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# A URYSOHN TYPE LEMMA FOR GROUPOIDS

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Abstract. Starting from the observation that through groupoids we can express in a unified way the notions of fundamental system of entourages of a uniform structure on a space $X$, respectively the system of neighborhoods of the unity of a topological group that determines its topology, we introduce in this paper a notion of $G$-uniformity for a groupoid $G$. The topology induced by a $G$-uniformity turns $G$ into a topological locally transitive groupoid.

We also prove a Urysohn type lemma for groupoids and obtain metrization theorems for groupoids unifying in two ways the Alexandroff–Urysohn Theorem and Birkhoff-Kakutani Theorem.

## 1. Introduction and preliminaries

The notion of groupoid is a natural generalization of the notion of group in the following sense: a groupoid is a set $G$ endowed with partially defined product operation $(x,y) \mapsto xy \ \left[: G^{(2)} \to G\right]$ (where $G^{(2)} \subset G \times G$) and an inversion operation $x \mapsto x^{-1} \ [: G \to G]$ satisfying the subsequent weaker versions of the group axioms:

**G1** If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then $(xy,z) \in G^{(2)}$, $(x,yz) \in G^{(2)}$ and $(xy)\,z = x\,(yz)$.

**G2** $(x^{-1})^{-1} = x$ for all $x \in G$.

**G3** For all $x \in G$, $(x,x^{-1}) \in G^{(2)}$, and if $(z,x) \in G^{(2)}$, then $(zx)\,x^{-1} = z$.

**G4** For all $x \in G$, $(x^{-1},x) \in G^{(2)}$, and if $(x,y) \in G^{(2)}$, then $x^{-1}(xy) = y$.

The maps $r$ and $d$ on $G$, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the range (target) map, respectively the domain (source) map. They have a common image called the unit space of $G$ and denoted $G^{(0)}$. The fibres of the range and the domain maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. Also for $u,v \in G^{(0)}$, $G^u_v = G^u \cap G_v$.

A topological groupoid is a groupoid $G$ together with a topology on $G$ such that the product operation $(x,y) \mapsto xy \ \left[: G^{(2)} \to G\right]$ (where $G^{(2)} \subset G \times G$ is endowed with the topology induced by the product topology on $G \times G$) and the inversion operation

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$x \mapsto x^{-1}$ [: $G \to G$] are continuous functions. A family $\{W_j\}_{j \in J}$ of neighborhoods of the unit space is said to be compatible with the topology of the $r$-fibres (respectively, $d$-fibres) if for every $u \in G^{(0)}$ and every open neighborhood $U$ of $u$, there is $j \in J$ such that $W_j \cap G^u \subset U \cap G^u$ and $u$ is in the interior of $W_j \cap G^u$ with respect to the topology on $G^u$ coming from $G$ (respectively, $W_j \cap G_u \subset U \cap G_u$ and $u$ is in the interior of $W_j \cap G_u$ with respect to the topology on $G_u$ coming from $G$).

Let us also recall that a uniform space is a set $X$ endowed with a uniform structure. A fundamental system of symmetric entourages of a uniform structure on $X$ is a nonempty family $\mathcal{W}$ of subsets of the Cartesian product $X \times X$ that satisfies the following conditions:

**U1** if $W$ is in $\mathcal{W}$, then $W$ contains the diagonal $\Delta = \{(x, x) : x \in X\}$.

**U2** if $W_1$ and $W_2$ are in $\mathcal{W}$, then there is $W_3 \in \mathcal{W}$ such that $W_3 \subset W_1 \cap W_2$.

**U3** if $W_1$ is in $\mathcal{W}$, then there exists $W_2$ in $\mathcal{W}$ such that, whenever $(x, y)$ and $(y, z)$ are in $W_2$, then $(x, z) \in W_1$.

**U4** if $W \in \mathcal{W}$, then $W = W^{-1} = \{(y, x) : (x, y) \in W\}$ ($W$ is a symmetric entourage).

The uniform space $X$ becomes a topological space by defining a subset $A \subset X$ to be open if and only if for every $x \in A$ there is $W_x \in \mathcal{W}$ such that $\{y : (x, y) \in W_x\} \subset A$.

The Cartesian product $X \times X$ can be viewed as a trivial groupoid $G$ under the operations: $(x, y)(y, z) = (x, z)$ and $(x, y)^{-1}$. In the settings of groupoids condition $U1$ can be written as "$G^{(0)} \subset W \subset G$ for all $W \in \mathcal{W}$" and condition $U3$ as "for every $W_1 \in \mathcal{W}$ there is $W_2 \in \mathcal{W}$ such that $W_2 W_2 \subset W_1$".

In this paper we work with a collection of subsets of a groupoid $G$ mimicking the properties of fundamental system of symmetric entourages of a uniform structure on $X$. Such a collection will be called in this paper $G$-uniformity. We prove that a $G$-uniformity induces a topology on $G$ that turns $G$ into a topological locally transitive groupoid. Let us recall that a topological locally transitive groupoid is a topological groupoid $G$ with the property that for all $u \in G^{(0)}$ the maps $r_u$ are open, where $r_u : G_u \to G^{(0)}$, $r_u(x) = r(x)$ for all $x \in G_u$ and $G_u$ is endowed with the topology coming from $G$ (see [12]). If we begin with a topological groupoid $(G, \tau)$ and with a $G$-uniformity given by a fundamental system of neighborhoods of the unit space, then the topology induced by de $G$-uniformity is finer than $\tau$ and coincides with $\tau$ if and only if $(G, \tau)$ is locally transitive. The main result of this paper is a Urysohn type lemma for groupoids (Theorem 2.5). The existence of a function with properties $1 - 3$ in Theorem 2.5 could also be obtained taking into account that a $G$-uniformity is a base for a uniform structure on $G$. However the topology defined by the $G$-uniformity do not necessarily coincides with the groupoid topology, even if the $G$-uniformity is given by a fundamental system of neighborhoods of the unit space. The construction in Theorem 2.5 allows us to get a function with additional properties. In particular, in the case of a topological groupoid with open range map and a $G$-uniformity given by a fundamental system of neighborhoods of the unit space, our construction allows us to put out a connection with the groupoid topology: the functions $f$ associated in Theorem 2.5 with open subsets of $G$ or with $G^{(0)}$ are upper semi-continuous on $G$ and their restrictions to the $r$-fibres as well as to the $d$-fibres of the groupoid are continuous functions. Thus these functions can be used to construct convolutions algebras as in [4] and possibly to extend the construction of a $C^*$-algebra associated to a topological locally compact groupoid with continuous Haar system introduced in [11]. Moreover the property 9 in Theorem 2.5 allows us to obtain metrization theorems for groupoids and thus to express in an unified way Alexandroff–Urysohn Theorem and Birkhoff-Kakutani Theorem as we explain below. Let us consider the following two theorems:

1.1. THEOREM. [Alexandroff–Urysohn Theorem] *A topological Hausdorff space $X$ is metrizable if and only if its topology is given by a uniformity with countable base. [1]*

1.2. THEOREM. [Birkhoff-Kakutani Theorem] *A topological group $G$ is metrizable if and only if there is a countable base for the topology at identity element in $G$. Furthermore, in such a case, the distance function may be taken to be either left-invariant or right-invariant. ([2], [6])*

Let us remark that the space $X$, respectively the group $G$ can be viewed as $r$-fibres (as well as $d$-fibres) of a groupoid ($X \times X$ in the first case and $G$ itself in the second case). We prove in this paper that the previous two results can be express in an unified way in the groupoid language:

1.3. THEOREM. *Let $G$ be a topological groupoid. Then there are left (respectively, right) invariant metrics compatible with the topology on $r$-fibres (respectively, the $d$-fibres) of the groupoid if and only if there is a countable $G$-uniformity $\{W_n\}_{n\in\mathbb{N}}$ compatible with the topology of the $r$-fibres (respectively, $d$-fibres) such that $\bigcap_{n\in\mathbb{N}} W_n = G^{(0)}$. (Proposition 3.14 and Proposition 3.15)*

The proof of this theorem is based on the construction of a function on $G$ satisfying the hypothesis of [8, Theorem 3.26]. This function is obtained as a particular case of Urysohn Lemma for groupoids (Theorem 2.5).

We also prove in this paper that:

1.4. THEOREM. *For a topological locally transitive groupoid $G$ the following statements are equivalent:*

(*a*) *$G$ is metrizable*

(*b*) *For every neighborhood $W$ of $G^{(0)}$ there is a neighborhood $W'$ of $G^{(0)}$ such that $W'W' \subset W$ and $G^{(0)}$ has a countable fundamental system $\{W_n\}_{n\in\mathbb{N}}$ of neighborhoods such that $\bigcap_{n\in\mathbb{N}} W_n = G^{(0)}$ and $\bigcap_{n\in\mathbb{N}} (r,d)(W_n) = diag\left(G^{(0)}\right)$.*

(*c*) *There is a countable $G$-uniformity $\{W_n\}_{n\in\mathbb{N}}$ compatible with the topology of the fibres such that $\bigcap_{n\in\mathbb{N}} W_n = G^{(0)}$ and $\bigcap_{n\in\mathbb{N}} (r,d)(W_n) = diag\left(G^{(0)}\right)$. Each $W_n$ may be taken to be a neighborhood of the unit space.*

*Moreover the distance function* $\rho$ *may be taken to satisfy the following properties:*

1. $\rho(x,y) = \rho(x^{-1}, y^{-1})$ *for all* $x, y \in G$.
2. $\rho(x, r(x)) = \rho(x, d(x))$ *for all* $x \in G$.
3. $\rho(xy, r(x)) \leq \rho(x, r(x)) + \rho(y, r(y))$ *for all* $(x,y) \in G^{(2)}$.
4. $\rho(x,y) \leq \rho(x^{-1}y, d(x))$ *for all* $x, y \in G$ *such that* $r(x) = r(y)$.
5. $\rho(d(x), d(y)) \leq 2\rho(x,y)$ *and* $\rho(r(x), r(y)) \leq 2\rho(x,y)$ *for all x,y* $\in G$. *(Theorem 3.16)*

## 2. Urysohn's lemma for groupoids

2.1. Definition. *Let* $G$ *be a groupoid. By a* $G$*-uniformity we mean a collection* $\{W\}_{W\in\mathcal{W}}$ *of subsets of* $G$ *satisfying the following conditions:*

1. $G^{(0)} \subset W \subset G$ *for all* $W \in \mathcal{W}$.
2. *If* $W_1$, $W_2 \in \mathcal{W}$, *then there is* $W_3 \subset W_1 \cap W_2$ *such that* $W_3 \in \mathcal{W}$.
3. *For every* $W_1 \in \mathcal{W}$ *there is* $W_2 \in \mathcal{W}$ *such that* $W_2W_2 \subset W_1$.
4. $W = W^{-1}$*for all* $W \in \mathcal{W}$.

2.2. Definition. *Let* $G$ *be a groupoid. Two* $G$*-uniformities* $\mathcal{W}$ *and* $\mathcal{W}'$ *are said to be equivalent if for every* $W \in \mathcal{W}$ *there is* $W' \in \mathcal{W}'$ *such that* $W' \subset W$ *and conversely, for every* $W' \in \mathcal{W}'$ *there is* $W \in \mathcal{W}$ *such that* $W \subset W'$.

Let $\mathcal{W}$ be a family of subsets of a groupoid $G$ satisfying conditions $1-4$ from Definition 2.1 and let

$$I = \left\{ \frac{1}{2^n},\ n \in \mathbb{N} \right\}$$

Let $W_0 \in \mathcal{W}$ and $W_1 \in \mathcal{W}$ be such that $W_1W_1 \subset W_0$. Inductively we can construct an $I$-indexed family $\{W_i\}_{i\in I}$. Suppose that for $W_i \in \mathcal{W}$ has already been built. Then according condition 3 in Definition 2.1, there is a $W_i' \in \mathcal{W}$ such that $W_i'W_i' \subset W_i$. Let $W_{i/2} = W_i'$. Thus we obtain an $I$-indexed family $\{W_i\}_{i\in I}$ satisfying the following properties:

1. $W_i \in \mathcal{W}$ for all $i \in I$.
2. $W_iW_i \subset W_{2i}$ for all $i \in I$, $i \leq \frac{1}{2}$.
3. $W_1W_1 \subset W_0$.

Hence $W_i \subset W_i W_i \subset W_{2i}$ for all $i \in I$, $i \leq \frac{1}{2}$ and
$...W_{1/2^n} \subset W_{1/2^n} W_{1/2^n} \subset W_{1/2^{n-1}} \subset W_{1/2^{n-1}} W_{1/2^{n-1}} \subset ...W_{1/2} \subset W_{1/2} W_{1/2} \subset W_1$
Let us note that:

1. If $i, j \in I$, then $i < j$ iff there is $p \in \mathbb{N}^*$ such that $j = 2^p i$.

2. If $i, j \in I$ and $i < j$, then $2i \leq j$.

3. If $i, j \in I$ and $i \leq j$, then $W_i \subset W_j$.

4. If $i_1$, $i_2$, ..., $i_k \in I$ and $i_k \leq i_{k-1} < i_{k-2} < ... < i_1 < 1$, then $W_{i_k} W_{i_{k-1}} ... W_{i_1} \subset W_{2i_1}$ and $W_{i_1} ... W_{i_{k-1}} W_{i_k} \subset W_{2i_1}$. Indeed,

$$\begin{aligned} W_{i_k} W_{i_{k-1}} W_{i_{k-2}} ... W_{i_1} &\subset W_{i_{k-1}} W_{i_{k-1}} W_{i_{k-2}} ... W_{i_1} \\ &\subset W_{2i_{k-1}} W_{i_{k-2}} ... W_{i_1} \\ &\subset W_{i_{k-2}} W_{i_{k-2}} ... W_{i_1} \\ &\subset ... \subset W_{2i_1} \end{aligned}$$

   Similarly, $W_{i_1} ... W_{i_{k-1}} W_{i_k} \subset W_{i_1} ... W_{i_{k-1}} W_{i_{k-1}} \subset W_{i_1} ... W_{i_{k-2}} W_{2i_{k-1}} \subset ... W_{2i_1}$.

5. If $i_1$, $i_2$, ..., $i_k$, $j_1$, $j_2$, ..., $j_m \in I$, $i_k < i_{k-1} < ... < i_1 \leq 1$, $j_m < j_{m-1} < ... < j_1 \leq 1$ and $i_k + i_{k-1} + ... + i_1 \leq j_m + j_{m-1} + ... + j_1$, then

$$W_{i_k} W_{i_{k-1}} ... W_{i_1} \subset W_{j_m} W_{j_{m-1}} ... W_{j_1}.$$

   Indeed, let us remark that $i_k + i_{k-1} + ... + i_1 = \frac{1}{2^{n_k}} + \frac{1}{2^{n_{k-1}}} + ... + \frac{1}{2^{n_1}}$ is the conversion into decimal system of the following number in base 2: $b_0, b_1 b_2 ... b_{n_k}$ where $b_i = 1$ if $i \in \{n_1, n_2, ..., n_k\}$ and $b_i = 0$ otherwise. Thus if $i_k + i_{k-1} + ... + i_1 = j_m + j_{m-1} + ... + j_1$, then $m = k$ and $i_k = j_k$, ..., $i_1 = j_1$. If $i_k + i_{k-1} + ... + i_1 < j_m + j_{m-1} + ... + j_1$, then there is $p \in \mathbb{N}^*$ such that $i_1 = j_1$, ..., $i_{p-1} = j_{p-1}$ and $i_p < j_p$. Hence

$$\begin{aligned} W_{i_k} W_{i_{k-1}} ... W_{i_p} W_{i_{p-1}} ... W_{i_1} &\subset W_{2i_p} W_{i_{p-1}} ... W_{i_1} \\ &\subset W_{j_p} W_{i_{p-1}} ... W_{i_1} \\ &= W_{j_p} W_{j_{p-1}} ... W_{j_1} \\ &\subset W_{j_m} W_{j_{m-1}} ... W_{j_1}. \end{aligned}$$

2.3. LEMMA. *Let $G$ be a groupoid, $\mathcal{W}$ be a $G$-uniformity (in the sense of Definition 2.1) and let*

$$I = \left\{ \frac{1}{2^n},\ n \in \mathbb{N} \right\}.$$

*Let us consider an $I$-indexed family $\{W_i\}_{i \in I}$ satisfying the following properties:*

*1. $W_i \in \mathcal{W}$ for all $i \in I$.*

*2.* $W_i W_i \subset W_{2i}$ *for all* $i \in I$, $i \leq \frac{1}{2}$.

*For* $W_{i_k}$, $W_{i_{k-1}}$, ..., $W_{i_1} \in \{W_i\}_{i\in I}$, *let us denote*

$$s\left(W_{i_k} W_{i_{k-1}} ... W_{i_1}\right) = i_k + i_{k-1} + ... + i_1.$$

*Let* $n \in \mathbb{N}^*$, *and* $i_1, i_2, ..., i_k \in I$ *be such that* $i_k < i_{k-1} < i_{k-2} < ... < i_1 < 1$. *Then there are* $j_1, j_2, ..., j_r \in I$ *such that*

1. $j_r < j_{r-1} < i_{r-2} < ... < j_1 \leq \max\left\{\frac{1}{2^{n-1}}, 2i_1\right\} \leq 1$
2. $W_{1/2^n} W_{i_k} W_{i_{k-1}} ... W_{i_1} \subset W_{j_r} W_{j_{r-1}} ... W_{j_1}$
3. $0 < s\left(W_{j_r} W_{j_{r-1}} ... W_{j_1}\right) - s\left(W_{i_k} W_{i_{k-1}} ... W_{i_1}\right) \leq \frac{1}{2^{n-1}}$

*Moreover* $j_1 < s\left(W_{i_k} W_{i_{k-1}} ... W_{i_1}\right) + \frac{1}{2^{n-1}}$ *and if* $j_r \neq \frac{1}{2^n}$, *then* $j_r \geq \frac{1}{2^{n-1}}$. *Also if* $\frac{1}{2^n} \leq i_k$, *then* $s\left(W_{j_r} W_{j_{r-1}} ... W_{j_1}\right) - s\left(W_{i_k} W_{i_{k-1}} ... W_{i_1}\right) \leq \frac{1}{2^n}$.

PROOF. Case 1: $\frac{1}{2^n} < i_k$. Obviously, $\frac{1}{2^n} < i_k < i_{k-1} < i_{k-2} < ... < i_1 < 1$ and $s\left(W_{1/2^n} W_{i_k} W_{i_{k-1}} ... W_{i_1}\right) = s\left(W_{i_k} W_{i_{k-1}} ... W_{i_1}\right) + \frac{1}{2^n}$.

Case 2: There is $m \in \{2, 3, ..., k\}$ such that $i_m = \frac{1}{2^n} < \frac{i_{m-1}}{2}$. Then

$$W_{1/2^n} W_{i_k} W_{i_{k-1}} ... W_{i_m} W_{i_{m-1}} ... W_{i_1} \subset W_{1/2^n} W_{2i_m} W_{i_{m-1}} ... W_{i_1}.$$

and we have

$$s\left(W_{1/2^n} W_{2i_m} W_{i_{m-1}} ... W_{i_1}\right) =$$

$$\begin{aligned} &= s\left(W_{i_k} ... W_{i_m} W_{i_{m-1}} ... W_{i_1}\right) - (i_k + ... + i_m) + 2i_m + \frac{1}{2^n} \\ &\leq s\left(W_{i_k} ... W_{i_m} W_{i_{m-1}} ... W_{i_1}\right) + i_m + \frac{1}{2^n} \\ &= s\left(W_{i_k} ... W_{i_m} W_{i_{m-1}} ... W_{i_1}\right) + \frac{2}{2^n}. \end{aligned}$$

Moreover $i_k + ... + i_m \leq \left(\frac{1}{2^{k-m}} + \frac{1}{2^{k-m}} + ... \frac{1}{2} + 1\right) i_m < 2i_m < 2i_m + \frac{1}{2^n}$. Consequently, $s\left(W_{1/2^n} W_{2i_m} W_{i_{m-1}} ... W_{i_1}\right) > s\left(W_{i_k} ... W_{i_m} W_{i_{m-1}} ... W_{i_1}\right)$.

Case 3: There is $m \in \{2, 3, ..., k\}$ such that $i_m = \frac{1}{2^n} = \frac{i_{m-1}}{2}$ and there is $q \in \{2, 3, ..., m-1\}$ such that $4i_q \leq i_{q-1}$. Let $p$ be the greatest element of the set

$$\{q : \ 2 \leq q \leq m-1, \ 4i_q \leq i_{q-1}\}.$$

Then $W_{1/2^n} W_{i_k} ... W_{i_1} \subset W_{1/2^n} W_{2i_m} W_{i_{m-1}} ... W_{i_1} \subset W_{1/2^n} W_{2i_p} W_{i_{p-1}} ... W_{i_1}$. Moreover

$$s\left(W_{2i_p} W_{i_{p-1}} ... W_{i_1}\right) = s\left(W_{i_{m-1}} ... W_{i_p} W_{i_{p-1}} ... W_{i_1}\right) - (i_{m-1} + .. + i_p) + 2i_p$$

$$\begin{aligned}
&= s\left(W_{i_{m-1}}...W_{i_p}W_{i_{p-1}}...W_{i_1}\right) - \left(i_{m-1} + 2i_{m-1} + ... + 2^{m-p-1}i_{m-1}\right) + 2^{m-p}i_{m-1} \\
&= s\left(W_{i_{m-1}}...W_{i_p}W_{i_{p-1}}...W_{i_1}\right) - i_{m-1}\left(2^{m-p} - 1\right) + i_{m-1}2^{m-p} \\
&= s\left(W_{i_k}...W_{i_p}W_{i_{p-1}}...W_{i_1}\right) - \left(i_k + ... + i_m\right) + i_{m-1},
\end{aligned}$$

and since $\frac{1}{2^{n-1}} = i_{m-1}$, it follows that

$$s\left(W_{1/2^n}W_{2i_p}W_{i_{p-1}}...W_{i_1}\right) = s\left(W_{2i_p}W_{i_{p-1}}...W_{i_1}\right) + \frac{1}{2^n}$$

$$\begin{aligned}
&= s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) - \left(i_k + ... + i_m\right) + i_{m-1} + \frac{1}{2^n} \\
&= s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) - \left(i_k + ... + i_{m+1}\right) - \frac{1}{2^n} + \frac{1}{2^{n-1}} + \frac{1}{2^n} \\
&= s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) - \left(i_k + ... + i_{m+1}\right) + \frac{1}{2^{n-1}}.
\end{aligned}$$

On the other hand, $i_k + ... + i_m \leq \left(\frac{1}{2^{k-m+1}} + \frac{1}{2^{k-m}} + ...\frac{1}{2}\right) i_{m-1} < i_{m-1}$ and therefore $s\left(W_{1/2^n}W_{2i_p}W_{i_{p-1}}...W_{i_1}\right) > s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right)$.

Case 4: There is $m \in \{2, 3, ..., k\}$ such that $i_m = \frac{1}{2^n} = \frac{i_{m-1}}{2} = \frac{i_{m-2}}{2^2} = ... = \frac{i_1}{2^{m-1}}$. Then $W_{1/2^n}W_{i_k}...W_{i_1} \subset W_{1/2^n}W_{2i_m}W_{i_{m-1}}...W_{i_1} \subset W_{1/2^n}W_{2i_1}$ and

$$\begin{aligned}
s\left(W_{1/2^n}W_{2i_1}\right) &= s\left(W_{i_m}...W_{i_1}\right) - \left(i_m + .. + i_1\right) + 2i_1 + \frac{1}{2^n} \\
&= s\left(W_{i_m}...W_{i_1}\right) - \left(i_m + 2i_m + ... + 2^{m-1}i_m\right) + 2^m i_m + \frac{1}{2^n} \\
&= s\left(W_{i_m}...W_{i_1}\right) - i_m\left(2^m - 1\right) + i_m 2^m + \frac{1}{2^n} \\
&= s\left(W_{i_k}...W_{i_{m+1}}W_{i_m}...W_{i_1}\right) - \left(i_k + ... + i_{m-1}\right) + i_m + \frac{1}{2^n} \\
&< s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^{n-1}}.
\end{aligned}$$

Also $s\left(W_{1/2^n}W_{2i_1}\right) = \frac{1}{2^n} + 2i_1 > 2i_1 > s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right)$ and

$$j_1 = 2i_1 < s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^{n-1}}.$$

Case 5: There is $m \in \{2, 3, ..., k\}$ such that $i_m < \frac{1}{2^n} < \frac{i_{m-1}}{2}$. Then

$$\begin{aligned}
W_{1/2^n}W_{i_k}W_{i_{k-1}}...W_{i_m}W_{i_{m-1}}...W_{i_1} &\subset W_{1/2^n}W_{2i_m}W_{i_{m-1}}...W_{i_1} \\
&\subset W_{1/2^n}W_{1/2^n}W_{i_{m-1}}...W_{i_1} \\
&\subset W_{1/2^{n-1}}W_{i_{m-1}}...W_{i_1}.
\end{aligned}$$

and

$$\begin{aligned}
s\left(W_{1/2^{n-1}}W_{i_{m-1}}...W_{i_1}\right) &= s\left(W_{i_k}...W_{i_m}W_{i_{m-1}}...W_{i_1}\right) - \left(i_k + ... + i_m\right) + \frac{1}{2^{n-1}} \\
&< s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^{n-1}}.
\end{aligned}$$

Moreover $i_k + ... + i_m \leq \left(\frac{1}{2^{k-m}} + \frac{1}{2^{k-m}} + ...\frac{1}{2} + 1\right) i_m < 2i_m \leq \frac{1}{2^n} < \frac{1}{2^{n-1}}$. Consequently, $s\left(W_{1/2^{n-1}} W_{i_{m-1}} ... W_{i_1}\right) > s\left(W_{i_k} ... W_{i_m} W_{i_{m-1}} ... W_{i_1}\right)$.

Case 6: There is $m \in \{2, 3, ..., k\}$ such that $i_m < \frac{1}{2^n} = \frac{i_{m-1}}{2}$ and there is $q \in \{2, 3, ..., m-1\}$ such that $4i_q \leq i_{q-1}$. If $p$ is the greatest element of the set

$$\{q : \ 2 \leq q \leq m-1, \ 4i_q \leq i_{q-1}\},$$

then

$$\begin{aligned} W_{1/2^n} W_{i_k} W_{i_{k-1}} ... W_{i_m} W_{i_{m-1}} ... W_{i_1} &\subset W_{1/2^n} W_{2i_m} W_{i_{m-1}} ... W_{i_1} \\ &\subset W_{1/2^n} W_{1/2^n} W_{i_{m-1}} ... W_{i_1} \\ &\subset W_{1/2^{n-1}} W_{i_{m-1}} ... W_{i_1} \\ &\subset W_{2i_p} W_{i_{p-1}} ... W_{i_1}. \end{aligned}$$

Moreover

$$s\left(W_{2i_p} W_{i_{p-1}} ... W_{i_1}\right) = s\left(W_{i_{m-1}} ... W_{i_p} W_{i_{p-1}} ... W_{i_1}\right) - (i_{m-1} + .. + i_p) + 2i_p$$

$$\begin{aligned} &= s\left(W_{i_{m-1}} ... W_{i_p} W_{i_{p-1}} ... W_{i_1}\right) - (i_{m-1} + 2i_{m-1} + ... + 2^{m-p-1} i_{m-1}) + 2^{m-p} i_{m-1} \\ &= s\left(W_{i_{m-1}} ... W_{i_p} W_{i_{p-1}} ... W_{i_1}\right) - i_{m-1}\left(2^{m-p} - 1\right) + i_{m-1} 2^{m-p} \\ &= s\left(W_{i_k} ... W_{i_m} W_{i_{m-1}} ... W_{i_1}\right) - (i_k + ... + i_m) + i_{m-1} \\ &= s\left(W_{i_k} W_{i_{k-1}} ... W_{i_1}\right) - (i_k + ... + i_m) + \frac{1}{2^{n-1}}. \end{aligned}$$

Hence

$$s\left(W_{2i_p} W_{i_{p-1}} ... W_{i_1}\right) < s\left(W_{i_k} W_{i_{k-1}} ... W_{i_1}\right) + \frac{1}{2^{n-1}}.$$

Since we have $i_k + ... + i_m \leq \left(\frac{1}{2^{k-m+1}} + \frac{1}{2^{k-m}} + ...\frac{1}{2}\right) i_{m-1} < i_{m-1}$, it follows that $i_{m-1} - (i_k + ... + i_m) > 0$. Thus $s\left(W_{2i_p} W_{i_{p-1}} ... W_{i_1}\right) > s\left(W_{i_k} ... W_{i_m} W_{i_{m-1}} ... W_{i_1}\right)$. We also have $y_r = 2i_p \geq \frac{1}{2^{n-1}}$.

Case 7: There is $m \in \{2, 3, ..., k\}$ such that $i_m < \frac{1}{2^n} = \frac{i_{m-1}}{2} = \frac{i_{m-2}}{2^2} = ... = \frac{i_1}{2^{m-1}}$. Then $W_{1/2^n} W_{i_k} ... W_{i_1} \subset W_{1/2^n} W_{2i_m} W_{i_{m-1}} ... W_{i_1} \subset W_{1/2^{n-1}} W_{i_{m-1}} ... W_{i_1} W_{1/2^n} W_{2i_1} \subset ... W_{2i_1}$ and

$$s\left(W_{2i_1}\right) = s\left(W_{i_{m-1}} ... W_{i_1}\right) - (i_{m-1} + .. + i_1) + 2i_1$$

$$\begin{aligned} &= s\left(W_{i_{m-1}} ... W_{i_1}\right) - (i_{m-1} + 2i_{m-1} + ... + 2^{m-2} i_{m-1}) + 2^{m-1} i_{m-1} \\ &= s\left(W_{i_{m-1}} ... W_{i_1}\right) - i_{m-1}\left(2^{m-1} - 1\right) + i_{m-1} 2^{m-1} + \frac{1}{2^n} \\ &= s\left(W_{i_k} ... W_{i_m} W_{i_{m-1}} ... W_{i_1}\right) - (i_k + ... + i_m) + i_{m-1} \\ &< s\left(W_{i_k} W_{i_{k-1}} ... W_{i_1}\right) + \frac{1}{2^{n-1}}. \end{aligned}$$

Also $s\left(W_{2i_1}\right) = 2i_1 > \left(\frac{1}{2^{k-1}} + ... + \frac{1}{2} + 1\right) i_1 \geq i_k + i_{k-1} + ...i_1 \geq s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right)$. Moreover $j_1 = 2i_1 = s\left(W_{2i_1}\right) < s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^{n-1}}$.

Case 8: $\frac{1}{2^n} = i_1$. We have

$$W_{1/2^n}W_{i_k}W_{i_{k-1}}...W_{i_1} \subset W_{1/2^n}W_{2i_1},$$

$$\begin{aligned} s\left(W_{1/2^n}W_{2i_1}\right) &= \frac{1}{2^n} + 2i_1 \leq s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^n} + i_1 \\ &= s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^{n-1}}. \end{aligned}$$

and $\frac{1}{2^n} + 2i_1 > 2i_1 > \left(\frac{1}{2^{k-1}} + ... + \frac{1}{2} + 1\right) i_1 \geq i_k + i_{k-1} + ...i_1 = s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right)$. We also have $j_1 = 2i_1 < s\left(W_{1/2^n}W_{2i_1}\right) \leq s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^{n-1}}$.

Case 9: $\frac{1}{2^n} > i_1$. We have

$$W_{1/2^n}W_{i_k}W_{i_{k-1}}...W_{i_1} \subset W_{1/2^n}W_{2i_1} \subset W_{1/2^n}W_{1/2^n} \subset W_{1/2^{n-1}},$$

$$s\left(W_{1/2^{n-1}}\right) = \frac{1}{2^{n-1}} < s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^{n-1}}$$

and $\frac{1}{2^{n-1}} > 2i_1 > \left(\frac{1}{2^{k-1}} + ... + \frac{1}{2} + 1\right) i_1 \geq i_k + i_{k-1} + ...i_1 = s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right)$. Moreover $j_1 = \frac{1}{2^{n-1}} < s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^{n-1}}$.

Let us also remark that if $\frac{1}{2^n} = i_k$ , then $W_{1/2^n}W_{i_k}W_{i_{k-1}}...W_{i_1} \subset W_{2i_m}W_{i_{m-1}}...W_{i_1}$, where $m$ is the greatest element of the set $\{q : 2 \leq q \leq k, 4i_q \leq i_{q-1}\}$ if the set is not empty or $m = 1$, otherwise. We have

$$\begin{aligned} s\left(W_{2i_m}W_{i_{m-1}}...W_{i_1}\right) &= s\left(W_{i_k}...W_{i_m}W_{i_{m-1}}...W_{i_1}\right) - (i_k + ... + i_m) + 2i_m \\ &= s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) - \left(1 + 2 + ... + 2^{k-m}\right) \frac{1}{2^n} + \frac{2^{k-m+1}}{2^n} \\ &= s\left(W_{i_k}W_{i_{k-1}}...W_{i_1}\right) + \frac{1}{2^n}. \end{aligned}$$

Moreover $i_k + ... + i_m \leq \left(\frac{1}{2^{k-m}} + \frac{1}{2^{k-m}} + ...\frac{1}{2} + 1\right) i_m < 2i_m$. Consequently,

$$s\left(W_{2i_m}W_{i_{m-1}}...W_{i_1}\right) > s\left(W_{i_k}...W_{i_m}W_{i_{m-1}}...W_{i_1}\right).$$

■

2.4. Remark. In the preceding lemma since $W_{1/2^n}W_{i_k}W_{i_{k-1}}...W_{i_1} \subset W_{j_r}W_{j_{r-1}}...W_{j_1}$, it follows that $\left(W_{1/2^n}W_{i_k}W_{i_{k-1}}...W_{i_1}\right)^{-1} \subset \left(W_{j_r}W_{j_{r-1}}...W_{j_1}\right)^{-1}$ and consequently,

$$W_{i_1}W_{i_2}...W_{i_k}W_{1/2^n} \subset W_{j_1}W_{j_2}...W_{j_r}.$$

2.5. Theorem. *Let $G$ be a groupoid, $\mathcal{W}$ be a $G$-uniformity (in the sense of Definition 2.1) and let $W \in \mathcal{W}$. Let us consider an $I = \left\{\frac{1}{2^n},\ n \in \mathbb{N}\right\}$-indexed subfamily $\mathcal{W}_I = \{W_i\}_{i\in I}$ of $\mathcal{W}$ as in Lemma 2.3 such that $W_1 \subset W$. Then for every subset $A$ of $G$ there is a function $f = f_{A,\mathcal{W}_I} : G \to [0,\ 1]$ satisfying the following conditions:*

1. *If $n \in \mathbb{N}$, $n \geq 2$, $x \in G$ and $y \in W_{1/2^n} x W_{1/2^n}$, then $|f(x) - f(y)| < \frac{1}{2^{n-2}}$.*
2. *$f(x) = 0$ for all $x \in A$.*
3. *$f(x) = 1$ for all $x \notin WAW$.*
4. *If $A = A^{-1}$, then $f(x) = f(x^{-1})$ for all $x \in G$.*
5. *If $G$ is endowed with a topology such that $W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}$ is open for all $i_1,\ i_2,\ ...,\ i_k \in I,\ i_k < i_{k-1} < ... < i_1 < 1$, then $f$ is upper semi-continuous.*
6. *For all $n \in \mathbb{N}$, $n \geq 2$, we have*
$$W_{1/2^{n+1}} A W_{1/2^{n+1}} \subset \left\{x : f(x) < \frac{1}{2^n}\right\} \subset W_{1/2^{n-1}} A W_{1/2^{n-1}}.$$
*In particular, if $A = G^{(0)}$, then*
$$W_{1/2^{n+1}} W_{1/2^{n+1}} \subset \left\{x : f(x) < \frac{1}{2^n}\right\} \subset W_{1/2^{n-1}} W_{1/2^{n-1}} \subset W_{1/2^{n-2}}$$
*for all $n \in \mathbb{N}$, $n \geq 2$.*
7. *If $A = G^{(0)}$, then $f(xy) \leq 3f(x) + f(y)$ for all $(x, y) \in G^{(2)}$.*
8. *If $A = G^{(0)}$, then $f(xy) \leq 2(f(x) + f(y))$ for all $(x, y) \in G^{(2)}$.*
9. *If $A = G^{(0)}$, then $f(x_1 x_2 ... x_n) \leq 3(f(x_1) + f(x_2) + ... + f(x_n))$ for all $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n \in G$ such that $d(x_i) = r(x_{i+1})$ for all $i \in \{1, 2, ..., n-1\}$.*
10. *If $A = G^{(0)}$ and for every $x \in G \setminus G^{(0)}$ there is $i_x \in I$ such that $x \notin W_{i_x}$ (or equivalently, $\bigcap\limits_n W_{1/2^n} = G^{(0)}$), then $f^{-1}(\{0\}) = G^{(0)}$.*

Proof. For each $x \in G$, let us define
$$i(x) = \inf \left\{ \begin{array}{r} i_k + i_{k-1} + ... + i_1 :\ i_1,\ i_2,\ ...,\ i_k \in I,\ i_k < i_{k-1} < ... < i_1, \\ x \in W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k} \end{array} \right\}$$
(with convention $\inf \emptyset = \infty$) and
$$f(x) = \min\{i(x), 1\}.$$

1. Let $x \in G$ and $y \in W_{1/2^n} x W_{1/2^n}$. If $i(x) \geq 1$ and $i(y) \geq 1$, then $f(x) = f(y) = 1$. Let us suppose that $i(x) < 1$ or $i(y) < 1$.

Case 1: $i(x) < 1$. Then there are $i_1, i_2, ..., i_k \in I$, $i_k < i_{k-1} < ... < i_1 < 1$ such that $x \in W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}$ and $i_k + i_{k-1} + ... + i_1 < i(x) + \frac{1}{2^n}$. By Lemma 2.3, there are $j_1, j_2, ..., j_r \in I$, $j_r < j_{r-1} < i_{r-2} < ... < j_1 \leq 1$ such that $W_{1/2^n} W_{i_k} W_{i_{k-1}} ... W_{i_1} \subset W_{j_r} W_{j_{r-1}} ... W_{j_1}$ and

$$0 < (j_r + ... + j_1) - (i_k + i_{k-1} + ... + i_1) < \frac{3}{2^n}$$

Hence

$$i_k + i_{k-1} + ... + i_1 \leq j_r + ... + j_1 < i(x) + \frac{1}{2^{n-2}}$$

and since

$$\begin{aligned} y &\in W_{1/2^n} x W_{1/2^n} \subset \\ &\subset W_{1/2^n} W_{i_k} W_{i_{k-1}} ... W_{i_1} A W_{i_1} ... W_{i_{k-1}} W_{i_k} W_{1/2^n} \\ &\subset W_{j_r} W_{j_{r-1}} ... W_{j_1} A W_{j_1} W_{j_2} ... W_{j_r} \end{aligned}$$

it follows that $i(y) < i(x) + \frac{1}{2^{n-2}}$. If $i(y) < 1$, then since $y \in W_{1/2^n} x W_{1/2^n}$ is equivalently to $x \in W_{1/2^n} y W_{1/2^n}$ it follows that $i(x) < i(y) + \frac{1}{2^{n-2}}$. Therefore $|f(x) - f(y)| = |i(x) - i(y)| < \frac{1}{2^{n-2}}$. If $i(y) \geq 1$, then $|f(x) - f(y)| = |i(x) - 1| = 1 - i(x) \leq i(y) - i(x) < \frac{1}{2^{n-2}}$.

Case 2: $i(y) < 1$. Since $y \in W_{1/2^n} x W_{1/2^n}$ is equivalently to $x \in W_{1/2^n} y W_{1/2^n}$, the case $i(y) < 1$ can be treated similarly as the case $i(x) < 1$.

2. Let us prove that $f(x) = 0$ for all $x \in A$. Since $A \subset W_{1/2^n} A W_{1/2^n}$ for all $n$, it follows that $i(x) = 0$, and consequently, $f(x) = 0$ for all $x \in A$.

3. Let us prove that $f(x) = 1$ for all $x \notin WAW$. Let $x \notin WAW$. By contradiction, let us suppose $f(x) < 1$. We necessarily have $i(x) < 1$, and hence there are $i_1, i_2, ..., i_k \in I$, $i_k < i_{k-1} < ... < i_1 < 1$ such that

$$\begin{aligned} x &\in W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k} \subset W_{2i_1} A W_{2i_1} \\ &\subset W_1 A W_1 \subset WAW \end{aligned}$$

This is in contradiction to the hypothesis $x \notin WAW$.

4. Since $A = A^{-1}$, it follows that

$$\left(W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}\right)^{-1} = W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}.$$

Thus $x \in W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}$ if and only if

$$x^{-1} \in W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}.$$

Therefore $f(x) = f(x^{-1})$ for all $x \in G$.

5. Let $\alpha \in \mathbb{R}$ and let us consider the set

$$U_\alpha = \{x \in G : f(x) < \alpha\}.$$

If $\alpha > 1$, then $U_\alpha = G$, hence $U_\alpha$ is an open set. Let us consider $\alpha \leq 1$ and let $x \in U_\alpha$. Then $f(x) < 1$. Thus $i(x) < 1$, and hence there are $i_1,\ i_2,\ ...,\ i_k \in I$, $i_k < i_{k-1} < ... < i_1 < 1$ such that

$$x \in W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}$$
$$i_k + i_{k-1} + ... + i_1 < \alpha.$$

For all $y \in W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}$ we have $i(y) < \alpha$. Consequently,

$$x \in W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k} \subset U_\alpha.$$

Therefore $U_\alpha$ is open.

6. If $x \in W_{1/2^{n+1}} A W_{1/2^{n+1}}$, then $i(x) \leq \frac{1}{2^{n+1}}$. Thus $f(x) \leq \frac{1}{2^{n+1}} < \frac{1}{2^n}$. If $f(x) < \frac{1}{2^n} < 1$, then $i(x) < \frac{1}{2^n}$ and there are $i_1,\ i_2,\ ...,\ i_k \in I$, $i_k < i_{k-1} < ... < i_1 < 1$ such that $x \in W_{i_k} W_{i_{k-1}} ... W_{i_1} A\ W_{i_1} ... W_{i_{k-1}} W_{i_k}$ and $i_k + i_{k-1} + ... + i_1 < i(x) + \frac{1}{2^n} < \frac{1}{2^{n-1}}$. Hence $i_1 < \frac{1}{2^{n-1}}$ and therefore $x \in W_{2i_1} A W_{2i_1} \subset W_{1/2^{n-1}} A W_{1/2^{n-1}}$.

7. Let $(x, y) \in G^{(2)}$. If $3f(x) + f(y) \geq 1$, then obviously, $f(xy) \leq 3f(x) + f(y)$. Let us suppose that $3f(x) + f(y) < 1$ or equivalently, $3i(x) + i(y) < 1$ (consequently, $i(x) < \frac{1}{3}$ and $i(y) < 1$). Let $\varepsilon > 0$ such that $\varepsilon < 1 - 3i(x) - i(y)$. Then there are $i_1,\ i_2,\ ...,\ i_k \in I$, $i_k < i_{k-1} < ... < i_1 \leq \frac{1}{4}$ such that $x \in W_{i_k} W_{i_{k-1}} ... W_{i_1} W_{i_1} ... W_{i_{k-1}} W_{i_k}$, $i_k + i_{k-1} + ... + i_1 < i(x) + \frac{\varepsilon}{3}$ and there are $j_1,\ j_2,\ ...,\ j_m \in I$, $j_m < j_{m-1} < ... < j_1 \leq \frac{1}{2}$ such that $y \in W_{j_m} W_{j_{m-1}} ... W_{j_1} W_{j_1} ... W_{j_{m-1}} W_{j_m}$, $j_m + j_{m-1} + ... + j_1 < i(y) + \frac{\varepsilon}{3}$. By Lemma 2.3, there are $q_1^1,\ q_2^1,\ ...,\ q_{r_1}^1 \in I$, $q_{r_1}^1 < q_{r_1-1}^1 < q_{r_1-2}^1 < ... < q_1^1 \leq 1$ such that $W_{i_k} W_{j_m} W_{i_{m-1}} ... W_{j_1} \subset W_{q_{r_1}^1} W_{q_{r_1-1}^1} ... W_{q_1^1}$,

$$0 < \left(q_{r_1}^1 + ... + q_1^1\right) - (j_m + j_{m-1} + ... + j_1) \leq 2i_k.$$

and $q_1^1 \leq j_m + j_{m-1} + ... + j_1 + 2i_k < i(y) + \frac{\varepsilon}{3} + 2i(x) + \frac{2\varepsilon}{3} < 1$. Repeatedly applying Lemma 2.3, for $p = 2, 3, ..., k$ there are $q_1^p,\ q_2^p,\ ...,\ q_{rp}^p \in I$, $q_{r_p}^p < q_{r_p-1}^p < q_{r_p-2}^p < ... < q_1^p \leq 1$ such that $W_{i_{k-p+1}} W_{q_{r_{p=1}}^{p=1}} W_{q_{r_{p-1}-1}^{p-1}} ... W_{q_1^{p-1}} \subset W_{q_{r_p}^p} W_{q_{r_p-1}^p} ... W_{q_1^p}$ ,

$$0 \leq \left(q_{r_p}^p + ... + q_1^p\right) - \left(q_{r_{p-1}}^{p-1} + ... + q_1^{p-1}\right) \leq 2i_{k-p+1}.$$

and

$$\begin{aligned} q_1^{p-1} &< q_{r_{p-1}}^{p-1} + ... + q_1^{p-1} + 2i_{k-p+1} \\ &< q_{r_{p-2}}^{p-2} + ... + q_1^{p-2} + 2i_{k-p} + 2i_{k-p+1} \\ &\quad ........ \\ &< j_m + j_{m-1} + ... + j_1 + 2i_k + ... + 2i_{k-p} + 2i_{k-p+1} \\ &< i(y) + \frac{\varepsilon}{3} + 2i(x) + \frac{2\varepsilon}{3} < 1. \end{aligned}$$

Applying again Lemma 2.3, there are $q_1^{k+1}$, $q_2^{k+1}$, ..., $q_{r_{k+1}}^{k+1} \in I$, $q_{r_{k+1}}^{k+1} < q_{r_{k+1}-1}^{k+1} < q_{r_{k+1}-2}^{k+1} < ... < q_1^{k+1} \leq 1$ such that $W_{i_1} W_{q_{r_k}^k} W_{q_{r_k-1}^k} ... W_{q_1^k} \subset W_{q_{rk+1}^{k+1}} W_{q_{r_{k+1}-1}^{k+1}} ... W_{q_1^{k+1}}$ and

$$0 < \left(q_{r_{k+1}}^{k+1} + ... + q_1^{k+1}\right) - \left(q_{r_k}^k + ... + q_1^k\right) < i_1$$

Moreover $q_{r_{k+1}}^{k+1} \geq i_1$. Hence $W_{i_1} W_{i_1} W_{i_2} ... W_{i_k} W_{j_m} W_{i_{m-1}} ... W_{j_1} \subset W_{q_{rk+1}^{k+1}} W_{q_{r_{k+1}-1}^{k+1}} ... W_{q_1^{k+1}}$ and

$$\begin{aligned} 0 &< \left(q_{r_{k+1}}^{k+1} + ... + q_1^{k+1}\right) - (j_m + j_{m-1} + ... + j_1) \\ &< 2(i_k + i_{k-1} + ... + i_1) + i_1. \end{aligned}$$

Thus $W_{i_k} ... W_{i_2} W_{i_1} W_{i_1} W_{i_2} ... W_{i_k} W_{j_m} W_{i_{m-1}} ... W_{j_1} \subset W_{i_k} ... W_{i_2} W_{q_{rk+1}^{k+1}} W_{q_{r_{k+1}-1}^{k+1}} ... W_{q_1^{k+1}}$. Consequently,

$$\begin{aligned} xy &\in W_{i_k} ... W_{i_2} W_{i_1} W_{i_1} W_{i_2} ... W_{i_k} W_{j_m} W_{j_{m-1}} ... W_{j_1} W_{j_1} ... W_{j_m} \\ &\subset W_{i_k} ... W_{i_2} W_{q_{rk+1}^{k+1}} W_{q_{r_{k+1}-1}^{k+1}} ... W_{q_1^{k+1}} W_{q_1^{k+1}} ... W_{q_{rk+1}^{k+1}} W_{i_2} ... W_{i_k} \end{aligned}$$

and $i_k < i_{k-1} < ... < i_2 < q_{r_{k+1}}^{k+1} < q_{r_{k+1}-1}^{k+1} < q_{r_{k+1}-2}^{k+1} < ... < q_1^{k+1} \leq 1$. Hence

$$i(xy) \leq i_k + i_{k-1} + ... + i_2 + \left(q_{r_{k+1}}^{k+1} + ... + q_1^{k+1}\right)$$

$$\begin{aligned} &< i_k + i_{k-1} + ... + i_2 + 2(i_k + i_{k-1} + ... + i_1) + i_1 + (j_m + j_{m-1} + ... + j_1) \\ &\leq 3(i_k + i_{k-1} + ... + i_1) + (j_m + j_{m-1} + ... + j_1) \\ &< 3i(x) + i(y) + \frac{4}{3}\varepsilon \end{aligned}$$

for all $\varepsilon > 0$. Therefore $i(xy) \leq 3i(x) + i(y)$ for all $(x, y) \in G^{(2)}$. Thus $f(xy) \leq 3f(x) + f(y)$ for all $(x, y) \in G^{(2)}$.

8. Let $(x, y) \in G^{(2)}$. We proved in 7 that

$$f(xy) \leq 3f(x) + f(y).$$

On the other hand we have $f(x^{-1}) = f(x)$, $f(y^{-1}) = f(y)$ and

$$f(xy) = f\left(y^{-1}x^{-1}\right) \underset{7}{\leq} 3f\left(y^{-1}\right) + f\left(x^{-1}\right) = 3f(y) + f(x).$$

Adding the last inequalities we obtain

$$2f(xy) \leq 3(f(x) + f(y)) + f(x) + f(y) = 4(f(x) + f(y)).$$

9. We prove the inequality by mathematical induction. For $n = 2$ is true, since by 7 we have $f(x_1x_2) \leq 3f(x_1) + f(x_2) \leq 3f(x_1) + 3f(x_2)$. Let us suppose that the inequality is true for some $n$ and let us prove that it is true for $n + 1$. Using 7 we obtain

$$\begin{aligned} & f(x_1x_2...x_nx_{n+1}) \underset{7}{\leq} 3f(x_1) + f(x_2...x_nx_{n+1}) \\ \leq \; & 3f(x_1) + 3(f(x_2) + f(x_3) + ... + f(x_n)). \end{aligned}$$

10. If $x \in G^{(0)}$ then by 2, $f(x) = 0$. Conversely, if $f(x) = 0$, then for all $n$, we have $i(x) < \frac{1}{2^n}$. Thus there are $i_1,\ i_2,\ ...,\ i_k \in I$, $i_k < i_{k-1} < ... < i_1 \leq \frac{1}{2^n}$ such that $x \in W_{i_k}W_{i_{k-1}}...W_{i_1}W_{i_1}...W_{i_{k-1}}W_{i_k}$ and $i_k + i_{k-1} + ... + i_1 < \frac{1}{2^n}$. Since $W_{i_k}W_{i_{k-1}}...W_{i_1} \subset W_{2i_1}$, it follows that $x \in W_{4i_1} \subset W_{1/2^{n-2}}$ for all $n \geq 2$. Thus $x \in \bigcap_n W_{1/2^n} = G^{(0)}$. ■

2.6. Proposition. *Let $G$ be a groupoid, $\mathcal{W}$ be a $G$-uniformity and $f : G \to [0,\ 1]$ be a function satisfying conditions 2, 4, 9 and 10 in Theorem 2.5 ($f$ associated to $A = G^{(0)}$). Then there is a function $f_{reg} : G \to [0,\ 1]$ satisfying the following conditions:*

1. $\frac{1}{3}f \leq f_{reg} \leq f$.
2. $f_{reg}(x) = f_{reg}(x^{-1})$ *for all* $x \in G$.
3. $f_{reg}(xy) \leq f_{reg}(x) + f_{reg}(y)$ *for all* $(x, y) \in G^{(2)}$.
4. $|f_{reg}(sxt) - f_{reg}(x)| \leq f_{reg}(s) + f_{reg}(t)$ *for all* $s, t, x \in G$ *with* $x \in G^{d(s)}_{r(t)}$.
5. $W_{1/2^{n+1}} \subset W_{1/2^{n+1}}W_{1/2^{n+1}} \subset \left\{x : f_{reg}(x) < \frac{1}{2^n}\right\} \subset W_{1/2^{n=3}}W_{1/2^{n-3}} \subset W_{1/2^{n-4}}$ *for all* $n \in \mathbb{N}$, $n \geq 2$.

Proof. In the spirit of [8, Theorem 3.26] let us define $f_{reg} : G \to [0,\ 1]$ by

$$f_{reg}(x) = \inf\left\{\sum_{i=1}^{n} f(x_i) : x_1x_2...x_n = x\right\} \text{ for all } x \in G.$$

Then $f_{reg}$ obviously satisfies conditions $1 - 3$.

4. Let $s, t, x \in G$ such that $x \in G^{d(s)}_{r(t)}$. Then $f_{reg}(sxt) \leq f_{reg}(s) + f_{reg}(x) + f_{reg}(t)$ and consequently, $f_{reg}(sxt) - f_{reg}(x) \leq f_{reg}(s) + f_{reg}(t)$. On the other hand $f_{reg}(x) = f_{reg}(s^{-1}sxtt^{-1}) \leq f_{reg}(s^{-1}) + f_{reg}(sxt) + f_{reg}(t^{-1}) = f_{reg}(s) + f_{reg}(sxt) + f_{reg}(t)$ and therefore $f_{reg}(x) - f_{reg}(sxt) \leq f_{reg}(s) + f_{reg}(t)$.

5. Let $x \in W_{1/2^{n+1}}W_{1/2^{n+1}}$. Then $f_{reg}(x) \leq f(x) < \frac{1}{2^n}$. Conversely, let $x$ be such that $f_{reg}(x) < \frac{1}{2^n}$. Then $\frac{1}{4}f(x) \leq \frac{1}{3}f(x) \leq f_{reg}(x) < \frac{1}{2^n}$. Hence $f(x) < \frac{1}{2^{n-2}}$ and therefore $x \in W_{1/2^{n-3}}W_{1/2^{n-3}}$. ■

## 3. A groupoid generalization of Alexandroff–Urysohn Theorem

As we remark in [5, p. 57], if $G$ is a topological groupoid whose unit space is a $T_1$-space (the points are closed in $G^{(0)}$), then the topologies of the $r$-fibres, as well as the topologies of the $d$-fibres, are determined by a fundamental system of neighborhoods $\{W\}_{W\in\mathcal{W}}$ of $G^{(0)}$. More precisely, for each $u \in G^0$ and each $x \in G^u$ (respectively, $x \in G_u$), $\{xW\}_{W\in\mathcal{W}}$ (respectively, $\{Wx\}_{W\in\mathcal{W}}$ ) is a local basis for $x$ with respect to the topology induced by $G$ on $G^u$ (respectively, $G_u$). We also prove in [5, p. 59] that if $\mathcal{W}$ satisfies the conditions imposed to a $G$-uniformity, then there is a topology denoted $\tau^r_{\mathcal{W}}$ (respectively, $\tau^d_{\mathcal{W}}$) on $G$ such that for all $x \in G$, $\mathcal{V}^r(x)$ (respectively, $\mathcal{V}^d(x)$) is a neighborhood basis for $x$, where

$$\mathcal{V}^r(x) = \{V \subset G : \text{ there is } W \in \mathcal{W} \text{ such that } xW \subset V\}.$$

respectively,

$$\mathcal{V}^d(x) = \{V \subset G : \text{ there is } W \in \mathcal{W} \text{ such that } Wx \subset V\}.$$

Unlike the case of a group, a groupoid $G$ (that isn't a group) is generally not a topological groupoid with respect to $\tau^r_{\mathcal{W}}$ or $\tau^d_{\mathcal{W}}$. That is why we define a new topology associated to a $G$-uniformity.

3.1. Definition. *Let $G$ be a groupoid endowed with a $G$-uniformity $\mathcal{W}$. The topology $\tau_{\mathcal{W}}$ induced by the $G$-uniformity $\mathcal{W}$ is the topology on $G$ defined in the following way: $A \in \tau_{\mathcal{W}}$ if and only if for every $x \in A$ there is $W_x \in \mathcal{W}$ such that $W_x x W_x \subset A$.*

For each $x \in G$ let us write

$$\mathcal{V}(x) = \{V \subset G : \text{ there is } W \in \mathcal{W} \text{ such that } WxW \subset V\}.$$

In order to see that $\tau_{\mathcal{W}}$ is indeed a topology it is enough to prove that for all $V \in \mathcal{V}(x)$, there is $U \in \mathcal{V}(x)$ such that $V \in \mathcal{V}(y)$ for all $y \in U$. Since $V \in \mathcal{V}(x)$, it follows that there is $W_x \in \mathcal{W}$ such that $W_x x W_x \subset V$ . Let $W'_x \in \mathcal{W}$ such that $W'_x W'_x \subset W_x$. If we take $U = W'_x x W'_x$, then for all $y \in U$ there is $s \in W'_x \cap G^{d(x)}$ and $t \in W'_x \cap G_{r(x)})$ such that $y = txs$ and

$$W'_x y W'_x = W'_x txs W'_x \subset W'_x W'_x x W'_x W'_x \subset W_x x W_x.$$

Alternatively, we can note that $\tau_{\mathcal{W}}$ is the topology on $G$ induced by the following uniform structure $\mathcal{U}_{\mathcal{W}}$ associated with the $G$-uniformity $\mathcal{W}$: $U \in \mathcal{U}_{\mathcal{W}}$ if and only if there is $W \in \mathcal{W}$ such that $\{x\} \times WxW \subset U$ for all $x \in G$.

Let us remark that for two equivalent $G$-uniformities $\mathcal{W}$ and $\mathcal{W}'$ in the sense of Definition 2.2 we have $\tau_{\mathcal{W}} = \tau_{\mathcal{W}'}$.

In [5] we introduced the notions of left uniform continuity on fibres and right uniform continuity on fibres reformulating the definition of left and right uniform continuity [3, Definition 3.1/p. 39] in the setting of a groupoid endowed with a family of subsets satisfying the conditions imposed to a $G$-uniformity. Let us define a new notion of uniform continuity with respect to a $G$-uniformity.

3.2. Definition. *Let $G$ be a groupoid endowed with a $G$-uniformity $\mathcal{W}$, $A \subset G$ and $E$ be a Banach space. The function $h : A \to E$ is said to be uniformly continuous on fibres (with respect to $\mathcal{W}$) if and only if for each $\varepsilon > 0$ there is $W_\epsilon \in \mathcal{W}$ such that:*

$$\|h(x) - h(sxt)\| < \varepsilon \text{ for all } s, t \in W_\varepsilon \text{ and } x \in A \cap G_{r(t)}^{d(s)} \text{ such that } sxt \in A.$$

3.3. Remark. The function $f$ defined in Theorem 2.5 as well as the function $f_{reg}$ in Proposition 2.6 are uniformly continuous on fibres with respect to the corresponding $G$-uniformity.

We will prove (Proposition 3.8) that if there is an appropriate connection between the $G$-uniformity and the topology of $G$, then the restrictions of a uniformly continuous on fibres function to $r$-fibres as well as to $d$-fibres are continuous functions.

3.4. Definition. *Let $G$ be a groupoid endowed with a topology $\tau$. Let $\{W_j\}_{j\in J}$ be a collection of subsets of $G$ such that for all $j \in J$, $G^{(0)} \subset W_j$ and $W_j = W_j^{-1}$. The collection $\{W_j\}_{j\in J}$ is said to be compatible with the topology of the $r$-fibres (respectively, $d$-fibres) if for every $u \in G^{(0)}$ and every open neighborhood $U$ of $u$, there is $j \in J$ such that $W_j \cap G^u \subset U \cap G^u$ and $u$ is in the interior of $W_j \cap G^u$ with respect to the topology on $G^u$ coming from $(G, \tau)$ (respectively, $W_j \cap G_u \subset U \cap G_u$ and $u$ is in the interior of $W_j \cap G_u$ with respect to the topology on $G_u$ coming from $(G, \tau)$).*

*The collection $\{W_j\}_{j\in J}$ is said to be compatible with the topology of the fibres if it is compatible with the topology of the $r$-fibres and $d$-fibres.*

3.5. Remark. If $G$ is groupoid endowed with a topology $\tau$ such that the inverse map is continuous, then a collection $\{W_j\}_{j\in J}$ is compatible with the topology of the $r$-fibres if and only if it is compatible with the topology of the $d$-fibres.

If $G$ is a topological groupoid and $G^{(0)}$ is a $T_1$-space (the points are closed in $G^{(0)}$), then any fundamental system of symmetric neighborhoods of $G^{(0)}$ is compatible with the topology of the fibres. Indeed, let $u \in G^{(0)}$. Since $G^{(0)}$ is a $T_1$-space, $G \setminus G^u$ is open for all $u$. If $U$ is an open subset of $G$ containing $u$, then $U \cup (G \setminus G^u)$ is an open neighborhood of $G^{(0)}$. Thus there is $W \in \mathcal{W}$ such that $W \subset U \cup (G \setminus G^u)$, and $W \cap G^u \subset U \cap G^u$.

If $G$ is a topological groupoid and $\{W_j\}_{j\in J}$ is compatible with the topology of the $r$-fibres (and hence to $d$-fibres), then the topologies of the $r$-fibres and $d$-fibres are determined by $\{W_j\}_{j\in J}$: for each $u \in G^0$ and each $x \in G^u$ (respectively, $x \in G_u$), $\{xW_j\}_{j\in J}$ (respectively, $\{W_jx\}_{j\in J}$ ) is a local basis for $x$ with respect to the topology induced by $G$ on $G^u$ (respectively, $G_u$).

3.6. Proposition. *If $G$ is a groupoid endowed with a topology such that for all $x \in G$ the map $y \mapsto xyx^{-1}$ $\left[: G_{d(x)}^{d(x)} \to G_{r(x)}^{r(x)}\right]$ is continuous at $d(x)$ and if $\mathcal{W}$ is compatible with the topology of the $r$- fibres or $d$-fibres, then for every $W_1 \in \mathcal{W}$ and $x \in G$ there is $W_2 \in \mathcal{W}$ such that $W_2 \cap G_{d(x)}^{d(x)} \subset x^{-1}W_1x$ (or equivalently, $xW_2x^{-1} \subset W_1$).*

Proof. Let $W_1 \in \mathcal{W}$ and $x \in G$. Since $xr(x)x^{-1} \in W_1 \cap G_{r(x)}^{r(x)}$, it follows that there is an open neighborhood $V$ of $d(x)$ such that $xVx^{-1} \subset W_1 \cap G_{r(x)}^{r(x)}$. Let $W_2 \in \mathcal{W}$ such that $W_2 \cap G^{d(x)} \subset V \cap G^{d(x)}$ or $W_2 \cap G_{d(x)} \subset V \cap G_{d(x)}$. Then $xW_2x^{-1} \subset xVx^{-1} \subset W_1$. ∎

A topological groupoid is said to be locally transitive (see [12]) if for all $u \in G^{(0)}$ the maps $r_u$ are open, where $r_u : G_u \to G^{(0)}$ is defined by $r_u(x) = r(x)$ for all $x \in G_u$ and $G_u$ is endowed with the topology coming from $G$. Hence the maps $d_u$ are open, where $d_u : G^u \to G^{(0)}$, $d_u(x) = d(x)$ for all $x \in G^u$ and $G^u$ is endowed with the topology coming from $G$. Topological groups and pair groupoids $X \times X$ ($X$ topological space) are topological locally transitive groupoids. More general any trivial groupoid $X \times G \times X$ ($X$ topological space and $G$ topological group) is locally transitive. Any transitive Polish groupoid with open range map is locally transitive [10] (see [9, p. 8] for transitive locally compact second countable groupoids with open range maps).

3.7. Proposition. *Let $G$ be a groupoid and $\mathcal{W}$ be a $G$-uniformity such that for every $W_1 \in \mathcal{W}$ and $x \in G$ there is $W_2 \in \mathcal{W}$ such that $W_2 \cap G_{d(x)}^{d(x)} \subset x^{-1}W_1x$ (or equivalently, $xW_2x^{-1} \subset W_1$). Then $G$ is a topological locally transitive groupoid with respect to the topology $\tau_{\mathcal{W}}$ induced by the $G$-uniformity $\mathcal{W}$ (in the sense of Definition 3.1). The topologies $\tau_{\mathcal{W}}^r$ and $\tau_{\mathcal{W}}^d$ are finer than $\tau_{\mathcal{W}}$. However the topologies induced by $\tau_{\mathcal{W}}^r$ and $\tau_{\mathcal{W}}$ on $r$-fibres (respectively, by $\tau_{\mathcal{W}}^d$ and $\tau_{\mathcal{W}}$ on $d$-fibres) coincide.*

Proof. Let us show that the inversion map and the product map are continuous with respect to $\tau_{\mathcal{W}}$. The fact that $(WxW)^{-1} = Wx^{-1}W$ ($x \in G$ and $W \in \mathcal{W}$) implies that the inversion is a homeomorphism. For all $W \in \mathcal{W}$, there is $W_1 \in \mathcal{W}$ such that $W_1W_1 \subset W$ and for all $y \in G$ there is $W_y \in \mathcal{W}$ such that $W_y \subset W$ and $W_y \cap G_{r(y)}^{r(y)} \subset yW_1y^{-1}$ or equivalently, $y^{-1}W_yy \subset W_1$. If $W_y' \in \mathcal{W}$ is such that $W_y'W_y' \subset W_y$ and $x \in G_{r(y)}$, then

$$W_y'xW_y'W_y'yW_y' \subset W_y'xyy^{-1}W_yyW_y' \subset W_y'xyW_1W_y' \subset WxyW,$$

Therefore the product map is continuous.

Obviously, the topologies $\tau_{\mathcal{W}}^r$ and $\tau_{\mathcal{W}}^d$ are finer than $\tau_{\mathcal{W}}$ ($xW \subset WxW$ and $Wx \subset WxW$). For every $u \in G^{(0)}$, $x \in G^u$ and $W \in \mathcal{W}$ there is $W_1 \in \mathcal{W}$ such that $W_1W_1 \subset W$ and there is $W_x \in \mathcal{W}$ such that $W_x \subset W_1$ and $W_x \cap G_{r(x)}^{r(x)} \subset xW_1x^{-1}$ or equivalently, $x^{-1}W_xx \subset W_1$. Thus

$$W_xxW_x \cap G^u = xx^{-1}W_xxW_x \subset xW_1W_x \subset xW.$$

Hence the topologies induced by $\tau_{\mathcal{W}}^r$ and $\tau_{\mathcal{W}}$ on $r$-fibres coincide. Similarly, the topologies induced by $\tau_{\mathcal{W}}^d$ and $\tau_{\mathcal{W}}$ on $d$-fibres coincide.

Let $u \in G^{(0)}$. In order to prove that $d_u : G^u \to G^{(0)}$ is open it suffices to note that if $x \in G^u$ and $W \in \mathcal{W}$, then

$$d_u(G^u \cap WxW) = d_u\left(G^{d(x)} \cap W\right) = (Wd(x)W) \cap G^{(0)}.$$

∎

3.8. PROPOSITION. *Let $(G,\tau)$ be a topological groupoid and $\mathcal{W}$ be a $G$-uniformity compatible with the topology of the fibres. Then:*

1. *The topology $\tau_{\mathcal{W}}$ (induced by the $G$-uniformity $\mathcal{W}$) is finer than $\tau$ (the original topology of $G$).*
2. *The topologies induced by $\tau$ and $\tau_{\mathcal{W}}$ on $r$-fibres (respectively, on $d$-fibres) coincide.*
3. *If $(G,\tau)$ is locally transitive, then the topology $\tau_{\mathcal{W}}$ coincides with $\tau$ on $G$.*

PROOF. 1. Let $U$ be an open subset of $G$ with respect to $\tau$ and let $x \in U$. Since $xd(x) \in U$, it follows that there is an open neighborhood $U_1 \in \tau$ of $x$ and an open neighborhood $V_1 \in \tau$ of $d(x)$ such that $U_1 V_1 \subset U$. Moreover since $r(x)x \in U_1$, it follows that there is an open neighborhood $V_2 \in \tau$ of $r(x)$ such that $V_2 x \subset U_1$. Hence $V_2 x V_1 \subset U$. Let $W_1 \in \mathcal{W}$ such that $W_1 \cap G^{d(x)} \subset V_1 \cap G^{d(x)}$, $W_2 \in \mathcal{W}$ such that $W_2 \cap G_{r(x)} \subset V_2 \cap G_{r(x)}$ and let $W \in \mathcal{W}$ such that $W \subset W_1 \cap W_2$. Then $WxW \subset V_2 x V_1 \subset U$. Thus $U$ is open with respect to $\tau_{\mathcal{W}}$.

2. Since $\tau_{\mathcal{W}}$ is finer than $\tau$, it suffices to prove that for all $u \in G^{(0)}$, $x \in G^u$ (respectively, $x \in G_u$) and all $W \in \mathcal{W}$, $WxW \cap G^u$ (respectively, $WxW \cap G_u$) is a neighborhood of $x$ with respect to the topology on $G^u$ (respectively, $G_u$) induced by $\tau$. Since the map $y \mapsto xy \left[: G^{d(x)} \to G^{r(x)}\right]$ (respectively, $y \mapsto yx \left[: G_{r(x)} \to G_{d(x)}\right]$) is a homeomorphism (with respect to $\tau$), it follows that $x\left(W \cap G^{d(x)}\right) = xW$ (respectively, $\left(W \cap G_{r(x)}\right)x = Wx$) is a neighborhood of $x$ in $G^{r(x)}$ (respectively, $G_{d(x)}$) with respect to the topology induced by $\tau$. Therefore $WxW \cap G^u \supset xW$ (respectively, $WxW \cap G_u \supset Wx$) is a neighborhood of $x$ with respect to the topology on $G^u$ (respectively, $G_u$) induced by $\tau$.

3. Let us assume that $(G,\tau)$ is locally transitive, or equivalently, that for all $u \in G^{(0)}$, $d_u : G^u \to G^{(0)}$ $(d_u(x) = d(x))$ is open. Since $\tau_{\mathcal{W}}$ is finer than $\tau$, in order to show that $\tau_{\mathcal{W}} = \tau$ it suffices to prove for all $x \in G$ and all $W \in \mathcal{W}$, $x$ is in the interior of $WxW$ with respect to $\tau$. For each $u \in G^{(0)}$ let $W^u$ be the interior of $G^u \cap W$ seen as a subset of the topological space $G^u$ and let $W_0 = \bigcup_{u \in G^{(0)}} W^u$ and $W_1 = \bigcup_{u \in G^{(0)}} (W^u)^{-1}$. Then $G^{(0)} \subset W_0 \subset W$ and $G^{(0)} \subset W_1 \subset W$. We prove that $W_1 x W_0$ is open with respect to $\tau$. Let $s \in W_1$, $t \in W_0$ and $(y_i)_i$ be a net in $G$ converging to $sxt$ (with respect to $\tau$). Then $d(y_i) \to d(t) = d_{r(t)}(t)$. Since $d_{r(t)} : G^{r(t)} \to G^{(0)}$ is open, we may pass to a subnet and assume that there are $t_i \in G^{r(t)}$ such that $t_i \to t$ and $d(t_i) = d(y_i)$ for all $i$. If $s_i = y_i t_i^{-1} x^{-1}$, then $s_i \to sxtt^{-1}x^{-1} = s$. Since $t_i \to t \in W_0 \cap G^{r(t)}$ and $s_i \to s \in W_1 \cap G_{r(x)}$, it follows that $s_i$ are eventually in $W_1$ and $t_i$ are eventually in $W_0$. Therefore $y_i = s_i x t_i$ is eventually in $W_1 x W_0$. Thus $x$ is in the interior of $WxW$. ■

3.9. PROPOSITION. *Let $G$ be a groupoid endowed with a pseudometric $\rho$ satisfying the following conditions:*

1. $\rho(x, r(x)) = \rho(x^{-1}, d(x))$ *for all* $x \in G$.
2. $\rho(xy, r(x)) \le \rho(y, r(y)) + \rho(x^{-1}, d(x))$ *for all* $(x,y) \in G^{(2)}$.

*For every* $n \in \mathbb{N}$ *let*

$$W_n := \left\{ x \in G : \rho\left(x, r\left(x\right)\right) < \frac{1}{2^n} \right\}.$$

*Then* $\mathcal{W} = \{W_n\}_n$ *is a* $G$*-uniformity compatible with the topology of* $r$*-fibres (induced by the pseudometric* $\rho$*).*

PROOF. Obviously, satisfies condition 1, 2 and 4 from Definition 2.1. Also let us note that $W_{n+1}W_{n+1} \subset W_n$ for all $n$ (since $\rho\left(xy, r\left(x\right)\right) \leq \rho\left(y, r\left(y\right)\right) + \rho\left(x^{-1}, d\left(x\right)\right) = \rho\left(y, r\left(y\right)\right) + \rho\left(x, r\left(x\right)\right)$ for all $(x, y) \in G^{(2)}$). Since for all $u$, $W_n \cap G^u = B\left(u, \frac{1}{2^n}\right) \cap G^u$, it follows that $\mathcal{W}$ is compatible with the topology of $r$-fibres. ∎

3.10. PROPOSITION. *Let* $G$ *be a groupoid endowed with a pseudometric* $\rho$ *satisfying the following conditions:*

1. $\rho\left(x, d\left(x\right)\right) = \rho\left(x^{-1}, r\left(x\right)\right)$ *for all* $x \in G$.
2. $\rho\left(xy, d\left(y\right)\right) \leq \rho\left(x, d\left(x\right)\right) + \rho\left(y^{-1}, r\left(y\right)\right)$ *for all* $(x, y) \in G^{(2)}$.

*For every* $n \in \mathbb{N}$ *let*

$$W_n := \left\{ x \in G : \rho\left(x, d\left(x\right)\right) < \frac{1}{2^n} \right\}.$$

*Then* $\mathcal{W} = \{W_n\}_n$ *is a* $G$*-uniformity compatible with the topology of* $d$*-fibres (induced by the pseudometric* $\rho$*).*

PROOF. The proof is similar to the proof of Proposition 3.9. ∎

3.11. DEFINITION. *Let* $G$ *be a groupoid endowed with a pseudometric* $\rho$ *satisfying conditions* 1 *and* 2 *in Proposition 3.9 or in Proposition 3.10. Then the* $G$*-uniformity constructed in Proposition 3.9 as well as the* $G$*-uniformity constructed in Proposition 3.10 will be called the* $G$*-uniformity associated to the pseudometric* $\rho$*.*

3.12. REMARK. If $\rho$ is a left invariant pseudometric on a groupoid $G$ (in the sense that $\rho\left(zx, zy\right) = \rho\left(x, y\right)$ for all $x, y, z \in G$ with $d\left(z\right) = r\left(x\right) = r\left(y\right)$), then $\rho\left(xy, r\left(x\right)\right) \leq \rho\left(x^{-1}, d\left(x\right)\right) + \rho\left(y, r\left(y\right)\right)$ for all $(x, y) \in G^{(2)}$ and $\rho\left(x, r\left(x\right)\right) = \rho\left(x^{-1}, d\left(x\right)\right)$ for all $x \in G$. Indeed, $\rho\left(xy, r\left(x\right)\right) = \rho\left(xy, xx^{-1}\right) = \rho\left(y, x^{-1}\right) \leq \rho\left(y, r\left(y\right)\right) + \rho\left(x^{-1}, r\left(y\right)\right) = \rho\left(y, r\left(y\right)\right) + \rho\left(x^{-1}, d\left(x\right)\right)$ for all $(x, y) \in G^{(2)}$. Also $\rho\left(x, r\left(x\right)\right) = \rho\left(xd\left(x\right), xx^{-1}\right) = \rho\left(d\left(x\right), x^{-1}\right)$ for all $x \in G$.

Also if $\rho$ is a right invariant pseudometric on a groupoid $G$ (in the sense that $\rho\left(xz, yz\right)$ $=\rho\left(x, y\right)$ for all $x, y, z \in G$ with $r\left(z\right) = d\left(x\right) = d\left(y\right)$), then $\rho\left(xy, d\left(x\right)\right) \leq \rho\left(x, d\left(x\right)\right) + \rho\left(y^{-1}, r\left(y\right)\right)$ for all $(x, y) \in G^{(2)}$and $\rho\left(x, d\left(x\right)\right) = \rho\left(x^{-1}, r\left(x\right)\right)$ for all $x \in G$.

3.13. REMARK. Any topological groupoid that is paracompact admits a fundamental system $\mathcal{W}$ of neighborhoods that is a $G$-uniformity compatible with the topology of fibres [10]. The same is true for a topological groupoid with paracompact unit space [5].

3.14. PROPOSITION. *Let $G$ be a groupoid and $\mathcal{W}=\{W_n\}_{n\in\mathbb{N}}$ be a countable $G$-uniformity. Then $G$ can be endowed with a pseudometric $\rho$ satisfying the following conditions:*

1. *$\rho$ is left invariant in the sense that $\rho(zx,zy)=\rho(x,y)$ for all $x,y,z\in G$ with $d(z)=r(x)=r(y)$.*
2. *$\rho$ induces a $G$-uniformity equivalent to $\mathcal{W}$.*
3. *For every $u\in G^{(0)}$ the restriction of $\rho$ to $G^u$ is compatible with the topology induced by $\tau^r_{\mathcal{W}}$ on $G^u$.*
4. *If $\bigcap\limits_{n\in\mathbb{N}} W_n=G^{(0)}$, then $\rho$ is a metric.*

PROOF. Let $I=\left\{\frac{1}{2^n},\ n\in\mathbb{N}\right\}$. Let $W_0\in\{W_n\}_{n\in\mathbb{N}}$ and $W_1'\in\mathcal{W}$ be such that $W_1'W_1'\subset W_0$ and $W_1'\subset W_1$. Inductively we construct an $I$-indexed family $\{W_i'\}_{i\in I}$. Suppose that for $W'_{1/2^n}\in\mathcal{W}$ has already been built. Then there is a $W''\in\mathcal{W}$ such that $W''W''\subset W'_{1/2^n}$ and $W''\subset W_{n+2}$. Let $W'_{1/2^{n+1}}=W''$. Thus we obtain an $I$-indexed family $\mathcal{W}'=\{W_i'\}_{i\in I}$ as in Theorem 2.5 and if $\bigcap\limits_{n\in\mathbb{N}} W_n=G^{(0)}$, then $G^{(0)}=\bigcap\limits_{i\in I} W_i'$. Let $f=f_{G^{(0)},\mathcal{W}'}$ be the function defined in Theorem 2.5 and $f_{reg}$ the function associated to $f$ in Proposition 2.6. Thus as in [8] we may define the following distance $\rho(x,y)=f_{reg}(x^{-1}y)$ if $r(x)=r(y)$ and $\rho(x,y)=1$ otherwise. Let $n\in\mathbb{N}$, $n\geq 4$ and $u\in G^{(0)}$. For $x\in G^u$ we have

$$\begin{aligned} B\left(x,\frac{1}{2^n}\right) &= \left\{y\in G^u : f_{reg}\left(x^{-1}y\right)<\frac{1}{2^n}\right\} \\ &\subset xW'_{1/2^{n-4}} \end{aligned}$$

On the other hand according Proposition 2.6 $W'_{1/2^{n+1}}\subset\left\{z: f_{reg}(z)<\frac{1}{2^n}\right\}$. Hence $xW'_{1/2^{n+1}}\subset\left\{y: f_{reg}(x^{-1}y)<\frac{1}{2^n}\right\}=B\left(x,\frac{1}{2^n}\right)$. Thus the topologies induced by $\tau^r_{\mathcal{W}}$ and the metric $\rho|_{G^u}$ on $G^u$ coincide. ∎

3.15. PROPOSITION. *Let $G$ be a groupoid and $\mathcal{W}=\{W_n\}_{n\in\mathbb{N}}$ be a countable $G$-uniformity. Then $G$ can be endowed with a pseudometric $\rho$ satisfying the following conditions:*

1. *$\rho$ is right invariant in the sense that $\rho(xz,yz)=\rho(x,y)$ for all $x,y,z\in G$ with $r(z)=d(x)=d(y)$.*
2. *$\rho$ induces a $G$-uniformity equivalent to $\mathcal{W}$.*
3. *For every $u\in G^{(0)}$ the restriction of $\rho$ to $G_u$ is compatible with the topology induced by $\tau^d_{\mathcal{W}}$ on $G_u$.*
4. *If $\bigcap\limits_{n\in\mathbb{N}} W_n=G^{(0)}$, then $\rho$ is a metric.*

PROOF. Similar as in the proof of Proposition 3.14 we may define the following distance $\rho(x,y)=f_{reg}(xy^{-1})$ if $d(x)=d(y)$ and $\rho(x,y)=1$ otherwise. ∎

3.16. Theorem. *A topological locally transitive groupoid. The following statements are equivalent:*

(*a*) *$G$ is metrizable*

(*b*) *$G$ is paracompact and $G^{(0)}$ has a countable fundamental system $\{W_n\}_{n\in\mathbb{N}}$ of neighborhoods such that $\bigcap_{n\in\mathbb{N}} W_n = G^{(0)}$ and $\bigcap_{n\in\mathbb{N}} (r,d)(W_n) = diag\left(G^{(0)}\right)$.*

(*c*) *For every neighborhood $W$ of $G^{(0)}$ there is a neighborhood $W'$ of $G^{(0)}$ such that $W'W' \subset W$ and $G^{(0)}$ has a countable fundamental system $\{W_n\}_{n\in\mathbb{N}}$ of neighborhoods such that $\bigcap_{n\in\mathbb{N}} W_n = G^{(0)}$ and $\bigcap_{n\in\mathbb{N}} (r,d)(W_n) = diag\left(G^{(0)}\right)$.*

(*d*) *There is a countable $G$-uniformity $\{W_n\}_{n\in\mathbb{N}}$ compatible with the topology of the fibres such that $\bigcap_{n\in\mathbb{N}} W_n = G^{(0)}$ and $\bigcap_{n\in\mathbb{N}} (r,d)(W_n) = diag\left(G^{(0)}\right)$. Each $W_n$ may be taken to be a neighborhood of the unit space.*

*Moreover the distance function $\rho$ may be taken to satisfy the following properties:*

1. *$\rho(x,y) = \rho(x^{-1}, y^{-1})$ for all $x, y \in G$.*
2. *$\rho(x, r(x)) = \rho(x, d(x))$ for all $x \in G$.*
3. *$\rho(xy, r(x)) \leq \rho(x, r(x)) + \rho(y, r(y))$ for all $(x,y) \in G^{(2)}$.*
4. *$\rho(x,y) \leq \rho(x^{-1}y, d(x))$ for all $x, y \in G$ such that $r(x) = r(y)$.*
5. *$\rho(d(x), d(y)) \leq 2\rho(x,y)$ and $\rho(r(x), r(y)) \leq 2\rho(x,y)$ for all $x$,$y \in G$.*

Proof. $(a) \Rightarrow (b)$. Let us assume that $G$ is a metrizable locally transitive topological groupoid. Then $G$ is paracompact topological groupoid. According to [10, p. 361-362], for each neighborhood $W$ of $G^{(0)}$, there is a neighborhood $W'$ of $G^{(0)}$ such that $W'W' \subset W$. Then the family $\mathcal{W}$ of symmetric neighborhoods of the unit space is a $G$-uniformity. By Proposition 3.8, the topology $\tau_{\mathcal{W}}$ induced by the $G$-uniformity $\mathcal{W}$ coincides with the topology of $G$. Applying [7, Metrization Theorem 13, p. 186] $G$ is pseudometrizable if and only if its uniformity has a countable base. Since a base for the uniform structure $\mathcal{U}_{\mathcal{W}}$ induced the topology $\tau_{\mathcal{W}}$ is $\{U_W\}_{W\in\mathcal{W}}$, where

$$U_W = \{(x,y) \in G \times G : y \in WxW\}$$

there is a countable family $\{W_n'\}_{n\in\mathbb{N}}$ such that each $W_n'$ is a neighborhood of $G^{(0)}$ and for each $W \in \mathcal{W}$ there is $n \in \mathbb{N}$ such that $U_{W_n'} \subset U_W$ or equivalently, $W_n'xW_n' \subset WxW$ for all $x \in G$. In particular, for each $W \in \mathcal{W}$ there is $n \in \mathbb{N}$ such that $W_n'W_n' \subset WW$. Since for each $W \in \mathcal{W}$, there is $W_1 \in \mathcal{W}$ such that $W_1W_1 \subset W$ and for $W_1$ there is $n_1 \in \mathbb{N}$ such that $W_{n_1}'W_{n_1}' \subset W_1W_1$, it follows that in fact for each for each $W \in \mathcal{W}$, there is

$n_1 \in \mathbb{N}$ such that $W'_{n_1} \subset W'_{n_1} W'_{n_1} \subset W$. Thus $\{W'_n\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of $G^{(0)}$. Since $G$ is Hausdorff, for each $x \notin G^{(0)}$ there is a neighborhood $V$ of $r(x)$ such that $x \notin V$. Furthermore $x \notin V \cup \left(G \setminus G^{r(x)}\right)$ and $V \cup \left(G \setminus G^{r(x)}\right)$ is a neighborhood of $G^{(0)}$. Thus $\bigcap_{W\in\mathcal{W}} W = G^{(0)}$ and therefore $\bigcap_{n\in\mathbb{N}} W'_n = G^{(0)}$. Let $u, v \in G^{(0)}$ be such that $u \neq v$. Since $G$ is Hausdorff, $G^u_v$ is closed and $G \setminus G^u_v$ is a neighborhood of $G^{(0)}$. Hence $\bigcap_{W\in\mathcal{W}} (r,d)(W) = diag\left(G^{(0)}\right)$. Consequently, $\bigcap_{n\in\mathbb{N}} (r,d)(W'_n) = diag\left(G^{(0)}\right)$.

$(b) => (c)$ Since $G$ is a paracompact topological groupoid, [10, p. 361-362], for each neighborhood $W$ of $G^{(0)}$, there is a neighborhood $W'$ of $G^{(0)}$ such that $W'W' \subset W$.

$(c) => (d)$ Let $\{W_n\}_{n\in\mathbb{N}}$ be a fundamental system of neighborhoods of $G^{(0)}$ such that $\bigcap_{n\in\mathbb{N}} W_n = G^{(0)}$ and $\bigcap_{n\in\mathbb{N}} (r,d)(W_n) = diag\left(G^{(0)}\right)$. Replacing $W_n$ with $W_n \cap W_n^{-1}$, we may assume that $W_n = W_n^{-1}$ for all $n$. Let $W'_0 = W_0$. Inductively we construct a $G$-uniformity $\{W'_n\}_{n\in\mathbb{N}}$ consisting in neighborhoods of $G^{(0)}$. Suppose a symmetric neighborhood $W'_n$ of $G^{(0)}$ has already been built. Let $W''$ be a symmetric neighborhood of $G^{(0)}$ such that $W''W'' \subset W'_n$. Let $W'_{n+1}$be a neighborhood of $G^{(0)}$ such that $W'_{n+1} \subset W'' \cap W_{n+1}$. Thus $\{W'_n\}_{n\in\mathbb{N}}$ is a $G$-uniformity. Moreover $\{W'_n\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of $G^{(0)}$. Therefore it is compatible with the topology of the fibres and $\bigcap_{n\in\mathbb{N}} W'_n = G^{(0)}$ as well as $\bigcap_{n\in\mathbb{N}} (r,d)(W'_n) = diag\left(G^{(0)}\right)$.

$(d) => (a)$. Let $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ be countable $G$-uniformity compatible with the topology of the fibres such that $\bigcap_{n\in\mathbb{N}} W_n = G^{(0)}$ and $\bigcap_{n\in\mathbb{N}} (r,d)(W_n) = diag\left(G^{(0)}\right)$. Let $I = \left\{\frac{1}{2^n},\ n \in \mathbb{N}\right\}$. Let $W_{n_0} \in \mathcal{W}$ be such that $W_{n_0} W_{n_0} \subset W_0$. Let $W'_0 = W_{n_0}$. Inductively we construct an $I$-indexed family $\{W'_i\}_{i\in I}$. Suppose that $W'_{1/2^n} \in \mathcal{W}$ has already been built. Since $\mathcal{W}$ is a $G$-uniformity, there is a $W_{m_n} \in \mathcal{W}$ such that $W_{m_n} W_{m_n} \subset W'_{1/2^n}$. Let $W'_{1/2^{n+1}} \in \mathcal{W}$ be such that $W'_{1/2^{n+1}} \subset W_{m_n} \cap W_{n+1}$. Thus we obtain an $I$-indexed family $\mathcal{W}' = \{W'_i\}_{i\in I}$ as in Theorem 2.5 that in addition satisfies $G^{(0)} = \bigcap_{i\in I} W'_i$ and $\bigcap_{i\in I} (r,d)(W'_i) = diag\left(G^{(0)}\right)$. Moreover $\mathcal{W}' = \{W'_i\}_{i\in I}$ is compatible with the topology of the fibres. Thus for every $x \in G$ and every $W'_i \in \mathcal{W}'$ there is $W'_{i_x} \in \mathcal{W}'$ such that $xW'_i x^{-1} \subset W'_i$. Let $f_{reg}$ be the function associated in Proposition 2.6 to $f = f_{G^{(0)},\mathcal{W}'}$, where $f = f_{G^{(0)},\mathcal{W}'}$ is the function constructed in Theorem 2.5. For all $x$,$y \in G$, let us define

$$\rho(x,y) := \frac{1}{2} \inf\left\{ f_{reg}\left(x^{-1}sy\right) + f_{reg}(s) :\ s \in G^{r(x)}_{r(y)} \right\},$$

if $G^{r(x)}_{r(y)} \neq \emptyset$ and $\rho(x,y) := 1$ otherwise. Let us note that $G^{r(y)}_{r(x)} \neq \emptyset$ if and only if

$G_{r(y)}^{r(x)} \neq \emptyset$ and

$$\begin{aligned}\rho(x,y) &= \frac{1}{2}\inf\left\{f_{reg}\left(x^{-1}sy\right)+f_{reg}(s):\ s\in G_{r(y)}^{r(x)}\right\}\\ &= \frac{1}{2}\inf\left\{f_{reg}\left(y^{-1}s^{-1}x\right)+f_{reg}\left(s^{-1}\right):\ s\in G_{r(y)}^{r(x)}\right\}\\ &= \frac{1}{2}\inf\left\{f_{reg}\left(y^{-1}tx\right)+f_{reg}(t):\ t\in G_{r(x)}^{r(y)}\right\}\\ &= \rho(y,x).\end{aligned}$$

Thus $\rho(x,y)=\rho(y,x)$.

Let us prove that if $r(x)=r(y)$, then $\rho(x,y)\leq\frac{1}{2}f_{reg}\left(x^{-1}y\right)$. Indeed,

$$\rho(x,y)\leq\frac{1}{2}\left(f_{reg}\left(x^{-1}r(x)\,y\right)+f_{reg}(r(x))\right)=\frac{1}{2}f_{reg}\left(x^{-1}y\right).$$

If $x=y$, then $\rho(x,y)\leq\frac{1}{2}f_{reg}\left(x^{-1}y\right)=0$.

Let $x,y,z\in G$ and let us prove that $\rho(x,z)\leq\rho(x,y)+\rho(y,z)$. If $\rho(x,y)=1$ or $\rho(y,z)=1$, then obviously, $\rho(x,z)\leq1\leq\rho(x,y)+\rho(y,z)$. If $\rho(x,y)<1$ and $\rho(y,z)<1$, then for every $\varepsilon>0$ there are $s_1=s_1(\varepsilon)\in G_{r(y)}^{r(x)}$ and $s_2=s_2(\varepsilon)\in G_{r(z)}^{r(y)}$ such that $\rho(x,y)>\frac{1}{2}f_{reg}\left(x^{-1}s_1y\right)+\frac{1}{2}f_{reg}(s_1)-\varepsilon$ and $\rho(y,z)>\frac{1}{2}f_{reg}\left(y^{-1}s_2x\right)+\frac{1}{2}f_{reg}(s_2)-\varepsilon$. Furthermore

$$\begin{aligned}\rho(x,z) &\leq \frac{1}{2}f_{reg}\left(x^{-1}s_1s_2z\right)+\frac{1}{2}f_{reg}(s_1s_2)\\ &\leq \frac{1}{2}f_{reg}\left(x^{-1}s_1yy^{-1}s_2z\right)+\frac{1}{2}f_{reg}(s_1)+\frac{1}{2}f_{reg}(s_2)\\ &\leq \frac{1}{2}f_{reg}\left(x^{-1}s_1y\right)+\frac{1}{2}f_{reg}\left(y^{-1}s_2z\right)+\frac{1}{2}f_{reg}(s_1)+\frac{1}{2}f_{reg}(s_2)\\ &< \rho(x,y)+\rho(y,z)+2\varepsilon.\end{aligned}$$

Therefore $\rho(x,z)\leq\rho(x,y)+\rho(y,z)$.

Let us show that if $\rho(x,y)=0$, then $x=y$. If $\rho(x,y)=0$, for every $n$ there is $s_n\in G_{r(y)}^{r(x)}$ such that $f_{reg}(s_n)<\frac{1}{2^n}$ and $f_{reg}\left(x^{-1}s_ny\right)<\frac{1}{2^n}$. Taking into account that $f_{reg}(s_n)<\frac{1}{2^n}$ applying Proposition 2.6, it follows that $s_n\in W'_{1/2^{n-4}}$ and consequently, $(r(x),r(y))=(r,d)(s_n)\in(r,d)\left(W'_{1/2^{n-4}}\right)$. Since $\bigcap\limits_{n\in\mathbb{N}}(r,d)\left(W'_{1/2^n}\right)=diag\left(G^{(0)}\right)$, it follows that $r(x)=r(y)$. Moreover since $f_{reg}\left(x^{-1}s_ny\right)<\frac{1}{2^n}$, it follows that $y\in W'_{1/2^{n-4}}xW'_{1/2^{n-4}}$ for all $n\geq4$. Let $W'_{x,n}\in\mathcal{W}'$ be such that $x^{-1}W'_{x,n}x\subset W'_{1/2^{n+1}}$ and $W'_{x,n}\subset W'_{1/2^{n+1}}$. We have $y\in W'_{x,n}xW'_{x,n}$. Consequently, $y\in xx^{-1}W'_{x,n}xW'_{x,n}\subset xW'_{1/2^{n+1}}W'_{1/2^{n+1}}\subset xW'_{1/2^n}$. Hence $x^{-1}y\in\bigcap\limits_{n\in\mathbb{N}}W'_{1/2^n}=G^{(0)}$. Thus $x=y$.

We have proved that $\rho$ is a metric on $G$. Let us prove that the topology defined by $\rho$ coincides with the topology induced by the $G$-uniformity $\mathcal{W}'$ and consequently, with the topology of $G$. Let $y \in B\left(x, \frac{1}{2^n}\right)$, $n \in \mathbb{N}$, $n \geq 6$. Then there is $s \in G_{r(y)}^{r(x)}$ such that $f_{reg}(s) < \frac{1}{2^{n-2}}$ and $f_{reg}(x^{-1}sy) < \frac{1}{2^{n-2}}$. By Proposition 2.6, it follows that $s \in W'_{1/2^{n-6}}$ and $x^{-1}sy \in W'_{1/2^{n-6}}$. Therefore $y \in W'_{1/2^{n-6}}xW'_{1/2^{n-6}}$ and $B\left(x, \frac{1}{2^n}\right) \subset W'_{1/2^{n-6}}xW'_{1/2^{n-6}}$. On the other hand for every $n$ and $x$, if $y \in W'_{1/2^n}xW'_{1/2^n}$, then there are $s, t \in W'_{1/2^n}$ such that $y = sxt$. Hence $f_{reg}(x^{-1}s^{-1}y) = f_{reg}(x^{-1}xt) = f_{reg}(t) < \frac{1}{2^{n-1}}$. Also $\rho(x,y) \leq \frac{1}{2}\left(f_{reg}(x^{-1}s^{-1}y) + f_{reg}(s^{-1})\right) = \frac{1}{2}\left(f_{reg}(t) + f_{reg}(s)\right) < \frac{1}{2^{n-3}}$. Therefore $W'_{1/2^n}xW'_{1/2^n} \subset B\left(x, \frac{1}{2^{n-3}}\right)$.

Let us prove that $\rho(x,y) = \rho(x^{-1}, y^{-1})$ for all $x$,$y \in G$. We have $G_{r(y)}^{r(x)} = \emptyset$ if and only if $G_{r(y^{-1})}^{r(x^{-1})} = \emptyset$. Thus if $G_{r(y)}^{r(x)} = \emptyset$, then $\rho(x,y) = 1 = \rho(x^{-1}, y^{-1})$. Let us assume that $G_{r(y)}^{r(x)} \neq \emptyset$. Then for every $\varepsilon > 0$ there is $s_\varepsilon \in G_{r(y)}^{r(x)}$ such that $\rho(x,y) > \frac{1}{2}f_{reg}(x^{-1}s_\varepsilon y) + \frac{1}{2}f_{reg}(s_\varepsilon) - \varepsilon$. Let $t = x^{-1}s_\varepsilon y$. Then $\rho(x^{-1}, y^{-1}) \leq \frac{1}{2}f_{reg}(xty^{-1}) + \frac{1}{2}f_{reg}(t) = \frac{1}{2}f_{reg}(s_\varepsilon) + \frac{1}{2}f_{reg}(x^{-1}s_\varepsilon y) \leq \rho(x,y) + \varepsilon$. Similarly, $\rho(x,y) \leq \rho(x^{-1}, y^{-1}) + \varepsilon$. Hence $\rho(x,y) = \rho(x^{-1}, y^{-1})$.

Let us show that $\rho(x, r(x)) = \rho(x^{-1}, d(x)) = \rho(x, d(x)) = \frac{1}{2}f_{reg}(x)$ for all $x \in G$. We have $\rho(x, r(x)) \leq \frac{1}{2}f_{reg}(x^{-1}r(x)) = \frac{1}{2}f_{reg}(x)$. For all $s \in G_{r(x)}^{r(x)}$ we have $\frac{1}{2}f_{reg}(x) = \frac{1}{2}f_{reg}(x^{-1}) = \frac{1}{2}f_{reg}(x^{-1}sr(x)s^{-1}) \leq \frac{1}{2}f_{reg}(x^{-1}sr(x)) + \frac{1}{2}f_{reg}^{1/2}(s^{-1})$. Thus $\rho(x, r(x)) = \frac{1}{2}f_{reg}(x)$.

Also $\rho(x^{-1}, d(x)) = \frac{1}{2}f_{reg}(x^{-1}) = \frac{1}{2}f_{reg}(x) = \rho(x, r(x))$. Moreover $\rho(x, d(x)) = \rho(x^{-1}, d(x))$ for all $x \in G$.

For all $(x,y) \in G^{(0)}$ we have $\rho(xy, r(x)) = \frac{1}{2}f_{reg}(xy) \leq \frac{1}{2}f_{reg}(x) + \frac{1}{2}f_{reg}(y) = \rho(x, r(x)) + \rho(y, r(y))$.

If $r(x) = r(y)$, then $\rho(x,y) \leq \frac{1}{2}f_{reg}^{1/2}(x^{-1}y) = \rho(x^{-1}y, d(x)) = \rho(y^{-1}x, d(y))$.

Let us prove that $\rho(d(x), d(y)) \leq 2\rho(x,y)$ and $\rho(r(x), r(y)) \leq 2\rho(x,y)$ for all $x$,$y \in G$. Obviously, if $G_{r(y)}^{r(x)} = \emptyset$, then $\rho(d(x), d(y)) = \rho(r(x), r(y)) = \rho(x,y) = 1$. If $G_{r(y)}^{r(x)} \neq \emptyset$, then for every $\varepsilon > 0$ there is $s_\varepsilon \in G_{r(y)}^{r(x)}$ such that $\rho(x,y) > \frac{1}{2}f_{reg}(x^{-1}s_\varepsilon y) + \frac{1}{2}f_{reg}(s_\varepsilon) - \varepsilon$. Let $t = x^{-1}s_\varepsilon y$. Then $\rho(d(x), d(y)) \leq \frac{1}{2}f_{reg}(t) + \frac{1}{2}f_{reg}(t) = f_{reg}(x^{-1}s_\varepsilon y) \leq f_{reg}(x^{-1}s_\varepsilon y) + f_{reg}(s_\varepsilon) < 2\rho(x,y) + 2\varepsilon$. Hence $\rho(d(x), d(y)) \leq 2\rho(x,y)$. We also have

$$\rho(r(x), r(y)) = \rho\left(d\left(x^{-1}\right), d\left(y^{-1}\right)\right) \leq \rho\left(x^{-1}, y^{-1}\right) = \rho(x,y)$$

for all $x$,$y \in G$. ∎

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