

# CORELATIONS ARE THE PROP FOR EXTRASPECIAL COMMUTATIVE FROBENIUS MONOIDS

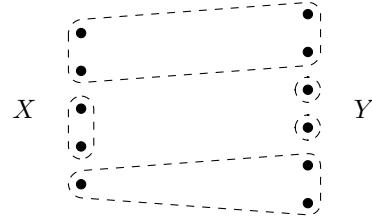
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**ABSTRACT.** Just as binary relations between sets may be understood as jointly monic spans, so too may equivalence relations on the disjoint union of sets be understood as jointly epic cospans. With the ensuing notion of composition inherited from the pushout of cospans, we call these equivalence relations *corelations*. We define the category of corelations between finite sets and prove that it is equivalent to the prop for extraspecial commutative Frobenius monoids. Dually, we show that the category of relations is equivalent to the prop for special commutative bimonoids. Throughout, we emphasise how corelations model interconnection.

## 1. Introduction

It is well-known that the category of relations between finite sets may be obtained as the category of isomorphism classes of jointly monic spans in the category of finite sets and functions. In this paper we investigate the dual notion: isomorphism classes of jointly epic cospans. These are known as corelations, and corelations from a set  $X$  to a set  $Y$  are characterised as partitions of the disjoint union  $X + Y$ .

Our slogan is ‘corelations model connection’. We understand a corelation as a partition of two sets into connected components, depicting examples as follows



Here we have a corelation from a set  $X$  of five elements to a set  $Y$  of six elements. Elements belonging to the same equivalence class of  $X + Y$  are grouped (‘connected’) by a dashed line.

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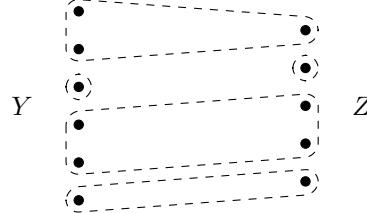
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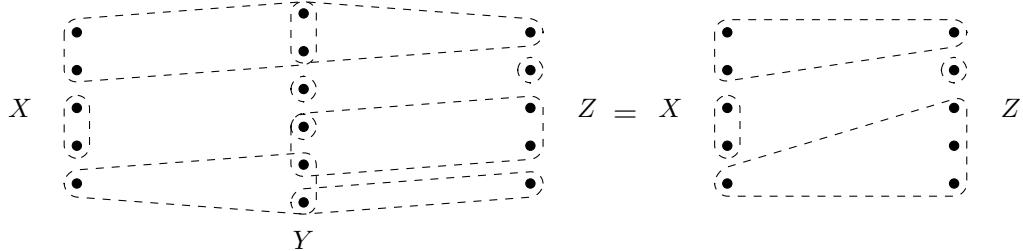
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Composition of corelations takes the transitive closure of the two partitions, before restricting the partition to the new domain and codomain. For example, suppose in addition to the corelation  $\alpha: X \rightarrow Y$  above we have another corelation  $\beta: Y \rightarrow Z$



Then the composite  $\beta \circ \alpha$  of our two corelations is given by



Informally, this captures the idea that two elements of  $X + Z$  are ‘connected’ if we may travel from one to the other staying within connected components of  $\alpha$  and  $\beta$ .

Another structure that axiomatises interconnection is the extraspecial commutative Frobenius monoid. An extraspecial commutative Frobenius monoid in a symmetric monoidal category is an object equipped with commutative monoid and cocommutative comonoid structures that together obey additional laws known as the Frobenius, special, and extra laws. Special commutative Frobenius monoids are well-known; the additional axiom here, the so-called extra law, requires that the unit composed with the counit is the identity on the unit for the monoidal product. We write this in string diagrams as

$$\bullet - \bullet =$$

Together, these axioms express the idea that connectivity is all that matters: not pairwise clustering, not multiple paths, not ‘extra’, interior components.

Corelations and extraspecial commutative Frobenius monoids are intimately related. To explicate this relationship, we will use the language of props. Recall that a prop<sup>1</sup> is a symmetric strict monoidal category with the natural numbers as objects and addition as the monoidal product. Also recall that a prop  $\mathcal{T}$  is termed the prop for an algebraic structure if, given another symmetric monoidal category  $\mathcal{C}$ , the symmetric monoidal functor category  $\mathcal{C}^{\mathcal{T}}$  is isomorphic to the category of the chosen algebraic structure in  $\mathcal{C}$ .

Considered as symmetric monoidal categories with monoidal product the disjoint union, Lack proved that the category of spans in the category of finite sets and functions

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<sup>1</sup>Also stylised PROP, for PROduct and Permutation category.

is equivalent as a symmetric monoidal category to the prop for bicommutative bimonoids, and that the category of cospans is equivalent as a symmetric monoidal category to the prop for special commutative Frobenius monoids [Lac04]. Note that the disjoint union of finite sets also gives a monoidal product on the category of corelations. Our main theorem is:

**1.1. THEOREM.** *The category of corelations is equivalent, as a symmetric monoidal category, to the prop for extraspecial commutative Frobenius monoids.*

Corelations and extraspecial commutative Frobenius monoids have been observed to play a key role in many frameworks relying on the interconnection of systems, including electrical circuits [BF], signal flow graphs [BE15, BSZ14], bond graphs [BC], linear time-invariant systems [FRS16], automata [RSW08], proofs [DP03], and matrices and other linear systems [Zan15]. The mutual characterisation of these structures provided by our main theorem clarifies and streamlines arguments in many of these applications. For example, the use of corelations provides a precise extraspecial commutative Frobenius monoid extension of the well-known ‘spider theorem’ characterising morphisms between tensor powers of a special commutative Frobenius monoid [CK16, CPP09]. Equivalently, this main theorem extends to the extraspecial case the characterisation of separable Frobenius monoidal functors as those that preserve connectivity of progressive plane string diagrams [MS10].

In independent but related work, Zanasi proves in his recent thesis [Zan15, §2.5] that the so-called prop of equivalence relations is the free prop on the theory of extraspecial commutative Frobenius monoids. As we do, Zanasi builds on Lack’s observation that category of cospans in the category of finite sets and functions is equivalent to the prop for special commutative Frobenius monoids [Lac04], as well as the observation of Bruni and Gadducci that cospans are related to equivalence relations [BG01]. Zanasi argues via a so-called ‘cube construction’, or fibred sum of props.

Similar results can also be found in [DP13] and [BGM02]. Došen and Petrić [DP13, §9] prove that the category of ‘split equivalences’ is isomorphic to the ‘equivalential Frobenius monad freely generated by a single object’. They argue this by constructing an auxiliary syntactic category isomorphic to the equivalential Frobenius monad freely generated by a single object, and then inducting on the terms of this new category. Bruni, Gadducci, and Montanari [BGM02, §5.1] sketch a slightly more general result relating so-called partition spaces and p-monoidal categories by exhibiting a normal form for morphisms in free p-monoidal categories.

Our novel approach through the understanding of corelations as jointly epic cospans permits a significantly simpler argument than the related three above, via a coequalizer of props. In doing so, our proof provides a clear narrative for the origin of the extra law and its relationship with other fundamental axioms. Moreover, such an approach is philosophically well motivated, and provides easy generalisation, such as the characterisation of linear relations as jointly epic cospans in the category of matrices over a field [Fon16], or linear time-invariant systems as jointly epic cospans in the category of matrices over a relevant Laurent polynomial ring [FRS16].

Ultimately, our work completes the beautiful picture

<b>spans</b>	<b>cospans</b>
bicommutative bimonoids	special commutative Frobenius monoids
<b>relations</b>	<b>corelations</b>
special bicommutative bimonoids	extraspecial commutative Frobenius monoids

pairing constructions on the category of finite sets and functions with important algebraic structures. Recall that the bimonoid laws and the Frobenius law describe the two major ways that a monoid and comonoid can interact. Their duality was proved by Lack [Lac04]: the bimonoid laws are derived from pullbacks of functions, while the Frobenius law is derived from pushouts. In this paper we move to the level of relations, demonstrating the importance of the heretofore overlooked extra law as the dual version of the special law.

**1.2. OUTLINE.** In the next two sections we introduce corelations and extraspecial commutative Frobenius monoids respectively. These are the stars of this paper, and our task will be to understand their relationship. To this end, in Section 4 we review the idea of a prop for an algebraic structure, and note that the category of corelations between finite sets is equivalent to a prop **Corel**. In Section 5, we then construct a prop **Th(ESCFM)** whose algebras are extraspecial commutative Frobenius monoids. We show in Section 6 that **Corel** and **Th(ESCFM)** are isomorphic, proving the main theorem. Finally, in Section 7 we outline the dual characterisation of the category of relations, and summarise the algebraic theories corresponding to spans, cospans, relations, and corelations.

## 2. Corelations

First we define corelations. Corelations arise as the dual of relations: recall that a binary relation from a set  $X$  to a set  $Y$  is a subset of the product  $X \times Y$ . A corelation is a quotient of the coproduct  $X + Y$ .

**2.1. DEFINITION.** A **corelation**  $\alpha: X \rightarrow Y$  between sets  $X$  and  $Y$  is a partition of  $X + Y$ .

Given another corelation  $\beta: Y \rightarrow Z$ , the composite  $\beta \circ \alpha: X \rightarrow Z$  is the restriction to  $X + Z$  of the finest partition on  $X + Y + Z$  that is coarser than both  $\alpha$  and  $\beta$  when restricted to  $X + Y$  and  $Y + Z$  respectively.

This composition is associative as both pairwise methods of composing corelations  $\alpha: X \rightarrow Y$ ,  $\beta: Y \rightarrow Z$ , and  $\gamma: Z \rightarrow W$  amount to finding the finest partition on  $X + Y + Z + W$  that is coarser than each of  $\alpha$ ,  $\beta$ , and  $\gamma$  when restricted to the relevant subset, and then restricting this partition to a partition on  $X + W$ ; reference to the motivating visualization makes this clear. Moreover, this composition has an identity: it is the partition of  $X + X$  such that each equivalence class comprises exactly two elements, an element  $x \in X$  considered as an element of both the first and then the second summand of  $X + X$ .

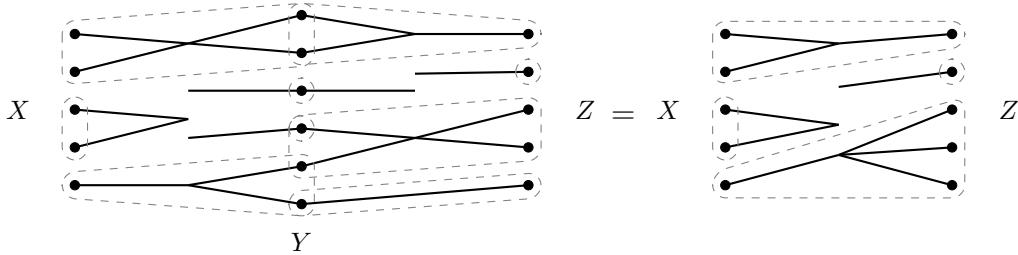
This allows us to define a category. We restrict our attention to corelations between finite sets.

**2.2. DEFINITION.** Let **Corel** be the symmetric monoidal category with objects finite sets, morphisms corelations between finite sets, and monoidal product disjoint union.

We shall freely abuse the notation **Corel** to refer to an equivalent skeleton. This is key for our main theorem: in Section 4 we show this skeleton is strict, and hence a prop.

Ellerman gives a detailed treatment of corelations from a logic viewpoint in [Ell14], while basic category theoretic aspects can be found in Lawvere and Rosebrugh [LR03]. Note that neither binary relations nor corelations are a generalisation of the other. A key property of corelations is that they form a compact category with the disjoint union of sets as monoidal product. This is not true of the category of relations.

Another way of visualising corelations and their composition is as terminals connected by junctions of ideal wires. We draw these by marking each equivalence class with a point (the ‘junction’), and then connecting each element of the domain and codomain to their equivalence class with a ‘wire’. Composition then involves collapsing connected junctions down to a point. The example from the introduction is represented as follows.



Again, the composition law captures the idea that connectivity is all that matters: as long as the wires are ‘ideal’, the exact path does not matter. The application to electrical circuits is discussed in detail in [BF].

This visualisation mimics the string diagrams defining extraspecial commutative Frobenius monoids.

### 3. Extraspecial commutative Frobenius monoids

We introduce extraspecial commutative Frobenius monoids in some detail, writing our axioms using the string calculus for monoidal categories introduced by Joyal and Street [JS91]. Diagrams will be read left to right, and we shall suppress the labels as we deal with a unique generating object and a unique generator of each type. While we expect that the algebraic structures below—monoids, comonoids, and so on—are familiar to most readers, we include the additional detail to underscore the similarity between the wire diagrams for corelations and string diagrams for extraspecial commutative Frobenius monoids. Again, we shall see that the laws defining this structure express the principle that connectivity is all that matters.

Recall that a **commutative monoid**  $(X, \mu, \eta)$  in symmetric monoidal category  $(\mathcal{C}, \otimes)$

is an object  $X$  of  $\mathcal{C}$  together with maps

$$\begin{array}{ccc} \text{---} & & \bullet \\ \mu: X \otimes X \rightarrow X & & \eta: I \rightarrow X \end{array}$$

obeying

$$\begin{array}{ccc} \text{---} = \text{---} & \bullet = \text{---} & \text{---} = \text{---} \\ (\text{associativity}) & (\text{unitality}) & (\text{commutativity}) \end{array}$$

where  $\times$  is the symmetry on  $X \otimes X$ . In addition to the ‘upper’ unitality law above, the mirror image ‘lower’ unitality law also holds, due to commutativity and the naturality of the symmetry.

Dually, a **cocommutative comonoid**  $(X, \delta, \epsilon)$  in  $\mathcal{C}$  is an object  $X$  together with maps

$$\begin{array}{ccc} \text{---} & & \bullet \\ \delta: X \rightarrow X \otimes X & & \epsilon: X \rightarrow I \end{array}$$

obeying

$$\begin{array}{ccc} \text{---} = \text{---} & \bullet = \text{---} & \text{---} = \text{---} \\ (\text{coassociativity}) & (\text{counitality}) & (\text{cocommutativity}) \end{array}$$

Given a monoid and comonoid on the same object, there are two well-known ways for them to interact: the bimonoid laws and the Frobenius law. We shall discuss both in this paper, but for now we restrict our attention to Frobenius structure.

**3.1. DEFINITION.** An **extraspecial commutative Frobenius monoid**  $(X, \mu, \eta, \delta, \epsilon)$  in a monoidal category  $(\mathcal{C}, \otimes)$  comprises a commutative monoid  $(X, \mu, \eta)$  and a cocommutative comonoid  $(X, \delta, \epsilon)$  that further obey

$$\begin{array}{ccc} \text{---} = \text{---} = \text{---} & \text{---} = \text{---} & \bullet = \bullet \\ (\text{the Frobenius law}) & (\text{the special law}) & (\text{the extra law}) \end{array}$$

While we write two equations for the Frobenius law, this is redundant: the equality of any two of the expressions implies the equality of all three. Note that a monoid and comonoid obeying the Frobenius law is commutative if and only if it is cocommutative. Thus while a commutative and cocommutative Frobenius monoid might more properly

be called a bicommutative Frobenius monoid, there is no ambiguity if we only say commutative.

The Frobenius law and the special law go back to Carboni and Walters, under the names  $S=X$  law and the diamond=1 law respectively [CW87]. The extra law is a more recent discovery, appearing first under this name in the work of Baez and Erbele [BE15], as the ‘bone law’ in [BSZ14, FRS16], and as the ‘irredundancy law’ in [Zan15].

Observe that each of these equations equate string diagrams that connect precisely the same elements of the domain and codomain. To wit, the associativity, coassociativity, and Frobenius laws show that the order in which we build a connected component through pairwise clustering is irrelevant, the special law shows that having multiple connections between points is irrelevant, and the extra law shows that ‘extra’ components not connected to the domain or codomain are irrelevant.

Our main theorem will show that these equations are exactly those required to have the converse: two morphisms built from the generators of an extraspecial commutative Frobenius monoid are equal if and only if their diagrams impose the same connectivity relations on the disjoint union of the domain and codomain. This formalises an extension of the well-known spider theorem for special commutative Frobenius monoids [CK16, CPP09].

The tight relationship between Frobenius structure and connectivity was also explored by McCurdy and Street [MS10]. Note, however, that in their work the connectivity preserved by separable Frobenius monoidal functors is the topological connectedness of the string diagram, whereas here connectivity refers to the equivalence relation imposed on the elements of the domain and codomain by the string diagram. The difference between these two perspectives, in algebraic terms, is the extra law.

## 4. Props for theories

Introduced by Mac Lane [Mac65] to generalise Lawvere’s algebraic theories approach to universal algebra [Law63], the theory of props provides a framework to discuss algebraic structures with multi-input multi-output operations.

**4.1. DEFINITION.** *A prop is a symmetric strict monoidal category having the natural numbers as objects and monoidal product given by addition. A morphism of props is a symmetric strict identity-on-objects monoidal functor.*

*If  $\mathcal{T}$  is a prop and  $\mathcal{C}$  is a symmetric monoidal category, we define an algebra of  $\mathcal{T}$  in  $\mathcal{C}$  to be a symmetric monoidal functor  $\mathcal{T} \rightarrow \mathcal{C}$ . A morphism of algebras of  $\mathcal{T}$  in  $\mathcal{C}$  is a monoidal natural transformation between them.*

Props allow us to study (one-sorted) symmetric monoidal theories, like those of monoids, commutative monoids, bimonoids, and Hopf monoids: models for these theories arise as algebras of certain props.

**4.2. DEFINITION.** *A symmetric monoidal theory  $T = (\Sigma, E)$  comprises a signature  $\Sigma$  and a set of equations  $E$ . A signature is a set of generators, where a generator is*

a formal symbol  $\sigma: n \rightarrow m$ . From a signature  $\Sigma$ , we may formally construct the set of  $\Sigma$ -terms. Defined inductively, a  **$\Sigma$ -term** takes one of the following forms:

- the empty term  $\emptyset: 0 \rightarrow 0$ , the unit  $\text{id}: 1 \rightarrow 1$ , the symmetry  $\times: 2 \rightarrow 2$ ;
- the generators  $\alpha: n \rightarrow m$  in  $\Sigma$ ;
- $\beta \circ \alpha: n \rightarrow p$ , where  $\alpha: n \rightarrow m$  and  $\beta: m \rightarrow p$  are  $\Sigma$ -terms; or
- $\alpha + \gamma: n + p \rightarrow m + q$ , where  $\alpha: n \rightarrow m$  and  $\gamma: p \rightarrow q$  are  $\Sigma$ -terms.

We call  $(n, m)$  the **type** of a  $\Sigma$ -term  $\tau: n \rightarrow m$ . An **equation** is then a pair of two  $\Sigma$ -terms with the same type.

A **model** for a symmetric monoidal theory in a symmetric strict monoidal category  $(\mathcal{C}, \otimes)$  is an object  $X$  together with morphisms  $\sigma_X: X^{\otimes n} \rightarrow X^{\otimes m}$  for every generator  $\sigma: n \rightarrow m$  in  $\Sigma$ , such that for every equation the two  $\Sigma$ -terms are equal interpreted as morphisms in  $\mathcal{C}$ . To be more precise, by interpret a  $\Sigma$ -term as a morphism in  $\mathcal{C}$ , we mean that we replace each generator  $\sigma$  in the  $\Sigma$ -term with the corresponding  $\sigma_X$ , each instance of  $+$  with  $\otimes$ , and take  $\circ$  to be composition in  $\mathcal{C}$ . Thus each  $\Sigma$ -term  $\tau$  of type  $(n, m)$  is interpreted as a morphism  $X^{\otimes n} \rightarrow X^{\otimes m}$  in  $\mathcal{C}$ .

Given a model on an object  $X$  and one on an object  $Y$ , a morphism of models from the former to the latter is a morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$  such that for every generator  $\sigma: n \rightarrow m$  we have  $f^{\otimes m} \circ \sigma_X = \sigma_Y \circ f^{\otimes n}: X^{\otimes n} \rightarrow Y^{\otimes m}$ .

If  $(\mathcal{C}, \otimes)$  is an arbitrary symmetric monoidal category, to talk of models in  $\mathcal{C}$  we may simply replace it with an equivalent symmetric strict monoidal category.

Many common algebraic structures can be expressed as symmetric monoidal theories, including all those discussed in the previous section. For example, the symmetric monoidal theory of commutative monoids has signature  $\{\mu: 2 \rightarrow 1, \eta: 0 \rightarrow 1\}$  and three equations: precisely those pairs of terms depicted in Section 3. Commutative monoids in a symmetric monoidal category are then models of this symmetric monoidal theory.

**4.3. DEFINITION.** We say that a prop  $\mathcal{T}$  is the **prop for** a symmetric monoidal theory  $T$  if for all symmetric monoidal categories  $\mathcal{C}$  the category of algebras of  $\mathcal{T}$  in  $\mathcal{C}$  is equivalent to the category of models of  $T$  in  $\mathcal{C}$ .

**4.4. EXAMPLE.** Write **FinSet** for the category of finite sets and functions, and also for its equivalent skeleton. This category inherits symmetric monoidal structure from the existence of finite coproducts, in this case the disjoint union of sets. Fixing a skeleton and, for example, utilising a total order on each set, one may choose the unitors and associator to be the identity, resulting a symmetric strict monoidal category [Bur93]. Thus we may consider **FinSet** to be a prop.

It is known that **FinSet** is the prop for commutative monoids [Gra01, Pir02]. Indeed, write  $m: 2 \rightarrow 1$  and  $e: 0 \rightarrow 1$  for the unique maps of their type in **FinSet**. Then given

a symmetric monoidal functor  $F : \mathbf{FinSet} \rightarrow \mathcal{C}$ , the tuple  $(F1, Fm, Fe)$  is a commutative monoid. Conversely, any commutative monoid  $(X, \mu, \eta)$  in  $\mathcal{C}$  gives rise to a functor  $\mathbf{FinSet} \rightarrow \mathcal{C}$  mapping 1 to  $X$ ,  $m$  to  $\mu$ , and  $e$  to  $\eta$ .

We may bootstrap on this construction to show that **Corel** has a strict skeleton, and so too may be considered a prop. First, recall that in any finitely cocomplete category  $\mathcal{C}$  we may construct a symmetric monoidal bicategory with the same objects and monoidal product, with morphisms cospans in  $\mathcal{C}$ , composition of morphisms given by pushout, and with 2-morphisms maps between apexes of cospans that commute over the feet [Ben67]. Decategorifying, we obtain a monoidal category **Cospan**( $\mathcal{C}$ ), where morphisms are isomorphism classes of cospans in  $\mathcal{C}$ .

Next, call a cospan  $X \rightarrow N \leftarrow Y$  **jointly epic** if the induced morphism  $X + Y \rightarrow N$  is an epimorphism. If monomorphisms in  $\mathcal{C}$  are preserved under pushout, we may construct a symmetric monoidal category **Corel**( $\mathcal{C}$ ) with objects again those of  $\mathcal{C}$ , but this time morphisms isomorphism classes of *jointly epic* cospans in  $\mathcal{C}$ , and composition taking the pushout of representative cospans, before corestricting to the jointly epic part [Mil00, JW96].<sup>2</sup> The unitors, associator, and symmetry are inherited from  $\mathcal{C}$ .

Our category **Corel** can be constructed in this way.

**4.5. PROPOSITION.** *The category **Corel** is isomorphic as a symmetric monoidal category to **Corel**( $\mathbf{FinSet}$ ).*

**PROOF.** By the universal property of the coproduct, corelations  $X + Y \rightarrow A$  are in one-to-one correspondence with jointly epic cospans  $X \rightarrow A \leftarrow Y$ . It is straightforward to check the notions of composition agree: consider the wire diagrams for corelations. ■

As equivalences preserve colimits, replacing **FinSet** with its strict skeleton thus shows that **Corel** also has a strict skeleton. We henceforth use **Corel** to refer to this equivalent prop. This allows us to restate our main theorem as follows.

**4.6. THEOREM.** ***Corel** is the prop for extraspecial commutative Frobenius monoids.*

To prove this theorem, we begin by giving a more explicit construction of the prop for extraspecial commutative Frobenius monoids.

## 5. Props from theories

If we consider the set  $\mathbb{N} \times \mathbb{N}$  as a discrete category, then a signature is a functor from  $\mathbb{N} \times \mathbb{N}$  to the category **Set** of sets and functions. Note that to each prop we may associate the so-called underlying signature  $\text{hom}(\cdot, \cdot) : \mathbb{N} \times \mathbb{N} \rightarrow \mathbf{Set}$ . The following important result

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<sup>2</sup> More generally, a category of corelations may be constructed from any finitely cocomplete category equipped with a  $(\mathcal{E}, \mathcal{M})$ -factorisation system such that  $\mathcal{M}$  is preserved under pushout [JW96]. In related papers, we have shown that this construction can be used to model interconnection of ‘black-boxed’ systems; that is, to model systems in which only the internal structure is obscured, leaving only the external behaviour [BF, FRS16, Fon16].

allows us to understand the category **PROP** of props; proofs can be found in both Baez, Coya, and Rebro [BCR] and Trimble [Tri14].

**5.1. PROPOSITION.** *The underlying signature functor  $U: \mathbf{PROP} \rightarrow \mathbf{Set}^{\mathbb{N} \times \mathbb{N}}$  is monadic.*

We write the left adjoint of this functor  $U$  as  $F: \mathbf{Set}^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathbf{PROP}$ , and call  $F\Sigma$  the **free prop** on the signature  $\Sigma$ . There is a surjection from the set of  $\Sigma$ -terms onto morphisms in the free prop [BCR].

Another important corollary of this theorem is that the category of props is cocomplete. In particular, this allows us to take coequalizers in the category of props. We use this to give an explicit construction of the prop for a symmetric monoidal theory.

Let  $(\Sigma, E)$  be a symmetric monoidal theory. Recall that each equation has a type, and abuse notation to write  $E$  also for the resulting signature. Then, as the morphisms in  $F\Sigma$  are  $\Sigma$ -terms and as  $U$  and  $F$  are adjoint, we may define functors  $\lambda, \rho: FE \rightarrow F\Sigma$  mapping each equation to the first element and the second element of the pair respectively. This allows us to build the prop for the theory.

**5.2. PROPOSITION.** *The prop for a symmetric monoidal theory  $(\Sigma, E)$  is the coequalizer of the diagram*

$$FE \rightrightarrows_{\rho} F\Sigma.$$

The intuition is that the coequalizer is the weakest prop that forces the ‘left-hand side’ (given by  $\lambda$ ) of each equation in  $E$  to equal the ‘right-hand side’ (given by  $\rho$ ). This implies an equivalence between algebras of the coequalizer prop and models of  $(\Sigma, E)$ .

Write **Th(ESCFM)** for the prop for extraspecial commutative Frobenius monoids constructed in this way. It remains to prove that this prop is isomorphic to **Corel**.

## 6. Corelations are the prop for extraspecial commutative Frobenius monoids

In the influential paper [Lac04], Lack develops the theory of distributive laws for props, and proves the following as an example. Note we write **Cospan** for **Cospan(FinSet)**.

**6.1. PROPOSITION.** ***Cospan** is isomorphic to the prop **Th(SCFM)** for special commutative Frobenius monoids.*

As the name suggests, a special commutative Frobenius monoid is a commutative monoid and cocommutative comonoid that further obey the Frobenius and special laws. Note that in **FinSet** there are unique maps  $0 \rightarrow 1$ ,  $1 \rightarrow 1$ , and  $2 \rightarrow 1$ . The isomorphism

acts as follows on the generators of  $\mathbf{Th}(\mathbf{SCFM})$ :

$$\begin{aligned}\alpha: \mathbf{Th}(\mathbf{SCFM}) &\longrightarrow \mathbf{Cospan}(\mathbf{FinSet}); \\ \text{---} \bullet &\longmapsto (2 \rightarrow 1 \leftarrow 1) \\ \bullet &\longmapsto (0 \rightarrow 1 \leftarrow 1) \\ \text{---} \curvearrowleft &\longmapsto (1 \rightarrow 1 \leftarrow 2) \\ \text{---} \bullet &\longmapsto (1 \rightarrow 1 \leftarrow 0).\end{aligned}$$

We use this to prove the main theorem. The guiding intuition is that to corestrict cospans to corelations is to impose the ‘extra’ condition upon a special commutative Frobenius monoid.

Our strategy will be to prove that  $\mathbf{Th}(\mathbf{ESCFM})$  and  $\mathbf{Corel}$  are coequalizers of isomorphic diagrams, and hence themselves isomorphic. First, we show how to construct  $\mathbf{Th}(\mathbf{ESCFM})$  as a coequalizer of props.

**6.2. LEMMA.** *The following is a coequalizer diagram:*

$$FE_{\text{Ex}} \rightrightarrows^{\lambda}_{\rho} \mathbf{Th}(\mathbf{SCFM}) \longrightarrow \mathbf{Th}(\mathbf{ESCFM}).$$

**PROOF.** Let  $(\Sigma, E_{\text{SCFM}})$  and  $(\Sigma, E_{\text{ESCFM}})$  be the theories of special commutative Frobenius monoids and extraspecial commutative Frobenius monoids respectively—note that they have the same set of generators,  $\Sigma$ . Write also  $(\Sigma, E_{\text{Ex}})$  for the theory of the ‘extra law’, so  $E_{\text{Ex}}$  contains just a single element  $\bullet: 0 \rightarrow 0$ . This has image  $\lambda_{\text{Ex}}(\bullet) = \bullet \dashv \bullet$  and  $\rho_{\text{Ex}}(\bullet) = \emptyset$  under the two canonical maps  $FE_{\text{Ex}} \rightrightarrows F\Sigma$ .

Now, by construction we have a map  $F\Sigma \rightarrow \mathbf{Th}(\mathbf{SCFM})$ , and composing this with  $\lambda_{\text{Ex}}$  and  $\rho_{\text{Ex}}$  gives  $\lambda$  and  $\rho$  respectively. Since an extraspecial commutative Frobenius monoid is a fortiori a special commutative Frobenius monoid, by construction we also have a map  $\mathbf{Th}(\mathbf{SCFM}) \rightarrow \mathbf{Th}(\mathbf{ESCFM})$ ; this is the unlabelled map above. As  $E_{\text{ESCFM}} = E_{\text{SCFM}} + E_{\text{Ex}}$ , it is straightforward to verify that the above diagram is a coequalizer diagram. ■

Next, we construct  $\mathbf{Corel}$  as a coequalizer.

**6.3. LEMMA.** *The following is a coequalizer diagram:*

$$FE_{\text{Ex}} \rightrightarrows^{\alpha \circ \lambda}_{\alpha \circ \rho} \mathbf{Cospan} \longrightarrow \mathbf{Corel}$$

**PROOF.** The map  $\mathbf{Cospan} \rightarrow \mathbf{Corel}$  is the canonical one corestricting each cospan to its jointly epic part. It is straightforward to check this is indeed a map of props; details can be found in [Fon16]. Now  $(\alpha \circ \lambda)(\bullet) = (0 \rightarrow 1 \leftarrow 0)$ , while  $(\alpha \circ \rho)(\bullet) = (0 \rightarrow 0 \leftarrow 0)$ . This implies the above diagram commutes from  $FE_{\text{Ex}}$  to  $\mathbf{Corel}$ . It remains to check the universal property.

Suppose that we have a prop  $\mathcal{T}$  such that the diagram

$$FE_{\text{Ex}} \xrightarrow[\alpha \circ \rho]{\alpha \circ \lambda} \mathbf{Cospans} \xrightarrow{A} \mathcal{T}$$

commutes from  $FE_{\text{Ex}}$  to  $\mathcal{T}$ . We must show there is a unique map  $A': \mathbf{Corel} \rightarrow \mathcal{T}$ .

As the map  $\mathbf{Cospans} \rightarrow \mathbf{Corel}$  is full, it is enough to show that each cospan  $(n \xrightarrow{f} a \xleftarrow{g} m)$  has the same image as its jointly epic part  $(n \xrightarrow{f'} \text{im}[f, g] \xleftarrow{g'} m)$  under  $A$ —we then have a unique and well-defined map  $A'$  sending each corelation to its image as a cospan under  $A$ . But this is straightforward:

$$\begin{aligned} A(n \xrightarrow{f} a \xleftarrow{g} m) &= A(n \xrightarrow{f'} \text{im}[f, g] \xleftarrow{g'} m) + A(0 \rightarrow (a - \text{im}[f, g]) \leftarrow 0) \\ &= A(n \xrightarrow{f'} \text{im}[f, g] \xleftarrow{g'} m) + (A \circ \alpha \circ \lambda)(\bullet^{+(a - \text{im}[f, g])}) \\ &= A(n \xrightarrow{f'} \text{im}[f, g] \xleftarrow{g'} m) + (A \circ \alpha \circ \rho)(\bullet^{+(a - \text{im}[f, g])}) \\ &= A(n \xrightarrow{f'} \text{im}[f, g] \xleftarrow{g'} m). \end{aligned}$$

This proves the lemma. ■

**PROOF OF THEOREM 4.6.** More explicitly now, our strategy is to show both  $\mathbf{Corel}$  and  $\mathbf{Th}(\mathbf{ESCFM})$  are coequalizers in the diagram

$$\begin{array}{ccccc} & & \mathbf{Th}(\mathbf{SCFM}) & \longrightarrow & \mathbf{Th}(\mathbf{ESCFM}) \\ FE_{\text{Ex}} & \xrightarrow{\lambda} & \downarrow \alpha & & \downarrow \\ & \xrightarrow{\rho} & & & \\ & \xrightarrow{\alpha \circ \lambda} & \mathbf{Cospans} & \longrightarrow & \mathbf{Corel} \end{array}$$

Lemma 6.2 shows the upper row is a coequalizer diagram, while Lemma 6.3 shows the lower row is too. As the two relevant triangles commute and the first vertical map is an isomorphism,  $\mathbf{Th}(\mathbf{ESCFM})$  and  $\mathbf{Corel}$  are coequalizers of isomorphic diagrams, and hence themselves isomorphic. ■

The so-called spider theorem is an immediate corollary.

**6.4. COROLLARY.** *Two morphisms in an extraspecial commutative Frobenius monoid are equal if and only if they map to the same corelation.*

## 7. Spans, cospans, relations, corelations

Lastly, we return to the big picture. The dual theorems are known for spans and relations [Lac04, WW], but the above method of proof provides a novel argument, and illuminates the duality. Recall bimonoids, sometimes also called bialgebras.

**7.1. DEFINITION.** A **bicommutative bimonoid**  $(X, \mu, \eta, \delta, \epsilon)$  in a monoidal category  $(\mathcal{C}, \otimes)$  comprises a commutative monoid  $(X, \mu, \eta)$  and a cocommutative comonoid  $(X, \delta, \epsilon)$  that further obey the extra law and the bimonoid laws

$$\begin{array}{ccc} \text{Diagram: two nodes connected by two strands forming a loop, with a dot at each node.} & = & \text{Diagram: two nodes connected by two strands forming a loop, with a dot at each node.} \\ \text{Diagram: two nodes connected by two strands forming a loop, with a dot at each node.} & = & \text{Diagram: two nodes connected by two strands forming a loop, with a dot at each node.} \\ \text{(the bimonoid laws)} & & \end{array}$$

Bimonoids can be understood as dual to Frobenius monoids: **Span(FinSet)** is the prop for bicommutative bimonoids. This fact goes back to Lack [Lac04]. Wadsley and Woods provide an alternative proof, via the fact that the category of matrices over a rig<sup>3</sup>  $R$  is the prop for bicommutative bimonoids equipped with an action of the rig  $R$  [WW, Theorem 5]. Choosing the rig of booleans, this also implies that **Rel**, the prop equivalent to the category of finite sets and relations, is the prop for special bicommutative bimonoids. The techniques of this paper can be co-opted to provide an alternative proof of this fact.

**7.2. THEOREM.** **Rel** is isomorphic to the prop for special bicommutative bimonoids.

**PROOF.** To sketch: Lack has already shown, using the distributive law arising from pull-backs in **FinSet**, that **Span(FinSet)** is isomorphic to the prop for bicommutative bimonoids. We may use this to set up isomorphic coequalizer diagrams in the category PROP to obtain both the prop for special bicommutative bimonoids and the prop **Rel**. The isomorphism arises from the observation that taking the jointly monic part of a span is equivalent to iteratively asserting that the span  $(1 \leftarrow 2 \rightarrow 1)$  may be replaced by the identity  $(1 \leftarrow 1 \rightarrow 1)$ , and that this manifests as the special law. ■

We conclude by displaying our table once again, bringing out the symmetry by annotating names with the suppressed aspects of their structure.

spans	cospans
extra bicommutative bimonoids	special bicommutative Frobenius monoids
relations	corelations
extraspecial bicommutative bimonoids	extraspecial bicommutative Frobenius monoids

## References

- [BC] J. C. Baez, B. Coya, A compositional framework for bond graphs, in preparation.
- [BCR] J. C. Baez, B. Coya, F. Rebro, Props in network theory, in preparation.

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<sup>3</sup>Also known as a semiring, a rig is a ring without the condition that additive inverses exist. That is, it is a **ring** without **negatives**.

- [BE15] J. C. Baez, J. Erbele, Categories in control, *Theory Appl. Categ.* **30**:836–881, 2015. Available at <http://www.tac.mta.ca/tac/volumes/30/24/30-24abs.html>.
- [BF] J. C. Baez, B. Fong, A compositional framework for passive linear circuits, preprint. Available as [arXiv:1504.05625](https://arxiv.org/abs/1504.05625).
- [Ben67] J. Bénabou, Introduction to bicategories, in *Reports of the Midwest Category Seminar*, eds. J. Bénabou *et al*, Springer Lecture Notes in Mathematics **47**:1–77, New York, 1967.
- [BSZ14] F. Bonchi, P. Sobociński, F. Zanasi, A categorical semantics of signal flow graphs, in *CONCUR 2014: Concurrency Theory*, eds. P. Baldan and D. Gorla, Springer Lecture Notes in Computer Science, **8704**:435–450, Berlin, 2014.
- [BG01] R. Bruni, F. Gadducci, Some algebraic laws for spans (and their connections with multi-relations), *Electron. Notes Comput. Sci.*, **44**(3):175–193, 2003. doi:[10.1016/S1571-0661\(04\)80937-X](https://doi.org/10.1016/S1571-0661(04)80937-X).
- [BGM02] R. Bruni, F. Gadducci, U. Montanari, Normal forms for algebras of connections, *Theor. Comput. Sci.*, **286**(2):247–292, 2002. doi:[10.1016/S0304-3975\(01\)00318-8](https://doi.org/10.1016/S0304-3975(01)00318-8).
- [Bur93] A. Burroni, Higher dimensional word problems with applications to equational logic, *Theor. Comput. Sci.*, **115**(1):43–62, 1993. doi:[10.1016/0304-3975\(93\)90054-W](https://doi.org/10.1016/0304-3975(93)90054-W).
- [CK16] B. Coecke, A. Kissinger, *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*, Cambridge University Press, 2016.
- [CPP09] B. Coecke, E. Paquette, D. Pavlovic, Quantum and classical structuralism, in *Semantic Techniques for Quantum Computation*, eds. I. Mackie and S. Gay, Cambridge University Press, pp.29–69, 2009. Also available as [arXiv:0904.1997](https://arxiv.org/abs/0904.1997).
- [CW87] A. Carboni, R. F. C. Walters, Cartesian bicategories I, *J. Pure Appl. Alg.* **49**:11–32, 1987.
- [DP03] K. Došen and Z. Petrić, Generality of proofs and its Brauerian representation, *J. Symbolic Logic* **68**(3):740–750, 2003. doi:[10.2178/jsl/1058448435](https://doi.org/10.2178/jsl/1058448435).
- [DP13] K. Došen and Z. Petrić, Syntax for split preorders, *Ann. Pur. Appl. Logic* **164**(4):443–481, 2013. doi:[10.1016/j.apal.2012.10.008](https://doi.org/10.1016/j.apal.2012.10.008).
- [Ell14] D. Ellerman, An introduction to partition logic, *Log. J. IGPAL* **22**(1):94–125, 2014. doi:[10.1093/jigpal/jzt036](https://doi.org/10.1093/jigpal/jzt036).
- [Fon16] B. Fong, *The Algebra of Open and Interconnected Systems*, DPhil thesis, University of Oxford, 2016. Available as [arXiv:1609.05382](https://arxiv.org/abs/1609.05382).

- [FRS16] B. Fong, P. Rapisarda, P. Sobociński, A categorical approach to open and interconnected systems, in *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '16, New York, NY, USA, July 5–8, 2016*, eds. M. Grohe, E. Koskinen, N. Shankar, pp.495–504, 2016. doi:10.1145/2933575.2934556.
- [Gra01] M. Grandis, Finite sets and symmetric simplicial sets, *Theory Appl. Categ.* **8**:244–252, 2001. Available at <http://www.tac.mta.ca/tac/volumes/8/n8/8-08abs.html>.
- [JW96] R. D. Jayawardene, O. Wyler, Categories of relations and functional relations, RR 96-194, Carnegie Mellon University, 1996.
- [JS91] A. Joyal, R. Street, The geometry of tensor calculus I, *Adv. Math.* **88**(1):55–112, 1991. doi:10.1016/0001-8708(91)90003-P.
- [Lac04] S. Lack, Composing PROPs, *Theory Appl. Categ.* **13**(9):147–163, 2004. Available at <http://www.tac.mta.ca/tac/volumes/13/9/13-09abs.html>.
- [Law63] F. W. Lawvere, Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories, *Reprints in Theory Appl. Categ.* **5**:1–121. Available at <http://www.tac.mta.ca/tac/reprints/articles/5/tr5abs.html>.
- [LR03] F. W. Lawvere, R. Rosebrugh, *Sets for Mathematics*, Cambridge University Press, 2003.
- [Mil00] S. Milius, *Relations in categories*, MA thesis, York University, Ontario, 2000.
- [Mac65] S. Mac Lane, Categorical algebra, *Bull. Amer. Math. Soc.* **71**(1):40–106, 1965. doi:10.1090/S0002-9904-1965-11234-4.
- [MS10] M. McCurdy, R. Street, What separable Frobenius monoidal functors preserve, *Cah. Top. Géom. Diff. Cat.* **51**(1):29–50, 2010.
- [Pir02] T. Pirashvili, On the PROP corresponding to bialgebras, *Cah. Top. Géom. Diff. Cat.* **43**(3):221–239, 2002. arXiv:math/0110014.
- [RSW08] R. Rosebrugh, N. Sabadini, R. F. C. Walters, Calculating colimits compositionally, in *Concurrency, Graphs and Models: Essays Dedicated to Ugo Montanari on the Occasion of His 65th Birthday*, eds. P. Degano, R. De Nicola and J. Meseguer, Springer Lecture Notes in Computer Science, **5065**:581–592, Berlin, 2008. doi:10.1007/978-3-540-68679-8\_36.
- [Tri14] T. Trimble, Multisorted Lawvere theories, version April 27 2014, nlab. Available at <http://ncatlab.org/toddtrimble/published/multisorted+Lawvere+theories>.

[WW] S. J. Wadsley, N. Woods, PROPs for linear systems, preprint, 2015.  
arXiv:1505.00048.

[Zan15] F. Zanasi, *Interacting Hopf Algebras—the Theory of Linear Systems*, Ph.D. thesis, École Normale Supérieure de Lyon, 2015. Available at <https://tel.archives-ouvertes.fr/tel-01218015/>.

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