A_3) & = [\phi_{g_4 g_3} \#_0 (\rho_{g_2} \rho_{g_1})] \#_1 [\phi_{g_4 g_3 g_2} \#_0 \rho_{g_1}] \#_1 \phi_{g_4 g_3 g_2 g_1}; \tag{19} \\
\text{corresponding to the 2-cells} & \\
\end{align*}

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]

The 2-arrow \( t_2(A_3) \)

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]

where \( \rho_a := \rho_{g_4 g_3 g_2}, \; \rho_b := \rho_{g_4 g_3} \). Then the composition \( A_1 \#_2 A_2 \#_2 A_3 \) of 3-isomorphisms is the left-hand side of the 3-cocycle condition (15), whose 2-source is \( s_2(A_1) \) in (17) and 2-target is \( t_2(A_3) \) in (19).

On the right-hand side of the 3-cocycle condition (15), the first 3-isomorphism is

\[ A'_1 = [(\rho_{g_4} \rho_{g_3}) \#_0 \phi_{g_2 g_1}] \#_1 \Phi_{g_4 g_3 g_2 g_1}, \]
corresponding to the 3-cell

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]

with 2-source \( s_2(A_1) \) in (17) and 2-target

\[
t_2(A'_1) = [(\rho_{g_4}\rho_{g_3})\#_0\phi_{g_2.g_1}]\#_1[\phi_{g_4.g_3} \#_0 \rho_{g_2.g_1}]\#_1 \phi_{g_4.g_3.g_2.g_1},
\]

(21)
corresponding to the left 2-cells in the following diagram:

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]

By the interchange law (6) for horizontal compositions, we can interchange 2-isomorphism (1) and (2) identically in the left 2-cells above to get the 2-isomorphism

\[
s_2(A'_2) = [\phi_{g_4.g_3} \#_0 (\rho_{g_2}\rho_{g_1})] \#_1 [\rho_{g_4.g_3} \#_0 \phi_{g_2.g_1}] \#_1 \phi_{g_4.g_3.g_2.g_1},
\]

(22)
corresponding to the right 2-cells above. The last 3-isomorphism is

\[
A'_2 = [\phi_{g_4.g_3} \#_0 (\rho_{g_2}\rho_{g_1})] \#_1 \Phi_{g_4.g_3.g_2.g_1}
\]
whose 2-target is exactly the 2-isomorphism $t_2(A_3)$ in (19)-(20).

It is not easy to draw several 3-cells corresponding to the composition of 3-arrows in a 3-category $\mathcal{C}$. For this reason, let us consider the associated 2-category $\mathcal{C}^+$ such that

$$(\mathcal{C}^+)_i := \mathcal{C}_{i+1},$$

and $i$-source and $i$-target are $s_{i+1}$ and $t_{i+1}$, $i = 0,1,2$, respectively. Functions $\tilde{\#}_p : \mathcal{C}_k^+ \times \mathcal{C}_k^+ \to \mathcal{C}_k^+$ are described by arrows $\#_{p+1} : \mathcal{C}_{k+1} \times \mathcal{C}_{k+1} \to \mathcal{C}_{k+1}$, and identities $\tilde{1} : \mathcal{C}_{k-1}^+ \to \mathcal{C}_k^+$ are defined in a similar manner. $\mathcal{C}^+$ is a strict 2-category since $\text{Hom}_\mathcal{C}(x,y)$ is a strict 2-category for any objects $x,y$ of $\mathcal{C}$, by the fact that a strict 3-category is a category enriched over the category of all small strict 2-categories. We also define $\mathcal{C}^{++}$ to be the category with

$$(\mathcal{C}^{++})_i := \mathcal{C}_{i+2},$$

and the $i$-source and $i$-target are now $s_{i+2}$ and $t_{i+2}$, $i = 0,1$, respectively. The function $\tilde{\#}_0 : \mathcal{C}_1^{++} \times \mathcal{C}_1^{++} \to \mathcal{C}_1^{++}$ becomes $\#_2 : \mathcal{C}_3 \times \mathcal{C}_3 \to \mathcal{C}_3$. $\mathcal{C}^{++}$ is a category by the same reason.

In the corresponding strict 2-category $\mathcal{C}^+$, 3-isomorphism $A_1$ in (16) is represented by the following 2-isomorphism:
Here the upper and lower boundaries in (23) (as 1-arrows in $\mathcal{C}^+$) represent the source $s_2(A_1)$ and target $t_2(A_1)$ in (17) (as 2-isomorphisms in $\mathcal{C}$) respectively. To draw the picture neatly, we omit the whiskering parts. Then the 3-cocycle condition (15) can be expressed simply as an identity of 2-isomorphisms in $\mathcal{C}^+$ as follows:

$$φ_{g_4g_3g_2g_1} \Rightarrow φ_{g_4g_3g_2} \Rightarrow φ_{g_4g_3} \Rightarrow φ_{g_4} \Rightarrow A_1 \Rightarrow φ_{g_2} \Rightarrow φ_{g_1} \Rightarrow A$$

(24)

where $φ_α := φ_{g_4g_3g_2g_1}$, $φ_β := φ_{g_4g_3g_2g_1}$. Here $\bullet$'s above represent 1-isomorphisms in $\mathcal{C}$. The 2-isomorphisms in (17), (18), (19), (21) and (22) are represented by 1-isomorphisms in (24). Now the 3-cocycle condition (24) can be viewed as the commutativity of the 2-isomorphisms in the boundary of the following cube in $\mathcal{C}^+$:

$$\begin{array}{c}
φ_{g_4g_3g_2g_1} \\
φ_{g_4g_3g_2} \\
φ_{g_4g_3} \\
φ_{g_4} \\
A_1 \\
φ_{g_2} \\
φ_{g_1} \\
A \\
\end{array}$$

(25)

2.7. Remark. (1) In the upper boundaries of diagrams in (24), the number of group elements in the second subscripts of $φ_{α,β}$'s is increasing: $g_1$, $g_2$, $g_3$, while in the lower boundaries it is the number of group elements in the first subscripts of $φ_{α,β}$'s which are increasing: $g_4$, $g_4g_3$, $g_4g_3g_2$.

(2) (24) or (25) is similar to the pentagon condition of bicategories, but here we actually have more complicated whiskering (cf. (23)).

Given a strict 2-category $\mathcal{V}$, there exists an associated 3-category $\mathcal{V}^*$ for which $\mathcal{V}_0^*$ consists of one object $\mathcal{V}$, $\mathcal{V}_1^*$ consists of all functors from $\mathcal{V}$ to $\mathcal{V}$, $\mathcal{V}_2^*$ consists of all pseudonatural transformations and $\mathcal{V}_3^*$ consists of all modifications. This is a 3-category. Because
only 3-representations of a group in a strict 3-category are developed, we have to consider a strict 3-subcategory $W$ of $V^*$ for a strict 2-categories $V$. We call a 3-representation of $G$ in such a strict 3-subcategory $W$ a strict 2-categorical action of $G$ on $V$. In particular, we have an endofunctor $\rho_g : V \to V$ for each $g \in G$, a pseudonatural transformation $\phi_{h,g} : \rho_h \#_0 \rho_g \Rightarrow \rho_{hg}$ for each $h, g \in G$, and a modification $\Phi_{g_3,g_2,g_1}$ (the associator in (12)) for each $g_3, g_2, g_1 \in G$. Here $\rho_h \#_0 \rho_g$ is the composition of functors:

$$\rho_h \#_0 \rho_g (w) := \rho_h (\rho_g (w))$$

for $w \in V$. By the definition of 3-representations, the endofunctor $\rho_g$, the pseudonatural transformation $\phi_{h,g}$ and the modification $\Phi_{g_3,g_2,g_1}$ must all be invertible in $W \subset V^*$.

For example, for the 2-category $V$ used in the 1-dimensional 3-representation in Subsection 3.8, its $V^*$ is a strict 3-category. For the general action of $G$ on a 2-category $V$, we need to develop 3-representation of a group in a Gray-category, since the semi-strictification of a 3-category is a Gray-category.

When a 2-category $V$ is viewed as a 3-category with only identity 3-arrow, a 3-representation of $G$ in $V$ is a 2-representation if the the associator 3-isomorphism in (12) is the identity, so that the 3-cocycle condition (15) holds trivially. This coincides with the definition of the 2-representation in the strict sense in section 2.2 of [13]. And for a category $V$, a 2-representation of $G$ in the 2-category $V^*$ is a categorical action of $G$ on $V$.

3. The 2-categorical traces of 3-representations

3.1. The 2-Categorical Trace of a 1-Endomorphism. Let $C$ be a 3-category, $x \in C$ and $A : x \to x$ be a 1-endomorphism. Then $A$ is an object of the 2-category $\text{Hom}_C(x, x)$. The 2-categorical trace of $A$ is defined as

$$\text{Tr}_2(A) = \text{Hom}_C(1_x, A),$$

which is a category. This is a subcategory of $C^{++}$.

Let $A : x \to x$ be a 1-endomorphism for $x \in C_0$, and let the 1-arrow $C : y \to x$ be a quasi-inverse to a 1-arrow $B : x \to y$. Then for any 2-arrow $\chi : 1_x \Rightarrow A$ in $\text{Tr}_2(A)_0$, the composition

$$1_y \xleftarrow{\chi} C \#_0 B \xrightarrow{C \#_0 1_x \#_0 B} C \#_0 A \#_0 B$$

defines a functor

$$\Psi(C, B, u) : \text{Tr}_2(A)_0 \longrightarrow \text{Tr}_2(C \#_0 A \#_0 B)_0,$$

$$(\chi : 1_x \Rightarrow A) \mapsto u \#_1 [C \#_0 \chi \#_0 B],$$
corresponding to the diagram

and for any 3-arrow $\gamma : \chi \Rightarrow \chi'$ in $\mathcal{T}_r(A)_1$, we have

$$\mathcal{T}_r(A)_1 \rightarrow \mathcal{T}_r(C\#_0 A\#_0 B)_1, \quad \gamma \mapsto u\#_1[C\#_0 \gamma \#_0 B],$$

3.2. Proposition. $\Psi(C, B, u) : \mathcal{T}_r(A) \rightarrow \mathcal{T}_r(C\#_0 A\#_0 B)$ is a functor.

Proof. For 2-arrows $\chi, \chi', \overline{\chi} : 1 \rightarrow A$ and 3-arrows $\gamma : \chi \Rightarrow \chi', \overline{\gamma} : \chi' \Rightarrow \overline{\chi}$, we have the composition $\gamma \#_2 \overline{\gamma} : \chi \Rightarrow \overline{\chi}$. Then by using repeatedly the compatibility condition (7) for compositions, we find

$$\Psi(C, B, u)(\gamma) \#_2 \Psi(C, B, u)(\overline{\gamma}) = \{u\#_1[C\#_0 \gamma \#_0 B]\} \#_2 \{u\#_1[C\#_0 \overline{\gamma} \#_0 B]\} = u\#_1[C\#_0 (\gamma \#_2 \overline{\gamma}) \#_0 B] = \Psi(C, B, u)(\gamma \#_2 \overline{\gamma}).$$

Thus $\Psi(C, B, u)$ is a functor. ■

3.3. The 2-Categorical Trace $\mathcal{T}_r\rho_f$. Let $\rho$ be a 3-representation of $G$ in a 3-category $\mathcal{C}$. Fix an object $x$ in $\mathcal{C}$ that $G$ acts on. For $f \in G$, let $\rho_f : x \rightarrow x$ be a 1-isomorphism in $\mathcal{C}$. Recall that $\mathcal{T}_r\rho_f$ is a category whose objects are 2-arrows with source $1_x$ and target $\rho_f$ and the morphisms are 3-arrows between them. In the sequel, we will use the notation $g^* := g^{-1}$ for simplicity. For any $g$ commuting with $f$ and a 2-arrow $\chi : 1 \rightarrow \rho_f$ in $(\mathcal{T}_r\rho_f)_0$, we define a 2-arrow $\psi_g(\chi) : 1 \rightarrow \rho_f$ by

$$\psi_g(\chi) := u_g\#_1[\rho_g\#_0 \chi \#_0 \rho_f^*] \#_1[\phi_{g,f}\#_0 \rho_f^*] \#_1\phi_{g,f}^* \phi_{g,f}^*.$$  \hfill (26)

This is given by the composition of 2-arrows in the following diagram

$$\mathcal{T}_r\rho_f(A)_1 \rightarrow \mathcal{T}_r\rho_f(C\#_0 A\#_0 B)_1, \quad \gamma \mapsto u\#_1[C\#_0 \gamma \#_0 B],$$

3.2. Proposition. $\Psi(C, B, u) : \mathcal{T}_r(A) \rightarrow \mathcal{T}_r(C\#_0 A\#_0 B)$ is a functor.

Proof. For 2-arrows $\chi, \chi', \overline{\chi} : 1 \rightarrow A$ and 3-arrows $\gamma : \chi \Rightarrow \chi', \overline{\gamma} : \chi' \Rightarrow \overline{\chi}$, we have the composition $\gamma \#_2 \overline{\gamma} : \chi \Rightarrow \overline{\chi}$. Then by using repeatedly the compatibility condition (7) for compositions, we find

$$\Psi(C, B, u)(\gamma) \#_2 \Psi(C, B, u)(\overline{\gamma}) = \{u\#_1[C\#_0 \gamma \#_0 B]\} \#_2 \{u\#_1[C\#_0 \overline{\gamma} \#_0 B]\} = u\#_1[C\#_0 (\gamma \#_2 \overline{\gamma}) \#_0 B] = \Psi(C, B, u)(\gamma \#_2 \overline{\gamma}).$$

Thus $\Psi(C, B, u)$ is a functor. ■

3.3. The 2-Categorical Trace $\mathcal{T}_r\rho_f$. Let $\rho$ be a 3-representation of $G$ in a 3-category $\mathcal{C}$. Fix an object $x$ in $\mathcal{C}$ that $G$ acts on. For $f \in G$, let $\rho_f : x \rightarrow x$ be a 1-isomorphism in $\mathcal{C}$. Recall that $\mathcal{T}_r\rho_f$ is a category whose objects are 2-arrows with source $1_x$ and target $\rho_f$ and the morphisms are 3-arrows between them. In the sequel, we will use the notation $g^* := g^{-1}$ for simplicity. For any $g$ commuting with $f$ and a 2-arrow $\chi : 1 \rightarrow \rho_f$ in $(\mathcal{T}_r\rho_f)_0$, we define a 2-arrow $\psi_g(\chi) : 1 \rightarrow \rho_f$ by

$$\psi_g(\chi) := u_g\#_1[\rho_g\#_0 \chi \#_0 \rho_f^*] \#_1[\phi_{g,f}\#_0 \rho_f^*] \#_1\phi_{g,f}^* \phi_{g,f}^*.$$  \hfill (26)

This is given by the composition of 2-arrows in the following diagram

$$\mathcal{T}_r\rho_f(A)_1 \rightarrow \mathcal{T}_r\rho_f(C\#_0 A\#_0 B)_1, \quad \gamma \mapsto u\#_1[C\#_0 \gamma \#_0 B],$$
where \( u_g = \phi_{g,g}^{-1} : 1_x \to \rho_g \rho_{g^*} \). For a 3-arrow \( \Theta : \chi \Rightarrow \chi' \), we define \( \psi_g(\Theta) \) as a 3-arrow whiskered by corresponding 2-isomorphisms in (27). In other words,

\[
\psi_g(\Theta) = u_g \#_1 [\rho_g \#_0 \Theta \#_0 \rho_{g^*}] \#_1 [(\phi_{g,f} \#_0 \rho_{g^*}) \#_1 \phi_{g,f,g^*}] : \psi_g(\chi) \Rightarrow \psi_g(\chi')
\]

is a 3-arrow corresponding to the diagram

\[
\begin{array}{c}
\xymatrix{ & 1_x \ar[dd]_{\rho_g} & \\
X \ar[rr]^\rho_g \ar[rru]_{\phi_g} & & X \ar[ll]_{\rho_g} \ar[uuu]_{\phi_{g,f,g^*}} \\
& X \ar[ur]_{\rho_{g^*}} & }
\end{array}
\]

in the 3-category \( \mathcal{C} \). Then \( \psi_g \) defines an endofunctor \( \psi_g \) on \( \text{Tr}_2 \rho_f \) by the proof of Proposition 3.2. Namely, we have

\[
\psi_g(\tilde{\Theta}) = \psi_g(\tilde{\Theta}) \#_0 \psi_g(\tilde{\Theta}')
\]

for any 3-arrow \( \Theta' : \chi' \Rightarrow \chi'' \), where \( \tilde{\#}_0 \) is the composition in the category \( \mathcal{C}^{++} \) (\( \#_0 = \#_2 \)).

In Section 3.4, we will construction a natural isomorphism \( \Gamma_{h,g} : \psi_h \circ \psi_g \Rightarrow \psi_{hg} \) for given \( g, h \in C_G(f) \). It gives us natural isomorphisms \( \Gamma_{g^*,g} : \psi_g^* \circ \psi_g \Rightarrow \psi_1 \) and \( \Gamma_{g,g^*} : \psi_g \circ \psi_g^* \Rightarrow \psi_1 \). Thus \( \psi_g \) for each \( g \in C_G(f) \) is an equivalence of the category \( \text{Tr}_2 \rho_f \).

3.4. THE ADJOIN 2-ISOMORPHISMS. For a 2-isomorphism \( \xymatrix{ x \ar@/^1pc/[rr]^\phi & y \ar@/^1pc/[ll]_{\phi} } \) in a 2-category \( \mathcal{V} \), we define the adjoint 2-isomorphism \( \phi^\dagger \) to be \( \xymatrix{ y \ar@/^1pc/[rr]^{\phi^\dagger} & x \ar@/^1pc/[ll]_{\phi} } \) by the composition of arrows

\[
\xymatrix{ y \ar[r]^{\chi_1^{-1}} & x \ar[u]_{\chi_2} & y \ar[l]_{\chi_2} \ar[r]^{\chi_1^{-1}} & x. }
\]

This is a 2-isomorphism with inverted 1-source and 1-target. This operation will be used later. See also section 2 of [20] for the definition of similar adjoint 2-arrows, but \( \phi^{-1} \) in (29) is replaced there by \( \phi \).
3.5. Proposition. (1) For any pair of 2-isomorphisms \( x \xrightarrow{\phi} y \) and \( x \xrightarrow{\psi} y \), we have \((\phi \#_1 \psi)^\dagger = \phi^\dagger \#_1 \psi^\dagger\).

(2) For any 1-isomorphism \( \chi_0 : z \rightarrow x \), we have \((\chi_0 \#_0 \phi)^\dagger = \phi^\dagger \#_0 \chi_0^{-1} \); and for 1-isomorphism \( \bar{\chi}_0 : y \rightarrow z \), we have \((\phi \#_0 \bar{\chi}_0)^\dagger = \bar{\chi}_0^{-1} \#_0 \phi^\dagger\).

(3) For a 2-isomorphism \( y \xrightarrow{\phi} z \), we have \((\phi \#_0 \bar{\phi})^\dagger = \bar{\phi}^\dagger \#_0 \phi^\dagger\), i.e., \( z \xrightarrow{\phi^\dagger} y \xrightarrow{\bar{\phi}^\dagger} x \).

Proof. (1) \((\phi^\dagger \#_1 \psi)^\dagger = (\phi^\dagger \#_1 \psi)^\dagger\) follows from

\[ y \xrightarrow{\chi_1^{-1}} x \xrightarrow{\phi} y \xrightarrow{\bar{\phi}^{-1}} x \xrightarrow{\psi} y \xrightarrow{\chi_2^{-1}} x \] by \( x \xrightarrow{\chi_2} y \xrightarrow{\bar{\chi}_2^{-1}} x \xrightarrow{\psi} y \) and the interchange law (6) for horizontal compositions.

(2) follows from the fact that \((\chi_0 \#_0 \phi)^\dagger\) is

\[ y \xrightarrow{\chi_1^{-1}} x \xrightarrow{\chi_0^{-1}} z \xrightarrow{\phi} x \xrightarrow{\psi} y \xrightarrow{\chi_2^{-1}} x \xrightarrow{\chi_0^{-1}} z \]

since \(\chi_0^{-1} \#_0 \chi_0\) is equal to the identity \(1_x\).

(3) Note that \(\phi \#_0 \bar{\phi} = (\chi_1 \#_0 \bar{\phi}) \#_1 (\phi \#_0 \bar{\chi}_2)\) by using the interchange law (6). We see that

\[(\phi \#_0 \bar{\phi})^\dagger = (\chi_1 \#_0 \bar{\phi})^\dagger \#_1 (\phi \#_0 \bar{\chi}_2)^\dagger = (\bar{\phi}^\dagger \#_0 \chi_1^{-1}) \#_1 (\chi_2^{-1} \#_0 \phi^\dagger) = \bar{\phi}^\dagger \#_0 \phi^\dagger\]

by using (1), (2) and the interchange law (6) again.

\[ \blacksquare \]

3.6. The categorical action of the centralizer of \( f \) on \( \text{Tr}_{2\rho_f} \). To construct a categorical action of the centralizer \( C_G(f) \) of \( f \) on the category \( \text{Tr}_{2\rho_f} \), let us write down the composition law for the functors \( \psi_h \) and \( \psi_g \),

\[ \psi_h \circ \psi_g : \text{Tr}_{2\rho_f} \rightarrow \text{Tr}_{2\rho_f}, \]
where $h, g \in C_G(f)$. For a fixed $\chi \in (\mathbb{T}_{2\rho} f)_0$ and $\Theta \in (\mathbb{T}_{2\rho} f)_1$, by using the definition (26)-(28) of $\psi_*$ twice, we see that $\psi_h \circ \psi_g(\chi) = \psi_h(\psi_g(\chi))$ is the composition of 2-arrows in $C$ in the following diagram:

and $\psi_h \circ \psi_g(\Theta) = \psi_h(\psi_g(\Theta))$ is a 3-arrow in $C$ defined similarly. Recall that we assume $\rho_g^1 = \rho_g 1_x$ and $\rho_h^1 = \rho_h 1_x$. The upper half part of (30) is the same as the lower half with $f$ replaced by $1_x$ and 2-isomorphisms inverted:

namely, we have $u_h = \phi_{h_1, h}^{-1} \#_1 [\phi_{h_1, 1}^{-1} \#_0 \rho_h^*]$ and similar identity for $u_g$. Note that $\phi_{h, 1}$ and $\phi_{g, 1}$ are identities by our assumptions in Remark 2.2 (1).

Now let us write down the natural isomorphism

$$\Gamma_{h,g} : \psi_h \circ \psi_g \longrightarrow \psi_{hg}$$

between functors on the category $\mathbb{T}_{2\rho} f$. The lower half of diagram (30) is
Here and in the following, for simplicity, we will use the notation

$$\rho_{g_1 g_2} := \rho_{g_1} \ldots g_2,$$

i.e., we omit the group elements between $g_1$ and $g_2$ in the sequence $h, g, f, g^*, h^*$ in diagram (32).

Recall that the associator 3-isomorphism $\Phi_{g_3, g_2, g_1}$ in (12)-(13) can be drawn in the form (14). By definition, the 3-isomorphism

$$\hat{\Lambda}_1 = \gamma_1 \#_1[\Phi_{h, g, f, g^*} \#_0 \rho_{h^*}] \#_1 \gamma_2,$$  \hspace{1cm} (33)

is the associator $\Phi_{h, g, f, g^*} \#_0 \rho_{h^*}$ whiskered by two 2-isomorphisms

$$\gamma_1 = [\rho_h \#_0 \phi_{g, f} \#_0 (\rho_g \rho_{h^*})] : x \xrightarrow{\rho_h} x \xrightarrow{\rho_g} x \xrightarrow{\rho_f} x \xrightarrow{\rho_{g^*}} x \xrightarrow{\rho_{h^*}} x,$$

$$\gamma_2 = \phi_{h^*, h^*} : x \xrightarrow{\rho_{h^*}} x \xrightarrow{\rho_h^*} x, \hspace{1cm} (34)$$

from above and below, respectively. This replaces the diagonal $\rho_{g g^*}$ of the dotted quadrilateral in diagram (32) by the wavy diagonal $\rho_{h^*}$ of the same quadrilateral in the following diagram:

$$x \xrightarrow{\rho_h} x \xrightarrow{\rho_g} x \xrightarrow{\rho_f} x \xrightarrow{\rho_{g^*}} x \xrightarrow{\rho_{h^*}} x.$$  \hspace{1cm} (35)

$\hat{\Lambda}_1$ in (33) is the following 3-isomorphism

$$x \xrightarrow{\chi} x \xrightarrow{\chi'} x,$$  \hspace{1cm} (36)

where $\chi$ is the 2-arrow corresponding to the dotted quadrilateral in diagram (32), $\chi'$ is the 2-arrow corresponding to the same quadrilateral in diagram (35) with the diagonal changed, and 2-arrows $\gamma_1$ and $\gamma_2$ are given by (34).
The 3-isomorphism
\[ \hat{\Lambda}_2 = \{ \Phi_{h,g,f} \circ (\rho_g \circ \rho_h) \} \circ \Phi_{h,g,f}^{-1}, \]  
(37)
as a whiskered associator (14), then changes the diagonal \( \rho_{gf} \) of the dotted-wavy quadrilateral in diagram (35) to the wavy diagonal \( \rho_{hg} \) of the same quadrilateral in the following diagram:

Similarly, the 3-isomorphism
\[ \hat{\Lambda}_3 = \{ \Phi_{h,g,f} \circ (\rho_g \circ \rho_h) \} \circ \Phi_{h,g,f}^{-1}, \]  
(39)
which is the whiskered associator \( \Phi_{h,g,f}^{-1} \), changes the diagonal \( \rho_{hg} \) of the dotted quadrilateral in diagram (38) to the wavy diagonal \( \rho_{g^*h^*} \) of the same quadrilateral in the following diagram:

Recall that the upper half of diagram (30) is the same as the lower half with \( f \) replaced by 1 and 2-isomorphisms inverted. So by the corresponding 3-isomorphisms, denoted by \( \hat{\Lambda}'_1, \hat{\Lambda}'_2, \hat{\Lambda}'_3 \), the upper half of (31) is changed to

```
\[
\hat{\Lambda}'_1 = \{ \Phi_{h,g,f} \circ (\rho_g \circ \rho_h) \} \circ \Phi_{h,g,f}^{-1},
\]
(41)
```

Note that
\[
\hat{\Phi}_{h,g}^{-1}
\]  
(42)
ON THE 3-REPRESENTATIONS OF GROUPS

and the part involving \( \rho_g^* \rho_h^* \) is also cancelled. As a result, the composition of (41) and (40), together with 2-arrow \( \chi : 1_x \longrightarrow \rho_f \), gives us the diagram (27) with \( g \) replaced by \( gh \). This is exactly \( \psi_{gh}(\chi) \). Therefore, the composition of suitable whiskered 3-isomorphisms \( \hat{\Lambda}_1', \hat{\Lambda}_2', \hat{\Lambda}_3', \hat{\Lambda}_1, \hat{\Lambda}_2 \) and \( \hat{\Lambda}_3 \) gives a natural isomorphism \( \Gamma_{h,g} : \psi_h \circ \psi_g \longrightarrow \psi_{hg} \) such that for \( \chi \in (\mathbb{T}\mathbb{r}_{2\rho_f})_0 \)

\[
\Gamma_{h,g}(\chi) : \psi_h(\psi_g(\chi)) \Longrightarrow \psi_{hg}(\chi)
\]
is a 3-isomorphism in \( \mathcal{C} \).

It is not easy to draw 3-arrows \( \hat{\Lambda}_j \)'s in the 3-category \( \mathcal{C} \). But in the 2-category \( \mathcal{C}^+ \), the first 3-arrow \( \hat{\Lambda}_1 \) in (33) can be drawn as the 2-isomorphism corresponding to the following diagram:

\[
\begin{array}{cccccc}
\rho_h \rho_g \rho_f^* \rho_g^* \rho_h^* & \xrightarrow{\rho_h \rho_g \rho_f \rho_g^* \rho_h} & \rho_h \rho_g \rho_f^* \rho_g^* \rho_h^* & \xrightarrow{\phi_{h, g^*} \rho_h^* \rho_g^* \rho_h} & \rho_h \rho_g \rho_f \rho_g^* \rho_h^* & \xrightarrow{\phi_{h, g^*} \rho_h} & \rho_h \rho_g \rho_f \rho_g^* \rho_h^* \\
\phi_{h, g^*} \rho_h^* \rho_g^* \rho_h^* & \xrightarrow{\phi_{h, g^*} \rho_h^* \rho_g^* \rho_h^*} & \phi_{h, g^*} \rho_h^* \rho_g^* \rho_h^* & \xrightarrow{\phi_{h, g^*} \rho_h} & \phi_{h, g^*} \rho_h^* \rho_g^* \rho_h^* & \xrightarrow{\phi_{h, g^*} \rho_h^* \rho_g^* \rho_h^*} & \phi_{h, g^*} \rho_h^* \rho_g^* \rho_h^* \\
\end{array}
\]

Here the upper path

\[
\rho_h \rho_g \rho_f \rho_g^* \rho_h^* \xrightarrow{\rho_h \rho_g \rho_f \rho_g^* \rho_h \rho_g^* \rho_h^*} \rho_h \rho_g \rho_f \rho_g^* \rho_h^* \xrightarrow{\rho_h \rho_g \rho_f \rho_g^* \rho_h \rho_g^* \rho_h^*} \ldots
\]
corresponds to the 2-isomorphisms in \( \mathcal{C} \) in (32) (the lower half of \( \psi_h(\psi_g(\chi)) \)), while the lower paths corresponds to the 2-isomorphisms in \( \mathcal{C} \) in (35) (the lower half of \( \psi_{hg}(\chi) \)). And the 2-isomorphism \( \hat{\Lambda}_1 \) corresponds to the 3-isomorphism in \( \mathcal{C} \) in (33). Since \( \mathbb{T}\mathbb{r}_{2\rho_f} \) is a subcategory of \( \mathcal{C}^{++} \), diagrams in the 2-category \( \mathcal{C}^+ \) are sufficient for our purpose. In the sequel, to simplify diagrams,

\[
\rho_h \cdots \rho_{g_1 \cdot g_2} \cdots \rho_{h^*}
\]
as an object in the 2-category \( \mathcal{C}^+ \). For simplicity, we also omit the whiskering part of 1-isomorphisms \( \phi_{*,*} \)'s in diagrams. The 3-isomorphisms \( \hat{\Lambda}_1 : (32) \implies (35) \), \( \hat{\Lambda}_2 : (35) \implies (38) \) and \( \hat{\Lambda}_3 : (38) \implies (40) \) in the 3-category \( \mathcal{C} \) correspond to 2-isomorphisms in the 2-category \( \mathcal{C}^+ \) in the following diagram:

\[
\mathcal{D}_f : \xrightarrow{} \rho_f \xrightarrow{\phi_{g, f}} \rho_{g f} \xrightarrow{\phi_{g f, g^*}} \rho_{g f g^*} \xrightarrow{\phi_{g f g^*, h^*}} \rho_{g f g^* h^*} \xrightarrow{\phi_{g f g^* h^*}} \rho_{g f g^* h^*} \xrightarrow{\phi_{g f g^* h^*}} \rho_{h h}, \quad (43)
\]
respectively. Just as for the upper half of diagram (30), diagram (31) is changed to diagram (41). In \(\mathcal{C}^+\), this is the composition of 2-isomorphisms given by the following diagram

\[
\begin{array}{c}
\rho_{h^*} \xrightarrow{\phi_{h^*}^{-1}} \rho_{h^*} \xrightarrow{\phi_{h^*}^{-1}} \rho_{g^*} \xrightarrow{\phi_{g^*}^{-1}} \rho_{g^*} \xrightarrow{\phi_{g^*}^{-1}} \rho_{g_1} \\
\rho_{h_1} \xrightarrow{\phi_{h_1}^{-1}} \rho_{h_1} \xrightarrow{\phi_{h_1}^{-1}} \rho_{g_1} \xrightarrow{\phi_{g_1}^{-1}} \rho_{g_1} \xrightarrow{\phi_{g_1}^{-1}} \rho_1 \xrightarrow{\phi_{g_1}^{-1}} \rho_1 \xrightarrow{\phi_{g_1}^{-1}} \rho_1 =: \mathcal{D}_1'
\end{array}
\]

where \(\hat{\Lambda}_j\) is the 2-isomorphism previously denoted by \(\hat{\Lambda}_j\) (with \(f\) replaced by 1), and \(\hat{\Lambda}_j\) (previously denoted by \(\hat{\Lambda}_j\)) is the 2-isomorphism adjoint to \(\hat{\Lambda}_j\), defined in §3.4. Recall that the adjoint 2-isomorphism is the inverse one with 1-source and 1-target inverted. We apply the adjoint operation to diagram (43) to get diagram (44), the mirror-symmetric diagram of (43), by using Proposition 3.5. Given \(\chi : 1_x \implies \rho_f\), we connect the diagrams (44) and (43) to get \(\Gamma_{h,g}(\chi)\) as a 2-isomorphism in \(\mathcal{C}^+\):

\[
\mathcal{D}_1' \xrightarrow{\chi} \mathcal{D}_f.
\]

For objects \(\chi, \chi' \in (\mathbb{T}r_2\rho_f)_0\) and a morphism \(\Theta : \chi \to \chi'\) in \((\mathbb{T}r_2\rho_f)_1\) (i.e., a 3-arrow in \(\mathcal{C}\)), \(\Gamma_{h,g}(\Theta)\) is also a 3-arrow. We connect diagrams (43) and (44) to get \(\Gamma_{h,g}(\Theta)\) as the following diagram in the 2-category \(\mathcal{C}^+\):

\[
\begin{array}{c}
\cdots \xrightarrow{\phi_{g_1}^{-1}} \cdots \\
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1} \\
\cdots \xrightarrow{\phi_{h_1}^{-1}} \cdots
\end{array}
\]

\[
\begin{array}{c}
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1} \\
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1} \\
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1}
\end{array}
\]

\[
\begin{array}{c}
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1} \\
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1} \\
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1}
\end{array}
\]

\[
\begin{array}{c}
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1} \\
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1} \\
\phi_{h_1}^{-1} \xrightarrow{\phi_{h_1}^{-1}} \phi_{g_1}^{-1}
\end{array}
\]

Note that \(\psi_h \circ \psi_g(\chi)\) in (30) is the upper boundary of diagram (46) and \(\psi_h(\chi')\) is the lower boundary of diagram (46). \(\Gamma_{h,g}(\chi)\) is the diagram (46) with the 2-arrow \(\Theta : \chi \implies \chi'\) deleted, but 1-arrow \(\chi : \rho_1 \to \rho_f\) remains, whereas \(\Gamma_{h,g}(\chi')\) is the diagram (46) with the 2-arrow \(\Theta : \chi \implies \chi'\) deleted, but 1-arrow \(\chi' : \rho_1 \to \rho_f\) remains. Applying the interchange law (6) to the diagram (46), we see that \(\Gamma_{h,g}\) is a natural isomorphism in the
category $\mathcal{T}_2\rho_f \subset C^+$, i.e. the diagram

\[
\begin{array}{c}
\psi_h \circ \psi_g(\chi) \quad \psi_h \circ \psi_g(\chi') \\
\downarrow \quad \downarrow \\
\psi_{hg}(\chi) \quad \psi_{hg}(\chi')
\end{array}
\]

is commutative, where the 2-arrows $\psi_h \circ \psi_g(\Theta)$ and $\psi_{hg}(\Theta)$ in $C^+$ are $\Theta$ whiskered by 1-isomorphisms corresponding to the upper and lower boundaries of diagram (46), respectively.

3.7. Theorem. $\{\psi_g, \Gamma_{h,g}\}_{g,h \in C_G(f)}$ is a categorical action of the centralizer $C_G(f)$ on the category $\mathcal{T}_2\rho_f$.

This theorem will be proved in Section 6 by checking the associative law (1) for $\Gamma_{h,g}$. The equation (1) for $\Gamma_{h,g}$ is in the usual order, not in the natural order which we assumed in Remark 2.1 (1).

3.8. 1-DIMENSIONAL 3-REPRESENTATIONS. We fix a field $k$ of characteristic 0 containing all roots of unity. Let $\mathcal{A}$ be a 2-category with only one object, 1-arrows and 2-arrows $\mathcal{A}_2 \cong k^*$. Fix a 3-cocycle $c$ satisfying the condition (2). Let $g^c$ be the strict 2-categorical action of $G$ on $\mathcal{A}$ as follows: $g^c$ is the identity functor for each $g \in G$;

\[
\phi_{h,g} : 1_A = g^c = g^c_{h,g} = g^c_{h,g} = 1_A
\]

is also the identity pseudounatural isomorphism for any $h,g \in G$; and

\[
\Phi_{g_3,g_2,g_1} : id = (g^c_{g_3} \#_0 g^c_{g_2,g_1} \#_1 g^c_{g_1}) = (g^c_{g_3} \#_0 g^c_{g_2} \#_1 g^c_{g_2,g_1}) = id,
\]

is a modification determined by the element $c(g_3,g_2,g_1) \in k^*$ for any $g_3,g_2,g_1 \in G$. Then the 3-cocycle condition (24) for $\Phi$ is reduced to the equation (2). The cohomology classes of 3-cocycles are classified by $H^3(G,k^*)$.

For $f \in G$, it is easy to see that $\mathcal{T}_2\rho_f^c$ is a category with a single object given by the identity pseudounatural isomorphism $\chi_0 : 1_A \rightarrow \rho_f^c = 1_A$, and morphisms $(\mathcal{T}_2\rho_f)_1 \cong k^*$ (an element of $k^*$ provides a modification). For $g \in C_G(f)$, $\psi_g : \mathcal{T}_2\rho_f^c \rightarrow \mathcal{T}_2\rho_f^c$ is the identity functor by the definitions (26)-(28). And

\[
\Gamma_{h,g} : \chi_0 = \psi_h \circ \psi_g(\chi_0) \longrightarrow \psi_{hg}(\chi_0) = \chi_0
\]

is a natural isomorphism given by the element (also denoted by $\Gamma_{h,g}$ by abuse of notations)

\[
\Gamma_{h,g} = \frac{c(h,gf,g^*)c(h,g,f)c(hf,g^*,h^*)^{-1}}{c(h,g,g^*)c(h,g,1)c(hg,g^*,h^*)^{-1}}.
\]
This element is obtained by replacing \( \Phi_{g_3,g_2,g_1} \) by the element \( c(g_3,g_2,g_1) \) and all other isomorphisms by 1 in \( \hat{A}_j \)'s in (33) (37) (39), and using the adjoint operation (29).

3.9. Proposition. \( \Gamma \) given by (48) is a 2-cocycle on the centralizer \( C_G(f) \).

This proposition will be proved in Section 6.1.

3.10. Remark. There exists a transgression map that maps a 3-cocycle \( c \) on a finite group \( G \) to a 2-cocycle on the inertia groupoid of \( G \) [26]. It is given by

\[
C_{h,g} := \frac{c(h,g,f)c(hgf^{-1}h^{-1},h,g)}{c(h,gfg^{-1},g)}
\]

for given \( f \in G \) (cf. Remark 3.17 in [14]). Note that for \( h, g \in C_G(f) \) we have \( C_{h,g} := c(h,g,f)c(f,h,g)/c(h,f,g) \). So our 2-cocycle \( \Gamma_{h,g} \) in (48) is different from the transgressed one. On the other hand, our 2-cocycle is only defined for elements which commute with a given element \( f \), not on the entire inertia groupoid of \( G \).

Let \( \varrho \) be a categorical action of a finite group \( G \) on a \( k \)-linear category \( W \). For a commuting pair of elements \( g \) and \( f \) in \( G \), the 2-character \( \chi_{\varrho}(f,g) \) of a categorical action \( \varrho \) is the joint trace of functors \( \varrho_f \) and \( \varrho_g \), i.e., the trace of the linear transformation induced by the functor \( \varrho_g \) on the categorical trace \( \text{Tr}_{\varrho_f} \) (a \( k \)-vector space, which we assume to be finite dimensional).

Now let \( \rho \) be a strict 2-categorical action of a finite group \( G \) on a \( k \)-linear 2-category \( V \). Then \( \text{Tr}_{\rho_f} \) is a \( k \)-linear category and \( \psi \) defines a categorical action of the centralizer of \( f \) in \( G \) on it by Theorem 3.7. If \( k, g, f \in G \) are pairwise commutative, we define the 3-character of the 2-categorical action \( \rho \) to be

\[
\chi_{\rho}(f,g,k) := \chi_{\psi}(g,k),
\]

the joint trace of functors \( \psi_g \) and \( \psi_k \) acting on the \( k \)-linear category \( \text{Tr}_{\rho_f} \), i.e., the trace of the linear transformation induced by the functor \( \psi_k \) on the \( k \)-vector space \( \text{Tr}_{\psi_g} \), which we assume to be finite dimensional.

By using the 2-character formula for 1-dimensional 2-representation in proposition 5.1 of [13], the 3-character of the 3-representation \( \rho^c \) for pairwise commutative \( k, g, f \in G \) is given by

\[
\chi_{\rho^c}(f,g,k) = \frac{\Gamma_{k,g}\Gamma_{k,g,k^{-1}}}{\Gamma_{k,1}\Gamma_{k,k^{-1}}},
\]

where the expressions \( \Gamma_{*,*} \)'s are defined by (48). It can also be derived from diagram (27).

4. The induced strict 2-categorical action on the induced 2-category

4.1. The induced 2-category. Let \( H \subseteq G \) be a subgroup of a finite group \( G \) and let \( \rho : H \to V^* \) be a strict 2-categorical action of \( H \) on a strict 2-category \( V \) (cf. definitions at the end of Section 2.4). \( \text{Ind}_H^G(V) \) is a strict 2-category where
• objects are maps $\vartheta : G \rightarrow V_0$ together with a 1-isomorphism

$$u_{g,h} : \vartheta(gh) \rightarrow \rho_{h^{-1}} \vartheta(g)$$

for each $g \in G, h \in H$, satisfying the condition:

1. $u_{g,1} : \vartheta(g) \rightarrow \rho_1 \vartheta(g)$ coincides with $\phi_1^{-1}[\vartheta(g)]$;
2. for each $g \in G, h_1, h_2 \in H$, we have a 2-isomorphism:

$$\vartheta(gh_1 h_2) \xrightarrow{u_{g, h_1, h_2}} \rho_{h_2^{-1}} \vartheta(gh_1) \xrightarrow{\rho_{h_2} u_{g, h_1}} \rho_{h_2 \rho h_1} \vartheta(g)$$

• 1-arrows $F : (\vartheta, u) \rightarrow (\vartheta', u')$ between objects;
• 2-arrows $\gamma : F \rightarrow \tilde{F}$.

For $k \in G$, the action $(\text{ind}_H^G \rho)_k$ on the 2-category $\text{Ind}_H^G(V)$ is given by

$$[(\text{ind}_H^G \rho)_k \vartheta](g) = \vartheta(k^{-1} g), \quad [(\text{ind}_H^G \rho)_k u]_{g,h} = u_{k^{-1} g, h},$$

for an object $(\vartheta, u)$ in $\text{Ind}_H^G(V)$. And $(\text{ind}_H^G \rho)_k(F)$ for a 1-arrows $F : (\vartheta, u) \rightarrow (\vartheta', u')$ and $(\text{ind}_H^G \rho)_k(\gamma)$ for a 2-arrow $\gamma : F \rightarrow \tilde{F}$ can be defined similarly. In general, each commutative diagram in the definition of the induced category in section 7.1 of [13] is replaced by a 2-isomorphism.

We will not write down the definition of the induced 2-category $\text{Ind}_H^G(V)$ explicitly. It is a little bit complicated. Since we only work on finite groups, we can simply identify $\text{Ind}_H^G(V)$ with $V^m$ as a 2-category, where $m$ is the index of $H$ in $G$. For a strict 2-category $V$, $V^m$ is also a strict 2-category with

- objects $V_0^m := \{(x_1, \ldots, x_m) : x_j \in V_0\}$,
- $p$-arrows $V_p^m := \{(\gamma_1, \ldots, \gamma_m) : (x_1, \ldots, x_m) \rightarrow (y_1, \ldots, y_m) ; V_p \ni \gamma_j : x_j \rightarrow y_j\}$,

$p = 1, 2$. The compositions are defined as

$$(\ldots, \gamma_j, \ldots) \#_p (\ldots, \gamma'_j, \ldots) := (\ldots, \gamma_j \#_p \gamma'_j, \ldots),$$

if $\gamma_j$ and $\gamma'_j$ are $p$-composable. The axioms for functions $\#_p$ and identities of $V^m$ are obviously satisfied. The identification $\text{Ind}_H^G(V) \cong V^m$ can be obtained by choosing a system of representatives

$$\mathcal{R} = \{r_1, \ldots, r_m\}$$
of left cosets of $H$ in $G$, and associating to each map $\vartheta : G \to V_0$ an object $(\vartheta(r_1), \ldots, \vartheta(r_m))$ in $V_0^m$.

Let $a_{jk} : V \to V$ be functors such that the $m \times m$ matrix $F = (a_{jk})$ has only one nonvanishing entry in each row or column. Then $F$ defines a strict functor from $V^m$ to $V^m$ by

$$F(\ldots, \delta_j, \ldots) = \left(\ldots, \sum_k a_{jk}(\delta_k), \ldots\right),$$

where we write $\sum_k a_{jk}(\delta_k)$ formally for $\delta_k \in V_p$, $p = 0, 1, 2$, since there exists only one term in this sum. But when the 2-category is $k$-linear, such sums exist. If $\tilde{F} = (\tilde{a}_{jk}) : V^m \to V^m$ is another such functor, then we have

$$(F \# \tilde{F})_{jk} := \sum_l a_{jl} \tilde{a}_{lk}.$$ 

Moreover, a pseudonatural transformation $\phi : F \to \tilde{F}$ is given by an $m \times m$ matrix $\phi = (\phi_{jk})$ with $\phi_{jk} : a_{jk} \to \tilde{a}_{jk}$ a pseudonatural transformation between functors on $V$. Let $\tilde{\phi} = (\tilde{\phi}_{jk}) : \tilde{F} \to \tilde{F}$ be another pseudonatural transformation. Then their composition is $\phi \#_1 \tilde{\phi} := (\phi_{jk} \#_1 \tilde{\phi}_{jk})$.

4.2. The induced strict 2-categorical action. Suppose that $\rho$ is a strict 2-categorical action of $H$ on the 2-category $V$. For $f \in G$, we define $(\text{ind}^G_H \rho)_f$ to be a functor from $V^m$ to $V^m$ as follows. It is an $m \times m$ matrix whose entries are functors from $V$ to $V$, i.e., the $(j, i)$-entry is

$$[(\text{ind}^G_H \rho)_f]_{ji} = \begin{cases} \rho_h, & \text{if } fr_i = r_j h, \text{ for } h \in H, \\ 0, & \text{otherwise}. \end{cases}$$

(51)

This corresponds to the fact that for a map $\vartheta : G \to V_0$, we have $[(\text{ind}^G_H \rho)_f(\vartheta)](r_j) = \vartheta(f^{-1}r_j)$ and $\vartheta(f^{-1}r_j) = \vartheta(r_j h^{-1}) \to \rho_h \vartheta(r_j)$. It is clear that only one entry in each row or column of the $m \times m$ matrix $(\text{ind}^G_H \rho)_f$ is nonvanishing. Then,

$$(\text{ind}^G_H \rho)_f(\ldots, \delta_j, \ldots) = \left(\ldots, \sum (\text{ind}^G_H \rho)_f(\delta_i), \ldots\right),$$

(52)

where $\delta_j \in V_p$, for $p = 0, 1, 2$.

For simplicity, from now on the induced object will be denoted by the hatted one, e.g. $\text{ind}^G_H \rho$ is denoted by $\hat{\rho}$. The composition functor $\hat{\rho}_{g_2}$ and $\hat{\rho}_{g_1}$ is defined as

$$(\hat{\rho}_{g_2} \hat{\rho}_{g_1})_{ki} = \begin{cases} \rho_{h_2} \rho_{h_1}, & \text{if } g_1 r_i = r_j h_1, g_2 r_j = r_k h_2, \text{ for some } h_1, h_2 \in H, \\ 0, & \text{otherwise}. \end{cases}$$

(53)

Thus $\hat{\rho}_{g_2} \hat{\rho}_{g_1}$ can be viewed as the product of two $m \times m$ matrices of functors. On the other hand,

$$(\hat{\rho}_{g_2 g_1})_{ki} = \rho_{h_2 h_1}$$

(54)
since $(g_2 g_1) r_i = r_k (h_2 h_1)$ by (53). We define the pseudonatural transformation (as a 2-isomorphism in $(\mathcal{V}^m)^*$)

$$\hat{\phi}_{g_2,g_1} : \hat{\rho}_{g_2} \hat{\rho}_{g_1} \longrightarrow \hat{\rho}_{g_2 g_1},$$

as the $m \times m$ matrix whose $(k,i)$-entry is the 2-isomorphism

$$(\hat{\phi}_{g_2,g_1})_{ki} = \phi_{h_2,h_1} : \rho_{h_2} \rho_{h_1} \longrightarrow \rho_{h_2 h_1}, \hspace{1cm} (55)$$

and all other entries vanish. For $g_1, g_2, g_3 \in G$, the 3-isomorphism in $(\mathcal{V}^m)^*$

$$\hat{\Phi}_{g_3,g_2,g_1} : [\hat{\rho}_{g_3} \#_0 \hat{\phi}_{g_2,g_1} ] \#_1 \hat{\phi}_{g_3,g_2,g_1} \longrightarrow [\hat{\phi}_{g_3,g_2} \#_0 \hat{\rho}_{g_1} ] \#_1 \hat{\phi}_{g_3,g_2,g_1}$$

is a modification. Write $g_3 r_k = r_l h_3$

for some $h_3 \in H$. Then we have

$$[\hat{\rho}_{g_3} \#_0 \hat{\phi}_{g_2,g_1} ]_{li} = \rho_{h_3} \#_0 \phi_{h_2,h_1} \hspace{1cm} \text{and} \hspace{1cm} [\hat{\phi}_{g_3,g_2} \#_0 \hat{\rho}_{g_1} ]_{li} = \phi_{h_4,h_2} \#_0 \rho_{h_1}. \hspace{1cm} (56)$$

etc.. We define $\hat{\Phi}_{g_3,g_2,g_1}$ as an $m \times m$ matrix whose $(l,i)$-entry is the modification (as a 3-isomorphism in $\mathcal{V}^*$)

$$(\hat{\Phi}_{g_3,g_2,g_1})_{li} = \Phi_{h_3,h_2,h_1} : [\rho_{h_3} \#_0 \phi_{h_2,h_1} ] \#_1 \phi_{h_4,h_2,h_1} \longrightarrow [\phi_{h_3,h_2} \#_0 \rho_{h_1} ] \#_1 \phi_{h_3,h_2,h_1},$$

and all other entries vanish.

For $g_4 \in G$, write

$$g_4 r_l = r_t h_4$$

for some $h_4 \in H$. The $(t,i)$-entry of the $m \times m$ matrix $[\hat{\rho}_{g_4} \#_0 \hat{\Phi}_{g_3,g_2,g_1} ] \#_1 \hat{\phi}_{g_4,g_3,g_2,g_1}$ is the modification

$$[\rho_{h_4} \#_0 \Phi_{h_3,h_2,h_1} ] \#_1 \phi_{h_4,h_3,h_2,h_1}$$

of $\mathcal{V}$, and similarly we obtain other terms in the 3-cocycle condition (15) for $\hat{\Phi}$. So the 3-cocycle condition (15) for $\hat{\Phi}$ is reduced to the 3-cocycle condition for $\Phi$. Note that functors or pseudonatural transformation or modification we consider are matrices, of which entries are in a strict 3-subcategory $\mathcal{W}$ of $\mathcal{V}^*$. It follows from the strictness of $\mathcal{W}$ that $\hat{\rho}$ is a strict 2-categorical action of $G$ on $\mathcal{V}^m \approx \text{Ind}_H^G(\mathcal{V})$.

5. The 3-character of the induced strict 2-categorical action

5.1. The 2-categorical trace of the induced strict 2-categorical action.

As above $\rho$ is a strict 2-categorical action of $H$ on the 2-category $\mathcal{V}$. Let $\mathcal{R}$ be a system of representatives of $G/H$. We have the decomposition

$$\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''.$$
where $\mathcal{R}' := \{ r \in \mathcal{R}; r^{-1}fr \in H \}$, $\mathcal{R}'' := \{ r \in \mathcal{R}; r^{-1}fr \notin H \}$. For a fixed element $f$ of $G$, the decomposition $[f]_G \cap H = [h_1]_H \cup \cdots [h_n]_H$ induces a decomposition

$$\mathcal{R}' = \bigcup_{i=1}^{n} \mathcal{R}_i \quad \text{with} \quad \mathcal{R}_i = \{ r \in \mathcal{R}; r^{-1}fr \in [h_i]_H \}. $$

For fixed $i$, we pick $r_i \in \mathcal{R}_i$ and write $h_i = r_i^{-1}fr_i$. For $r \in \mathcal{R}_i$, we have $r^{-1}fr = h^{-1}h_i h$ for some $h \in H$. From now on, by replacing $r$ by $rh^{-1}$ in the representatives of $\mathcal{R}_i \subset G/H$, we can assume $r^{-1}fr = h_i$ for all $r \in \mathcal{R}_i$. 

(57)

Denote

$$m_i := |\mathcal{R}_i|, \quad m' := |\mathcal{R}'| = \sum_{i=1}^{n} m_i, \quad m'' := |\mathcal{R}''|, \quad m := m' + m''.$$

It follows from the definition (51)-(52) of $\hat{\rho}_f$ that

$$\hat{\rho}_f = \begin{pmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0n} \\ A_{10} & A_{11} & 0 & \cdots & 0 \\ A_{20} & 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n0} & 0 & 0 & \cdots & A_{nn} \end{pmatrix}, \quad A_{ii} = \begin{pmatrix} \rho_{h_i} & & \\ & \ddots & \vdots \\ & \vdots & \rho_{h_i} \end{pmatrix}_{m_i \times m_i},$$

(58)

where $i = 1, \ldots, n$, and $A_{00}$ is an off-diagonal $m'' \times m''$ matrix. So an object of $\mathbb{T}r_2 \hat{\rho}_f$ is a pseudonatural transformation $\chi : 1_{\mathcal{V}^{m}} \to \hat{\rho}_f$ of the form

$$\chi = \begin{pmatrix} 0_{m'' \times m''} \\ \vdots \\ D_i \\ \vdots \\ 0_{m'' \times m''} \end{pmatrix}, \quad D_i = \begin{pmatrix} \chi_{m_1 + \cdots + m_{i-1} + 1} \\ \vdots \\ \chi_{m_1 + \cdots + m_i} \end{pmatrix},$$

(59)

where $\chi_{m_1 + \cdots + m_{i-1} + \alpha} : 1_{\mathcal{V}} \to \rho_{h_i}$ is an object of $\mathbb{T}r_2 \rho_{h_i}$, $\alpha = 1, \ldots, m_i$. Also morphisms in $\mathbb{T}r_2 \hat{\rho}_f$ are diagonal. So we have

$$\mathbb{T}r_2 \hat{\rho}_f = \bigoplus_{i=1}^{n} (\mathbb{T}r_2 \rho_{h_i})^{m_i}.$$
5.2. Lemma. ([13], Lemma 7.7) *Left multiplication with \( r_i^{-1} \) maps \( R_i \) into a system of representatives of \( C_G(h_i)/C_H(h_i) \).

For \( g \in C_G(f) \) and \( r \in R_i \), we write

\[ gr = \tilde{r}h, \]

for some \( \tilde{r} \in R \) and \( h \in H \). Also, \( r \) is uniquely determined by \( \tilde{r} \) for fixed \( g \). Then

\[ \tilde{r}^{-1}fr = hr^{-1}g^{-1}frh^{-1} = hh_ih^{-1} \]

by (57). Hence \( \tilde{r} \in R_i \) and so \( \tilde{r}^{-1}fr = h_i \) by assumption (57). It follows that \( h \in C_H(h_i) \).

5.2. Lemma.

\[ \text{Left multiplication with } r_i^{-1} \text{ maps } R_i \text{ into a system of representatives of } C_G(h_i)/C_H(h_i). \]

For \( g \in C_G(f) \) and \( r \in R_i \), we write

\[ gr = \tilde{r}h, \]

for some \( \tilde{r} \in R \) and \( h \in H \). Also, \( r \) is uniquely determined by \( \tilde{r} \) for fixed \( g \). Then

\[ \tilde{r}^{-1}fr = hr^{-1}g^{-1}frh^{-1} = hh_ih^{-1} \]

by (57). Hence \( \tilde{r} \in R_i \) and so \( \tilde{r}^{-1}fr = h_i \) by assumption (57). It follows that \( h \in C_H(h_i) \).

Then

\[ gr = \tilde{r}h \quad \text{gives} \quad (\hat{\rho}_g)_{\tilde{r}r} = \rho_h, \]
\[ fr = rh_i \quad \text{gives} \quad (\hat{\rho}_f)_{rr} = \rho_{h_i}, \]
\[ g^{-1}\tilde{r} = rh^{-1} \quad \text{gives} \quad (\hat{\rho}_g^{-1})_{\tilde{r}\tilde{r}} = \rho_{h^*}, \]

and all other entries vanish. Thus

\[ (\hat{\rho}_g\hat{\rho}_f\hat{\rho}_g^*)_{\tilde{r}\tilde{r}} = \rho_h\rho_{h_i}\rho_{h^*} \]

and all other entries in the last \((m' \times m')\)-block vanish (see (58)).

We denote by \( \hat{\psi} \) the categorical action of the centralizer \( C_G(f) \) of \( f \) on the category \( \text{Tr}_2\hat{\rho}_f \). By definition (26), \( \hat{\psi}_g \) for \( g \in C_G(f) \) is an invertible functor as follows. For a pseudonatural transformation \( \text{diag}(\ldots, \chi, \ldots) = \chi : 1_{\mathcal{V}} \to \hat{\rho}_f \) in (59), \( \hat{\psi}_g(\chi) \) is a pseudonatural transformation given by

\[ \text{diag}(1_{\mathcal{V}}, \ldots, 1_{\mathcal{V}}) \overset{\hat{\phi}_g^{-1}}{\longrightarrow} \hat{\rho}_g\hat{\rho}_g^* \overset{\hat{\rho}_g\#\#\hat{\rho}_g^*\hat{\rho}_g^*}{\longrightarrow} \hat{\rho}_g\hat{\rho}_f\hat{\rho}_g^* \overset{\hat{\phi}_g\hat{\rho}_f\hat{\rho}_g^*}{\longrightarrow} \hat{\rho}_{gf}\hat{\rho}_g^* \overset{\hat{\phi}_{gf}\hat{\rho}_g^*}{\longrightarrow} \hat{\rho}_{gf}\hat{\rho}_g^* = \hat{\rho}_f, \]

where the first \( m'' \) diagonal terms of \( \hat{\psi}_g(\chi) \) must vanish, and other diagonal terms are

\[ (\hat{\phi}_{gg,g}^{-1})_{\tilde{r}\tilde{r}} = \phi_{h,h^* : 1_{\mathcal{V}} \to \rho_{h}\rho_{h^*}}, \]
\[ (\hat{\rho}_g\#\hat{\rho}_g^*\hat{\rho}_g^*\hat{\rho}_g^*)_{\tilde{r}\tilde{r}} = (\hat{\rho}_g)_{\tilde{r}r}#(\hat{\rho}_g)_{\tilde{r}r}#(\hat{\rho}_g^*)_{\tilde{r}r} = \phi_{h,h^* : \rho_{h}\rho_{h^*} \to \rho_{h}\rho_{h^*}}, \]
\[ (\hat{\phi}_{gg,f}\hat{\rho}_g^*)_{\tilde{r}\tilde{r}} = \phi_{h,h^* : \rho_{h}\rho_{h^*} \to \rho_{h}\rho_{h^*}}, \]
\[ (\hat{\phi}_{gg,f}\hat{\rho}_g^*)_{\tilde{r}\tilde{r}} = \phi_{h,h^* : \rho_{h}\rho_{h^*} \to \rho_{h}\rho_{h^*} = \rho_{h^*}.} \]

All other entries vanish by definitions (54)-(55). Therefore, \( \hat{\psi}_g(\chi) \) is a diagonal \( m \times m \) matrix of pseudonatural transformations, whose \((\tilde{r}, \tilde{r})\)-entry for \( \tilde{r} \in R' \) is

\[ (\hat{\psi}_g(\chi))_{\tilde{r}\tilde{r}} = \phi_{h,h^* : 1_{\mathcal{V}} \to \rho_{h^*} = 1_{\mathcal{V}} \to \rho_{h_i}, \]

and vanishes for all \( \tilde{r} \in R'' \).

Now denote by \( \psi^{(i)} \) the categorical action of the centralizer \( C_H(h_i) \) on the category \( \text{Tr}_2\rho_{h_i} \), which is constructed from the strict 2-categorical action \( \rho \) of \( H \) on \( \mathcal{V} \). Recall that
by definition (26), we have a functor $\psi_h^{(i)}$ for each $h \in C_H(h_i)$. For $h \in C_H(h_i)$ and a pseudonatural transformation $\omega : 1_V \to \rho_{h_i}$, the pseudonatural transformation $\psi_h^{(i)}(\omega)$ is again by definition (26) the composition of the following pseudonatural transformations between functors:

$$1_V \xrightarrow{\phi_h^{(i)}h^*} \rho_h \rho_{h^*} \xrightarrow{\rho_h \omega \# \alpha \rho_h^*} \rho_h \rho_{h_i} \rho_{h^*} \xrightarrow{\phi_{h_i} \rho_h \rho_{h_i}^*} \rho_{h_i} \rho_{h^*} \xrightarrow{\phi_{h_i} h^*} \rho_{h_i} \rho_{h^*} = \rho_{h_i}.$$  

Then we see that (61) can be written as

$$\left( \tilde{\psi}_g (\chi) \right)_{\tilde{r} r} = \psi_h^{(i)}(\chi) : 1_V \to \rho_{h_i}, \quad (62)$$

with $r, \tilde{r} \in R_i$ and $h$ determined by (60). Namely, the resulting $\tilde{r}$-th diagonal term is the image of the $r$-th diagonal term under the action of the functor $\psi_h^{(i)}$.

Note that we have the identification

$$\text{Ind}^{C_G(h_i)}_{C_H(h_i)} \mathbb{T} \rho_{h_i} \cong (\mathbb{T} \rho_{h_i})^{m_i}, \quad (63)$$

since $\mid C_G(h_i)/C_H(h_i) \mid = m_i$ by Lemma 5.2, and that (60) is equivalent to

$$(r_i^{-1} g_r r_i)(r_i^{-1} r) = (r_i^{-1} r) h, \quad h \in C_H(h_i). \quad (64)$$

The coset $C_G(h_i)/C_H(h_i)$ are represented by $r_i^{-1} r$ for $r \in R_i$ by Lemma 5.2 again, and an element of $C_G(h_i)$ can always be written as $r_i^{-1} g_r$ for some $g \in C_G(f)$. As above we denote by $\tilde{\psi}^{(i)}$ the induced action of the centralizer $C_G(h_i)$ of $h_i$ on the category $\text{Ind}^{C_G(h_i)}_{C_H(h_i)} \mathbb{T} \rho_{h_i}$.

Recall the definition (51)-(52) of the induced action. So the action of $r_i^{-1} g_r \in C_G(h_i)$ on the induced category (63) is given by the functor $\tilde{\psi}^{(i)}_r r_i^{-1} g_r$ on $(\mathbb{T} \rho_{h_i})^{m_i}$ with

$$\left( \tilde{\psi}^{(i)}_r r_i^{-1} g_r (\chi) \right)_{r_i^{-1} \tilde{r}, r_i^{-1} r} = \psi_h^{(i)}(\chi)_{r_i^{-1} r} : 1_V \to \rho_{h_i}, \quad (65)$$

for $\chi \in (\mathbb{T} \rho_{h_i})^{m_i}$, where $h$ is given by (64), and all other entries vanish. Here we use the expressions $r_i^{-1} r$ as indices of the components of $(\mathbb{T} \rho_{h_i})^{m_i}$. Comparing (62) with (65), we find that the action of $g \in C_G(f)$ on $(\mathbb{T} \rho_{h_i})^{m_i}$ coincides with the induced action of $r_i^{-1} g_r \in C_G(h_i)$ on it, and so the action of the centralizer $C_G(f)$ on $\mathbb{T} \rho_{h_i}$ decomposes into actions on

$$\bigoplus_i (\mathbb{T} \rho_{h_i})^{m_i} = \bigoplus_i \text{Ind}^{C_G(h_i)}_{C_H(h_i)} \mathbb{T} \rho_{h_i}. \quad (66)$$

Recall that the initia groupoid $\Lambda(G)$ of a group $G$ has as objects, the elements of $G$, and for two such elements $u$ and $v$, there is one morphism in $\Lambda(G)$ from $u$ to $v$ for every $g \in G$ such that $v = g u g^{-1}$. Note that the initia groupoid $\Lambda(G)$ is equivalent to the groupoid with the set of objects consisting of the conjugacy classes $[g_i]$ and the set of morphisms consisting of $g : [g_i] \to [g_i]$ for $g \in C_G(g_i)$. Therefore the above result can be summarized as follows.
5.3. **Theorem.** Let \( V \) be a \( k \)-linear \( 2 \)-category. The \( 2 \)-categorical trace \( \mathbb{T}_2 \) takes induced strict \( 2 \)-categorical action into the induced categorical action of the associated initial groupoids, i.e. \( (3) \) holds.

5.4. **Remark.** Even for the categorical action, Section 4 above and the present subsection provide some details not written down explicitly in section 7.2 of [13].

5.5. **The 3-character formula.** Recall the 2-character formula for an induced categorical action.

5.6. **Theorem.** ([18], Corollary 7.6) Let \( \varrho \) be a categorical action of a subgroup \( H \) of a finite group \( G \) on a \( k \)-linear category \( W \). Suppose that \( \mathbb{T}_2 \varrho_h \) is finite dimensional for each \( h \in H \). Then the \( 2 \)-character of the induced categorical action of \( G \) is given by

\[
\chi_{\text{ind}}(f,g) = \frac{1}{|H|} \sum_{s \in G} \chi_{\varrho}(s^{-1}fs, s^{-1}gs) \tag{67}
\]

for \( g \in C_G(f) \).

We now state:

5.7. **Theorem.** Let \( H \) be a subgroup of a finite group \( G \) and let \( \rho \) be a strict \( 2 \)-categorical action of \( H \) on the \( 2 \)-category \( \mathcal{V} \). Let \( \psi \) be the categorical actions of the centralizers on the \( 2 \)-categorical trace. Suppose that \( \mathbb{T}_2 \psi_h \) is finite dimensional for each \( h \in H \). Then the \( 3 \)-character of the induced strict \( 2 \)-categorical action of \( G \) is given by

\[
\chi_{\text{ind}}(f,g,k) = \frac{1}{|H|} \sum_{s \in G} \chi_{\varrho}(s^{-1}fs, s^{-1}gs, s^{-1}ks) \tag{68}
\]

for \( f, g \) and \( k \) pairwise commutative.

**Proof.** By the decomposition (66) of the action of \( C_G(f) \) on \( \mathbb{T}_2 \hat{\rho}_f \) and (62)-(65), we have

\[
\chi_{\text{ind}}(f,g,k) = \sum_{i=1}^{n} \chi_{\hat{\psi}^{(i)}}(r^{-1}_i gr_i, r^{-1}_i kr_i).
\]

Now apply Theorem 5.6 to the categorical action \( \hat{\psi}^{(i)} \) (65) of \( C_G(h_i) \), which is induced from the categorical action \( \psi^{(i)} \) of \( C_H(h_i) \) on \( \mathbb{T}_2 \rho_{h_i} \), to get

\[
\chi_{\text{ind}}(f,g,k) = \sum_{i=1}^{n} \frac{1}{|C_H(h_i)|} \sum_{t \in C_G(h_i)} \chi_{\hat{\psi}^{(i)}}(t^{-1}r^{-1}_i gr_i t, t^{-1}r^{-1}_i kr_i t).
\]

Recall that \( \psi^{(i)} \) is the categorical action of \( C_H(h_i) \) on \( \mathbb{T}_2 \rho_{h_i} \), constructed from the strict \( 2 \)-categorical action \( \rho \) of \( H \). So we have

\[
\chi_{\psi^{(i)}}(t^{-1}r^{-1}_i gr_i t, t^{-1}r^{-1}_i kr_i t) = \chi_{\rho}(h_i, t^{-1}r^{-1}_i gr_i t, t^{-1}r^{-1}_i kr_i t)
\]
by the definition of the 3-character (49) for the strict 2-categorical action $\rho$ of group $H$. Moreover, the decomposition of the action of $C_G(f)$ on $\mathbb{T}_{r_2} \hat{\rho}_f$ in Section 5.1 is independent of the choice of $h_i \in [h_i]_H$, conjugacy class of $h_i$ in $H$. Therefore,

$$\chi_{\text{ind}}(f, g, k) = \sum_{h \in H} \frac{1}{|[h]_H| \cdot |C_H(h)|} \sum_{s^{-1}f^g = h, s \in G, \ s^{-1}(g, k) \in C_H(h) \times C_H(h)} \chi_{\hat{\rho}}(h, s^{-1}gs, s^{-1}ks).$$

Here we have used the fact that $h_i = s^{-1}f^g = s^{-1}r_i h_i r_i^{-1} s$ if and only if $r_i^{-1} s \in C_G(h_i)$. Note that for $s \in G$, we have $s^{-1}gs$ (resp. $s^{-1}ks$) $\in H$ if and only if $s^{-1}gs$ (resp. $s^{-1}ks$) $\in C_H(h)$ since $g$ and $k$ commute with $f = shs^{-1}$. The 3-character formula (68) follows.

6. The categorical action of the centralizer of $f$ on $\mathbb{T}_{r_2} \hat{\rho}_f$

6.1. A model: the 1-dimensional case. Let us prove by using the condition (2) for 3-cocycles repeatedly that the expression $\Gamma$ given in (48) is a 2-cocycle on the centralizer $C_G(f)$. This proof corresponds step by step to that of the general case carried out in Section 6.4.

Proof of Proposition 3.9. By the definition of $\Gamma_{*,*}$ in (48), we see that

$$\Gamma_{h, g} \Gamma_{k, hg} = \frac{\Pi_f}{\Pi_1},$$

where

$$\Pi_f := c(h, gf, g^*) c(h, g, f) c(hgf, g^*, h^*)^{-1} \cdot c(k, hf, g^*h^*) c(k, h, f) c(kf, g^*, h^*)^{-1},$$

(69)

and $\Pi_1$ is just $\Pi_f$ with $f$ replaced by 1. Similarly, we have

$$\Gamma_{k, h} \Gamma_{kh, g} = \frac{\Pi'_f}{\Pi'_1},$$

where

$$\Pi'_f = c(k, hf, h^*) c(k, h, f) c(khf, h^*, k^*)^{-1} \cdot c(kh, g, f) c(kf, g^*, (kh)^*)^{-1},$$

(70)

and $\Pi'_1$ is just $\Pi'_f$ with $f$ replaced by 1.

Apply the 3-cocycle condition (2) to the product of the two boldface terms in (70) with $g_4 = k, g_3 = h, g_2 = gf, g_1 = g^*$ to get

$$\Pi'_f = c(h, gf, g^*) c(k, hgf, g^*) c(k, h, gf)$$

$$\cdot c(k, hf, h^*) c(khf, h^*, k^*)^{-1} c(kh, g, f) c(kf, g^*, (kh)^*)^{-1}.$$  

(71)
Here the second line above is the right-hand side of (70) with the two boldface terms deleted. Note that $kfg^* = khf$. Apply the 3-cocycle condition (2) to the product of the two boldface terms in (71) with $g_4 = k_{f}, g_3 = g^*, g_2 = h^*, g_1 = k^*$ to get

$$
\Pi'_f = c(g^*, h^*, k^*)^{-1}c(k_{f}, g^*h^*, k^*)^{-1}c(k_{f}, g^*, h^*)^{-1}c(h, g, g^*)c(k, g^*)c(k, h, h^*)c(k, h, g^*)c(k, h, f),
$$

(72)

Here the second line above is the right-hand side of (71) with the two boldface terms deleted. Apply the 3-cocycle condition (2) to the product of the three boldface terms in (72) with $g_4 = k, g_3 = h_{f}, g_2 = g^*, g_1 = h^*$ to get

$$
\Pi'_f = c(k, h_{f}, g^*h^*)c(h_{f}, g^*, h^*)^{-1}c(g^*, h^*, k^*)^{-1}c(k_{f}, g^*h^*, k^*)^{-1}c(h, g, g^*)c(k, h, g^*)c(k, h, f),
$$

(73)

by $khf = kf$ and $h_{fg}^* = hf$. Here the second line above is the right-hand side of (72) with the three boldface terms deleted. Apply the 3-cocycle condition (2) to the product of the two boldface terms in (73) with $g_4 = k, g_3 = h, g_2 = g, g_1 = f$ to get

$$
\Pi'_f = c(h, g, f) \cdot c(k, h, g, f) c(k, h, g) \cdot c(k, h, f) c(k, h^*, f) c(k, h^*, g) c(k, h, g^*) c(k, h^*, g^*)^{-1} c(k_{f}, g^*h^*, k^*)^{-1} c(h, g, g^*). \tag{74}
$$

For $f = 1$ in (74), we see that $\Pi'_f$ also has the product $c(k, h, g)c(g^*, h^*, k^*)^{-1}$ of the two boldface terms, which is independent of $f$. They are cancelled in $\Pi'_f/\Pi'_1$. So we get

$$
\frac{\Pi'_f}{\Pi'_1} = \frac{\Pi_f}{\Pi_1}
$$

by comparing (74) and (69). Proposition 3.9 is proved.

6.2. The natural isomorphism $\Gamma_{k, h, g}^\#(\psi_k \circ \Gamma_{h, g})$. Let us write down the natural isomorphism $\Gamma_{k, h, g}^\#(\psi_k \circ \Gamma_{h, g})$. For a fixed $\chi \in (\mathbb{T}r_2 \rho_f)_0$, by using the definition of compositions in (30) twice, we see that $\psi_k \circ \psi_h \circ \psi_{g}(\chi)$ is the composition of the following 2-arrows:

![Diagram](75)
Let us calculate the 3-isomorphism

$$[\Gamma_{k,hg}(\psi_k \circ \Gamma_{h,g})(\chi)] : \psi_k \circ \psi_h \circ \psi_g(\chi) \Rightarrow \psi_{kgh}(\chi)$$

(76)

for a fixed 2-arrow $\chi \in \mathrm{Tr}_2 \rho_f \subset C^{++}$. We consider the lower half part of (75) first. The 3-isomorphism

$$\Lambda_1 = \diamond \#_1[\rho_k \#_0 \Phi_{h,gf,g} \#_0(\rho_h^*, \rho_k^*)] \#_1 \diamond,$$

(77)

the associator $\Phi_{h,gf,g}$ (14) whiskered by 2-isomorphisms $\diamond$ which we do not write down explicitly, changes the diagonal $\rho_{gg}^*$ of the dotted quadrilateral in (75) to the wavy diagonal $\rho_{hf}$ of the same quadrilateral in the following diagram:

This is a 3-arrow as (36). The 2-arrows outside the quadrilateral are fixed as the whiskering parts. The 3-isomorphism

$$\Lambda_2 = \diamond \#_1[\rho_k \#_0 \Phi_{h,gf,g} \#_0(\rho_h^*, \rho_k^*)] \#_1 \diamond,$$

(79)

as a whiskered associator (14), changes the diagonal $\rho_{gf}$ of the above dotted-wavy quadrilateral to the wavy diagonal $\rho_{hg}$ of the same quadrilateral in the following diagram:

The 3-isomorphism

$$\Lambda_3 = \diamond \#_1[\rho_k \#_0 \Phi_{h,hf,h}^{-1} \#_0(\rho_h^*, \rho_k^*)] \#_1 \diamond,$$

(81)
as a whiskered associator (14), changes the diagonal $\rho_{hg}^*$ of the above dotted quadrilateral to the wavy diagonal $\rho_{g^*h^*}$ of the same quadrilateral in the following diagram:

(82)

\[
\begin{array}{cccccccc}
\rho_k & \rho_h & \rho_g & \rho_f & \rho_g^* & \rho_h^* & \rho_k^* & \rho_j^* \\
\end{array}
\]

Note that the diagrams (78), (80) and (82) are exactly the diagrams (35), (38) and (40) by adding from below to each of these the arrows:

By definition, the composition $\Lambda_1 \#_2 \Lambda_2 \#_2 \Lambda_3$ is the 3-isomorphism

\[
\psi_k \circ \Gamma_{h,g}(\chi) : \psi_k \circ \psi_h \circ \psi_g(\chi) \Rightarrow \psi_k \circ \psi_{hg}(\chi)
\]

corresponding to the lower half of (75).

The 3-isomorphism

\[
\Lambda_4 = \Diamond \#_1 [\Phi_{k,h^f,g^*h^*} \#_0 \rho_{k^*}] \#_1 \Diamond,
\]
as a whiskered associator (14), changes the diagonal $\rho_{hh}^*$ of the dotted-wavy quadrilateral in (82) to the wavy diagonal $\rho_{kf}^*$ of the same quadrilateral in the following diagram:

(83)

\[
\begin{array}{cccccccc}
\rho_k & \rho_h & \rho_g & \rho_f & \rho_g^* & \rho_h^* & \rho_k^* & \rho_j^* \\
\end{array}
\]

The 3-isomorphism

\[
\Lambda_5 = \Diamond \#_1 [\Phi_{k,hg,f} \#_0 (\rho_{g^*h^*} \rho_{k^*})] \#_1 \Diamond,
\]
as a whiskered associator (14), changes the diagonal \( \rho_{hf} \) of the above dotted-wavy quadrilateral to the wavy diagonal \( \rho_{kg} \) of the same quadrilateral in the following diagram:

\[
\begin{array}{c}
\bullet \xrightarrow{\rho_k} \bullet \xrightarrow{\rho_h} \bullet \xrightarrow{\rho_g} \bullet \xrightarrow{\rho_f} \bullet \xrightarrow{\rho_g^*} \bullet \xrightarrow{\rho_h^*} \bullet \xrightarrow{\rho_k^*} \bullet
\end{array}
\]

The 3-isomorphism

\[
\Lambda_6 = \Diamond \#_4 \Phi^{-1}_{kf,g^*h^*,k^*},
\]

as a whiskered associator (14), changes the diagonal \( \rho_{kh^*} \) of the above dotted quadrilateral to the wavy diagonal \( \rho_{g^*k^*} \) of the same quadrilateral in the following diagram:

\[
\begin{array}{c}
\bullet \xrightarrow{\rho_k} \bullet \xrightarrow{\rho_h} \bullet \xrightarrow{\rho_g} \bullet \xrightarrow{\rho_f} \bullet \xrightarrow{\rho_g^*} \bullet \xrightarrow{\rho_h^*} \bullet \xrightarrow{\rho_k^*} \bullet
\end{array}
\]

The composition \( \Lambda_4 \#_2 \Lambda_5 \#_2 \Lambda_6 \) is the 3-isomorphism

\[
\Gamma_{k,hg}(\chi) : \psi_k \circ \psi_{hg}(\chi) \equiv \psi_{khg}(\chi)
\]

corresponding to the lower half of (75).

In the 2-category \( \mathcal{C}^+ \), the composition \( \Lambda_1 \#_2 \cdots \#_2 \Lambda_6 \) of 3-isomorphisms corresponds to the following diagram \( \mathcal{D}_f :=
\]

\[
\begin{array}{c}
\rho_f \xrightarrow{\phi_{g,f}} \rho_g \xrightarrow{\phi_{g,f,g^*}} \rho_{gg^*} \xrightarrow{\phi_{h,g,g^*}^*} \rho_{hh^*} \xrightarrow{\phi_{h,h,h^*}} \rho_{hh^*} \xrightarrow{\phi_{k,h,k^*}} \rho_{kk^*} \xrightarrow{\phi_{k,k,k^*}} \rho_{kk^*}
\end{array}
\]
where the symbol \( = \) in this diagram follows from the interchange law (6) for a horizontal composition: the commutativity of \( \phi_{k,h}f \) and \( \phi_{g,*h,*} \). Note that the part involving \( \Lambda_1, \Lambda_2, \Lambda_3 \) is just the diagram (43). Let \( D_f^l \) be the corresponding diagram in \( C^+ \) with \( f \) replaced by \( 1 \), by using adjoint operations as in (44). Then as in (45), the 2-isomorphism in \( C^+ \) corresponding to the morphism \( [\Gamma_{k,hg}(\psi_k \circ \Gamma_{h,g})](\chi) \) in \( \mathcal{T}_\rho f \) is

\[
D_f^l \xrightarrow{\chi} D_f^l.
\] (87)

6.3. The natural isomorphism \( \Gamma_{k,hg}(\Gamma_{k,h} \circ \psi_g) \). To calculate \( \Gamma_{k,h} \circ \psi_g \), we fix the part

\[
\begin{array}{cccc}
X & \rho_g & X & \rho_f & X & \rho_g^* & X \\
\rho_g f & \rho_g f & \rho_g f & \rho_g f & \rho_g f & \rho_g f & \rho_g f
\end{array}
\]

in the lower half of (75), which corresponds to \( \psi_g \). The 3-isomorphism

\[
\tilde{\Lambda}_1 = \Diamond \#_1 \Phi_{k,hg,*h,*} \#_1 \Diamond,
\] (88)

as a whiskered associator (14), changes the 1-isomorphism \( \rho_{hh^*} \) in the lower part of (75) to the wavy diagonal \( \rho_{kg^*} \) of the same quadrilateral in the following diagram:

\[
\begin{array}{cccc}
X & \rho_k & X & \rho_h & X & \rho_g & X & \rho_f & X & \rho_g^* & X & \rho_h^* & X & \rho_k^* & X \\
\rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h \\
\rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*}
\end{array}
\]

The 3-isomorphism

\[
\tilde{\Lambda}_2 = \Diamond \#_1 [\Phi_{k,hg,*h,*} \#_0 (\rho_{h^*} \rho_{k^*})] \#_1 \Diamond,
\] (89)

as a whiskered associator (14), changes the diagonal \( \rho_{kg^*} \) of the above dotted-wavy quadrilateral to the wavy diagonal \( \rho_{kh} \) of the same quadrilateral in the following diagram:

\[
\begin{array}{cccc}
X & \rho_k & X & \rho_h & X & \rho_g & X & \rho_f & X & \rho_g^* & X & \rho_h^* & X & \rho_k^* & X \\
\rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h & \rho_k h \\
\rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*} & \rho_{kg^*}
\end{array}
\]
The 3-isomorphism
\[ \tilde{\Lambda}_3 = \Diamond \#_1 \Phi^{-1}_{kh^*, k^*}, \] (90)
as a whiskered associator (14), changes the diagonal \( \rho_{kh} \) of the above dotted quadrilateral to the wavy diagonal \( \rho_{h^*k^*} \) of the same quadrilateral in the following diagram:

![Diagram](image)

The composition \( \tilde{\Lambda}_1 \#_2 \tilde{\Lambda}_2 \#_2 \tilde{\Lambda}_3 \) is the 3-isomorphism
\[ \Gamma_{k,h} \circ \psi_g : \psi_h \circ \psi_g (\chi) \mapsto \psi_{kh} \circ \psi_g (\chi), \]
corresponding to the lower half of (75).

The 3-isomorphism
\[ \tilde{\Lambda}_4 = \Diamond \#_1 \Phi_{kh, gf, g^*} \#_0 (\rho_{g^*} \rho_{h^*} \rho_{k^*}) \] (92)
as a whiskered associator (14), changes the diagonal \( \rho_{g^*} \) of the dotted quadrilateral in (91) to the wavy diagonal \( \rho_{kf} \) of the same quadrilateral in the following diagram:

![Diagram](image)

The 3-isomorphism
\[ \tilde{\Lambda}_5 = \Diamond \#_1 \Phi_{kh, g, f} \#_0 (\rho_{g^*} \rho_{h^*} \rho_{k^*}) \] (93)
as a whiskered associator (14), changes the diagonal \( \rho_{gf} \) of the above dotted quadrilateral to the wavy diagonal \( \rho_{kg} \) of the same quadrilateral in the following diagram:

![Diagram](image)
At last, the 3-isomorphism
\[ \tilde{\Lambda}_6 = \Diamond \#_1 \Phi_{k_f,g}^{-1} \cdot h^* \cdot k^*, \]
(94)
as a whiskered associator (14), changes the diagonal \( \rho_{k^*} \) of the above dotted quadrilateral to the wavy diagonal \( \rho_{g^* k^*} \) of the same quadrilateral in the following diagram:

The composition \( \tilde{\Lambda}_4 \#_2 \tilde{\Lambda}_5 \#_2 \tilde{\Lambda}_6 \) is the 3-isomorphism \( \Gamma_{kh,g}(\chi) : \psi_{kh} \circ \psi_{g}(\chi) \Rightarrow \psi_{khg}(\chi) \) corresponding to the lower half of (75).

The composition of \( \tilde{\Lambda}_1 \#_2 \cdots \#_2 \tilde{\Lambda}_6 \) in the 2-category \( C^+ \) is the following diagram \( D_f : = \)

Let \( D^r \) be the corresponding diagram in \( C^+ \) with \( f \) replaced by 1, by using adjoint operations as in (44). Then the 2-isomorphism in \( C^+ \) corresponding to the morphism \( [\Gamma_{kh,g} \# (\Gamma_{k,h} \circ \psi_{g})](\chi) \) in \( \mathbb{T} r_2 \rho_f \) is

6.4. THE PROOF OF THE ASSOCIATIVITY. Let us show the identity (1), i.e., that diagrams \( D^l \Rightarrow D^l_f \) in (87) and \( D^r \Rightarrow D^r_f \) in (96) are identical in the 2-category \( C^+ \), by using the 3-cocycle identity (24) repeatedly. This proof corresponds to that of the 1-dimensional case in Section 6.1 step by step.

Apply the 3-cocycle identity (24) to the dotted diagram in (95) with \( g_4 = k, g_3 = \)
where $g_2 = gf$, $g_1 = g^*$ to get wavy isomorphisms in the following diagram

$$h, g_2 = gf, g_1 = g^*$$

Note that $\tilde{\Lambda}_3$ in (90) and $\tilde{\Lambda}_6$ in (94) are the inverse of associators. Apply the 3-cocycle identity, the inverse version of (24) (the lower and upper boundaries are exchanged), to the above dotted diagram with $g_4 = k_f, g_3 = g^*, g_2 = h^*, g_1 = k^*$ to get wavy isomorphisms in the following:

$$\hat{\Lambda}$$

Note that the commutative cube in (25) implies the following identity.

$$\phi_a := \phi_{g4g3g2g1}, \phi_b := \phi_{g4g3g2}, \phi_c := \phi_{g3g2g1}.$$
side is the left (this 2-isomorphism is inverted), top and front faces of the cube. Apply this identity to the dotted diagram in (98) with \( g_4 = k, g_3 = h_f, g_2 = g^*, g_1 = h^* \) to get wavy isomorphisms in the following:

\[
\begin{align*}
\rho_f & \twoheadrightarrow \phi_{g,f} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{h,g} \twoheadrightarrow \rho_{h^*} \twoheadrightarrow \phi_{k,h} \twoheadrightarrow \rho_{k^*} \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \rho_{k^*,h^*} \\
\phi_{k,h} & \twoheadrightarrow \phi_{k,h} \twoheadrightarrow \phi_{h,f} \twoheadrightarrow \rho_{h^*} \twoheadrightarrow \phi_{h,f} \twoheadrightarrow \rho_{h^*} \twoheadrightarrow \phi_{h,f} \twoheadrightarrow \rho_{h^*} \\
\rho_{k^*} & \twoheadrightarrow \phi_{g,k} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{g,k} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{g,k} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{g,k} \\
\end{align*}
\]

(99)

Apply the 3-cocycle identity (24) to the above dotted diagram with \( g_4 = k, g_3 = h, g_2 = g, g_1 = f \) to get wavy isomorphisms in the following diagram \( \mathcal{D}_f := \)

\[
\begin{align*}
\rho_f & \twoheadrightarrow \phi_{g,f} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{h,g} \twoheadrightarrow \rho_{h^*} \twoheadrightarrow \phi_{k,h} \twoheadrightarrow \rho_{k^*} \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \rho_{k^*,h^*} \\
\phi_{k,h} & \twoheadrightarrow \phi_{k,h} \twoheadrightarrow \phi_{h,f} \twoheadrightarrow \rho_{h^*} \twoheadrightarrow \phi_{h,f} \twoheadrightarrow \rho_{h^*} \twoheadrightarrow \phi_{h,f} \twoheadrightarrow \rho_{h^*} \\
\rho_{k^*} & \twoheadrightarrow \phi_{g,k} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{g,k} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{g,k} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{g,k} \\
\end{align*}
\]

(100)

With \( f \) replaced by 1, by using adjoint operations as in (44), the diagram \( \mathcal{D}_f^1 \) corresponding to the upper half is identically changed to the following diagram \( \mathcal{D}_1 := \)

\[
\begin{align*}
\rho_{k^*} & \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \rho_{k^*} \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \rho_{k^*} \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \rho_{k^*} \twoheadrightarrow \phi_{k^*,h^*} \\
\phi_{k^*,h^*} & \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \phi_{k^*,h^*} \twoheadrightarrow \phi_{k^*,h^*} \\
\rho_{k^*} & \twoheadrightarrow \phi_{g,k} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{g,k} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{g,k} \twoheadrightarrow \rho_{g^*} \twoheadrightarrow \phi_{g,k} \\
\end{align*}
\]

(101)

where \( \Xi_j^1 \) is the adjoint of \( \Xi_j, j = 1, 2 \). Then the whole diagram \( \mathcal{D}^r \cong \mathcal{D}_1^r \) in (96) is
identically changed to
\[ \widetilde{D}^r_1 \xrightarrow{\chi} \widetilde{D}^r_f, \]

namely,

\[ \begin{align*}
\ldots \quad \rho_1 & \quad \chi \\
\text{\includegraphics[width=0.8\textwidth]{diagram1.png}} \nonumber
\end{align*} \]  

(102)

Note that (100) is exactly \( D_f \) in (86) with two extra 2-isomorphisms \( \Xi_1 \) and \( \Xi_2 \). But by definition, the 2-isomorphisms \( \Xi_1 \) and \( \Xi_2^\dagger \) are the associators (13) corresponding to the 3-isomorphisms in \( \mathcal{C} \), which change

\[ \begin{align*}
\ldots \quad \rho_{hg} & \quad \rho_{kh} \\
\text{\includegraphics[width=0.8\textwidth]{diagram2.png}} \nonumber
\end{align*} \]

and we have

\[ \begin{align*}
\ldots \quad \rho_k & \quad \chi \\
\text{\includegraphics[width=0.8\textwidth]{diagram3.png}} \nonumber
\end{align*} \]

by cancellation (42). So \( \Xi_1 \) and \( \Xi_2^\dagger \) are cancelled. More precisely, as a 3-isomorphism, \( \Xi_1^\dagger \#_0 \chi \#_0 \Xi_1 \) is

\[ (\Xi_1^\dagger \#_0 \chi) \#_0 (\Xi_1 \#_0 \chi) = (\Xi_1^\dagger \#_1 \Xi_1) \#_0 \chi = 1_{\rho_{kg}} \#_0 \chi, \]

and we have

\[ (\phi_{k,hg}^{-1} \#_0 \phi_{h,g}^{-1}) \#_0 (\phi_{h,g} \#_0 \phi_{k,hg}) \#_0 \chi = (\phi_{k,hg}^{-1} \#_0 \phi_{h,g}^{-1}) \#_0 \chi \#_0 (\phi_{h,g} \#_0 \phi_{k,hg}) \]
in the 2-category $\mathcal{C}^+$, up to whiskering, by the interchange law. Namely, $\mathcal{D}^r_1 \xrightarrow{\chi} \mathcal{D}^r_f$ in (102) is identical to

$$
\begin{array}{ccc}
\cdots \cdots \rho_1 & \xrightarrow{\chi} & \rho_f \\
\downarrow \phi_{h,g}^{-1} & & \downarrow \phi_{h,g} \\
\cdots \cdots \rho_{hg} & & \cdots \cdots \rho_{hg} \\
& \downarrow \phi_{k,hg}^{-1} \downarrow \phi_{k,hg} & & \\
\cdots \cdots \rho_{k,g} & & \cdots \cdots \rho_{k,g} \\
\end{array}
$$

Similarly, the 2-isomorphisms $\Xi^2_2$ in (100) and $\Xi^1_2$ in (101) are also cancelled. The resulting diagram is exactly the diagram $\mathcal{D}^l_1 \xrightarrow{\phi} \mathcal{D}^l_f$ in (87). This completes the proof of Theorem 3.7.

References


Department of Mathematics, Zhejiang University, Zhejiang 310027, P. R. China
Email: wang@zju.edu.cn

This article may be accessed at http://www.tac.mta.ca/tac/
THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Full text of the journal is freely available from the journal’s server at http://www.tac.mta.ca/tac/. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

INFORMATION FOR AUTHORS The typesetting language of the journal is \TeX, and \LaTeX2e is required. Articles in PDF format may be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at http://www.tac.mta.ca/tac/.

MANAGING EDITOR. Robert Rosebrugh, Mount Allison University: rrosebrugh@mta.ca

\TeXNICAL EDITOR. Michael Barr, McGill University: barr@math.mcgill.ca

ASSISTANT \TeX EDITOR. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne: gavin_seal@fastmail.fm

TRANSMITTING EDITORS.
Clemens Berger, Université de Nice-Sophia Antipolis: cberger@math.unice.fr
Richard Blute, Université d’Ottawa: rblute@uottawa.ca
Lawrence Breen, Université de Paris 13: breen@math.univ-paris13.fr
Ronald Brown, University of North Wales: ronnie.profbrown(at)btinternet.com
Valeria de Paiva: Nuance Communications Inc: valeria.depaiva@gmail.com
Ezra Getzler, Northwestern University: getzler(at)northwestern(dot)edu
Kathryn Hess, Ecole Polytechnique Fédérale de Lausanne: kathryn.hess@epfl.ch
Martin Hyland, University of Cambridge: M.Hyland@dpmms.cam.ac.uk
Anders Kock, University of Aarhus: kock@imf.au.dk
Stephen Lack, Macquarie University: steve.lack@mq.edu.au
F. William Lawvere, State University of New York at Buffalo: wlawvere@buffalo.edu
Tom Leinster, University of Edinburgh: Tom.Leinster@ed.ac.uk
Ieke Moerdijk, Radboud University Nijmegen: i.moerdijk@math.ru.nl
Susan Niefield, Union College: niefiels@union.edu
Robert Paré, Dalhousie University: pare@mathstat.dal.ca
Jiri Rosicky, Masaryk University: rosicky@math.muni.cz
Giuseppe Rosolini, Università di Genova: rosolini@disi.unige.it
Alex Simpson, University of Ljubljana: Alex.Simpson@fmf.uni-lj.si
James Stasheff, University of North Carolina: jds@math.upenn.edu
Ross Street, Macquarie University: street@math.mq.edu.au
Walter Tholen, York University: tholen@mathstat.yorku.ca
Myles Tierney, Université du Québec à Montréal: tierney.myles4@gmail.com
R. J. Wood, Dalhousie University: rjwood@mathstat.dal.ca