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TRANSFINITE LIMITS IN TOPOS THEORY

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ABSTRACT. For a coherent site we construct a canonically associated enlarged coherent site, such that cohomology of bounded below complexes is preserved by the enlargement. In the topos associated to the enlarged site transfinite compositions of epimorphisms are epimorphisms and a weak analog of the concept of the algebraic closure exists. The construction is a variant of the work of Bhatt and Scholze on the pro-étale topology.

1. Introduction

In [5] B. Bhatt and P. Scholze construct a so called pro-étale enlargement of the usual étale topos of a scheme, see also [2, Tag 0965] and [16]. A characteristic feature of the pro-étale topos is that certain limits have better exactness properties than in the usual étale topos, while the cohomology of classical étale sheaves does not change. This turns out to be useful when working with unbounded derived categories.

In this paper we propose a variant of the theory of Scholze and Bhatt–Scholze which works for a coherent site and we give two applications to the calculation of hypercohomology and to the existence of a left adjoint of the pullback of sheaves along a closed immersion of schemes in the Nisnevich and étale topology.

Consider a coherent topos $\mathbf{E}$, as defined in [1, Exp. VI], for example the étale topos of a quasi-compact and quasi-separated scheme. The key property we are interested in is whether in the topos a transfinite composition of epimorphisms is an epimorphisms. More precisely we say that $\mathbf{E}$ is α-transfinite if the following property holds:

For an ordinal $\lambda \le \alpha$ and for a functor $F : \lambda^{\mathrm{op}} \to \mathbf{E}$ with the property that

- for any ordinal $1 \le i+1 < \lambda$ the morphism $F_{i+1} \to F_i$ is an epimorphism and
- for any limit ordinal $\mu < \lambda$ the natural morphism

  $$F_\mu \xrightarrow{\sim} \lim_{i<\mu} F_i$$

  is an isomorphism

we ask that

$$\lim_{i<\lambda} F_i \xrightarrow{\sim} F_0$$

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is an epimorphism. Here the ordinal λ as an ordered set is identified with the associated category.

The property $\aleph_0$-transfinite is studied in [5] under the name replete. The topos of sets is α-transfinite for all cardinals α, while the standard topoi that show up in algebraic geometry, for example the small étale topos, are usually not $\aleph_0$-transfinite. So it is natural to try to make them transfinite in a minimal way.

In our first main theorem, Theorem 4.1, we construct for any coherent site $\mathbf{C}$ which is admissible in the sense of Definition 3.6 and for any infinite cardinal α a new coherent site $\langle\alpha\rangle\mathbf{C}$ and a continuous functor preserving finite limits $\pi_\alpha^{\mathbf{C}} : \mathbf{C} \to \langle\alpha\rangle\mathbf{C}$ such that the topos $\langle\alpha\rangle\mathbf{E} = \mathrm{Sh}(\langle\alpha\rangle\mathbf{C})$ is α-transfinite and the associated morphism of topoi

$$((\pi_\alpha^{\mathbf{E}})_*, (\pi_\alpha^{\mathbf{E}})^*) : \langle\alpha\rangle\mathbf{E} \to \mathbf{E} = \mathrm{Sh}(\mathbf{C}),$$

has the property that $(\pi_\alpha^{\mathbf{E}})^*$ is fully faithful and preserves cohomology of bounded below complexes.

In our second main theorem, Theorem 4.2, we show that for large α the topos $\langle\alpha\rangle\mathbf{E}$ is generated by weakly contractible objects. Here following [5] we call an object C of $\mathbf{E}$ weakly contractible if any epimorphism $D \to C$ in $\mathbf{E}$ splits. In some sense this means that the topoi $\langle\alpha\rangle\mathbf{E}$ 'stabilize' for α large. Note that in category theory it is more common to use the word projective instead of weakly contractible.

The main difference between our construction and the construction in [5] for the étale topos is that we work with a topology, which we call transfinite topology, which sits between the usual étale topology and the pro-étale topology and in some sense captures properties of both. The precise relation is explained in Section 9 for the Zariski topos.

Concretely our construction works as follows. We consider the pro-category pro^α-$\mathbf{C}$ of pro-objects whose index category is bounded by α. We define in Section 5 the transfinite topology on pro^α-$\mathbf{C}$ as the weakest topology such that the canonical functor

$$\mathbf{C} \to \mathrm{pro}^\alpha\text{-}\mathbf{C}$$

is continuous and such that a transfinite composition of covering morphisms in pro^α-$\mathbf{C}$ is a covering morphism. Then the site $\langle\alpha\rangle\mathbf{C}$ is just pro^α-$\mathbf{C}$ with the transfinite topology.

In order to motivate the construction of this paper we explain in Section 8 why classical Cartan-Eilenberg hypercohomology of unbounded complexes can be recovered as the derived cohomology on the enlarged topos $\langle\alpha\rangle\mathbf{E}$.

Another motivation stems from the fact that, roughly speaking, in the world of transfinite enlarged topoi the pullback functor of sheaves i^* for a morphism of schemes $i : Y \to X$ tends to have a left adjoint in the setting of Grothendieck's six functor formalism. This was observed for the pro-étale topology in [5, Rmk. 6.1.6] and the argument in our setting is very similar. Concretely, we show that for X quasi-compact and separated and for a closed immersion $i : Y \to X$ the pullback functor

$$i^* : D_\Lambda(\langle\alpha\rangle X_{\mathrm{t}}) \to D_\Lambda(\langle\alpha\rangle Y_{\mathrm{t}})$$

on derived categories of sheaves of Λ-modules has a left adjoint if α is large. Here t stands for the small Nisnevich or étale topology on the category of affine, étale schemes over X or Y.

Notation. A category is called small if up to isomorphism its objects form a set and not only a class. When we say topos we mean a Grothendieck topos. For topos theory we follow the notation of [12].

A coherent site is a small category having finite limits together with a topology generated by finite coverings. For a subcanonical site $\mathbf{C}$ we write $\mathbf{y} : \mathbf{C} \to \mathrm{Sh}(\mathbf{C})$ for the Yoneda embedding.

A partially ordered set $(S, \leq)$ is considered as a category with a unique morphism $s_1 \to s_2$ if $s_1 \leq s_2$ and no morphisms form s_1 to s_2 otherwise.

By a 2-category we mean a $(2,1)$-category, i.e. all 2-morphisms are invertible, 2-functors between 2-categories are allowed to be lax. So the formalism of ∞-categories is applicable and we freely use notions from [10].

We use Zermelo–Fraenkel set theory including the axiom of choice. We do not use the concept of universes as applied in [1].

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2. Preliminaries on towers and limits

In this section we summarize some properties of pro-categories and diagrams indexed by ordinals, which we call towers.

Pro-categories. As a general reference for pro-categories see for example [8]. In this paper we need to bound the cardinalities of the index categories; however the basic arguments essentially stay the same as in the existing literature, so we do not give any proofs.

Let $\mathbf{C}$ and I be a categories and let α be an infinite cardinal. We call I an α-category if the system of all morphisms $\mathrm{Mor}(I)$ of I forms a set of cardinality at most α. For a α-category I and a functor $F : I \to \mathbf{C}$ we call $\lim_{i \in I} F(i)$ an α-limit if it exists. There is a corresponding notion for a functor to preserve α-limits.

Note that the formation of α-limits can be 'decomposed' into equalizers and products indexed by sets I with $\operatorname{card} I \leq \alpha$ [11, Sec. V.2].

On can associate with $\mathbf{C}$ its pro-category $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ indexed by cofiltered α-categories. The objects of $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ are the functors

$$F : I \to \mathbf{C} \tag{1}$$

where I is cofiltered α-category. For $F : I \to \mathbf{C}$ and $G : J \to \mathbf{C}$ objects of $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ the set of morphisms from F to G is given by

$$\mathrm{Mor}_{\mathrm{pro}^\alpha\text{-}\mathbf{C}}(F, G) = \lim_{j\in J} \operatorname*{colim}_{i\in I} \mathrm{Mor}_{\mathbf{C}}(F(i), G(j)). \tag{2}$$

Each object of $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ has a level representation $F : I \to \mathbf{C}$ with I a cofinite directed set with $\operatorname{card} I \leq \alpha$. This follows from the proof of [1, Prop. I.8.1.6]. All cofiltered α-limits exist in $\mathrm{pro}^\alpha\text{-}\mathbf{C}$, see [8, Thm. 4.1]. If $\mathbf{C}$ has finite limits $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ has all α-limits.

Pro-categories can be characterized by the following universal property. Let $\mathbf{Cat}^{fl}$ be the 2-category whose objects are small categories having finite limits, whose 1-morphisms are functors preserving finite limits and whose 2-morphisms are all natural equivalences. Let $\mathbf{Cat}^{l}$ be the 2-category whose objects are all small categories having α-limits, whose 1-morphisms are functors preserving α-limits and whose 2-morphisms are natural equivalences.

2.1. Proposition. *The canonical 2-functor $\mathbf{Cat}^{f} \to \mathbf{Cat}^{fl}$ is right adjoint to the 2-functor mapping $\mathbf{C} \mapsto \mathrm{pro}^\alpha\text{-}\mathbf{C}$.*

For the notion of adjointness in higher category theory see for example [10, Sec. 5.2]. Proposition 2.1 is equivalent to the statement that there is a natural equivalence of groupoids

$$\mathrm{Mor}_{\mathbf{Cat}^{fl}}(\mathbf{C}, \mathbf{D}) \xrightarrow{\sim}_{\phi} \mathrm{Mor}_{\mathbf{Cat}^{f}}(\mathrm{pro}^\alpha\text{-}\mathbf{C}, \mathbf{D}) \tag{3}$$

for $\mathbf{C} \in \mathbf{Cat}^{fl}$ and $\mathbf{D} \in \mathbf{Cat}^{l}$. This equivalence is given as follows. For $F \in \mathrm{Mor}_{\mathbf{Cat}^{fl}}(\mathbf{C}, \mathbf{D})$ and $(C_i)_{i\in I} \in \mathrm{pro}^\alpha\text{-}\mathbf{C}$ set

$$\phi(F)((C_i)_{i\in I}) = \lim_{i\in I} F(C_i).$$

Towers. For an ordinal λ consider a functor $F : \lambda^{\mathrm{op}} \to \mathbf{C}$. We usually denote such a functor by $F = (F_i)_{i<\lambda}$. For an ordinal $\mu \leq \lambda$ write

$$F_{<\mu} = \lim_{i<\mu} F_i \tag{4}$$

if the limit exists. We call F a λ-tower (or just tower) if for any limit ordinal $\mu < \lambda$ the limit (4) exists and if the natural morphism

$$F_\mu \to F_{<\mu}$$

is an isomorphism.

We say that the tower $F : \lambda^{\mathrm{op}} \to \mathbf{C}$ has a certain property $\mathcal{P}$ if all the morphisms $F_{i+1} \to F_i$ have the property $\mathcal{P}$ for $1 \leq i + 1 < \lambda$. We call $F_{<\lambda} \to F_0$ the transfinite composition of the tower $(F_i)_{i<\lambda}$ if the limit exists. By a morphism of towers we mean a natural transformation of functors.

Let α be a cardinal and $\mathbf{C}$, $\mathbf{D}$ categories having all α-limits. We say that a functor $u : \mathbf{D} \to \mathbf{C}$ preserves α-transfinite limits if u maps λ-towers to λ-towers for $\lambda \leq \alpha$.

Let $F = (F_i)_{i<\lambda}$ be a tower and $\pi_0 : E_0 \to F_\mu$ a morphism for some ordinal $\mu < \lambda$. If fiber products exist in $\mathbf{C}$ we define the pullback tower $E = \pi_0^* F$ by

$$E_i = \begin{cases} E_0 & \text{if } i \leq \mu \\ E_0 \times_{F_\mu} F_i & \text{if } i > \mu \end{cases}$$

There is a natural morphism of towers $\pi : E \to F$.

Let $F = (F_i)_{i<\lambda}$ and $G = (G_j)_{j<\mu}$ be two towers. If $F_{<\lambda} \cong G_0$ we consider the concatenation of towers $((F \circ G)_k)_{k<\lambda+\mu}$ with

$$(F \circ G)_k = \begin{cases} F_k & \text{if } k < \lambda \\ G_j & \text{if } k = \lambda + j \end{cases}$$

The concatenation of two towers can be generalized to the concatenation of a family of towers indexed by an ordinal. We leave the details to the reader.

If we are given a symmetric monoidal structure $\odot : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which preserves limits of towers and we are given two towers $F = (F_i)_{i<\lambda}$ and $G = (G_j)_{j<\mu}$ we consider the tower $((F \odot G)_k)_{k<\max(\lambda,\mu)}$. Without loss of generality let $\lambda \leq \mu$. Then, assuming $F_{<\lambda}$ exists, $F \odot G$ is defined by

$$(F \odot G)_k = \begin{cases} F_k \odot G_k & \text{if } k < \min(\lambda, \mu) \\ F_{<\lambda} \odot G_k & \text{if } k \geq \lambda \end{cases}$$

For example we can use the categorical product for $\odot$ if it exists.

3. Transfinite sites and topoi

In this section we study sites and topoi in which certain limits indexed by ordinal numbers behave well. More precisely we call a topos transfinite if transfinite compositions of epimorphisms are epimorphisms, in the sense of towers as in Section 2. The $\aleph_0$-transfinite topoi are the same as the replete topoi of Bhatt and Scholze [5, Sec. 3].

Let α be an infinite cardinal and let $\mathbf{E}$ be a topos.

3.1. Definition. *We say that $\mathbf{E}$ is α-transfinite if for any ordinal $\lambda \leq \alpha$ and for any λ-tower $(E_i)_{i<\lambda}$ of epimorphisms, i.e. with $E_{i+1} \to E_i$ an epimorphism for all $1 \leq i+1 < \lambda$, the transfinite composition*

$$E_{<\lambda} = \lim_{i<\lambda} E_i \to E_0$$

is an epimorphism. We say that $\mathbf{E}$ is transfinite if it is α-transfinite for all cardinals α.

3.2. Example. The topos of sets **Set** is transfinite. For a group G the topos $\mathbf{B}G$ of G-sets is transfinite.

More generally, any topos with enough weakly contractible objects in the sense of [5, Def. 3.2.1] is transfinite.

3.3. Definition. *We call an object C of a topos $\mathbf{E}$ weakly contractible, if any epimorphism $D \to C$ splits in $\mathbf{E}$, i.e. if there is a morphism $C \to D$ such that the composition $C \to D \to C$ is the identity. We say that a topos $\mathbf{E}$ has enough weakly contractible objects if for any object C of $\mathbf{E}$ there is an epimorphism $D \to C$ with D weakly contractible.*

Note that a small coproduct of weakly contractible objects in a topos is weakly contractible.

As any epimorphism splits in **Set**, the topos of sets has enough weakly contractible objects. The referee points out that the following proposition is a classical fact about categories with enough projective objects: in fact in such a category a morphism $f : C \to E$ is an epimorphism if and only if the induced map $\mathrm{Mor}(P, C) \to \mathrm{Mor}(P, E)$ is surjective for any projective object P. Therefore Proposition 3.4 is reduced to Example 3.2. As the latter might not be well-known to a geometer and as we need a variant of the proof in Corollary 5.6, we give a detailed argument below.

3.4. Proposition. *Let $\mathbf{E}$ be a topos with enough weakly contractible objects. Then $\mathbf{E}$ is transfinite.*

Proof. Let $F = (F_i)_{i<\lambda}$ be a tower of epimorphisms in $\mathbf{E}$. Choose a weakly contractible E_0 and an epimorphism $\pi_0 : E_0 \to F_0$. Let $\pi : E \to F$ be the pullback tower along π_0. As the pullback of an epimorphism is an epimorphism in a topos the tower E consists of epimorphisms. In the commutative diagram

$$\begin{tikzcd} F_{<\lambda} \arrow[d, "\mathbf{1}"'] & E_{<\lambda} \arrow[l] \arrow[d, "\mathbf{3}"] \\ F_0 & E_0 \arrow[l, "\mathbf{2}"'] \end{tikzcd}$$

the morphisms **2** and **3** are epimorphisms by Claim 3.5. So as **1** is dominated by an epimorphism it is itself an epimorphism. ∎

3.5. Claim. *The morphism $E_{<\lambda} \xrightarrow{\mathbf{3}} E_0$ splits. In particular it is an epimorphism.*

Proof of claim. We successively construct a compatible family of splittings $(E_0 \xrightarrow{s_i} E_i)_{i<\mu}$ for $\mu \leq \lambda$. Compatible means that the diagram

$$\begin{tikzcd} E_0 \arrow[r, "s_i"] \arrow[rr, bend right, "s_j"'] & E_i \arrow[r] & E_j \end{tikzcd}$$

commutes for all $j < i < \mu$. Assume the family of splittings has been constructed for some $\mu < \lambda$. If μ is a successor ordinal use the weak contractibility of E_0 to find s_μ such that the diagram

$$\begin{tikzcd} & E_\mu \arrow[d] \\ E_0 \arrow[ur, "s_\mu"] \arrow[r, "s_{\mu-1}"'] & E_{\mu-1} \end{tikzcd}$$

commutes. If μ is a limit ordinal let

$$s_\mu = \lim_{i<\mu} s_i : E_0 \to E_\mu$$

be the morphism obtained from the system $(s_i)_{i<\mu}$ by the universal property of the inverse limit and the isomorphism $E_\mu \xrightarrow{\sim} \lim_{i<\mu} E_i$.

By this successive construction we can assume that there is a system of splittings $(E_0 \xrightarrow{s_i} E_i)_{i<\lambda}$. The morphism $\lim_{i<\lambda} s_i$ is a splitting of **3**. ∎

Another way, beside finding enough weakly contractible objects, to show that a topos is transfinite, is to find a site defining the topos in which transfinite compositions of coverings are coverings. We will make this precise in the following.

3.6. DEFINITION. *A coherent site* $\mathbf{C}$ *is called admissible if its topology is subcanonical and for a finite family of objects* $(C_i)_{i\in I}$ *the coproduct* $C = \coprod_{i\in I} C_i$ *exists and* $\{C_i \to C \mid i \in I\}$ *is a covering. We furthermore assume that in* $\mathbf{C}$ *there is a strict initial object and coproducts are disjoint and stable under pullback, see [12, App.] and Definition 7.3.*

3.7. LEMMA. *The following are equivalent for a coherent subcanonical site* $\mathbf{C}$*:*

(i) $\mathbf{C}$ *is admissible.*

(ii) $\mathbf{C}$ *has a strict instal object* $\emptyset$ *and the essential image of the Yoneda functor*

$$\mathbf{y} : \mathbf{C} \to \mathbf{y}(\emptyset)/\operatorname{Sh}(\mathbf{C})$$

is closed under finite coproducts in the comma category $\mathbf{y}(\emptyset)/\operatorname{Sh}(\mathbf{C})$.

Working with admissible sites instead of coherent sites is no real restriction as the following lemma shows. For a site $\mathbf{C}$ we denote by $\mathbf{ay} : \mathbf{C} \to \operatorname{Sh}(\mathbf{C})$ the composition of the Yoneda embedding and the sheafification.

3.8. LEMMA. *For any coherent site* $\mathbf{C}$ *let* $\overline{\mathbf{C}}$ *be the smallest strictly full subcategory of* $\operatorname{Sh}(\mathbf{C})$ *which contains the essential image of* $\mathbf{ay}$ *and which is closed under finite coproducts and finite limits. Then* $\overline{\mathbf{C}}$ *with the epimorphic coverings is admissible and the continuous functor* $\mathbf{ay} : \mathbf{C} \to \overline{\mathbf{C}}$ *induces an equivalence of topoi.*

Recall that a morphism $E \to D$ in $\mathbf{C}$ is called a covering morphism if the sieve generated by $E \to D$ is a covering sieve.

3.9. Definition. *An admissible site* $\mathbf{C}$ *is called* α*-transfinite if* α*-limits exist in* $\mathbf{C}$ *and if transfinite compositions of* λ*-towers of covering morphisms* ($\lambda \leq \alpha$) *are covering morphisms, i.e. we assume that for a* λ*-tower* $(F_i)_{i<\lambda}$ *in* $\mathbf{C}$ *with* $F_{i+1} \to F_i$ *a covering morphism for all* $i+1 < \lambda$ *that* $F_{<\lambda} \to F_0$ *is a covering morphism.*

3.10. Proposition. *The topos* $\mathrm{Sh}(\mathbf{C})$ *associated with an* α*-transfinite site* $\mathbf{C}$ *is* α*-transfinite.*

Proof. Let $(F_i)_{i<\lambda}$ be a tower of epimorphisms in $\mathrm{Sh}(\mathbf{C})$ ($\lambda \leq \alpha$). Choose a family $(C_r)_{r\in R}$ of objects in $\mathbf{C}$ and an epimorphism $\pi_1 : \coprod_{r\in R} \mathbf{y}(C_r) \to F_0$. Recall that $\mathbf{y} : \mathbf{C} \to \mathrm{Sh}(\mathbf{C})$ denotes the Yoneda embedding. For simplicity of notation we assume that $R = \{0\}$ consists of only one element. Choose a family $(C_r^{(1)})_{r\in R_1}$ of elements of $\mathbf{C}$ and an epimorphism

$$\coprod_{r\in R_1} \mathbf{y}(C_r^{(1)}) \to \mathbf{y}(C_0) \times_{F_0} F_1. \tag{5}$$

As $\mathbf{y}(C_0)$ is quasi-compact there is a finite subset $\tilde{R}_1 \subset R_1$ such that the composite morphism

$$\coprod_{r\in \tilde{R}_1} \mathbf{y}(C_r^{(1)}) \to \mathbf{y}(C_0) \times_{F_0} F_1 \to \mathbf{y}(C_0) \tag{6}$$

is an epimorphism. As the Yoneda functor is fully faithful, this morphism is induced by a covering morphism $C_1 = \coprod_{r\in\tilde{R}_1} C_r^{(1)} \to C_0$, see [12, III.7 Cor. 7]. We get a commutative diagram

$$\begin{tikzcd} \mathbf{y}(C_1) \arrow[r] \arrow[d] & F_1 \arrow[d] \\ \mathbf{y}(C_0) \arrow[r] & F_0 \end{tikzcd}$$

which we are going to extend successively to the morphism of towers (8).

For doing so we assume now that for $\mu < \lambda$ we have constructed a tower $(C_i)_{i<\mu}$ of covering morphisms in $\mathbf{C}$ and a morphism of towers $\pi_\mu : (\mathbf{y}(C_i))_{i<\mu} \to F|_\mu$. If μ is a successor ordinal we proceed as above to find a covering morphism $C_\mu \to C_{\mu-1}$ and an extension of π_μ to a morphism of towers

$$\pi_{\mu+1} : (\mathbf{y}(C_i))_{i\leq\mu} \to F|_{\mu+1}. \tag{7}$$

If μ is a limit ordinal we let $C_\mu = \lim_{i<\mu} C_i$ and we let the morphism $\mathbf{y}(C_\mu) = \lim_{i<\mu} \mathbf{y}(C_i) \to F_\mu$ be the inverse limit of the morphism of towers π_μ. This defines the required extension as in (7) in the case of a limit ordinal μ.

In the end this successive construction produces a tower of covering morphisms $(C_i)_{i<\lambda}$ and a morphism of towers

$$\pi : (\mathbf{y}(C_i))_{i<\lambda} \to F. \tag{8}$$

The morphism $C_{<\lambda} = \lim_{i<\lambda} C_i \to C_0$ is the composition of a tower of covering morphisms, so is a covering morphism itself, because $\mathbf{C}$ is α-transfinite. In the commutative diagram

$$\begin{tikzcd} \mathbf{y}(C_{<\lambda}) \arrow[r] \arrow[d, "\mathbf{1}"'] & F_{<\lambda} \arrow[d, "\mathbf{3}"] \\ \mathbf{y}(C_0) \arrow[r, "\mathbf{2}"] & F_0 \end{tikzcd}$$

the morphism **1** is the Yoneda image of a covering morphism and therefore an epimorphism. As also **2** is an epimorphism, we see that **3** is dominated by an epimorphism and so is an epimorphism itself. ∎

3.11. Example. For an infinite cardinal α let $\mathbf{Aff}_\alpha$ be the category of affine schemes $\operatorname{Spec} R$ with $\operatorname{card}(R) \leq \alpha$. We endow $\mathbf{Aff}_\alpha$ with the fpqc-topology. Recall that the fpqc-topology on $\mathbf{Aff}_\alpha$ is generated by coverings $\{U_i \to U \mid i \in I\}$ with I finite, $U_i \to U$ flat and such that

$$\coprod_{i \in I} U_i \to U$$

is surjective. Clearly, the site $\mathbf{Aff}_\alpha^{\mathrm{fpqc}}$ is α-transfinite, so by Proposition 3.10 the fpqc-topos $\mathrm{Sh}(\mathbf{Aff}_\alpha^{\mathrm{fpqc}})$ is α-transfinite.

4. Main theorems

Let α be an infinite cardinal. Let $\mathbf{Si}$ be the 2-category in the sense of [11, XII.3] whose objects are admissible sites $\mathbf{C}$ (Definition 3.6), whose 1-morphisms are continuous functors $\mathbf{C} \to \mathbf{D}$ preserving finite limits and whose 2-morphisms are the natural equivalences. Similarly, we consider the 2-subcategory $\mathbf{Si}_\alpha$ of $\mathbf{Si}$ whose objects are the α-transfinite sites (Definition 3.9) whose 1-morphisms are the continuous functors preserving α-limits and whose 2-morphisms are all natural equivalences as above.

4.1. Theorem. *For an infinite cardinal α the canonical functor of 2-categories $\mathbf{Si}_\alpha \to \mathbf{Si}$ admits a left adjoint*

$$\langle \alpha \rangle : \mathbf{Si} \to \mathbf{Si}_\alpha.$$

For $\mathbf{C}$ admissible let $\mathbf{E} = \mathrm{Sh}(\mathbf{C})$ and $\langle \alpha \rangle \mathbf{E} = \mathrm{Sh}(\langle \alpha \rangle \mathbf{C})$ be the associated topoi. The induced morphism of topoi $\pi_\alpha : \langle \alpha \rangle \mathbf{E} \to \mathbf{E}$ has the property that π_α^* is fully faithful and preserves cohomology of bounded below complexes of abelian sheaves.

For the precise meaning of adjointness between 2-categories in our sense see [10, Sec. 5.2]. The proof of Theorem 4.1 is given in the following two sections. In Section 5 we define the site $\langle \alpha \rangle \mathbf{C}$ as the category of pro-objects $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ with the so called transfinite topology. In Proposition 6.3 we show that this site is admissible. The adjointness property is then obvious from the definition. The fact that $\langle \alpha \rangle$ is fully faithful is immediate from Lemma 6.5. The preservation of cohomology is shown in Proposition 6.6.

Unfortunately, we do not know whether the topos $\langle \alpha \rangle \mathbf{E}$ depends on the site C or only on the topos $\mathbf{E}$. Roughly speaking Theorem 4.1 means that for any admissible site $\mathbf{C}$ we get a tower of topoi

$$\cdots \to \langle \aleph_\lambda \rangle \mathbf{E} \to \cdots \to \langle \aleph_1 \rangle \mathbf{E} \to \langle \aleph_0 \rangle \mathbf{E} \to \mathbf{E} \tag{9}$$

indexed by all ordinals λ, such that the higher up we get the topoi become 'more transfinite'. In fact our second main theorem tells us, see Corollary 4.3, that from some point on the topoi in the tower (9) in fact are transfinite.

4.2. Theorem. *For any admissible site* $\mathbf{C}$ *there is a cardinal* β *such that for all cardinals* $\alpha \geq \beta$ *the topos* $\langle \alpha \rangle(\mathbf{E}) = \mathrm{Sh}(\langle \alpha \rangle(\mathbf{C}))$ *has enough weakly contractible objects. More precisely, in* $\langle \alpha \rangle(\mathbf{E})$ *there exists a generating set of coherent, weakly contractible objects.*

Recall that an object C of $\mathbf{E}$ is quasi-compact if any covering family has a finite subfamily which is covering. The object C is called coherent if it is quasi-compact and for any quasi-compact objects S, T of $\mathbf{E}$ and any two morphisms $S \to C$, $T \to C$ the object $S \times_C T$ is quasi-compact [1, Exp. VI.1].

The proof of Theorem 4.2 is given in the first part of Section 7. Using Proposition 3.4 we deduce:

4.3. Corollary. *For any admissible site* $\mathbf{C}$ *there is a cardinal* β *such that for all cardinals* $\alpha \geq \beta$ *the topos* $\langle \alpha \rangle \mathbf{E}$ *is transfinite.*

4.4. Remark. The cardinal β in Theorem 4.2 and Corollary 4.3 can be chosen to be $\mathrm{card}(\mathrm{Mor}(\mathbf{C}))$. More precisely β can be chosen in such a way that the admissible site $\mathbf{C}$ is β-small. For the notion of smallness see Definition 5.2.

5. The pro-site of a coherent site

Let $\mathbf{C}$ be a coherent site and let α be an infinite cardinal. We are going to construct two topologies on the pro-category $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ defined in Section 2, such that the embedding of categories $\mathbf{C} \to \mathrm{pro}^\alpha\text{-}\mathbf{C}$ is continuous, i.e. maps coverings to coverings. Recall that this embedding also preserves finite limits.

Weak topology. The weak topology on $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ is defined as the weakest topology such that the functor $\mathbf{C} \to \mathrm{pro}^\alpha\text{-}\mathbf{C}$ is continuous.

Clearly, for any covering morphism $V \to W$ in $\mathbf{C}$ and for a morphism $U \to W$ in $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ the base change $V \times_W U \to U$ is a covering morphism in the weak topology. We call such weak covering morphisms distinguished. Similarly, if $\{W_i \to W \mid i \in I\}$ is a finite covering in $\mathbf{C}$ the family $\{W_i \times_W U \to U \mid i \in I\}$ is a weak covering in $\mathrm{pro}^\alpha\text{-}\mathbf{C}$, which we call distinguished.

One can give an explicit level representation of the distinguished weak coverings. Let $F : I \to \mathbf{C}$ be an object of $\mathrm{pro}^\alpha\text{-}\mathbf{C}$. We assume that I has a final element $i_\circ$ and that there is given a covering $\{C_w \to F(i_\circ) \mid w \in W\}$ in the site $\mathbf{C}$. Let $F_w : I \to \mathbf{C}$ be the functor given by $F_w(i) = F(i) \times_{F(i_\circ)} C_w$. Then

$$\{F_w \to F \mid w \in W\} \tag{10}$$

is a distinguished covering in $\text{pro}^{\alpha}\text{-}\mathbf{C}$ and all distinguished coverings are of this form up to isomorphism.

5.1. Proposition. *For a coherent site* $\mathbf{C}$ *the weak topology on* $\text{pro}^{\alpha}\text{-}\mathbf{C}$ *is coherent and has as a basis the coverings which have level representations of the form* (10)*, i.e. the distinguished weak coverings.*

Proof. We have to show that the system of distinguished weak coverings defines a basis $\mathcal{B}$ for a topology on $\text{pro}^{\alpha}\text{-}\mathbf{C}$. Clearly, an isomorphism is a covering in $\mathcal{B}$ and the pullback of a covering in $\mathcal{B}$ exists and is itself a covering in $\mathcal{B}$ by definition.

The property we have to check is that the composition of coverings from $\mathcal{B}$ is a covering in $\mathcal{B}$. More precisely, let $\{F_w \to F \mid w \in W\}$ be a covering in $\mathcal{B}$ of the form (10), i.e. with a level representation indexed by the cofiltered α-category I with final element $i_\circ$. Given coverings $\{G_{w,v} \to F_w \mid v \in W_w\}$ in $\mathcal{B}$ for $w \in W$ we have to show that the composite morphisms

$$\{G_{w,v} \to F \mid w \in W, v \in W_w\} \tag{11}$$

form a covering in $\mathcal{B}$. Changing the level representation (here we use that W is finite) we can assume that the $G_{w,v}$ are also indexed by I and that $G_{w,v}(i) = F_w(i) \times_{F_w(i_\circ)} D_{w,v}$ for all $i \in I$. Here $\{D_{w,v} \to F_w(i_\circ) \mid v \in W_w\}$ are coverings in $\mathbf{C}$. So (11) is level equivalent to the pullback of the covering $\{D_{w,v} \to F(i_\circ) \mid w \in W, v \in W_w\}$ along $F \to F(i_\circ)$ and therefore is a covering in $\mathcal{B}$. ∎

5.2. Definition. *Let* α *be an infinite cardinal. We say that a site* $\mathbf{C}$ *is* α*-small if for any object* C *in* $\mathbf{C}$ *there is a set of covering morphisms* $K(C)$ *of* C *with* $\operatorname{card} K(C) \leq \alpha$ *such that for any covering morphism* $E \to C$ *in* $\mathbf{C}$ *there is* $D \to C$ *in* $K(C)$ *and a factorization* $D \to E \to C$*.*

Clearly, any coherent site whose underlying category is an α-category is α-small.

5.3. Proposition. *If the coherent site* $\mathbf{C}$ *is* α*-small the pro-site* $\text{pro}^{\alpha}\text{-}\mathbf{C}$ *with the weak topology is also* α*-small.*

Proof. Consider $F : I \to \mathbf{C}$ in $\text{pro}^{\alpha}\text{-}\mathbf{C}$ with I a directed set with $\operatorname{card} I \leq \alpha$. For every $i \in I$ let K_i be a set of covering morphisms of $F(i)$ in $\mathbf{C}$ as in Definition 5.2. By Proposition 5.1 the set of cardinality at most α of covering morphisms

$$\{D \times_{F(i)} F \to F \mid i \in I, (D \to F(i)) \in K_i\}$$

satisfies the condition of Definition 5.2 for the weak topology. ∎

In the next lemma we collect for later reference a few fact about coproducts in $\text{pro}^{\alpha}\text{-}(\mathbf{C})$.

5.4. Lemma. *Assume* $\mathbf{C}$ *is an admissible site, see Definition 3.6.*

(i) A strict initial object in $\mathbf{C}$ *defines a strict initial object in* $\mathrm{pro}^{\alpha}\text{-}(\mathbf{C})$.

(ii) $\mathrm{pro}^{\alpha}\text{-}(\mathbf{C})$ *has finite coproducts which are disjoint and stable under pullback. Furthermore, finite coproducts of towers are towers.*

(iii) For a finite coproduct $U = \coprod_{i\in I} U_i$ *in* $\mathrm{pro}^{\alpha}\text{-}(\mathbf{C})$ *the family* $\{U_i \to U \mid i \in I\}$ *is a distinguished weak covering.*

(iv) For a distinguished weak covering $\{U_i \to U \mid i \in I\}$ *in* $\mathrm{pro}^{\alpha}\text{-}(\mathbf{C})$ *the morphism* $\coprod_{i\in I} U_i \to U$ *is a distinguished weak covering morphism.*

(v) For a finite family of distinguished weak covering morphisms $V_i \to U_i$ *in* $\mathrm{pro}^{\alpha}\text{-}(\mathbf{C})$ $(i \in I)$ *the morphism* $\coprod V_i \to \coprod U_i$ *is a distinguished weak covering morphism.*

Proof.
(ii): Use that finite coproducts commute with cofiltered α-limits in $\mathrm{pro}^{\alpha}\text{-}(\mathbf{C})$ by [8, Thm. 6.1].
(iii): Choose common level representations $(U_i(j))_{j\in J}$ of the U_i $(i \in I)$ such that J has the final element $j_\circ$. We know that $(\coprod_i U_i(j))_{j\in J}$ is a level representation for U, which we fix. As $U_i(j) \xrightarrow{\sim} U_i(j_\circ) \times_{U(j_\circ)} U(j)$ is an isomorphism (use the strict initial object), we see that $\{U_i \to U \mid i \in I\}$ is the pullback of the covering $\{U_i(j_\circ) \to U(j_\circ) \mid i \in I\}$ in $\mathbf{C}$. ∎

Transfinite topology. The transfinite topology on $\mathrm{pro}^{\alpha}\text{-}\mathbf{C}$ is the weakest topology such that the functor $\mathbf{C} \to \mathrm{pro}^{\alpha}\text{-}\mathbf{C}$ is continuous and such that λ-transfinite compositions of covering morphisms are covering morphisms $(\lambda \le \alpha)$. The latter means that if $(F_i)_{i<\lambda}$ is a tower in $\mathrm{pro}^{\alpha}\text{-}\mathbf{C}$ with $\lambda \le \alpha$ such that $F_{i+1} \to F_i$ is a covering morphism for all $i + 1 < \lambda$ the morphism

$$F_{<\lambda} = \lim_{i<\lambda} F_i \to F_0$$

is a covering morphism.

The category $\mathrm{pro}^{\alpha}\text{-}\mathbf{C}$ with the transfinite topology is denoted $\langle\alpha\rangle\mathbf{C}$. In Proposition 6.3 we show that $\langle\alpha\rangle\mathbf{C}$ is admissible if $\mathbf{C}$ is admissible. This will complete the proof of the adjointness part of Theorem 4.1 in view of Proposition 2.1.

A key step is to give an explicit presentation of the transfinite topology for an admissible site $\mathbf{C}$, see Definition 3.6. For this consider transfinite coverings in $\mathrm{pro}^{\alpha}\text{-}\mathbf{C}$ of the following form. We call a morphism $\tilde{U} \to U$ in $\mathrm{pro}^{\alpha}\text{-}\mathbf{C}$ a distinguished transfinite covering morphism if it is an λ-transfinite composition $(\lambda \le \alpha)$ of distinguished weak covering morphisms. The families of the form

$$\{U_w \to \tilde{U} \to U \mid w \in W\} \tag{12}$$

with $\tilde{U} \to U$ a distinguished transfinite covering morphism and $\{U_w \to \tilde{U} \mid w \in W\}$ a distinguished weak covering (W finite) are transfinite coverings, called distinguished transfinite coverings.

5.5. PROPOSITION. *If* $\mathbf{C}$ *is an admissible site the transfinite topology on* pro^{α}-$\mathbf{C}$ *is coherent and has as a basis the distinguished transfinite coverings, i.e. the coverings of the form* (12).

PROOF. First we show that the coverings (12) form a basis $\mathcal{B}$ for a topology. The only nontrivial part is to check that the composition of coverings in $\mathcal{B}$ is in $\mathcal{B}$.

Let

$$\{U_w \to \tilde{U} \to U \mid w \in W\}$$

be in $\mathcal{B}$ and for all $w \in W$ let

$$\{U_{w,v} \to \tilde{U}_w \to U_w \mid v \in W_w\}$$

be in $\mathcal{B}$.

The morphism $\coprod_{w\in W} U_w \to \tilde{U}$ is a distinguished weak covering morphism by Lemma 5.4(iv). In pro^{α}-($\mathbf{C}$) finite coproducts of towers are towers and finite coproducts of distinguished weak covering morphisms are distinguished weak covering morphisms by Lemma 5.4(ii) and (v). So by concatenation of towers we get that the composition

$$\coprod_{w\in W} \tilde{U}_w \to \coprod_{w\in W} U_w \to \tilde{U} \to U$$

is a distinguished transfinite covering morphism. As

$$\{U_{w,v} \to \coprod_{w'\in W} \tilde{U}_{w'} \mid w \in W, v \in W_w\}$$

is a distinguished weak covering we have shown that

$$\{U_{w,v} \to U \mid w \in W, v \in W_w\}$$

is in $\mathcal{B}$.

In order to finish the proof of Proposition 5.5 we have to show that λ-transfinite compositions of covering morphisms with respect to the topology defined by $\mathcal{B}$ are covering morphisms in the same topology ($\lambda \le \alpha$). By an argument very similar to the proof of Proposition 3.10 one is reduced to showing that for a tower $(U_i)_{i<\lambda}$ of distinguished transfinite covering morphisms the composition $U_{<\lambda} \to U_0$ is a distinguished transfinite covering morphism. By assumption for any $i+1 < \lambda$ we can find a tower $(U_{i,j})_{j<\lambda_i}$ of distinguished weak covering morphisms such that $U_{i,0} = U_i$ and $\lim_{j<\lambda_i} U_{i,j} = U_{i+1}$. By transfinite concatenation of the towers $(U_{i,j})_{j\in\lambda_i}$ we get a tower $(U'_k)_{k<\lambda'}$ of distinguished weak covering morphisms indexed by the ordinal $\lambda' = \sum_{i<\lambda} \lambda_i$ with $U'_0 = U_0$ and $U'_{<\lambda'} = U_{<\lambda}$. So $U_{<\lambda} \to U_0$ is a distinguished transfinite covering morphism. ∎

We say that an object U of a site is weakly contractible if any covering morphism $V \to U$ of the site splits. Clearly, if the site is subcanonical this is equivalent to saying that the sheaf $\mathbf{y}(U)$ is weakly contractible in the associated topos in the sense of Definition 3.3.

5.6. Corollary. *Let* $\mathbf{C}$ *be an admissible site and let* $U \in \mathrm{pro}^\alpha\text{-}\mathbf{C}$ *be weakly contractible for the weak topology. Then* U *is also weakly contractible for the transfinite topology.*

The proof of Corollary 5.6 is very similar to the proof of Proposition 3.4, so we omit the details. It is sufficient to show that a distinguished transfinite covering morphism $\lim_{i<\lambda} U_i \to U$ splits. Here $U_{i+1} \to U_i$ are distinguished weak covering morphisms for all i. Such a splitting can be constructed successively over i as splittings $U \to U_i$.

6. Pro-covering morphisms

Let $\mathbf{C}$ be an admissible site and α an infinite cardinal. In this section we collect a few results which are related to the concept of pro-covering morphism.

6.1. Definition. *A morphism* $f : V \to U$ *in* $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ *is a pro-covering morphism if* f *has a level representation by covering morphisms in* $\mathbf{C}$*.*

6.2. Lemma. *A distinguished covering morphism in the weak and in the transfinite topology is a pro-covering morphism.*

Proof. The case of the weak topology is trivial by the description (10) of distinguished weak covering morphisms.

Let $(U_i)_{i<\lambda}$ $(\lambda \le \alpha)$ be a tower of distinguished weak covering morphisms. We want to show that $U_{<\lambda} \to U_0$ is a pro-covering morphism. Without loss of generality λ is not a limit ordinal. We argue by contradiction. If the composition is not a pro-covering morphism there exists a smallest ordinal $\mu < \lambda$ such that $U_\mu \to U_0$ is not a pro-covering morphism.

If μ is a successor ordinal $U_\mu \to U_{\mu-1} \to U_0$ is a composition of a distinguished weak covering morphism and a pro-covering morphism, so it is a pro-covering morphism, contradiction.

If μ is a limit ordinal

$$U_\mu \xrightarrow{\sim} \lim_{i<\mu} U_i \to U_0$$

is a cofiltered limit of pro-covering morphisms, so is a pro-covering morphism by [8, Cor. 5.2], contradiction. ∎

6.3. Proposition. *For* $\mathbf{C}$ *admissible the site* $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ *with the weak and the transfinite topology is admissible.*

By what is shown in Section 5 the site $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ with both topologies is coherent. The site $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ is subcanonical by Lemmas 6.2 and 6.4. It has finite coproducts with the requested properties by Lemma 5.4.

6.4. Lemma. *For a pro-covering morphism $f : V \to U$ and for an object W in* pro^α-$\mathbf{C}$

$$\mathrm{Mor}(U, W) \to \mathrm{Mor}(V, W) \rightrightarrows \mathrm{Mor}(V \times_U V, W)$$

is an equalizer.

Proof. Without loss of generality we can assume $W \in \mathbf{C}$. Let $(V_i \xrightarrow{f_i} U_i)_{i \in I}$ be a level representation of f with f_i a covering morphism in $\mathbf{C}$. For each $i \in I$ we get an equalizer

$$\mathrm{Mor}(U_i, W) \to \mathrm{Mor}(V_i, W) \rightrightarrows \mathrm{Mor}(V_i \times_{U_i} V_i, W). \tag{13}$$

Taking the colimit over $i \in I$ in (13) and using the fact that in the category of sets filtered colimits commute with finite limits [11, IX.2] we finish the proof of Lemma 6.4. ∎

Let $\pi : \mathbf{C} \to \mathrm{pro}^\alpha$-$(\mathbf{C})$ be the canonical functor.

6.5. Lemma. *For a sheaf K on $\mathbf{C}$ the sheaf $\pi^* K$ on* pro^α-$\mathbf{C}$ *with the weak or transfinite topology is given on $U = (U_i)_{i \in I}$ by*

$$\pi^* K(U) = \operatorname*{colim}_{i \in I} K(U_i).$$

Proof. For any sheaf L on $\mathbf{C}$ consider the presheaf

$$L^\sharp : U = (U_i)_{i \in I} \mapsto \operatorname*{colim}_{i \in I} L(U_i)$$

on pro^α-$\mathbf{C}$. For a pro-covering morphism $W \to U$

$$L^\sharp(U) \to L^\sharp(W) \to L^\sharp(W \times_U W)$$

is an equalizer, because in **Set** finite limits commute with filtered colimits. So by Lemma 6.2 it follows that $L^\sharp$ is a sheaf.

By [1, Prop. I.5.1] the presheaf pullback of K to a presheaf on pro^α-$\mathbf{C}$ is given by

$$U \mapsto \operatorname*{colim}_{(V, f_V) \in U/\pi} K(V) \tag{14}$$

where U/π is the comma category whose objects consist of $V \in \mathbf{C}$ and a morphism $U \to \pi(V)$ in pro^α-$\mathbf{C}$. As the objects $(U_i, U \to U_i)$ are cofinal in this comma category we see that the presheaf (14) coincides with the sheaf $K^\sharp$, which finishes the proof by [1, Prop. III.1.3]. ∎

We conclude this section with an application of the notion of pro-covering morphism to derived categories. For a commutative unital ring Λ let $D^+_\Lambda(\mathbf{C})$ be the derived category of bounded below complexes of Λ-modules. The following proposition is a variant of [5, Cor. 5.1.6].

6.6. Proposition. *For both the weak and the transfinite topology and for $K \in D^+_\Lambda(\mathbf{C})$ the natural transformation*

$$K \to \mathbf{R}\pi_*\pi^* K$$

is an equivalence.

Proof. One easily reduces to the case of an injective sheaf K in $\mathrm{Sh}_\Lambda(\mathbf{C})$. As $K \xrightarrow{\sim} \pi_*\pi^* K$ is an isomorphism by Lemma 6.5, we have to show that

$$H^j(U_{\mathrm{pro}^\alpha\text{-}\mathbf{C}}, \pi^* K) = 0 \quad \text{ for all } j > 0 \text{ and } U \text{ in } \mathrm{pro}^\alpha\text{-}\mathbf{C}$$

in the weak and in the transfinite topology, because this implies that $\pi_*\pi^* K \to \mathbf{R}\pi_*\pi^* K$ is a quasi-isomorphism by [1, Prop. V.5.1]. By [1, Prop. V.4.3] it suffices to show that Čech cohomology

$$\check{H}^j(U_{\mathrm{pro}^\alpha\text{-}\mathbf{C}}, K) = \operatorname*{colim}_{f \in \mathbf{Cov}_{\mathrm{pro}^\alpha\text{-}\mathbf{C}}(U)} \check{H}^j(f, K)$$

vanishes for $j > 0$. Here $\mathbf{Cov}_{\mathrm{pro}^\alpha\text{-}\mathbf{C}}(U)$ is the category of distinguished covering morphisms of U in the weak resp. transfinite topology. For simplicity of notation we do not distinguish between K and $\pi^* K$. As the distinguished covering morphisms are pro-covering morphisms by Lemma 6.2, $f \in \mathbf{Cov}_{\mathrm{pro}^\alpha\text{-}\mathbf{C}}(U)$ has a level representation of the form $(V_i \xrightarrow{f_i} U_i)_{i\in I}$ with covering morphisms f_i in $\mathbf{C}$. Again by [1, Prop. V.4.3] and using injectivity of K as a sheaf on $\mathbf{C}$ we obtain the vanishing of

$$\check{H}^j(f, K) = \operatorname*{colim}_{i \in I} \check{H}^j(f_i, K) = 0 \quad \text{ for } j > 0.$$

∎

7. Weakly contractible objects

Proof of existence. In this subsection we prove Theorem 4.2. Consider the topos $\mathbf{E} = \mathrm{Sh}(\mathbf{C})$, where $\mathbf{C}$ is an admissible site. Let β be an infinite cardinal such that $\mathbf{C}$ is β-small, see Definition 5.2. The site $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ with the weak topology is α-small for $\alpha \geq \beta$ by Proposition 5.3. We are going to show that under this condition for any object U in $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ there is a transfinite covering morphism $\mathcal{P}^\infty(U) \to U$ such that $\mathcal{P}^\infty(U)$ is weakly contractible in the weak topology. Then by Corollary 5.6 $\mathcal{P}^\infty(U)$ is also weakly contractible in the transfinite topology and this clearly implies that $\langle\alpha\rangle\mathbf{E} = \mathrm{Sh}(\langle\alpha\rangle\mathbf{C})$ has a generating set of coherent weakly contractible objects.

So consider $\alpha \geq \beta$. Choose for each U in $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ a set of cardinality at most α of generating covering morphisms $K(U)$ as in Definition 5.2. Let $\mathcal{P}(U)$ be the product $\prod_{(V\to U)\in K(U)}(V \to U)$ in the comma category $\mathrm{pro}^\alpha\text{-}\mathbf{C}/U$.

7.1. Claim. *For each U in $\mathrm{pro}^\alpha\text{-}\mathbf{C}$ the morphism $\mathcal{P}(U) \to U$ is a transfinite covering morphism in $\mathrm{pro}^\alpha\text{-}\mathbf{C}$.*

PROOF. Let $\lambda \le \alpha$ be an ordinal such that there is a bijection

$$\iota : \{ i < \lambda \mid i \text{ is successor ordinal } \} \xrightarrow{\sim} K(U).$$

We successively construct a tower of weak covering morphisms $(V_i)_{i<\lambda}$ with $V_0 = U$ and $V_{<\lambda} = \mathcal{P}(U)$. Assume V_j has already been defined for all $j < i$. If $i < \lambda$ is a successor ordinal set

$$V_i = V_{i-1} \times_U \iota(i).$$

If $i < \lambda$ is a limit ordinal set

$$V_i = \lim_{j<i} V_j.$$

■

For a positive integer i let $\mathcal{P}^i(U)$ be the i-fold application of $\mathcal{P}$, i.e.

$$\mathcal{P}^i(U) = \mathcal{P}(\mathcal{P}^{i-1}(U)) \quad \text{for } i > 1,$$

and let $\mathcal{P}^\infty(U)$ be $\lim_{i\in\mathbb{N}} \mathcal{P}^i(U)$. By concatenation of towers we see that $\mathcal{P}^\infty(U) \to U$ is a λ-transfinite composition of weak covering morphisms ($\lambda \le \alpha$).

7.2. CLAIM. *The object $\mathcal{P}^\infty(U)$ of pro^α-$\mathbf{C}$ is weakly contractible in the weak topology.*

PROOF. Let $V' \to \mathcal{P}^\infty(U)$ be a distinguished weak covering morphism. There exists a positive integer i and a distinguished weak covering morphism $V \to \mathcal{P}^i(U)$ such that

$$V' \cong V \times_{\mathcal{P}^i(U)} \mathcal{P}^\infty(U).$$

By the definition of $\mathcal{P}$ there is a factorization

$$\mathcal{P}^{i+1}(U) \to V \to \mathcal{P}^i(U)$$

of the canonical morphism $\mathcal{P}^{i+1}(U) \to \mathcal{P}^i(U)$, which induces a splitting of $V' \to \mathcal{P}^\infty(U)$. ■

DISJOINT COVERING TOPOLOGY.

7.3. DEFINITION. *We call a small category $\mathbf{D}$ a dc-category if finite coproducts exist in $\mathbf{D}$ and furthermore finite coproducts are disjoint and stable under pullback, see [12, App.]. The finite coverings of the form $\{V_i \to V \mid i \in I\}$ with $V = \coprod_{i\in I} V_i$ define a basis for a topology on $\mathbf{D}$, which we call the dc-topology.*

7.4. LEMMA. *Let $\mathbf{C}$ be an admissible site.*

(i) The full subcategory of weakly contractible objects $\tilde{\mathbf{C}}$ in $\mathbf{C}$ forms a dc-category and the functor $\tilde{\mathbf{C}} \to \mathbf{C}$ is continuous, see [1, Def. III.1.1].

(ii) If there are enough weakly contractible objects in $\mathbf{C}$, i.e. if for any object U in $\mathbf{C}$ there is a covering morphism $V \to U$ with $V \in \mathbf{C}$ weakly contractible the restriction of sheaves induces an equivalence of categories between $\mathrm{Sh}(\mathbf{C})$ and $\mathrm{Sh}(\tilde{\mathbf{C}})$. Here $\tilde{\mathbf{C}}$ has the dc-topology.

To show Lemma 7.4(ii) one uses the comparison lemma [12, App., Cor. 4.3].

For a ring Λ and a topos $\mathbf{E}$ let $\mathrm{Mod}_\Lambda(\mathbf{E})$ be the category of Λ-modules in $\mathbf{E}$.

7.5. Lemma. *Let* $\mathbf{E}$ *be a topos and let* U *be a weakly contractible object in* $\mathbf{E}$*. The additive functor from* $\mathrm{Mod}_\Lambda(\mathbf{E})$ *to* Λ*-modules*

$$F \mapsto \Gamma(U,F) = \mathrm{Mor}_{\mathbf{E}}(U,F)$$

is exact.

8. Example: Cartan–Eilenberg hypercohomology

Let C and D be admissible sites. Let $f : \mathbf{D} \to \mathbf{C}$ be a continuous functor preserving finite limits. For a commutative unital ring Λ let $\mathrm{Mod}_\Lambda(\mathbf{C})$ be the category of sheaves of Λ-modules on $\mathbf{C}$ and let $D_\Lambda(\mathbf{C})$ be its derived category.

In geometry one is often interested in studying the right derived functor $\mathbf{R}f_* : D_\Lambda(\mathbf{C}) \to D_\Lambda(\mathbf{D})$. It was shown by Joyal and Spaltenstein [17] that this right derived functor always exists abstractly, see for example [6] for a modern account. However, it has good 'geometric' properties only for complexes bounded below or under some condition of finite cohomological dimension. These problematic aspects of the right derived functor are discussed in the framework of homotopy theory in [10, Sec. 6.5.4].

As an alternative to the derived functor one can use the older concept of Cartan–Eilenberg hypercohomology pushforward

$$\mathbb{H}f_* : D_\Lambda(\mathbf{C}) \to D_\Lambda(\mathbf{D})$$

defined for a complex K^* as $\mathbb{H}f_*(K^*) = f_*(\mathrm{Tot} I^{**})$, where $K^* \to I^{**}$ is a Cartan–Eilenberg injective resolution [7, Sec. XVII.1] and where $(\mathrm{Tot} I^{**})_n = \prod_{i+j=n} I^{i,j}$. In this form Cartan–Eilenberg hypercohomology is studied in [18, App.]. In fact, in [7] the direct sum instead of the direct product is used, but this does not seem to be appropriate for cohomology. Cartan–Eilenberg hypercohomology is equivalent to hypercohomology calculated using the Godement resolution, see [19, App.].

For admissible sites we can give a universal characterization of Cartan–Eilenberg hypercohomology in terms of derived functors. Let

$$\pi_\alpha^{\mathbf{C}} : \mathbf{C} \to \langle\alpha\rangle\mathbf{C}$$

be the canonical functor. We denote the induced functor $\langle\alpha\rangle\mathbf{D} \to \langle\alpha\rangle\mathbf{C}$ by f^α.

8.1. Proposition. *For coherent sites* $\mathbf{C}$ *and* $\mathbf{D}$ *and an infinite cardinal* α *the diagram*

$$\begin{tikzcd}
D_\Lambda(\langle\alpha\rangle\mathbf{C}) \arrow[r, "\mathbf{R}f^\alpha_*"] & D_\Lambda(\langle\alpha\rangle\mathbf{D}) \arrow[d, "\mathbf{R}(\pi_\alpha^{\mathbf{D}})_*"] \\
D_\Lambda(\mathbf{C}) \arrow[u, "(\pi_\alpha^{\mathbf{C}})^*"] \arrow[r, "\mathbb{H}f_*"'] & D_\Lambda(\mathbf{D})
\end{tikzcd}$$

commutes up to canonical equivalence.

PROOF. It is sufficient to show that for a complex K of sheaves of Λ-modules on $\mathbf{C}$ there is a quasi-isomorphism

$$\mathbf{R}(\pi_\alpha^{\mathbf{C}})_*(\pi_\alpha^{\mathbf{C}})^*K \simeq \mathrm{Tot} I^{**},$$

where $K^* \to I^{**}$ is a Cartan–Eilenberg injective resolution as above.

8.2. CLAIM. *The functor $(\pi_\alpha^{\mathbf{C}})_*$ maps a Cartan–Eilenberg injective resolution of $(\pi_\alpha^{\mathbf{C}})^*K$ on $\langle\alpha\rangle\mathbf{C}$ to a Cartan–Eilenberg injective resolutions of K.*

PROOF OF CLAIM. Note that $(\pi_\alpha^{\mathbf{C}})_*$ preserves injective sheaves and products because it is a right adjoint of the exact functor $(\pi_\alpha^{\mathbf{C}})^*$. One easily reduces the proof of the claim to the case in which K is in $\mathrm{Mod}_\Lambda(\mathbf{C})$. Let $(\pi_\alpha^{\mathbf{C}})^*K \to I^*$ be an injective resolution of K. Then by Proposition 6.6 the pushforward $(\pi_\alpha^{\mathbf{C}})_*I$ is an injective resolution of $K \simeq (\pi_\alpha^{\mathbf{C}})_*(\pi_\alpha^{\mathbf{C}})^*K$, so the claim follows. ∎

Using the claim Proposition 8.1 follows immediately from [18, Thm. A.3]. Here we use that countable products are exact in $\mathrm{Mod}_\Lambda(\langle\alpha\rangle\mathbf{C})$, see [5, Prop. 3.1.9], which is sufficient in the proof of [18, Thm. A.3]. ∎

9. Example: transfinite Zariski topos

In this section we explain how the construction of Section 5 applied to the Zariski topos of an affine scheme $X = \mathrm{Spec}\,(R)$ relates to the method of Bhatt–Scholze [5, Sec. 2]. The comparison in the étale case is very similar. We fix an infinite cardinal α with $\alpha \geq \mathrm{card}(R)$

THE CATEGORY $\mathbf{Aff}_X$. Let $\mathbf{Aff}_X$ be the category of affine schemes $\mathrm{Spec}\,(A)$ over X with $\mathrm{card}(A) \leq \alpha$. The Zariski topology on $\mathbf{Aff}_X$ has a basis given by coverings

$$\{\mathrm{Spec}\,(A[\frac{1}{f_i}]) \to \mathrm{Spec}\,(A) \mid i = 1, \dots, n\}$$

where $f_1, \dots, f_n \in A$ generate the unit ideal in A.

THE SITE $\mathbf{Aff}_X^{\mathrm{Zar}}$. Let $\mathbf{Aff}_X^{\mathrm{Zar}}$ be the full subcategory of $\mathbf{Aff}_X$ whose objects are of the form $\mathrm{Spec}\,\prod_{i=1}^n R[1/f_i]$ with $f_1, \dots, f_n \in R$ and whose morphisms are scheme morphisms over X. We endow $\mathbf{Aff}_X^{\mathrm{Zar}}$ with the Zariski topology. In [5, Sec. 2.2] the objects of $\mathbf{Aff}_X^{\mathrm{Zar}}$ are called Zariski localizations of R.

Clearly, the associated topos $\mathrm{Sh}(\mathbf{Aff}_X^{\mathrm{Zar}})$ is equivalent to the usual Zariski topos of X. Moreover, $\mathbf{Aff}_X^{\mathrm{Zar}}$ is admissible, see Definition 3.6.

THE CATEGORY $\mathbf{Aff}_X^{\mathrm{pro}}$. The functor

$$\lim : \mathrm{pro}^\alpha\text{-}\mathbf{Aff}_X^{\mathrm{Zar}} \to \mathbf{Aff}_X \tag{15}$$

which maps a pro-system to its inverse limit is fully faithful. For any affine scheme $\mathrm{Spec}\,A$ in the image of the functor (15) Bhatt–Scholze say that A is an ind-Zariski localizations of $\mathrm{Spec}\,R$. We write the image of the functor (15) as $\mathbf{Aff}_X^{\mathrm{pro}}$.

Topologies on $\mathbf{Aff}_X^{\mathrm{pro}}$. The topology on $\mathbf{Aff}_X^{\mathrm{pro}}$ induced by the Zariski topology on $\mathbf{Aff}_X$ is isomorphic to the weak topology on pro^α-$\mathbf{Aff}_X^{\mathrm{Zar}}$ via the equivalence induced by (15).

In [5] and [2, Tag 0965] the pro-étale topology is studied. There is an obvious analog in the Zariski word, the pro-Zariski topology, defined as follows:

$$\{U_i \xrightarrow{\pi_i} U \mid i \in I\}$$

is a pro-Zariski covering if I is finite, $\coprod_{i\in I} U_i \to U$ is surjective and π_i induces an isomorphism $\mathcal{O}_{U,\pi_i(x)} \to \mathcal{O}_{U_i,x}$ for all $x \in U_i$.

We get the following relations between topologies on $\mathbf{Aff}_X^{\mathrm{pro}}$

$$(\text{Zariski topology}) \subset (\text{transfinite topology}) \subset (\text{pro-Zariski topology}).$$

9.1. Question. *Does there exist an analog of the pro-Zariski topology on* pro^α-$\mathbf{C}$ *for a general admissible site* $\mathbf{C}$. *This pro-topology should be stronger than the transfinite topology. For example one might try to define the requested pro-topology as generated by coverings* $\{U_w \to U \mid w \in W\}$ *with* W *finite and with* $\coprod_{w\in W} U_w \to U$ *a pro-covering morphism which induces a surjection on topos points.*

The category $\mathbf{Aff}_X^{\mathrm{oil}}$. One problem of the pro-category $\mathbf{Aff}_X^{\mathrm{pro}}$ is that its definition is not local on X. This is the reason why in [5] and [2, Tag 0965] the weakly étale morphisms and in the Zariski case the isomorphisms of local rings morphisms are used. A similar technique, which is related to the pro-étale topology of rigid spaces as defined in [16], can be used in our case in order to replace $\mathbf{Aff}_X^{\mathrm{pro}}$ by a more local definition.

Consider the full subcategory of $\mathbf{Aff}_X$ consisting of universally open morphisms $f : Y \to X$ which identify local rings, i.e. for any point $y \in Y$ the map $f^* : \mathcal{O}_{X,f(y)} \xrightarrow{\sim} \mathcal{O}_{Y,y}$ is an isomorphism.

9.2. Lemma. *For a functor* $F : I \to \mathbf{Aff}_X^{\mathrm{oil}}$ *such that* I *is a cofiltered* α*-category and such that all transition maps* $F(i) \to F(j)$ *are surjective the limit* $Y = \lim_{i\in I} F(i)$ *taken in* $\mathbf{Aff}_X$ *is an object of* $\mathbf{Aff}_X^{\mathrm{oil}}$.

Proof. We show that $Y \to X$ is open. Any affine open subscheme $U \subset Y$ is the preimage of some affine open $U_i \subset F(i)$ for some $i \in I$. Note that $Y \to F(i)$ is surjective, because the fibres of the transition maps in the system F are finite and nonempty and a cofiltered limits of finite nonempty sets is nonempty. So the image of U in X is the same as the image of U_i and therefore is open. ∎

If not mentioned otherwise we endow $\mathbf{Aff}_X^{\mathrm{oil}}$ with the transfinite topology, i.e. the weakest topology containing the Zariski coverings and such that a λ-transfinite composition of covering morphisms is a covering morphism ($\lambda \le \alpha$). This topology has an explicit description similar to Proposition 5.5.

For U in $\mathbf{Aff}_X^{\mathrm{oil}}$ the weakly contractible object $\mathcal{P}^\infty(U)$ as defined in Section 7 exists in $\mathbf{Aff}_X^{\mathrm{oil}}$. So the site $\mathbf{Aff}_X^{\mathrm{oil}}$ has similar properties as $\mathbf{Aff}_X^{\mathrm{pro}}$ with the transfinite topology. In fact both are closely related as we show now.

9.3. PROPOSITION. *For any* $Y \to X$ *in* $\mathbf{Aff}_X^{\mathrm{oil}}$ *there is a* λ*-transfinite composition* ($\lambda \leq \alpha$) *of surjective Zariski localizations* $\tilde{Y} \to Y$ *such that* $\tilde{Y} \to X$ *is in* $\mathbf{Aff}_X^{\mathrm{pro}}$.

In particular there is an equivalence of topoi

$$\mathrm{Sh}(\mathbf{Aff}_X^{\mathrm{oil}}) \cong \mathrm{Sh}(\mathbf{Aff}_X^{\mathrm{oil}} \cap \mathbf{Aff}_X^{\mathrm{pro}}),$$

where both sites have the transfinite topology.

PROOF (SKETCH). Composing $Y \to X$ with the transfinite composition of surjective Zariski localizations $\mathcal{P}^\infty(Y) \to Y$ we can assume without loss of generality that Y is weakly contractible.

Consider the following data: $Y = \coprod_{i \in I} V_i$ is a finite decomposition into open and closed affine subschemes and $U_i \subset X$ is an open affine subscheme such that $f(V_i) \subset U_i$. The set of such data forms a directed set $\mathbf{J}$ under the ordering by refinement. Then

$$Y \xrightarrow{\sim} \lim_{\mathbf{J}} \coprod_{i \in I} U_i$$

is an isomorphism. ■

10. Example: a left adjoint to i^*

Let $i : Y \to X$ be a closed immersion of separated, quasi-compact schemes. Consider the category of schemes U together with an étale, affine morphism $U \to X$. We write X_{t} for this category endowed either with the Nisnevich (t = Nis) or étale (t = ét) topology, similarly for Y. Clearly, X_{t} and Y_{t} are admissible.

Let Λ be a commutative ring. Consider the pullback functor

$$i^* : D_\Lambda(Y_{\mathrm{t}}) \to D_\Lambda(X_{\mathrm{t}}) \tag{16}$$

on unbounded derived categories of complexes of Λ-modules.

Our aim in this section is to show that the analogous pullback functor in the transfinite Nisnevich and transfinite étale topology has a left adjoint. A similar result for the pro-étale topology has been observed in [5, Rmk. 6.1.6]. Before discussing the transfinite case we discuss why in the classical case the functor (16) has no left adjoint in general.

Recall that the derived categories in (16) have small products. For $(K_i)_{i \in I}$ a family of complexes Λ-modules in X_{t} the infinite product of these complexes in $D_\Lambda(X_{\mathrm{t}})$ is calculated by first replacing the K_i by K-injective complexes as in [17], see e.g. [6], and then taking degreewise products of sheaves. The following example shows that in general the functor (16) does not preserve infinite products, in particular it cannot have a left adjoint.

10.1. EXAMPLE. For a prime p write the henselization of $\mathbb{Z}_{(p)}$ as an filtered direct limit of étale $\mathbb{Z}_{(p)}$-algebras

$$\mathbb{Z}_{(p)}^h = \operatorname*{colim}_{j \in J} A_j.$$

We consider the closed immersion

$$i : \mathrm{Spec}\,(\mathbb{F}_p) \to \mathrm{Spec}\,(\mathbb{Z}_{(p)}).$$

By $\mathbb{Z}[A_j]$ we denote the étale sheaf of free abelian groups on $X_{\text{ét}} = \mathrm{Spec}\,(\mathbb{Z}_{(p)})_{\text{ét}}$ represented by $\mathrm{Spec}\,(A_j)$. Then the homotopy limit in the sense of [13, Sec. 1.6] taken in the triangulated category $D_{\mathbb{Z}}(X_{\text{ét}})$

$$K = \mathrm{holim}_{j\in J}\, \mathbb{Z}[A_j]$$

has vanishing étale cohomology sheaf in degree zero. However $i^*\mathbb{Z}[A_j]$ is the constant sheaf $\mathbb{Z}$, so

$$\mathrm{holim}_{j\in J}\, i^*\mathbb{Z}[A_j] = \mathbb{Z}$$

is not quasi-isomorphic to i^*K.

10.2. Theorem. *Let X be quasi-compact and separated. There exists an infinite cardinal β such that for $\alpha \geq \beta$ the functors*

$$i^* : \mathrm{Sh}(\langle\alpha\rangle X_{\mathrm{t}}) \to \mathrm{Sh}(\langle\alpha\rangle Y_{\mathrm{t}}) \tag{17}$$

$$i^* : D_\Lambda(\langle\alpha\rangle X_{\mathrm{t}}) \to D_\Lambda(\langle\alpha\rangle Y_{\mathrm{t}}) \tag{18}$$

have left adjoints.

Proof. Choose β such that for any open affine subscheme $\mathrm{Spec}\,(A) \to X$ we have $\beta \geq \mathrm{card}(A)$. Then according to Theorem 4.2 and Remark 4.4 there exists a generating set of coherent, weakly contractible objects in the topoi $\mathrm{Sh}(\langle\alpha\rangle X_{\mathrm{t}})$ and $\mathrm{Sh}(\langle\alpha\rangle Y_{\mathrm{t}})$.

10.3. Lemma. *Any coherent topos $\mathbf{E}$ as a small cogenerating set.*

Proof. By Deligne's theorem [1, Sec. IX.11] any coherent topos $\mathbf{E}$ has a set of points

$$(p_{j,*}, p_j^*) : \mathbf{Set} \to \mathbf{E} \qquad (j \in J)$$

such that all p_j together induce a faithful functor

$$(p_j^*)_{j\in J} : \mathbf{E} \to \prod_{j\in J} \mathbf{Set}.$$

The set of objects $p_{j,*}(\{1,2\})$ $(j \in J)$ is cogenerating. ∎

10.4. Lemma. *The triangulated categories $D_\Lambda(\langle\alpha\rangle X_{\mathrm{t}})$ and $D_\Lambda(\langle\alpha\rangle Y_{\mathrm{t}})$ are compactly generated.*

Proof. For simplicity of notation we restrict to $D_\Lambda(\langle\alpha\rangle X_{\mathrm{t}})$. For U a coherent, weakly contractible object in $\mathrm{Sh}(\langle\alpha\rangle X_{\mathrm{t}})$ the sheaf of free Λ-modules $\Lambda[U]$ represented by U is a compact object of the triangulated category $D_\Lambda(\langle\alpha\rangle X_{\mathrm{t}})$. In fact the global section functor $\Gamma(U, -)$ preserves exact complexes by Lemma 7.5. Furthermore, taking sections over a coherent object preserves direct sums of Λ-modules [1, Thm. VI.1.23].

Let $\mathcal{W}$ be a set of such coherent, weakly contractible objects U which generate the topos $\mathrm{Sh}(\langle\alpha\rangle X_{\mathrm{t}})$. Then the set of compact objects $\{\Lambda[U] \mid U \in \mathcal{W}\}$ generates the triangulated category $D_\Lambda(\langle\alpha\rangle X_{\mathrm{t}})$. ∎

By the special adjoint functor theorem [11, Sec. V.8] the existence of a left adjoint to (17) follows once we show that the functor (17) preserves small products. Indeed, coherent topoi satisfy the conditions of the special adjoint functor theorem by Lemma 10.3 and general properties of topoi.

By [9, Prop. 5.3.1] and by Lemma 10.4 the existence of the left adjoint to (18) follows if we show that (18) preserves small products.

In order to prove that our two functors i^* preserve small products we can assume without loss of generality that X is affine. In this case the fact that (17) preserves products is immediate from Lemma 10.5. The argument for the functor (18) is given after the proof of Lemma 10.5.

10.5. Lemma. *For α as above and for X affine the functor*

$$i^* : \mathrm{Sh}(\langle\alpha\rangle X_{\mathrm{t}}) \to \mathrm{Sh}(\langle\alpha\rangle Y_{\mathrm{t}})$$

has a left adjoint, denoted $i_\flat$, which satisfies

(i) $i^* \circ i_\flat \simeq \mathrm{id}$,

(ii) $i_\flat$ *maps weakly contractible objects to weakly contractible objects.*

Proof of Lemma 10.5. For $V \to Y$ affine étale there exists an affine étale scheme $U' \to X$ such that $U' \times_X Y \cong V$, see [2, Tag 04D1]. Let U be the henselization of U' along V, see [15, Ch. XI]. The resulting affine scheme U together with the isomorphism $U \times_X Y \cong V$ is unique up to unique isomorphism and depends functorially on V.

Taking inverse limits defines a fully faithful functor from $\langle\alpha\rangle X_{\mathrm{t}}$ to the category of affine schemes over X. And the scheme U constructed above lies in the essential image of this functor. Without loss of generality we will identify U with an object of $\langle\alpha\rangle X_{\mathrm{t}}$.

So the map $V \mapsto U$ extends to a functor $i_\flat^{\mathrm{pre}} : Y_{\mathrm{t}} \to \langle\alpha\rangle X_{\mathrm{t}}$ which we can extend by continuity to a functor

$$i_\flat^{\mathrm{pre}} : \langle\alpha\rangle Y_{\mathrm{t}} \to \langle\alpha\rangle X_{\mathrm{t}}. \tag{19}$$

which is left adjoint to the pullback functor $U \mapsto U \times_X Y$.

By [1, Prop. I.5.1] the pullback along i in the sense of presheaves maps a presheaf F on $\langle\alpha\rangle X_{\mathrm{t}}$ to the presheaf

$$V \mapsto \operatorname*{colim}_{(U,f_U)} F(U), \tag{20}$$

on $\langle\alpha\rangle Y_{\mathrm{t}}$, where (U, f_U) runs through the comma category of all pairs in which U is in $\langle\alpha\rangle X_{\mathrm{t}}$ and f_U is a map $V \to U \times_X Y$ in $\langle\alpha\rangle Y_{\mathrm{t}}$. But clearly for given V the object $i_\flat^{\mathrm{pre}} V$ in $\langle\alpha\rangle X_{\mathrm{t}}$ together with the isomorphism $i_\flat^{\mathrm{pre}} V \times_X Y \cong V$ is an initial element in the comma category of these pairs. So the presheaf pullback of F is given by

$$V \mapsto F(i_\flat^{\mathrm{pre}} V). \tag{21}$$

Let $\tilde{Y}_{\mathrm{t}}$ be the full subcategory of $\langle\alpha\rangle Y_{\mathrm{t}}$ given by the weakly contractible objects. Note that according to the first part of Section 7, the objects of $\tilde{Y}_{\mathrm{t}}$ generate $\langle\alpha\rangle Y_{\mathrm{t}}$. The restriction of the functor (19) to $\tilde{Y}_{\mathrm{t}}$ with the dc-topology, see Definition 7.3, is continuous in the sense of [1, Def. III.1.1]. To see this, note that decompositions into disjoint unions of $V \in \tilde{Y}_{\mathrm{t}}$ can be lifted to $i_\flat^{\mathrm{pre}} V$ by [15, Prop. XI.2.1]. Now [1, Prop. III.1.2] tells that we get an adjoint pair of functors

$$i_\flat^{\mathrm{res}} : \mathrm{Sh}(\tilde{Y}_{\mathrm{t}}) \rightleftharpoons \mathrm{Sh}(\langle\alpha\rangle X_{\mathrm{t}}) : i^*_{\mathrm{res}}.$$

By Lemma 7.4 and by what is explained above equation (21), the right adjoint i^*_{res} is just the composition of the sheaf pullback

$$i^* : \mathrm{Sh}(\langle\alpha\rangle X_{\mathrm{t}}) \to \mathrm{Sh}(\langle\alpha\rangle Y_{\mathrm{t}}) \tag{22}$$

composed with the restriction to $\tilde{Y}_{\mathrm{t}}$. However the latter restriction

$$\mathrm{Sh}(\langle\alpha\rangle Y_{\mathrm{t}}) \xrightarrow{\sim} \mathrm{Sh}(\tilde{Y}_{\mathrm{t}})$$

is an equivalence of categories according to Lemma 7.4, so we obtain a left adjoint $i_\flat$ to (22) as stated in Lemma 10.5.

Property (i) of the Lemma is immediate from the construction of $i_\flat$ and (ii) follows abstractly from the adjointness property. ∎

Consider a family of complexes of sheaves of Λ-modules $(K_j)_{j\in J}$ on $\langle\alpha\rangle X_{\mathrm{t}}$. Note that, because there are enough weakly contractible objects in $\mathrm{Sh}(\langle\alpha\rangle X_{\mathrm{t}})$, small products of complexes of sheaves of Λ-modules on $\langle\alpha\rangle X_{\mathrm{t}}$ preserve quasi-isomorphisms by Lemma 7.5. So we have to show that

$$i^*(\prod_{j\in J} K_j) \to \prod_{j\in J} i^* K_j$$

is a quasi-isomorphism, where the product is just the degreewise product of sheaves. Using compact generators $\Lambda[U]$ of $D_\Lambda(\langle\alpha\rangle Y_{\mathrm{t}})$, see Lemma 10.4, it suffices to show that

$$\Gamma(U, i^*(\prod_{j\in J} K_j)) \to \prod_{j\in J} \Gamma(U, i^* K_j) \tag{23}$$

is a quasi-isomorphism of complexes of Λ-modules. By the adjunction of Lemma 10.5 this is equivalent to showing that

$$\Gamma(i_\flat U, \prod_{j\in J} K_j) \to \prod_{j\in J} \Gamma(i_\flat U, K_j)$$

is a quasi-isomorphism, which is obvious. ∎

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