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LAX PULLBACK COMPLEMENTS AND PULLBACKS OF SPANS

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ABSTRACT. The formation of the "strict" span category Span($\mathcal{C}$) of a category $\mathcal{C}$ with pullbacks is a standard organizational tool of category theory. Unfortunately, limits or colimits in Span($\mathcal{C}$) are not easily computed in terms of constructions in $\mathcal{C}$. This paper shows how to form the pullback in Span($\mathcal{C}$) for many, but not all, pairs of spans, given the existence of some specific so-called lax pullback complements in $\mathcal{C}$ of the "left legs" of at least one of the two given spans. For some types of spans we require the ambient category to be adhesive to be able to form at least a weak pullback in Span($\mathcal{C}$). The existence of all lax pullback complements in $\mathcal{C}$ along a given morphism is equivalent to the exponentiability of that morphism. Since exponentiability is a rather restrictive property of a morphism, the paper first develops a comprehensive framework of rules for individual lax pullback complement diagrams, which resembles the set of pasting and cancellation rules for pullback diagrams, including their behaviour under pullback. We also present examples of lax pullback complements along non-exponentiable morphisms, obtained via lifting along a fibration.

1. Introduction

Completing a given diagram $U \xrightarrow{u} S \xrightarrow{s} A$ to a pullback square

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "s'"'] & S \arrow[d, "s"] \\ P \arrow[r, "p"'] & A \end{tikzcd}$$

in a (co)universal manner was described in [Tholen, 1983] as finding a *pullback complement (pbc) of u along s*. There has been interest in pbcs for almost four decades, especially by theoretical computer scientists; see, for example, [Ehrig, Kreowski, 1979] for an early contribution, and [Shir Ali Nasab, Hosseini, 2015] for a fairly recent account. Of particular interest are pbcs when u is the projection

$$Y \times S \longrightarrow S$$

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along some object Y, in which case (P, p) is called a *partial product of Y over s*. First considered in [Pasynkov, 1965], the universal property of a partial product was indicated in [Dyckhoff, 1984] in its original topological context, before being put into a categorical setting in [Tholen, 1983]. A fundamental example of [Pasynkov, 1965] is the recursive presentation of the n-sphere S^n as a partial product of the 2-point discrete space over the embedding $I^n \setminus S^{n-1} \hookrightarrow I^n$, with I^n denoting the n-dimensional cube.

For the given morphism s, [Dyckhoff, Tholen, 1987] characterized the existence of all partial products over s as the *exponentiability* of s in its ambient category. This powerful categorical property had been systematically explored in [Niefield, 1982], where the characterization of exponentiable topological spaces of [Day, Kelly, 1970] was extended to a characterization of exponentiable continuous maps (see also [Niefield, 2002, Richter, 2002, Richter, Tholen, 2001]). The more restrictive condition whereby, given s, the pbc of u along s exists for all morphisms u (not just for the product projections onto S) was also characterized in [Dyckhoff, Tholen, 1987]: s must be an exponentiable *mono*morphism.

These characterizations, as well as the proofs of the stability properties of exponentiable morphisms under composition and pullback as established in [Niefield, 1982], rely on well-known composition and cancellation arguments for adjoint functors and thereby presume the existence of partial products of *all* objects Y over s, or of pbcs of *all* morphisms u composable with s. This approach, however, leaves open the question whether, similarly to the composition and cancellation rules for pullback diagrams, there is a "calculus" for individually given partial products or pullback complements which, as a secondary consequence, is sufficiently strong to yield the desired stability properties, after quantification over all objects Y or morphisms u.

When s is monic, [Tholen, 1986] provided an account of such a calculus for pbcs. The first goal of this paper is the provision of a more comprehensive calculus, under minimal restrictions on s. To this end we extend the notion of pbc of u along s to that of a *lax pullback complement (lax pbc)* of u along s. While this notion just "rephrases" the (co)universal property of exponentiability in an easy diagrammatic way, it still provides the opportunity to paste individual diagrams and explore to which extent a lax pbc is being preserved under diagram pasting or cancellation, or under pullback. Having recalled the standard functorial presentation of exponentiable morphisms in Section 2, we present such an exploration in the first half of this paper (Sections 3-5). The most intricate result in this context is given by Theorem 5.2; it explains how to form the lax pbc along the pullback t of some morphism s, given the existence of a couple of partial products over s. Through quantification the Theorem then gives the known pullback stability of exponentiable morphisms in a finitely complete category.

The second part of the paper is devoted to the construction of pullbacks in the "strict" span category $\mathrm{Span}(\mathcal{C})$ of spans (= pairs of morphisms with common domain) in $\mathcal{C}$, and to the discussion of some characteristic examples. Originally introduced in [Bénabou, 1967] as an example of a bicategory and studied extensively ever since (see in particular [Dawson, Paré, Pronk, 2004, Dawson, Paré, Pronk, 2010]), little seems to be known about the existence of limits (or, by selfduality, of colimits) in the one-dimensional category $\mathrm{Span}(\mathcal{C})$, with the exception of the rather easily established fact that, *when* $\mathcal{C}$ *is (finitely) lextensive* (see [Carboni, Lack, Walters, 1993]), Span($\mathcal{C}$) *has (finite) products and coproducts*. Our principal result for spans (Theorem 6.1) guarantees the existence of pullbacks of *total* spans (= spans whose left leg is iso) along spans whose left leg admits a certain lax pullback complement in $\mathcal{C}$ and, hence, along spans with exponentiable left leg. When $\mathcal{C}$ is *adhesive* (see [Lack, Sobociński, 2005, Garner, Lack, 2012]), using a pullback complement we can also construct *weak* pullbacks of *cototal* spans (= spans whose right leg is iso): see Theorem 7.3. In Section 8 we present some known and new examples of lax pullback complements, taking advantage of a "lifting result" for lax pullback complements along fibrations (Theorem 8.2), which allows us to construct lax pbcs in $\mathcal{V}$-Cat (in the context of [G.M. Kelly, 1982]) and $(\mathbb{T},\mathcal{V})$-Cat (in the context of [Hofmann, Seal, Tholen, 2014]. Section 9 gives an alternative categorical description of lax pullback complements as coreflections.

2. Short review of exponentiable morphisms

Recall that an object A in a category $\mathcal{C}$ that has all products with A is *exponentiable* if the endofunctor $(-)\times A$ of $\mathcal{C}$ has a right adjoint. A morphism $s : S \longrightarrow A$ in $\mathcal{C}$, such that all pullbacks along s exist in $\mathcal{C}$, is *exponentiable* if it is exponentiable when considered as an object in the comma category $\mathcal{C}/A$. There are various equivalent functorial ways of expressing the exponentiability of s; for that, consider the following commutative diagram of functors:

$$\begin{tikzcd} \mathcal{C}/A \arrow[rr, "s^*"] \arrow[d, "(-)\times s"'] \arrow[drr, "(-)\times_A S" near end] & & \mathcal{C}/S \arrow[d, "\mathrm{dom}_S"] \arrow[dll, "s_!"' near start] \\ \mathcal{C}/A \arrow[rr, "\mathrm{dom}_A"'] & & \mathcal{C} \end{tikzcd}$$

Here, s^* is pulling back along s; its (trivially existing) left adjoint $s_!$ is given by composition with s, so that $s_! s^* = (-)\times s$ becomes the product-with-s functor in the comma category $\mathcal{C}/A$; dom_S and dom_A denote the respective domain (or forgetful) functors to $\mathcal{C}$.

For $\mathcal{C}$ finitely complete, exponentiability of s is known to be equivalently described by the left adjointness of each of the three functors departing from the upper left-hand corner of the above diagram (see [Niefield, 1982]). For the sake of completeness and ease of subsequent reference, we sketch the main arguments of the proof.

2.1. Proposition. *In a finitely complete category* $\mathcal{C}$*, each of the following statements characterizes the morphism* $s : S \longrightarrow A$ *as exponentiable in* $\mathcal{C}$*:*

(*i*) $s^* : \mathcal{C}/A \longrightarrow \mathcal{C}/S$ *has a right adjoint;*

(*ii*) $(-)\times s : \mathcal{C}/A \longrightarrow \mathcal{C}/A$ *has a right adjoint;*

(*iii*) $(-)\times_A S : \mathcal{C}/A \longrightarrow \mathcal{C}$ *has a right adjoint.*

PROOF. $(i) \Rightarrow (ii)$: The composition of the adjunctions $s_! \dashv s^*$ and $s^* \dashv s_*$ gives the adjunction $(-) \times s = s_! s^* \dashv s^* s_*$. $(ii) \Rightarrow (iii)$: The functor dom_A has a trivial right adjoint which assigns to an object B the projection $B \times A \longrightarrow A$; its composition with $(-) \times s$ gives the functor $(-) \times_A S$. $(iii) \Rightarrow (i)$: Since dom_S (like dom_A) has a trivial right adjoint, and since $\mathrm{dom}_S s^* = (-) \times_A S$, the assertion of (i) follows from a Dubuc-style adjoint-triangle argument, as laid out in [Niefield, 1982], which we recall next. ■

2.2. LEMMA. *For a category $\mathcal{A}$ with equalizers, a functor $F : \mathcal{A} \longrightarrow \mathcal{C}/S$ has a right adjoint if the composite functor $\mathrm{dom}_S F : \mathcal{A} \longrightarrow \mathcal{C}$ has a right adjoint.*

PROOF. The to-be-constructed right adjoint $G : \mathcal{C}/S \longrightarrow \mathcal{A}$ of F assigns to every $\mathcal{C}$-morphism $u : U \longrightarrow S$, considered as an object of $\mathcal{C}/S$, the domain of the equalizer

$$G(u) \longrightarrow JU \underset{\sigma_{JU}}{\overset{Ju}{\rightrightarrows}} JS\ ;$$

here J is the given right adjoint of $\mathrm{dom}_S F$, and $\sigma : 1_{\mathcal{A}} \longrightarrow \triangle JS$ is the natural transformation corresponding to $\kappa F : \mathrm{dom}_S F \longrightarrow \triangle S$ by adjunction, with $\kappa : \mathrm{dom}_S \longrightarrow \triangle S$ the "structure transformation" given by $\kappa_u = u$. ■

2.3. REMARK. As observed in [Dyckhoff, Tholen, 1987], *for s exponentiable, the right adjoint s_* of s^* is fully faithful if, and only if, s is a monomorphism.* This is clear because, by general adjunction rules, the right adjoint s_* of s^* is fully faithful precisely when the (trivially existing) left adjoint $s_!$ of s^* is fully faithful, and it is elementary to show that that is the case precisely when s is monic.

The following consequence is again taken from [Niefield, 1982]:

2.4. COROLLARY. *The class of exponentiable morphisms in a finitely complete category contains all isomorphisms and is closed under composition and stable under pullback.*

PROOF. Exponentiability of isomorphisms holds trivially. Closure under composition follows from pullback pasting: $(r \cdot s)^* \simeq s^* r^*$. Finally, for a pullback diagram

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "t"'] & S \arrow[d, "s"] \\ B \arrow[r, "h"'] & A \end{tikzcd}$$

pullback pasting also shows that, up to isomorphism, $(-) \times_B U$ is the composite functor

$$\mathcal{C}/B \xrightarrow{h_!} \mathcal{C}/A \xrightarrow{(-)\times_A S} \mathcal{C}.$$

Hence, with Lemma 2.2, exponentiability of s implies exponentiability of t. ■

3. Lax pullback complements and partial products

Given a morphism $s : S \longrightarrow A$ in a category $\mathcal{C}$, such that all pullbacks along s exist in $\mathcal{C}$, we now embark on a diagrammatic description of co-universal arrows with respect to the functors s^* and $(-) \times_A S$, which leads us to generalizing the notion of pullback complement and revisiting the categorical notion of partial product of [Tholen, 1983].

3.1. Definition.

1. *For a morphism* $u : U \longrightarrow S$ *in* $\mathcal{C}$*, one calls a morphism* $p : P \longrightarrow A$ *in* $\mathcal{C}$ *a* lax pullback complement (lax pbc) of u along s, *writing* $p \cong s_*(u)$, *if there is a morphism* e *with* $s^*(p) \cong u \cdot e$*, so that one has a pullback diagram*

$$\begin{tikzcd} P \times_A S \arrow[r, "e"] \arrow[d, "s'"'] & U \arrow[r, "u"] & S \arrow[d, "s"] \\ P \arrow[rr, "p"'] & & A \end{tikzcd} , \tag{1}$$

   satisfying the following universal property: for any morphisms $q : Q \longrightarrow A$ *and* d *in* $\mathcal{C}$ *with* $s^*(q) \cong u \cdot d$*, so that one has a pullback diagram*

$$\begin{tikzcd} Q \times_A S \arrow[r, "d"] \arrow[d, "s''"'] & U \arrow[r, "u"] & S \arrow[d, "s"] \\ Q \arrow[rr, "q"'] & & A \end{tikzcd} , \tag{2}$$

   there is a unique morphism $h : Q \longrightarrow P$ *with* $p \cdot h = q$ *and* $e \cdot (h \times 1_S) = d$*. We also say that diagram* (1) *is a* lax pbc diagram *in this situation.*

2. *A lax pullback complement of* u *along* s *is a* pullback complement (pbc) of u along s*, if the morphism* e *in diagram* (1) *may be chosen to be an isomorphism; that is:* $p \cong s_*(u)$ *and* $s^*(p) \cong u$*. We also say that diagram* (1) *is a* pbc diagram *in this case.*

3. *For an object* Y *in* $\mathcal{C}$ *one calls* $(P,\ p : P \longrightarrow A)$*, or just* p*, a* partial product of Y over s*, if there is a morphism* $e : P \times_A S \longrightarrow Y$ *such that, for any morphisms* $q : Q \longrightarrow A$ *and* $d : Q \times_A S \longrightarrow Y$ *, there is a unique morphism* $h : Q \longrightarrow P$

rendering the diagram

$$
\begin{tikzcd}
 & P\times_A S \arrow[rr] \arrow[dl, "e"'] \arrow[dd, "s'"] & & S \arrow[dd, "s"] \\
Y & & Q\times_A S \arrow[ll, "d"'] \arrow[ul, dotted, "h\times 1_S"'] \arrow[ur] \arrow[dd] & \\
 & P \arrow[rr, "p"] & & A \\
 & & Q \arrow[ur, "q"'] \arrow[ul, dotted, "h"] &
\end{tikzcd}
\tag{3}
$$

commutative.

3.2. Remark.

(1) The morphisms p and e of diagram (1) are (up to isomorphism) uniquely determined by s and u. In fact, the properties stated say exactly that e is an s^*-couniversal arrow for $u \in \mathrm{ob}(\mathcal{C}/S)$:

$$
\begin{tikzcd}
p & s^*(p) \arrow[r, "e"] & u \\
q \arrow[u, dotted, "h"] & s^*(q) \arrow[u, "s^*(h)"] \arrow[ur, "d"'] \arrow[ur, phantom, "\circlearrowleft", pos=0.7, shift left=3]
\end{tikzcd}
$$

We use the notation $e \cong \varepsilon^s_u$, thus emphasizing the role of e as a counit when s^* has a right adjoint.

(2) A lax pullback complement along a *mono*morphism s is, in fact, a pullback complement, as we show in Proposition 3.4 below. The condition that s be monic becomes necessary for a lax pbc along s to be a pbc along s if sufficiently many lax pbcs exist; see Remark 2.3.

(3) The morphism $e : P \times_A S \longrightarrow Y$ of diagram (3) giving a partial product of Y over s is precisely a $((-) \times_A S)$-couniversal arrow for Y:

$$
\begin{tikzcd}
P & P\times_A S \arrow[r, "e"] & Y \\
Q \arrow[u, dotted, "h"] & Q\times_A S \arrow[u, "h\times 1_S"] \arrow[ur, "d"'] \arrow[ur, phantom, "\circlearrowleft", pos=0.7, shift left=3]
\end{tikzcd}
$$

In conjunction with Proposition 2.1, Remark 3.2 extends the characterization of exponentiability in terms (lax) pullback complements and partial products given in [Dyckhoff, Tholen, 1987], as follows:

3.3. Corollary. *In a finitely complete category $\mathcal{C}$, the following statements for a morphism $s : S \longrightarrow A$ are equivalent:*

(*i*) *s is exponentiable;*

(*ii*) *the lax pullback complement of u along s exists for every morphism u precomposable with s;*

(*iii*) *the partial product of Y over s exists for every object Y.*

Furthermore, lax pullback complements along the exponentiable morphism s are pullback complements if, and only if, s is a monomorphism.

We now turn to the "pointwise" version of this Corollary, investigating the relationship between individually given lax pbcs and partial products, as well as the question of conditions that can produce pbcs from lax pbcs. With respect to the last question, making good on Remark 3.2(2), we can state more comprehensively:

3.4. Proposition. *In a category with pullbacks along s, if p is a lax pullback complement of u along s, such that at least one of s, u or ε^s_u is monic, then p is a pullback complement of $u \cdot \varepsilon^s_u$ along s; that is: if $p \cong s_*(u)$ with s, u or ε^s_u monic, then $p \cong s_*(u \cdot \varepsilon^s_u)$. When s is monic, ε^s_u is actually an isomorphism, making the lax pbc diagram (1) a pbc diagram.*

Proof. In order to check the required universal property for proving $p \cong s_*(u \cdot\ e)$, given that $p \cong s_*(u)$ and $e \cong \varepsilon^s_u$, we consider the diagram

$$\begin{tikzcd}
Q \times_A S \arrow[r, "g"] \arrow[d, "s''"'] & P \times_A S \arrow[r, "u\cdot e"] \arrow[d, "s'"'] & S \arrow[d, "s"] \\
Q \arrow[rr, bend right, "q"'] & P \arrow[r, "p"'] & A
\end{tikzcd}$$

whose outer part is a pullback diagram, and obtain a unique morphism $h : Q \longrightarrow P$ with $p \cdot h = q$ and $e \cdot (h \times 1_S) = e \cdot g$. We must now show $h \times 1_S = g$ or, equivalently, $h \cdot s'' = s' \cdot g$. When e is monic, this is immediate. In the case that u is monic, we first consider the following diagram on the left, with all new squares being pullbacks (of pullbacks of s):

$$\begin{tikzcd}
\cdot \arrow[r, "g'"] \arrow[d, "\bar{s}"'] & \cdot \arrow[r, "t"] \arrow[d, "r"'] & P \times_A S \arrow[r, "u\cdot e"] \arrow[d, "s'"'] & S \arrow[d, "s"] \\
Q \times_A S \arrow[r, "g"'] \arrow[rrr, bend right, "q\cdot s''"'] & P \times_A S \arrow[r, "s'"'] & P \arrow[r, "p"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "g''"] \arrow[d, "\bar{s}"'] & \cdot \arrow[r, "h\times 1_S"] \arrow[d, "s''"'] & P \times_A S \arrow[r, "u\cdot e"] \arrow[d, "s'"'] & S \arrow[d, "s"] \\
Q \times_A S \arrow[r, "s''"'] \arrow[rrr, bend right, "q\cdot s''"'] & Q \arrow[r, "h"'] & P \arrow[r, "p"'] & A
\end{tikzcd}$$

Since

$$\begin{aligned} s\cdot u\cdot e\cdot t\cdot g' &= p\cdot s'\cdot g\cdot \bar{s} = s\cdot u\cdot e\cdot g\cdot \bar{s} = s\cdot u\cdot e\cdot (h\times 1_S)\cdot \bar{s} \\ &= p\cdot s'\cdot (h\times 1_S)\cdot \bar{s} = p\cdot h\cdot s''\cdot \bar{s} = q\cdot s''\cdot \bar{s}, \end{aligned}$$

there is a unique morphism g'' making the diagram on the right commute and its upper horizontal composite arrow coincide with the corresponding composite arrow of the left diagram. Now this last property implies $e\cdot(h\times 1_S)\cdot g'' = e\cdot t\cdot g'$ since u is monic, which enables us to conclude the desired equality $h\cdot s'' = s'\cdot g$, thanks to the universal property of diagram (1):

$$\begin{tikzcd}
{} & P\times_A S \arrow[r, "e"] \arrow[dd, "s'"'] & U \arrow[r, "u"] & S \arrow[dd, "s"] \\
{} \arrow[ur, shift left, "(h\times 1_S)\cdot g''"] \arrow[ur, shift right, "t\cdot g'"'] \arrow[urr, "e\cdot(h\times 1_S)\cdot g''"'] \arrow[dd, "\bar{s}"'] & & & \\
 & P \arrow[rr, "p"] & & A \\
{} \arrow[ur, shift left, "h\cdot s''"] \arrow[ur, shift right, "s'\cdot g"'] \arrow[urrr, "q\cdot s''"'] & & &
\end{tikzcd}$$

Finally, in the case that s is a monomorphism, exploiting the universal property of diagram (1) with the pullback diagram

$$\begin{tikzcd}
U \arrow[r, "u"] \arrow[d, "1_U"'] & S \arrow[d, "s"] \\
U \arrow[r, "s\cdot u"'] & A
\end{tikzcd}$$

one routinely shows that e must be an isomorphism. ■

For a "pointwise" version of the equivalence $(ii) \Leftrightarrow (iii)$ of Corollary 3.3, we now clarify how individual lax pbcs are being constructed from specific partial products, and conversely.

3.5. Proposition. *Let $s : S \longrightarrow A$ be a morphism in the category $\mathcal{C}$ that has all pullbacks along s.*

(1) *For an object Y such that the product $Y\times S$ exists, a partial product of Y over s exists if, and only if, a lax pullback complement of p_2 along s exists, with $p_2 : Y\times S \longrightarrow S$ the second product projection.*

(2) *For a morphism $u : U \longrightarrow S$, a lax pullback complement of u along s may be constructed from partial products of U and of S over s, provided that $\mathcal{C}$ has equalizers.*

PROOF. (1) is easy (and well known, [Tholen, 1983]): with the trivial observation that arrows $P \times_A S \longrightarrow Y \times S$ correspond to pairs of arrows $P \times_A S \longrightarrow Y$, $P \times_A S \longrightarrow S$, of which the latter factors as

$$P \times_A S \longrightarrow Y \times S \xrightarrow{p_2} S ,$$

one sees immediately that this composite serves as the upper arrow in diagram (1) if, and only if, the pair of arrows belongs to diagram (3), with each satisfying the respective universal properties.

The proof of (2) requires a "pointwise" version of Lemma 2.2, which we provide next; in there, one simply considers $\mathcal{A} = \mathcal{C}/A$ and $F = s^*$ to obtain the assertion of (2). ■

3.6. LEMMA. *Let $\mathcal{A}$ be a category with equalizers and $F : \mathcal{A} \longrightarrow \mathcal{C}/S$ be a functor. For the morphism $u : U \longrightarrow S$ in $\mathcal{C}$, with given* $\mathrm{dom}_S F$*-couniversal arrows for U and S in $\mathcal{C}$, there exists an F-couniversal arrow for u when u is considered as an object in $\mathcal{C}/S$.*

PROOF. In what follows, for a morphism $g : u \longrightarrow v$ in $\mathcal{C}/S$, we will denote its dom_S-image again by g and write $g : \mathrm{dom}_S u \longrightarrow \mathrm{dom}_S v$ in $\mathcal{C}$. Now, let $\alpha_U : \mathrm{dom}_S FC \longrightarrow U$ and $\alpha_S : \mathrm{dom}_S FD \longrightarrow S$ be the given $\mathrm{dom}_S F$-couniversal arrows for U and S, respectively, with objects C, D in $\mathcal{A}$. Then there is a unique morphism $f : C \longrightarrow D$ in $\mathcal{A}$ such that $\alpha_S \cdot Ff = u \cdot \alpha_U$. There is also a unique morphism $\sigma_C : C \longrightarrow D$ such that $\alpha_S \cdot F\sigma_C = \kappa_{FC}$, with $\kappa : \mathrm{dom_S} \longrightarrow \triangle S$ denoting the "structure transformation" already mentioned in the proof of Lemma 2.2. Now form the equalizer diagram

$$E \xrightarrow{e} C \underset{\sigma_C}{\overset{f}{\rightrightarrows}} D$$

in $\mathcal{A}$ and consider the composite morphism

$$\mathrm{dom}_S FE \xrightarrow{Fe} \mathrm{dom}_S FC \xrightarrow{\alpha_U} U$$

in $\mathcal{C}$. By naturality of κ, one has

$$u \cdot \alpha_U \cdot Fe = \alpha_S \cdot Ff \cdot Fe = \alpha_S \cdot F\sigma_C \cdot Fe = \kappa_{FC} \cdot Fe = \kappa_{FE},$$

in $\mathcal{C}$, so that the composite $FE \xrightarrow{\alpha_U \cdot Fe} u$ is actually a morphism in $\mathcal{C}/S$; we show that it serves as an F-couniversal arrow for u.

Indeed, for a morphism $x : FA \longrightarrow u$ in $\mathcal{C}/S$ with A in $\mathcal{A}$, one has a unique morphism $y : A \longrightarrow C$ in $\mathcal{A}$ making

$$\begin{tikzcd}
\mathrm{dom}_S FC \arrow[r, "\alpha_U"] & U \\
\mathrm{dom}_S FA \arrow[u, "Fy"] \arrow[ur, "x"'] &
\end{tikzcd}$$

commute in $\mathcal{C}$. The naturality of κ implies

$$\alpha_S \cdot F(f \cdot y) = \alpha_S \cdot Ff \cdot Fy = u \cdot \alpha_U \cdot Fy = u \cdot x = \kappa_{FA}, \quad \text{and}$$

$$\alpha_S \cdot F(\sigma_C \cdot y) = \alpha_S \cdot F\sigma_C \cdot Fy = \kappa_{FC} \cdot Fy = \kappa_{FA},$$

which, by couniversality of α_S, implies $f \cdot y = \sigma_C \cdot y$. Now the equalizer gives us a unique morphism $z : A \longrightarrow E$ in $\mathcal{A}$ with $e \cdot z = y$, and we obtain the desired commutative diagram in $\mathcal{C}/S$:

$$\begin{tikzcd} FE \arrow[r, "\alpha_U \cdot Fe"] & u \\ FA \arrow[u, "Fz"] \arrow[ur, "x"'] & \end{tikzcd} \tag{4}$$

Finally, if $z' : A \longrightarrow E$ is another morphism making diagram (4) commute, then $\alpha_U \cdot F(e \cdot z') = x = \alpha_U \cdot Fy$, and so $e \cdot z' = y$. Therefore, $z' = z$. ∎

4. A first calculus of lax pullback complements

In this section we prove some stability and cancellation properties for lax pullback complement diagrams under vertical and horizontal pasting, and under pullback; these generalize known properties for pullback complement diagrams, as established in [Tholen, 1986]. Since adjoints compose, so that the class of exponentiable morphisms in a category $\mathcal{C}$ with sufficiently many pullbacks is closed under composition (Corollary 2.4), it is not surprising that any existing vertically pasted lax pbc diagrams give again a lax pbc diagram. But one also has a cancellation property; more precisely:

4.1. Proposition. (*Vertical Pasting*) *Consider the commutative diagram*

$$\begin{tikzcd} P \times_A T \arrow[r, "e'"] \arrow[d, "1_P \times t"'] \arrow[dr, phantom, "(c)"] & U \times_S T \arrow[r, "d"] \arrow[d, "t'"'] \arrow[drr, phantom, "(b)"] & V \arrow[r, "v"] & T \arrow[d, "t"] \\ P \times_A S \arrow[r, "e"] \arrow[d, "s'"'] \arrow[drrr, phantom, "(a)"] & U \arrow[rr, "u"] & & S \arrow[d, "s"] \\ P \arrow[rrr, "p"'] & & & A \end{tikzcd} \tag{5}$$

in which $(a), (b), (c)$ are all pullback diagrams. Let (d) denote the horizontal pasting of (b) with (c), and (e) its vertical pasting with (a), i.e., (e) denotes the outer rectangle of diagram (5). Then:

(1) *If (a) and (b) are lax pbc diagrams, then (e) is also a lax pbc diagram; that is: if $p \cong s_*(u)$ and $u \cong t_*(v)$, then $p \cong (s \cdot t)_*(v)$.*

(2) *If (a) and (e) are lax pbc diagrams, and if s monic, then (b) is also a lax pbc diagram; that is: $s_*(u) \cong (s \cdot t)_*(v)$ with u monic implies $u \cong t_*(v)$.*

PROOF. (1) As indicated above, this statement follows from the two-step procedure to finding a co-universal arrow with respect to the composite of two functors, namely s^* and t^*.

(2) Since s is monic, the morphism e of the lax pbc diagram (a) must be an isomorphism and therefore make the pullback (c) collapse. Consequently, (a) is actually a pbc diagram, and the statement to be proved represents only a slight generalization of the corresponding statement of [Tholen, 1986] where also (e) was assumed to be a pbc diagram. We can therefore leave the routine proof of the generalized statement to the reader. ∎

We now turn to the horizontal pasting of lax pbc diagrams and their cancellation properties.

4.2. PROPOSITION. (*Horizontal Pasting*) *Consider the diagram*

$$\begin{tikzcd}
Q \times_P (P \times_A S) \arrow[r, "d"] \arrow[ddd, "s''"'] \arrow[dddrr, phantom, "(b)"] & (P \times_A S) \times_U V \arrow[r, "e'"] \arrow[drr, "j'"'] \arrow[drrr, phantom, "(c)"] & V \arrow[drrr, "j"] & & & \\
& & & P \times_A S \arrow[r, "e"] \arrow[dd, "s'"] \arrow[ddrr, phantom, "(a)"] & U \arrow[r, "u"] & S \arrow[dd, "s"] \\
& & & & & \\
Q \arrow[rrr, "q"'] & & & P \arrow[rr, "p"'] & & A
\end{tikzcd} \tag{6}$$

formed by the pullback diagrams $(a),(b),(c)$, and let (d) be their pasting, i.e., the pullback diagram formed by the peripheral arrows of diagram (6).

(1) *If (a) and (b) are lax pbc diagrams, then (d) is also a lax pbc diagram; that is: $p \cong s_*(u)$ and $q \cong s'_*(j')$ implies $p \cdot q \cong s_*(u \cdot j)$.*

(2) *Conversely, if (d) is a lax pbc diagram, either with p monic, or with e' monic and (a) a lax pbc diagram, then (b) is also a lax pbc diagram.*

PROOF. We just prove (1); for (2) one proceeds similarly. Let

$$\begin{tikzcd}
\tilde{P} \times_A S \arrow[r, "g"] \arrow[d, "\tilde{s}"'] & V \arrow[r, "u \cdot j"] & S \arrow[d, "s"] \\
\tilde{P} \arrow[rr, "\tilde{p}"'] & & A
\end{tikzcd}$$

be a pullback diagram. The lax pbc diagram (a) gives a unique morphism h such that

$$\begin{tikzcd}
& P \times_A S \arrow[r, "e"] \arrow[dd, "s'"'] & U \arrow[r, "u"] & S \arrow[dd, "s"] \\
\tilde{P} \times_A S \arrow[ur, dotted, "h \times 1_S"] \arrow[urr, "j \cdot g"] \arrow[dd, "\tilde{s}"'] & & & \\
& P \arrow[rr, "p"] & & A \\
\tilde{P} \arrow[ur, dotted, "h"] \arrow[urrr, "\tilde{p}"'] & & &
\end{tikzcd}$$

commutes. With i the morphism rendering

$$\begin{tikzcd}
\tilde{P} \times_A S \arrow[dr, dotted, "i"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h \times_A 1_S"'] & & \\
& (P \times_A S) \times_U V \arrow[r, "e'"] \arrow[d, "j'"'] & V \arrow[d, "j"] \\
& P \times_A S \arrow[r, "e"'] & U
\end{tikzcd}$$

commutative, the lax pbc diagram (b) gives a unique morphism $\bar{h}$ making

$$\begin{tikzcd}
& Q \times_P (P \times_A S) \arrow[r, "d"] \arrow[dd, "s''"'] & (P \times_A S) \times_U V \arrow[r, "j'"] & P \times_A S \arrow[dd, "s'"] \\
\tilde{P} \times_A S \arrow[ur, dotted, "\bar{h} \times 1_S"] \arrow[urr, "i"'] \arrow[dd, "\tilde{s}"'] & & & \\
& Q \arrow[rr, "q"] & & P \\
\tilde{P} \arrow[ur, dotted, "\bar{h}"] \arrow[urrr, "h"'] & & &
\end{tikzcd} \tag{7}$$

commute. So, we have $p \cdot q \cdot \bar{h} = p \cdot h = \tilde{p}$ and $e' \cdot d \cdot (\bar{h} \times 1_S) = e' \cdot i = g$, and it is now a routine exercise to check that $\bar{h}$ is uniquely determined by these two equations. ∎

Finally, let us show that pullbacks of lax pbc diagrams are lax pbc diagrams, a property that must not be confused with the pullback stability of the class of exponentiable morphisms, which we consider in generalized form in Section 5 below.

4.3. Proposition. (*Pulling Back*) *Every pullback of a lax pullback complement diagram is a lax pullback complement diagram; that is: if the back face of the commutative diagram* (8) *is a lax pbc diagram, with the right, bottom and both top faces being pullbacks, then* *its front face is also a lax pbc diagram.*

$$\begin{tikzcd}
& E \arrow[rr, "e"] \arrow[dd, "s'"' near start] & & U \arrow[rr, "u"] & & S \arrow[dd, "s"] \\
E' \arrow[ur, "l''"] \arrow[rr, "e'"] \arrow[dd, "t'"'] & & U' \arrow[ur, "l'"] \arrow[rr, "u'"] & & T \arrow[ur, "l"] \arrow[dd, "t" near start] & \\
& P \arrow[rrrr, "p"] & & & & A \\
P' \arrow[ur, "h'"] \arrow[rrrr, "p'"'] & & & & B \arrow[ur, "h"'] &
\end{tikzcd} \tag{8}$$

PROOF. By hypothesis, one has $E \cong P \times_A S$, and the standard pasting and cancellation properties of pullback diagrams make also the front face a pullback diagram, so that $E' \cong P' \times_B T \cong P' \times_A S$. Now let

$$\begin{tikzcd}
Q' \arrow[r, "g"] \arrow[d, "\bar{t}"'] & U' \arrow[r, "u'"] & T \arrow[d, "t"] \\
Q \arrow[rr, "q"'] & & B
\end{tikzcd}$$

be a pullback diagram. Then $Q' \cong Q \times_A S$, and the back face of (8) being a lax pbc diagram gives a unique morphism r making the following diagram commute:

$$\begin{tikzcd}
& P \times_A S \arrow[rr, "e"] \arrow[dd, "s'"'] & & U \arrow[r, "u"] & S \arrow[dd, "s"] \\
Q \times_A S \arrow[ur, dotted, "\bar{r}=r\times 1_S"] \arrow[urrr, "l'\cdot g"] \arrow[dd, "\bar{t}"'] & & & & \\
& P \arrow[rrr, "p"] & & & A \\
Q \arrow[ur, dotted, "r"] \arrow[urrrr, "hq"'] & & & &
\end{tikzcd} \tag{9}$$

We then have the unique morphisms j and $\bar{j}$ rendering commutative the diagrams

$$\begin{tikzcd}
Q' \arrow[dr, dotted, "\bar{j}"] \arrow[drr, bend left, "\bar{r}"] \arrow[ddr, bend right, "g"'] & & \\
& E' \arrow[r, "l''"] \arrow[d, "e'"'] & E \arrow[d, "e"] \\
& U' \arrow[r, "l'"'] & U
\end{tikzcd}
\quad
\begin{tikzcd}
Q \arrow[dr, dotted, "j"] \arrow[drr, bend left, "r"] \arrow[ddr, bend right, "q"'] & & \\
& P' \arrow[r, "h'"] \arrow[d, "p'"'] & P \arrow[d, "p"] \\
& B \arrow[r, "h"'] & A
\end{tikzcd}
\quad
\begin{tikzcd}
Q' \arrow[dr, shift left, "j\bar{t}"] \arrow[dr, shift right, "t'\bar{j}"'] \arrow[drr, bend left, "r\bar{t}"] \arrow[ddr, bend right, "q\bar{t}"'] & & \\
& P' \arrow[r, "h'"] \arrow[d, "p'"'] & P \arrow[d, "p"] \\
& B \arrow[r, "h"'] & A
\end{tikzcd} \tag{10}$$

Consequently, also the diagram

$$\begin{tikzcd}
 & P' \times_B T \arrow[r, "e'"] \arrow[dd, "t'"'] & U' \arrow[r, "u'"] & T \arrow[dd, "t"] \\
Q \times_B T \arrow[ur, "\bar{j}=j\times 1_T"] \arrow[urr, "g"'] \arrow[dd, "\bar{t}"'] & & & \\
 & P' \arrow[rr, "p'"] & & B \\
Q \arrow[ur, "j"] \arrow[urrr, "q"'] & & &
\end{tikzcd} \tag{11}$$

commutes, and it is easy to verify that j is the only morphism rendering it commutative. ∎

4.4. Corollary. (*Pulling Back Partial Products*) *Let the category* $\mathcal{C}$ *have pullbacks along* $h : B \longrightarrow A$ *, as well as the product* $Y \times S$*. Then, if* (P, p) *is a partial product of* Y *over* $s : S \longrightarrow A$ *, then* $(P \times_A B, h^*(p))$ *is a partial product of* Y *over* $h^*(s) : B \times_A S \longrightarrow B$ *.*

Proof. According to Proposition 3.5(1), in diagram (8) we consider for u the projection $Y \times S \longrightarrow S$. Then u' is the projection $Y \times T \longrightarrow T$, with the existence of the product guaranteed as a pullback of $Y \times S$ along h, and Proposition 4.3 applies. ∎

4.5. Corollary. *Consider the commutative diagram*

$$\begin{tikzcd}
P \times_A S \arrow[r, "e"] \arrow[d, "t'"'] & V \arrow[r, "j"] \arrow[d, phantom, "(c)"] & U \arrow[r, "u"] \arrow[d, "t"'] \arrow[dr, phantom, "(a)"] & S \arrow[dd, "s"] \\
Q \arrow[rr, "q"] \arrow[d, "h'"'] & {} \arrow[d, phantom, "(b)"] & B \arrow[dr, "h"] & \\
P \arrow[rrr, "p"'] & {} & & A
\end{tikzcd} \tag{12}$$

in which (a) *and* (b) *are pullback diagrams, and let* (d) *denote the outer pullback diagram of* (12)*. Then, if* (d) *is a lax pbc diagram and* u *is monic, then* (c) *is also a lax pbc diagram, that is: if* $p \cong s_*(u \cdot j)$ *with* u *monic, then* $q \cong t_*(j)$*.*

Proof. One can re-draw Diagram (12) as

$$\begin{tikzcd}
& P \times_A S \arrow[rr, "e"] \arrow[dd, "h' \cdot t'"] & & V \arrow[rr, "u \cdot j"] & & S \arrow[dd, "s"] \\
P \times_A S \arrow[ur, "1_{P \times_A S}"] \arrow[rr, "e"] \arrow[dd, "t'"'] & & V \arrow[ur, "1_V"] \arrow[rr, "j"] & & U \arrow[ur, "u"'] \arrow[dd, "t"] & \\
& P \arrow[rrrr, "p"] & & & & A \\
Q \arrow[ur, "h'"] \arrow[rrrr, "q"'] & & & & B \arrow[ur, "h"'] &
\end{tikzcd}$$

Now Proposition 4.3 implies that, under the given hypotheses, (c) is a lax pbc diagram. ■

5. Pullback stability

Throughout this section, let

$$\begin{tikzcd}
U \arrow[r, "u"] \arrow[d, "t"'] & S \arrow[d, "s"] \\
B \arrow[r, "h"'] & A
\end{tikzcd} \tag{13}$$

be a pullback diagram in $\mathcal{C}$. For any morphism $j : V \longrightarrow U$, we would like to construct a lax pbc of j along t, given the existence of *certain* lax pbcs along s. In Proposition 4.3 we did so in the particular case that j is obtained as a pullback of a morphism whose lax pbc along s exists. Before addressing this question in full generality, we first note that, when u is monic, Proposition 4.3 actually covers the general case, as follows.

5.1. Proposition. *If, in the pullback diagram (13), u is monic, then a lax pbc of j along t is obtained by pulling back along h the given lax pbc of $u \cdot j$ along s; that is: $t_*(j) \cong h^*(s_*(u \cdot j))$.*

Proof. Since u is monic, any morphism j is a pullback of $u \cdot j$ along u. Proposition 4.3 may therefore be applied to the diagram

$$\begin{tikzcd}
& P \times_A S \arrow[rr, "e"] \arrow[dd, "s'"] & & V \arrow[rr, "u \cdot j"] & & S \arrow[dd, "s"] \\
P \times_A S \arrow[ur, "1_{P \times_A S}"] \arrow[rr, "e"] \arrow[dd, "t'"'] & & V \arrow[ur, "1_V"] \arrow[rr, "j"] & & U \arrow[ur, "u"] \arrow[dd, "t"] & \\
& P \arrow[rrrr, "p"] & & & & A \\
P \times_A B \arrow[ur, "h'"] \arrow[rrrr, "p'"'] & & & & B \arrow[ur, "h"'] &
\end{tikzcd}$$

■

We now turn to the question which lax pbcs along s are needed for us to be able to construct a lax pbc of a morphism $j : V \longrightarrow U$ along t, without assuming u to be monic? Analyzing the arguments proving Corollary 2.4, we can prove the following "pointwise" version of the pullback stability of exponentiable morphisms, keeping in mind the diagram

$$\begin{tikzcd}
\mathcal{C}/B \arrow[r, "h_!"] \arrow[d, "t^*"'] \arrow[dr, phantom, "\cong"] & \mathcal{C}/A \arrow[d, "s^*"] \\
\mathcal{C}/U \arrow[r, "u_!"'] \arrow[dr, "\mathrm{dom}_U"'] & \mathcal{C}/S \arrow[d, "\mathrm{dom}_S"] \\
& \mathcal{C}
\end{tikzcd}$$

5.2. Theorem. *For the pullback diagram (13) in the finitely complete category $\mathcal{C}$, the lax pullback complement of a morphism $j : V \longrightarrow U$ along t exists if the partial products p and q of, respectively, U and V over s exist in $\mathcal{C}$, and it may then be presented as $t_*(j) \cong h^*(q) \cdot e$, for some regular monomorphism e.*

Proof. According to Proposition 3.5(2), one starts by constructing the partial products of U and V over t which, by Corollary 4.4, may be obtained by pulling back along h the given partial products $(P, p = s_*(p_2))$ and $(Q, q = s_*(q_2))$ of, respectively, U and V over s. In the largely self-explanatory diagram

$$\begin{tikzcd}
& & & & & V \times S \arrow[r, phantom, ""] \arrow[dr, "j\times 1_S"] \arrow[drr, "q_2"] \arrow[dd, "q_1"] & & \\
& & Q \times_A S \arrow[urrr, "\varepsilon^s_{q_2}"] \arrow[rr, "\bar{j}\times 1_S"'] \arrow[ddd, "s''"'] & & P \times_A S \arrow[rr, "\varepsilon^s_{p_2}"] \arrow[ddd, "s'"'] & & U \times S \arrow[r, "p_2"'] \arrow[dd, "p_1"] & S \arrow[ddd, "s"] \\
& & & & & V \arrow[dr, "j"] & & \\
E \times_A U \arrow[r, "e\times 1_U"] \arrow[dd, "t'''"'] & Q \times_A U \arrow[uur, "u''"] \arrow[urrrr, "q_1\cdot\varepsilon^s_{q_2}\cdot u''"] \arrow[rr, shift left, "\bar{j}\times 1_U"] \arrow[rr, shift right, "\sigma\times 1_U"'] \arrow[dd, "t''"'] & & P \times_A U \arrow[uur, "u'"] \arrow[rrr, "t^*(h^*(p))"] \arrow[dd, "t'"'] & & & U \arrow[uur, "u"'] \arrow[dd, "t"'] & \\
& & Q \arrow[rr, "\bar{j}"] & & P \arrow[rrr, "p"] & & & A \\
E \arrow[r, "e"] & Q \times_A B \arrow[ur, "h''"'] \arrow[rr, shift left, "\bar{j}\times 1_B"] \arrow[rr, shift right, "\sigma"'] & & P \times_A B \arrow[ur, "h'"'] \arrow[rrr, "h^*(p)"] & & & B \arrow[ur, "h"'] &
\end{tikzcd}$$

with product projections p_1, p_2, q_1, q_2, the morphisms s', s'', t', t'', t''' and h', h'' are all iteratively obtained as pullbacks of s, t and h, respectively, and with the morphisms u', u'' then induced by the pullback property. Note that, while generally $u \cdot p_1 \neq p_2$, all other parts of the diagram commute to the extent one expects them to do when forming equalizers, with the morphisms $\bar{j}, \sigma$ and e defined as prescribed by the proof of Lemma 3.6. Specifically, the morphism $\bar{j} : Q \longrightarrow P$ is induced by the $\mathcal{C}/S$-morphism $j \times 1_S : q_2 \longrightarrow p_2$, satisfying

$$p \cdot \bar{j} = q, \qquad \varepsilon^s_{p_2} \cdot (\bar{j} \times 1_S) = (j \times 1_S) \cdot \varepsilon^s_{q_2}.$$

The morphism $\sigma : Q \times_A B \longrightarrow P \times_A B$ satisfies

$$h^*(p) \cdot \sigma = h^*(q), \qquad t^*(h^*(p)) \cdot (\sigma \times 1_U) = t^*(h^*(q)),$$

and e is the equalizer of $\bar{j} \times 1_B$ and σ. Now Lemma 3.6 confirms that the morphism

$$h^*(q) \cdot e = h^*(p) \cdot \sigma \cdot e : E \longrightarrow B$$

serves as a lax pbc of j along t, showing $t_*(j) \cong h^*(q) \cdot e$, as in the lax pbc diagram

$$\begin{tikzcd}
E \times_B U \arrow[r, "q_1 \cdot \varepsilon^s_{q_2} \cdot u'' \cdot (e \times 1_U)"] \arrow[d, "t'''"'] & V \arrow[r, "j"] & U \arrow[d, "t"] \\
E \arrow[rr, "h^*(q) \cdot e"'] & & B\,.
\end{tikzcd}$$

■

6. Pullbacks of total morphisms in the "strict" span category

For a category $\mathcal{C}$ with pullbacks, we consider the (ordinary) "strict" span category $\mathrm{Span}(\mathcal{C})$, having the same objects as $\mathcal{C}$; morphisms $[g, f] : A \longrightarrow B$ are isomorphism classes of pairs of $\mathcal{C}$-morphisms with common domain, with $(g, f) \cong (g', f')$ referring to the existence of an isomorphism i with $g = g' \cdot i$ and $f = f' \cdot i$. The composite of the morphisms $A \xrightarrow{[g,f]} B \xrightarrow{[k,h]} C$ in $\mathrm{Span}(\mathcal{C})$ is, of course, given by pullback:

$$[k, h] \cdot [g, f] = [g \cdot f^*(k), h \cdot k^*(f)].$$

We say that the morphism $[g, f]$ is *total* if g is an isomorphism in $\mathcal{C}$; without loss of generality, one may then assume $g = 1$. We show that the existence of their pullbacks in $\mathrm{Span}(\mathcal{C})$ depends on the existence of certain lax pullback complements in $\mathcal{C}$, as follows.

6.1. Theorem. *The pullback of a total morphism $[1_C, t] : C \longrightarrow B$ along a morphism $[g, f] : A \longrightarrow B$ exists in* $\mathrm{Span}(\mathcal{C})$ *if, and only if, the lax pullback complement $p \cong g_*(f^*(t)) : P \longrightarrow A$ of $f^*(t)$ along g exists in $\mathcal{C}$, and in that case the* $\mathrm{Span}(\mathcal{C})$-*pullback is described by*

$$[g, f]^*([1_C, t]) \cong [1_P, p] \text{ and } [1_C, t]^*([g, f]) \cong [p^*(g), t^*(f) \cdot \varepsilon_g^{t^*(f)}].$$

PROOF. For the "if"-part, we can assume $p \cong g_*(t')$, with $t' \cong f^*(t)$, as in the diagram

$$\begin{tikzcd}
P \arrow[dd, "p"'] & P \times_A D \arrow[l, "g'"'] \arrow[d, "e=\varepsilon_g^{t'}"] & \\
 & D \times_B C \arrow[l, phantom, "lax\,pbc"] \arrow[r, "f'"] \arrow[d, "t'"'] \arrow[dr, phantom, "pb"] & C \arrow[d, "t"] \\
A & D \arrow[l, "g"] \arrow[r, "f"'] & B
\end{tikzcd}$$

We will show that this diagram actually represents a pullback diagram in Span($\mathcal{C}$); that is: $[g,f]^*([1_C,t]) \cong [1_P,p]$. First, with the pullback diagrams

$$\begin{tikzcd}
P \times_A D \arrow[r, "t' \cdot e"] \arrow[d, "g'"'] \arrow[dr, phantom, "pb"] & D \arrow[r, "f"] \arrow[d, "g"] & B \\
P \arrow[r, "p"'] \arrow[d, "1_P"'] & A & \\
P & &
\end{tikzcd}
\qquad
\begin{tikzcd}
P \times_A D \arrow[r, "f' \cdot e"] \arrow[d, "1_{P \times_A D}"'] \arrow[dr, phantom, "pb"] & C \arrow[r, "t"] \arrow[d, "1_C"] & B \\
P \times_A D \arrow[r, "f' \cdot e"'] \arrow[d, "g'"'] & C & \\
P & &
\end{tikzcd}$$

and $t \cdot f' \cdot e = f \cdot t' \cdot e$, we obtain $[g,f] \cdot [1_P,p] = [1_C,t] \cdot [g', f' \cdot e]$. Next, let $[r,s]$ and $[u,v]$ be any morphisms in Span($\mathcal{C}$) with $[1_C,t] \cdot [u,v] = [g,f] \cdot [r,s]$, so that we have the pullback diagrams

$$\begin{tikzcd}
X \times_A D \arrow[r, "g^*(s)"] \arrow[d, "s^*(g)"'] \arrow[dr, phantom, "pb"] & D \arrow[r, "f"] \arrow[d, "g"] & B \\
X \arrow[r, "s"'] \arrow[d, "r"'] & A & \\
Z & &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "1_Y"'] \arrow[dr, phantom, "pb"] & C \arrow[r, "t"] \arrow[d, "1_C"] & B \\
Y \arrow[r, "v"'] \arrow[d, "u"'] & C & \\
Z & &
\end{tikzcd}$$

where, without loss of generality, we may assume $Y = X \times_A D$, $t \cdot v = f \cdot g^*(s)$ and $r \cdot s^*(g) = u$. Let i be the unique morphism making

$$\begin{tikzcd}
Y \arrow[dr, dotted, "i"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "g^*(s)"'] & & \\
 & D \times_B C \arrow[r, "f'"] \arrow[d, "t'"'] & C \arrow[d, "t"] \\
 & D \arrow[r, "f"'] & B
\end{tikzcd}$$

commutative. The lax pbc now produces a unique morphism h rendering the diagram

$$\begin{tikzcd}
 & P\times_A D \arrow[r, "e"] \arrow[dd, "g'"'] & D\times_B C \arrow[r, "t'"] & D \arrow[dd, "g"] \\
X\times_A D \arrow[ur, "h\times 1_D"] \arrow[urr, "i"] \arrow[dd, "s^*(g)"'] & & & \\
 & P \arrow[rr, "p"] & & A \\
X \arrow[ur, "h"] \arrow[urrr, "s"'] & & &
\end{tikzcd}$$

commutative. We have therefore established the commutative Span($\mathcal{C}$)-diagram

$$\begin{tikzcd}
Z \arrow[dr, "{[r,h]}"] \arrow[drr, bend left, "{[u,v]}"] \arrow[ddr, bend right, "{[r,s]}"'] & & \\
 & P \arrow[r, "{[g',f'\cdot e]}"] \arrow[d, "{[1_P,p]}"] & C \arrow[d, "{[1_C,t]}"] \\
 & A \arrow[r, "{[g,f]}"'] & B
\end{tikzcd} \tag{14}$$

Finally, suppose that $[l,k] : Z \longrightarrow P$ is another Span($\mathcal{C}$)-morphism rendering diagram (14) commutative in lieu of $[r,h]$. We may then assume $p \cdot k = s$ and $l = r$, so that only $k = h$ remains to be shown, and for that it suffices to confirm the equality $e \cdot (k \times 1_D) = i$. But considering the diagram

$$\begin{tikzcd}
X\times_A D \arrow[r, dotted, "k\times 1_D"] \arrow[rr, bend left, "g^*(s)"] \arrow[d, "s^*(g)"'] & P\times_A D \arrow[r, "t'\cdot e"] \arrow[d, "g'"'] & D \arrow[d, "g"] \\
X \arrow[r, "k"'] \arrow[rr, bend right, "s"'] & P \arrow[r, "p"'] & A
\end{tikzcd}$$

we first derive $t' \cdot e \cdot (k \times 1_D) = g^*(s)$, and from $[g', f' \cdot e] \cdot [r,k] = [u,v]$ we obtain $f' \cdot e \cdot (k \times 1_D) = v$. Hence, $e \cdot (k \times 1_D)$ satisfies the defining equalities of i, so that these morphisms must indeed coincide.

For the "only if" part of the assertion of the Theorem, we suppose to be given the pullback diagram

$$\begin{tikzcd}
P \arrow[r, "{[g',f'']}"] \arrow[d, "{[1_P,p]}"'] & C \arrow[d, "{[1_C,t]}"] \\
A \arrow[r, "{[g,f]}"'] & B
\end{tikzcd}$$

in $\mathrm{Span}(\mathcal{C})$ where, by Remark 6.4(1) below, we may indeed assume that the pullback of $[1,t]$ along $[g,f]$ is total. The commutativity of the diagram implies $f\cdot g^*(p) = t\cdot f''$. Hence, the pullback diagram

$$\begin{tikzcd} D\times_B C \arrow[r, "f'"] \arrow[d, "t'"'] & C \arrow[d, "t"] \\ D \arrow[r, "f"'] & B \end{tikzcd}$$

in $\mathcal{C}$ gives us a unique morphism $e : P\times_A D \longrightarrow D\times_B C$ such that $f'\cdot e = f''$ and $t'\cdot e = g^*(p)$. Now, the confirmation that

$$\begin{tikzcd} P\times_A D \arrow[r, "e"] \arrow[d, "g'"'] & D\times_B C \arrow[r, "t'"] & D \arrow[d, "g"] \\ P \arrow[rr, "p"'] & & A \end{tikzcd}$$

is a lax pbc diagram in $\mathcal{C}$ can be safely left to the reader. ∎

We formulate some immediate consequences of Theorem 6.1.

6.2. Corollary. *In the diagram*

$$\begin{tikzcd} E \arrow[d, "\bar t"'] \arrow[dr, phantom, "pbc"] & D\times_B C \arrow[l, "\bar t^*(g)"'] \arrow[r, "t^*(f)"] \arrow[d, "f^*(t)"] \arrow[dr, phantom, "pb"] & C \arrow[d, "t"] \\ A & D \arrow[l, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

let $\bar t$ be a pullback complement of $f^*(t)$ along g in $\mathcal{C}$. Then

$$[1_C, t]^*([g,f]) \cong [\bar t^*(g), t^*(f)]$$

in $\mathrm{Span}(\mathcal{C})$.

Should $[g,f]$ be *cototal*, so that f is an isomorphism and, therefore, $[g,f]\cong[g,1_B]$, the above formula shows that also $[1_C,t]^*([g,1_B])$ must be cototal.

6.3. Corollary. *For every morphism $[g,f]$ in* $\mathrm{Span}(\mathcal{C})$ *with g exponentiable in $\mathcal{C}$, the pullback of any total morphism $[1_C,t]$ along $[g,f]$ exists in* $\mathrm{Span}(\mathcal{C})$ *and is total again. Should $[g,f]$ be cototal with g an exponentiable monomorphism, its pullback along $[1_C,t]$ exists and is cototal again.*

6.4. Remark.

(1) Unlike the stability statement for cototal morphisms, the assertion of the Corollary that the pullback of a total morphism in $\mathrm{Span}(\mathcal{C})$ (once its existence has been secured) is total again cannot surprise. In fact, since any morphism $[s,t] : E \longrightarrow B$ in $\mathrm{Span}(\mathcal{C})$ factors trivially as $[s,t] = [1_C, t] \cdot [s, 1_C]$, that is, as a cototal morphism followed by a total morphism, and since the orthogonality condition holds quite trivially as well, (cotal, total) is an orthogonal factorization system in $\mathrm{Span}(\mathcal{C})$. Therefore, the class of total morphisms must be stable under pullback (and, in fact, under all existing limits) in $\mathrm{Span}(\mathcal{C})$.

(2) With the (cototal, total)-factorization, in order to construct a pullback of a *partial morphism* $[s,t]$ (that is, of a span $[s,t]$ with s monic in $\mathcal{C}$) along $[g,h]$, after the construction of a pullback of $[1_C, t]$ along $[g,h]$ as in Theorem 6.1 it suffices to construct a pullback of $[s, 1_C]$ along

$$[1_C, t]^*([g,f]) \cong [p^*(g), t^*(f) \cdot \varepsilon_g^{t^*(f)}] =: [\check{g}, \check{f}] = [1_{\check{D}}, \check{f}] \cdot [\check{g}, 1_{\check{D}}],$$

where (in the notation of Theorem 6.1) $\check{D} = P \times_A D$. For that one may first apply Theorem 6.1 again and construct the pullback of the cototal morphism $[s, 1_C]$ along the total morphism $[1_{\check{D}}, \check{f}]$ and obtain the cototal morphism $[\check{s}, 1_{\check{D}}] := [1_{\check{D}}, \check{f}]^*([s, 1_C])$, and it is then clear by pullback composition and cancellation laws that the pullback $[g,f]^*([s,t])$ exists if, and only if, the pullback $[\check{g}, 1_{\check{D}}]^*([\check{s}, 1_{\check{D}}])$ exists in $\mathrm{Span}(\mathcal{C})$.

With Remark 6.4(2) we obtain:

6.5. Corollary. *For $\mathcal{C}$ finitely complete, the following are equivalent:*

(i) *for every morphism $[g,f]$ in* $\mathrm{Span}(\mathcal{C})$ *with g exponentiable in $\mathcal{C}$, the pullback of every partial morphism $[s,t]$ along $[g,f]$ with s exponentiable exists in* $\mathrm{Span}(\mathcal{C})$;

(ii) *for every cototal morphism $[g, 1_B]$ in* $\mathrm{Span}(\mathcal{C})$ *with g exponentiable in $\mathcal{C}$, the pullback of every cototal partial morphism $[s, 1_B]$ along $[g, 1_B]$ exists in* $\mathrm{Span}(\mathcal{C})$.

7. Weak pullbacks of cototal morphisms in the span category

Unfortunately, condition (ii) of Corollary 6.5 turns out to be non-trivial. In what follows, we succeed in constructing only *weak* pullbacks of cototal morphisms, and even for that we need additional provisions on the category, which we provide first. (As usual, a weak pullback satisfies the universal property of a pullback diagram only weakly, so that factoring arrows will always exist but may not be uniquely determined.)

Recall that a category $\mathcal{C}$ with pullbacks is *adhesive* (see [Lack, Sobociński, 2005]) if it has pushouts of monomorphisms and if such pushout diagrams are *van Kampen squares*; this means in particular that the pushout of a monomorphism is monic again, and that any pullback of such a pushout square is again a pushout square. Moreover, when forming the pushout of a monomorphism, the resulting square is both a pushout and a pullback diagram. In fact, a stronger result holds, as stated in [Shir Ali Nasab, Hosseini, 2015] with reference to [Shir Ali Nasab, 2009]; we supply a proof here:

7.1. PROPOSITION. *For a pushout square*

$$\begin{tikzcd} \cdot \arrow[r, "m"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "n"'] & \cdot \end{tikzcd} \tag{15}$$

in an adhesive category $\mathcal{C}$ with m monic, n is a pullback complement of m along g.

PROOF. Since $\mathcal{C}$ is adhesive, the pushout square is a pullback. Given any pullback square

$$\begin{tikzcd} \cdot \arrow[r, "k"] \arrow[d, "l"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "h"'] & \cdot \end{tikzcd}$$

with the same right arrow g, such that k factors as $k = m \cdot e$, one trivially has $m^*(k) \cong e$ and $k^*(m) \cong 1$, since m is monic. Furthermore, pulling back n along h gives a unique morphism l' with $h^*(n) \cdot l' = l$ and $n^*(h) \cdot l' = f \cdot e$. In this way we present the square

$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "l'"'] & \cdot \arrow[d, "l"] \\ \cdot \arrow[r, "h^*(n)"'] & \cdot \end{tikzcd} \tag{16}$$

as a pullback of (15) along h, and adhesiveness makes it a pushout square. Consequently, $h^*(n)$ is an isomorphism, and $x := n^*(h) \cdot (h^*(n))^{-1}$ provides the desired factorization $n \cdot x = h, x \cdot l = f \cdot e$. ∎

7.2. COROLLARY. *For a span $[g', f']$, assume that f' is the pullback of some morphism f along the monomorphism m in the adhesive category $\mathcal{C}$. Then there is a morphism g in $\mathcal{C}$ such that $[g', f']$ is the pullback of $[g, f]$ along the monomorphism $[1, m]$ in $\mathrm{Span}(\mathcal{C})$.*

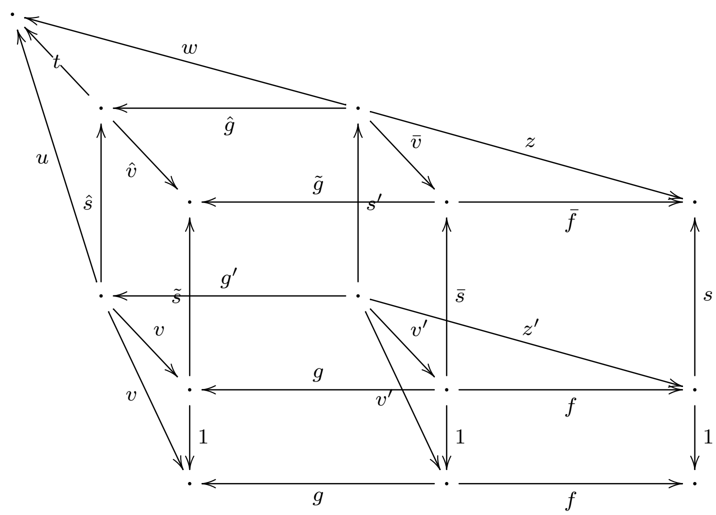
PROOF. With g the pushout of g' along $m' \cong f^*(m)$ in $\mathcal{C}$, we obtain the diagram

$$\begin{tikzcd} C \arrow[d, "n"'] & F \arrow[l, "g'"'] \arrow[r, "f'"] \arrow[d, "m'"] \arrow[dl, phantom, "po"] \arrow[dr, phantom, "pb"] & D \arrow[d, "m"] \\ A & E \arrow[l, "g"] \arrow[r, "f"'] & B, \end{tikzcd}$$

in which n is a pullback complement of m' along g, by Proposition 7.1. Now Corollary 6.2 implies that $[g', f'] \cong [1_C, m]^*([g, f])$. ∎

7.3. Theorem. *In an adhesive category $\mathcal{C}$, let the pullback complement of f along the monomorphism s exist. Then, for any morphism g with the same domain as f, a weak pullback of $[g, f]$ along $[s, 1]$ exists in* $\mathrm{Span}(\mathcal{C})$.

Proof. By hypothesis, we can form the pullback complement $\bar{f}$ of f along s and then the pushout $\tilde{s}$ of the monomorphism $\bar{s} := f^*(s)$ along g. Denoting by $\tilde{g}$ the pushout of g along $\bar{s}$, we obtain the front-face squares of the diagram



both of which are pullback diagrams, by adhesiveness of $\mathcal{C}$. Trivially then,

$$[g, f] \cdot [\tilde{s}, 1] = [s, 1] \cdot [\tilde{g}, \bar{f}],$$

and we claim that the corresponding square in $\mathrm{Span}(\mathcal{C})$ is a weak pullback diagram. Indeed, given $[u, v], [w, z]$ with $[g, f] \cdot [u, v] = [s, 1] \cdot [w, z]$, we obtain pullbacks $v' := g^*(v), g' := v^*(g)$ and $z' := s^*(z), s' := z^*(s)$ with the same domain, satisfying $u \cdot g' = w \cdot s'$ and $f \cdot v' = z'$. Therefore, the pbc $\bar{f}$ of f and the pullback diagram

$$\begin{tikzcd} \cdot \arrow[r, "z'"] \arrow[d, "s'"'] & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "z"'] & \cdot \end{tikzcd}$$

give the arrow $\bar{v}$ with $\bar{f} \cdot \bar{v} = z$ and $\bar{v} \cdot s' = \bar{s} \cdot v'$, where $s' \cong \bar{v}^*(\bar{s})$, by pullback cancellation. We can now form the pushout $\hat{s}$ of s' along g'. The ensuing pushout diagram

$$\begin{tikzcd} \cdot \arrow[r, "g'"] \arrow[d, "s'"'] & \cdot \arrow[d, "\hat{s}"] \\ \cdot \arrow[r, "\hat{g}"'] & \cdot \end{tikzcd}$$

gives us morphisms t and $\hat{v}$ determined by

$$t \cdot \hat{s} = u, \; t \cdot \hat{g} = w \quad \text{and} \quad \hat{v} \cdot \hat{s} = \tilde{s} \cdot v, \; \hat{v} \cdot \hat{g} = \tilde{g} \cdot \bar{v},$$

respectively. By adhesiveness of $\mathcal{C}$, the last two equalities frame two pullback diagrams, showing that $[u, v]$ and $[w, z]$ factor as

$$[\tilde{s}, 1] \cdot [t, \hat{v}] = [u, v] \quad \text{and} \quad [\tilde{g}, \bar{f}] \cdot [t, \hat{v}] = [w, z],$$

as desired. ∎

With Remark 6.4(2) we conclude:

7.4. Corollary. *In a finitely complete and adhesive category $\mathcal{C}$, any two* Span($\mathcal{C}$)*-morphisms $[g, h], [s, t]$ with common codomain, s monic and both, g and s, exponentiable in $\mathcal{C}$, allow for the formation of a weak pullback diagram*

$$\begin{tikzcd} \cdot \arrow[r, "{[\breve{g},\check{f}]}"] \arrow[d, "{[\check{s},\check{t}]}"'] & \cdot \arrow[d, "{[s,t]}"] \\ \cdot \arrow[r, "{[g,f]}"'] & \cdot \end{tikzcd}$$

in Span($\mathcal{C}$)*, with $\check{s}$ monic in $\mathcal{C}$.*

7.5. Remark. For existence criteria for pullbacks in the (non-full) subcategory Par($\mathcal{C}$) of Span($\mathcal{C}$) given by the *partial morphisms* in $\mathcal{C}$ we refer the reader to [Shir Ali Nasab, Hosseini, 2017]. In a forthcoming paper, we plan to extend the methods used there and by [Cockett, Lack, 2002] from partial morphisms to spans.

8. Examples, lifting lax pullback complements along fibrations

By definition, in a locally cartesian closed category $\mathcal{C}$, every morphism s is exponentiable and therefore guarantees the existence of a lax pullback complement along s of any morphism u precomposable with s. This applies in particular to every (quasi)topos and, therefore, to every presheaf category $\mathcal{C} = \mathsf{Set}^{\mathcal{D}^{\mathrm{op}}}$, for any small category $\mathcal{D}$, including the category Set itself. If $\mathcal{C}$ fails to be locally Cartesian closed, the task of characterizing the exponentiable morphisms usually becomes challenging. But even when characterizations are known (for which we refer to the extensive literature, including [Niefield, 1982, Johnstone, 1993, Mantovani, 1988, Tholen, 2000, Niefield, 2001, Richter, Tholen, 2001, Niefield, 2002, Richter, 2002]), the effective construction of lax pbcs may remain cumbersome. Hence, our list of examples concentrates on this aspect, rather than on mere existence statements.

Let us also point out that, although referred to here only in the ordinary sense, the constructions given are known to carry over to the 2-categorical context when the ambient category is considered as a 2-category in a natural fashion.

8.1. Example.

(1) *The lax pbc diagram* (1) *of* $u : U \longrightarrow S$ *along* $s : S \longrightarrow A$ *in* Set *may be described as*

$P = \{(x, k) \mid x \in A,\ k : s^{-1}x \longrightarrow U,\ u \cdot k = (\ s^{-1}x \hookrightarrow S\ )\},\ \ p(x, k) = x,$
$P \times_A S \cong \{(y, k) \mid y \in S, (s(y), k) \in P\},\quad e(y, k) = k(y),\ s'(y, k) = (s(y), k).$

(2) *When extending the presentation of lax pbcs from* Set *to* $\mathsf{Set}^{\mathcal{D}^{op}}$, *it is convenient to use the fact that the slices of presheaf categories are presheaf categories themselves. Thus, for* $A : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ *we use the category equivalence*

$$\mathsf{Set}^{\mathcal{D}^{\mathrm{op}}}/A \longrightarrow \mathsf{Set}^{(\int_{\mathcal{D}} A)^{\mathrm{op}}}\ ,\quad (\ s : S \longrightarrow A\ ) \longmapsto (s^* : (i, x) \mapsto s_i^{-1}x\ ),$$

where $\int_{\mathcal{D}} A$ *is the element category of* A*; it has objects* (i, x) *with* $i \in \mathrm{ob}\mathcal{D}$ *and* $x \in Ai$*, and a morphism* $\delta : (j, z) \longrightarrow (i, x)$ *is a* $\mathcal{D}$*-morphism* $\delta : j \longrightarrow i$ *with* $(A\delta)(x) = z$*. The category equivalence assigns to a morphism* $u : t \longrightarrow s$ *in* $\mathsf{Set}^{\mathcal{D}^{\mathrm{op}}}/A$ *the natural transformation* $\widetilde{u} : t^* \longrightarrow s^*$, *with* $\widetilde{u}_{(i,x)} : t_i^{-1}x \longrightarrow s_i^{-1}x$ *denoting the restriction of the map* u_i*. For* $u : U \longrightarrow S$ *and* $s : S \longrightarrow A$ *in* $\mathsf{Set}^{\mathcal{D}^{op}}$, *considering* u *as a morphism* $s \cdot u \longrightarrow s$ *in* $\mathsf{Set}^{\mathcal{D}^{\mathrm{op}}}/A$*, so that* $\widetilde{u} : (s \cdot u)^* \longrightarrow s^*$, *we may now describe the lax pbc diagram* (1)*, as follows:*

$Pi = \{(x, \kappa) \mid x \in Ai,\ \kappa : (\int_{\mathcal{D}} A)(\text{-}, (i, x)) \times s^* \longrightarrow (s \cdot u)^*,\ \widetilde{u} \cdot \kappa = \mathrm{pr}_2\},\ p_i(x, \kappa) = x,$

$(P \times_A S)i \cong \{(y, \kappa) \mid y \in Si, (s_i(y), \kappa) \in Pi\}, e_i(y, \kappa) = \kappa_{(i,x)}(x, 1_{(i,x)}), s_i'(y, \kappa) = (x, \kappa),$
where $x = s_i(y)$.

(3) *As a special case of* (2) *we consider the category* Gph *of directed graphs, that is: the category* $\mathsf{Set}^{\mathcal{D}^{\mathrm{op}}}$, *where* $\mathcal{D}$ *is the category* $\{\ 0 \underset{c}{\overset{d}{\rightrightarrows}} 1\ \}$*. Presenting its objects in the form* $A = (\ A_1 \underset{c_A}{\overset{d_A}{\rightrightarrows}} A_0\ )$, *we describe the set of vertices of the domain* P *of the lax pbc* $p : P \longrightarrow A$ *of* $u : U \longrightarrow S$ *along* $s : S \longrightarrow A$ *as in* Set *by*

$$P_0 = \{(x, k_0) \mid x \in A_0,\ k_0 : s_0^{-1}x \longrightarrow U_0\ ,\ u_0 \cdot k_0 = (\ s_0^{-1}x \hookrightarrow S_0\ )\};$$

the edges of P *are quadruples* (a, k_1, k_d, k_c) *with* $a \in A_1$ *and maps*

$$k_1 : s_1^{-1}a \longrightarrow U_1\ ,\ \ k_d : s_0^{-1}(d_A a) \longrightarrow U_0\ ,\ \ k_c : s_0^{-1}(c_A a) \longrightarrow U_0\ ,$$

satisfying $d_U \cdot k_1 = k_d \cdot \widetilde{d}_S$, $c_U \cdot k_1 = k_c \cdot \widetilde{c}_S$ *(with* $\widetilde{d}_S, \widetilde{c}_S$ *restrictions of* d_S, c_S*), and*

$$u_1 \cdot k_1 = (s_1^{-1}a \hookrightarrow S_1),\ u_0 \cdot k_d = (\ s_0^{-1}(d_A a) \hookrightarrow S_0),\ u_0 \cdot k_c = (\ s_0^{-1}(c_A a) \hookrightarrow S_0).$$

The domain and codomain maps $d_P, c_P : P_1 \longrightarrow P_0$ *assign to the edge* (a, k_1, k_d, k_c) *the vertices* $(d_A a, k_d), (c_A a, k_c)$*, respectively.*

(4) *In the category* Ord *of (pre)ordered sets, a monotone map* $s : S \longrightarrow A$ *is exponentiable if it has the* interpolation lifting property *(also known as being* convex*), that is: whenever* $y \leq z$ *in* S *and* $s(y) \leq b \leq s(z)$ *in* A*, there must exist* $a \in S$ *with* $y \leq a \leq z$ *and* $s(a) = b$*. The lax pbc* $p : P \longrightarrow A$ *of* $u : S \longrightarrow A$ *along* s *may be constructed as in* Set*, except that for* P *one considers only those pairs* (x, k) *for which the map* k *is monotone; they are ordered componentwise by*

$$(x, k) \leq (x', k') \Longleftrightarrow x \leq x' \text{ and } \forall y \in s^{-1}x, y' \in {s'}^{-1}x' \, (y \leq y' \Rightarrow ky \leq k'y').$$

The construction may be restricted to the category Pos *of separated objects in* Ord *(known as partially ordered sets), and it may be extended to the category* Cat *of small categories and their functors. The characterization of exponentiable morphisms in* Cat *goes back to [Giraud, 1964]; being rediscovered in [Conduché, 1972], they are commonly known as* Conduché fibrations*.*

(5) *For the characterization of exponentiable morphisms* $s : S \longrightarrow A$ *in the category* Top *of topological spaces we refer to [Niefield, 2002]. These maps include all perfect (= proper and separated) maps, all local homeomorphisms, all continuous maps with locally compact domain* S *and Hausdorff codomain* A*, and all locally closed embeddings (= inclusion maps for which* S *is the intersection of an open set and a closed set of* A*). For such maps, we may again construct a lax pbc* $p : P \longrightarrow A$ *of* $u : U \longrightarrow S$ *along* s *as in* Set*, except that we must consider for* P *only the pairs* (x, k) *for which* k *is continuous. The topology of* P *is generated by the sets*

$$\langle W, H \rangle = \{(x, k) \in P \,|\, k^{-1}(W_x) \in H_x\},$$

where W *is an open subset of* U*,* $W_x = W \cap (s \cdot u)^{-1}(x)$*,* $H \subset \bigcup_{x \in A} \mathcal{O}(s^{-1}(x))$ *and* $H_x = H \cap \mathcal{O}(s^{-1}(x))$*, subject to the conditions that* H_x *be Scott open for all* $x \in A$*, and that for all* $V \in \mathcal{O}(S)$ *the set* $\{x \in A \,|\, V_x \in H_x\}$ *be open in* A*.*

Rather than restricting the choice of morphisms s, next we give a construction of lax pullback complements which, while restricting the choice of morphisms u, puts no restriction on s. Specifically, we show that a fibred category $\mathcal{C}$ over $\mathcal{B}$ with well-behaved pullbacks, although usually quite far from being locally Cartesian closed even for a "good" base category $\mathcal{B}$, still allows for the formation of lax pbcs of all Cartesian morphisms u along any morphism s. In fact, without any hypotheses on the category $\mathcal{B}$ one can prove:

8.2. THEOREM. *Let* $F : \mathcal{C} \longrightarrow \mathcal{B}$ *be a fibration and* $s : S \longrightarrow A$ *be a morphism in* $\mathcal{C}$*, such that* $\mathcal{C}$ *has all pullbacks of* s *which are being preserved by* F*. Then, for every* F*-Cartesian morphism* $u : U \longrightarrow S$ *in* $\mathcal{C}$*, if a lax pullback complement of* Fu *along* Fs *exists in* $\mathcal{B}$*, also a lax pullback complement of* u *along* s *exists in* $\mathcal{C}$ *and is preserved by* F*.*

PROOF. By hypothesis, we have the lax pbc diagram

$$\begin{tikzcd} R \times_{FA} FS \arrow[r, "a"] \arrow[d, "b"'] & FU \arrow[r, "Fu"] & FS \arrow[d, "Fs"] \\ R \arrow[rr, "r"'] & & FA \end{tikzcd} \tag{17}$$

in $\mathcal{B}$. With $p : P \longrightarrow A$ an F-Cartesian lifting of r we can form the pullback diagram

$$\begin{tikzcd} P \times_A S \arrow[r, "p'"] \arrow[d, "s'"'] & S \arrow[d, "s"] \\ P \arrow[r, "p"'] & A \end{tikzcd} \tag{18}$$

in $\mathcal{C}$ and assume, without loss of generality, that F maps it to (17), so that $Fs' = b, Fp' = Fu \cdot a$. The F-Cartesian morphism u then determines a morphism $e : P \times_S A \longrightarrow U$ with $Fe = a$, $u \cdot e = p'$. We claim that, with this factorization of p', (18) becomes a lax pbc diagram in $\mathcal{C}$.

Indeed, given any pullback diagram (2), the lax pbc diagram (17) determines a morphism $k : FQ \longrightarrow FP$ with $Fp \cdot k = Fq$, $k \cdot Fs'' = Fs' \cdot (k \times_{FA} 1_{FS})$ which, by F-Cartesianness of p, lifts to a morphism $h : Q \longrightarrow P$ with $Fh = k$, $p \cdot h = q$, and then $h \cdot s'' = s' \cdot (h \times 1_S)$, $e \cdot (h \times 1_S) = d$ follows. A routine check shows that h is uniquely determined by the last three equations. ∎

For the principal application of the Theorem we consider a monoidal category $\mathcal{V}$ with pullbacks and let $\mathcal{C} = \mathcal{V}\text{-}\mathsf{Cat}$ be the (ordinary) category of small $\mathcal{V}$-categories and their $\mathcal{V}$-functors. Then $\mathcal{V}\text{-}\mathsf{Cat}$ has pullbacks whose object sets are formed by pullback in Set and whose hom-objects are formed by pullback in $\mathcal{V}$. Consequently, the functor

$$\mathrm{ob} : \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Set}$$

preserves pullbacks, and quite trivially, ob is a fibration, with the ob-Cartesian morphisms characterized as the fully faithful $\mathcal{V}$-functors. Hence, we obtain:

8.3. COROLLARY. *For a monoidal category $\mathcal{V}$ with pullbacks and composable $\mathcal{V}$-functors u and s, where u is fully faithful, a lax pullback complement of u along s exists in $\mathcal{V}$-Cat, and the functor* ob *preserves it.*

8.4. EXAMPLE.

(1) *Choosing $\mathcal{V}$ to be* Set *with its Cartesian product, or* AbGrp *with its tensor product over $\mathbb{Z}$, one obtains lax pbcs in the categories* Cat *and* AddCat *of small (additive) categories for all fully faithful (additive) functors along any other (additive) functor.*

(2) *Choosing for $\mathcal{V}$ a quantale, such as the two-element chain* $\mathsf{2}$ *or the Lawvere interval* $[0,\infty]$*, the Corollary gives lax pbcs in the categories* Ord *and* Met *of pre-ordered sets and generalized metric spaces for all morphisms u leaving the respective structure invariant ("order isomorphisms" and "isometries", respectively).*

(3) *If, in addition to a quantale $\mathcal{V}$, we are given a monad $\mathbb{T}$ on* Set *that comes with a lax extension to the category* $\mathcal{V}$-Rel *of sets and their $\mathcal{V}$-valued relations, then, as in [Hofmann, Seal, Tholen, 2014], one can form the category* $(\mathbb{T},\mathcal{V})$-Cat *of small* $(\mathbb{T},\mathcal{V})$*-categories and their* $(\mathbb{T},\mathcal{V})$*-functors. Being topological, the forgetful functor* $\mathrm{ob} : (\mathbb{T},\mathcal{V})\text{-}\mathsf{Cat} \longrightarrow \mathsf{Set}$ *certainly satisfies the conditions of Theorem 8.2, which guarantees the existence of lax pbcs for all* ob*-Cartesian* $(\mathbb{T},\mathcal{V})$*-functors. In particular, for* $\mathcal{V} = \mathsf{2}$ *or* $[0,\infty]$ *and $\mathbb{T}$ the ultrafilter monad of* Set *with its appropriate lax extension to* Rel*, thus producing the categories* Top *and* App *of topological and approach spaces, respectively, we obtain lax pbcs for all morphisms* $u : U \longrightarrow S$ *for which U carries the initial (="weak") structure induced by S.*

9. Lax pullback complements as coreflections

We finally note that there is an alternative universal characterization of lax pullback complements, as follows. Let $\mathcal{S}$ be a class of morphisms in a category $\mathcal{C}$ containing all isomorphisms. We consider $\mathcal{S}$ as a full subcategory of the category $\mathcal{C}^2$ of morphisms of $\mathcal{C}$ and assume that all pullbacks of morphisms in $\mathcal{S}$ exist in $\mathcal{C}$ and belong to $\mathcal{S}$ again. Consequently, the codomain functor

$$\mathrm{cod} : \mathcal{S} \longrightarrow \mathcal{C}$$

is a fibration whose right adjoint embeds $\mathcal{C}$ fully into the category $\mathcal{S}$. The cod-Cartesian morphisms $(p,q) : t \longrightarrow s$ in $\mathcal{S}$ are given by pullback diagrams

$$\begin{tikzcd} T \arrow[r, "q"] \arrow[d, "t"'] & S \arrow[d, "s"] \\ B \arrow[r, "p"'] & A \end{tikzcd}$$

in $\mathcal{C}$, with vertical arrows in the class $\mathcal{S}$. We denote by $\mathrm{Cart}(\mathcal{S})$ the (non-full) subcategory of $\mathcal{S}$ formed by all such pullback diagrams; it contains $\mathcal{C}$ as a full reflective subcategory.

In turn, the category $\mathrm{Cart}(\mathcal{S})$ gets fully and reflectively embedded into the category $\mathrm{Fact}(\mathcal{S})$ whose objects are composable pairs (s,u) in $\mathcal{C}$ with s in the class $\mathcal{S}$, and whose morphisms $(p,q,r) : (t,v) \longrightarrow (s,u)$ are commutative diagrams

$$\begin{tikzcd}
V \arrow[r, "r"] \arrow[d, "v"'] & U \arrow[d, "u"] \\
T \arrow[r, "q"] \arrow[d, "t"'] & S \arrow[d, "s"] \\
B \arrow[r, "p"'] & A
\end{tikzcd}$$

such that the lower part is a pullback diagram in $\mathcal{C}$. The full embedding

$$E : \mathrm{Cart}(\mathcal{S}) \longrightarrow \mathrm{Fact}(\mathcal{S}), \quad s \longmapsto (s, 1),$$

is right adjoint to the obvious forgetful functor (which, like cod, is a fibration, provided that $\mathcal{C}$ has all pullbacks). One can now prove (see also [Tholen, 1983]):

9.1. Proposition. *A lax pullback complement of u along $s \in \mathcal{S}$ exists in $\mathcal{C}$ if, and only if, the* $\mathrm{Fact}(\mathcal{S})$*-object (s, u), has a coreflection into* $\mathrm{Cart}(\mathcal{S})$*. Consequently, all morphisms in the class $\mathcal{S}$ are exponentiable in $\mathcal{C}$ if, and only if,* $\mathrm{Cart}(\mathcal{S})$ *lies (not only reflectively but also) coreflectively in* $\mathrm{Fact}(\mathcal{S})$*.*

Proof. Let diagram (1) describe a lax pullback complement, so that we have the commutative diagram

$$\begin{tikzcd}
P \times_A S \arrow[r, "e"] \arrow[d, "1_{P\times_A S}"'] & U \arrow[d, "u"] \\
P \times_A S \arrow[r, "u\cdot e"] \arrow[d, "s'"'] & S \arrow[d, "s"] \\
P \arrow[r, "p"'] & A
\end{tikzcd} \tag{19}$$

We show that $(p, u\cdot e, e) : E(s') \longrightarrow (s, u)$ is a coreflection. Indeed, given any morphism $(q, q', d) : E(s'') \longrightarrow (s, u)$, there is a unique morphism h such that

$$\begin{tikzcd}
& P \times_A S \arrow[rr, "e"] \arrow[dd, "s'"] & & U \arrow[r, "u"] & S \arrow[dd, "s"] \\
Q \times_A S \arrow[ur, dotted, "h\times 1_S"] \arrow[urrr, "d"] \arrow[dd, "s''"'] & & & & \\
& P \arrow[rrr, "p"] & & & A \\
Q \arrow[ur, dotted, "h"] \arrow[urrrr, "q"'] & & & &
\end{tikzcd} \tag{20}$$

is commutative. Thus, $(p, u\cdot e, e)\cdot E(h) = (q, q', d)$ with $E(h) = (h, h\times 1_S, h\times 1_S)$. Clearly, h is uniquely determined.

Conversely, for a coreflection $(p, p', e) : E(s') \longrightarrow (s, u)$ of (s, u) into $\mathrm{Cart}(\mathcal{S})$, one necessarily has $p' = u \cdot e$ and confirms that p is a lax pbc of u along s, as above. ∎

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