

## ON THE NORMALLY ORDERED TENSOR PRODUCT AND DUALITY FOR TATE OBJECTS

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ABSTRACT. This paper generalizes the normally ordered tensor product from Tate vector spaces to Tate objects over arbitrary exact categories. We show how to lift bi-right exact monoidal structures, duality functors, and construct external Homs. We list some applications: (1) Adèles of a flag can be written as ordered tensor products; (2) Intersection numbers can be interpreted via these tensor products; (3) Pontryagin duality uniquely extends to  $n$ -Tate objects in locally compact abelian groups.

### 1. Introduction

Over a field  $k$ , the tensor product  $k[[s]] \otimes k[[t]]$  is much smaller than  $k[[s, t]]$ . This behaviour is usually not wanted and resolved by using a *completed* tensor product. The corresponding problem for Laurent series  $k((s))$  and  $k((t))$  is more difficult. It is asymmetric due to

$$k((s))((t)) \neq k((t))((s)). \quad (1.1)$$

This issue is also well-understood: Formal Laurent series can be axiomatized as Tate vector spaces, and Beilinson and Drinfeld introduced the *normally ordered tensor product*, which has the desired property  $k((s)) \overrightarrow{\otimes} k((t)) = k((s))((t))$  in [BD04, §3.6.1], [Bei08].

We generalize this construction to Tate objects over arbitrary exact categories:

1.1. THEOREM. *Let  $\mathcal{C}$  be an exact category with a bi-exact monoidal structure  $\otimes$ .*

1. *Then for all  $n, m \geq 0$ , there exists a canonical bi-exact functor*

$$-\overrightarrow{\otimes}- : n\text{-Tate}(\mathcal{C}) \times m\text{-Tate}(\mathcal{C}) \longrightarrow (n+m)\text{-Tate}(\mathcal{C}). \quad (\diamond)$$

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The associativity constraint for  $\otimes$  naturally induces an associativity constraint for  $\overrightarrow{\otimes}$ .

2. For  $\mathcal{C} := \mathbf{Vect}_f(k)$ , the realization functor from Tate objects in  $\mathcal{C}$  to topological  $k$ -vector spaces sends  $\overrightarrow{\otimes}$ -products to the normally ordered tensor products of [BD04, §3.6.1].
3. If  $\mathcal{C}$  is idempotent complete, closed monoidal and  $\underline{\mathit{Hom}}_{\mathcal{C}}(-, -)$  is bi-exact as well, then for all  $n, m \geq 0$ , there exists a canonical bi-exact functor

$$\underline{\mathit{Hom}}(-, -) : n\text{-Tate}(\mathcal{C})^{op} \times m\text{-Tate}(\mathcal{C}) \longrightarrow (n + m)\text{-Tate}(\mathcal{C}) .$$

Moreover, there is a form of a Hom-tensor adjunction.

For precise statements, see §2 and §4. We prove more, but we defer the relevant statements to the main body of the paper. Even if  $\otimes$  is symmetric,  $\overrightarrow{\otimes}$  will not be symmetric, reflecting the issue in (1.1). This failure of symmetry allows one to define a different product  $\overrightarrow{\otimes}_{\sigma}$  for every  $(n, m)$ -shuffle  $\sigma$ . We discuss this in more detail in §2.

For general exact categories  $\mathcal{C}$  new problems arise. Notably, it is rare that a monoidal structure  $\otimes$  is bi-exact. To this end, we introduce the full sub-category of ‘flat Tate objects’,  $\mathit{Tate}^b(\mathcal{C})$ , in §2 and show that bi-right exact monoidal structures extend to flat Tate objects as in  $(\diamond)$ . As the category of vector spaces is split exact, none of these problems occur in the situation of [BD04, §3.6.1]. Next, we address duality. We show that Tate objects can also be realized inside Pro-Ind objects, even though they are usually viewed as living inside Ind-Pro objects:

- 1.2. THEOREM. (Prop. 3.1) Suppose  $\mathcal{C}$  is idempotent complete. Then there is a canonical fully exact embedding

$$\mathit{Tate}^{el}(\mathcal{C}) \hookrightarrow \mathit{Pro}^a \mathit{Ind}^a \mathcal{C} .$$

The essential image of this embedding consists of all such admissible Pro-Ind-objects which admit a lattice, i.e. a Pro sub-object with an Ind-object quotient.

Moreover, duality functors always lift to Tate categories:

- 1.3. THEOREM. (Prop. 3.5) Suppose  $\mathcal{C}$  is idempotent complete. Every exact equivalence  $\mathcal{C}^{op} \xrightarrow{\sim} \mathcal{C}$  extends to an exact equivalence

$$\mathit{Tate}^{el}(\mathcal{C})^{op} \xrightarrow{\sim} \mathit{Tate}^{el}(\mathcal{C}) .$$

This duality restricts to exact equivalences

$$\mathit{Ind}^a(\mathcal{C})^{op} \xrightarrow{\sim} \mathit{Pro}^a(\mathcal{C}) \quad \text{and} \quad \mathit{Pro}^a(\mathcal{C})^{op} \xrightarrow{\sim} \mathit{Ind}^a(\mathcal{C}) .$$

These results form the foundation of a calculus of tensor products, Homs and duality in Tate categories.

*Summary of Applications*

Section 6 of the paper is devoted to adelic geometry. Suppose  $X/k$  is an integral scheme of finite type. Adèles of a flag of points  $\Delta = (\eta_0 > \dots > \eta_n)$  in a scheme will be denoted by  $A(\Delta, \mathcal{O}_X)$ . The latter has a canonical structure as an object in  $n\text{-Tate}(k)$ . In fact, we will show that it is a normally ordered tensor product:

1.4. THEOREM. (Cor. 5.5) *Suppose  $X/k$  is an integral scheme of finite type and pure dimension  $n$ . If  $\Delta$  is a saturated flag, then*

$$A(\Delta, \mathcal{O}_X) = {}_{\eta_n}\Lambda_{\eta_{n-1}} \overrightarrow{\otimes} \dots \overrightarrow{\otimes} {}_{\eta_1}\Lambda_{\eta_0} \quad \text{for} \quad {}_x\Lambda_y := \operatorname{colim}_{\mathcal{F} \subseteq \mathcal{O}_y} \lim_j \mathcal{F}/\mathcal{I}_x^j,$$

where  ${}_x\Lambda_y$  are flat Tate objects in  $\operatorname{Coh}_{\overline{\{x\}}}(X)$ ,  $\mathcal{I}_x$  is the ideal sheaf of  $\overline{\{x\}}$ , and  $\mathcal{F}$  runs through the coherent sub-sheaves of  $\mathcal{O}_y$ .

This has the following application:

Recall that on a curve  $\pi : X \rightarrow \mathbb{F}_q$  the action of any idèle representative of a line bundle  $L$  on the adèles rescales any choice of a Haar measure by  $q^{\deg(L)}$ . This stays true over an arbitrary base field  $k$  if one works additively and replaces the Haar measure by a renormalized dimension theory as in Kapranov [Kap01]. Phrased in terms of Tate objects: Given any element  $f \in \mathcal{O}_y^\times$ , it acts through multiplication by  $f$  on  ${}_x\Lambda_y$  as a Tate object, and any such automorphism defines a canonical  $K$ -theory class

$$[f] \in K_1(\operatorname{Tate}^b(\operatorname{Coh}_{\overline{\{x\}}}(X))) \circlearrowleft_{{}_x\Lambda_y} \tag{1.2}$$

We properly introduce this notation in Definition 5.8. There is an exact push-forward functor

$$\pi_* : \operatorname{Tate}^b(\operatorname{Coh}_0(X)) \longrightarrow \operatorname{Tate}(k),$$

where  $\operatorname{Coh}_0(X)$  are coherent sheaves of zero-dimensional support. Using this notation, recall the following classical degree formula:

1.5. THEOREM. (WEIL) *Let  $\pi : X \rightarrow k$  be an integral smooth proper curve with generic point  $\eta_0$  and  $(f_{\nu\mu})_{\nu\mu} \in H^1(X, \mathbb{G}_m)$  an alternating Čech representative of a line bundle  $L$  in a finite open cover  $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ ,  $I$  totally ordered. For any  $x \in X$ , let  $\alpha(x)$  be the smallest element of  $I$  such that  $x \in U_{\alpha(x)}$ . Then*

$$\deg(L) = - \sum_{\eta_1} \pi_* [f_{\alpha(\eta_1)\alpha(\eta_0)}] \circlearrowleft_{\eta_1\Lambda_{\eta_0}}$$

and the sum has only finitely many non-zero summands. The right-hand side defines an element of  $K_1(\operatorname{Tate}(k)) \cong \mathbb{Z}$ , and this integer is the degree of  $L$ . Here the sum runs over all closed points  $\eta_1 \in X$ .

We provide a higher-dimensional generalization. We define an external product in  $K$ -theory such that the external product of  $n$  classes  $[f]$  as in (1.2) lies in  $K_n$  of a flat  $n$ -Tate category. Suppose  $\pi : X \rightarrow k$  is a purely  $n$ -dimensional integral smooth proper variety. Let  $L_1, \dots, L_n$  be line bundles which are represented by alternating Čech representatives

$$f^q = (f_{\rho,\nu}^q)_{\rho,\nu \in I} \in H^1(X, \mathbb{G}_m) \quad (\text{for } q = 1, \dots, n)$$

in a finite open cover  $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ ,  $I$  totally ordered. As before, for any  $x \in X$ , let  $\alpha(x)$  be the smallest element of  $I$  such that  $x \in U_{\alpha(x)}$ .

1.6. THEOREM. (Theorem 6.6) Under these assumptions, the sum

$$(-1)^{\frac{n(n+1)}{2}} \sum_{\Delta=(\eta_0 > \dots > \eta_n)} \pi_* [f_{\alpha(\eta_n)\alpha(\eta_{n-1})}^n] \overset{\rightarrow}{\otimes} \dots \overset{\rightarrow}{\otimes} [f_{\alpha(\eta_1)\alpha(\eta_0)}^1]$$

has only finitely many non-zero summands, defines an element in  $K_n(n\text{-Tate}(k)) \cong \mathbb{Z}$ , and this integer is the intersection multiplicity  $L_1 \cdots L_n$ .

Here the sum runs over all saturated chains  $\Delta$  in  $X$ . For  $n = 1$  this is Weil’s degree formula.

In Section 7, we discuss an entirely different application: We develop the relation between our duality mechanism on Tate categories and Pontryagin Duality for locally compact abelian (LCA) groups, and we show how our formalism extends the Pontryagin Duality of LCA groups to a duality for  $n$ -Tate objects in LCA groups.

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## 2. Tensor products

Let  $k$  be a field. The inclusion

$$k((t_1)) \otimes_k k((t_2)) \hookrightarrow k((t_1))((t_2))$$

allows us to view the 2-variable Laurent series as a completion of the tensor product of  $k((t_1)) \otimes_k k((t_2))$ . This is an instance of a general phenomenon for Tate objects. We begin by developing the theory of bi-exact tensor products. We then extend this to consider tensor products which are only right exact.

### BI-EXACT TENSOR PRODUCTS.

2.1. PROPOSITION.

1. Let  $\mathcal{C}$  be an exact category with a bi-exact monoidal structure  $\otimes$ . Let  $n$  and  $m$  be natural numbers. Then there exists a bi-exact functor

$$-\overrightarrow{\otimes}- : n\text{-Tate}(\mathcal{C}) \times m\text{-Tate}(\mathcal{C}) \longrightarrow (n+m)\text{-Tate}(\mathcal{C}).$$

The associativity constraint for  $\otimes$  determines an associativity constraint for  $\overrightarrow{\otimes}$  in a natural fashion, and the unit object for  $\otimes$  in  $\mathcal{C}$  gives the unit object for  $\overrightarrow{\otimes}$ . Further, if  $\otimes$  is faithful in both variables, then so is  $\overrightarrow{\otimes}$ .

2. For  $\mathcal{C} = \mathbf{Vect}_f(k)$ , denote by

$$\tau : n\text{-Tate}_{\mathbb{N}_0}(k) \longrightarrow \mathbf{Vect}_{top}(k),$$

the natural functor from countable  $n$ -Tate spaces to the category  $\mathbf{Vect}_{top}(k)$  of complete, separated topological  $k$ -vector spaces with linear topologies. Let  $V \in n\text{-Tate}_{\mathbb{N}_0}(k)$  and  $W \in m\text{-Tate}_{\mathbb{N}_0}(k)$ . Then there exists a canonical natural isomorphism

$$\tau(V \overrightarrow{\otimes} W) \cong \tau(V) \overrightarrow{\otimes} \tau(W)$$

where the tensor product on the right is the one considered by Beilinson in [Bei08] (with the same notation).<sup>1</sup>

We deduce the proposition from the following lemma, which we will use repeatedly throughout.

- 2.2. LEMMA.  $F : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$  be a bi-exact functor. Then  $F$  extends canonically to bi-exact functors

$$F : \mathbf{Tate}(\mathcal{C}) \times \mathcal{D} \longrightarrow \mathcal{E} \quad \text{and} \quad F : \mathcal{C} \times \mathbf{Tate}(\mathcal{D}) \longrightarrow \mathbf{Tate}(\mathcal{E}).$$

If  $F$  is faithful in both variables, the extensions of  $F$  are as well.

PROOF. By the universal property of idempotent completion, it suffices to construct bi-exact functors

$$F : \mathbf{Tate}^{el}(\mathcal{C}) \times \mathcal{D} \longrightarrow \mathbf{Tate}^{el}(\mathcal{E}) \quad \text{and} \quad F : \mathcal{C} \times \mathbf{Tate}^{el}(\mathcal{D}) \longrightarrow \mathbf{Tate}^{el}(\mathcal{E}).$$

For the first, let  $V \in \mathbf{Tate}^{el}(\mathcal{C})$  and let  $X \in \mathcal{D}$ . Let  $V : I \longrightarrow \mathbf{Pro}^a(\mathcal{C})$  be an elementary Tate diagram representing  $V$ . For each  $V_i$ , let  $V_i : J_i \longrightarrow \mathcal{C}$  be an admissible Pro-diagram representing  $V_i$ . By the bi-exactness of  $F$ , we have an admissible Pro-diagram

$$F(V_i, X) : J_i \longrightarrow \mathcal{E}$$

for each  $i \in I$ , and thus an elementary Tate diagram

$$F(V, X) : I \longrightarrow \mathbf{Pro}^a(\mathcal{E})$$

$$i \mapsto \lim_{J_i} F(V_{i,j}, X)$$

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<sup>1</sup>See also [BD04, Section 3.6.1].

as well. Define  $F(V, X)$  to be the elementary Tate object determined by this diagram. This defines the functor  $F(-, -)$  on objects. The definition for morphisms follows from the straightening construction (cf. [BGW16c, Lemma 3.9]); further, we immediately see that if  $F$  is faithful in both variables, so is its extension. To complete the proof, it remains to show that this functor is exact. But this follows by the straightening construction for exact sequences (cf. [BGW16c, Proposition 3.12]), so we are done with the first extension.

A similar construction defines  $F: \mathcal{C} \rightarrow \text{Tate}^{el}(\mathcal{D}) \rightarrow \text{Tate}^{el}(\mathcal{E})$ . In detail, let  $V \in \mathcal{C}$  and let  $W \in \text{Tate}^{el}(\mathcal{D})$ . Let  $W: I \rightarrow \text{Pro}^a(\mathcal{D})$  be an elementary Tate diagram representing  $W$ . For each  $i \in I$ , let  $W_i: J_i \rightarrow \mathcal{D}$  be an admissible Pro-diagram representing  $W_i$ . Then the bi-exactness of  $F$  guarantees that we have an admissible Pro-diagram

$$F(V, W_i): J_i \rightarrow \mathcal{E},$$

and an admissible Pro-object in  $\mathcal{E}$  denoted by the same. The straightening argument for exact sequences and the inductive hypothesis shows that the assignment  $W_i \mapsto F(V, W_i)$  is exact, so we obtain an elementary Tate diagram

$$F(V, W): I \rightarrow \text{Pro}^a(\mathcal{E})$$

and thus an object  $F(V, W) \in \text{Tate}^{el}(\mathcal{E})$ . This constructs the functor on objects. The definition for morphisms again follows from the straightening construction for morphisms, which also shows the faithfulness in both variables. And, by the straightening argument for exact sequences, we see that it is exact in both variables as claimed.  $\blacksquare$

**PROOF OF PROPOSITION 2.1.** By the universal property of idempotent completion, it suffices to construct a bi-exact functor

$$-\overset{\rightarrow}{\otimes}-: n\text{-Tate}^{el}(\mathcal{C}) \times m\text{-Tate}^{el}(\mathcal{C}) \rightarrow (n+m)\text{-Tate}^{el}(\mathcal{C}),$$

This follows from Lemma 2.2 by induction on  $n$  and  $m$ . The key point is that we first conduct all extensions in the first variable, then conduct all extensions in the second variable. The case  $n = m = 0$  is trivial. Now suppose that we have constructed

$$-\overset{\rightarrow}{\otimes}-: (n-1)\text{-Tate}(\mathcal{C}) \times \mathcal{C} \rightarrow (n-1)\text{-Tate}(\mathcal{C}).$$

Then by Lemma 2.2, we obtain a bi-exact functor

$$-\overset{\rightarrow}{\otimes}-: n\text{-Tate}(\mathcal{C}) \times \mathcal{C} \rightarrow n\text{-Tate}(\mathcal{C}).$$

Now suppose we have constructed

$$-\overset{\rightarrow}{\otimes}-: n\text{-Tate}(\mathcal{C}) \times (m-1)\text{-Tate}(\mathcal{C}) \rightarrow (n+m-1)\text{-Tate}(\mathcal{C}).$$

Then by Lemma 2.2, we obtain a bi-exact functor

$$-\overset{\rightarrow}{\otimes}-: n\text{-Tate}(\mathcal{C}) \times m\text{-Tate}(\mathcal{C}) \rightarrow (n+m)\text{-Tate}(\mathcal{C}).$$

Denote by  $\alpha$  the associativity constraint for  $\otimes$ . We now show that, given an  $n$ -Tate object  $U$ , an  $m$ -Tate object  $V$ , and an  $\ell$ -Tate object  $W$ , then  $\alpha$  determines a natural isomorphism

$$\alpha: U \overrightarrow{\otimes} (V \overrightarrow{\otimes} W) \xrightarrow{\cong} (U \overrightarrow{\otimes} V) \overrightarrow{\otimes} W.$$

This follows from the above construction of  $\overrightarrow{\otimes}$  by induction on  $n$ ,  $m$  and  $\ell$ . Indeed, for the base case of  $n = m = \ell = 0$ , there is nothing to show. Further, the construction above immediately implies that an associativity constraint for  $(n, 0, 0)$  determines an associativity constraint for  $(n+1, 0, 0)$ . Similarly, it implies that an associativity constraint for  $(n, m, 0)$  determines an associativity constraint for  $(n, m+1, 0)$ . It remains to show that an associativity constraint for  $(n, m, \ell)$  determines one for  $(n, m, \ell+1)$ . As above, let

$$W: I \longrightarrow \mathbf{Pro}^a((\ell-1)\text{-Tate}^{el}(\mathcal{C}))$$

be an elementary Tate diagram representing  $W$ . For each  $i \in I$ , let

$$W_i: J_i \longrightarrow (\ell-1)\text{-Tate}^{el}(\mathcal{C})$$

be an admissible Pro-diagram representing  $W_i$ . By the inductive hypothesis, for each  $i$ ,  $\alpha$  determines a natural isomorphism of admissible Pro-diagrams

$$\begin{array}{ccc} J_i & \xlongequal{\quad} & J_i \\ \downarrow (U \overrightarrow{\otimes} V) \overrightarrow{\otimes} W_i & \xRightarrow{\cong} & \downarrow U \overrightarrow{\otimes} (V \overrightarrow{\otimes} W_i) \\ & & (n+m+\ell-1)\text{-Tate}^{el}(\mathcal{C}) \end{array} .$$

This determines a natural isomorphism of elementary Tate diagrams

$$\begin{array}{ccc} I & \xlongequal{\quad} & I \\ \downarrow (U \overrightarrow{\otimes} V) \overrightarrow{\otimes} W & \xRightarrow{\cong} & \downarrow U \overrightarrow{\otimes} (V \overrightarrow{\otimes} W) \\ & & \mathbf{Pro}^a((n+m+\ell-1)\text{-Tate}^{el}(\mathcal{C})) \end{array} ,$$

and thus a natural isomorphism of elementary Tate objects

$$\alpha: (U \overrightarrow{\otimes} V) \overrightarrow{\otimes} W \xrightarrow{\cong} U \overrightarrow{\otimes} (V \overrightarrow{\otimes} W)$$

which completes the induction step. A similar induction proves that the unit object for  $(\mathcal{C}, \otimes)$  provides a unit object for  $\overrightarrow{\otimes}$  (with the unitality constraints for  $\overrightarrow{\otimes}$  determined by those for  $\otimes$  under an analogous induction as the above).

It remains to prove the comparison of topological tensor products. Let  $V \in n\text{-Tate}_{\mathbb{N}_0}(k)$  and  $W \in m\text{-Tate}_{\mathbb{N}_0}(k)$ . We construct, by induction on  $n$  and  $m$ , a natural isomorphism

$$\tau(V) \overrightarrow{\otimes} \tau(W) \xrightarrow{\cong} \tau(V \overrightarrow{\otimes} W).$$

For  $m = 0$ , the existence of this isomorphism follows immediately from the definition of the topology on both sides. We now show that a natural isomorphism for  $(n, m)$  determines one for  $(n, m + 1)$ . Let

$$W : \mathbb{N} \longrightarrow \mathbf{Pro}^a(m\text{-Tate}_{\mathbb{N}_0}(k))$$

be an elementary Tate diagram representing  $W$ , and for each  $i \in \mathbb{N}$ , let

$$W_i : \mathbb{N} \longrightarrow m\text{-Tate}_{\mathbb{N}_0}(k)$$

be an admissible Pro-diagram representing  $W_i$ . By definition, we have

$$\tau(V \overrightarrow{\otimes} W) \cong \operatorname{colim}_{i \in \mathbb{N}} \lim_{j \in \mathbb{N}} \tau(V \overrightarrow{\otimes} W_{i,j})$$

in  $\mathbf{Vect}_{top}(k)$ . Further, by inductive hypothesis, for all  $i \in \mathbb{N}$ , we have a natural isomorphism of diagrams

$$\begin{array}{ccc} \mathbb{N} & \xlongequal{\quad} & \mathbb{N} \\ \tau(V) \overrightarrow{\otimes} \tau(W_i) \searrow & \cong \Rightarrow & \searrow \tau(V \overrightarrow{\otimes} W_i) \\ & \mathbf{Vect}_{top}(k) & \end{array} .$$

This determines a natural isomorphism of diagrams

$$\begin{array}{ccc} \mathbb{N} & \xlongequal{\quad} & \mathbb{N} \\ \lim_{j \in \mathbb{N}} \tau(V) \overrightarrow{\otimes} \tau(W_{i,j}) \searrow & \cong \Rightarrow & \searrow \tau(V \overrightarrow{\otimes} W_i) \\ & \mathbf{Vect}_{top}(k) & \end{array} ,$$

and thus a natural isomorphism

$$\operatorname{colim}_{i \in \mathbb{N}} \lim_{j \in \mathbb{N}} \tau(V) \overrightarrow{\otimes} \tau(W_{i,j}) \xrightarrow{\cong} \tau(V \overrightarrow{\otimes} W).$$

It remains to construct a natural isomorphism

$$\tau(V) \overrightarrow{\otimes} \tau(W) \xrightarrow{\cong} \operatorname{colim}_{i \in \mathbb{N}} \lim_{j \in \mathbb{N}} \tau(V) \overrightarrow{\otimes} \tau(W_{i,j}).$$

Denote by

$$\delta : \mathbf{Vect}_{top}(k) \longrightarrow \mathbf{Vect}_{top}(k)$$

the functor which sends a topological vector space to the underlying discrete vector space. Because  $\otimes_k$  preserves arbitrary colimits, we have a natural map of discrete  $k$ -vector spaces

$$\delta\tau(V) \otimes_k \delta\tau(W) \longrightarrow \operatorname{colim}_{i \in \mathbb{N}} \lim_{j \in \mathbb{N}} \delta\tau(V) \otimes_k \delta\tau(W_{i,j}).$$

By definition, the tensor product  $\tau(V) \overrightarrow{\otimes} \tau(W)$  is obtained by completing  $\delta\tau(V) \otimes_k \delta\tau(W)$  with respect to the topology generated by subsets of the form  $P \otimes_k w$  and  $P \otimes Q$  for  $P \subset \tau(V)$  open,  $w \in W$ , and  $Q \subset \tau(W)$  open. By completing with respect to subsets of the form  $P \otimes_k w$ , we obtain

$$\tau(V) \otimes_k \delta\tau(W).$$

By definition, an open subset  $Q \subset \tau(W)$  arises as  $Q \cong \operatorname{colim}_{i \in \mathbb{N}} Q_i$  for  $Q_i \subset \tau(W_i)$  open. Therefore, completing with respect to the subsets  $P \otimes_k Q$  determines a natural isomorphism

$$\tau(V) \overrightarrow{\otimes} \tau(W) \xrightarrow{\cong} \operatorname{colim}_{i \in \mathbb{N}} \tau(V) \overrightarrow{\otimes} \tau(W_i).$$

Similarly, we see that for all  $i$ , we have a natural isomorphism

$$\tau(V) \overrightarrow{\otimes} \tau(W_i) \xrightarrow{\cong} \lim_{j \in \mathbb{N}} \tau(V) \overrightarrow{\otimes} \tau(W_{i,j}).$$

Combining these, we obtain a natural isomorphism

$$\tau(V) \overrightarrow{\otimes} \tau(W) \xrightarrow{\cong} \operatorname{colim}_{i \in \mathbb{N}} \lim_{j \in \mathbb{N}} \tau(V) \overrightarrow{\otimes} \tau(W_{i,j})$$

as claimed. ■

An  $n$ -Tate object is a linear algebraic analogue of an  $n$ -dimensional scheme  $X$  equipped with a complete flag of closed subschemes

$$\xi = (X = Z_n \supset Z_{n-1} \cdots \supset Z_1 \supset Z_0)$$

with  $\dim Z_i = i$ . The tensor product of Proposition 2.1 corresponds to the product of schemes with flags

$$((X_1, \xi_1), (X_2, \xi_2)) \mapsto (X_1 \times X_2, Z_{1,n} \times Z_{2,m} \supset \cdots \supset Z_{1,n} \times Z_{2,0} \supset \cdots \supset Z_{1,0} \times Z_{2,0}).$$

However, we also obtain a flag in  $X_1 \times X_2$  for any  $(n, m)$ -shuffle. A similar phenomenon exists for Tate objects. To state this, we need to recall a fact about shuffles.

**2.3. LEMMA.** *Let  $\sigma$  be an  $(n, m)$ -shuffle, and let  $\tau$  be an  $(n + m, \ell)$ -shuffle. Then there is a unique  $(m, \ell)$ -shuffle  $\tau'$  and a unique  $(n, m + \ell)$ -shuffle  $\sigma'$  such that*

$$\tau \circ (\sigma \sqcup \operatorname{id}_{\{1 < \dots < \ell\}}) = \sigma' \circ (\operatorname{id}_{\{1 < \dots < n\}} \sqcup \tau').$$

**PROOF.** Given  $\sigma$  and  $\tau$ , we define  $\tau'$  to be the  $(m, \ell)$ -shuffle obtained from the linearly ordered set  $I$  consisting of the images, under  $\tau \circ (\sigma \sqcup 1_{\{1 < \dots < \ell\}})$ , of  $\{1 < \dots < m\}$  and  $\{1 < \dots < \ell\}$  in  $\{1 < \dots < n + m + \ell\}$ . Similarly, define  $\sigma'$  to be the  $(n, m + \ell)$ -shuffle obtained by identifying the linearly ordered set  $I$  with  $\{1 < \dots < m + \ell\}$  and considering the images of  $I$  and  $\{1 < \dots < n\}$  in  $\{1 < \dots < n + m + \ell\}$ . ■

2.4. PROPOSITION. *Let  $\mathcal{C}$  be an exact category with a bi-exact symmetric monoidal structure  $\otimes$ . Let  $n$  and  $m$  be natural numbers, and let  $\sigma$  be an  $(n, m)$ -shuffle. Then there exists a bi-exact functor*

$$-\overrightarrow{\otimes}_{\sigma}- : n\text{-Tate}(\mathcal{C}) \times m\text{-Tate}(\mathcal{C}) \longrightarrow (n+m)\text{-Tate}(\mathcal{C}).$$

Further, given  $n, m$ , and  $\ell$ , an  $(n, m)$ -shuffle  $\sigma$  and an  $(n+m, \ell)$ -shuffle  $\tau$ , then the associativity constraint for  $(\mathcal{C}, \otimes)$  determines a natural isomorphism

$$(-\overrightarrow{\otimes}_{\sigma}-)\overrightarrow{\otimes}_{\tau}- \cong -\overrightarrow{\otimes}_{\sigma'}(-\overrightarrow{\otimes}_{\tau'}-)$$

where  $\tau'$  and  $\sigma'$  are as in the previous lemma.

2.5. REMARK. In the notation of the previous proposition, the functor  $\overrightarrow{\otimes}$  of Proposition 2.1 corresponds to the trivial  $(n, m)$ -shuffle, i.e. the map

$$\{1 < \dots < n\} \sqcup \{1 < \dots < m\} \longrightarrow \{1 < \dots < n < 1 \dots < m\}.$$

PROOF. We construct the functors  $-\overrightarrow{\otimes}_{\sigma}-$  by induction on  $(n, m)$ . For  $n, m \leq 1$ , the tensor products were constructed in Proposition 2.1. Now suppose we have constructed  $\overrightarrow{\otimes}_{\sigma}$  for any  $(\ell, 1)$ -shuffle with  $(\ell, 1) < (n, 1)$ . Let  $\sigma$  be an  $(n, 1)$ -shuffle. If  $\sigma$  is the trivial shuffle

$$\sigma : \{1 < \dots < n\} \sqcup \{1\} \longrightarrow \{1 < \dots < n < 1\}$$

then the tensor product was constructed in Proposition 2.1. If  $\sigma(\{1\}) < n$ , we obtain the functor  $-\overrightarrow{\otimes}_{\sigma}-$  by applying Lemma 2.2 to the functor

$$-\overrightarrow{\otimes}_{\sigma'}- : (n-1)\text{-Tate}(\mathcal{C}) \times \mathcal{C} \longrightarrow (n-1)\text{-Tate}(\mathcal{C})$$

where  $\sigma'$  is the  $(n-1, 1)$ -shuffle given by restricting  $\sigma$  to  $\{1 < \dots < n-1\} \sqcup \{1\} \subset \{1 < \dots < n\} \sqcup \{1\}$ .

Now suppose we have defined  $\overrightarrow{\otimes}_{\sigma}$  for all  $(k, \ell)$ -shuffles with  $(k, \ell) < (n, m)$  in reverse lexicographical ordering. Let  $\sigma$  be an  $(n, m)$ -shuffle. Suppose  $\sigma(m) = n+m$ . Denote by  $\sigma'$  the  $(n, m-1)$ -shuffle given by restricting  $\sigma$  to  $\{1 < \dots < n\} \sqcup \{1 < \dots < m-1\}$ . We now obtain the functor  $-\overrightarrow{\otimes}_{\sigma}-$  by applying Lemma 2.2 to the functor

$$-\overrightarrow{\otimes}_{\sigma'}- : n\text{-Tate}(\mathcal{C}) \times (m-1)\text{-Tate}(\mathcal{C}) \longrightarrow (n+m-1)\text{-Tate}(\mathcal{C}).$$

If  $n+m \notin \sigma(\{1 < \dots < m\})$ , then we define  $\overrightarrow{\otimes}_{\sigma}$  by applying Lemma 2.2 to the functor

$$-\overrightarrow{\otimes}_{\sigma'}- : (n-1)\text{-Tate}(\mathcal{C}) \times m\text{-Tate}(\mathcal{C}) \longrightarrow (n+m-1)\text{-Tate}(\mathcal{C})$$

where now  $\sigma'$  is the  $(n-1, m)$ -shuffle obtained by restricting  $\sigma$  to  $\{1 < \dots < n-1\} \sqcup \{1 < \dots < m\}$ .

It remains to establish the associativity. This follows by inductive argument analogous to the one which showed the associativity of the tensor product of Proposition 2.1. ■

We can summarize the above structure as follows. Define

$$\infty\text{-Tate}(\mathcal{C}) := \bigcup_n n\text{-Tate}(\mathcal{C}).$$

This is a filtered exact category, and we can grade the objects in the natural fashion, i.e. the objects in degree  $n$  consist of those objects which lie in the complement  $n\text{-Tate}^{el}(\mathcal{C}) \setminus (n - 1)\text{-Tate}^{el}(\mathcal{C})$ . We can now define a “shuffle product”

$$\overrightarrow{\otimes}_{all}: \infty\text{-Tate}(\mathcal{C}) \times \infty\text{-Tate}(\mathcal{C}) \longrightarrow \infty\text{-Tate}(\mathcal{C})$$

by stipulating that on degree  $(n, m)$  it is given by

$$V \overrightarrow{\otimes}_{all} W := \bigoplus_{(n,m) \text{ shuffles}} V \overrightarrow{\otimes}_\sigma W.$$

As an immediate corollary of the above, we have

2.6. **COROLLARY.** *Let  $\mathcal{C}$  be an exact category with a bi-exact symmetric monoidal product  $\otimes$ . Then the product  $\overrightarrow{\otimes}_{all}$  defines a bi-exact symmetric monoidal product on  $\infty\text{-Tate}(\mathcal{C})$ , which is compatible with the filtration. Further, the fully faithful embedding  $\mathcal{C} \hookrightarrow \infty\text{-Tate}(\mathcal{C})$  is symmetric monoidal.*

2.7. **REMARK.** The category  $\infty\text{-Tate}(\mathcal{C})$  is a categorical analogue of the “shuffle algebra” on a vector space  $V$ . For finite dimensional  $V$ , the shuffle algebra is in fact a Hopf algebra, and is the graded dual of the universal enveloping algebra of the free Lie algebra on  $V$ . However, it would be surprising if an analogous story holds for  $\infty\text{-Tate}(\mathcal{C})$ . In particular, an analogue of the co-multiplication would involve associating a pair of Tate objects to each 2-Tate object; we currently are unaware of any natural construction of this sort.

2.8. **REMARK.** In [Bei08], Beilinson considers a “chiral” pseudo-tensor structure  $\overrightarrow{\otimes}^{ch}$  on the category of topological vector spaces. The symmetric monoidal structure  $\overrightarrow{\otimes}_{all}$  refines  $\overrightarrow{\otimes}^{ch}$  in the following sense: for any finite collection of countable 1-Tate spaces  $V_i$ , we have

$$\tau(\overrightarrow{\otimes}_{all} V_i) \cong \overrightarrow{\otimes}^{ch} \tau(V_i).$$

**RIGHT EXACT TENSOR PRODUCTS.** We now consider the situation when  $\mathcal{C}$  has only a right exact tensor product. We show that we still obtain a good category of “flat” admissible Pro-objects, and, from this, good categories of “flat”  $n$ -Tate objects on which we can define bi-exact tensor products. A primary motivation for this is to be able to work with adically complete modules, which are not, in general, Pro-objects in a category of flat modules.

2.9. DEFINITION. Let  $\mathcal{C}$  be an exact category. We say that a tensor product  $\otimes$  is bi-right exact if, for any  $A \in \mathcal{C}$  and for any short exact sequence

$$0 \longrightarrow X \hookrightarrow^i Y \twoheadrightarrow^p Z \longrightarrow 0,$$

the maps

$$\begin{aligned} A \otimes Y &\xrightarrow{1 \otimes p} A \otimes Z \\ Y \otimes A &\xrightarrow{p \otimes 1} Z \otimes A \end{aligned}$$

are admissible epics, and the natural maps

$$\begin{aligned} A \otimes X &\longrightarrow \ker(1 \otimes p) \\ X \otimes A &\longrightarrow \ker(p \otimes 1) \end{aligned}$$

are admissible epics as well.

A bi-right exact tensor product  $\otimes$  on  $\mathcal{C}$  canonically extends to a bi-right exact tensor product

$$\hat{\otimes}: \text{Pro}^a(\mathcal{C}) \times \text{Pro}^a(\mathcal{C}) \longrightarrow \text{Pro}^a(\mathcal{C}).$$

2.10. DEFINITION. Define the category  $\text{Pro}^{a,b}(\mathcal{C})$  of flat admissible Pro-objects to be the full sub-category of  $\text{Pro}^a(\mathcal{C})$  consisting of all  $X \in \text{Pro}^a(\mathcal{C})$  such that the functors

$$\begin{aligned} -\hat{\otimes} X &: \text{Pro}^a(\mathcal{C}) \longrightarrow \text{Pro}^a(\mathcal{C}) \\ X \hat{\otimes} - &: \text{Pro}^a(\mathcal{C}) \longrightarrow \text{Pro}^a(\mathcal{C}) \end{aligned}$$

are exact.

2.11. PROPOSITION. Let  $\mathcal{C}$  be an exact category with a bi-right exact tensor product  $\otimes$ . Then:

1. The sub-category  $\text{Pro}^{a,b}(\mathcal{C}) \subset \text{Pro}^a(\mathcal{C})$  is closed under extensions. In particular,  $\text{Pro}^{a,b}(\mathcal{C})$  is a fully exact sub-category of  $\text{Pro}^a(\mathcal{C})$ .
2. The functors

$$\begin{aligned} -\hat{\otimes} - &: \text{Pro}^a(\mathcal{C}) \times \text{Pro}^{a,b}(\mathcal{C}) \longrightarrow \text{Pro}^a(\mathcal{C}) \\ -\hat{\otimes} - &: \text{Pro}^{a,b}(\mathcal{C}) \times \text{Pro}^a(\mathcal{C}) \longrightarrow \text{Pro}^a(\mathcal{C}) \end{aligned}$$

are bi-exact, and restrict to a bi-exact functor

$$-\hat{\otimes} - : \text{Pro}^{a,b}(\mathcal{C}) \times \text{Pro}^{a,b}(\mathcal{C}) \longrightarrow \text{Pro}^{a,b}(\mathcal{C}).$$

PROOF. We first show that  $\mathrm{Pro}^{a,b}(\mathcal{C})$  is closed under extensions. Let

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 \quad (2.1)$$

be a short exact sequence of admissible Pro-objects with  $X$  and  $Z$  in  $\mathrm{Pro}^{a,b}(\mathcal{C})$ . Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad (2.2)$$

be any short exact sequence in  $\mathrm{Pro}^a(\mathcal{C})$ . The category  $\mathrm{Lex}(\mathcal{C})$  of left exact functors to abelian groups is a Grothendieck abelian category, see [BGW16c, Definition 2.20]. In particular, it has enough injectives. By duality,  $\mathrm{Lex}(\mathcal{C}^{op})^{op}$  has enough projectives. By [BGW16c, Theorem 4.2],  $\mathrm{Pro}^a(\mathcal{C})$  is a fully exact sub-category of  $\mathrm{Lex}(\mathcal{C}^{op})^{op}$ . We can therefore choose a projective resolution in  $\mathrm{Lex}(\mathcal{C}^{op})^{op}$

$$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0$$

of the sequence (2.2). Tensoring with (2.1), we get a  $3 \times 3$  diagram with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_\bullet \hat{\otimes} X & \longrightarrow & A_\bullet \hat{\otimes} Y & \longrightarrow & A_\bullet \hat{\otimes} Z \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_\bullet \hat{\otimes} X & \longrightarrow & B_\bullet \hat{\otimes} Y & \longrightarrow & B_\bullet \hat{\otimes} Z \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_\bullet \hat{\otimes} X & \longrightarrow & C_\bullet \hat{\otimes} Y & \longrightarrow & C_\bullet \hat{\otimes} Z \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The left and right columns consist solely of acyclic complexes because  $X$  and  $Z$  are flat admissible Pro-objects. Thus, by the exactness of the above diagram, the middle column consists solely of acyclic complexes. The same argument applies to tensoring on the left. We conclude that  $Y \in \mathrm{Pro}^{a,b}(\mathcal{C})$ .

It remains to show the bi-exactness. By the definition of  $\mathrm{Pro}^{a,b}(\mathcal{C})$ , the functor

$$-\hat{\otimes}- : \mathrm{Pro}^a(\mathcal{C}) \times \mathrm{Pro}^{a,b}(\mathcal{C}) \longrightarrow \mathrm{Pro}^a(\mathcal{C}) \quad (2.3)$$

is exact in the first variable. It remains to show that it is exact in the second variable. To see this, observe that the functor  $-\hat{\otimes}-$  is (by construction) the restriction of the right Kan extension of  $\otimes$  to  $\mathrm{Lex}(\mathcal{C}^{op})^{op}$ . Because  $\mathrm{Lex}(\mathcal{C}^{op})^{op}$  has enough projectives, to compute  $-\hat{\otimes}^{\mathbb{L}}-$ , we can resolve in either variable. In particular, given  $A \in \mathrm{Pro}^a(\mathcal{C})$  and a short exact sequence in  $\mathrm{Pro}^{a,b}(\mathcal{C})$

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

we obtain a short exact sequence of complexes

$$0 \longrightarrow A_{\bullet} \hat{\otimes} X \longrightarrow A_{\bullet} \hat{\otimes} Y \longrightarrow A_{\bullet} \hat{\otimes} Z \longrightarrow 0$$

for any projective resolution  $A_{\bullet}$  of  $A$ . However, because the functors  $-\hat{\otimes}X$ ,  $-\hat{\otimes}Y$ , and  $-\hat{\otimes}Z$  are exact, each of the complexes in the above sequence is acyclic. In particular, they are all equivalent to their  $H_0$ , and we have an exact sequence

$$0 \longrightarrow A \hat{\otimes} X \longrightarrow A \hat{\otimes} Y \longrightarrow A \hat{\otimes} Z \longrightarrow 0.$$

We conclude that the functor (2.3) is bi-exact, and the same argument shows the bi-exactness of

$$-\hat{\otimes}- : \text{Pro}^{a,b}(\mathcal{C}) \times \text{Pro}^a(\mathcal{C}) \longrightarrow \text{Pro}^a(\mathcal{C}).$$

Finally, that  $A \hat{\otimes} B \in \text{Pro}^{a,b}(\mathcal{C})$  if  $A, B \in \text{Pro}^{a,b}(\mathcal{C})$  follows directly from the associativity of  $\hat{\otimes}$ . ■

2.12. DEFINITION. *Let  $\mathcal{C}$  be an exact category with a bi-right exact tensor product. Define the category  $\text{Tate}^{el,b}(\mathcal{C})$  of flat elementary Tate objects to be the full sub-category of  $\text{Tate}^{el}(\mathcal{C})$  consisting of all elementary Tate objects which can be represented by an admissible Ind-diagram in  $\text{Pro}^{a,b}(\mathcal{C})$ .*

2.13. PROPOSITION. *The sub-category  $\text{Tate}^{el,b}(\mathcal{C}) \subset \text{Tate}^{el}(\mathcal{C})$  is closed under extensions. In particular,  $\text{Tate}^{el,b}(\mathcal{C})$  is a fully exact sub-category of  $\text{Tate}^{el}(\mathcal{C})$ .*

PROOF. Let

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be a short exact sequence of elementary Tate objects with  $X, Z \in \text{Tate}^{el,b}(\mathcal{C})$ . By straightening exact sequences [BGW16c, Proposition 3.12], we can represent this as the colimit of an admissible Ind-diagram of short exact sequences in  $\text{Pro}^a(\mathcal{C})$

$$0 \longrightarrow X_i \longrightarrow Y_i \longrightarrow Z_i \longrightarrow 0$$

with  $X_i$  and  $Z_i$  in  $\text{Pro}^{a,b}(\mathcal{C})$ . Because  $\mathcal{C} \subset \text{Pro}^a(\mathcal{C})$  is closed under extensions ([BGW16c, Theorem 4.2(2)]), the  $Y_i$  are lattices of  $Y$ . Further, because  $\text{Pro}^{a,b}(\mathcal{C}) \subset \text{Pro}^a(\mathcal{C})$  is closed under extensions (Proposition 2.11), the  $Y_i$  are flat admissible Pro-objects, i.e.  $Y \in \text{Tate}^{el,b}(\mathcal{C})$ . ■

2.14. DEFINITION. *Define the category  $\text{Tate}^b(\mathcal{C})$  of flat Tate objects to be the idempotent completion of  $\text{Tate}^{el,b}(\mathcal{C})$ . For  $n > 1$ , define the category  $n\text{-Tate}^{el,b}(\mathcal{C})$  of flat elementary  $n$ -Tate objects to be the category of elementary  $(n-1)$ -Tate objects in  $\text{Tate}^b(\mathcal{C})$ . Define the category  $n\text{-Tate}^b(\mathcal{C})$  of flat  $n$ -Tate objects to be the idempotent completion of  $n\text{-Tate}^{el,b}(\mathcal{C})$  (equivalently, the category of  $(n-1)$ -Tate objects in  $\text{Tate}^b(\mathcal{C})$ ).*

2.15. PROPOSITION. *Let  $\mathcal{C}$  be an exact category with a bi-right exact tensor product  $\otimes$ . Then for any  $n, m$ , there exists a bi-exact functor*

$$-\overset{\rightarrow}{\otimes}- : n\text{-Tate}^b(\mathcal{C}) \times m\text{-Tate}(\mathcal{C}) \longrightarrow (n+m)\text{-Tate}(\mathcal{C})$$

which restricts to a bi-exact functor

$$-\overset{\rightarrow}{\otimes}- : n\text{-Tate}^b(\mathcal{C}) \times m\text{-Tate}^b(\mathcal{C}) \longrightarrow (n+m)\text{-Tate}^b(\mathcal{C}).$$

PROOF. We prove these statements by an induction on  $n$  and  $m$  which follows the same logic, *mutatis mutandis*, as the proofs of Lemma 2.2 and Proposition 2.1. We leave the details to the interested reader.  $\blacksquare$

### 3. Duality

Let  $V$  be an elementary Tate object in  $\mathcal{C}$ , and consider the diagram

$$\begin{array}{c} Gr(V) \longrightarrow \text{Ind}^a(\mathcal{C}) \\ L \mapsto V/L \end{array}$$

When  $\mathcal{C}$  is idempotent complete, the poset  $Gr(V)$  is co-filtered by [BGW16c, Theorem 6.7], and the above defines an admissible Pro-diagram in  $\text{Ind}^a(\mathcal{C})$ . In this section, we will show that the assignment

$$V \mapsto \lim_{Gr(V)} V/L$$

defines a fully exact embedding

$$\Phi : \text{Tate}^{el}(\mathcal{C}) \longrightarrow \text{Pro}^a(\text{Ind}^a(\mathcal{C})).$$

This embedding sends Pro-objects to Pro-objects and Ind-objects to Ind-objects. From this, we will deduce that if  $\mathcal{C}$  is idempotent complete, then  $\text{Ind}^a(\mathcal{C})$  is right s-filtering in  $\text{Tate}^{el}(\mathcal{C})$  (thus clearing up a loose end in [BGW16c]). More important, we deduce that if  $\mathcal{C}$  has an exact duality, then so does  $\text{Tate}^{el}(\mathcal{C})$ .

3.1. PROPOSITION. *Let  $\mathcal{C}$  be idempotent complete. Then the assignment above defines a fully exact embedding*

$$\Phi : \text{Tate}^{el}(\mathcal{C}) \hookrightarrow \text{Pro}^a(\text{Ind}^a(\mathcal{C})). \quad (3.1)$$

*The essential image of this embedding consists of all admissible Pro-Ind objects which admit a lattice, or equivalently, by [BGW16c, Theorem 5.6], the essential image is the sub-category*

$$\text{Tate}^{el}(\mathcal{C}^{op})^{op} \subset \text{Ind}^a(\text{Pro}^a(\mathcal{C}^{op}))^{op} = \text{Pro}^a(\text{Ind}^a(\mathcal{C})).$$

*Further, this embedding restricts to the canonical embeddings*

$$\text{Ind}^a(\mathcal{C}) \hookrightarrow \text{Pro}^a(\text{Ind}^a(\mathcal{C}))$$

and

$$\text{Pro}^a(\mathcal{C}) \hookrightarrow \text{Pro}^a(\text{Ind}^a(\mathcal{C})).$$

PROOF. We start by showing that the assignment above is functorial. Indeed, let

$$f: V \longrightarrow W$$

be a map of elementary Tate objects. Let  $\Delta^1$  denote the category with two objects 0 and 1 and one non-identity morphism  $0 \longrightarrow 1$ . Recall that the inclusion

$$\text{Tate}^{el}(\text{Fun}(\Delta^1, \mathcal{C})) \hookrightarrow \text{Fun}(\Delta^1, \text{Tate}^{el}(\mathcal{C}))$$

is an exact equivalence (by the straightening construction). Under this equivalence, we can view the map  $f$  as an elementary Tate object in  $\text{Fun}(\Delta^1, \mathcal{C})$ . Note that a lattice in  $V \xrightarrow{f} W$  consists of a lattice  $L$  of  $V$  and a lattice  $L'$  of  $W$  fitting into a commuting square

$$\begin{array}{ccc} L & \longrightarrow & L' \\ \downarrow & & \downarrow \\ V & \xrightarrow{f} & W \end{array}$$

The idempotent completeness of  $\mathcal{C}$  implies that  $\text{Fun}(\Delta^1, \mathcal{C})$  is also idempotent complete. Therefore, the Sato Grassmannian  $Gr(f: V \rightarrow W)$  is also co-directed, and the assignment above determines a map of Pro-Ind objects

$$\lim_{Gr(f: V \rightarrow W)} (V/L \longrightarrow W/L'). \tag{3.2}$$

The projection  $Gr(f: V \rightarrow W) \longrightarrow Gr(V)$  determines a map

$$\lim_{Gr(f: V \rightarrow W)} V/L \longrightarrow \lim_{Gr(V)} V/L.$$

It remains to show that this map is an isomorphism, or equivalently that the projection  $Gr(f: V \rightarrow W) \longrightarrow Gr(V)$  is cofinal. However, this is obvious, as the projection is surjective, since every lattice of  $V$  factors through some lattice of  $W$  (because Pro-objects are left filtering in Tate objects). We can therefore define the functor  $\Phi$  on morphisms by specifying that  $f$  is sent to the map

$$\lim_{Gr(V)} V/L \xrightarrow{\cong} \lim_{Gr(f: V \rightarrow W)} V/L \longrightarrow \lim_{Gr(W)} W/L'$$

where the last map is the composite of (3.2) with the map of limits induced from the projection  $Gr(f: V \rightarrow W) \longrightarrow Gr(W)$ . The associativity of this assignment now follows by similar lines as above, and unitality is immediate; i.e. we have indeed defined a functor.

We now show that the functor we have constructed is a fully exact embedding, i.e. an exact equivalence onto its essential image. For this, we observe that, after passing to

opposite categories, [BGW16c, Theorem 5.6] shows that the category

$$\begin{aligned} \text{Tate}^{el}(\mathcal{C}^{op})^{op} &\subseteq \text{Ind}^a(\text{Pro}^a(\mathcal{C}^{op}))^{op} \\ &=: \text{Pro}^a(\text{Pro}^a(\mathcal{C}^{op})^{op}) \\ &=: \text{Pro}^a(\text{Ind}^a(\mathcal{C})) \end{aligned}$$

is the smallest full sub-category of  $\text{Pro}^a(\text{Ind}^a(\mathcal{C}))$  which contains  $\text{Pro}^a(\mathcal{C})$ ,  $\text{Ind}^a(\mathcal{C})$  and is closed under extensions. In particular, the construction of the functor (3.1) shows that it factors through the sub-category  $\text{Tate}^{el}(\mathcal{C}^{op})^{op}$ . Because  $\mathcal{C}$  is idempotent complete if and only if  $\mathcal{C}^{op}$  is, we have also defined a functor

$$\begin{aligned} \Phi^{op}: \text{Tate}^{el}(\mathcal{C}^{op})^{op} &\longrightarrow \text{Pro}^a(\text{Ind}^a(\mathcal{C}^{op}))^{op} \\ &=: \text{Ind}^a(\text{Ind}^a(\mathcal{C}^{op})^{op}) \\ &=: \text{Ind}^a(\text{Pro}^a(\mathcal{C})). \end{aligned}$$

Unwinding the definitions, we see that  $\Phi^{op}$  and  $\Phi$  are inverse equivalences.

It remains to show that  $\Phi$  preserves exact sequences. To see this, recall the equivalence  $\mathcal{E}\text{Tate}^{el}(\mathcal{C}) \simeq \text{Tate}^{el}(\mathcal{E}\mathcal{C})$  [BGW16c, Proposition 5.12], and observe that  $\mathcal{E}\mathcal{C}$  is idempotent complete if and only if  $\mathcal{C}$  is. The construction above therefore defines an equivalence

$$\mathcal{E}\text{Tate}^{el}(\mathcal{C}) \simeq \text{Tate}^{el}(\mathcal{E}\mathcal{C}) \xrightarrow[\simeq]{\Phi} \text{Tate}^{el}(\mathcal{E}\mathcal{C}^{op})^{op} \simeq (\mathcal{E}\text{Tate}^{el}(\mathcal{C}^{op}))^{op} \subset \mathcal{E}(\text{Pro}^a(\text{Ind}^a(\mathcal{C}))),$$

i.e.  $\Phi$  is fully exact. ■

**3.2. COROLLARY.** *Let  $\mathcal{C}$  be idempotent complete, and let  $V \in \text{Tate}^{el}(\mathcal{C})$ . Then the intersection  $\lim_{Gr(V)} L$  of all lattices in  $V$  is the zero object.*

**PROOF.** A cone on  $Gr(V)$  is equivalent to a map  $f: X \rightarrow V$  such that  $f$  factors through  $L \hookrightarrow V$  for every lattice  $L \in Gr(V)$ . The definition of the functor (3.1) shows that  $\Phi(f)$  is the zero map. But,  $\Phi$  is fully faithful, so therefore  $f$  is the zero map and  $0 \rightarrow V$  is the terminal cone on  $Gr(V)$ . ■

**3.3. REMARK.** We interpret the corollary as follows. Tate objects are a categorical abstraction of locally linearly compact topological vector spaces. In such a vector space, lattices form a basis of neighborhoods of the origin; in particular the intersection of all lattices is the origin itself. The corollary shows that the same holds for lattices in arbitrary Tate objects, provided the category  $\mathcal{C}$  is idempotent complete.

**3.4. COROLLARY.** *If  $\mathcal{C}$  is idempotent complete, then  $\text{Ind}^a(\mathcal{C})$  is right  $s$ -filtering in  $\text{Tate}^{el}(\mathcal{C})$ .*

**PROOF.** The sub-category  $\text{Ind}^a(\mathcal{C})$  is right filtering in  $\text{Tate}^{el}(\mathcal{C})$  by [BGW16c, Proposition 5.10 (2)]. Further, because  $\text{Ind}^a(\mathcal{C})$  is right special in  $\text{Pro}^a(\text{Ind}^a(\mathcal{C}))$  it is right special in  $\text{Tate}^{el}(\mathcal{C})$ . ■

3.5. PROPOSITION. *Let  $\mathcal{C}$  be idempotent complete. An exact equivalence  $\mathcal{C}^{op} \xrightarrow{\cong} \mathcal{C}$  extends to an exact equivalence*

$$\text{Tate}^{el}(\mathcal{C})^{op} \xrightarrow{\cong} \text{Tate}^{el}(\mathcal{C}).$$

*This duality restricts to exact equivalences*

$$\text{Ind}^a(\mathcal{C})^{op} \xrightarrow{\cong} \text{Pro}^a(\mathcal{C})$$

and

$$\text{Pro}^a(\mathcal{C})^{op} \xrightarrow{\cong} \text{Ind}^a(\mathcal{C}).$$

*If  $\mathcal{D}, \mathcal{D}'$  are full sub-categories such that the equivalence restricts to  $\mathcal{D}^{op} \rightarrow \mathcal{D}'$  and  $\mathcal{D}'^{op} \rightarrow \mathcal{D}$ , then this property is preserved:*

$$\text{Tate}^{el}(\mathcal{D})^{op} \xrightarrow{\cong} \text{Tate}^{el}(\mathcal{D}') \quad \text{Tate}^{el}(\mathcal{D}')^{op} \xrightarrow{\cong} \text{Tate}^{el}(\mathcal{D}).$$

PROOF. The duality is given by the composition of exact equivalences

$$\text{Tate}^{el}(\mathcal{C})^{op} \xrightarrow[\cong]{\Phi} \text{Tate}^{el}(\mathcal{C}^{op}) \xrightarrow{\cong} \text{Tate}^{el}(\mathcal{C})$$

where the first equivalence is that induced by  $\Phi$ , and the second is that induced by the duality on  $\mathcal{C}$  (cf. [BGW16c, Proposition 5.16]). The statement about restrictions is now immediate. ■

3.6. COROLLARY. *Suppose  $\mathcal{C}$  is an exact category with an exact equivalence  $\mathcal{C}^{op} \xrightarrow{\cong} \mathcal{C}$ . For every  $n \geq 0$ , there is a canonical exact equivalence*

$$n\text{-Tate}(\mathcal{C}^{ic})^{op} \xrightarrow{\cong} n\text{-Tate}(\mathcal{C}^{ic}).$$

PROOF. This follows by induction. Firstly, if  $(-)^{\vee} : \mathcal{C}^{op} \xrightarrow{\cong} \mathcal{C}$  is an exact equivalence, then the idempotent completion  $\mathcal{C}^{ic}$  also carries such an exact equivalence  $(\mathcal{C}^{ic})^{op} \xrightarrow{\cong} \mathcal{C}^{ic}$ , by sending  $(X, p)$  to  $(X^{\vee}, p^{\vee})$ , where  $p^{\vee}$  is the idempotent induced on the dual [Sch10, §5.1]. By Proposition 3.5 for every idempotent complete exact category, we get an equivalence on elementary Tate objects,  $\text{Tate}^{el}(\mathcal{C})^{op} \xrightarrow{\cong} \text{Tate}^{el}(\mathcal{C})$ . Taking  $\text{Tate}^{el}(\mathcal{C})$  for  $\mathcal{C}$  and repeating the above  $n$  times yields the claim. ■

## 4. External Homs

We now consider extensions of internal homs in  $\mathcal{C}$  to higher Tate objects. If  $U \in n\text{-Tate}(\mathcal{C})$  and  $V \in m\text{-Tate}(\mathcal{C})$ , we will construct an  $n + m$ -Tate object  $\underline{\text{Hom}}(V, W)$ , which we think of as an “external hom”.<sup>2</sup> We then explain in what sense the external hom  $\underline{\text{Hom}}(U, -)$  provides a right adjoint to  $U \overset{\rightarrow}{\otimes} -$  when  $U$  is a (higher) Tate object. If  $\mathcal{C}$  is in fact a rigid tensor category, we also show that, in a natural sense,  $- \overset{\rightarrow}{\otimes} U^{\vee}$  is right adjoint to  $- \overset{\rightarrow}{\otimes} U$ .

<sup>2</sup>These homs are ‘external’ in a sense analogous to the external tensor product of modules over different  $k$ -algebras.

4.1. PROPOSITION. *Let  $\mathcal{C}$  be an idempotent complete, closed monoidal, exact category in which both  $- \otimes -$  and  $\underline{Hom}_{\mathcal{C}}(-, -)$  are bi-exact. Then there exists a bi-exact functor*

$$\underline{Hom}(-, -): n\text{-Tate}(\mathcal{C})^{op} \times m\text{-Tate}(\mathcal{C}) \longrightarrow (n + m)\text{-Tate}(\mathcal{C}).$$

4.2. REMARK. The construction of the functors  $\underline{Hom}$  follows by an induction similar to the proof of Proposition 2.1. The key is that the diagrams defining the Tate objects in the first variable sandwich the diagrams defining the Tate objects in the second variable, i.e.

$$\text{hom}_{\text{Tate}(\mathcal{C})}(\text{colim}_I \lim_{J_i} V_{ij}, \text{colim}_K \lim_{L_k} W_{kl}) = \lim_I \text{colim}_K \lim_{L_k} \text{colim}_{J_i} \text{hom}_{\mathcal{C}}(V_{ij}, W_{kl}).$$

In particular, while the logic is very similar, we cannot just quote Lemma 2.2.

PROOF. By the universal property of idempotent completeness, it suffices to construct  $\underline{Hom}$  for elementary Tate objects. We construct the functors using three related inductions on  $(n, m)$ . First, we show that the construction for  $(n, n)$  implies it for  $(n + m, n)$  for all  $m$ . Second, we show that the construction for  $(n, n)$  implies it for  $(n, n + m)$  for all  $m$ . And last, we show that the construction for  $(n - 1, n - 1)$  implies it for  $(n, n)$ .

For the base case  $n = m = 0$ , there is nothing to show. Now suppose that we have constructed  $\underline{Hom}(-, -)$  for  $(n + m, n)$ . Let  $W \in n\text{-Tate}^{el}(\mathcal{C})$  and let  $V \in (n + m + 1)\text{-Tate}^{el}(\mathcal{C})$  be represented by an elementary Tate diagram

$$V: I \longrightarrow \text{Pro}^a((n + m)\text{-Tate}^{el}(\mathcal{C})).$$

For  $i \in I$ , let  $V_i$  be represented by an admissible Pro-diagram

$$V_i: J_i \longrightarrow (n + m)\text{-Tate}^{el}(\mathcal{C}).$$

Then we define

$$\underline{Hom}(V, W) := \lim_{I^{op}} \text{colim}_{J_i^{op}} \underline{Hom}(V_{i,j}, W) \in \text{Pro}^a(\text{Ind}^a((2n + m)\text{-Tate}^{el}(\mathcal{C}))).$$

This is invariant under different choices of representing diagrams. By the straightening construction for morphisms, we see the assignment  $(V, W) \mapsto \underline{Hom}(V, W)$  is indeed functorial. And, by the inductive hypothesis and the straightening construction for exact sequences, we see that it is bi-exact. In order to show that it factors through the inclusion

$$(2n + m + 1)\text{-Tate}^{el}(\mathcal{C}) \hookrightarrow \text{Pro}^a(\text{Ind}^a((2n + m)\text{-Tate}^{el}(\mathcal{C}))),$$

it suffices to exhibit a co-lattice of  $\underline{Hom}(V, W)$ . And indeed, the construction shows that

$$\underline{Hom}(V, W) \twoheadrightarrow \underline{Hom}(L, W)$$

is a co-lattice, for any lattice  $L \hookrightarrow V$ . This completes the first induction.

The second induction follows by a similar argument. Suppose we have constructed  $\underline{Hom}(-, -)$  for  $(n, n + m)$ . Let  $V \in n\text{-Tate}^{el}(\mathcal{C})$  and let  $W \in (n + m + 1)\text{-Tate}^{el}(\mathcal{C})$  be represented by an elementary Tate diagram

$$W: I \longrightarrow \text{Pro}^a((n + m)\text{-Tate}^{el}(\mathcal{C})).$$

For  $i \in I$ , let  $V_i$  be represented by an admissible Pro-diagram

$$W_i: J_i \longrightarrow (n + m)\text{-Tate}^{el}(\mathcal{C}).$$

Then we define

$$\underline{Hom}(V, W) := \text{colim}_I \lim_{J_i} \underline{Hom}(V, W_{i,j}) \in \text{Ind}^a(\text{Pro}^a((2n + m)\text{-Tate}^{el}(\mathcal{C}))).$$

As above, this assignment is functorial and bi-exact. We see that it takes values in the category  $2n + m + 1\text{-Tate}$  objects by observing that for any lattice  $L \hookrightarrow W$ , the inclusion

$$\underline{Hom}(V, L) \hookrightarrow \underline{Hom}(V, W)$$

is a lattice.

It remains to show the third induction. Suppose that we have constructed  $\underline{Hom}(-, -)$  for  $(n - 1, n - 1)$ . Let  $V, W \in n\text{-Tate}^{el}(\mathcal{C})$ , and let

$$V: I \longrightarrow \text{Pro}^a((n - 1)\text{-Tate}^{el}(\mathcal{C})) \quad \text{and} \quad W: L \longrightarrow \text{Pro}^a((n - 1)\text{-Tate}^{el}(\mathcal{C}))$$

be elementary Tate diagrams representing  $V$  and  $W$ . For each  $i \in I$  and  $\ell \in L$ , let

$$V_i: J_i \longrightarrow \text{Pro}^a((n - 1)\text{-Tate}^{el}(\mathcal{C}))$$

and

$$W_\ell: K_\ell \longrightarrow \text{Pro}^a((n - 1)\text{-Tate}^{el}(\mathcal{C}))$$

be admissible Pro-diagrams representing  $V_i$  and  $W_\ell$ . Then we define

$$\begin{aligned} \underline{Hom}(V, W) &:= \lim_{I^{op}} \text{colim}_L \lim_{K_\ell} \text{colim}_{J_i^{op}} \underline{Hom}(V_{i,j}, W_{\ell,k}) \\ &\in \text{Pro}^a(\text{Ind}^a(\text{Pro}^a(\text{Ind}^a((2n - 2)\text{-Tate}^{el}(\mathcal{C}))))). \end{aligned}$$

This assignment is invariant under different choices of representing diagrams. By the straightening construction for morphisms, we see the assignment  $V \mapsto \underline{Hom}(V, W)$  is indeed functorial. And, by the inductive hypothesis and the straightening construction for exact sequences, we see that it is bi-exact.

It remains to show that it factors through the sub-category of  $2n\text{-Tate}$  objects. For any lattice  $L \hookrightarrow V$ , we obtain, by the bi-exactness of  $\underline{Hom}(-, -)$ , an exact sequence

$$\underline{Hom}(V/L, W) \hookrightarrow \underline{Hom}(V, W) \twoheadrightarrow \underline{Hom}(L, W).$$

Because the sub-category of elementary  $2n$ -Tate objects is closed under extensions in

$$\mathrm{Pro}^a(\mathrm{Ind}^a(\mathrm{Pro}^a(\mathrm{Ind}^a((2n-2)\text{-Tate}^{el}(\mathcal{C}))))))$$

by [BGW16c, Theorem 5.6] and Proposition 3.1, it suffices to show that the kernel and cokernel terms of this exact sequence are  $2n$ -Tate objects. We see this directly, as follows. Let

$$V/L: I \longrightarrow (n-1)\text{-Tate}(\mathcal{C})$$

be an admissible Ind-diagram representing  $V/L$ . Then, by definition, we have

$$\begin{aligned} \underline{Hom}(V/L, W) &\cong \lim_{I^{op}} \underline{Hom}((V/L)_i, W) \\ &\in \mathrm{Pro}^a((2n-1)\text{-Tate}^{el}(\mathcal{C})) \\ &\subset (2n)\text{-Tate}^{el}(\mathcal{C}) \end{aligned}$$

where the assertion that this is a Pro-object follows by inductive hypothesis and the first statement we showed. Similarly, let

$$L: J \longrightarrow (n-1)\text{-Tate}(\mathcal{C})$$

be an admissible Pro-diagram representing  $L$ . Let

$$W: A \longrightarrow \mathrm{Pro}^a((n-1)\text{-Tate}(\mathcal{C}))$$

be an elementary Tate diagram representing  $W$ , and for each  $a \in A$  let

$$W_a: B_a \longrightarrow (n-1)\text{-Tate}(\mathcal{C})$$

be an admissible Pro-diagram representing  $W_a$ . Then, for each  $a \in A$ , we have

$$\begin{aligned} \underline{Hom}(L, W_a) &\cong \lim_{B_a} \mathrm{colim}_{J^{op}} \underline{Hom}(L_j, W_{a,b}) \\ &\in \mathrm{Pro}^a(\mathrm{Ind}^a((2n-2)\text{-Tate}(\mathcal{C}))) \\ &\subset \mathrm{Pro}^a((2n-1)\text{-Tate}(\mathcal{C})). \end{aligned}$$

Thus, the assignment  $a \mapsto \underline{Hom}(L, W_a)$  defines an admissible Ind-diagram in  $\mathrm{Pro}^a((2n-1)\text{-Tate}(\mathcal{C}))$ . To see that it is in fact an elementary Tate diagram, we observe that, for each  $a < a'$ , we have an exact sequence

$$\underline{Hom}(L, W_a) \hookrightarrow \underline{Hom}(L, W_{a'}) \twoheadrightarrow \underline{Hom}(L, W_a/W_{a'}),$$

Because  $W_a/W_{a'} \in (n-1)\text{-Tate}(\mathcal{C})$  and  $L \in \mathrm{Pro}^a((n-1)\text{-Tate}(\mathcal{C}))$ , we conclude, by inductive hypothesis, that the cokernel  $\underline{Hom}(L, W_a/W_{a'})$  is a  $(2n-1)$ -Tate object, and therefore that

$$\underline{Hom}(L, W) \cong \mathrm{colim}_A \underline{Hom}(L, W_a)$$

is an elementary  $2n$ -Tate object as claimed. ■

4.3. PROPOSITION. *Let  $\mathcal{C}$  be an idempotent complete, closed monoidal exact category in which both  $- \otimes -$  and  $\underline{Hom}_{\mathcal{C}}(-, -)$  are bi-exact.*

1. *Given  $U \in \mathcal{C}$ , and  $V, W \in n\text{-Tate}(\mathcal{C})$ , there exists a canonical natural isomorphism*

$$\text{hom}_{n\text{-Tate}(\mathcal{C})}(U \overrightarrow{\otimes} V, W) \cong \text{hom}_{n\text{-Tate}(\mathcal{C})}(V, \underline{Hom}(U, W)).$$

2. *Given  $V \in \mathcal{C}$ , and  $U, W \in n\text{-Tate}(\mathcal{C})$ , there exists a canonical natural isomorphism*

$$\text{hom}_{n\text{-Tate}(\mathcal{C})}(U \overrightarrow{\otimes} V, W) \cong \text{hom}_{2n\text{-Tate}(\mathcal{C})}(V, \underline{Hom}(U, W)).$$

PROOF. The proof follows by induction on  $n$  and formal manipulation of limits and colimits. For  $n = 0$ , there is nothing to show. Now suppose we have shown the statement for  $n$ . In the first case, suppose  $U \in \mathcal{C}$  and let  $V, W \in (n + 1)\text{-Tate}(\mathcal{C})$  be represented by elementary Tate diagrams

$$V: I \longrightarrow \text{Pro}^a(n\text{-Tate}(\mathcal{C})) \quad \text{and} \quad W: A \longrightarrow \text{Pro}^a(n\text{-Tate}(\mathcal{C}))$$

and for each  $i \in I$  and  $a \in A$ , let  $V_i$  be represented by an admissible Pro-diagram

$$V_i: J_i \longrightarrow n\text{-Tate}(\mathcal{C})$$

and let  $W_a$  be represented by an admissible Pro-diagram

$$W_a: B_a \longrightarrow n\text{-Tate}(\mathcal{C}).$$

Then we have

$$\begin{aligned} \text{hom}_{(n+1)\text{-Tate}(\mathcal{C})}(\mathcal{C})(U \overrightarrow{\otimes} V, W) &\cong \lim_{I^{op}} \text{colim}_A \lim_{B_a} \text{colim}_{J_i^{op}} \text{hom}_{n\text{-Tate}(\mathcal{C})}(U \overrightarrow{\otimes} V_{i,j}, W_{a,b}) \\ &\cong \lim_{I^{op}} \text{colim}_A \lim_{B_a} \text{colim}_{J_i^{op}} \text{hom}_{n\text{-Tate}(\mathcal{C})}(V_{i,j}, \underline{Hom}(U, W_{a,b})) \\ &\cong \text{hom}_{(n+1)\text{-Tate}(\mathcal{C})}(V, \underline{Hom}(U, W)). \end{aligned}$$

Similarly, if  $V \in \mathcal{C}$  and  $U, W \in (n + 1)\text{-Tate}(\mathcal{C})$  are represented by elementary Tate diagrams

$$U: I \longrightarrow \text{Pro}^a(n\text{-Tate}(\mathcal{C})) \quad \text{and} \quad W: A \longrightarrow \text{Pro}^a(n\text{-Tate}(\mathcal{C}))$$

and for each  $i \in I$  and  $a \in A$ , let  $U_i$  be represented by an admissible Pro-diagram

$$U_i: J_i \longrightarrow n\text{-Tate}(\mathcal{C})$$

and let  $W_a$  be represented by an admissible Pro-diagram

$$W_a: B_a \longrightarrow n\text{-Tate}(\mathcal{C}).$$

Then we have

$$\begin{aligned} \mathrm{hom}_{(n+1)\text{-Tate}(\mathcal{C})}(\mathcal{C})(U \overrightarrow{\otimes} V, W) &\cong \lim_{I^{op}} \mathrm{colim}_A \lim_{B_a} \mathrm{colim}_{J_i^{op}} \mathrm{hom}_{n\text{-Tate}(\mathcal{C})}(U_{i,j} \overrightarrow{\otimes} V, W_{a,b}) \\ &\cong \lim_{I^{op}} \mathrm{colim}_A \lim_{B_a} \mathrm{colim}_{J_i^{op}} \mathrm{hom}_{2n\text{-Tate}(\mathcal{C})}(V, \underline{\mathrm{Hom}}(U_{i,j}, W_{a,b})) \\ &\cong \mathrm{hom}_{(2n+2)\text{-Tate}(\mathcal{C})}(V, \underline{\mathrm{Hom}}(U, W)). \end{aligned}$$

■

4.4. PROPOSITION. *Let  $\mathcal{C}$  be an idempotent complete rigid tensor category in which the tensor product is bi-exact, i.e. suppose there exists an exact duality*

$$(-)^\vee : \mathcal{C}^{op} \xrightarrow{\cong} \mathcal{C}$$

along with a natural isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(-, -) \cong (-)^\vee \otimes -.$$

1. For  $U \in \mathcal{C}$  and  $V \in n\text{-Tate}(\mathcal{C})$ , there is a canonical natural isomorphism

$$V \overrightarrow{\otimes} U^\vee \cong \underline{\mathrm{Hom}}(U, V),$$

where  $V^\vee$  denotes the extension of the duality  $(-)^\vee$  to  $n\text{-Tate}(\mathcal{C})$  as in Proposition 3.5.

2. For  $V \in n\text{-Tate}(\mathcal{C})$  and  $U, W \in m\text{-Tate}(\mathcal{C})$ , there is a canonical natural isomorphism

$$\mathrm{hom}_{(n+m)\text{-Tate}(\mathcal{C})}(U \overrightarrow{\otimes} V, W) \cong \mathrm{hom}_{(n+m)\text{-Tate}(\mathcal{C})}(U, W \overrightarrow{\otimes} V^\vee).$$

PROOF. Both of these statements follow from induction and formal manipulation of limits and colimits. For the first statement, for  $n = 0$ , there is nothing to show. Now suppose we have shown the first statement for  $n$ . Let  $U \in \mathcal{C}$  and let  $V \in (n+1)\text{-Tate}(\mathcal{C})$  be represented by an elementary Tate diagram

$$V : I \longrightarrow \mathrm{Pro}^a(n\text{-Tate}(\mathcal{C}))$$

and for each  $i \in I$ , let  $V_i$  be represented by an admissible Pro-diagram

$$V_i : J_i \longrightarrow n\text{-Tate}(\mathcal{C}).$$

Then we have

$$\begin{aligned} V \overrightarrow{\otimes} U^\vee &\cong \mathrm{colim}_I \lim_{J_i} V_{i,j} \overrightarrow{\otimes} U^\vee \\ &\cong \mathrm{colim}_I \lim_{J_i} \underline{\mathrm{Hom}}(U, V_{i,j}) \\ &\cong \underline{\mathrm{Hom}}(U, V). \end{aligned}$$

This completes the induction step and the proof of the first statement.

For the second statement, the case for  $n = 0$  and arbitrary  $m$  follows from the first statement combined with Proposition 4.3. Now suppose we have shown the second statement for  $n$  and  $m$ . Let  $U, W \in m\text{-Tate}(\mathcal{C})$  and let  $V \in (n + 1)\text{-Tate}(\mathcal{C})$  be represented by an elementary Tate diagram

$$V: I \longrightarrow \text{Pro}^a(n\text{-Tate}(\mathcal{C}))$$

and for each  $i \in I$ , let  $V_i$  be represented by an admissible Pro-diagram

$$V_i: J_i \longrightarrow n\text{-Tate}(\mathcal{C}).$$

Then we have

$$\begin{aligned} \text{hom}_{(n+m+1)\text{-Tate}(\mathcal{C})}(U \overrightarrow{\otimes} V, W) &\cong \lim_{I^{op}} \text{colim}_{J_i^{op}} \text{hom}_{(n+m)\text{-Tate}(\mathcal{C})}(U \overrightarrow{\otimes} V_{i,j}, W) \\ &\cong \lim_{I^{op}} \text{colim}_{J_i^{op}} \text{hom}_{(n+m)\text{-Tate}(\mathcal{C})}(U, W \otimes V_{i,j}^\vee) \\ &\cong \text{hom}_{(n+m+1)\text{-Tate}(\mathcal{C})}(U, W \otimes V^\vee). \end{aligned}$$

This completes the induction and the proof of the second statement. ■

4.5. REMARK.

1. For  $U \in n\text{-Tate}(\mathcal{C})$ ,  $V \in m\text{-Tate}(\mathcal{C})$  and  $W \in \ell\text{-Tate}(\mathcal{C})$  with  $n \geq m + \ell$ , the abelian groups

$$\text{hom}_{(n+m)\text{-Tate}(\mathcal{C})}(U \overrightarrow{\otimes} V, W)$$

and

$$\text{hom}_{n\text{-Tate}(\mathcal{C})}(U, W \overrightarrow{\otimes} V^\vee)$$

are not isomorphic in general. It suffices to take  $n = 2$ ,  $m = \ell = 1$  to see this.

2. For  $U \in n\text{-Tate}(\mathcal{C})$ ,  $V \in m\text{-Tate}(\mathcal{C})$  and  $W \in \ell\text{-Tate}(\mathcal{C})$  with  $\ell \geq n + m$ , the abelian groups

$$\text{hom}_{\ell\text{-Tate}(\mathcal{C})}(U \overrightarrow{\otimes} V, W)$$

and

$$\text{hom}_{(m+\ell)\text{-Tate}(\mathcal{C})}(U, W \overrightarrow{\otimes} V^\vee)$$

are not isomorphic in general. Similarly, it suffices to take  $n = m = 1$  and  $\ell = 2$  to see this.

### 5. Formal tubular neighbourhoods

One of the most prominent applications of Tate categories is the study of the geometry of a variety using adelic methods. In this section, we shall show that the Beilinson adèles along a flag of points in a variety can be written as a normally ordered tensor product. In the literature this seems to have gone unnoticed so far.

Recall that  $n\text{-Tate}(k)$  is a shorthand for  $n\text{-Tate}(\text{Vect}_f(k))$ , the category of  $n$ -Tate objects over the category of finite-dimensional  $k$ -vector spaces, and note that because the tensor product of vector spaces is bi-exact, the categories of flat (higher) Tate objects and all (higher) Tate objects coincide.

As we have seen in §2, every bi-right exact functor  $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  induces a bi-exact monoidal structure on flat Tate objects, i.e.

$$-\overset{\rightarrow}{\otimes}- : n\text{-Tate}^b(\mathcal{C}) \times m\text{-Tate}^b(\mathcal{C}) \longrightarrow (n+m)\text{-Tate}^b(\mathcal{C})$$

and following Waldhausen [Wal85, §1.5] every bi-exact functor induces a pairing in algebraic  $K$ -theory:

5.1. DEFINITION. *On the level of  $K$ -theory groups, this takes the form*

$$\overset{\rightarrow}{\otimes} : K_p(n\text{-Tate}^b(\mathcal{C})) \times K_q(m\text{-Tate}^b(\mathcal{C})) \longrightarrow K_{p+q}((n+m)\text{-Tate}^b(\mathcal{C})).$$

We will also call this product “ $\overset{\rightarrow}{\otimes}$ ” the external product in the  $K$ -theory of flat Tate objects.

*Conventions for geometry and adèles:* All morphisms of schemes will tacitly be assumed to be separated. For us a *variety* is a scheme of finite type over a field. If  $X$  is a scheme and  $x, y \in X$  scheme points, we write  $x \geq y$  if  $y \in \overline{\{x\}}$ , and  $x > y$  if additionally  $x \neq y$ . Write  $S(X)_\bullet$  for the simplicial set of flags of points in a scheme,  $A(K, \mathcal{F})$  for the Parshin–Beilinson adèles for a subset  $K_\bullet \subseteq S(X)_\bullet$ , and a quasi-coherent sheaf  $\mathcal{F}$  (see [Bei80, §2], [BGW16a, §2.1] for details). If  $X$  is a purely  $n$ -dimensional scheme, an element  $\Delta = (\eta_0 > \dots > \eta_n) \in S(X)_n$  with  $\text{codim}_X \overline{\{\eta_i\}} = i$  will be called a *saturated flag*.

5.2. DEFINITION. *Let  $X$  be an integral Noetherian scheme. For scheme points  $x, y \in X$ , we define a 1-Tate object in coherent sheaves supported on the Zariski closure  $\overline{\{x\}}$ :*

$${}_x\Lambda_y := \text{colim}_{\mathcal{F} \subseteq \mathcal{O}_y} \lim_j \mathcal{F}/\mathcal{I}_x^j \in 1\text{-Tate}^b(\text{Coh}_{\overline{\{x\}}}(X)), \tag{5.1}$$

where  $\mathcal{F}$  runs through all coherent sub-sheaves of the quasi-coherent sheaf  $\mathcal{O}_y$ ,  $j \in \mathbb{Z}_{\geq 1}$ , and  $\mathcal{I}_x$  denotes the ideal sheaf of  $\overline{\{x\}}$ . This defines an exact functor

$$\begin{aligned} {}_x\Omega_y : \text{Coh } X &\longrightarrow 1\text{-Tate}^b(\text{Coh}_{\overline{\{x\}}}(X)) \\ \mathcal{G} &\longmapsto \mathcal{G} \overset{\rightarrow}{\otimes} {}_x\Lambda_y, \end{aligned}$$

where  $\overrightarrow{\otimes}$  is the tensor product with  $\mathcal{G}$ , viewed as a 0-Tate object. More explicitly, this just means that

$${}_x\Omega_y(\mathcal{G}) = \operatorname{colim}_{\mathcal{F} \subseteq \mathcal{O}_y} \lim_j (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}) / \mathcal{I}_x^j$$

with the same notation as in Equation 5.1.

PROOF. Let us justify why this definition makes sense: Firstly, each  $\mathcal{F}/\mathcal{I}_x^j$  is a coherent sheaf with support contained in  $\overline{\{x\}}$  since  $j \geq 1$ . Thus, for every coherent sub-sheaf  $\mathcal{F}$  of  $\mathcal{O}_y$ ,  $\lim_j \mathcal{F}/\mathcal{I}_x^j$  defines an object in  $\operatorname{Pro}^a(\operatorname{Coh}_{\overline{\{x\}}}(X))$ . By the Artin–Rees Lemma this object is actually flat in the sense of Definition 2.10. Hence,  ${}_x\Lambda_y$  is an Ind-object of flat admissible Pro-objects. For any scheme point  $y$ , the morphism  $t : \operatorname{Spec} \mathcal{O}_y \rightarrow X$  gives rise to a natural morphism  $\mathcal{O}_X \rightarrow t_* t^* \mathcal{O}_X \cong \mathcal{O}_y$ . On stalks this morphism amounts to a localization, i.e. the kernel  $\ker(\mathcal{O}_X \rightarrow \mathcal{O}_y)$  consists only of torsion elements of the multiplicative system of the localization, and since  $X$  is an integral scheme, no non-zero such can exist. We may therefore regard  $\mathcal{O}_X$  as a subsheaf of  $\mathcal{O}_y$ . Then the concrete choice  $\mathcal{F} := \mathcal{O}_X$  gives rise to the short exact sequence

$$0 \longrightarrow \lim_j \mathcal{O}_X / \mathcal{I}_x^j \longrightarrow {}_x\Lambda_y \longrightarrow \operatorname{colim}_{\mathcal{F} \subseteq \mathcal{O}_y} \mathcal{F} / \mathcal{O}_X \longrightarrow 0$$

in the category of admissible Ind-Pro objects. As the first term is a Pro-object and the last an Ind-object, it follows that  ${}_x\Lambda_y$  is a (flat) Tate object, and thus lies in  $1\text{-Tate}^b(\operatorname{Coh}_{\overline{\{x\}}}(X))$ . The exactness of the functor  ${}_x\Omega_y$  follows from the flatness of  $\mathcal{O}_y$  over  $\mathcal{O}_X(U)$  for any affine open  $U$  containing  $y$ , and again the Artin–Rees Lemma. ■

5.3. REMARK. Following a suggestion of the referee, let us point out that one may think of this definition as

$${}_x\Lambda_y \sim \mathcal{O}_{X, \widehat{\{x\} \cap y}},$$

at least philosophically.

5.4. THEOREM. *Let  $k$  be a field. Suppose  $X/k$  is a purely  $n$ -dimensional integral Noetherian  $k$ -scheme and  $\Delta = (\eta_0 > \dots > \eta_n)$  a saturated flag. Then the diagram of functors*

$$\begin{array}{ccc} \operatorname{Coh} X & \times \cdots \times & \operatorname{Coh} X \xrightarrow{\otimes} \operatorname{Coh} X \\ \downarrow \eta_m \Omega_{\eta_{m-1}} & & \eta_1 \Omega_{\eta_0} \downarrow \qquad \qquad \qquad A(\Delta, -) \downarrow \\ 1\text{-Tate}^b(\operatorname{Coh}_{\{\eta_m\}}(X)) & \times \cdots \times & 1\text{-Tate}^b(\operatorname{Coh}_{\{\eta_1\}}(X)) \xrightarrow{\overrightarrow{\otimes}} n\text{-Tate}^b(\operatorname{Coh}_{\{\eta_m\}}(X)) \end{array}$$

is commutative. Here  $A(\Delta, -)$  denotes the Beilinson adèles (as in [Bei80]).

Before proving this result, we note the most important consequence: The Beilinson adèles along a flag can functorially be written as a normally ordered tensor product:

5.5. COROLLARY. (TUBULAR DECOMPOSITION) *With the same assumptions as in the theorem, we have an isomorphism*

$$A(\Delta, \mathcal{O}_X) \cong {}_{\eta_n} \Lambda_{\eta_{n-1}} \overrightarrow{\otimes} \cdots \overrightarrow{\otimes} {}_{\eta_1} \Lambda_{\eta_0}$$

in the category of  $n$ -Tate objects of coherent sheaves with support in the closed point  $\eta_n$ .

PROOF. We prove the theorem. The horizontal arrow in the top row is the tensor product of  $\mathcal{O}_X$ -modules. This is a right  $n$ -polyexact functor, i.e. for any bracketing  $((\dots)\dots)$  decomposing it into a concatenation of  $n - 1$  functors in two arguments, all these two-argument functors are bi-right exact. This holds as the individual tensor product

$$\text{Coh } X \times \text{Coh } X \longrightarrow \text{Coh } X$$

is bi-right exact. The horizontal arrow in the bottom row is the external product of flat Tate objects, induced from the tensor product of  $\mathcal{O}_X$ -modules with support,

$$\text{Coh}_{Z_1} X \times \text{Coh}_{Z_2} X \longrightarrow \text{Coh}_{Z_1 \cap Z_2} X$$

for  $Z_1, Z_2$  arbitrary closed subsets of  $X$ . As this is also bi-right exact, the results of §2 apply. Thus, all ingredients of the diagram are well-defined. Next, we need to check commutativity. Suppose we start from the object  $\mathcal{G}_{n-1} \times \cdots \times \mathcal{G}_0$  in the product category in the upper left corner of the diagram. For  $0 \leq r < n$  we let

$$T^r := {}_{\eta_{r+1}} \Lambda_{\eta_r} = \text{colim}_{\mathcal{F}_r \subseteq \mathcal{O}_{\eta_r}} \lim_{j_{r+1}} \frac{\mathcal{G}_r \otimes \mathcal{F}_r}{\mathcal{I}_{\eta_{r+1}}^{j_{r+1}}} \in 1\text{-Tate}^b(\text{Coh}_{\{\eta_{r+1}\}}(X)),$$

where  $\mathcal{F}_r$  runs through the coherent sub-sheaves of  $\mathcal{O}_{\eta_r}$ . Unravel the construction of the tensor product following §2. This yields

$$T^{n-1} \overrightarrow{\otimes} \cdots \overrightarrow{\otimes} T^0 = \text{colim}_{\mathcal{F}_0 \subseteq \mathcal{O}_{\eta_0}} \lim_{j_1} \cdots \text{colim}_{\mathcal{F}_{n-1} \subseteq \mathcal{O}_{\eta_{n-1}}} \lim_{j_n} (\mathcal{G}_{n-1} \otimes \cdots \otimes \mathcal{G}_0) \otimes \frac{\mathcal{F}_{n-1}}{\mathcal{I}_{\eta_n}^{j_n}} \otimes \cdots \otimes \frac{\mathcal{F}_0}{\mathcal{I}_{\eta_1}^{j_1}}, \tag{5.2}$$

where the tensor products on the right are those of  $\text{Mod}_{\mathcal{O}_X}$ . On the other hand, following the inductive definition of the adèles as an  $n$ -Tate object (see for example [BGW16c] or [BGW16b]), we get

$$\begin{aligned} A(\eta_0 > \cdots > \eta_n, \mathcal{G}_{n-1} \otimes \cdots \otimes \mathcal{G}_0) &= \text{colim}_{\mathcal{F}_0 \subseteq \mathcal{O}_{\eta_0}} \lim_{j_1} A(\eta_2 > \cdots > \eta_n, \mathcal{F}_0 \otimes \mathcal{O}_{\eta_1} / \mathcal{I}_{\eta_1}^{j_1}) \\ &= \cdots = \text{colim}_{\mathcal{F}_0 \subseteq \mathcal{O}_{\eta_0}} \lim_{j_1} \cdots \text{colim}_{\mathcal{F}_{n-1} \subseteq \mathcal{O}_{\eta_{n-1}}} \lim_{j_n} (\mathcal{G}_{n-1} \otimes \cdots \otimes \mathcal{G}_0) \otimes \frac{\mathcal{F}_0}{\mathcal{I}_{\eta_1}^{j_1}} \otimes \cdots \otimes \frac{\mathcal{F}_{n-1}}{\mathcal{I}_{\eta_n}^{j_n}}, \end{aligned}$$

which is of course literally the same object, thanks to the symmetry of the tensor product in  $\mathcal{O}_X$ -module sheaves. If  $\mathcal{G}, \mathcal{G}'$  are coherent sheaves with supports in  $Z$  and  $Z'$ , then  $\mathcal{G} \otimes \mathcal{G}'$  has support in  $Z \cap Z'$  and thus this  $n$ -Tate object actually lies in  $n\text{-Tate}^b(\text{Coh}_W(X))$  with  $W = \overline{\{\eta_n\}} \cap \cdots \cap \overline{\{\eta_1\}} = \overline{\{\eta_n\}}$ . While this was just a verification of the commutativity on the level of objects, all our steps were natural in all objects  $\mathcal{G}_{n-1}, \dots, \mathcal{G}_0$ , so morphisms between objects get induced compatibly as well. The corollary follows by evaluation of the object  $\mathcal{O}_X \times \cdots \times \mathcal{O}_X$  from the upper left corner in the lower right corner in the two compatible ways. ■

5.6. COROLLARY. *With the same assumptions as in the theorem, the restriction to vector bundles in the top row*

$$\begin{array}{ccc}
 \text{VB } X & \times \cdots \times & \text{VB } X \xrightarrow{\quad \otimes \quad} \text{VB } X \\
 \downarrow \eta_m \Omega_{\eta_{m-1}} & & \eta_1 \Omega_{\eta_0} \downarrow \qquad \qquad \qquad A(\Delta, -) \downarrow \\
 1\text{-Tate}^b(\text{Coh}_{\{\eta_m\}}(X)) & \times \cdots \times & 1\text{-Tate}^b(\text{Coh}_{\{\eta_1\}}(X)) \xrightarrow[\otimes]{} n\text{-Tate}^b(\text{Coh}_{\{\eta_m\}}(X))
 \end{array}$$

is a commutative diagram whose horizontal functors are  $n$ -polyexact and downward functors exact.

PROOF. The downward functors are always exact, even for coherent sheaves. The bottom horizontal arrow is exact as we work with flat Tate objects. The top horizontal functor now is exact by the local flatness of vector bundles. ■

5.7. REMARK. We may also read

$$L := \text{colim}_{\mathcal{F} \subseteq \mathcal{O}_y} \lim_j \mathcal{F}/\mathcal{I}_x^j$$

as a  $k$ -algebra if we decide to carry out the limit and colimit. We get a canonical  $k$ -algebra homomorphism  $\mathcal{O}_y \rightarrow L$ . In particular, every unit  $f \in \mathcal{O}_y^\times$  acts by multiplication on  $L$ . As any such multiplication is compatible with the Ind- and Pro-limit, it also determines an automorphism of  ${}_x\Lambda_y$  as an object in  $1\text{-Tate}^b(\text{Coh}_{\{x\}}(X))$ . Finally, every automorphism of an object in an exact category defines a canonical element in the  $K_1$ -group of this category. Thus, we get a canonical group homomorphism

$$[-] : \mathcal{O}_y^\times \longrightarrow K_1(1\text{-Tate}^b(\text{Coh}_{\{x\}}(X))).$$

$\circlearrowleft$   ${}_x\Lambda_y$

5.8. DEFINITION. *In the situation of the previous remark, write*

$$[f] \in K_1(\text{Tate}^b(\text{Coh}_{\{x\}}(X)))$$

$\circlearrowleft$   ${}_x\Lambda_y$

for the image of an element  $f \in \mathcal{O}_y^\times$ .

## 6. Adèles

6.1. MOTIVATION/THE CLASSICAL CASE. Let us recall the relation between the degree of a line bundle on a curve and the adèles. To this end, let  $k$  be a field,  $\pi : X \rightarrow k$  an integral smooth proper curve and  $L$  a line bundle on  $X$ . Using the Weil uniformization of the Picard group, the isomorphism class of  $L$  has a unique representative in

$$\text{Pic } X = k(X)^\times \backslash \mathbf{A}^\times / \mathbf{O}^\times, \tag{6.1}$$

where  $\mathbf{A}$  denotes the adèles, so that  $\mathbf{A}^\times$  are the idèles,  $\mathbf{O}$  the integral adèles and  $k(X)$  the rational function field. The multiplicative group  $\mathbf{A}^\times$  acts on the adèles via multiplication,

$$\mathbf{A}^\times \curvearrowright \mathbf{A},$$

but the adèles can also be regarded as a 1-Tate object of finite-dimensional  $k$ -vector spaces, and this action induces an automorphism of this 1-Tate object. Like any automorphism, this pins down a unique element in the  $K_1$ -group of the category. In other words, we get a group homomorphism

$$\mathbf{A}^\times \longrightarrow K_1(\mathbf{Tate}(k)) \cong \mathbb{Z}. \tag{6.2}$$

It is a classical computation that this morphism sends any representative of the line bundle, as in Equation 6.1, to the degree of the line bundle:

6.2. THEOREM. (WEIL) *This morphism sends a line bundle  $L$  to its degree  $\deg L$ .*

6.3. ELABORATION. Of course, classically this fact has been formulated without  $K$ -theory. Indeed, (6.2) can be made concrete as follows: Choose a splitting  $\mathbf{A} \simeq E \oplus \mathbf{O}$  as  $k$ -vector spaces. Now, let  $f \in \mathbf{A}^\times$  act. Then for any  $k$ -vector subspace  $E' \subseteq \mathbf{A}$  such that  $E, fE \subseteq E'$  and such that both subspaces are of finite codimension in  $E'$ , take  $\dim(E'/E) - \dim(E'/fE) \in \mathbb{Z}$ . To be sure that this makes sense, one has to prove that  $E'$  exists and the resulting integer is independent of its choice. The  $K$ -theoretic approach sweeps these technicalities under the rug.

Now, for the present article, we modify the viewpoint: There is an exact push-forward functor

$$\pi_* : \mathbf{Tate}^b(\mathbf{Coh}_0(X)) \longrightarrow \mathbf{Tate}(k),$$

where  $\mathbf{Coh}_0(X)$  are coherent sheaves of zero-dimensional support. In particular, if  $x$  denotes a closed point, all  $[f]$  in Definition 5.8, taking values in the  $K$ -theory of the category  $\mathbf{Tate}^b(\mathbf{Coh}_{\{x\}}(X))$  also define classes in the  $K$ -theory of  $\mathbf{Coh}_0(X)$ . Line bundles are often given as Čech cocycle representatives in  $H^1(X, \mathbb{G}_m)$ . The corresponding reformulation of (6.2) becomes:

6.4. PROPOSITION. (WEIL - REPHRASED) *Let  $\pi : X \rightarrow k$  be an integral smooth proper curve with generic point  $\eta_0$  and  $(f_{\nu\mu})_{\nu\mu} \in H^1(X, \mathbb{G}_m)$  an alternating Čech representative of a line bundle  $L$  in a finite open cover  $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ ,  $I$  totally ordered. For any  $x \in X$ , let  $\alpha(x)$  be the smallest element of  $I$  such that  $x \in U_{\alpha(x)}$ . Then*

$$\deg(L) = - \sum_{\eta_1} \pi_* [f_{\alpha(\eta_1)\alpha(\eta_0)}]_{\curvearrowright_{\eta_1} \mathbf{A}_{\eta_0}} \tag{6.3}$$

*and the sum has only finitely many non-zero summands. The right-hand side defines an element of  $K_1(\mathbf{Tate}(k)) \cong \mathbb{Z}$ , and this integer is the degree of  $L$ . Here the sum runs over all closed points  $\eta_1 \in X$ .*

This is the same statement as in Weil’s theorem. The sum in (6.3) corresponds to making the computation in (6.2) locally at all points of the curve. As our line bundle is

given on a concrete finite open cover, it turns out to suffice to work on these opens. We will not give a separate proof for the above proposition as it will be the one-dimensional special case of Theorem 6.6.

Let us discuss how to generalize these ideas to surfaces. To begin, we consider an easy geometric example, and also just locally. In a classical local field, say

$$\mathcal{K}_{X,x} \simeq \kappa(x)((t)) \quad \text{for} \quad \mathcal{K}_{X,x} := \text{Frac } \mathcal{O}_{X,x}$$

on a curve  $X$ , the local multiplicity of a line bundle is a measure of compatibility between a given  $k$ -vector space splitting  $\mathcal{K}_{X,x} \simeq A \oplus \mathcal{O}_{X,x}$  and a local idèle component acting on it. This corresponds to the valuation of the idèle at this point. For example, if

$$f_{\alpha(\eta_1)\alpha(\eta_0)} = t^r$$

for some  $r \geq 0$  in  $\mathcal{K}_{X,x} \simeq \kappa(x)((t))$ , then the multiplicity is

$$\dim(\mathcal{O}_{X,x}/t^r \mathcal{O}_{X,x}) = r.$$

Now suppose  $\pi : X \rightarrow k$  is an integral smooth proper surface over a field. Let  $L$  and  $L'$  be line bundles, given by

$$f = (f_{\rho,\nu})_{\rho,\nu \in I} \in H^1(X, \mathbb{G}_m) \quad \text{and} \quad g = (g_{\rho,\nu})_{\rho,\nu \in I} \in H^1(X, \mathbb{G}_m)$$

in a joint finite open cover  $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ ,  $I$  totally ordered. The analogous local consideration lets us look at an adèle of the surface. That is, a saturated flag  $\Delta = (\eta_0 > \eta_1 > \eta_2)$  and its 2-local field, e.g.,

$$A(\Delta, \mathcal{O}_X) \simeq \kappa((s))((t))$$

for a suitable finite field extension  $\kappa/k$ . If, say, locally  $f_{\rho,\nu} = t^r$  and  $g_{\rho,\nu} = s^m$  for some  $r, m \geq 0$ , then clearly we would expect the correct multiplicity to be  $r \cdot m$ , inspired by the classical formula for proper intersections

$$\dim_k \frac{\mathcal{O}_{X,\eta_2}}{(t^r, s^m)}$$

for divisors which are locally cut out by the equations  $t^r$  and  $s^m$  in a neighbourhood of the closed point  $\eta_2$ . Note that this situation gets more complicated when  $r, s \in \mathbb{Z}$ , possibly negative, since then  $(t^r, s^m)$  might no longer define an ideal in  $\mathcal{O}_{X,x}$ . Moreover, there is no reason why we can assume that  $f_{\rho,\nu}$  is an expression only in the variable  $t$ , and  $g_{\rho,\nu}$  only in the variable  $s$ . It nonetheless seems plausible that we can make this work in general by phrasing it in terms of a suitable operation which “mixes” the disruption of a splitting caused by  $f_{\rho,\nu}$  and the disruption caused by  $g_{\rho,\nu}$  simultaneously. This should be the rôle of a suitable concept of tensor product – and indeed of the normally ordered tensor product in the case at hand. Specifically, for our surface, we will get the finite sum

$$- \sum_{\Delta=(\eta_0>\eta_1>\eta_2)} \pi_* \left[ f_{\alpha(\eta_2)\alpha(\eta_1)} \right]_{\circlearrowleft_{\eta_2} \Lambda_{\eta_1}} \otimes \left[ g_{\alpha(\eta_1)\alpha(\eta_0)} \right]_{\circlearrowleft_{\eta_1} \Lambda_{\eta_0}}$$

in Theorem 6.6 below. It has only finitely many non-zero summands, and defines an element in  $K_2(2\text{-Tate}(k)) \cong \mathbb{Z}$ , and this integer is the intersection multiplicity  $L \cdot L'$ . The  $\overrightarrow{\otimes}$ -product denotes the product of  $K$ -theory classes induced to the 2-Tate category. It contains  ${}_{\eta_2}\Lambda_{\eta_1} \overrightarrow{\otimes} {}_{\eta_1}\Lambda_{\eta_0}$ , which is precisely the 2-Tate object coming from the adèles along the flag  $\Delta$ .

After these introductory comments, let us work out these ideas in a rigorous fashion.

6.5. ADELIC MULTIPLICITY FORMULA. We are ready to prove an adelic variant of an intersection multiplicity formula. It recasts the adelic intersection pairing of Parshin [Par83, §2] in a new light.

6.6. THEOREM. *Let  $k$  be a field. Suppose  $\pi : X \rightarrow k$  is a purely  $n$ -dimensional integral smooth proper variety. Let  $L_1, \dots, L_n$  be line bundles which are represented by alternating Čech representatives*

$$f^q = (f_{\rho,\nu}^q)_{\rho,\nu \in I} \in H^1(X, \mathbb{G}_m) \quad (\text{for } q = 1, \dots, n)$$

in a finite open cover  $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ ,  $I$  totally ordered. For any  $x \in X$ , let  $\alpha(x)$  be the smallest element of  $I$  such that  $x \in U_{\alpha(x)}$ . Recall our notation

$$\begin{matrix} [f] \\ \circlearrowleft_x \Lambda_y \end{matrix}$$

for  $K_1$ -classes from Definition 5.8 and the external product in  $K$ -theory from Definition 5.1. Using this notation, the sum

$$(-1)^{\frac{n(n+1)}{2}} \sum_{\Delta=(\eta_0 > \dots > \eta_n)} \pi_* \left[ f_{\alpha(\eta_n)\alpha(\eta_{n-1})}^n \right]_{\circlearrowleft_{\eta_n}\Lambda_{\eta_{n-1}}} \overrightarrow{\otimes} \dots \overrightarrow{\otimes} \left[ f_{\alpha(\eta_1)\alpha(\eta_0)}^1 \right]_{\circlearrowleft_{\eta_1}\Lambda_{\eta_0}}$$

has only finitely many non-zero summands, defines an element in  $K_n(n\text{-Tate}(k)) \cong \mathbb{Z}$ , and this integer is the intersection multiplicity  $L_1 \cdots L_n$ .

We shall address the proof in the next subsection.

6.7. ELABORATION. Let us give a few explanations to make the statement of the formula more easily digestible:

1. Each  $f_{\rho,\nu}^\bullet$  is a function defined on some open  $U_\rho \cap U_\nu$ . It is in particular defined in the local ring underlying Definition 5.8. As a result, we get a  $K_1$ -class  $[f_{\rho,\nu}^\bullet]$  in the Tate category of  $\text{Coh}_{\{-\}}(X)$ .
2. Then we use the external product in  $K$ -theory of Definition 5.1. As a result, we get a class in  $K_n$  of an  $n$ -Tate category. Since the support of coherent sheaves, when tensored, gets intersected, this will be the  $n$ -Tate category of coherent sheaves of support in some zero-dimensional set. In particular, thanks to  $\pi_*$ , we can send this to an  $n$ -Tate vector space over  $k$ .

3. Using the canonical isomorphism  $K_n(n\text{-Tate}(k)) \cong \mathbb{Z}$ , we can read such a class as an integer and add them.

The reader may wonder about the rôle of the normally ordered tensor product behind the scenes here. We allow ourselves to jump ahead a little. Indeed, its rôle is two-fold: On the one hand, “ $\overrightarrow{\otimes}$ ” is constructed from the ordinary tensor product of  $\text{Coh}(X)$ , and this is relevant in the same way as in Proposition 6.12. However, secondly, the reader will notice the absence of a counterpart of the boundary map  $\partial_\bullet$  of Proposition 6.12 in the formula of Theorem 6.6. This has to do with the second rôle of the normally ordered tensor product: The underlying object of this product is precisely the adèles  $A(\Delta, \mathcal{O}_X)$  of the relevant flag. Being realized in precisely this fashion in the  $n$ -Tate category basically encodes the datum of the boundary maps of Proposition 6.12. Although mathematically unrelated, this is philosophically analogous to the Uniqueness Principle, Proposition 6.16: Loosely speaking, after completing, the object is so rigid that the datum of all its valuations is uniquely encoded in it.

To prove agreement with the usual intersection multiplicity, we need to decide which approach for defining the latter we pick. We choose a viewpoint based on Chow groups. The intersection multiplicity can be defined by multiplying the divisors of the line bundles in the Chow ring. The latter is naturally connected to considerations in  $K$ -theory by the Bloch–Quillen formula.

Let us recall how this works in detail, also in order to set up the notation for our proof of agreement.

6.8. INTERSECTION PAIRING. Let  $X$  be a scheme of finite type over a field  $k$ . Let  $\mathcal{K}_p^M$  denote the  $p$ -th Milnor  $K$ -theory sheaf: We recall its definition. For every Zariski open  $U \subseteq X$ , define

$$\mathcal{K}_p^M(U) := \ker \left( \prod_{x \in U^0} K_p^M(\kappa(x)) \xrightarrow{d} \prod_{x \in U^1} K_{p-1}^M(\kappa(x)) \right), \tag{6.4}$$

where  $U^i$  denotes the set of points  $x \in U$  with  $\text{codim}_X \overline{\{x\}} = i$ ,  $\kappa(x)$  denotes the residue field at  $x$  and  $K_i^M(-)$  denotes the  $i$ -th Milnor  $K$ -group of a field; we recall  $d$  below. There is a natural restriction map to smaller Zariski opens, making  $\mathcal{K}_p^M$  a sheaf of abelian groups for the Zariski topology.

See Remark 6.9 for an alternative, and perhaps simpler-looking, definition. If one replaces each occurrence of a Milnor  $K$ -group by a Quillen  $K$ -group, this gives the corresponding definition of Quillen  $K$ -theory sheaves  $\mathcal{K}_p$ .

The sheaf  $\mathcal{K}_p^M$  has a Zariski resolution, a quasi-isomorphism to a flasque complex of sheaves concentrated in degrees  $[0, p]$ , namely

$$\mathcal{K}_p^M(U) \xrightarrow{\sim} \left[ \prod_{x \in U^0} K_p^M(\kappa(x)) \xrightarrow{d} \cdots \xrightarrow{d} \prod_{x \in U^j} K_{p-j}^M(\kappa(x)) \xrightarrow{d} \cdots \right]_{0,p}.$$

We may call this the *Gersten complex*. In fact, Equation 6.4 is a definition of the sheaf  $\mathcal{K}_p^M$  modelled after this resolution, instead of the more traditional approach of Remark

6.9. The differential  $d$  is of the following shape:

$$d = \sum_{x,y} \partial_y^x,$$

where  $x \in U^j$  is any codimension  $j$  point of the scheme, and  $y \in \overline{\{x\}}^1$  any point of codimension one in the closure  $\overline{\{x\}}$ . Each  $\partial_y^x$  in turn is a map

$$\partial_y^x : K_*^M(\kappa(x)) \longrightarrow K_{*-1}^M(\kappa(y)),$$

which is defined as follows: The local ring  $\mathcal{O}_{\overline{\{x\}},y}$  is one-dimensional since  $y$  is of codimension one in  $\overline{\{x\}}$ . Take its normalization, i.e. the integral closure  $\mathcal{O}'_{\overline{\{x\}},y}$  of  $\mathcal{O}_{\overline{\{x\}},y}$  inside  $\kappa(x)$ . As we work in the context of a variety of finite type over a field, the normalization is a finite ring extension and as  $\mathcal{O}_{\overline{\{x\}},y}$  is local, this implies that  $\mathcal{O}'_{\overline{\{x\}},y}$  is a semi-local ring inside  $\kappa(x)$ . If  $\mathfrak{m}_1, \dots, \mathfrak{m}_r$  denotes its maximal ideals, each localization  $(\mathcal{O}'_{\overline{\{x\}},y})_{\mathfrak{m}_i}$  then is a discrete valuation ring. Write  $v_i$  for its normalized valuation. Then define

$$\begin{aligned} \partial_y^x : K_*^M(\kappa(x)) &\longrightarrow K_{*-1}^M(\kappa(y)) & (6.5) \\ \beta &\longmapsto \sum_{i=1}^r N_{\kappa(\mathfrak{m}_i)/\kappa(y)} \partial_{v_i}(\beta), \end{aligned}$$

where  $\partial_v$  is the Milnor  $K$ -boundary map of the discrete valuation  $v$ , and  $N_{-/}$  denotes the norm. The finiteness of the integral closure also implies that the residue fields  $\kappa(\mathfrak{m}_i)$  of the valuations  $v_i$  are finite over  $\kappa(y)$ .

The same discussion is valid for Quillen  $K$ -theory (and indeed much more generally, see [Ros96]).

Let  $\underline{\mathbb{Z}}$  denote the locally constant Zariski sheaf with values in  $\mathbb{Z}$ . There is a natural  $\underline{\mathbb{Z}}$ -graded commutative  $\underline{\mathbb{Z}}$ -algebra structure on the Milnor  $K$ -sheaves,

$$\mathcal{K}_p^M \otimes_{\underline{\mathbb{Z}}} \mathcal{K}_q^M \longrightarrow \mathcal{K}_{p+q}^M, \tag{6.6}$$

and analogously on the ordinary  $K$ -theory sheaves  $\mathcal{K}_p$ . Thus, these can be promoted to be viewed as sheaves of graded rings  $\mathcal{K}_*^M$  resp.  $\mathcal{K}_*$ . Finally, there is a morphism of sheaves of graded  $\underline{\mathbb{Z}}$ -algebras

$$\mathcal{K}_*^M \longrightarrow \mathcal{K}_*, \tag{6.7}$$

induced from the canonical morphism  $K_p^M(F) \rightarrow K_p(F)$  for every field  $F$ .

6.9. REMARK. If the base field  $k$  is infinite, there is an alternative definition for the Milnor  $K$ -theory sheaf, given by

$$\mathcal{K}_*^M(U) := T_{\mathbb{Z}}(\mathcal{O}_X(U)^\times) / \langle x \otimes (1-x) \mid \text{for all } x \text{ with } x, 1-x \in \mathcal{O}_X(U)^\times \rangle,$$

where  $T_{\mathbb{Z}}$  denotes the free tensor algebra as a  $\mathbb{Z}$ -module. This definition is equivalent to the one of Equation 6.4 for smooth varieties over infinite fields, as was proven by van der

Kallen and more generally Kerz, see [Ker10, Proposition 10]. In fact, it also works for finite base fields once they have sufficiently large cardinality, cf. loc. cit.

The cup product of sheaf cohomology, in conjunction with the ring structure of the Milnor  $K$ -theory sheaves  $\mathcal{K}_p^M \otimes_{\mathbb{Z}} \mathcal{K}_q^M \rightarrow \mathcal{K}_{p+q}^M$ , induces a product

$$H^i(X, \mathcal{K}_p^M) \otimes_{\mathbb{Z}} H^j(X, \mathcal{K}_q^M) \longrightarrow H^{i+j}(X, \mathcal{K}_p^M \otimes_{\mathbb{Z}} \mathcal{K}_q^M) \longrightarrow H^{i+j}(X, \mathcal{K}_{p+q}^M).$$

Applied to  $n$  classes in cohomological degree one, this becomes

$$H^1(X, \mathcal{K}_1^M) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} H^1(X, \mathcal{K}_1^M) \longrightarrow H^n(X, \mathcal{K}_n^M)$$

and in view of the algebra morphism of (6.7), we can map this compatibly to its counterpart for Quillen  $K$ -theory.

6.10. ČECH COHOMOLOGY. Let  $X$  be any topological space. Let  $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$  ( $I$  any index set) be an open cover of  $X$  and  $\mathcal{F}$  a sheaf of abelian groups. We denote intersections in the given open cover by

$$U_{\alpha_0 \dots \alpha_r} := \bigcap_{i=0, \dots, r} U_{\alpha_i} \quad (\text{for } \alpha_0, \dots, \alpha_r \in I).$$

Then we have the Čech cohomology groups, which we denote by  $\check{H}^p(\mathfrak{U}, \mathcal{F})$ . They are defined as the cohomology groups of the complex

$$\check{C}^p(\mathfrak{U}, \mathcal{F}) := \prod_{\alpha_0 \dots \alpha_r \in I^{p+1}} \mathcal{F}(U_{\alpha_0 \dots \alpha_r}), \quad \delta : \check{C}^p(\mathfrak{U}, \mathcal{F}) \rightarrow \check{C}^{p+1}(\mathfrak{U}, \mathcal{F}).$$

There is a sub-complex of *alternating chains*, requiring additionally

$$f_{\alpha_{\pi(0)} \dots \alpha_{\pi(r)}} = \text{sgn}(\pi) f_{\alpha_0 \dots \alpha_r}$$

for arbitrary permutations  $\pi$  on  $r + 1$  letters, as well as  $f_{\alpha_0 \dots \alpha_r} = 0$  whenever any two indices agree,  $\alpha_i = \alpha_j$  for  $i \neq j$ . This sub-complex of alternating chains is quasi-isomorphic to the whole complex. In particular, we can restrict to working with alternating chains whenever this appears convenient.

6.11. DEFINITION. Given an open cover  $\mathfrak{U}$  of  $X$ , a disjoint decomposition consists of pairwise disjoint subsets  $\{(\Sigma_\alpha)_{\alpha \in I} \mid \Sigma_\alpha \subseteq U_\alpha\}$  such that  $X = \bigcup_{\alpha \in I} \Sigma_\alpha$ . Given this datum, and  $x \in X$  is a point,  $\alpha(x)$  denotes the unique index in  $I$  such that  $x \in \Sigma_{\alpha(x)}$  holds.

For all sheaves that we shall be working with, the sheaf cohomology  $H^p(X, \mathcal{F})$  arises as the colimit of  $\check{H}^p(\mathfrak{U}, \mathcal{F})$  over refinements of covers.

6.12. PROPOSITION. *Suppose  $X/k$  is a purely  $n$ -dimensional smooth proper variety and  $L_1, \dots, L_n$  line bundles which are represented by Čech representatives*

$$f^q = (f_{\rho, \nu}^q)_{\rho, \nu \in I} \in H^1(X, \mathbb{G}_m) \quad (\text{for } q = 1, \dots, n)$$

*in a finite open cover  $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ . Then for every disjoint decomposition  $(\Sigma_\alpha)_{\alpha \in I}$ , the intersection multiplicity  $L_1 \cdots L_n$  equals*

$$\sum_{\eta_n, \dots, \eta_0} [\kappa(\eta_n) : k] \cdot \partial_{\eta_n}^{\eta_{n-1}} \cdots \partial_{\eta_1}^{\eta_0} \{f_{\alpha(\eta_0)\alpha(\eta_1)}^1, f_{\alpha(\eta_1)\alpha(\eta_2)}^2, \dots, f_{\alpha(\eta_{n-1})\alpha(\eta_n)}^n\} \in \mathbb{Z},$$

*where the sum runs over all chains of points  $\eta_0 > \dots > \eta_n$ .*

PROOF. The degree one case of the Bloch–Quillen formula is the classical isomorphism  $H^1(X, \mathbb{G}_m) \cong \text{CH}^1(X)$ , identifying isomorphism classes of line bundles with Weil divisor classes. The intersection multiplicity  $L_1 \cdots L_n$  is, by definition, the push-forward of the product of the Weil divisor classes in the Chow ring to the base field, i.e. for the proper structure morphism  $s : X \rightarrow \text{Spec } k$ ,

$$L_1 \cdots L_n = s_*([L_1] \smile \cdots \smile [L_n]). \tag{6.8}$$

The Bloch–Quillen formula is compatible with the intersection product,

$$\begin{array}{ccc} \text{CH}^p(X) \otimes_{\mathbb{Z}} \text{CH}^q(X) & \longrightarrow & \text{CH}^{p+q}(X) \\ \downarrow & & \downarrow \\ H^p(X, \mathcal{K}_p^M) \otimes_{\mathbb{Z}} H^q(X, \mathcal{K}_q^M) & \longrightarrow & H^{p+q}(X, \mathcal{K}_{p+q}^M) \end{array}$$

and so we may equivalently evaluate the sheaf cup product of the  $H^1$ -classes in  $H^1(X, \mathbb{G}_m)$ , giving a cohomology class in  $H^n(X, \mathcal{K}_n^M)$ . Finally, the isomorphism  $H^n(X, \mathcal{K}_n^M) \xrightarrow{\sim} \text{CH}^n(X)$ , or equivalently to  $\text{CH}_0(X)$ , is made explicit by [Bra13, Theorem 2] and yields

$$h_{\eta_n} = \sum_{\eta_{n-1}, \dots, \eta_0} \partial_{\eta_n}^{\eta_{n-1}} \cdots \partial_{\eta_1}^{\eta_0} \{f_{\alpha(\eta_0)\alpha(\eta_1)}^1, f_{\alpha(\eta_1)\alpha(\eta_2)}^2, \dots, f_{\alpha(\eta_{n-1})\alpha(\eta_n)}^n\}$$

for the zero cycle  $h = \sum_{\eta_n \in X^n} h_{\eta_n}[\eta_n]$ , which is the output of the aforementioned isomorphism (the symbol  $\partial_*^*$  denotes the boundary map in Milnor  $K$ -theory, see *loc. cit.* or Equation 6.5). Here the sum runs over all chains  $\eta_0 > \dots > \eta_{n-1}$  and  $\alpha$  is as in the statement of our claim. Note that the set of codimension  $n$  points  $X^n$  is the same as the set of closed points, given that  $X$  is of pure dimension  $n$ . Returning to Equation 6.8, it thus remains to compute the push-forward  $s_*(h)$ , which amounts to the formula of our claim, and the additional sum index  $\eta_n$  stems from the summation over all closed points which appear in the evaluation of  $s_*$ , [Ros96, §3]. This finishes the proof. ■

6.13. PROOF OF THE THEOREM. Now we address the proof of Theorem 6.6.

6.13.1. STEP 1: HIGHER VALUATIONS (ALGEBRAIC). We shall use the abbreviation DVF for discrete valuation fields.

6.14. DEFINITION. *Let  $k$  be a field.*

1. *We call the field  $k$  a 0-DVF with last residue field  $k$ .*
2. *We call a field  $F$  an  $n$ -DVF with last residue field  $k$  if it is a DVF such that its residue field comes equipped with the structure of an  $(n - 1)$ -DVF with last residue field  $k$ . Write  $\mathcal{O}_1$  for the ring of integers of  $F$ .*

*We may write  $F$  as a pair  $(F, \mathcal{E})$ , where  $\mathcal{E}$  refers to the datum of the  $n$ -DVF structure, if we want to stress the choice of the latter.*

Given this datum, it inductively determines a sequence of fields and their rings of integers. We get the following diagram

$$\begin{array}{ccc}
 & F & \\
 & \uparrow & \\
 & \mathcal{O}_1 & \twoheadrightarrow k_1 \\
 & & \uparrow \\
 & & \mathcal{O}_2 & \twoheadrightarrow k_2 \\
 & & & \uparrow \\
 & & & \vdots
 \end{array} \tag{6.9}$$

where  $\mathcal{O}_i$  denotes the rings of integers, and  $k_i$  the respective residue fields. Each upward arrow denotes the inclusion of the ring of integers into its field of fractions, whereas each rightward arrow denotes a quotient map  $\mathcal{O}_i \twoheadrightarrow \mathcal{O}_i/\mathfrak{m}_i$ , where  $\mathfrak{m}_i$  denotes the maximal ideal. So all the rings of integers and these maps canonically come with  $\mathcal{E}$ .

6.15. DEFINITION. *Let  $k$  be a field.*

1. *We call the field  $k$  a 0-local field with last residue field  $k$ .*
2. *We call a field  $F$  an  $n$ -local field with last residue field  $k$  if it is a complete discrete valuation field such that its residue field comes equipped with the structure of an  $(n - 1)$ -local field.*

In particular, an  $n$ -local field is a particular example of an  $n$ -DVF. The only difference in the definition is that an  $n$ -local field is complete with respect to the topology induced from the discrete valuation, whereas there is no such requirement for an  $n$ -DVF. The contrast is quite strong however. We recall a classical fact due to Parshin:

6.16. PROPOSITION. (UNIQUENESS PRINCIPLE) *If a (not necessarily topologized) field  $F$  admits the structure of an  $n$ -local field with last residue field  $k$ , then this structure is unique. Call it  $\mathcal{E}_{\text{local}}$ .*

For a proof see [BGW16a, Cor. 1.3, (a)]. In particular, it is not necessary to specify the higher local field structures on the residue fields  $k_i$  manually.

6.17. EXAMPLE. There is a field inclusion  $k(s, t) \hookrightarrow k((s))((t))$ . The right field is a 2-local field with last residue field  $k$ ; concretely Figure 6.9 takes the shape

$$\begin{array}{ccc}
 k((s))((t)) & & \\
 \uparrow & & \\
 k((s))[[t]] & \longrightarrow & k((s)) \\
 & & \uparrow \\
 & & k[[s]] \longrightarrow k
 \end{array}$$

Step by step, we may restrict the discrete valuations to the field  $k(s, t)$  and its residue fields, giving  $k(s, t)$  the structure of a 2-DVF. There is a similar field inclusion  $k(s, t) \hookrightarrow k((t))((s))$ . Proceeding analogously, this equips  $k(s, t)$  with two *different* 2-DVF structures. On the contrary  $k((s))((t))$  only carries the  $t$ -adic valuation and  $k((t))((s))$  only the  $s$ -adic. Indeed, the sequence  $x_n := s^{-n}t^n$  in  $k(s, t)$  converges to zero in  $k((s))((t))$ , but diverges in  $k((t))((s))$ .

For each  $n$ -DVF  $(F, \mathcal{E})$ , there is a *boundary map*  $\partial_v$  in  $K$ -theory

$$\partial_v : K_*(F) \longrightarrow K_{*-1}(k_1).$$

The map  $\partial_v$  can be constructed as follows: The open-closed complement decomposition of  $\text{Spec } \mathcal{O}_1$  into its generic and special point gives rise to a fiber sequence in  $K$ -theory, the localization sequence

$$\text{Spec } \mathcal{O}_1/\mathfrak{m}_1 \hookrightarrow \text{Spec } \mathcal{O}_1 \hookrightarrow \text{Spec } F, \quad K(k_1) \longrightarrow K(\mathcal{O}_1) \longrightarrow K(F), \quad (6.10)$$

and then  $\partial_v$  denotes the connecting homomorphism of the long exact sequence of homotopy groups associated to a fiber sequence.

6.18. REMARK. Under the map from Milnor to Quillen  $K$ -theory, this map  $\partial_v$  is compatible with the corresponding boundary map in Milnor  $K$ -theory, [Ros96].

As we have just explained, we get a map  $K_*(F) \longrightarrow K_{*-1}(k_1)$  for each  $n$ -DVF  $F$ , but since  $k_1$  is by definition an  $(n - 1)$ -DVF, we may inductively concatenate these boundary maps all down to the last residue field, ‘jumping down’ all the steps of the staircase in Figure 6.9.

6.19. DEFINITION. Let  $(F, \mathcal{E})$  be an  $n$ -DVF with last residue field  $k_n$  and such that  $k_n$  is a finite extension of  $k$ . We call the morphism

$$V_{F, \mathcal{E}} : K_*(F) \xrightarrow{\partial} K_{*-1}(k_1) \xrightarrow{\partial} \cdots \xrightarrow{\partial} K_{*-n}(k_n) \xrightarrow{N_{k_n/k}} K_{*-n}(k)$$

the (algebraic) higher valuation of  $(F, \mathcal{E})$ . Here  $N_{k_n/k}$  denotes the norm map along the field extension  $k_n/k$ .

6.19.1. STEP 2: HIGHER VALUATIONS (VIA IND-PRO METHODS). Before we proceed, we need to recall the fundamental delooping result of Sho Saito [Sai15]. There is a canonical equivalence

$$D : \Omega K(n\text{-Tate}(\mathcal{C})) \longrightarrow K((n - 1)\text{-Tate}(\mathcal{C})) \tag{6.11}$$

for idempotent complete exact categories  $\mathcal{C}$  (where  $\Omega$  denotes the desuspension of a spectrum, i.e. the degree shift inverse to the suspension). Such an equivalence was first constructed in loc. cit. by Saito, but for us it is most convenient to define it by  $D := \text{Index}$ , with the latter constructed as in [BGW17]<sup>3</sup>.

6.20. EXAMPLE. Naturally, the Laurent series  $k((s))$  can be interpreted as an object in  $1\text{-Tate}(k)$ . Moreover, every automorphism of an object in an exact category  $\mathcal{C}$  induces a canonical element in  $K_1(\mathcal{C})$ . Thus, multiplication by a series  $f \in k((s))$  defines an element  $[f] \in K_1(1\text{-Tate}(k))$ . Using the explicit description of [BGW18] the value of  $D([f]) \in K_0(k)$  is computed as follows: Pick two lattices  $L_1, L_2 \hookrightarrow k((s))$  such that

$$L_1 \subseteq L_2 \quad \text{and} \quad fL_1 \subseteq L_2$$

and then take the  $K_0$ -class of  $[L_2/f_1L_1] - [L_2/L_1] \in K_0(k)$ . Since  $K_0(k) \cong \mathbb{Z}$ , and this isomorphism is just the dimension of the underlying vector space, we get

$$[L_2/f_1L_1] - [L_2/L_1] = v(f) \cdot [k],$$

where  $v(f) \in \mathbb{Z}$  is the  $s$ -adic valuation, and  $[k]$  the class of a one-dimensional  $k$ -vector space. Compare this to Elaboration 6.3.

We turn to geometry. Let  $k$  be a field and  $\pi : X \rightarrow k$  an integral variety of pure dimension  $n$ . Let us write  $F := k(X)$  for its rational function field. If  $\Delta := \{(\eta_0 > \cdots > \eta_n)\} \in S(X)_n$  denotes a saturated flag, we have the Parshin–Beilinson adèle ring

$$A(\Delta, \mathcal{O}_X) = A(\{\eta_0 > \cdots > \eta_n\}, \mathcal{O}_X).$$

We may regard this solely as a ring. Then there is a canonical isomorphism of  $k$ -algebras

$$A(\Delta, \mathcal{O}_X) \cong \prod_{i=1}^r F_i$$

<sup>3</sup>In [BGW18] it is proven that this equivalence is essentially the same as Saito’s equivalence, so we favour using the map of [BGW17] to avoid logically relying on this comparison result.

for some finite  $r$ , and each  $F_i$  is an  $n$ -local field with last residue field a finite extension of  $k$ . For a proof, see [Yek92, §3] or [BGW16a, Theorem 4.2]. Moreover, there is a natural ring homomorphism

$$\text{diag} : F \longrightarrow A(\Delta, \mathcal{O}_X), \tag{6.12}$$

frequently called the *diagonal embedding*.

In particular, each  $F_i$  comes equipped with a canonical algebraic higher valuation as in Definition 6.19, using the single and unique  $n$ -DVF structure  $\mathcal{E}_{\text{local}}$ , Proposition 6.16. Thus, we obtain a canonical homomorphism

$$V_{\text{alg}}(\Delta) : K_n(F) \xrightarrow{\text{diag}} \prod_{i=1}^r K_n(F_i) \xrightarrow{V_{F_i, \mathcal{E}_{\text{local}}}} K_0(k).$$

On the other hand, instead of viewing the adèles as a ring, they also carry the structure of an  $n$ -Tate object in finite-dimensional  $k$ -vector spaces, [BGW16c, Theorem 7.10], alongside an exact functor

$$\text{real} : \text{Coh}(X) \longrightarrow n\text{-Tate}^{el}(\text{Coh}_0(X)),$$

where  $\text{Coh}_0(X)$  denotes the abelian category of coherent sheaves of zero-dimensional support. This functor is known as the *Tate realization*.

Once the support is zero-dimensional, it is finite over the target under the structure map  $\pi : X \rightarrow k$ . Hence, the push-forward is an exact functor  $\pi_* : \text{Coh}_0(X) \rightarrow \mathbf{Vect}_f(k)$ , and by functoriality of  $n$ -Tate categories, [BGW16c, Theorem 7.2, (3)], we obtain the exact functor composed as

$$\text{Coh}(X) \xrightarrow{\text{real}} n\text{-Tate}^{el}(\text{Coh}_0(X)) \xrightarrow{\pi_*} n\text{-Tate}^{el}(k).$$

This remains true if we replace  $X$  by a smaller Zariski open neighbourhood of the generic point of  $X$ , so that we may take the colimit. This only affects the source of the functor and we obtain an exact functor  $\mathbf{Vect}_f(F) \rightarrow n\text{-Tate}^{el}(\mathbf{Vect}_f(k))$ , as the function field  $F$  is the stalk at the generic point.

This leads to a second map

$$V_{\text{Tate}}(\Delta) : K_n(F) \xrightarrow{\text{real}} K_n(n\text{-Tate}^{el}(k)) \xrightarrow{D \circ \dots \circ D} K_0(k), \tag{6.13}$$

where  $D$  denotes the equivalence of Equation 6.11.

The following holds for all saturated flags  $\Delta \in S(X)_n$ .

6.21. LEMMA. *We have  $V_{\text{Tate}}(\Delta) = V_{\text{alg}}(\Delta)$ .*

PROOF. Note that the definition of  $V_{\text{alg}}$  essentially hinges on the concatenation of  $n$  boundary maps, while the definition of  $V_{\text{Tate}}$  uses a concatenation of  $n$  index maps  $D$ . Thus, we will show the compatibility of a single boundary map with a single index map. Once this is done, use this argument inductively for all  $n$  steps. Concretely: in the case

at hand, if  $F'$  is a  $d$ -local field with ring of integers  $\mathcal{O}_1$ , use [BGW17, Theorem 1.1] with the choice  $X := \text{Spec } \mathcal{O}_1$ , and  $Z$  the closed sub-scheme defined by the special point, i.e.,  $Z := \text{Spec } \mathcal{O}_1/\mathfrak{m}_1$ . Then the open complement is  $U := \text{Spec } F'$ , so we are in the situation of Equation 6.10. The functor  $\mathbf{T}_Z$  of loc. cit. is compatible with the Tate realization functor of the adèles, and the index equivalence  $i$  of loc. cit. relies on the map  $\text{Index}$  of loc. cit., which is what we have taken as our definition of the map  $D$ . ■

6.22. REMARK. Example 6.20 demonstrates the previous lemma in the simplest possible case: The evaluation of  $D$  yields the same as the boundary map  $\partial_1 : K_1(k((s))) \rightarrow K_0(k)$ , since the latter is nothing but the valuation map.

6.22.1. STEP 3: COMPARISON WITH GEOMETRICALLY DEFINED MULTIPLICITY. Next, we will compare the algebraic higher valuation with the maps  $\partial_y^x$  which appear in the Gersten complex:

We define

$$V_{Ger} : K_n(F) \longrightarrow K_0(k)$$

by the formula

$$\beta \longmapsto \sum_{\Delta=(\eta_n, \dots, \eta_0)} [\kappa(\eta_n) : k] \cdot \partial_{\eta_n}^{\eta_{n-1}} \dots \partial_{\eta_1}^{\eta_0} \beta,$$

where  $\partial_*^*$  is defined as in the description of the differential of the Gersten complex, i.e., as in Equation 6.5. The sum runs over all saturated flags  $\Delta \in S(X)_n$ . (It is easy to see that all but finitely many summands are zero, so this is well-defined)

Note that since  $X$  is integral, it has only a single generic point, so we must always have  $\overline{\{\eta_0\}} = X$  for all flags  $\Delta$ , and correspondingly  $\kappa(\eta_0) = F$ .

To bridge between  $V_{alg}$  and  $V_{Ger}$ , we need to recall two standard facts regarding the functoriality of  $\partial_v$ :

Firstly, the push-forward compatibility of the boundary map with finite field extensions: Suppose  $L$  is a discrete valuation field with valuation  $v$  and residue field  $\kappa(v)$ . Let  $L'/L$  be a finite extension, and  $A'$  the integral closure of the valuation ring of  $L$  inside  $L'$ . Suppose  $A'$  is a finite  $A$ -module (this holds for example if  $A$  is excellent). Then  $A'$  is semi-local and if  $w$  runs through the discrete valuations extending  $v$ , the diagram

$$\begin{array}{ccc} K_*(L') & \xrightarrow{\oplus \partial_w} & \bigoplus_w K_{*-1}(\kappa(w)) \\ N_{L'/L} \downarrow & & \downarrow \Sigma N_{\kappa(w)/\kappa(v)} \\ K_*(L) & \xrightarrow{\partial_v} & K_{*-1}(\kappa(v)) \end{array} \tag{6.14}$$

commutes (cf. [Ros96, R3b]).

Secondly, there is also a pullback compatibility of the boundary map along arbitrary field extensions: Suppose  $L$  is a discrete valuation field with valuation  $v$  and  $L'/L$  an

arbitrary field extension; write  $i : L \hookrightarrow L'$  for the inclusion. Suppose  $w$  is a discrete valuation of  $L'$  extending  $v$  with ramification index  $e$ . Then

$$\begin{CD} K_*(L') @>\partial_w>> K_{*-1}(\kappa(w)) \\ @V{i}VV @VVV \\ K_*(L) @>e \cdot \partial_v>> K_{*-1}(\kappa(v)) \end{CD} \tag{6.15}$$

commutes, and the upward arrow on the right is induced from  $i$  by the fact that the valuation  $w$  extends  $v$  (cf. [Ros96, R3a]).

6.23. LEMMA. *We have  $V_{Ger} = \sum_{\Delta \in S(X)_n} Valg(\Delta)$ .*

PROOF. (Step 1) Note that the above map  $V_{Ger}$  is a concatenation of boundary maps

$$\partial_y^x : K_*(\kappa(x)) \longrightarrow K_{*-1}(\kappa(y)),$$

where  $x, y$  are points of the scheme, and  $\kappa(-)$  the respective residue fields. Unravelling each summand in

$$\begin{aligned} \partial_y^x : K_*(\kappa(x)) &\longrightarrow K_{*-1}(\kappa(y)) \\ \beta &\longmapsto \sum_{i=1}^r N_{\kappa(\mathfrak{m}_i)/\kappa(y)} \partial_{v_i}(\beta) \end{aligned} \tag{6.16}$$

individually, and using the norm compatibility of Diagram 6.14 to move all norms jointly all to the left, we can rewrite

$$\begin{aligned} \partial_{\eta_n}^{\eta_{n-1}} \dots \partial_{\eta_1}^{\eta_0} \beta & \\ &= \sum_{(v_1, \dots, v_n)} N_{\kappa(v_n)/k(\eta_n)} \partial_{v_n} \dots \partial_{v_2} \partial_{v_1} \beta, \end{aligned} \tag{6.17}$$

where  $v_1, \dots, v_n$  are the discrete valuations which stem from the individual valuations of the maximal ideals  $\mathfrak{m}_i \subseteq \kappa(x)$  for  $x = \eta_0, \dots, \eta_{n-1}$  when unravelling  $\partial_{\eta_n}^{\eta_{n-1}} \dots \partial_{\eta_1}^{\eta_0}$  using Formula 6.16 (it may be helpful to compare this to the explicit explanation and description of  $\partial_*^*$  which we have given around Equation 6.5). These discrete valuations  $v_1, \dots, v_n$  actually define the structure of an  $n$ -DVF on the function field  $\kappa(\eta_0) = F$ , since  $\eta_0$  is the single generic point of  $X$ . We write  $\mathcal{E}_{v_1, \dots, v_n}$  to denote this  $n$ -DVF structure with last residue field  $\kappa(v_n)$ .

(Step 2) Now compare the right-hand side of Equation 6.17 with Definition 6.19. As norms along finite field extensions and the boundary map at a valuation  $\partial_v$  are compatible (Diagram 6.14), we get

$$\begin{aligned} V_{Ger} : K_n(F) &\longrightarrow K_0(k), \\ \beta &\longmapsto \sum_{\Delta} \sum_{v_1, \dots, v_n} V_{F, \mathcal{E}_{v_1, \dots, v_n}} \end{aligned} \tag{6.18}$$

relying on Definition 6.19. Here the sum runs over (1) all summands  $\Delta$  in the definition of  $V_{Ger}$ , and (2) over all the valuation chains  $v_1, \dots, v_n$ , which arise from the finitely many choices of maximal ideals  $\mathfrak{m}_*$  in the individual unravelling steps of Equation 6.16 (for that one fixed  $\Delta$  of the first sum).

Note that the only difference with  $V_{alg}$ , i.e.,

$$V_{alg}(\Delta) : K_n(F) \xrightarrow{\text{diag}} \prod_{i=1}^r K_n(F_i) \xrightarrow{V_{F_i, \mathcal{E}_{\text{local}}}} K_0(k),$$

is that instead of using  $F$  and the  $n$ -DVF structure  $\mathcal{E}$ , the map  $V_{alg}(\Delta)$  first goes through the field extension  $F \hookrightarrow F_i$ , and then uses the (canonical)  $n$ -DVF structure  $\mathcal{E}_{\text{local}}$  on the latter. The valuations of  $F_i$  are precisely the ones lying over the valuations  $v_1, \dots, v_n$  of  $\mathcal{E}_{v_1, \dots, v_n}$  (actually: given the Uniqueness Principle, Proposition 6.16,  $F_i$  only has one single  $n$ -local field structure, so we cannot go wrong here anyway). However, since the boundary maps along valuations are compatible with field extensions (Diagram 6.15), both results agree. We merely factor the evaluation of a concatenated boundary map over  $n$ -local field completions. Thus, Equation 6.18 transforms into

$$\beta \longmapsto \sum_{\Delta} V_{alg}(\Delta),$$

confirming our claim. ■

6.23.1. STEP 4: END OF THE PROOF. Now we can prove our formula for the intersection multiplicity.

PROOF OF THEOREM 6.6. As the cover is indexed by the totally ordered set  $I$ , we obtain a disjoint decomposition  $(\Sigma_\alpha)$  in the sense of Definition 6.11 by letting  $\Sigma_\alpha$  be the set of those scheme points  $x \in U_\alpha$  such that for no strictly smaller  $\alpha' < \alpha$  we have  $x \in U_{\alpha'}$ .

Consider the adelic expression which we aim to evaluate. Making the tacit identification  $K_n(n\text{-Tate}(k)) \cong \mathbb{Z}$  explicit, it reads:

$$(-1)^{\frac{n(n+1)}{2}} \sum_{\Delta=(\eta_n < \dots < \eta_0)} \pi_*(D \circ \dots \circ D)_{A_X(\Delta, \mathcal{O}_X)} \left( \begin{matrix} [f_{\alpha(\eta_n)\alpha(\eta_{n-1})}^n]_{\circlearrowleft \eta_n \Lambda_{\eta_{n-1}}} \otimes \dots \otimes [f_{\alpha(\eta_1)\alpha(\eta_0)}^1]_{\circlearrowleft \eta_1 \Lambda_{\eta_0}} \end{matrix} \right). \tag{6.19}$$

Firstly, note that we are using the external product of  $K$ -theory classes  $K_1 \times \dots \times K_1 \rightarrow K_n$  here, as introduced in Definition 5.1. By Corollary 5.6 this product commutes with taking the product in the ordinary  $K$ -theory of the variety  $X$  first, and then performing the Tate realization. We use that this is not just true for  $X$ , but also holds on all its open subsets; in particular on the opens  $U_*$  where the  $f_{*,*}^*$  of the open cover  $\mathfrak{U}$  are defined. As

$$\begin{matrix} [f] \\ \circlearrowleft (-) \end{matrix}$$

denotes the Tate realization, we may replace the term in the second line, Equation 6.19, by

$$\text{diag}^* \{f_{\alpha(\eta_n)\alpha(\eta_{n-1})}^n, \dots, f_{\alpha(\eta_1)\alpha(\eta_0)}^1\},$$

where  $\{-, \dots, -\}$  now refers to the product in the  $K$ -theory of the open  $U_*$  in  $\mathfrak{U}$ , where the  $f_{*,*}^*$  are defined. As alternating Čech cocycles satisfy  $f_{\nu\mu} = -f_{\mu\nu}$  (or  $f_{\nu\mu} = f_{\mu\nu}^{-1}$ , written for  $\mathbb{G}_m$ ), we may further rewrite this as

$$(-1)^n \{f_{\alpha(\eta_{n-1})\alpha(\eta_n)}^n, \dots, f_{\alpha(\eta_0)\alpha(\eta_1)}^1\}.$$

Since the product in  $K$ -theory is graded-commutative, we may re-arrange these terms as

$$\text{Eq. (6.19)} = \sum_{\Delta=(\eta_n < \dots < \eta_0)} (D \circ \dots \circ D)_{A_X(\Delta, \mathcal{O}_X)} \text{diag}^* \{f_{\alpha(\eta_0)\alpha(\eta_1)}^1, \dots, f_{\alpha(\eta_{n-1})\alpha(\eta_n)}^n\},$$

and the factor  $(-1)^{n(n-1)/2}$ , which had remained from the previous step, cancels, as this is the sign of the permutation reversing the order of  $n$  letters.

Next, we recognize that this expression is exactly of the form  $V_{Tate}$  as in Equation 6.13, so that

$$= \sum_{\Delta \in S(X)_n} V_{Tate}(\Delta)(\{f_{\alpha(\eta_0)\alpha(\eta_1)}^1, \dots, f_{\alpha(\eta_{n-1})\alpha(\eta_n)}^n\}).$$

By Lemma 6.21 this equals

$$= \sum_{\Delta \in S(X)_n} V_{alg}(\Delta)(\{f_{\alpha(\eta_0)\alpha(\eta_1)}^1, \dots, f_{\alpha(\eta_{n-1})\alpha(\eta_n)}^n\})$$

and by Lemma 6.23 this equals  $V_{Ger}(\{f_{\alpha(\eta_0)\alpha(\eta_1)}^1, \dots, f_{\alpha(\eta_{n-1})\alpha(\eta_n)}^n\})$ . And using the definition of  $V_{Ger}$ , we unravel this expression as

$$= \sum_{\eta_n, \dots, \eta_0} [\kappa(\eta_n) : k] \cdot \partial_{\eta_n}^{\eta_n-1} \dots \partial_{\eta_1}^{\eta_0} \{f_{\alpha(\eta_0)\alpha(\eta_1)}^1, \dots, f_{\alpha(\eta_{n-1})\alpha(\eta_n)}^n\}.$$

Now use Proposition 6.12 and we conclude that this equals the intersection multiplicity  $L_1 \cdots L_n$ . ■

## 7. Pontryagin duality

Let us also address a different application of our methods: The classical inspiration for Tate categories are the Tate vector spaces. For a number of applications, especially when one only needs  $n$ -Tate objects for  $n = 1$ , one can work with variants of topological  $k$ -vector spaces instead. And if  $k$  happens to be a *finite* field, one can even work with locally compact abelian (LCA) groups. A discussion in this spirit is given in [FHK], §4.1. Indeed, adèles for curves over finite fields are usually considered in the category of LCA groups and Weil’s degree formula from the Introduction would classically be phrased in this context. We briefly discuss this:

Let  $\mathbf{HAb}$  be the category of Hausdorff topological abelian groups. Morphisms are continuous group homomorphisms. Let  $\mathbf{LCA}$  be the category of locally compact abelian topological groups.

Let  $\mathbb{T}$  denote the standard torus group (i.e.  $U(1)$  with its usual topology, or equivalently  $\mathbb{R}/\mathbb{Z}$  with the quotient topology). For  $G \in \mathbf{HAb}$  the *dual group* is defined as  $G^\vee := \text{Hom}(G, \mathbb{T})$ , the group of continuous group homomorphisms, equipped with the compact-open topology. There is a canonical continuous homomorphism  $\eta_G : G \rightarrow G^{\vee\vee}$  and  $G$  is called *reflexive* if  $\eta_G$  is an isomorphism in  $\mathbf{HAb}$ . The central duality result for  $\mathbf{LCA}$  groups is the following:

**7.1. THEOREM. (PONTRYAGIN DUALITY)** *All groups in  $\mathbf{LCA}$  are reflexive and there is a canonical exact equivalence of exact categories*

$$\mathbf{LCA}^{op} \xrightarrow{\sim} \mathbf{LCA}.$$

*Suppose  $\mathbf{LCA}^{op} \xrightarrow{F} \mathbf{LCA}$  is any equivalence of categories. Then  $F$  is naturally equivalent to the Pontryagin Duality functor.*

The duality is due to Pontryagin and van Kampen. The uniqueness is due to Roeder [Roe71, Theorem 5].

Next, besides the duality provided by Pontryagin’s Duality Theorem, one would like to have a tensor product and internal Homs. This is not readily possible:

**7.2. REMARK. (THE PROBLEM WITH INTERNAL HOMS)** Moskowitz established that if  $\text{Hom}(G, H)$  is locally compact for all choices of  $H$  (resp.  $G$ ), then  $G$  must be a finitely generated discrete group (resp.  $H$  must be compact without small sub-groups) [Mos67, Theorem 4.3]. Thus, one cannot hope to equip the entire category  $\mathbf{LCA}$  with internal Homs.

In [HS07] Hoffmann and Spitzweck present a different approach to equip  $\mathbf{LCA}$  with a closed monoidal structure.

**7.3. DEFINITION.** *An  $\mathbf{LCA}$  group  $A$  has*

1. *finite  $\mathbb{Z}$ -rank if  $\text{Hom}(A, \mathbb{R})$  is a finite-dimensional real vector space,*
2. *finite  $\mathbb{S}^1$ -rank if  $\text{Hom}(\mathbb{R}, A)$  is a finite-dimensional real vector space,*
3. *finite  $p$ -rank if multiplication by  $p$ ,  $(-) \cdot p : A \rightarrow A$  is a strict morphism whose kernel and cokernel are finite abelian groups.*

*We say that  $A$  has finite ranks if its  $\mathbb{Z}$ -rank is finite, its  $\mathbb{S}^1$ -rank is finite, and for all prime numbers  $p$  its  $p$ -rank is finite. Let  $\mathbf{FLCA}$  be the full sub-category of  $\mathbf{LCA}$  of groups with finite ranks. (see [HS07, Definition 2.5])*

The dual group of a finite ranks group will have finite ranks again, so that Pontryagin Duality restricts to the category  $\mathbf{FLCA}$  and its opposite.

7.4. THEOREM. (HOFFMANN–SPITZWECK) *The additive categories*

$$\text{FLCA} \subset \text{LCA} \subset \text{HAb}$$

*are each quasi-abelian. In particular, they are exact categories in a natural way. The category FLCA moreover carries a closed symmetric monoidal structure with a bi-right exact tensor product*

$$- \otimes - : \text{FLCA} \times \text{FLCA} \longrightarrow \text{FLCA}.$$

*Finally, FLCA is a fully exact sub-category of LCA.*

PROOF. See [HS07], Proposition 1.2 and Corollary 2.10 for proofs that the categories are quasi-abelian, Proposition 3.14 for the closed monoidal structure. The internal Hom is shown to be bi-left exact in §3 *loc. cit.* and the bi-right exactness of the tensor product gets deduced from this, Remark 4.3. *loc. cit.* Finally, FLCA is fully exact in LCA by Proposition 2.9 *loc. cit.* ■

Using Corollary 3.6 and the closed monoidal structure of Theorem 7.4, we obtain:

7.5. THEOREM. (GENERALIZED PONTRYAGIN DUALITY) *For all  $n \geq 0$ , there is a canonical exact equivalence of exact categories*

$$n\text{-Tate}(\text{LCA})^{op} \xrightarrow{\sim} n\text{-Tate}(\text{LCA}) \quad n\text{-Tate}(\text{FLCA})^{op} \xrightarrow{\sim} n\text{-Tate}(\text{FLCA}).$$

*For any  $n, m \geq 1$ , there exists a bi-exact functor*

$$-\overset{\rightarrow}{\otimes}- : n\text{-Tate}^b(\text{FLCA}) \times m\text{-Tate}^b(\text{FLCA}) \longrightarrow (n+m)\text{-Tate}^b(\text{FLCA}).$$

7.6. REMARK. These categories might be the appropriate candidate when one is interested in a category which contains both  $n$ -Tate objects as well as the classical adèles of a number field along with its archimedean places.

Classical Pontryagin Duality exchanges certain sub-categories, e.g.,

$$\begin{array}{ll} \text{discrete} & \leftrightarrow \text{compact} \\ \text{(topological) } p\text{-torsion} & \leftrightarrow \text{(topological) } p\text{-torsion} \\ \text{compact metrizable} & \leftrightarrow \text{countable} \\ \text{compact connected} & \leftrightarrow \text{torsion-free discrete.} \end{array}$$

By Proposition 3.5 our extension of an equivalence  $\mathcal{C}^{op} \xrightarrow{\sim} \mathcal{C}$  to the Tate category (1) preserves the property to exchange such sub-categories, and (2) exchanges Ind- and Pro-objects. Applying this observation to LCA and the aforementioned dual pairs, we get a whole panorama of duality assertions, all from our general principles. For example, in Tate(LCA) we will have

$$\begin{array}{ll} \text{pro-discrete} & \leftrightarrow \text{ind-compact} \\ \text{pro-(topological) } p\text{-torsion} & \leftrightarrow \text{ind-(topological) } p\text{-torsion} \\ \text{Tate-(compact metrizable)} & \leftrightarrow \text{Tate-countable} \\ \text{Tate-(compact connected)} & \leftrightarrow \text{Tate-(torsion-free discrete).} \end{array}$$

Of course this list is not exhaustive, and for  $n$ -Tate categories, we obtain all analogous variations for  $n$  prefixes Ind-, Pro- or Tate.

7.7. **REMARK.** Barr has also constructed complete and co-complete categories containing  $\mathbf{LCA}$  and all of whose groups are reflexive [Bar77]. This might be the largest category which extends Pontryagin Duality within the context of topological groups. A classification of the reflexive objects in  $\mathbf{HAb}$  appears to be very complicated, see Hernández’ Theorem [Her01, Theorem 3].

We briefly comment on the question to what extent the category  $\mathbf{LCA}$  might itself have full sub-categories resembling full sub-categories of Tate object categories. For example, parts of the category  $\mathbf{LCA}$  have the shape of Ind- and Pro-objects themselves. This was first isolated by Roeder:

7.8. **PROPOSITION.** *Let  $\mathbf{LCA}_{comp}$  denote the full sub-category of compact LCA groups, and  $\mathbf{LCA}_{disc}$  the full sub-category of discrete groups. Moreover,*

1. *let  $C_0$  be the full sub-category of  $\mathbf{LCA}$  of groups of the shape  $\mathbb{T}^n \oplus$  (finite abelian),*
2. *let  $A_0$  be the full sub-category of  $\mathbf{LCA}$  of groups of the shape  $\mathbb{Z}^n \oplus$  (finite abelian),*

*Then there are equivalences of categories*

$$\mathbf{LCA}_{comp} \xrightarrow{\sim} \mathbf{Pro}^a(C_0) \quad \text{and} \quad \mathbf{LCA}_{disc} \xrightarrow{\sim} \mathbf{Ind}^a(A_0).$$

This is [Roe74, Prop. 5, Prop. 7], along with an inspection of Roeder’s definition of the Ind- and Pro-category, which turns out to be compatible with ours.

The categories  $C_0$  and  $A_0$  are Pontryagin duals. Many variations of this theme are possible, e.g. cardinality constraints. Thus, if  $X_0$  denotes the full sub-category of  $\mathbf{LCA}$  of groups of the shape  $\mathbb{T}^n \oplus \mathbb{Z}^m \oplus$  (finite abelian), then  $\mathbf{Tate}^{el}(X_0)$  is an exact category containing full sub-categories equivalent to copies of  $\mathbf{LCA}_{comp}$  (contained in the Pro-objects) and  $\mathbf{LCA}_{disc}$  (contained in the Ind-objects).

It is natural to ask whether the underlying Ind- and Pro-objects can be realized as topological groups. The category  $\mathbf{LCA}$  is neither complete, nor cocomplete, so we cannot carry out Ind- or Pro-limits in  $\mathbf{LCA}$  itself. However, this does not rule out the possibility to work in a larger category of reflexive topological groups. Indeed, Kaplan showed that Ind- and Pro-limits of  $\mathbf{LCA}$  groups, indexed over  $\mathbb{N}$ , exist in  $\mathbf{HAb}$  and the dual group of such a Pro-limit is the corresponding Ind-limit of the dual groups [Kap50, §5].

7.9. **PROPOSITION.** *There is a natural transformation from the duality functor of Proposition 3.5 to dualization on  $\mathbf{HAb}$ ,*

$$\begin{array}{ccc} \mathbf{Ind}_{\mathbb{N}_0}^a(\mathbf{LCA})^{op} & \xrightarrow{\sim} & \mathbf{Pro}_{\mathbb{N}_0}^a(\mathbf{LCA}) \\ \downarrow & & \downarrow \\ \mathbf{HAb}^{op} & \xrightarrow{G \mapsto G^\vee} & \mathbf{HAb}, \end{array}$$

*where each downward arrow refers to carrying out the Ind- resp. Pro-limit. If we restrict the bottom row to the essential images of the top row, this diagram extends to a natural equivalence of duality functors.*

PROOF. The category  $\mathbf{LCA}$  is quasi-abelian and thus an idempotent complete exact category. Pontryagin Duality induces an equivalence  $\mathbf{LCA}^{op} \xrightarrow{\sim} \mathbf{LCA}$  and so Proposition 3.5 applies. Now apply Kaplan’s duality theorem [Kap50, §5, Duality Theorem]. ■

## A. Construction of a higher Haar-type torsor

There is a recurring dream in the literature, hoping to establish a good analogue of harmonic analysis for the adèles of a scheme. Unless the scheme is one-dimensional and has only finite fields as residue fields, the underlying topological space of the adèles fails to be locally compact for any reasonable choice of a topology on it. This rules out the existence of a Haar measure. New ideas are needed.

However, whenever a Haar measure exists, it pins down a torsor, encoding the fact that the Haar measure is only unique up to multiplication with a positive scalar. We will now give a fairly general construction of the higher generalization of this torsor, following the idea of [Kap01] – albeit without addressing the construction of any form of a generalized measure from which it could stem.

The papers [OP08], [OP11] and [Zhu14] contain interesting constructions in this respect. In the case of varieties over a field  $k$ , one can work with the  $C_n$ -categories of loc. cit., which are a form of Tate categories. However, when working with number fields or arithmetic surfaces, one really needs local field factors of the shape  $\mathbb{R}, \mathbb{C}$ , which exist in  $\mathbf{LCA}$ , but not in a category like  $\mathbf{Tate}(\mathbb{F}_q)$ . As a solution, the paper [OP11] introduces categories “ $C_n^{\text{ar}}$ ” for  $n = 0, 1, 2$ . The category  $\mathbf{LCA}(2)$  of Zhu [Zhu14] and Liu–Zhu [LZ17] is another related candidate. Let  $\mathbf{FLCA}_{\mathbb{N}}$  be the full sub-category of  $\mathbf{FLCA}$  of second countable groups. This is a quasi-abelian category by the method of [HS07], and moreover closed under Pontryagin Duality. We consider the category

$$\mathcal{C}_n^{\text{Ar}} := n\text{-Tate}(\mathbf{FLCA}_{\mathbb{N}}),$$

which is a quite natural variation/generalization of the categories of [OP11] (note however that the indexing by  $n$  has incompatible meanings! Our  $\mathcal{C}_0^{\text{Ar}}$  is philosophically closer to their  $C_1^{\text{ar}}$  than  $C_0^{\text{ar}}$ ). As an advantage, by its very construction as a Tate category, all the tools of the paper [BGW16c] are available for  $\mathcal{C}_n^{\text{Ar}}$ . The approach of [OP11] sets up the categories  $C_0^{\text{ar}}, C_1^{\text{ar}}, C_2^{\text{ar}}$  by several individual constructions. In particular, we automatically have a notion of lattices for objects in our  $\mathcal{C}_n^{\text{Ar}}$ , we know that any pairs of lattices have a common over- and under-lattice, the categories are exact and using the tools of the present article, we get a normally ordered tensor product on the flat Tate objects.

Moreover, we can compute the  $K$ -theory in terms of the  $K$ -theory of  $\mathbf{FLCA}_{\mathbb{N}}$  using Saito’s delooping theorem [Sai15]. Unfortunately, the  $K$ -theory of  $\mathbf{FLCA}_{\mathbb{N}}$  is not known at the moment. If  $\mathbf{LCA}_{\mathbb{N}}$  denotes the full exact sub-category of  $\mathbf{LCA}$  of second countable groups, Clausen [Cla17] has computed the entire  $K$ -theory spectrum  $K(\mathbf{LCA}_{\mathbb{N}})$ . Most notably,

$$K_0(\mathbf{LCA}_{\mathbb{N}}) = 0 \quad \text{and} \quad K_1(\mathbf{LCA}_{\mathbb{N}}) \cong \mathbb{R}_{>0}^{\times},$$

and the latter isomorphism has the following explicit description: If  $G \in \text{LCA}_{\mathbb{N}}$  is an LCA group, pick a Haar measure  $\mu$  on it. Now, every automorphism  $\gamma$  of  $G$  canonically determines an element in  $K_1(\text{LCA}_{\mathbb{N}})$ , giving a map

$$C : \text{Aut}_{\text{LCA}_{\mathbb{N}}}(G) \longrightarrow K_1(\text{LCA}_{\mathbb{N}}) \xrightarrow{\sim} \mathbb{R}_{>0}^{\times} \tag{A.1}$$

by Clausen’s computation. At the same time,  $\mu \circ \gamma$  is again a bi-invariant measure on  $G$ , and by the uniqueness of Haar measures up to rescaling by a positive constant, there is a unique  $c_{\gamma} \in \mathbb{R}_{>0}^{\times}$  such that  $(\mu \circ \gamma)(U) = c_{\gamma} \cdot \mu(U)$  holds for all measurable sets  $U \subseteq G$ . This construction defines a further map  $c : \text{Aut}_{\text{LCA}_{\mathbb{N}}}(G) \rightarrow \mathbb{R}_{>0}^{\times}$ . According to Clausen’s computation, both maps agree, i.e.  $C = c$ . Following Weil,  $c_{\gamma}$  is called the *module* of an automorphism, see [Wei74, Ch. I, §2].

We may now generalize this to construct a type of higher Haar-type torsor for objects in the categories  $\mathcal{C}_n^{\text{Ar}}$ . Saito’s delooping theorem provides a canonical equivalence

$$K(\mathcal{C}_n^{\text{Ar}}) \xrightarrow{\sim} B^n K(\text{FLCA}_{\mathbb{N}}) \tag{A.2}$$

and since  $\text{FLCA}_{\mathbb{N}} \hookrightarrow \text{LCA}_{\mathbb{N}}$  is a fully exact sub-category, the functor of inclusion is exact and thus we have an induced map in  $K$ -theory,

$$K(\text{FLCA}_{\mathbb{N}}) \longrightarrow K(\text{LCA}_{\mathbb{N}}).$$

The truncation of the  $K$ -theory spectrum of  $\text{LCA}_{\mathbb{N}}$  to degrees  $[0, 1]$  yields the Eilenberg-Mac Lane spectrum of the group  $\mathbb{R}_{>0}^{\times}$  (taken with the discrete topology) in degree 1. Thus, composing with the truncation map, the above map induces a map

$$K(\mathcal{C}_n^{\text{Ar}}) \longrightarrow B^n K(\text{LCA}_{\mathbb{N}}) \longrightarrow B^n \tau_{\leq 1} K(\text{LCA}_{\mathbb{N}}) \xrightarrow{\sim} B^{n+1} \mathbb{R}_{>0}^{\times}. \tag{A.3}$$

Suppose  $G \in \mathcal{C}_n^{\text{Ar}}$ . The generalization of the above construction with automorphisms is the canonical map

$$B \text{Aut}_{\mathcal{C}_n^{\text{Ar}}}(G) \longrightarrow \Omega^{\infty} K(\mathcal{C}_n^{\text{Ar}}),$$

where the left-hand side is the classifying space of the group  $\text{Aut}_{\mathcal{C}_n^{\text{Ar}}}(G)$ , and the right-hand side the infinite loop space attached to the  $K$ -theory spectrum of  $\mathcal{C}_n^{\text{Ar}}$ , which happens to be connective. In the special case  $n = 0$ , this induces on the level of the fundamental group  $\pi_1$  the map  $C : \text{Aut}_{\mathcal{C}_n^{\text{Ar}}}(G) \rightarrow \mathbb{R}_{>0}^{\times}$  discussed above. Now we pre-compose the map of Equation A.3 with this construction, giving the following:

**A.1. DEFINITION.** *For every  $n \geq 0$  and every object  $G \in \mathcal{C}_n^{\text{Ar}}$ , there is a canonical map of spaces*

$$B \text{Aut}_{\mathcal{C}_n^{\text{Ar}}}(G) \longrightarrow B^{n+1} \mathbb{R}_{>0}^{\times}.$$

*In particular, this defines a canonical group cohomology class*

$$H \in H_{\text{grp}}^{n+1}(\text{Aut}_{\mathcal{C}_n^{\text{Ar}}}(G), \mathbb{R}_{>0}^{\times}),$$

*which we call the (cohomology class of the) higher Haar torsor on  $G$ .*

To construct  $H$ , we use the standard fact from homotopy theory that for any group  $A$  and abelian group  $Z$ , group cohomology has the description

$$H_{\text{grp}}^n(A, Z) = \pi_0 \text{map}(BA, B^n Z),$$

i.e. group cohomology classes correspond to homotopy classes of maps from  $BA$  to  $B^n Z$ . As the above constructions produce such a map, this pins down a cohomology class.

A.2. REMARK. (HAAR-TYPE TORSOR) Having constructed the cohomology class  $H$ , one can also speak of the actual higher Haar-type torsor, instead of just the map classifying it. For this one needs to specify what the word “higher torsor” should mean. For example, following the approach of Saito to higher  $K$ -theory torsors [Sai14], one can use the framework of [NSS15]. This Haar-type torsor thus fits into the pattern of Saito’s general construction.

A.3. REMARK. Unlike [OP11], we only construct higher Haar-type torsor candidates, but we make no attempt here to construct some generalized measure theory such that our Haar torsor would arise from it. Even having the higher Haar torsor available, this does not clarify in any way what, for example, a generalized Schwartz–Bruhat function should be.

A.4. PROPOSITION. (AGREEMENT) For  $n = 0$ ,  $\mathcal{C}_0^{\text{Ar}} = \text{FLCA}_{\mathbb{R}}$  is the category of second countable LCA groups of finite ranks à la Hoffmann–Spitzweck [HS07]. For every such group  $G$ , the Haar torsor of Definition A.1 is a classical  $\mathbb{R}_{>0}^{\times}$ -torsor and

$$H \in H^1(\text{Aut}(G), \mathbb{R}_{>0}^{\times})$$

is just the map  $c : \text{Aut}(G) \rightarrow \mathbb{R}_{>0}^{\times}$ ,

$$\gamma \longmapsto \frac{\mu(\gamma U)}{\mu(U)},$$

where  $\mu$  is any Haar measure on  $G$ , and  $U$  any measurable sub-group of  $G$  of positive measure.

PROOF. The property  $\mathcal{C}_0^{\text{Ar}} = \text{FLCA}_{\mathbb{R}}$  holds by construction. The class  $H$  agrees with  $C$  of Equation A.1, and by Clausen’s computation in [Cla17],  $C = c$ , giving the claim. As a result, the torsor associated to  $H$  by the classical theory of torsors (or compatibly [NSS15]), is isomorphic to the torsor of Haar measures coming from classical harmonic analysis. ■

A.5. REMARK. A description of Saito’s equivalence in Equation A.2 by an explicit simplicial model based on lattices in Tate objects is given in [BGW18]. Thus, if one wants to study the higher Haar torsors in an explicit fashion, this would be a first step.

A.6. EXAMPLE. As a concrete example, we get a Haar measure analogue of the tame symbol. Let us spell out the details: For  $n = 1$ , every object  $G \in \mathcal{C}_1^{\text{Ar}}$  comes with a canonical central extension

$$1 \longrightarrow \mathbb{R}_{>0}^\times \longrightarrow \widehat{\text{Aut}(G)} \longrightarrow \text{Aut}(G) \longrightarrow 1$$

of its automorphism group, as defined by  $H \in H^2(\text{Aut}(G), \mathbb{R}_{>0}^\times)$ . As part of this structure for any two commuting automorphisms  $f, g$  of  $G$ , we get an element

$$\langle f, g \rangle \in \mathbb{R}_{>0}^\times.$$

When restricting to an abelian sub-group  $A \subseteq \text{Aut}(G)$  for example, all its elements pairwise commute, and then  $\langle -, - \rangle$  becomes a bi-linear pairing

$$\langle -, - \rangle : A \times A \longrightarrow \mathbb{R}_{>0}^\times.$$

If we read the symbols  $F := \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$  as referring to their corresponding objects in FLCA, we can consider the following object in  $\mathcal{C}_1^{\text{Ar}}$ ,

$$F((t)) := \text{colim}_i \lim_j t^{-i} F[t]/t^j.$$

Clearly  $F((t))$  (now viewed as a ring) acts by multiplication on  $F((t))$ , giving the abelian subgroup  $F((t))^\times$  inside the automorphism group of  $F((t)) \in \mathcal{C}_1^{\text{Ar}}$ . Then we get the pairing

$$F((t))^\times \times F((t))^\times \longrightarrow \mathbb{R}_{>0}^\times \quad (f, g) \longmapsto \left| \frac{f^{v(g)}}{g^{v(f)}}(0) \right|, \tag{A.4}$$

where  $v(-)$  refers to the  $t$ -valuation, so that  $f^{v(g)}/g^{v(f)}$  is a formal power series with non-zero constant coefficient in  $F$ , and  $|\cdot|$  refers to the real absolute value for  $F = \mathbb{R}$ , the square of the complex absolute value  $|\cdot|^2$  for  $F = \mathbb{C}$ , or the  $p$ -adic absolute value for  $F = \mathbb{Q}_p$ . Moreover, if  $\mathbb{A}$  denotes the classical adèles of a number field or a curve over a finite field, viewed as an LCA group, we get

$$\mathbb{A}((t))^\times \times \mathbb{A}((t))^\times \longrightarrow \mathbb{R}_{>0}^\times \quad (f, g) \longmapsto \prod_w \left| \frac{f^{v(g)}}{g^{v(f)}}(0) \right|, \tag{A.5}$$

where  $w$  runs over the places of the number field/function field, and all but finitely many factors in this product are equal to one. To prove the assertions, use the following trick: In the cases  $F := \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$  use that there is an exact functor from finite-dimensional  $F$ -vector spaces to  $\text{FLCA}_{\mathbb{N}}$ ,

$$\text{Vect}_f(F) \longrightarrow \text{FLCA}_{\mathbb{N}},$$

which sends a finite-dimensional  $F$ -vector space to its additive group, equipped with the topology coming from  $F$ . In the cases at hand, this is always a second countable LCA group. This functor is easily checked to be exact. Thus, the functor

$$n\text{-Tate}(\text{Vect}_f(F)) \longrightarrow n\text{-Tate}(\text{FLCA}_{\mathbb{N}}) = \mathcal{C}_n^{\text{Ar}}$$

is exact. This has a great advantage for our computation: If we know the result for  $n$ -Tate  $(\mathbf{Vect}_f(F))$ , we can simply make the computation there and then map it to  $\mathbf{FLCA}_{\mathbb{N}}$ . As is shown in [BGW17] or [BGW18], for  $n$ -Tate  $(\mathbf{Vect}_f(F))$ , the delooping map is compatible with the boundary map in  $K$ -theory of the exact sequence of exact categories

$$\mathrm{Coh}_{\overline{\{(t)\}}} F[[t]] \longrightarrow \mathrm{Coh} F[[t]] \longrightarrow \mathrm{Coh} F((t))$$

and on the level relevant for our computation, this is just the ordinary tame symbol

$$(-1)^{v(f)v(g)} f^{v(g)} g^{-v(f)}(0) \in F^\times,$$

which lives in  $K_1(F) \cong K_1(\mathrm{Coh}_{\overline{\{(t)\}}} F[[t]])$ . Once we have this element in  $F^\times$ , the induced map  $K_1(F) \longrightarrow K_1(\mathbf{LCA}_{\mathbb{N}})$  is

$$\begin{aligned} F^\times &\longrightarrow \mathbb{R}_{>0}^\times \\ \alpha &\longmapsto |\alpha|_{\mathbb{R}} \quad (\text{resp. } |\alpha|_{\mathbb{C}}, \text{ resp. } |\alpha|_p). \end{aligned}$$

This observation is due to André Weil, and in a sense lies at the roots of the subject: For all the fields which occur in the classical adèles, namely locally compact topological fields, one can uniquely reconstruct their natural notions of absolute value (resp. its square) by studying the module of automorphisms with respect to any Haar measure. We refer to [Wei74, Chapter 1], which is the ultimate treatment using this perspective. This proves our claim for  $F := \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$ . We leave the case of the adèles to the reader.

**A.7. EXAMPLE.** Following Clausen, we also get vanishing statements in the context of the previous example. If we let  $F := \mathbb{Z}$  or  $\mathbb{T}$  as an object of  $\mathbf{FLCA}$ , the corresponding Haar central extensions on

$$\mathbb{Z}((t)) \quad \text{resp.} \quad \mathbb{T}((t))$$

are trivial, i.e. they split. This can be seen as follows: Just as the above computation factored the classifying map to  $\mathbf{LCA}_{\mathbb{N}}$  over  $\mathbf{Vect}_f(F)$ , we may now factor over  $\mathbf{LCA}_{disc, \mathbb{N}}$  resp.  $\mathbf{LCA}_{comp, \mathbb{N}}$ , and the  $K$ -theory of these categories is contractible. Indeed, these are an Ind- and a Pro-category themselves, by a mild variation of Proposition 7.8.

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