CLASSIFYING TANGENT STRUCTURES USING WEIL ALGEBRAS

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Abstract. At the heart of differential geometry is the construction of the tangent bundle of a manifold. There are various abstractions of this construction, and of particular interest here is that of Tangent Structures.

Tangent Structure is defined via giving an underlying category $\mathcal{M}$ and a tangent functor $T$ along with a list of natural transformations satisfying a set of axioms, then detailing the behaviour of $T$ in the category $\text{End}(\mathcal{M})$. However, this axiomatic definition at first seems somewhat disjoint from other approaches in differential geometry.

The aim of this paper is to present a perspective that addresses this issue. More specifically, this paper highlights a very explicit relationship between the axiomatic definition of Tangent Structure and the Weil algebras (which have a well established place in differential geometry).

1. Introduction

The starting point for the notion of tangent structure is that given a smooth manifold $M$, we can construct the tangent space $TM$, which to each point $x \in M$ attaches the vector space $T_x M$ of all tangents to $M$ at $x$. The functoriality of this construction is used to capture the idea of differentiation of maps between more abstract spaces.

$T$ being a functor (moreover an endofunctor over the category under consideration) allows us to talk about "tangent space" to an arbitrary category. There is also a more specific, technical meaning of "Tangent Structure" given by Rosicky in [Rosicky, 1984] and by Cockett and Cruttwell in [Cockett and Cruttwell, 2014].

Weil algebras, on the other hand, have a well established history in the world of differential geometry. For instance, Kolar et al give discussion of Weil algebras and Weil functors in [Kolar et al., 2010] (Section 35), and of course there is the work of Weil [Weil, 1953]. Indeed, there are also the ideas of synthetic differential geometry (SDG), which define tangent spaces and related structures through the use of infinitesimals (given as the spectrum of corresponding Weil algebras; for instance, see [Kock, 2006] for more details).

There are, as we shall see, strong connections between these two seemingly different concepts. Furthermore, it will turn out that the tangent functor $T$ is closely related to
a particular Weil algebra in a very meaningful way. We shall begin with a brief look at
Tangent Structure, then discuss Weil algebras and some of their properties.

We will then introduce (co)graphs and show that they are a surprisingly useful tool
in characterising not only the objects, but also the morphisms of a category \textbf{Weil\textsubscript{1}} (a
particular subcategory of \textbf{Weil}) we shall be using in our discussion.

More specifically, we will show that each object of \textbf{Weil\textsubscript{1}} corresponds canonically to
a particular graph (moreover, what we shall call \textit{piecewise complete} graphs), and further
that each morphism \( f: A \rightarrow B \) of such Weil algebras can be described using cliques and
independent sets. These observations then provide a language for a methodical process
to “construct” any such map using a collection of generating maps.

We will conclude with Theorem 14.1, which states that to give a tangent structure (in
the sense of [Cockett and Cruttwell, 2014]) over a category \( \mathcal{M} \) is to give a functor
\[
F: \textbf{Weil\textsubscript{1}} \rightarrow [\mathcal{M}, \mathcal{M}]
\]
satisfying certain axioms.

One final observation we will make is that we can in fact remove the requirement of
the codomain of \( F \) needing to be an endofunctor category \([\mathcal{M}, \mathcal{M}]\), and instead replace
it with an arbitrary monoidal category \((G, \Box, I)\). This then more clearly exhibits \textbf{Weil\textsubscript{1}}
as what one might call the “initial” tangent structure.

2. Tangent Structure

Tangent Structure is defined by Rosický [Rosický, 1984] using (internal) bundles of abelian
groups, but we will be following the more general definition of Cockett-Cruttwell [Cockett
and Cruttwell, 2014] using (internal) bundles of commutative monoids. More explicitly,
this requires that the tangent bundle \( TM \) sitting over a smooth manifold \( M \) is a commu-
tative monoid, referred to as an \textit{additive bundle}.

In this section, we shall give said definition of Tangent Structure below in Definition
2.6. However, we first have the following:

2.1. \textbf{Definition.} Given a category \( C \), a commutative monoid \textit{in} \( C \) consists of

1. An object \( C \) such that finite powers of \( C \) exist (the terminal object we shall call \( t \));

2. A pair of maps \( \eta: t \rightarrow C \) and \( \mu: C \times C \rightarrow C \) such that the following diagrams
   commute

\[
\begin{array}{ccc}
C \times (C \times C) & \xrightarrow{\alpha} & (C \times C) \times C \\
\downarrow_{1 \times \mu} & & \downarrow_{\mu} \\
C \times C & \xrightarrow{\mu} & C
\end{array}
\quad
\begin{array}{ccc}
t \times C & \xrightarrow{\eta \times 1} & C \times C \\
\downarrow_{\cong} & & \downarrow_{\cong} \\
C \times t & \xrightarrow{1 \times \eta} & C \times C
\end{array}
\]
where $\alpha: C \times (C \times C) \to (C \times C) \to C$ is the obvious associativity map, and $\mu$ agrees with the symmetry map $s: C \times C \to C \times C$,

so that the diagram

$\begin{array}{ccc}
C \times C & \xrightarrow{s} & C \times C \\
\downarrow{\mu} & & \downarrow{\mu} \\
C & \downarrow{\mu}
\end{array}$

also commutes.

2.2. Remark. Often, commutative monoids are considered in categories with all finite products, but we shall not be assuming this.

2.3. Definition. If $A$ is an object in a category $\mathcal{M}$, then an additive bundle over $A$ is a commutative monoid in the slice category $\mathcal{M}/A$. Explicitly, this consists of

1. A map $p: X \to A$ such that pullback powers of $p$ exist, the $n$th pullback power denoted by $X^{(n)}$ and projections $\pi_i: X^{(n)} \to X$ for $i \in \{1, \ldots, n\}$;
2. Maps $+: X^{(2)} \to X$ and $\eta: A \to X$ with $p \circ + = p \circ \pi_1 = p \circ \pi_2$ and $p \circ \eta = \text{id}$ which are associative, commutative, and unital.

2.4. Remark. We will note here that the notation used in [Cockett and Cruttwell, 2014] for the $n$th pullback power is instead $X_n$.

2.5. Definition. Suppose $p: X \to A$ and $q: Y \to B$ are additive bundles. An additive bundle morphism is a pair of maps $f: X \to Y$ and $g: A \to B$ such that the following diagrams commute.

2.6. Definition. Given a category $\mathcal{M}$, a tangent structure $\mathbb{T} = (T, p, \eta, +, l, c)$ consists of

1. (tangent functor) a functor $T: \mathcal{M} \to \mathcal{M}$ and a natural transformation $p: T \Rightarrow 1_\mathcal{M}$ such that pullback powers $T^{(n)}$ of $p$ exist and the composites $T^m$ of $p$ preserve these pullbacks for all $m \in \mathbb{N}$;
2. (tangent bundle) natural transformations $+: T^{(2)} \Rightarrow T$ and $\eta: 1_\mathcal{M} \Rightarrow T$ making $p: T \Rightarrow 1_\mathcal{M}$ into an additive bundle;
3. **(vertical lift)** a natural transformation \( l: T \Rightarrow T^2 \) such that

\[
(l, \eta): (p, +, \eta) \to (Tp, T+, T\eta)
\]

is an additive bundle morphism;

4. **(canonical flip)** a natural transformation \( c: T^2 \Rightarrow T^2 \) such that

\[
(c, id_T): (Tp, T+, T\eta) \to (pT, +T, \eta T)
\]

is an additive bundle morphism;

where the natural transformations \( l \) and \( c \) satisfy

1. **(coherence of \( l \) and \( c \))** \( c^2 = id, \ c \circ l = l, \) and the following diagrams commute

\[
\begin{array}{cccccc}
T & \xrightarrow{l} & T^2 & \xrightarrow{Tl} & T^3 & \xrightarrow{cT} & T^3 \\
\downarrow{l} & & \downarrow{Tl} & & \downarrow{cT} & & \downarrow{cT} \\
T^2 & \xrightarrow{Tl} & T^3 & & & & \\
\end{array}
\]

2. **(universality of vertical lift)** the following is an equaliser diagram

\[
\begin{array}{cccccc}
T(2) & \xrightarrow{(T+) \circ (l \times_T \eta T)} & T^2 & \xrightarrow{Tp \circ \eta T} & T \\
\end{array}
\]

where \( (T+) \circ (l \times_T \eta T) \) is the composite

\[
\begin{array}{cccccc}
T & \xrightarrow{l} & T^2 & \xrightarrow{Tl} & T^3 & \xrightarrow{cT} & T^3 \\
\downarrow{\pi_1} & & \downarrow{\pi_1} & & \downarrow{cT} & & \downarrow{cT} \\
T(2) & \xrightarrow{Tl} & T^3 & & & & \\
\downarrow{\pi_2} & & \downarrow{\pi_2} & & \downarrow{\pi_2} & & \downarrow{\pi_2} \\
T & \xrightarrow{\eta T} & T^2 \\
\end{array}
\]

2.7. **Remark.** We will note here that \( l: T \Rightarrow T^2 \) and \( p: T \Rightarrow 1_M \) do not form a comonad. However, there is a canonical way to make \( T \) a monad (detailed in [Cockett and Cruttwell, 2014]).

We may then refer to the pair \((\mathcal{M}, \mathbb{T})\) as a **tangent category**.
3. Weil Algebras

We now introduce Weil algebras. For the purposes of this paper, we will always be using commutative, unital algebras. We shall initially define Weil algebras over a field, but ultimately we are interested in working over a commutative rig; recall a rig is a commutative monoid equipped with (a unital) multiplication.

Traditionally, Weil algebras are defined over a field, and we may at first naively use some more general structure (say an arbitrary ring). The problem in doing so is that the notion of “Weil algebra” in complete generality becomes somewhat difficult to define in a consistent and coherent manner.

For the purposes of this paper, however, we will only be interested in Weil algebras with presentations of a particular form (we will describe this in detail in 6). As such, when we restrict to these presentations, we will be able to work unhindered over a rig (rather than a field).

In particular, if we take $k$ to be (the commutative ring) $\mathbb{Z}$, we will ultimately recover the abelian group bundles of [Rosický, 1984], while (the commutative rig) $\mathbb{N}$ corresponds to the additive bundles of [Cockett and Cruttwell, 2014]. Later on in our discussion, we will also be interested in the rig $2$ (which we shall formally introduce in Definition 7.1).

We shall begin by defining Weil algebras over some given field $k$.

3.1. Definition. A Weil algebra $B$ is an augmented (commutative and unital) algebra with a finite dimensional underlying $k$-vector space, for which all elements of the augmentation ideal are nilpotent.

Equivalently, we can say that a Weil algebra is simply a finite dimensional local algebra with residue field $k$.

3.2. Remark. The equivalence arises from the fact that the augmentation ideal $\ker(\varepsilon)$ (for augmentation $\varepsilon : B \to k$) is the unique maximal ideal of $B$.

A morphism between Weil algebras $B$ and $C$ is simply an augmented algebra homomorphism, i.e. an algebra map

$$f : B \to C$$

that is compatible with the augmentations, i.e. we have a commuting diagram

$$\begin{array}{ccc}
B & \xrightarrow{f} & C \\
\varepsilon_B \downarrow & & \varepsilon_C \\
k & \xleftarrow{\varepsilon_C} & C
\end{array}$$

From here onwards, we shall simply refer to these augmented algebra homomorphisms as maps.

3.3. Definition. Let Weil be the category with objects the Weil algebras and morphisms the maps described above.
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3.4. Remark. The category Weil is a full subcategory of \textbf{AugAlg} (= \textbf{Alg}/k, the category of augmented algebras).

3.5. Remark. We will (soon) further restrict Weil to a full subcategory Weil_1 in order to discuss tangent structure.

It is often convenient to give a Weil algebra \( B \) via a presentation
\[
B = k[b_1, \ldots, b_m]/Q_B ,
\]
where we quotient the free algebra \( k[b_1, \ldots, b_m] \) by the list of terms in \( Q_B \).

3.6. Remark. This is always possible, since each Weil algebra is a finitely generated and commutative algebra, and such algebras always have a presentation of this form.

3.7. Example.

1. \( k[x]/x^2 \) is the Weil algebra with \( \{1, x\} \) as a basis for the underlying \( k \)-module and equipped with the obvious multiplication, but with \( x^2 \) identified as 0.

2. \( k[x]/x^3 \) is the Weil algebra with \( \{1, x, x^2\} \) as a basis for the underlying \( k \)-module and equipped with the obvious multiplication, but with \( x^3 \) identified with 0.

3. \( k[x,y]/x^2, y^2 \) is the Weil algebra with \( \{1, x, y, xy\} \) as a basis for the underlying \( k \)-module and equipped with the obvious multiplication, but with \( x^2 \) and \( y^2 \) each identified with 0.

We also note the following:

1. We shall always use presentations for which the augmentation \( \varepsilon: B \to k \) sends each generator \( b_i \) to 0.

2. Recall that for a linear map \( h: X \to Y \) between vector spaces, it suffices to define how \( h \) acts on basis elements of \( V \). Analogously, for an augmented algebra homomorphism \( f: B \to C \), it suffices to define how \( f \) acts on generators (then check that it is suitably compatible with the relations).

3. For Weil algebras \( A = k[a_1, \ldots, a_m]/Q_A \) and \( B = k[b_1, \ldots, b_n]/Q_B \) and a map \( f: A \to B \), \( f(a_i) \) is a polynomial in the generators \( b_1, \ldots, b_n \) with no constant term.

Now that we have defined the category \textbf{Weil}, we shall establish some facts about this category. We begin with the following:

1. The category \textbf{AugAlg} has all limits and colimits.

2. Coproducts in \textbf{AugAlg} are given by \( \otimes \).

which are well established and we shall not prove.

3.8. Lemma. \( k \) is a zero object of \textbf{Weil}.

Proof. For each Weil algebra \( A \), the augmentation \( \varepsilon_A: A \to k \) and the unit \( \eta_A: k \to A \) are the unique maps to and from \( k \) respectively. \qed
3.9. Proposition. The category $\text{Weil}$ has all finite products.

Proof. Since $k$ is a zero object, it is the nullary product. For arbitrary Weil algebras $A$ and $B$, begin by taking the pullback

\[
\begin{array}{ccc}
A \times_k B & \longrightarrow & B \\
\downarrow & & \downarrow \varepsilon_B \\
A & \longrightarrow & k
\end{array}
\]

(or equivalently, the product) in $\text{AugAlg}$. Since both $A$ and $B$ are finitely dimensional and have nilpotent augmentation ideals, then the same is true of $A \times_k B$. Thus it is also a Weil algebra.

Thus $\text{Weil}$ has all finite products. $
$

3.10. Definition. Let $\text{NilAugAlg}$ be the full subcategory of $\text{AugAlg}$ containing all augmented algebras whose augmentation ideals are nilpotent.

3.11. Proposition. The category $\text{NilAugAlg}$ has all finite limits.

Proof. Let $A$ be a finite category and consider an arbitrary diagram

\[ R: A \to \text{NilAugAlg} \]

Since $\text{AugAlg}$ has all limits, we can form a limiting cone

\[ A \xrightarrow{\Delta X} \text{NilAugAlg} \xleftarrow{\gamma} \text{AugAlg} \]

But since $A$ is finite, the (finite) set $\{\gamma_a \mid a \in A\}$ is jointly monic and each $Ra$ is nilpotent, then the augmentation ideal of $X$ is necessarily nilpotent, and so $X \in \text{NilAugAlg}$.

Thus $\text{NilAugAlg}$ has all finite limits. $
$

3.12. Definition. For each $n \in \mathbb{N}$, let $W_n$ be the Weil algebra $k[x]/x^{n+1}$.

3.13. Proposition. The set $\{W_n \mid n \in \mathbb{N}\}$ forms a strong generator for $\text{NilAugAlg}$.

Proof. We want to show that the set of functors

\[ \text{NilAugAlg}(W_n, -): \text{NilAugAlg} \to \text{Set} \]

for all $n \in \mathbb{N}$ jointly reflect isomorphisms.

Let $f: A \to B$ be an arbitrary map of $\text{NilAugAlg}$ for which

\[ \text{NilAugAlg}(W_n, f): \text{NilAugAlg}(W_n, A) \to \text{NilAugAlg}(W_n, B) \]
is an isomorphism for all $n \in \mathbb{N}$.

Let $\alpha$ be an element of $A$ with $f(\alpha) = 0$. In particular, $\alpha$ is an element of the augmentation ideal $\ker(\varepsilon_A)$. Since this is nilpotent, then we can define

$$r = \max\{s \in \mathbb{N} \mid \alpha^s \neq 0\}.$$ 

Note also that $\alpha^{r+1} = 0$. As such, we may define a map $g: W_r \to A$ given as $g(x) = \alpha$. Further, let $z: W_r \to A$ be the zero map (i.e. $z(x) = 0$).

Now, we have $g, z \in \text{NilAugAlg}(W_r, A)$. Moreover, we clearly have $f \circ g = f \circ z$. But since $\text{NilAugAlg}(W_r, f)$ is an isomorphism, then we must have $g = z$, i.e. $\alpha = 0$.

$\therefore \ker(f) = \{0\}$.

Now, let $\beta$ be an arbitrary element of $\ker(\varepsilon_B)$. Since $\ker(\varepsilon_B)$ is nilpotent, then we can define

$$\rho = \max\{\sigma \in \mathbb{N} \mid \beta^\sigma \neq 0\}.$$ 

Note also that $\beta^{\rho+1} = 0$. As such, we may define a map $\gamma: W_\rho \to B$ given as $\gamma(x) = \beta$.

But now we have $\gamma \in \text{NilAugAlg}(W_\rho, B)$, and since $\text{NilAugAlg}(W_\rho, f)$ is an isomorphism, then there is a unique map $h: W_\rho \to A$ such that

$$W_\rho \xrightarrow{h} A \xrightarrow{f} B.$$ 

commutes. This shows that $f$ is surjective on elements. But this means that $f$ is an isomorphism in $\text{Vect}$.

Thus $f$ is an isomorphism in $\text{NilAugAlg}$. Since $\text{NilAugAlg}$ has all equalisers, then the set $\{W_n \mid n \in \mathbb{N}\}$ forms a strong generator for $\text{NilAugAlg}$.

In particular, since each $W_n \in \text{Weil}$, this then says that the inclusion $I: \text{Weil} \hookrightarrow \text{NilAugAlg}$ preserves and reflects any existing (finite) limits.

**3.14. Proposition.** For an arbitrary $A \in \text{Weil}$, the functor $A \otimes -: \text{Weil} \to \text{Weil}$ preserves finite connected limits.

**Proof.** Consider the diagram

$$\begin{array}{ccc}
\text{Weil} & \xrightarrow{A \otimes -} & \text{Weil} \\
\downarrow & & \downarrow \\
\text{NilAugAlg} & \xrightarrow{A \otimes -} & \text{NilAugAlg} \\
\downarrow & & \downarrow \\
\text{AugAlg} & \xrightarrow{A \otimes -} & \text{AugAlg}.
\end{array}$$

The inclusions all preserve and reflect (finite) limits, and $A \otimes -: \text{AugAlg} \to \text{AugAlg}$ preserves connected limits. $\blacksquare$
3.15. Proposition. The category \textbf{Weil} has all finite coproducts, and moreover, coproduct is given by $\otimes$.

Proof. (Finite) coproducts in \textbf{AugAlg} are given by $\otimes$, and since \textbf{Weil} is a full subcategory of \textbf{AugAlg}, it remains only to show that \textbf{Weil} is closed under (finite) $\otimes$.

Further, as $k$ is a zero object, then it is the nullary coproduct. Now, since Weil algebras are finitely dimensional, then any finite coproduct of them must also be finitely dimensional. The nilpotency of the augmentation ideal is immediate.

3.16. Lemma. Let $A$ and $B$ be Weil algebras with presentations

$$
A = k[a_1, \ldots, a_m]/Q_A \\
B = k[b_1, \ldots, b_n]/Q_B.
$$

Then:

- The product $A \times B$ has presentation

$$
A \times B = k[a_1, \ldots, a_m, b_1, \ldots, b_n]/Q_A \cup Q_B \cup \{a_ib_j \mid \forall i, j\};
$$

- The coproduct $A \otimes B$ has presentation

$$
A \otimes B = k[a_1, \ldots, a_m, b_1, \ldots, b_n]/Q_A \cup Q_B.
$$

Proof. The proof is immediate.

Finally, let us define $W$ to be the Weil algebra $k[x]/x^2$. Then, the $n^{th}$ power and copower of $W$, denoted $W^n$ and $nW$ respectively, have presentations

$$
W^n = k[x_1, \ldots, x_n]/\{x_ix_j \mid \forall i \leq j\}
$$

$$
nW = k[x_1, \ldots, x_n]/\{x_i^2 \mid \forall i\}.
$$

3.17. Definition. For Weil algebras $A$, $B$ and $C$, the pullback

$$
\begin{array}{ccc}
A \otimes (B \times C) & \xrightarrow{A \otimes \pi_B} & A \otimes B \\
\downarrow^{A \otimes \pi_C} & & \downarrow^{A \otimes \varepsilon_B} \\
A \otimes C & \xrightarrow{A \otimes \varepsilon_C} & A
\end{array}
$$

is a foundational pullback.
3.18. **Remark.** Foundational pullbacks are a direct application of Proposition 3.14 to Proposition 3.9, with products regarded as pullbacks over the zero object \( k \).

The facts established above assume \( k \) is a field. However, we are more interested in \( k = \mathbb{N}, \mathbb{Z} \) and \( 2 \) (which, again, we define in Definition 7.1). The notion of Weil algebras in this slightly higher level of generality then becomes somewhat muddled. However, the subcategory \( \text{Weil}_1 \) of \( \text{Weil} \) we will use in our discussion will always consist of objects having a presentation of the form

\[
k[x_1, \ldots, x_n]/\{c_i c_j \mid \forall c_i \sim c_j\},
\]

for a symmetric, reflexive relation \( \sim \) (although not all such presentations will yield an object of \( \text{Weil}_1 \)), and we will still refer to these as Weil algebras. In particular, such Weil algebras all have finitely generated and free underlying \( k \)-modules.

In particular, Proposition 3.14 still holds when restricting to \( \text{Weil}_1 \) for these more general \( k \) using the same arguments.

As we stated at the beginning of this section, the more general \( k \) is needed in order to make our comparison with the definitions of [Rosický, 1984] and [Cockett and Cruttwell, 2014]. To reiterate, taking \( k = \mathbb{Z} \) (as a ring) will ultimately return the abelian group bundles of [Rosický, 1984]. However, we are more interested in taking \( k = \mathbb{N} \) to ultimately obtain the commutative monoid bundles of [Cockett and Cruttwell, 2014]. We will also consider \( k = 2 \) (Definition 7.1), as this shall provide a convenient tool for our calculations.

4. **Tangent Structure and Weil algebras**

The tangent functor \( T \) is closely related to the Weil algebra \( W = k[x]/x^2 \). For instance, the tangent functor in synthetic differential geometry (see [Kock, 2006]) is the representable functor \( (\_)^D \), where \( D = \text{Spec}(W) \).

Here, we will begin to describe a different relationship between \( \text{Weil} \) and tangent structure. Regard coproduct \( \otimes \) as a monoidal operation on \( \text{Weil} \) (with unit \( k \)).

4.1. **Proposition.** The (endo)functor

\[
W \otimes \_ : \text{Weil} \to \text{Weil}
\]

*can be used to define a Tangent Structure on \( \text{Weil} \).*

**Proof.** With \( T = W \otimes \_ \), we first give the natural transformations required in order to have a tangent structure on \( \text{Weil} \). The names for the morphisms used below will be deliberately chosen to coincide with those of tangent structure.
Natural transformation | Explanation
--- | ---
Projection | \( \varepsilon_W \_ : T \Rightarrow \text{id}_{\text{Weil}} \)
Addition | \( + \_ : T^{(2)} \Rightarrow T \)
Unit | \( \eta_W \_ : \text{id}_{\text{Weil}} \Rightarrow T \)
Vertical lift | \( l \_ : T \Rightarrow T^2 \)
Canonical flip | \( c \_ : T^2 \Rightarrow T^2 \)

\( \varepsilon_W : W \rightarrow k \) is the augmentation for \( W \)
\( T^{(2)} \) is the functor \( W^2 \_ \)
\( + : W^2 \rightarrow W; \ x_1, x_2 \mapsto x \)
\( \eta_W : k \rightarrow W \) is the (multiplicative) unit for \( W \)
\( T^2 = T \circ T \) is the functor \( 2W \_ \)
\( l : W \rightarrow 2W; \ x \mapsto x_1x_2 \)
\( c : 2W \rightarrow 2W; \ x_i \mapsto x_{3-i}, \) for \( i = 1, 2 \)

With these choices of natural transformations as well as the facts established in Section 3 (so that \( (W \_)^n = (nW \_ \) preserves the required pullbacks), it is a very routine exercise to verify that this does in fact define a Tangent Structure on \( \text{Weil}. \)

We will also note that the following
\[
W^2 \xrightarrow{(W \_ +) \circ (l \times_W (\eta_W \_ W))} 2W \xrightarrow{W \_ \varepsilon_W \circ \eta_W \_ \varepsilon_W} W
\]
is an equaliser in \( \text{Weil} \) (the universality of vertical lift equaliser in Definition 2.6).

Note that the map \( (W \_ +) \circ (l \times_W (\eta_W \_ W)) \), which we will denote as \( v \), is given as
\[
k[x_1, x_2]/x_1^2, x_2, x_1x_2 \rightarrow k[y_1, y_2]/y_1^2, y_2^2
\]
\[
x_1 \mapsto y_1y_2
\]
\[
x_2 \mapsto y_2.
\]

The map \( W \_ \varepsilon_W : k[y_1, y_2]/y_1^2, y_2^2 \rightarrow k[z]/z^2 \) sends \( y_1 \) to \( z \) and \( y_2 \) to 0, and \( \eta_W \_ \varepsilon_W : k[y_1, y_2]/y_1^2, y_2^2 \rightarrow k[z]/z^2 \) sends both \( y_1 \) and \( y_2 \) to 0.

This Tangent Structure on \( \text{Weil} \) relies on the object \( W \), its (finite product) powers \( W^n \) and tensors of these. With this in mind, it makes sense to give the following definition:

4.2. Definition. Let \( \text{Weil}_1 \) be the category consisting of:

1. Objects: The closure of the set \( \{W^n \mid n \in \mathbb{N}\} \) under finite \( \otimes \).
2. Morphisms: All algebra homomorphisms compatible with units and augmentations.

4.3. Remark. This definition is valid for \( k = \mathbb{N}, \mathbb{Z} \) or 2 as well.

4.4. Remark. If we wish to use a particular \( k \) when discussing \( \text{Weil}_1 \), we shall use “\( k \)-” as a prefix, e.g. \( \mathbb{N} \text{-Weil}_1 \).

Recall that as a consequence of Lemma 3.16, the (finite product) power \( W^n \) would have presentation
\[
k[x_1, \ldots, x_n]/\{x_ix_j \mid \forall i \leq j \}.
\]
and that the presentation for a tensor $A \otimes B$ took a particular form. As such, a tensor $\bigotimes_{i=1}^{m} W^{n_i}$ of powers of $W$ would have a certain presentation that we will not try to describe explicitly right now (we shall see this in 6).

In general, however, such objects will have a presentation

$$k[x_1, \ldots, x_n]/\{x_i x_j | \forall x_i \sim x_j\}$$

for some symmetric, reflexive relation $\sim$ (although not all symmetric, reflexive relations will yield an object of $\text{Weil}_1$). Since we will always require $x_i^2 = 0$ in these presentations, there is no loss of information if we omit the corresponding relation $x_i \sim x_i$ and take $\sim$ to merely be symmetric (and in fact, anti-reflexive).

However, such symmetric relations can be thought of as graphs.

4.5. Remark. We treat the relations as anti-reflexive so that the corresponding graph will not have loops.

5. Graphs

We begin by defining some basic concepts relating to graphs that we will need to use. These are all, for the most part, standard definitions that can be found in any introductory graph theory textbook (for example, see [Bondy and Murty, 1991]). The notation, however, seems to vary depending on the text.

5.1. Definition. A graph $G$ is a pair of sets $(V, E)$, with $V$ a finite set of “vertices” of $G$, and $E$ a set of unordered pairs of distinct vertices, called the “edges” of $G$.

5.2. Example. $G = \left( \{1, 2, 3, 4, 5, 6\}, \{(1, 2), (1, 3), (1, 6), (2, 3), (4, 5)\} \right)$ is the graph

5.3. Remark. In more formal graph theory terms, we are actually describing simple (undirected edges, no loops and at most one edge between any pair of vertices) finite graphs.

5.4. Definition. For graphs $G = (V, E)$ and $G' = (V', E')$, a graph homomorphism $h : G \rightarrow G'$ is a function $h : V \rightarrow V'$ such that for distinct $u, v \in V$,

$$(u, v) \in E \Rightarrow (f(u), f(v)) \in E' \text{ or } f(u) = f(v).$$
5.5. Definition. Let $\text{Gph}$ be the category of graphs and graph homomorphisms.

5.6. Definition. For a non-empty graph $G = (V, E)$, we will say it is connected if for any two distinct vertices $u$ and $v$, there exist vertices $v_1, \ldots, v_s \in V$ with $(v_i, v_{i+1}) \in E$ for each $i$, with $v_1 = u$ and $v_s = v$.

5.7. Definition. Given a graph $G = (V, E)$, the complement of $G$ is the graph $G^c = (V, E^c)$, where for any two distinct $u, v \in V$,

$$(u, v) \in E \iff (u, v) \notin E^c.$$  

We now define two important binary operations on graphs. Let graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be given.

5.8. Definition. The disjoint union of $G_1$ and $G_2$, denoted as $G_1 \otimes G_2$, is the graph

$$G_1 \otimes G_2 = (V_1 \sqcup V_2, E_1 \sqcup E_2) ;$$

where $\sqcup$ denotes disjoint union.

Or, put simply, it is the graph given by simply placing $G_1$ adjacent to $G_2$ without adding or removing any edges.

5.9. Definition. The graph join of $G_1$ and $G_2$, denoted $G_1 \times G_2$, is the graph

$$G_1 \times G_2 = (V_1 \sqcup V_2, \tilde{E})$$

where $\tilde{E} = E_1 \sqcup E_2 \sqcup (V_1 \times V_2)$.

Or, put simply, it is the graph given by taking $G_1 \otimes G_2$, then adding in an edge from each vertex in $G_1$ to each vertex in $G_2$. Equivalently, it can be defined as

$$G_1 \times G_2 = (G_1^c \otimes G_2^c)^c$$

5.10. Remark. The notation $G_1 \times G_2$ is in no way intended to suggest the product of $G_1$ and $G_2$ in the category $\text{Gph}$ of graphs.

5.11. Remark. The use of $\otimes$ and $\times$ to denote the operations of disjoint union and graph join respectively do not coincide with the notation used in graph theory. Graph union is often denoted as $G_1 \cup G_2$ or $G_1 + G_2$. Further, the graph join, sometimes called “graph sum”, is denoted $G_1 \lor G_2$, (to add to the confusion, some texts denote this as $G_1 + G_2$; moreover the meaning of “graph sum” can also vary depending on the literature). However, the notation $\{\otimes, \times\}$ was chosen in place of $\{\cup, \lor\}$ to correspond with the notation for coproduct and product of Weil algebras.

5.12. Definition. A graph $G$ is said to be complete if every pair $(u, v)$ of distinct vertices has an edge joining them (i.e. $(u, v) \in E$ for all $u \neq v$).

Equivalently, $G$ is the graph join of an appropriate number of instances of the single point graph.
5.13. Definition. A graph $G$ is said to be discrete if the edge set $E$ is empty. Equivalently, $G$ is the disjoint union of an appropriate number of instances of the single point graph.

Equivalently again, $G$ is discrete iff its complement $G^c$ is complete.

5.14. Remark. In graph theory literature, sometimes discrete graphs are also called “edgeless graphs” or “null graphs”.

5.15. Definition. We will give an iterative definition of cograph (complement-reducible graph) as follows:

1. The empty graph (empty vertex set) and one point graph are cographs.

2. If $G_1$ and $G_2$ are cographs, so are $G_1 \times G_2$ and $G_1 \otimes G_2$.

5.16. Remark. Cographs are not in any way a dual notion to graphs. The prefix “co-” is an abbreviation of “complement reducible”.

In fact, cographs have been studied extensively by graph theorists, and there are various equivalent characterisations of them (for instance, see [Corneil et al., 1981]).

5.17. Remark. For example, given a graph $G$, the following are equivalent:

1. $G$ is a cograph;

2. $G$ does not contain the graph $P_4$ (the path graph with four vertices) as a full subgraph (We shall not define $P_4$ explicitly, but instead simply note that the definition can be found in any introductory graph theory text).

6. Graphs and Weil algebras

In 4, we defined the category $\text{Weil}_1$ (Definition 4.2, and noted that each object of this category can be regarded as a graph.

Let us formalise this by first giving the following definition:

6.1. Definition. The functor $\kappa: \text{Gph} \to \text{Weil}$ is defined as follows:

1. On objects: For a graph $G = (V, E)$, $\kappa(G)$ is the Weil algebra $k[v_1, \ldots, v_m]/Q_E$, where $V = \{v_1, \ldots, v_m\}$, $v_i^2 \in Q_E$ for all $i$ and for $i \neq j$, $v_iv_j \in Q_E \Leftrightarrow (v_i, v_j) \in E$.

2. On morphisms: For a graph homomorphism $h: G \to G'$, $\kappa h: \kappa(G) \to \kappa(G')$ is given as

   $$(\kappa h)(v_i) = h(v_i)$$

   for all $i$;

where we use the underlying function $h: V \to V'$ on the vertex sets.
6.2. Remark. We shall leave as an exercise to the reader to verify that $\kappa h$ is indeed a valid morphism of Weil algebras, and that this definition of $\kappa$ is functorial, i.e. that it preserves identities and composition.

Conversely, we have the following:

6.3. Definition. Given a Weil algebra $X$ with presentation of the form

$$X = k[x_1, \ldots, x_n]/\{x_i x_j \mid \forall x_1 \sim x_j\},$$

let $\Gamma_X$ denote the graph induced by $\sim$; namely the graph with vertices the generators $x_1, \ldots, x_n$ and an edge between $x_i$ and $x_j$ (for $i \neq j$) whenever $x_i \sim x_j$.

6.4. Remark. With this convention, for a Weil algebra $X$ with presentation as described above, it is easy to see that $\kappa(\Gamma_X) = X$, and for a graph $G$, we have $\Gamma_{\kappa(G)} = G$.

For example, we have

<table>
<thead>
<tr>
<th>Weil algebra</th>
<th>Presentation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$k[\ ]$</td>
<td></td>
</tr>
<tr>
<td>$W$</td>
<td>$k[x]/x^2$</td>
<td>1</td>
</tr>
<tr>
<td>$2W$</td>
<td>$k[x_1, x_2]/x_1^2, x_2^2$</td>
<td>1 2</td>
</tr>
<tr>
<td>$W^2$</td>
<td>$k[x_1, x_2]/x_1^2, x_2^2, x_1 x_2$</td>
<td>1 2</td>
</tr>
<tr>
<td>$3W$</td>
<td>$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2$</td>
<td>2 3</td>
</tr>
<tr>
<td>$W^2 \otimes W$</td>
<td>$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2, x_1 x_2$</td>
<td>2 3</td>
</tr>
<tr>
<td>$W \times 2W$</td>
<td>$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2, x_1 x_2, x_1 x_3$</td>
<td>2 3</td>
</tr>
<tr>
<td>$W^3$</td>
<td>$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2, x_1 x_2, x_1 x_3, x_2 x_3$</td>
<td>2 3</td>
</tr>
</tbody>
</table>
6.5. **Remark.** The object $W \times 2W$ is not contained in the category $\text{Weil}_1$, but we shall include it in the table anyway.

6.6. **Proposition.** For graphs $G$ and $G'$, we have:

1. $\kappa(G) \otimes \kappa(G) = \kappa(G \otimes G')$;
2. $\kappa(G) \times \kappa(G) = \kappa(G \times G')$.

**Proof.** This is a direct consequence of Lemma 3.16, Definition 5.8 and Definition 5.9. □

To require precisely those Weil algebras given as the closure of $\{W^n\}_{n \in \mathbb{N}}$ under $\otimes$ is thus to ask for those that correspond to disjoint unions of complete graphs.

6.7. **Definition.** We shall refer to such graphs as being piecewise complete (p.c. graphs). Note that p.c. graphs are a subset of the cographs (as defined in Definition 5.15).

6.8. **Remark.** Although we are in this chapter interested in p.c. graphs, we shall often speak in greater generality by discussing cographs.

Now that we have a description for the objects of our subcategory $\text{Weil}_1$, we may now revisit the idea mentioned towards the end of Section 3; namely that $k$ need not be a field and give formal discussion of the morphisms.

7. The morphisms of $\text{Weil}_1$

We introduced the category $\text{Weil}_1$ in Definition 4.2, and noted that it was worded in such a way that $k$ need not be a field. We noted towards the end of Section 3 (and at a few other points) that we have a particular interest in the case where $k = \mathbb{N}$.

However, we shall for now take $k = 2$ as a tool to help deal with our immediate calculations.

7.1. **Definition.** Let 2 be the rig $\{0, 1\}$, with the usual multiplication, and addition given by max; in particular $1 + 1 = 1$.

We shall begin by showing that using the maps $\{\varepsilon_W, +, \eta, l, c\}$, composition, $\otimes$ and the universal property of foundational pullbacks (as given in Definition 3.17), we can construct (in some appropriate sense) any map of $2\text{-Weil}_1$.

7.2. **Remark.** We will not need the universal property of $\otimes$ (the coproduct), but rather we shall consider $2\text{-Weil}_1$ as a monoidal category with respect to $\otimes$ (with $k$ as the unit).

We will need some extra constructions of graphs before we begin.

7.3. **Definition.** A clique $U$ of $G$ is a (possibly empty) subset of $V$ for which any two distinct vertices in $U$ have an edge between them (or equivalently, the full subgraph of $G$ induced by $U$ is complete).
7.4. Definition. Conversely, an independent set $U$ of $G$ is a (possibly empty) subset of $V$ for which no two distinct vertices in $U$ have an edge between them (or equivalently, the full subgraph of $G$ induced by $U$ is discrete).

7.5. Remark. Given a graph $G$, an independent set $U$ of $G$ is also a clique of $G^c$.

We can actually use these notions of cliques and independent sets to form new graphs from existing ones.

7.6. Definition. Given a graph $G = (V, E)$, define $\text{Ind}(G)$ to be the graph given by:

1. Vertices: the independent sets of $G$;
2. Edges: given any two distinct independent sets $U_1$ and $U_2$ of $G$, there is an edge between them in $\text{Ind}(G)$ when there exist $x \in U_1$ and $y \in U_2$ such that either there is an edge between $x$ and $y$ in $G$ or $x = y$ (i.e. $U_1 \cap U_2 \neq \emptyset$).

7.7. Definition. Given a graph $G = (V, E)$, define $\text{Cl}(G)$ to be the graph given by:

1. Vertices: the cliques of $G$;
2. Edges: given any two distinct cliques $U_1$ and $U_2$ of $G$, there is an edge between them in $\text{Cl}(G)$ whenever their union $U_1 \cup U_2$ is also a clique of $G$ (note that there is no requirement for $U_1$ and $U_2$ to be disjoint).

7.8. Remark. In defining the graph $\text{Cl}(G)$, there is often the additional requirement that cliques $U_1$ and $U_2$ are disjoint for there to be an edge between them. If that were the case, then we would have

$$\text{Ind}(G) = (\text{Cl}(G^c))^c$$

7.9. Remark. As defined here, $\text{Cl}: \text{Gph} \to \text{Gph}$ is functorial and moreover can be made into a monad. We shall not be needing this fact, so we shall not prove it.

7.10. Definition. Given a graph $G = (V, E)$, define $\text{Ind}_+(G)$ to be the full subgraph of $\text{Ind}(G)$ where the vertices are the non-empty independent sets of $G$.

These tools now allow us to canonically express morphisms in $\textbf{2-Weil}_1$ in a pictorial manner. Recall that to define a map between (Weil) algebras, it suffices to define how the map acts on each of the generators. So, let a map $f: A \to B$ in $\textbf{2-Weil}_1$ be given, where $A$ and $B$ have presentations

$$A = 2[a_1, \ldots, a_m]/Q_A \quad \text{and} \quad B = 2[b_1, \ldots, b_n]/Q_B.$$ 

Then, for each generator $a_i$ of $A$, we can express $f(a_i)$ (uniquely) as a sum

$$f(a_i) = \sum_{b \in B} \alpha^{(i)}_{\frac{1}{2}}b;$$
the sum being across all non-zero monomials \( b \) of \( B \) in the generators \( \{ b_1, \ldots, b_n \} \), and \( \alpha^{(i)}_2 \in 2 \) is a constant (taking value 0 or 1).

In fact, since we are using a presentation for which \( \varepsilon_A(a_i) = 0 \) for all \( i \), the sum can in fact skip the trivial monomial (i.e. the constant).

We may also try to express \( f \) pictorially.

7.11. Example. Consider the map \( f: W \to 3W \) given by \( x \mapsto y_1y_2 + y_1y_3 \). We can represent this in graph form:

\[
\begin{align*}
1 & \quad 2 \\
2 & \quad 3
\end{align*}
\]

where each term of \( f(x) \) is represented by circling the vertices that generate the term (so the term \( y_1y_2 \) is represented by the ellipse encompassing the vertices 1 and 2). Note in particular that \( \{1, 2\} \) and \( \{1, 3\} \) are independent sets of \( \Gamma_{3W} \).

We also note that we label the vertices 1, 2 and 3 instead of \( y_1, y_2 \) and \( y_3 \) for convenience.

This suggests that we can express the map \( f \) using the language of graphs.

In the following discussion, we shall not necessarily restrict the discussion to only the p.c. graphs, but rather implicitly refer to all graphs.

7.12. Proposition. For the Weil algebra \( B = 2[b_1, \ldots, b_n]/Q_B \) with corresponding (p.c.) graph \( \Gamma_B \), the set of non-zero monomials \( b \) of \( B \) in the generators \( \{ b_1, \ldots, b_n \} \) are (canonically) in bijection with the independent sets of \( \Gamma_B \).

Proof. Since each generator \( b_i \) of \( B \) squares to zero, then each non-zero monomial \( b \) can be expressed (uniquely) as

\[
\prod_{i \in I} b_i ;
\]

for some appropriate subset \( I \subseteq \{ b_1, \ldots, b_n \} \). Since \( b \neq 0 \), then for distinct \( i, j \in I \), we must have \( b_ib_j \neq 0 \), i.e. \( b_ib_j \notin Q_B \). This equivalently means there is no edge between the vertices \( b_i \) and \( b_j \) in \( \Gamma_B \). \( I \) is thus a (possibly empty) independent set of \( \Gamma_B \).

The reverse direction for the bijection is then obvious. 

7.13. Remark. Using Proposition 7.12, we can equivalently say that to give a non-constant monomial \( b \) is to give a vertex of \( \text{Ind}_+(\Gamma_B) \).

As such, we may now express \( f(a_i) \) (uniquely) as

\[
f(a_i) = \sum_{U \in \text{Ind}_+(\Gamma_B)} \alpha^{(i)}_U b_U
\]
over the non-empty independent sets \( U \) of \( \Gamma_B \)

7.14. **Notation.** For a graph \( G \), let a *circle* \( U \) of \( G \) simply mean an independent set of \( G \), but regarded pictorially as some shape encompassing the relevant vertices.

We may use this idea to express \( f: A \to B \) pictorially: start by taking the generator \( a_1 \). Then take the graph \( \Gamma_B \) for \( B \), and for each \( U \) with \( \alpha_U^1 = 1 \), we add onto \( \Gamma_B \) a circle corresponding to \( U \), and we do this for all \( U \) with \( \alpha_U^1 = 1 \). Then repeat this process for each generator \( a_i \), but (say) using a different colour for each different generator.

7.15. **Example.** The map \( f: 2W \to 3W \) given by \( x_1 \mapsto y_1y_2 + y_2y_3 \) and \( x_2 \mapsto y_1 + y_1y_3 \) may be represented as

![Diagram](image)

where \( f(x_1) \) is represented in red and \( f(x_2) \) is represented in blue.

7.16. **Notation.** For a map \( f: A \to B \), let \( \{U\}_f \) denote the graph \( \Gamma_B \) together with a set \( \{(U,i) \mid \forall \alpha_U^i = 1 \} \), all of this regarded pictorially as a set of coloured circles on \( \Gamma_B \).

7.17. **Remark.** For a map \( f: W \to B \), we will simply refer to a circle \((U,i)\) of \( \{U\}_f \) as \( U \) (i.e. we omit the index \( i \)).

So, to any map \( f \) we can associate a graph with coloured circles. However, not all sets of circles on the graph \( \Gamma_B \) are permissible.

In order to investigate this idea further, we begin with the following:

7.18. **Proposition.** Consider maps of the form \( f: W \to B \). To give such an \( f \) is to give a clique of \( \text{Ind}_+(\Gamma_B) \).

**Proof.** Let \( x \) be the generator of \( W \). Recall from Proposition 7.12 that each summand (monomial) of \( f(x) \) is a (non-empty) independent set of \( \Gamma_B \), i.e. a vertex of \( \text{Ind}_+(\Gamma_B) \). We may thus regard \( f(x) \) as some subset \( X_f \) of the vertices of \( \text{Ind}_+(\Gamma_B) \).

Let distinct \( U_1, U_2 \in X_f \) be given (i.e. two distinct monomials of \( f(x) \)). Then, since \( x^2 = 0 \), either

1. \( U_1 \cap U_2 \neq \phi \) (so that they have a common vertex which becomes squared in the product \( b_{U_1}b_{U_2} \)), or
2. there exists \( b_i \in U_1 \) and \( b_j \in U_2 \) (with \( i \neq j \)) such that \( (b_ib_j) \) is an edge of \( \Gamma_B \).
In either case, each of the above conditions is equivalent to the independent sets $U_1$ and $U_2$ having an edge joining them in $\text{Ind}_+(\Gamma_B)$. $X$ is thus a clique of $\text{Ind}_+(\Gamma_B)$. In particular, $f(x)$ corresponds to a vertex of $\text{Cl}(\text{Ind}_+(\Gamma_B))$.

Conversely, given a clique $X$ of $\text{Ind}_+(\Gamma_B)$, there is the obvious polynomial $p_X(b_1, \ldots, b_n)$ corresponding to $X$, and it is routine to check that $f_X(x) = p_X(b_1, \ldots, b_n)$ defines a valid morphism $f_X: W \to B$.

7.19. Notation. For convenience, we shall let $\chi(\_)$ denote $\text{Cl}(\text{Ind}_+(\_))$.

We can take this one step further:

7.20. Proposition. To give a map $f: A \to B$ is to give a graph homomorphism $\tilde{f}: \Gamma_A \to \chi(\Gamma_B)$.

Proof. We know from Proposition 7.18 that each $f(a_i)$ corresponds to a vertex of $\chi(\Gamma_B)$ (we may view this as pre-composition with $\theta_i: W \to A$, with $\theta_i(x) = a_i$).

This gives us a function from the set $\{a_1, \ldots, a_m\}$ of vertices of $\Gamma_A$ to the set of vertices of $\chi(\Gamma_B)$. We now verify that this function yields a valid graph homomorphism.

Suppose $a_i$ and $a_j$ are two distinct vertices of $\Gamma_A$ with an edge joining them.

$$(a_i, a_j) \text{ is an edge of } \Gamma_A$$
$$\implies a_ia_j = 0 \text{ in } A$$
$$\implies f(a_i)f(a_j) = 0 \text{ in } B .$$

This tells us that if $b_i$ and $b_j$ are each a monomial from $f(a_i)$ and $f(a_j)$ respectively, then $b_ib_j = 0$. Using the same idea as the proof for Proposition 7.18, this says that there is an edge joining $b_i$ and $b_j$ in $\text{Ind}_+(\Gamma_B)$.

This is true for all such pairs of monomials, and so $f(a_i)$ and $f(a_j)$, viewed as cliques in $\text{Ind}_+(\Gamma_B)$, together (i.e. taking the union of the two cliques) give a clique. As such, when viewed as vertices of $\chi(\Gamma_B)$, there is an edge joining $f(a_i)$ and $f(a_j)$.

Thus $f: A \to B$ yields a unique graph homomorphism $\tilde{f}: \Gamma_A \to \chi(\Gamma_B)$.

The reverse direction is then obvious.

These ideas actually allow us to prove an interesting fact about $\chi$.

7.21. Proposition. $\chi$ defines an endofunctor on the category $\textbf{Gph}$, and moreover, $\chi$ is canonically a monad.

Proof. We first exhibit $\chi$ as an endofunctor. It is already well defined on objects. Let $G = (V, E)$ and $G' = (V', E')$ be arbitrary graphs and $h: G \to G'$ some chosen graph homomorphism.

Define $\chi(h): \chi(G) \to \chi(G')$ as follows:

1. For a vertex $v \in V$, regarded as the singleton clique of the singleton independent set (so that it is a vertex of $\chi(G)$), define $(\chi h)(v) = h(v)$ (where $h(v) \in V'$ is regarded as a vertex of $\chi(G')$ in the same way).
2. For a non-empty independent set $U$ of $G$ (hence a vertex of Ind$_+ (G)$), and thus a singleton clique) viewed as a vertex of $\chi (G)$, define $(\chi h)(U)$ as

$$
\begin{cases}
\bigcup_{v \in U} h(v) &; \text{if the function } h \text{ restricted to domain } U \text{ is injective,}

\text{The empty clique} &; \text{otherwise}
\end{cases}
$$

if $\bigcup_{v \in U} h(v)$ does indeed define an independent set of $G'$, we again regard it as a singleton clique of Ind$_+ (G')$, hence a vertex in $\chi (G')$.

3. For a clique $C$ of Ind$_+ (G)$, define $\chi (C)$ as the clique of Ind$_+ (G')$ consisting of all $(\chi h)(U)$ not the empty clique, for all (non-empty) independent sets $U \in C$.

We leave as an exercise to the reader to show that this will preserve identities and composition, so that $\chi$ is functorial.

To show $\chi$ is a monad, we first give the unit $\eta: 1_{\text{Gph}} \Rightarrow \chi$ by its components; $\eta_G: G \rightarrow \chi (G)$ sends each vertex $v \in V$ to the singleton clique of the singleton independent set $\{\{v\}\}$.

Using Proposition 7.20, it is easy to see that each $\eta_G: G \rightarrow \chi (G)$ corresponds to the identity $id_{\kappa (G)}: \kappa (G) \rightarrow \kappa (G)$.

The multiplication $\mu: \chi^2 \Rightarrow \chi$ has components $\mu_G: \chi^2 (G) \rightarrow \chi (G)$ given as follows:

Recall that

1. Vertices of $G$ correspond to generators of $\kappa (G)$ (Definition 6.1);

2. Non-empty independent sets $U$ of $G$ correspond to non-constant, non-zero monomials of $\kappa (G)$ (Proposition 7.12), and an edge in Ind$_+ (G)$ is equivalent to the corresponding monomials multiply to zero;

3. Cliques of such independent sets are polynomials squaring to zero (Proposition 7.18), and an edge in $\chi (G)$ means that the product of the two corresponding polynomials yields zero.

Using Definition 7.7 and Definition 7.10, we can then see that

1. A non-empty independent set of such a clique (i.e. a vertex of Ind$_+ (\chi (G))$) then corresponds to a set $X$ of polynomials for which the product of all polynomials in this set $X$ is not zero, or $X$ contains only the zero polynomial itself (taking the empty clique as a singleton). An edge between $X$ and $Y$ in this graph corresponds to there being polynomials $p \in X$ and $q \in Y$ such that $pq = 0$ in $\kappa (G)$;

2. A (possibly empty) clique of such an independent set (i.e. a vertex of $\chi^2 (G)$) is a family $\varrho$ of such sets of polynomials such that for any two distinct sets $X$ and $Y$ of this family, there are polynomials $p \in X$ and $q \in Y$ such that $pq = 0$ in $\kappa (G)$, and an edge between vertices $\varrho$ and $\sigma$ says that the union of the two families is also such a family.
Then, to give $\mu_G: \chi^2(G) \to \chi(G)$ is to associate each family of sets of polynomials to a polynomial squaring to zero. Let $\varrho$ be one such family. Let $X \in \varrho$, and suppose $X = \{p_1, \ldots, p_r\}$, where each $p_i$ is a polynomial of $\kappa(G)$ squaring to zero.

With this notation, we define $\mu_G(\varrho)$ to be the polynomial

$$\sum_{X \in \varrho} \left( \prod_{p_i \in X} p_i \right).$$

Explicitly, for each set $X \in \varrho$, multiply together all the polynomials in this set (recall that unless $X$ contains only the zero polynomial, then this product is non-zero). Then add up all such resultant polynomials across all $X \in \varrho$.

Now, each polynomial $p_i$ squares to zero, so each product

$$\prod_{p_i \in X} p_i$$

squares to zero. Since $\varrho$ is a clique of $\text{Ind}_+ \chi(G)$, then any two sets $X, Y \in \varrho$ therefore are joined by an edge. As such, there exists $p \in X$ and $q \in Y$ with $pq = 0$ in $\kappa(G)$. As such, the product

$$\left( \prod_{p_i \in X} p_i \right) \left( \prod_{q_j \in Y} q_j \right)$$

must be zero. This is true for all pairs $X, Y \in \varrho$.

Thus, the polynomial $\mu_G(\varrho)$ squares to zero (hence is a vertex of $\chi(G)$).

Finally, suppose $\sigma$ is another vertex such that $(\varrho, \sigma)$ is an edge of $\chi^2(G)$. This means that $\varrho \cup \sigma$ is another family. As such, $\mu_G(\varrho \cup \sigma)$ is well defined and moreover squares to zero. In particular, this means that $\mu_G(\varrho)\mu_G(\sigma) = 0$ in $\kappa(G)$.

Therefore there must be an edge between $\mu_G(\varrho)$ and $\mu_G(\sigma)$.

We shall leave verifying the axioms of the monad as an exercise for the reader. 

Since $\chi$ is a monad, we can then consider the Kleisli category $\text{Gph}_\chi$. Moreover, we can then define $\text{Gph}_\chi'$ as the full subcategory whose objects are precisely the p.c. graphs.

7.22. PROPOSITION. There exists an equivalence of categories

$$F: \text{Gph}_\chi' \to \text{2-Weil}_1.$$

PROOF. The functor $F$ is defined as:

1. On objects: $F(G) = \kappa(G)$

2. On morphisms: A map $h: G \to G'$ of $\text{Gph}_\chi'$ is a graph homomorphism $h': G \to \chi(G')$, and this corresponds to a unique map $h: \kappa(G) \to \kappa(G')$ of $\text{2-Weil}_1$ (Proposition 7.20). Thus, take $F(h) = \overline{h}$.

Using Proposition 7.20, $F$ is clearly full and faithful. Using Definition 6.1, Definition 6.3, and the fact that $\Gamma_{\kappa(G)} = G$, then $F$ is essentially surjective. 

8. Construction of maps

We shall show in this section that using the set \( \{ \varepsilon_W, +, \eta_W, l, c \} \) (as defined in Section 4), composition, \( \otimes \) and the universal property of foundational pullbacks (as given in Definition 3.17) of 2-\texttt{Weil}_1, we are able to “construct” (in some appropriate sense) any map \( f: A \to B \) of 2-\texttt{Weil}_1. We begin by expressing the maps \( \{ \varepsilon_W, +, \eta_W, l, c \} \) in the form \( \{ U \}_f \) in Table 1 below:

<table>
<thead>
<tr>
<th>Map</th>
<th>Action on Generators</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon_W: W \to 2 )</td>
<td>( x_1 \mapsto 0 )</td>
<td>( \square )</td>
</tr>
<tr>
<td>( \text{id}_W: W \to W )</td>
<td>( x \mapsto x )</td>
<td>( \circ )</td>
</tr>
<tr>
<td>( +: W^2 \to W )</td>
<td>( x_1 \mapsto x, x_2 \mapsto x )</td>
<td>( \square )</td>
</tr>
<tr>
<td>( \eta_W: 2 \to W )</td>
<td>( (2 \text{ has no generators}) )</td>
<td>( \square )</td>
</tr>
<tr>
<td>( l: W \to 2W )</td>
<td>( x \mapsto x_1x_2 )</td>
<td>( \circ )</td>
</tr>
<tr>
<td>( c: 2W \to 2W )</td>
<td>( x_1 \mapsto x_2, x_2 \mapsto x_1 )</td>
<td>( \square )</td>
</tr>
</tbody>
</table>

Table 1:

Pictorially, given \( \{ U \}_f \) for some map \( f: A \to B \), we can naively interpret ‘post-composition’ with the above maps as follows:

- \( \varepsilon_W \) corresponds to deleting a particular vertex in \( \Gamma_B \) as well as any circles that go through that vertex.
• + corresponds to taking two vertices in \( \Gamma_B \) joined by an edge and collapsing them to a single vertex. Circles that had contained either vertex (but not both) now contain the collapsed vertex instead.

• \( \eta_W \) corresponds to adding a new vertex to \( \Gamma_B \), but has no effect on any of the existing circles.

• \( l \) corresponds to taking a single vertex of \( \Gamma_B \) and splitting it into two vertices without an edge joining them, and any circle \( U \) that contained the original vertex now contain both of the new vertices.

• \( c \) corresponds to switching labels of (unjoined) vertices, and does nothing to the circles themselves.

These ideas will become clearer in subsequent discussion. We shall now precisely define what it means to say that a map \( f: A \to B \) is "constructible".

8.1. Definition. Let \( \Xi \) be a set given iteratively as follows:

1. The maps \( \varepsilon_W, +, \eta_W, l, c \) are contained in \( \Xi \).

2. \( \Xi \) contains all identities.

3. For all \( n \in \mathbb{N} \), each projection \( \pi_i: W^n \to W \) is contained in \( \Xi \).

4. If \( f: X \to Y \) and \( g: Y \to Z \) are both in \( \Xi \), then their composite \( g \circ f: X \to Z \) is also in \( \Xi \). Equivalently, \( \Xi \) is closed under composition.

5. If \( f: X \to Y \) and \( g: A \to B \) are both in \( \Xi \), then their tensor \( f \otimes g: X \otimes A \to Y \otimes B \) is also in \( \Xi \). Equivalently, \( \Xi \) is closed under tensor.

6. For a foundational pullback

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \to & D
\end{array}
\]

in \( 2\text{-Weil}_1 \), and for a commuting square

\[
\begin{array}{ccc}
X & \to & B \\
\downarrow^f & & \downarrow^g \\
C & \to & D
\end{array}
\]

with \( X \in 2\text{-Weil}_1 \) and \( f, g \in \Xi \), then the uniquely induced map \( h: X \to A \) is also in \( \Xi \).

8.2. Definition. For a map \( f: A \to B \) of \( 2\text{-Weil}_1 \), we shall say that \( f \) is constructible if \( f \in \Xi \).
8.3. Lemma. For any Weil algebra $A \in 2\text{-Weil}_1$, the unit $\eta_A$ and augmentation $\varepsilon_A$ are both constructible.

Proof. The lemma is by definition true for $A = W$. We then simply note that $\varepsilon_{W^n} = \varepsilon_W \circ \pi_i$ (for any $i$) and $\eta_{W^n}$ (induced using $\eta_W$ and product diagrams regarded as foundational pullbacks) are both constructible, and for $X, Y \in 2\text{-Weil}_1$ with $\eta_X, \eta_Y, \varepsilon_X, \varepsilon_Y$ constructible, then $\eta_{X \otimes Y} = \eta_X \otimes \eta_Y$, $\varepsilon_{X \otimes Y} = \varepsilon_X \otimes \varepsilon_Y$ are also constructible.

8.4. Corollary. Any zero map $z: A \to B$ is constructible, for all $A, B \in 2\text{-Weil}_1$.

Proof. For given $A, B \in 2\text{-Weil}_1$, the zero map $z: A \to B$ is the composite

$$A \xrightarrow{\varepsilon_A} k \xrightarrow{\eta_B} B.$$ 

8.5. Lemma. The only (non-trivial) products in $2\text{-Weil}_1$ are the product powers $W^n$.

Proof. For arbitrary $X, Y \in 2\text{-Weil}_1$, the graph $\Gamma_{X \times Y}$ for their product would need to be connected. The only connected p.c. graphs are the complete ones, and so $X \times Y = W^n$ for some $n$.

8.6. Lemma. For arbitrary $0 < n' < n$ in $\mathbb{N}$, all projections

$$\pi': W^n \to W^{n'}$$

are constructible.

Proof. Let $\pi': W^n \to W^{n'}$ be a given projection. Without loss of generality, suppose $\pi'$ preserves the first $n'$ generators of $W^n$. Since each product can be regarded as a foundational pullback (Definition 3.17), $\pi'$ is then constructed as $id_{W^{n'}} \times \varepsilon_{W^{n-n'}}$.

8.7. Corollary. Let

$$\begin{array}{ccc}
A \otimes W^{m+n} & \xrightarrow{A \otimes \pi_1} & A \otimes W^m \\
A \otimes \pi_2 & & A \otimes \varepsilon_{W^m} \\
A \otimes W^n & \xrightarrow{A \otimes \varepsilon_{W^n}} & A
\end{array}$$

be an arbitrary foundational pullback (recall from Lemma 8.5 that the only products in $2\text{-Weil}_1$ are product powers).

Then, each of the four maps in this pullback diagram are constructible.

Proof. This is an immediate consequence of Definition 8.1, Lemma 8.3 and Lemma 8.6.
Clearly, any map that is constructible by definition must live in \(2\text{-Weil}_1\). We shall now sequentially build up in a different manner the maps of \(\Xi\) and show that in fact all maps \(f: A \rightarrow B\) of \(2\text{-Weil}_1\) are constructible.

8.8. Lemma. Any map \(f: W \rightarrow nW\) with precisely one circle is constructible.

Let us begin with an example.

8.9. Example. The map \(f: W \rightarrow 5W\) given by \(x \mapsto x_1x_3x_4\) may be represented as

\[
\begin{array}{ccc}
2 & & 5 \\
1 & 3 & 4
\end{array}
\]

Define a map \(\tilde{f}\) as the composite

\[
W \xrightarrow{t} 2W \xrightarrow{W \otimes t} 3W
\]

\[
x \mapsto x_1x_2 \mapsto x_1x_2x_3 .
\]

Clearly \(\tilde{f}\) is constructible.

Then \(\{U\}_f\) is

\[
\begin{array}{ccc}
2 & & \\
1 & 3
\end{array}
\]

i.e. the single circle includes all 3 vertices.

Now define a map \(g\) as the map

\[
W \otimes \eta_W \otimes W \otimes \eta_W : 3W \rightarrow 5W
\]

\[
x_1 \mapsto y_1 \\
x_2 \mapsto y_3 \\
x_3 \mapsto y_4 .
\]

Clearly, \(g\) is constructible.

Then the composite \(g \circ \tilde{f}\) is precisely the original map \(f\). Thus \(f\) is constructible.

We generalise this idea to prove Lemma 8.8.
Proof. Let \( f : W \to nW \) with precisely one circle \( U \) be given. Let \( r = |U| \). Define \( \tilde{f} \) as the composite
\[
W \xrightarrow{t} 2W \xrightarrow{W \otimes I} \cdots \xrightarrow{(r-1)W \otimes I} rW
\]
Clearly, \( \tilde{f} \) is constructible.

In an analogous manner to Example 8.9, define a constructible map \( g : rW \to nW \) with \( g \circ \tilde{f} = f \). Thus \( f \) is constructible. \( \blacksquare \)

8.10. Lemma. All maps \( f : W \to nW \) are constructible.

Proof. If there are no circles in \( \{U\}_f \) (i.e. \( x \mapsto 0 \)), then the \( f \) is given by (say) the composite
\[
W \xrightarrow{\varepsilon} k \xrightarrow{\eta} W \xrightarrow{W \otimes \eta} \cdots \xrightarrow{(n-1)W \otimes \eta} nW
\]
i.e. the zero map, hence \( f \) is constructible.

If \( f \) has one circle, we apply Lemma 8.8.

If \( f \) has more than one circle, then we prove this by induction. Let \( S(m) \) be the statement “All maps \( f : W \to nW \) with \( m \) circles or fewer are constructible, for all \( n \in \mathbb{N} \).

We know \( S(1) \) is true. Suppose that \( S(r) \) is true for some \( r \in \mathbb{N} \).

Let a map \( f : W \to nW \) with precisely \( r+1 \) circles be given. Explicitly, this means that \( f(x) \) is a polynomial in the generators of \( nW \) (which we shall call \( y_1, \ldots, y_n \)) with precisely \( r+1 \) monomial summands.

Recall that for \( f \) to be a valid map, since the codomain is \( nW \) (or equivalently, the corresponding graph \( \Gamma_{nW} \) is discrete), then any two distinct summands of \( f(x) \) must have (at least) one generator \( y_i \) in common. Let \( t \) and \( t' \) be distinct summands, and without loss of generality, suppose \( y_n \) is a common generator.

Now define a map
\[
f' : W \to (n-1)W \otimes W^2
\]
where \( W^2 = 2[y_n, \tilde{y}_n]/y_n^2, \tilde{y}_n^2, y_n \tilde{y}_n \), with \( f'(x) \) having the same expression as \( f(x) \), except that the \( y_n \) in term \( t' \) is replaced with \( \tilde{y}_n \). It is a routine task to check that this is a valid map. Furthermore, the composite
\[
W \xrightarrow{f'} (n-1)W \otimes W^2 \xrightarrow{(n-1)W \otimes +} nW
\]
will return the original map \( f \). Clearly, the map \( (n-1)W \otimes + \) is constructible, so it suffices to show that \( f' \) is constructible.

But the codomain of \( f' \), \( (n-1)W \otimes W^2 \), is the pullback
\[
(n-1)W \otimes W^2 \xrightarrow{(n-1)W \otimes \pi_1} nW
\]
\[
(n-1)W \otimes \pi_2
\]
\[
nW \xrightarrow{(n-1)W \otimes \varepsilon_W} (n-1)W \]
and moreover, this is a foundational pullback.

Thus, to prove that \( f' \) is constructible, it suffices to prove that each of the composites

\[
((n - 1)W \otimes \pi_i) \circ f' : W \to nW; \ i \in \{1, 2\}
\]

is constructible. But each of these composites have a number of circles strictly fewer than \( r + 1 \). Since we assumed that \( S(r) \) was true, then both these composites are constructible, hence \( f \) is constructible.

Thus \( S(r + 1) \) is true.

As such, all maps \( f : W \to nW \) are constructible.

We can actually prove Lemma 8.10 more directly. Suppose we have an arbitrary map \( f : W \to nW \) with \( \{U_f\} \) given. For each \( i \in \{1, \ldots, n\} \), let \( m_i \) be the number of circles containing vertex \( i \) (or equivalently, the number of terms of \( f(x) \) containing the generator \( y_i \)). Then, in a similar manner as before, we can define a map

\[
f' : W \to W^{m_1} \otimes \cdots \otimes W^{m_n}
\]

in such a way that \((+_{m_1} \otimes \cdots \otimes +_{m_n}) \circ f' = f\). Here, since + is an associative and commutative operation, then \( +_m : W^m \to W \) is well defined, and \( +_0 \) is the nullary sum \( \eta_W \).

Clearly, the map \(+_{m_1} \otimes \cdots \otimes +_{m_n} \) is constructible. As for \( f' : W \to W^{m_1} \otimes \cdots \otimes W^{m_n} \), we have the following

8.11. Proposition. For each \( n \in \mathbb{N} \), the object \( W^{m_1} \otimes \cdots \otimes W^{m_n} \) is a limit of a (canonical) connected diagram of \( sW \)'s.

Proof. This is clearly true for \( n = 1 \) (as \( W^m \) is the \( m \)-fold pullback of \( \varepsilon_W \)).

Suppose this is true for \( n = r \) for some \( r \in \mathbb{N} \), i.e. for a given \( V = W^{m_1} \otimes \cdots \otimes W^{m_r} \), there is a (canonical) connected diagram

\[
\mathcal{D} \xrightarrow{D} \text{2-Weil}_1
\]

with \( Dd = s_dW \) for all \( d \in \mathcal{D} \) and \( \lim D = V \) (the limit calculated using iterations of foundational pullbacks).

Consider \( V \otimes W^{m_{r+1}} \) for some \( m_{r+1} \in \mathbb{N}_{>0} \). Let

\[
\mathcal{C} \xrightarrow{C} \text{2-Weil}_1
\]

be the \( m_{r+1} \)-fold connected diagram of \( \varepsilon_W \)'s (i.e. \( \lim C = W^{m_{r+1}} \)).

We know from Proposition 3.14 that \( V \otimes - \) preserves \( W^{m_{r+1}} \) as a limit, i.e.

\[
\lim \left( \mathcal{C} \xrightarrow{G} \text{2-Weil}_1 \xrightarrow{V \otimes} \text{2-Weil}_1 \right) = V \otimes W^{m_{r+1}}.
\]

For each \( c \in \mathcal{C} \), by again using Proposition 3.14 as well as the symmetry of \( \otimes \), we have

\[
\lim \left( \mathcal{D} \xrightarrow{H} \text{2-Weil}_1 \xrightarrow{\otimes Cc} \text{2-Weil}_1 \right) = V \otimes Cc,
\]

and note that each \( Dd \otimes Cc \) is of the form \( sW \) for some \( s \in \mathbb{N} \).
8.12. Lemma. Let an arbitrary object \( A = 2[a_1, \ldots, a_n]/Q_A \) of 2-\textbf{Weil}_1 be given. Then any map \( f: A \to W \) given as \( f(a_i) = x \) for some fixed \( i \) and \( f(a_j) = 0 \) for all \( j \neq i \) is constructible.

**Proof.** Since \( A \in 2\text{-}\textbf{Weil}_1 \), then \( \Gamma_A \) is a p.c. graph, or more generally, a cograph. We then show that \( f \) is constructible recursively as follows:

1) If \( \Gamma_A = \{ \bullet \} \) (the one point graph), then \( f \) is the identity and is thus constructible.

2) If \( \Gamma_A = G \otimes H \) with \( a_i \in H \), then \( f \) is the composite

\[
A = \kappa(G) \otimes \kappa(H) \xrightarrow{\varepsilon_{\kappa(G)} \otimes \kappa(H)} \kappa(H) \xrightarrow{f'} W ;
\]

for a unique map \( f' \), and it thus suffices to show that \( f' \) is constructible.

3) If \( \Gamma_A = G \times H \) with \( a_i \in H \), then \( f \) is the composite

\[
A = \kappa(G) \times \kappa(H) \xrightarrow{\pi_{\kappa(H)}} \kappa(H) \xrightarrow{f'} W ;
\]

for a unique map \( f' \), and it thus suffices to show that \( f' \) is constructible.

\( \square \)

8.13. Lemma. Every map \( f: A \to nW \) with no intersecting circles is constructible.

**Proof.** Let \( \mathcal{A} \) be the full subcategory of 2-\textbf{Weil}_1 consisting of all objects \( A \) with the property that any map \( A \to nW \) with no intersecting circles is constructible.

By Lemma 8.3, we have \( 2 \in \mathcal{A} \) (since 2 is a zero object, the only map to any \( A \) is the unit \( \eta_A \)), and by Lemma 8.10, we have \( W \in \mathcal{A} \).

For arbitrary \( m \in \mathbb{N}_{\geq 2} \), let an arbitrary map \( f: W^m \to nW \) with no intersecting circles be given. If \( f \) is the zero map, then by Corollary 8.4, it is constructible. Suppose then that \( a \) is a generator of \( W_n \) for which \( f(a) \neq 0 \). Let \( a' \) be any other generator of \( W^m \).

Now, since \( aa' = 0 \) by construction, then \( f(aa') = f(a)f(a') = 0 \).

But since the codomain of \( f \) is \( nW \) and \( f \) has no intersecting circles, then we must have \( f(a') = 0 \). This is true for all generators of \( W^m \) (other than \( a \), of course). But this means that \( f \) factors through the appropriate projection \( \pi: W^m \to W \) preserving \( a \) (the other map being one of the form described in Lemma 8.8), thus \( f \) is constructible. Thus \( W^m \in \mathcal{A} \) for all \( m \in \mathbb{N} \).

Now suppose that \( A_1 \) and \( A_2 \) are arbitrary objects of \( \mathcal{A} \). Let an arbitrary map \( f: A_1 \otimes A_2 \to nW \) with no intersecting circles is given. Then, with some appropriate post-composition with \( c \)'s, we can write \( f = f_1 \otimes f_2 \), for an appropriate pair \( f_1: A_1 \to rW \) and \( f_2: A_2 \to (n-r)W \) neither of which have intersecting circles. Thus \( f \) is constructible. Thus we have \( A_1 \otimes A_2 \in \mathcal{A} \).

Now, since \( \mathcal{A} \) is a full subcategory of 2-\textbf{Weil}_1 containing \( W^m \forall m \in \mathbb{N} \) and is closed under \( \otimes \), then \( \mathcal{A} \) is just 2-\textbf{Weil}_1 itself. Thus any map \( A \to nW \) with no intersecting circles is constructible.

\( \square \)
8.14. Lemma. Every map $f : A \to nW$ is constructible.

Proof. Let an arbitrary map $f : A \to nW$ be given. Using an analogous idea to that described in the proof of Lemma 8.10, we can construct a map

$$f' : A \to W^{m_1} \otimes \cdots \otimes W^{m_n}$$

as follows:

1) For each generator $a_i$ of $A$, take the polynomial $f(a_i)$ in the generators $z_1, \ldots, z_n$ of $nW$.

2) Let $m_j$ be the total number of terms across all the polynomials $f(a_1)$ containing $z_j$ for $j = 1, \ldots, n$.

3) Define the map $f' : A \to W^{m_1} \otimes \cdots \otimes W^{m_n}$ by specifying each $f'(a_i)$ to be $f(a_i)$, but in such a way that each generator of $W^{m_1} \otimes \cdots \otimes W^{m_n}$ is used exactly once (in a similar fashion to the proof for Lemma 8.10).

8.15. Example. Consider the map $f : 2W \to 3W$ given as

$$x_1 \mapsto y_1y_2 + y_1y_3$$
$$x_2 \mapsto y_2y_3.$$ 

Noting that each generator $y_i$ appears in exactly two monomials, then we have the map $f' : 2W \to W^2 \otimes W^2 \otimes W^2$ given as

$$x_1 \mapsto y_1y_2 + y_1'y_3$$
$$x_2 \mapsto y_2'y_3.$$ 

Then $f$ is the composite

$$A \xrightarrow{f'} W^{m_1} \otimes \cdots \otimes W^{m_n} \xrightarrow{+m_1 \otimes \cdots \otimes +m_n} nW,$$

and so it suffices to show $f'$ is constructible. But now, for each projection

$$\pi = \pi_{i_1} \otimes \cdots \otimes \pi_{i_n} : W^{m_1} \otimes \cdots \otimes W^{m_n} \to nW,$$

the composite $\pi \circ f' : A \to nW$ has no intersecting circles, and is thus constructible using Lemma 8.13, and we use a series of foundational pullbacks to recover $f'$.

Before we introduce Theorem 8.17, we shall also require the following lemma:

8.16. Lemma. Let $G$ be a cograph (recall that each p.c. graph is also a cograph) with at least one edge (and hence at least two vertices). Then $G$ can be expressed as $(G_1 \times G_2) \otimes H$, where $G_1$ and $G_2$ are non-empty cographs ($H$ may be empty).

Proof. Let $e$ be a chosen edge of $G$. Let $G'$ be the connected component of $G$ containing the edge $e$. Clearly, we can express $G$ as a disjoint union $G' \otimes H$ (with $H$ possibly empty).

Now, since $G'$ contains an edge, it cannot be the one point graph. Since it is connected, it cannot be expressed (non-trivially) as $G_1 \otimes G_2$. Since it is a cograph, then by Definition 5.15, it can be expressed non-trivially as $G_1 \times G_2$. 


We now have the following:

8.17. **Theorem.** *Every map* \( f : A \to B \) *in* \( \mathbf{2-Weil}_1 \) *is constructible.*

**Proof.** Consider the Weil algebra \( B \). If \( \Gamma_B \) has any edges then, using Lemma 8.16, it can be expressed (non-trivially) as \( (G_1 \times G_2) \otimes H \) (with \( H \) possibly being the empty graph). Correspondingly, \( B = (\kappa(G_1) \times \kappa(G_2)) \otimes \kappa(H) \) and we thus have the foundational pullback

\[
\begin{array}{ccc}
B & \xrightarrow{\pi_1 \otimes \kappa(H)} & \kappa(G_1) \otimes \kappa(H) \\
\downarrow{\pi_2 \otimes \kappa(H)} & & \downarrow{\varepsilon_1 \otimes \kappa(H)} \\
\kappa(G_2) \otimes \kappa(H) & \xrightarrow{\varepsilon_2 \otimes \kappa(H)} & \kappa(H)
\end{array}
\]

and so \( f : A \to B \) is uniquely induced by the pair \( (\pi_i \otimes \kappa(H)) \circ f; \ i = 1, 2 \). As such, it suffices to show that each of these is constructible.

Note now that the graphs \( G_i \otimes H \) for the codomains each have strictly fewer edges than \( \Gamma_B \). As such, we repeat this process until the codomains are all of the form \( nW \), then directly apply Lemma 8.14.

9. **Obtaining coefficients outside** \( \mathbf{2} \)

We gave Theorem 8.17, which said that every map \( f : A \to B \) is constructible. However, this was for the case of \( \mathbf{2-Weil}_1 \), and so we limit the permissible maps by restricting the coefficients to being either 0 or 1.

Consider \( \mathbf{k-Weil}_1 \) for an arbitrary rig \( k \). For each \( t \in k \), define \( \hat{g}_t : W \to W \) to be the map given as \( \hat{g}_t(x) = tx \). Note that \( \hat{g}_0 \) is the zero map and \( \hat{g}_1 \) is the identity \( \text{id}_W \).

Define \( \Xi_k \) in the same way as Definition 8.1, with the added condition that \( \hat{g}_t \) is contained in \( \Xi_k \) for all \( t \in k \). Define the notion of a *constructible* morphism in the same way.

9.1. **Proposition.** *Every map* \( g : A \to B \) *of* \( \mathbf{k-Weil}_1 \) *is constructible.*

**Proof.** *(Sketch)* Consider first (the analogue of) Lemma 8.8. Suppose we had a map \( f : W \to nW \) with \( f(x) \) given by a single monomial (with some arbitrary coefficient \( r \in k \)). Let \( f' : W \to nW \) be the map with \( f'(x) \) being the same monomial, but with coefficient one. Clearly, \( f' \) is constructible.

Then, the composite

\[
W \xrightarrow{\hat{g}_r} W \xrightarrow{f'} W
\]

yields \( f \), and so \( f \) is constructible.

From there, the proofs for (the analogues of) Lemma 8.10 through to Lemma 8.14 as well as Theorem 8.17 are identical.
Recall, however, that we are ultimately interested in $\mathbb{N}$-$\text{Weil}_1$. We begin with the following:

**9.2. Proposition.** Let $\psi^\dagger : \mathbb{N} \to \mathbb{2}$ be the rig morphism

$$
\psi^\dagger(n) = \begin{cases} 
0 & ; n = 0 \\
1 & ; \text{otherwise}
\end{cases}.
$$

The canonical functor

$$
\psi : \mathbb{N}$-\text{Weil}_1 \to \mathbb{2}$-\text{Weil}_1
$$
induced by the rig morphism above is bijective on objects and full.

Here, $\psi$ sends each object $\mathbb{N}[x_1, \ldots, x_r]/\mathbb{Q}$ of $\mathbb{N}$-$\text{Weil}_1$ to its counterpart $\mathbb{2}[x_1, \ldots, x_r]/\mathbb{Q}$ in $\mathbb{2}$-$\text{Weil}_1$. There is analogous action of $\psi$ on morphisms.

**Proof.** Bijectivity on objects follows immediately from the fact that Definition 4.2 defines the objects of $k$-$\text{Weil}_1$ independently from the choice of $k$.

For any morphism $f : A \to B$ of $\mathbb{2}$-$\text{Weil}_1$, there is a corresponding map $g : A \to B$ in $\mathbb{N}$-$\text{Weil}_1$ given by the same action on generators as $f$. Clearly, we then have $\psi g = f$.

Let us now work with $\mathbb{N}$-$\text{Weil}_1$. Let $\Xi$ be defined as in Definition 8.1 (i.e. we do not explicitly include in $\Xi$ the maps $\hat{g}_t$ for all $t \in \mathbb{N}$). We first have the following:

**9.3. Lemma.** For each $t \in \mathbb{N}$, the map $\hat{g}_t$ is constructible.

**Proof.** First we note again that $\hat{g}_0$ is the zero map and $\hat{g}_1$ is the identity, and thus both are constructible.

We shall show that all $\hat{g}_t$’s are constructible by induction. Let $S(t)$ be the statement “$\hat{g}_t$ is constructible”. We have established that $S(0)$ and $S(1)$ are true. Suppose $S(r)$ is true.

We then have

\[ W \xrightarrow{\hat{g}_r} W^2 \xrightarrow{\pi_1} W \]

\[ \xrightarrow{\pi_2} W \xrightarrow{\xi_W} \mathbb{2}, \]

so the map $h$ is constructible. Then, the map $\hat{g}_{r+1}$ is clearly the composite

\[ W \xrightarrow{h} W^2 \xrightarrow{\pi_1} W \xrightarrow{\pi_2} W \rightarrow W^2 \rightarrow W, \]

so that $\hat{g}_{r+1}$ is also constructible, so that $S(r+1)$ is true.
9.4. Remark. We may try to give a similar construction in \(2\text{-}\text{Weil}_1\), but note that for all \(t \in \mathbb{N}_{>0}\), we will have \(\hat{g}_t = id_W\), since \(1 + 1 = 1\) in \(2\).

With these “coefficient maps” \(\hat{g}_t\) being constructible along with Theorem 8.17, we now have the following:

9.5. Proposition. Every map \(g: A \to B\) of \(\mathbb{N}\text{-}\text{Weil}_1\) is constructible.

Proof. This is a direct consequence of Proposition 9.1 and Lemma 9.3.

10. Instructions for assembly

In Lemmas 8.10 and 8.14, there was an element of choice involved; namely given a map \(f: A \to nW\) (for the case of Lemma 8.14, say), the corresponding map \(f': A \to W^{m_1} \otimes \cdots \otimes W^{m_n}\) required a choice as to which circle would correspond to which projection. Ultimately, this choice is inconsequential as different choices are (up to isomorphism) equivalent.

However, for the purposes of what we wish to do, we will assume that for each \(f: A \to nW\), there is some pre-determined choice that has already been made regarding the corresponding map \(f'\).

This then implicitly equips each map \(f: A \to B\) of \(2\text{-}\text{Weil}_1\) (and hence \(\mathbb{N}\text{-}\text{Weil}_1\)) with a set of instructions for its construction.

11. The map \(\Omega\)

We will now describe a certain construction \(\Omega\), a map in \(\mathbb{N}\text{-}\text{Weil}_1\), which we shall require in order to prove Proposition 12.13 later.

Let \(s \in \mathbb{N}\) be given. For an arbitrary map \(g: B \to sW\), recall that \(g\) decomposes (as described in the proof of Lemma 8.14) as

\[
\begin{array}{c}
W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \\
\downarrow +_{\beta} \\
sW.
\end{array}
\]

By 10, the particular decomposition is fixed (i.e. \(g'\) is uniquely determined by \(g\)).

One way we can view this decomposition involves the slice category \(\mathbb{N}\text{-}\text{Weil}_1/sW\); the pair \((g, +_{\beta})\) (again, \(g'\) is uniquely determined by \(g\)) can be seen as an object of the arrow category \(\left(\mathbb{N}\text{-}\text{Weil}_1/sW\right)^2\). We shall now extend this to a functor

\[
\tau: \mathbb{N}\text{-}\text{Weil}_1/sW \to \left(\mathbb{N}\text{-}\text{Weil}_1/sW\right)^2
\]

whose composite with the domain functor \(d: \left(\mathbb{N}\text{-}\text{Weil}_1/sW\right)^2 \to \mathbb{N}\text{-}\text{Weil}_1/sW\) is the identity.
This amounts to giving, for each arrow 

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
\downarrow{\downarrow{sW}} & & \\
\end{array}
\]

of \(N\text{-Weil}_1/sW\), a morphism 

\[
\Omega: W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \rightarrow W^{\delta_1} \otimes \cdots \otimes W^{\delta_s},
\]

such that the diagram 

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h'} & & \downarrow{g'} \\
W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} & \xrightarrow{\Omega} & W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} \\
\downarrow{+\beta} & & \downarrow{+\beta} \\
\downarrow{sW} & & \\
\end{array}
\]

commutes, and satisfying the evident functoriality conditions.

11.1. Remark. We are, of course, taking \(h: A \rightarrow sW\) to decompose as 

\[
\begin{array}{ccc}
W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} & \xrightarrow{h'} & sW \\
\downarrow{+\delta} & & \\
A & \xrightarrow{h} & \\
\end{array}
\]

It now remains to specify \(\Omega\).

Since \(W^{\beta_1} \otimes \cdots \otimes W^{\beta_s}\) is a limit (Proposition 8.11), it then suffices to define each map \(\Omega_{(r_1,\ldots,r_s)}\) as shown below 

\[
\begin{array}{ccc}
W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} & \xrightarrow{1} & sW \\
\downarrow{\Omega_{(r_1,\ldots,r_s)}} & & \\
W^{\beta_1} \otimes \cdots \otimes W^{\beta_s} & \xrightarrow{(r_1,\ldots,r_s)=\pi_1 \otimes \cdots \otimes \pi_s} & \\
\end{array}
\]

where the \(\Omega_{(r_1,\ldots,r_s)}\) are suitably compatible.

But to give \(\Omega_{(r_1,\ldots,r_s)}\), it suffices to say where each generator of \(W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}\) is sent. Let \(y_1\) be a generator of \(W^{\delta_1}\) (without loss of generality, let \(\alpha_1 \geq 1\)). We shall refer to the generators of \(sW\) as \(z_1, \ldots, z_s\). Observe that \(+\beta(y_1) = z_1\).

Recall from 10 the construction of \(h': A \rightarrow W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}\). There is a unique circle \((U_1, a)\) for some generator \(a\) of \(A\) with \(y_1 \in U_1\) (and correspondingly, a unique circle \((U_1, a)\) of \(h\) as well with \(z_1 \in U_1\)). Recall also that \(h = g \circ f\). Let 

\[
h(a) = U_1 + U_2 + \ldots ,
\]
where each $U_i$ is a monomial in the generators $z_1, \ldots, z_s$. Similarly, let

$$f(a) = V_1 + V_2 + \ldots,$$

where each $V_i$ is a monomial in the generators $\{b_j\}$ of $B$.

Then (ignoring coefficients), since $g$ preserves addition and multiplication, we can express $(g \circ f)(a)$ as

$$(g \circ f)(a) = g(f(a))$$

$$= g(V_1 + V_2 + \ldots)$$

$$= g(V_1) + g(V_2) + \ldots$$

$$= \left[ \prod_{b_j \in V_1} g(b_j) \right] + \left[ \prod_{b_j \in V_2} g(b_j) \right] + \ldots.$$

But this needs to be equal to $h(a)$. In particular, $U_1$ must be somewhere in the expression for $(g \circ f)(a)$. Without loss of generality, suppose $U_1$ is contained in the first term

$$\prod_{b_j \in V_1} g(b_j).$$

Now, for each $b_j \in V_1$, we must be able to choose precisely one circle $Q_j$ in such a way that

$$\bigcup_{b_j \in V_1} Q_j = U_1$$

with the $Q_j$’s pairwise distinct. This is because for each $b_j \in V_1$, $g(b_j)$ is a polynomial in the generators $z_1, \ldots, z_n$. Then, if the product of these polynomials (which in turn is another polynomial) is to contain a particular monomial (namely $U_1$), then this monomial must have arisen as the product of one monomial from each of the factor polynomials.

Moreover, since $z_1 \in U_1$, then we also have $z_1 \in Q_j$ for a unique $j$. Take $j = 1$ so that $Q_1$ is one of the terms of the polynomial $g(b_1)$.

$\Rightarrow Q_1$ is a circle of $g$ corresponding to $b_1$

$\Rightarrow$ In $g': B \to W^{\beta_1} \otimes \cdots \otimes W^{\beta_n}$, $\exists!$ generator $v$ of $W^{\beta_1}$ corresponding to the circle $Q_1$

$\Rightarrow$ Define $\Omega_{(r_1, \ldots, r_s)}(y_1) = \begin{cases} z_1 & (r_1, \ldots, r_s) \text{ preserves } v \text{(in particular, } r_1 \text{ preserves } v) \\ 0 & \text{otherwise} \end{cases}$

and repeat for all generators of $W^{\beta_1} \otimes \cdots \otimes W^{\beta_n}$.

In particular, note that since $\Omega$ can only assign a generator from any $W^{\beta_i}$ to a generator of the corresponding $W^{\beta_i}$, then we have $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_s$, for appropriate maps $\Omega_i: W^{\beta_i} \to W^{\beta_i}$.

11.2. Remark. We shall note here that in full formality, we should use the label $\Omega_{f,g}$ (or something to this effect), but we shall not be doing this.
12. Back to Tangent Structure

We defined the category $k$-$\text{Weil}_1$ in Definition 4.2 and in Section 8, we defined the notion of a constructible morphism (Definition 8.2) and showed that any map of $2$-$\text{Weil}_1$ was constructible (Theorem 8.17). We then said in Proposition 9.5 that in fact any map of $\mathbb{N}$-$\text{Weil}_1$ was constructible, and moreover in 10 we noted that each map $g: A \rightarrow B$ was equipped with a set of instructions for its construction.

We shall conclude by linking these ideas about Weil algebras back to Tangent Structures in an explicit manner.

We wish to construct a functor

$$F: \mathbb{N}$\text{-}\text{Weil}_1 \rightarrow \text{End}(\mathcal{M})$$

with certain properties (which we shall specify in due course). However, we first need to establish some facts.

Suppose that a given category $\mathcal{M}$ is equipped with a Tangent Structure $\mathcal{T}$ (in the sense of Definition 2.6). Regard $\text{End}(\mathcal{M})$ as a monoidal category with respect to the operation of composition $\circ$, and with unit the identity functor $1_{\mathcal{M}}$.

12.1. Notation. To avoid confusion, when we want to regard composition as a monoidal operation in $\text{End}(\mathcal{M})$, we will use concatenation if the meaning is clear (otherwise we will explicitly use $\otimes$), and save $\circ$ for actual composition. For example, if we have natural transformations $\alpha: R \Rightarrow S$ and $\beta: S \rightarrow U$ in $\text{End}(\mathcal{M})$, then $\beta \circ \alpha$ denotes the composite

$$R \overset{\alpha}{\longrightarrow} S \overset{\beta}{\longrightarrow} U$$

whereas $\beta \alpha$ denotes the natural transformation

$$S \circ R \overset{\beta \alpha}{\longrightarrow} U \circ S .$$

12.2. Definition. Let

$$F_0: \text{ob}(\mathbb{N}$\text{-}\text{Weil}_1) \rightarrow \text{ob}(\text{End}(\mathcal{M}))$$

be the function given as $F_0(\mathbb{N}) = 1_\mathcal{M}$, $F_0(W^m) = T^{(m)}$ for all $m \in \mathbb{N}$, and then recursively, if $A, B \in \mathbb{N}$-$\text{Weil}_1$ with $F_0(A) = R$, $F_0(B) = S$, then $F_0(A \otimes B) = R \circ S$.

12.3. Proposition. For any foundational pullback

$$\begin{array}{ccc}
A \otimes (B \times C) & \overset{A \otimes \pi_B}{\longrightarrow} & A \otimes B \\
\downarrow A \otimes \pi_C & & \downarrow A \otimes \varepsilon_B \\
A \otimes C & \overset{A \otimes \varepsilon_C}{\longrightarrow} & A
\end{array}$$
in \( \mathbb{N} \)-Weil\(_1\) (recall from Lemma 8.5 that the only products in \( \mathbb{N} \)-Weil\(_1\) are the powers \( W^n \) of \( W \)), we have a corresponding pullback

\[
\begin{array}{c}
\xymatrix{
F_0(A \otimes (B \times C)) \ar[r] \ar[d] & F_0(A \otimes B) \ar[d] \\
F_0(A \otimes C) \ar[r] & F_0(A)
}\end{array}
\]

in \( \text{End}(\mathcal{M}) \), which we may also equivalently express as

\[
\begin{array}{c}
\xymatrix{
F_0(A)F_0(B \times C) \ar[r] \ar[d] & F_0(A)F_0(B) \ar[d] \\
F_0(A)F_0(C) \ar[r] & F_0(A)
}\end{array}
\]

We shall also refer to these as foundational pullbacks (in \( \text{End}(\mathcal{M}) \)).

**Proof.** The square in \( \text{End}(\mathcal{M}) \) being a pullback is a direct consequence of the axioms of \( T \).

12.4. **Definition.** Let \( \Psi \) be a collection of pairs \((f, \alpha)\), where \( f: X \rightarrow Y \) is a morphism in \( \mathbb{N} \)-Weil\(_1\) and \( \alpha : F_0(X) \Rightarrow F_0(Y) \) is a morphism in \( \text{End}(\mathcal{M}) \) (i.e. a natural transformation), given as follows:

- We begin with the following pairs:
  - Each element of \( \{\varepsilon_W, \eta_W, +, l, c\} \) is paired with its obvious counterpart \( \{p, +, \eta, l, c\} \).
  - For each object \( A \in \mathbb{N} \)-Weil\(_1\), the pair \((\text{id}_A, \text{id}_{F_0(A)})\).
  - For any given foundational pullback in \( \mathbb{N} \)-Weil\(_1\), each map in this pullback is paired with its obvious counterpart in the corresponding pullback in \( \text{End}(\mathcal{M}) \) (in the sense of Proposition 12.3).

This gives us a starting point for \( \Psi \). Recall from 10 that any map \( h: A \rightarrow B \) of \( \mathbb{N} \)-Weil\(_1\) is equipped with a (finite) sequential set of instructions for its construction. We then iteratively add to \( \Psi \) as follows:

- Let \( f, g \) and \( h \) be maps in \( \mathbb{N} \)-Weil\(_1\), and suppose we already have pairs \((f, \alpha), (g, \beta) \in \Psi \).
  - If the final step of the instructions of \( h \) was to obtain \( h \) as the composite \( g \circ f \), then we add to \( \Psi \) the pair \((h, \beta \circ \alpha)\). That is, we close \( \Psi \) under certain compositions.
  - If the final step of the instructions of \( h \) was to obtain \( h \) as the tensor \( g \otimes f \), then we add to \( \Psi \) the pair \((h, \beta \alpha)\). That is, we close \( \Psi \) under certain tensors.
• If the final step of the instructions of $h$ was to (uniquely) induce $h$ using $f$ and $g$ as

$$A \xrightarrow{h} B \xrightarrow{g} B_1 \xrightarrow{\downarrow} B_2 \xrightarrow{\downarrow} C;$$

where the pullback square is a foundational one, then we consider the diagram

$$\begin{array}{ccc}
F_0(A) & \xrightarrow{\alpha} & F_0(B) \\
\downarrow{\beta} & & \downarrow{\downarrow} \\
F_0(B_2) & \xrightarrow{} & F_0(C)
\end{array}$$

in $\text{End}(\mathcal{M})$ (where we use the foundational pullback in $\text{End}(\mathcal{M})$ corresponding to the one above).

If the exterior commutes, then by the universal property of the pullback, a unique map $\gamma: F_0(A) \to F_0(B)$ will be induced. In that case, add to $\Psi$ the pair $(h, \gamma)$.

If the exterior does not commute, then we will say that “$h$ does not have a pairing in $\Psi$”.

12.5. Definition. Let $\Phi$ be the collection of all maps $h: A \to B$ in $\mathbb{N-Weil}_1$ which do not have a pairing in $\Psi$.

12.6. Lemma. Each coefficient map $\hat{g}_t$ is paired with some (unique) natural transformation $\lambda_t$ in $\Psi$.

Proof. We know that $\lambda_0$ is the composite

$$T \xrightarrow{p} 1_{\mathcal{M}} \xrightarrow{\eta} T$$

(since this was how $\hat{g}_0$ was constructed) and $\lambda_1$ is the identity $id_T$ (since $\hat{g}_1$ was the identity).

Since each $\hat{g}_t$ is then constructed recursively as

$$\begin{array}{ccc}
W & \xrightarrow{\pi_1} & W^{t-1} \\
\downarrow{id_W} & & \downarrow{\pi_2} \\
W & \xrightarrow{+} & W^2 \xrightarrow{\pi_1} W
\end{array},$$

then each $\lambda_t$ is constructed recursively in the same way.
Clearly, $\Psi$ and $\Phi$ are mutually exclusive and exhaustive collections. For now, let us focus on $\Psi$.

12.7. Notation. When describing pairs in $\Psi$, if $f$ is a map in $\text{N-Weil}_1$, then we will use $\tilde{f}$ to denote the corresponding natural transformation in $\text{End}(\mathcal{M})$, i.e. we have $(f, \tilde{f}) \in \Psi$, as we shall see below.

We will now sequentially show that the pairings of $\Psi$ “preserve” (arbitrary) composition, i.e. if we have arbitrary pairings $(f, \tilde{f}), (g, \tilde{g}), (h, \tilde{h}) \in \Psi$ such that $h = g \circ f$ in $\text{N-Weil}_1$, then we have $\tilde{h} = \tilde{g} \circ \tilde{f}$ in $\text{End}(\mathcal{M})$.

Explicitly, suppose we have

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{h=gof} & & & & \\
\end{array}
$$

in $\text{N-Weil}_1$. We wish to show that

$$
\begin{array}{ccc}
F_0A & \xrightarrow{\tilde{f}} & F_0B & \xrightarrow{\tilde{g}} & F_0C \\
\downarrow{\tilde{h}} & & & & \\
\end{array}
$$

commutes in $\text{End}(\mathcal{M})$, for all $(f, \tilde{f}), (g, \tilde{g}), (h = g \circ f, \tilde{h}) \in \Psi$.

As with Section 8, we shall begin with the most basic case for “preservation” of composition by the pairings in $\Psi$, and then sequentially build our way up to the general case.

12.8. Proposition. For all $f : qW \to rW$ and $g : rW \to sW$, neither of which having intersecting circles, the diagram

$$
\begin{array}{ccc}
T^q & \xrightarrow{\tilde{f}} & T^r & \xrightarrow{\tilde{g}} & T^s \\
\downarrow{\tilde{h}} & & & & \\
\end{array}
$$

commutes in $\text{End}(\mathcal{M})$.

Proof. First, since $f$ and $g$ have no intersecting circles, then $h$ also has no intersecting circles.

Since $f$ has domain $qW$ and has no intersecting circles, then it can be expressed in the form (modulo some appropriate post-composition with $c'$s)

$$
f = f_1 \otimes \cdots \otimes f_q \otimes \eta_{qW} : W \otimes \cdots \otimes W \otimes k \to \xi_1 W \otimes \cdots \otimes \xi_q W \otimes q'W
$$

(and $f$ is constructed as such); where each $f_i$ is has a single circle and is either given as $\varepsilon_W$ if $\xi_i = 0$, or constructed as the composite

$$
W \xrightarrow{\hat{g}_{a_i}} W \xrightarrow{t} \cdots \xrightarrow{(\xi_{i-1})W \otimes l} \xi_i W
$$

(as described in the proof of Lemma 8.8), for an appropriate coefficient map $\hat{g}_{a_i}$. 
12.9. Remark. Note that we also have

\[
\left( \sum_{i=1}^{q} \xi_i \right) + q' = m.
\]

An analogous fact is true for \( g \) and \( h \). The natural transformations \( \tilde{f}, \tilde{g} \) and \( \tilde{h} \) are then constructed in a corresponding manner.

Now, it can be shown that for all \( c, d \in \mathbb{N} \), the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\lambda} & T \\
\downarrow{\lambda_{cd}} & & \downarrow{\lambda_dT} \\
T & \xrightarrow{I} & T^2
\end{array}
\]

commutes in \( \text{End}(M) \) (recall that each \( \lambda_t \) is paired with the coefficient map \( \hat{g}_t \) in \( \Psi \)). Together with the fact that the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{I} & T^2 \\
\downarrow{I} & & \downarrow{\mu} \\
T^2 & \xrightarrow{TT} & T^3
\end{array}
\]

commutes in \( \text{End}(M) \) (an axiom of \( \mathbb{T} \)), then we have \( \tilde{g} \circ \tilde{f} = \tilde{h} \).

Thus \( \tilde{h} = \tilde{g} \circ \tilde{f} \).

12.10. Proposition. For all \( f : A \to rW \) and \( g : rW \to sW \), neither of which having intersecting circles, the diagram

\[
\begin{array}{ccc}
F_0A & \xrightarrow{\tilde{f}} & T^r \\
\downarrow{\tilde{h}} & & \downarrow{\tilde{g}} \\
& & T^s
\end{array}
\]

commutes in \( \text{End}(M) \).

Proof. First, we note that if \( f \) and \( g \) do not have intersecting circles, then neither does \( h \).

Consider \( f : A \to rW \). Using the arguments from the proof of Lemma 8.13, since \( f \) has no intersecting circles, it must factor through some particular projection \( \pi : A \to qW \) of \( A \) (as the final step in its construction), and the same is true for \( h \).

Correspondingly, \( \alpha \) and \( \gamma \) both factor through the corresponding projection \( \pi : F_0A \to T^q \).

As such, it suffices to assume \( A = qW \), and so \( F_0A = T^q \). Then, we can apply Proposition 12.8 directly.

\[\therefore \, \tilde{h} = \tilde{g} \circ \tilde{f} \]
12.11. Proposition. For all arbitrary $f: A \rightarrow rW$, and $g: rW \rightarrow sW$ with no intersecting circles, the diagram

$$
\begin{array}{c}
F_0A \xrightarrow{\tilde{f}} T^r \xrightarrow{\tilde{g}} T^s \\
\downarrow \tilde{h} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua
in End(\(\mathcal{M}\)). The same is true for \(\pm \vartheta\).

Firstly, this means that for the map \(h\), there are precisely \(\delta_1\) circles (say \(U_1, \ldots, U_{\delta_1}\)) containing the generator \(z_1\). But given what we’ve established about \(g\), and noting that \(h = g \circ f\), then the \(z_1\) term in each of these \(U_i\) must arise as a result of the generator \(y_1\) (since \(\psi(z_1) = y_1\)). More explicitly, to each circle \(U_i\) of \(h\) containing \(z_1\) we can associate a unique circle of \(f\) containing \(y_1\).

Conversely, for each circle \(V_j\) of \(f\) containing \(y_1\), we have \(g(V_j) \neq 0\) (moreover, \(g(V_j)\) is a single circle) and \(z_1 \in g(V_j)\). Therefore the number of circles of \(f\) containing \(y_1\) (namely \(\vartheta_1\)) is the same as the number of circles of \(h\) containing \(z_1\) (namely \(\alpha_1\)). Thus, if \(\psi(z_i) = y_j\), then \(\delta_i = \vartheta_j\).

We then define a map

\[
\Lambda: W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} \to W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}
\]

induced using

\[
\begin{array}{ccc}
W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} & \xrightarrow{t} & \Pi \Lambda \xrightarrow{g} \Pi \delta_1 \otimes \cdots \otimes W^{\delta_s}
\end{array}
\]

where, for each fixed projection \(\pi = (a_1, \ldots, a_s)\), \(t\) is determined as follows:

- Consider \(\overline{\pi} \circ h': A \to sW\). If this is the zero map, then \(t\) is also the zero map.

- If not, this means that there is at least one circle \(U\) of \(h\) (and hence \(h'\)) with each of its generators preserved by \(\overline{\pi}\). Moreover, if \(h\) has multiple circles, then they must be disjoint and each corresponds to a different generator of \(A\) (see proof of Lemma 8.14).

Without loss of generality, assume there is only one such circle \(U\). Regard \(U\) as a subset of \(\{z_1, \ldots, z_s\}\). Then we know \(\psi(U)\) (the image of \(U\) under \(\psi\)) is the unique circle of \(f\) corresponding to \(U\). Choose \(t\) (in the unique way) so that this circle \(\psi(U)\) of \(f\) is preserved, but sends any \(y_j \notin \psi(U)\) to 0.

We shall also note that there is a corresponding natural transformation \(\widetilde{t}\) (i.e. \((t, \widetilde{t}) \in \Psi\), we shall not prove this).

Now, it is fairly routine (albeit tedious) to show that \(\Lambda\) is paired with some unique natural transformation \(\widetilde{\Lambda}\) in \(\Psi\) (i.e. that it exists). It can also be shown that the diagram

\[
\begin{array}{ccc}
W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} & \xrightarrow{\Lambda} & W^{\delta_1} \otimes \cdots \otimes W^{\delta_s}
\end{array}
\]

induced using

\[
\begin{array}{ccc}
W^{\vartheta_1} \otimes \cdots \otimes W^{\vartheta_r} & \xrightarrow{rW} & sW
\end{array}
\]

where, for each fixed projection \(\pi = (a_1, \ldots, a_s)\), \(t\) is determined as follows:
commutes in $\mathbb{N}$-Weil$_1$ (and that the corresponding diagram commutes in End($\mathcal{M}$)). We now have the following diagram

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.8]

\node (A) at (0,0) {$F_0A$};
\node (B) at (0,3) {$\cdots$};
\node (C) at (3,0) {$\cdots T(\delta_s)$};
\node (D) at (6,0) {$T^s$};
\node (E) at (0,6) {$T^r$};
\node (F) at (6,6) {$\cdots T(\delta_r)$};

\draw[->] (A) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (E) -- (F);
\draw[->] (E) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (E) -- (A);
\draw[->] (A) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (E) -- (F);
\draw[->] (F) -- (A);
\end{tikzpicture}
\end{array}
\]

and to show the exterior commutes, it suffices to show that

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.8]

\node (A) at (0,0) {$FA$};
\node (B) at (0,3) {$\cdots T(\delta_s)$};
\node (C) at (3,0) {$T(\delta_i)$};
\node (D) at (6,0) {$\cdots T(\delta_r)$};
\node (E) at (0,6) {$T(\delta_1)$};
\node (F) at (6,6) {$\cdots T(\delta_r)$};

\draw[->] (A) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (E) -- (F);
\draw[->] (E) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (E) -- (A);
\draw[->] (A) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (E) -- (F);
\draw[->] (F) -- (A);
\end{tikzpicture}
\end{array}
\]

commutes.

Since $T(\delta_1) \otimes \cdots \otimes T(\delta_s)$ is a limit with projections $(a_1, \ldots, a_s)$, and since $\Lambda$ (and hence $\tilde{\Lambda}$) was given using Diagram (1), then it suffices to show that

\[\overline{a} \circ \tilde{\Lambda} \circ \tilde{f}' = \overline{a} \circ \tilde{h}'\]

for all projections $\overline{a} = (a_1, \ldots, a_s)$.

Consider the diagram

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.8]

\node (A) at (0,0) {$T(\delta_1) \otimes \cdots T(\delta_i)$};
\node (B) at (3,0) {$T(\delta_1) \otimes \cdots T(\delta_s)$};
\node (C) at (0,3) {$T^r$};
\node (D) at (3,3) {$T^r$};
\node (E) at (0,6) {$T^r$};
\node (F) at (3,6) {$\cdots T(\delta_r)$};

\draw[->] (A) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (E) -- (F);
\draw[->] (E) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (E) -- (A);
\draw[->] (A) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\draw[->] (E) -- (F);
\draw[->] (F) -- (A);
\end{tikzpicture}
\end{array}
\]

(again, we leave as an exercise to the reader to verify that all the necessary pairs in $\Psi$ exist and are well defined).

To show the commutativity of the exterior, we first note that the lower left triangle and right square commute by construction, and further that it is routine to check that the top triangle commutes. So all that remains is to verify the commutativity of the innermost triangle.

But since $t \circ f'$ by definition has no intersecting circles, then we can apply Proposition 12.10 directly.

\[\therefore \tilde{h} = \tilde{g} \circ \tilde{f}\]
12.12. Proposition. For all arbitrary \( f: A \to B \), and \( g: B \to sW \) with no intersecting circles, the diagram

\[ F_0A \xrightarrow{\tilde{f}} F_0B \xrightarrow{\tilde{g}} T^s \xrightarrow{\tilde{h}} \]

commutes in \( \text{End}(\mathcal{M}) \).

Proof. Using Lemma 8.16 and Proposition 6.6, we can see that if the graph \( \Gamma_B \) contains any edges, then \( B \) is part of a (foundational) pullback

\[
B = B' \otimes (B_1 \times B_2) \xrightarrow{B' \otimes \pi_1} B' \otimes B_1 \xrightarrow{B' \otimes \varepsilon_{B_1}} B' \otimes B_2 \xrightarrow{B' \otimes \varepsilon_{B_2}} B' .
\]

Further, recall the proof of Lemma 8.13. Since \( g: B \to sW \) has no intersecting circles, then if \( \Gamma_B \) has any edges, we know that \( g \) must then factorise through one of \( B \)'s (foundational) projections, say as

\[
B \xrightarrow{B' \otimes \pi_1} B' \otimes B_1 \xrightarrow{\gamma} sW
\]

(and moreover, this would be in the instructions for its construction). The same is thus true of \( \tilde{g} \). We shall simply denote the projection as \( \pi_1 \) for convenience.

We now have

\[
F_0A \xrightarrow{\tilde{f}} F_0B \xrightarrow{\tilde{g}} T^s \xrightarrow{\tilde{h}}
\]

and note that the map \( \tilde{\pi}_1: F_0B \to F_0B'F_0B_1 \) is part of a foundational pullback in \( \text{End}(\mathcal{M}) \) (Proposition 12.3).

As such, this tells us that \( \tilde{\pi}_1 \circ \tilde{f} = \tilde{\pi}_1 \circ f \) (Definition 12.4). It thus suffices to show the commutativity of

\[
F_0A \xrightarrow{\tilde{f}} F_0B \xrightarrow{\tilde{g}} T^s .
\]

But we know that since \( g \) has no intersecting circles, then neither does \( \gamma \), and we can thus repeat this iteratively until there are no more edges in \( B \), i.e. we have \( B = rW \) and apply Proposition 12.11 directly. \( \blacksquare \)
12.13. Proposition. For all arbitrary \( f: A \to B \) and \( g: B \to sW \), the diagram

\[
\begin{array}{ccc}
F_0A & \xrightarrow{\tilde{f}} & F_0B \\
& \searrow{\tilde{g}} & \searrow{T^s} \\
& \nearrow{\tilde{h}} & \\
\end{array}
\]

commutes in \( \text{End}(\mathcal{M}) \).

Proof. Recall from the proof of Lemma 8.14 that \( g \) factorises as the composite

\[
\begin{array}{cccc}
W^{\beta_1} \otimes \cdots \otimes W^{\beta_n} & \xrightarrow{\tilde{g}'} & sW \\
\downarrow{g} & & \\
B & \xrightarrow{\tilde{g}} & T^n \\
\end{array}
\]

(and is constructed as such). Thus, \( \tilde{g} \) is constructed as the corresponding composite

\[
\begin{array}{cccc}
T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} & \xrightarrow{\tilde{g}'} & T^n \\
\downarrow{+\beta} & & \\
F_0B & \xrightarrow{\tilde{g}} & T^n \\
\end{array}
\]

Recall that we also said \( \tilde{h} \) was constructed as the composite

\[
\begin{array}{cccc}
T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} & \xrightarrow{\tilde{h}'} & T^s \\
\downarrow{+\delta} & & \\
F_0A & \xrightarrow{\tilde{h}} & T^s \\
\end{array}
\]

We now have the following diagram

\[
\begin{array}{ccccccc}
F_0B & \xrightarrow{\tilde{g}'} & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} & \xrightarrow{+\beta} & T^n \\
\downarrow{+\beta} & & & & \\
F_0B & \xrightarrow{\tilde{g}} & T^n \\
\end{array}
\]

for which we wish to show the commutativity of the exterior.

We already know the bottom triangle as well as top right triangle commute by construction. We begin with the innermost square

\[
\begin{array}{cccccc}
F_0A & \xrightarrow{\tilde{g} \circ f} & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} & \xrightarrow{+\beta} & T^n \\
\downarrow{\tilde{h}'} & & & & \\
F_0A & \xrightarrow{\tilde{h}'} & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} & \xrightarrow{+\delta} & T^s \\
\end{array}
\]

and

\[
\begin{array}{ccccccc}
F_0A & \xrightarrow{\tilde{g} \circ f} & T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)} & \xrightarrow{+\beta} & T^n \\
\downarrow{\tilde{h}'} & & & & \\
F_0A & \xrightarrow{\tilde{h}'} & T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} & \xrightarrow{+\delta} & T^s \\
\end{array}
\]

for which we wish to show the commutativity of the exterior.
and introduce the map $\tilde{\Omega}$ (for $\Omega$ defined in 11, again, we shall not explicitly show explicitly the existence of $\tilde{\Omega}$). Recall that

$$\Omega: W^{\delta_1} \otimes \cdots \otimes W^{\delta_s} \rightarrow W^{\beta_1} \otimes \cdots \otimes W^{\beta_n}$$

assigned each generator of the domain to a particular generator in the codomain, and moreover we have $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_s$ for $\Omega_i: W^{\delta_i} \rightarrow W^{\beta_i}$.

First, it now becomes rather routine to show that

$$+_{\delta} = +_{\beta} \circ \tilde{\Omega}$$

(i.e. the lower triangle in Diagram (3)). To show $\tilde{g'} \circ f = \tilde{\Omega} \circ \tilde{h'}$ in $\text{End}(\mathcal{M})$, note first that $g' \circ f = \Omega \circ h'$ in $\mathbb{N}$-$\text{Weil}_1$ by design. So equivalently, we can show that

$$F_0 A \xrightarrow{h'} T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \xrightarrow{\tilde{\Omega}} T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_s)}$$

commutes. But recall that $T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)}$ is a limit (constructed as iterations of foundational pullbacks in $\text{End}(\mathcal{M})$) with projections $\pi = (r_1, \ldots, r_s)$. As such, it suffices to show the commutativity of

$$F_0 A \xrightarrow{\tilde{g'}} T^{(\delta_1)} \otimes \cdots \otimes T^{(\delta_s)} \xrightarrow{\tau_0 \tilde{\Omega}} T^s$$

for each $\pi$.

But noting that $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_s$ and $\pi = \pi_{r_1} \otimes \cdots \otimes \pi_{r_s}$, and noting the form of each $\pi_{r_i} \circ \Omega_i: T^{(\alpha_1)} \otimes \cdots \otimes T^{(\alpha_n)} \rightarrow T$ from 11, then the commutativity of the upper triangle in Diagram (3) above is immediate.

Hence, all that remains is to show the commutativity of the upper left triangle of Diagram (2), namely the commutativity of

$$F_0 A \xrightarrow{\tilde{f}} T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_s)} \xrightarrow{\tau_0 g'} T^s .$$

Again, since $T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_n)}$ is a limit, it suffices to show the commutativity of

$$F_0 B \xrightarrow{\tilde{f}} F_0 A \xrightarrow{g' \circ f} T^{(\beta_1)} \otimes \cdots \otimes T^{(\beta_s)} \xrightarrow{\tau_0 g'} T^s .$$
for each projection $\tau$.

Finally, note that by definition, each map
\[
\tau \circ g' : B \to nW
\]
has no intersecting circles, so we may apply Proposition 12.12 directly. ■

12.14. Proposition. For all arbitrary $f : A \to B$ and $g : B \to C$, the diagram
\[
\begin{array}{ccc}
F_0 A & \xrightarrow{\tilde{f}} & F_0 B & \xrightarrow{\tilde{g}} & F_0 C \\
& \searrow h & \downarrow & \swarrow & \\
& & F_0 C & & \\
\end{array}
\]

commutes in $\text{End}(\mathcal{M})$.

Proof. using the same argument as in the proof of Proposition 12.12, if the graph $\Gamma_C$ contains any edges, then $C$ is part of a (foundational) pullback
\[
\begin{array}{c}
C = C' \otimes (C_1 \times C_2) \\
\end{array}
\]

Correspondingly, $F_0 C$ is part of the pullback
\[
\begin{array}{c}
F_0 C \xrightarrow{\pi_1} F_0 C' F_0 C_1 \\
\downarrow \pi_2 \\
F_0 C' F_0 C_2 \xrightarrow{} F_0 C' .
\end{array}
\]

We now have
\[
\begin{array}{ccc}
F_0 A & \xrightarrow{\tilde{f}} & F_0 B & \xrightarrow{\tilde{g}} & F_0 C \\
& \searrow h & \downarrow & \swarrow & \\
& & F_0 C_2 & & \\
\end{array}
\]
in $\text{End}(\mathcal{M})$, and using the fact that $\pi_i \circ \tilde{g} = \tilde{\pi}_i \circ \tilde{g}$ for $i = 1, 2$ (and a corresponding fact for $h$), it suffices to show the commutativity of
\[
\begin{array}{ccc}
F_0 A & \xrightarrow{\tilde{f}} & F_0 B & \xrightarrow{\tilde{\pi}_i \circ \tilde{g}} & F_0 C_i \\
& \searrow \tilde{\pi}_i \circ h & \downarrow & \swarrow & \\
& & F_0 C_i & & \\
\end{array}
\]
for each $i$. Using this argument iteratively, it suffices to assume the graph $\Gamma_C$ has no edges, i.e. $C = sW$ and apply Proposition 12.13 directly. ■
We have now shown that the pairings of the collection $\Psi$ "preserve" arbitrary compositions. We now need to consider the collection $\Phi$ (Definition 12.5).

13. The Problem with Pullbacks

As we mentioned in Definition 12.4, we may have maps $f, g, h \in \mathbb{N}\text{-Weil}_1$ for which the final step of the instructions for $h$ is to uniquely induce it using $f$ and $g$ as

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow & & \downarrow \\
B & \xrightarrow{\pi_1} & B_1 \\
\downarrow & & \downarrow \\
B_2 & \xrightarrow{\pi_2} & C,
\end{array}
\]

but the exterior of

\[
\begin{array}{ccc}
F_0A & \xrightarrow{\tilde{f}} & F_0B \\
\downarrow & & \downarrow \\
F_0B & \xrightarrow{\pi_1} & F_0B_1 \\
\downarrow & & \downarrow \\
F_0B_2 & \xrightarrow{\pi_2} & F_0C
\end{array}
\]

does not commute, and so we cannot construct $\tilde{h}$, and said that $\tilde{h}$ is not well defined. We then defined $\Phi$ to be the set of such "undefined" maps (Definition 12.5). We will now show that this set $\Phi$ is in fact empty.

13.1. Proposition. The collection $\Phi$ is empty.

Proof. Suppose that $\Phi$ is non-empty. Then for each $f \in \Phi$ (with $f: A \to B$ a map in $\mathbb{N}\text{-Weil}_1$), let $n(f)$ be the number of vertices in the graph $\Gamma_B$. Finally, let $N(\Phi) = \{n(f) \mid \forall f \in \Phi\}$.

Since $N(\Phi)$ is a non-empty subset of $\mathbb{N}$, then by the well ordering principle, it has a least element. Choose a map $h: A \to B$ corresponding to this least element. Further, suppose that the cograph for this codomain has at least one edge (if $\Gamma_B$ has no edges, i.e. $B = nW$, then we construct $\tilde{h}$ directly using the methods in the proof of 8.14).

We then have the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f=\pi_1h} & B_1 \otimes C \\
\downarrow & \downarrow & \downarrow \\
(B_1 \times B_2) \otimes C & \xrightarrow{\pi_1} & B_1 \otimes C \\
\downarrow & \downarrow & \downarrow \\
B_2 \otimes C & \xrightarrow{e \otimes C} & C
\end{array}
\]
and noting that since $\Gamma_B$ has at least one edge, then $\Gamma_{B_1 \otimes C}$ and $\Gamma_{B_2 \otimes C}$ each have strictly fewer vertices in their respective cographs than $\Gamma_B$. Thus, $\tilde{f}$ and $\tilde{g}$ are both well defined.

We wish to show the commutativity of

$$
\begin{array}{ccc}
F_0A & \xrightarrow{\tilde{f}} & F_0B_1 \otimes F_0C \\
\downarrow{\tilde{g}} & & \downarrow \\
F_0B_2 \otimes C & \xrightarrow{\tilde{h}} & F_0C
\end{array}
$$

so that $\tilde{h}$ can be induced using the foundational pullback in $\text{End}(\mathcal{M})$.

Let $\psi = (\varepsilon_1 \otimes C) \circ f : A \to C$ in $\text{N-Weil}_1$, i.e. the composite

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B_1 \otimes C \\
\downarrow{\psi} & & \downarrow{\varepsilon_1 \otimes C} \\
\downarrow & & \downarrow \\
C & & C
\end{array}
$$

Since $\Gamma_C$ has strictly fewer vertices than $\Gamma_B$, then $\tilde{\psi}$ is also well defined. But by Proposition 12.14, each of the triangles in the diagram

$$
\begin{array}{ccc}
F_0A & \xrightarrow{\tilde{f}} & F_0B_1 \otimes F_0C \\
\downarrow{\tilde{g}} & & \downarrow \\
F_0B_2 \otimes C & \xrightarrow{\tilde{h}} & F_0C
\end{array}
$$

commute in $\text{End}(\mathcal{M})$, and thus the exterior commutes.

Therefore $\tilde{h}$ is well defined. Thus the original assumption is incorrect, i.e. $\Phi$ is an empty set.

What we have shown then is that $F_0$ and the pairings of $\Psi$ together define precisely a functor.

14. The Functor $F$ and the universality of $\text{N-Weil}_1$

We now have the following:

14.1. Theorem. Suppose we have a given category $\mathcal{M}$. Regard $\text{End}(\mathcal{M})$ as a monoidal category with respect to composition and $\text{N-Weil}_1$ as monoidal with respect to coproduct.

Then to give a Tangent Structure $\mathbb{T}$ to $\mathcal{M}$ is equivalent (up to isomorphism) to giving a strong monoidal functor $F : \text{N-Weil}_1 \to \text{End}(\mathcal{M})$ satisfying the following conditions:
1) Given a product \( A = A_1 \times A_2 \) in \( \mathbb{N}\text{-Weil}_1 \), regarded as a pullback of the augmentations, and an arbitrary Weil algebra \( B \in \mathbb{N}\text{-Weil}_1 \), then \( F \) preserves the pullback

\[
\begin{array}{ccc}
B \otimes A & \xrightarrow{B \otimes \pi_1} & B \otimes A_1 \\
\downarrow & & \downarrow \\
B \otimes A_2 & \xrightarrow{B \otimes \varepsilon_2} & B
\end{array}
\]

i.e. it preserves all “foundational pullbacks” of \( \mathbb{N}\text{-Weil}_1 \) (as defined in Definition 3.17).

2) The equaliser

\[
W^2 \xrightarrow{v} 2W \xrightarrow{W \otimes \varepsilon_W} W
\]

as given in 4 is preserved.

Proof. Given such a functor \( F \), the corresponding Tangent Structure is given as

\[
\mathbb{T} = (FW, F\varepsilon_W, F\eta_W, F+, Fl, Fc)
\]

and it can be readily verified that this satisfies all the necessary conditions to be a Tangent Structure.

Conversely, suppose we have a Tangent Structure \( \mathbb{T} \). Then \( F_0 : ob(\mathbb{N}\text{-Weil}_1) \to ob(\text{End}(\mathcal{M})) \) and \( \Psi \) give us our assignations for objects and morphisms, and Propositions 12.14 and 13.1 together give functoriality.

Moreover, \( F_0 \) actually makes \( F \) monoidal (see Definition 12.2). \( F \) being strong monoidal as well as the preservation of foundational pullbacks is then a direct consequence of the fact that we are using composition as the monoidal structure of \( \text{End}(\mathcal{M}) \) together with Proposition 12.3.

Finally, preservation of the equaliser

\[
W^2 \xrightarrow{v} 2W \xrightarrow{W \otimes \varepsilon_W} W
\]

is trivial, since it is a condition of \( \mathbb{T} \) that the corresponding fork in \( \text{End}(\mathcal{M}) \) is also an equaliser. \( \qed \)

We have thus shown that to equip a category \( \mathcal{M} \) with a Tangent Structure \( \mathbb{T} \) is equivalent to giving (up to a suitable isomorphism) a strong monoidal functor \( F : \mathbb{N}\text{-Weil}_1 \to \text{End}(\mathcal{M}) \) satisfying some extra properties.

As such, \( \mathbb{N}\text{-Weil}_1 \) becomes an initial Tangent Structure in the sense that it characterises any Tangent Structure \( \mathbb{T} \) via this functor \( F \).

We also note that this functor \( F \) only required that \( \text{End}(\mathcal{M}) \) was a monoidal category (with respect to composition and with unit \( 1_{\mathcal{M}} \)) and that certain pullbacks were preserved. As a result, we make the following generalisation.
14.2. Definition. Let \((\mathcal{G}, \Box, I)\) be a monoidal category. Regard the category \(\mathbb{N}\text{-Weil}_1\) as monoidal with respect to coproduct and having unit \(\mathbb{N}\). A Tangent Structure \(\mathcal{G}\) internal to \(\mathcal{G}\) is a strong monoidal functor

\[ F : (\mathbb{N}\text{-Weil}_1, \otimes, \mathbb{N}) \to (\mathcal{G}, \Box, I) \]

satisfying the following conditions:
1) \(F\) preserves foundational pullbacks
2) The equaliser

\[
\begin{array}{ccc}
W^2 & \xrightarrow{v} & 2W \\
\downarrow & & \downarrow \eta_{W \otimes \varepsilon_W}
\end{array}
\]

\[
\xrightarrow{\eta_W \circ (\varepsilon_W \otimes \varepsilon_W)} W
\]
is preserved

14.3. Corollary. A Tangent Structure on \(\mathcal{M}\) (in the sense of Theorem 14.1) is the same as a Tangent Structure internal to \(\text{End}(\mathcal{M})\) (in the sense of Definition 14.2).

In fact, Definition 14.2 actually gives a universal property of the category \(\mathbb{N}\text{-Weil}_1\) in relation to Tangent Structures. One way we might express this is that Tangent Structures are simply models of \(\mathbb{N}\text{-Weil}_1\) (regarded as a theory).

References


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