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LINEAR DISTRIBUTIVITY WITH NEGATION, STAR-AUTONOMY, AND HOPF MONADS

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Abstract. We show that a Hopf monad on a $*$-autonomous category lifts $*$-autonomous structure to the category of algebras precisely when there is an algebra structure on the dualizing object. Our proof is based on Pastro's characterization of $*$-autonomous (co)monads as linearly distributive (co)monads with negation.

1. Introduction

As observed by Moerdijk [8], for a monad on a monoidal category, to give a comonoidal (also known as opmonoidal) structure to the monad is precisely to give a monoidal structure to the category of algebras of the monad such that the forgetful functor strictly preserves the monoidal structure. Bruguières and Virelizier [5] later identified additional conditions on comonoidal monads on autonomous categories (monoidal categories with duals) such that they lift the autonomous structure to the category of algebras, and they called such a comonoidal monad a **Hopf monad**. Hopf monads and their algebras can be seen as generalizations of Hopf algebras (with invertible antipodes) and their modules. In fact, Hopf monads in [5] are defined as comonoidal monads equipped with an antipode given by certain natural transformations. Later, Bruguières, Lack and Virelizier [4] introduced Hopf monads on arbitrary monoidal categories, by simplifying and generalizing the notion of Hopf monads on autonomous categories. Now a Hopf monad on a monoidal category is a comonoidal monad T such that the induced maps (called **fusion operators**) $\mathsf{T}(A \otimes \mathsf{T}B) \to \mathsf{T}A \otimes \mathsf{T}\mathsf{T}B \to \mathsf{T}A \otimes \mathsf{T}B$ and $\mathsf{T}(\mathsf{T}A \otimes B) \to \mathsf{T}\mathsf{T}A \otimes \mathsf{T}B \to \mathsf{T}A \otimes \mathsf{T}B$ are invertible (Definition 4.1). It has been shown that a comonoidal monad on a monoidal closed category lifts the monoidal closed structure to the category of algebras exactly when it is a Hopf monad [4].

On the other hand, Pastro and Street [10] considered the conditions on monoidal comonads (the dual of comonoidal monads) on **$*$-autonomous categories** [1, 2] (also known as **Grothendieck-Verdier categories** [3]) for lifting the $*$-autonomous struc-

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ture to the category of coalgebras, and called such monoidal comonads **$*$-autonomous comonads**. By dualizing their result, one obtains the notion of a **$*$-autonomous monad** on a $*$-autonomous category, which can be described as a comonoidal monad that lifts the $*$-autonomous structure to the category of algebras of said monad.

Since a $*$-autonomous category is monoidal closed, a $*$-autonomous monad is necessarily a Hopf monad (Corollary 4.5). However, (as briefly mentioned in [4]) the converse is not true. Here is a simple counterexample:

1.1. Example. Consider real numbers $\mathbb{R}$ with the usual order $\leq$ and fix a real number r. Then the poset $(\mathbb{R}, \leq)$, regarded as a category, is symmetric $*$-autonomous, with the monoidal structure given by $x \otimes y = x + y$ and $\top = 0$, and the dualizing object defined as $\bot = r$. The dual monoidal structure ("par") is therefore given by $x \parr y = x + y - r$, while the internal hom from x to y is $y - x$. Now let $\mathsf{T} : \mathbb{R} \to \mathbb{R}$ be the "ceiling function" sending x to the least integer y such that $x \leq y$. Then T, regarded as a functor, is a monad on $(\mathbb{R}, \leq)$. It is comonoidal with respect to $(\otimes, \top)$ as $\mathsf{T}0 = 0$ and $\mathsf{T}(x + y) \leq \mathsf{T}x + \mathsf{T}y$ hold. Moreover T is a Hopf monad since $\mathsf{T}(x + \mathsf{T}y) = \mathsf{T}(\mathsf{T}x + y) = \mathsf{T}x + \mathsf{T}y$ holds. The category $(\mathbb{R}, \leq)^{\mathsf{T}}$ of the algebras of T is the integers $(\mathbb{Z}, \leq)$, and the symmetric monoidal closed structure on $(\mathbb{R}, \leq)$ is lifted to $(\mathbb{Z}, \leq)$. However, it lifts the $*$-autonomous structure only when r carries a T-algebra structure, that is, when r is an integer.

This example shows that a Hopf monad may not necessarily lift a dualizing object to the category of its algebras. It also suggests that a Hopf monad is $*$-autonomous (in the dual sense of Pastro and Street) when there is an algebra structure on the dualizing object. In this paper, we show that this is the case.

Rather than directly working with the notion of $*$-autonomous (co)monads, we follow Pastro's approach [9] based on linearly distributive categories [6]. Since a $*$-autonomous category is none other than a **linearly distributive category with negation** [6], it makes sense to first identify (co)monads for lifting linearly distributive structure and then put additional conditions for lifting negation. In this way Pastro characterized $*$-autonomous comonads as **linearly distributive comonads with negation** (Definition 3.2). We find Pastro's characterization more suitable for our purpose, and use his techniques in [9] in our proof of Theorem 5.8.

The rest of this paper is organized as follows. In Section 2, we recall the notion of linearly distributive categories with negation. Section 3 gives the definition of linearly distributive monads with negation which are the dual of Pastro's linearly distributive comonads with negation. In Section 4, we recall Hopf monads and their basic results. In Section 5, we prove our main result and provide some examples.

Conventions: To simplify working in monoidal categories, we will be working with *strict* monoidal categories and so we will suppress the associator and unitor isomorphisms.

2. Linearly Distributive Categories With Negation

2.1. Definition. A **linearly distributive category** [6] is a septuple $(\mathbb{X}, \otimes, \top, \parr, \bot, \partial_l, \partial_r)$ consisting of:

- **(i)** A monoidal category $(\mathbb{X}, \otimes, \top)$
- **(ii)** A monoidal category $(\mathbb{X}, \parr, \bot)$
- **(iii)** Two natural transformations, called respectively the left and right distributors, $\partial_l : A \otimes (B \parr C) \to (A \otimes B) \parr C$ and $\partial_r : (A \parr B) \otimes C \to A \parr (B \otimes C)$

such that a number of coherence diagrams [6] commute.

2.2. Definition. A **negation** [6] on a linearly distributive category $(\mathbb{X}, \otimes, \top, \parr, \bot, \partial_l, \partial_r)$ is a sextuple $(\mathsf{S}, \mathsf{S}', \alpha, \beta, \alpha', \beta')$ which consists of:

- **(i)** Two contravariant functors $\mathsf{S} : \mathbb{X}^{op} \to \mathbb{X}$ and $\mathsf{S}' : \mathbb{X}^{op} \to \mathbb{X}$
- **(ii)** Four maps (called the evaluation and coevaluation maps):

$$\alpha : \mathsf{S}A \otimes A \to \bot \qquad \beta : \top \to A \parr \mathsf{S}A$$
$$\alpha' : A \otimes \mathsf{S}'A \to \bot \qquad \beta' : \top \to \mathsf{S}'A \parr A$$

such that the four triangle identities [6] are satisfied.

There are a number of equivalent ways of defining $*$-autonomous categories [2, 3, 10, 9]. Here we shall recall one of them (as found in [9]):

2.3. Definition. A **$*$-autonomous category** is a monoidal category $(\mathbb{X}, \otimes, \top)$ equipped with an adjoint equivalence $\mathsf{S} \dashv \mathsf{S}' : \mathbb{X}^{\mathrm{op}} \xrightarrow{\simeq} \mathbb{X}$ such that there is a bijection

$$\mathbb{X}(A \otimes B, \mathsf{S}C) \cong \mathbb{X}(A, \mathsf{S}(B \otimes C)) \tag{1}$$

natural in A, B and C.

2.4. Theorem. *[6, 9] The notions of linearly distributive categories with negation and $*$-autonomous categories coincide.*

Proof. For a $*$-autonomous category $(\mathbb{X}, \otimes, \top, \mathsf{S}, \mathsf{S}')$, we have the dual tensor $A \parr B = \mathsf{S}'(\mathsf{S}B \otimes \mathsf{S}A) \cong \mathsf{S}(\mathsf{S}'B \otimes \mathsf{S}'A)$ and $\bot = \mathsf{S}\top \cong \mathsf{S}'\top$. The relevant data $\partial_l, \partial_r, \alpha, \beta, \alpha'$ and β' are routinely derived from the adjoint equivalence $\mathsf{S} \vdash \mathsf{S}'$ and the natural bijection (1), making $\mathbb{X}$ a linearly distributive category with negation. Conversely, for a linearly distributive category $(\mathbb{X}, \otimes, \top, \parr, \bot, \partial_l, \partial_r)$ with negation $(\mathsf{S}, \mathsf{S}', \alpha, \beta, \alpha', \beta')$, it is not hard to show that S and S' give an adjoint equivalence, and that the natural bijection (1) exists. Moreover these constructions are mutually inverse. ■

3. Linearly Distributive Monads with Negation

3.1. Definition. *[11]* Let $\mathsf{F} : \mathbb{X}^{op} \to \mathbb{X}$ be a contravariant functor and (T, μ, η) a monad on $\mathbb{X}$ with multiplication $\mu_A : \mathsf{TT}A \to \mathsf{T}A$ and unit $\eta_A : A \to \mathsf{T}A$. Then a **distributive law of F over** (T, μ, η) is a natural transformation $\lambda : \mathsf{TFT}A \to \mathsf{F}A$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathsf{FT}A & \xrightarrow{\eta} & \mathsf{TFT}A \\ & \searrow^{\mathsf{F}(\eta)} & \downarrow{\lambda} \\ & & \mathsf{F}A \end{array} \qquad \begin{array}{ccccc} \mathsf{TTFT}A & \xrightarrow{\mathsf{TTF}(\mu)} & \mathsf{TTFTT}(A) & \xrightarrow{\mathsf{T}(\lambda)} & \mathsf{TFT}A \\ \downarrow{\mu} & & & & \downarrow{\lambda} \\ \mathsf{TFT}A & & \xrightarrow{\lambda} & & \mathsf{F}A \end{array} \tag{2}$$

This induces a contravariant functor $\overline{\mathsf{F}} : (\mathbb{X}^{\mathsf{T}})^{op} \to \mathbb{X}^{\mathsf{T}}$ defined on objects as:

$$\overline{\mathsf{F}}(A,\ \mathsf{T}A \xrightarrow{\mathsf{a}} A\ ) := (\mathsf{F}A,\ \mathsf{TF}A \xrightarrow{\mathsf{TF(a)}} \mathsf{TFT}A \xrightarrow{\lambda} \mathsf{F}A) \tag{3}$$

and on maps as $\overline{\mathsf{F}}(f) = \mathsf{F}(f)$.

3.2. Definition. A **linearly distributive monad with negation** on a linearly distributive category $(\mathbb{X}, \otimes, \top, \parr, \bot, \partial_l, \partial_r)$ with negation $(\mathsf{S}, \mathsf{S}', \alpha, \beta, \alpha', \beta')$ is a nonuple $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1, \mathsf{n}_2, \mathsf{n}_1, \nu, \nu')$ consisting of:

- **(i)** A comonoidal monad $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1)$ on $(\mathbb{X}, \otimes, \top)$ with comonoidality structure $\mathsf{m}_{2A,B} : \mathsf{T}(A \otimes B) \to \mathsf{T}A \otimes \mathsf{T}B$ and $\mathsf{m}_1 : \mathsf{T}\top \to \top$
- **(ii)** A comonoidal monad $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ on $(\mathbb{X}, \parr, \bot)$ with comonoidality structure $\mathsf{n}_{2A,B} : \mathsf{T}(A \parr B) \to \mathsf{T}A \parr \mathsf{T}B$ and $\mathsf{n}_1 : \mathsf{T}\bot \to \bot$
- **(iii)** A distributive law ν of S over (T, μ, η)
- **(iv)** A distributive law ν' of S' over (T, μ, η)

such that (the dual of) the coherence diagrams from Pastro's paper [9] commute. As noted by Pastro, these coherence conditions are equivalent to requiring that ∂_l, ∂_r, α, β, α', β' are all T-algebra morphisms (whenever their parameters are T-algebras).

3.3. Theorem. *[9] The category of algebras of a linearly distributive monad with negation is a linearly distributive category with negation and the forgetful functor strictly preserves the structure.*

As mentioned in the introduction, we find the dual notion of Pastro's linearly distributive comonad with negation [9] to be more suitable for proving our main result (Theorem 5.8), rather than Pastro and Street's notion of a $*$-autonomous comonad [10]. Of course, however, these two notions coincide and the proofs of Section 5 could be done with either.

3.4. Theorem. *[9] The notions of linearly distributive monads with negation and $*$-autonomous monads coincide.*

4. Hopf Monads

4.1. Definition. Let $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1)$ be a comonoidal monad on a monoidal category $(\mathbb{X}, \otimes, \top)$. The left and right **fusion operators** [4] are respectively the two natural transformations $\mathsf{h}_l : \mathsf{T}(A \otimes \mathsf{T}B) \to \mathsf{T}A \otimes \mathsf{T}B$ and $\mathsf{h}_r : \mathsf{T}(\mathsf{T}A \otimes B) \to \mathsf{T}A \otimes \mathsf{T}B$ defined as follows:

$$\mathsf{h}_l := \ \mathsf{T}(A \otimes \mathsf{T}B) \xrightarrow{\mathsf{m}_2} \mathsf{T}A \otimes \mathsf{T}\mathsf{T}B \xrightarrow{1 \otimes \mu} \mathsf{T}A \otimes \mathsf{T}B \tag{4}$$

$$\mathsf{h}_r := \ \mathsf{T}(\mathsf{T}A \otimes B) \xrightarrow{\mathsf{m}_2} \mathsf{T}\mathsf{T}A \otimes \mathsf{T}B \xrightarrow{\mu \otimes 1} \mathsf{T}A \otimes \mathsf{T}B \tag{5}$$

A **Hopf monad** [4] is a comonoidal monad whose fusion operators are natural isomorphisms.

4.2. Lemma. *The following diagrams commute:*

$$\begin{array}{ccc} A \otimes \mathsf{T}B & \xrightarrow{\eta \otimes 1} & \mathsf{T}A \otimes \mathsf{T}B \\ & {\scriptstyle \eta} \searrow & \downarrow {\scriptstyle \mathsf{h}_l^{-1}} \\ & & \mathsf{T}(A \otimes \mathsf{T}B) \end{array} \qquad \begin{array}{ccccc} \mathsf{T}\mathsf{T}A \otimes \mathsf{T}B & & \xrightarrow{\mu \otimes 1} & & \mathsf{T}A \otimes \mathsf{T}B \\ \downarrow {\scriptstyle \mathsf{h}_l^{-1}} & & & & \downarrow {\scriptstyle \mathsf{h}_l^{-1}} \\ \mathsf{T}(\mathsf{T}A \otimes \mathsf{T}B) & \xrightarrow[\mathsf{T}(\mathsf{h}_l^{-1})]{} & \mathsf{T}\mathsf{T}(A \otimes \mathsf{T}B) & \xrightarrow[\mu]{} & \mathsf{T}(A \otimes \mathsf{T}B) \end{array} \tag{6}$$

$$\begin{array}{ccc} \mathsf{T}A \otimes B & \xrightarrow{1 \otimes \eta} & \mathsf{T}A \otimes \mathsf{T}B \\ & {\scriptstyle \eta} \searrow & \downarrow {\scriptstyle \mathsf{h}_r^{-1}} \\ & & \mathsf{T}(\mathsf{T}A \otimes B) \end{array} \qquad \begin{array}{ccccc} \mathsf{T}A \otimes \mathsf{T}\mathsf{T}B & & \xrightarrow{1 \otimes \mu} & & \mathsf{T}A \otimes \mathsf{T}B \\ \downarrow {\scriptstyle \mathsf{h}_r^{-1}} & & & & \downarrow {\scriptstyle \mathsf{h}_r^{-1}} \\ \mathsf{T}(\mathsf{T}A \otimes \mathsf{T}B) & \xrightarrow[\mathsf{T}(\mathsf{h}_r^{-1})]{} & \mathsf{T}\mathsf{T}(\mathsf{T}A \otimes B) & \xrightarrow[\mu]{} & \mathsf{T}(\mathsf{T}A \otimes B) \end{array} \tag{7}$$

$$\begin{array}{ccc} \mathsf{T}(A \otimes B) & \xrightarrow{\mathsf{m}_2} & \mathsf{T}A \otimes \mathsf{T}B \\ & {\scriptstyle \mathsf{T}(1 \otimes \eta)} \searrow & \downarrow {\scriptstyle \mathsf{h}_l^{-1}} \\ & & \mathsf{T}(A \otimes \mathsf{T}B) \end{array} \qquad \begin{array}{ccc} \mathsf{T}A \otimes \mathsf{T}\mathsf{T}B & \xrightarrow{\mathsf{h}_l^{-1}} & \mathsf{T}(A \otimes \mathsf{T}\mathsf{T}B) \\ \downarrow {\scriptstyle 1 \otimes \mu} & & \downarrow {\scriptstyle \mathsf{T}(1 \otimes \mu)} \\ \mathsf{T}A \otimes \mathsf{T}B & \xrightarrow[\mathsf{h}_l^{-1}]{} & \mathsf{T}(A \otimes \mathsf{T}B) \end{array} \tag{8}$$

$$\begin{array}{ccc} \mathsf{T}(A \otimes B) & \xrightarrow{\mathsf{m}_2} & \mathsf{T}A \otimes \mathsf{T}B \\ & {\scriptstyle \mathsf{T}(\eta \otimes 1)} \searrow & \downarrow {\scriptstyle \mathsf{h}_r^{-1}} \\ & & \mathsf{T}(\mathsf{T}A \otimes B) \end{array} \qquad \begin{array}{ccc} \mathsf{T}\mathsf{T}A \otimes \mathsf{T}B & \xrightarrow{\mathsf{h}_r^{-1}} & \mathsf{T}(\mathsf{T}A \otimes \mathsf{T}B) \\ \downarrow {\scriptstyle \mu \otimes 1} & & \downarrow {\scriptstyle \mathsf{T}(\mu \otimes 1)} \\ \mathsf{T}A \otimes \mathsf{T}B & \xrightarrow[\mathsf{h}_r^{-1}]{} & \mathsf{T}(\mathsf{T}A \otimes B) \end{array} \tag{9}$$

Proof. They follow from the identities on fusion operators found in [4] (Proposition 2.6). ∎

4.3. Theorem. *[5, 4]Let* T *be a comonoidal monad on an autonomous category* $\mathbb{X}$*. Then* T *is a Hopf monad if and only if the category* $\mathbb{X}^{\mathsf{T}}$ *is autonomous and the forgetful functor strictly preserves the structure.*

4.4. Theorem. *[4] Let* T *be a comonoidal monad on a monoidal closed category* $\mathbb{X}$*. Then* T *is a Hopf monad if and only if the category* $\mathbb{X}^{\mathsf{T}}$ *is monoidal closed and the forgetful functor strictly preserves the structure.*

Since $*$-autonomous categories (linearly distributive categories with negation) are monoidal closed [2], as an immediate corollary to the theorem above we have:

4.5. Corollary. *A* $*$*-autonomous monad (linearly distributive monad with negation) on a* $*$*-autonomous category (linearly distributive category with negation) is a Hopf monad.*

5. Main Result

In this section we will show that every Hopf monad on $\otimes$ such that $\bot$ is a T-algebra induces a linearly distributive monad with negation. So for the remainder of this section, let $(\mathbb{X}, \otimes, \top, \parr, \bot, \partial_l, \partial_r)$ be a linearly distributive category with negation $(\mathsf{S}, \mathsf{S}', \alpha, \beta, \alpha', \beta')$ and let $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1)$ be a Hopf monad on $(\mathbb{X}, \otimes, \top)$ equipped with a map $\mathsf{n}_1 : \mathsf{T}\bot \to \bot$ such that $(\bot, \mathsf{n}_1)$ is a T-algebra.

Define the maps $\phi : \mathsf{TST}A \otimes A \to \bot$ and $\phi' : A \otimes \mathsf{TS'T}A \to \bot$ respectively as follows:

$$\phi := \mathsf{TST}A \otimes A \xrightarrow{1\otimes\eta} \mathsf{TST}A \otimes \mathsf{T}A \xrightarrow{\mathsf{h}_l^{-1}} \mathsf{T}(\mathsf{ST}A \otimes \mathsf{T}A) \xrightarrow{\mathsf{T}(\alpha)} \mathsf{T}\bot \xrightarrow{\mathsf{n}_1} \bot \tag{10}$$

$$\phi' := A \otimes \mathsf{TS'T}A \xrightarrow{\eta\otimes 1} \mathsf{T}A \otimes \mathsf{TS'T}A \xrightarrow{\mathsf{h}_r^{-1}} \mathsf{T}(\mathsf{T}A \otimes \mathsf{S'T}A) \xrightarrow{\mathsf{T}(\alpha')} \mathsf{T}\bot \xrightarrow{\mathsf{n}_1} \bot \tag{11}$$

Similarly, define the maps $\Phi : \mathsf{TTSTT}A \otimes A \to \bot$ and $\Phi' : A \otimes \mathsf{TTS'TT}A \to \bot$ respectively as:

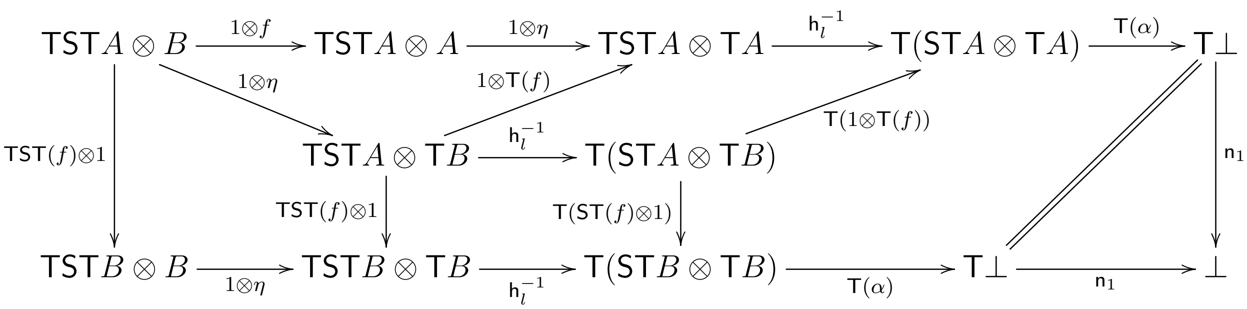
$$\Phi := \mathsf{TTSTT}A \otimes A \xrightarrow{1\otimes\eta} \mathsf{TTSTT}A \otimes \mathsf{T}A \xrightarrow{\mathsf{h}_l^{-1}} \mathsf{T}(\mathsf{TSTT}A \otimes \mathsf{T}A) \xrightarrow{\mathsf{T}(\phi)} \mathsf{T}\bot \xrightarrow{\mathsf{n}_1} \bot \tag{12}$$

$$\Phi' := A \otimes \mathsf{TTS'TT}A \xrightarrow{\eta\otimes 1} \mathsf{T}A \otimes \mathsf{TTS'TT}A \xrightarrow{\mathsf{h}_r^{-1}} \mathsf{T}(\mathsf{T}A \otimes \mathsf{TS'TT}A) \xrightarrow{\mathsf{T}(\phi')} \mathsf{T}\bot \xrightarrow{\mathsf{n}_1} \bot \tag{13}$$

5.1. Lemma. *The following diagrams commute for a morphism* $f : B \to A$*:*

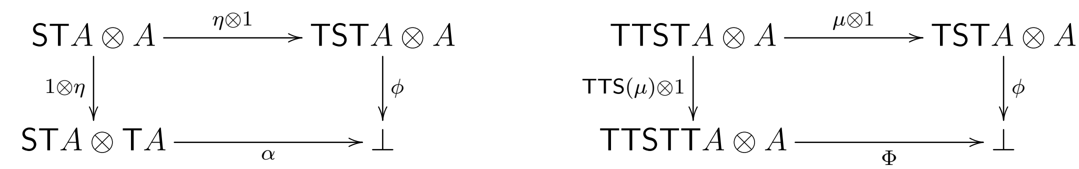
$$\begin{array}{ccc} \mathsf{TST}A \otimes B & \xrightarrow{1\otimes f} & \mathsf{TST}A \otimes A \\ {\scriptstyle \mathsf{TST}(f)\otimes 1}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathsf{TST}B \otimes B & \xrightarrow[\phi]{} & \bot \end{array} \qquad \begin{array}{ccc} B \otimes \mathsf{TS'T}A & \xrightarrow{f\otimes 1} & A \otimes \mathsf{TS'T}A \\ {\scriptstyle 1\otimes\mathsf{TS'T}(f)}\downarrow & & \downarrow{\scriptstyle \phi'} \\ B \otimes \mathsf{TS'T}B & \xrightarrow[\phi']{} & \bot \end{array} \tag{14}$$

Proof. The commutativity of the left diagram is shown as follows:

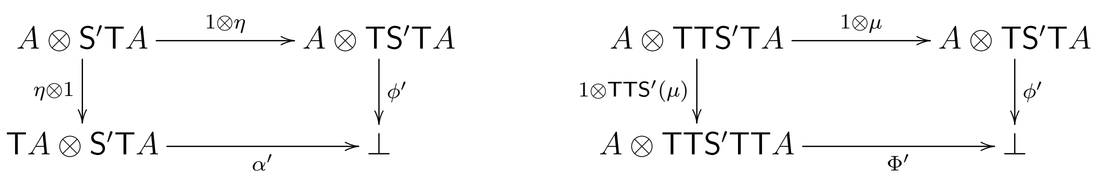


And similar proof shows that the right diagram commutes as well. ∎

5.2. Lemma. *The following diagrams commute:*

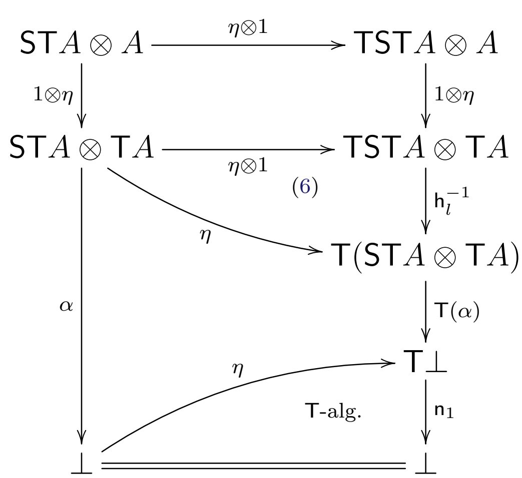


(15)

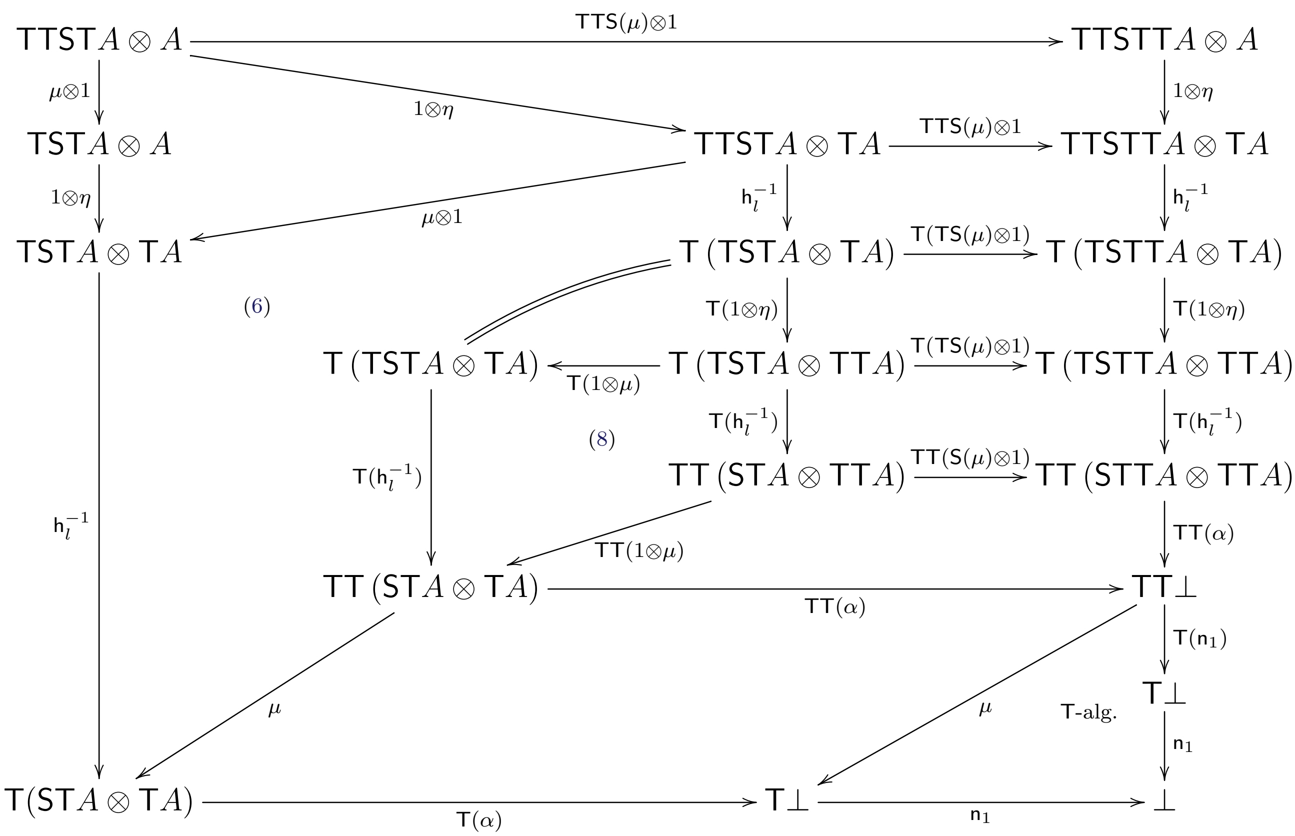


(16)

Proof. That ϕ satisfies the left diagram of (15) follows from commutativity of the following diagram:



That ϕ satisfies the right diagram of (15) follows from commutativity of the following diagram:



Similar arguments are used to show that ϕ' satisfies both diagrams of (16). ∎

Define the natural transformations $\nu : \mathsf{TST}A \to \mathsf{S}A$ and $\nu' : \mathsf{TS'T}A \to \mathsf{S'}A$ respectively as follows:

$$\nu := \mathsf{TST}A \xrightarrow{1\otimes\beta} \mathsf{TST}A \otimes (A \parr \mathsf{S}A) \xrightarrow{\partial_l} (\mathsf{TST}A \otimes A) \parr \mathsf{S}A \xrightarrow{\phi\parr 1} \mathsf{S}A \tag{17}$$

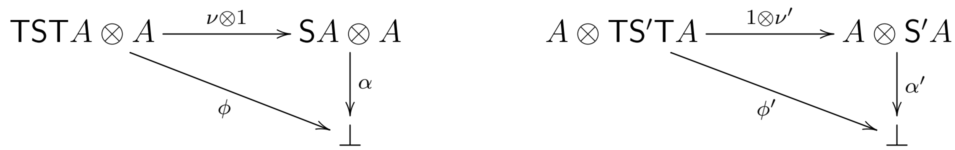
$$\nu' := \mathsf{TS'T}A \xrightarrow{\beta'\otimes 1} (\mathsf{S'}A \parr A) \otimes \mathsf{TS'T}A \xrightarrow{\partial_r} \mathsf{S'}A \parr (A \otimes \mathsf{TS'T}A) \xrightarrow{1\parr\phi'} \mathsf{S'}A \tag{18}$$

These will be our distributive laws for our linearly distributive monad with negation.

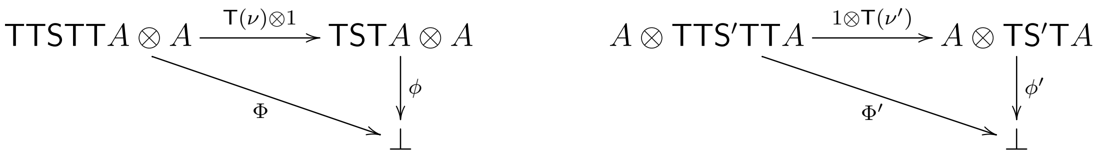
5.3. Lemma. *ν and ν' are natural transformations.*

Proof. Naturality of ν and ν' follows from (14), which we leave to the reader to check for themselves. ∎

5.4. Lemma. *The following diagrams commute:*



(19)



(20)

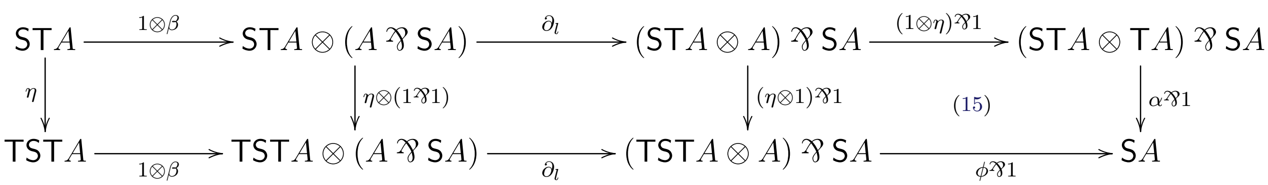
Proof. These follow from the triangle identities of a linearly distributive category with negation, which again we will leave to the reader to check for themselves. ■

5.5. Proposition. *The natural transformation ν (resp. ν') is a distributive law of S (resp. S') over (T,μ,η).*

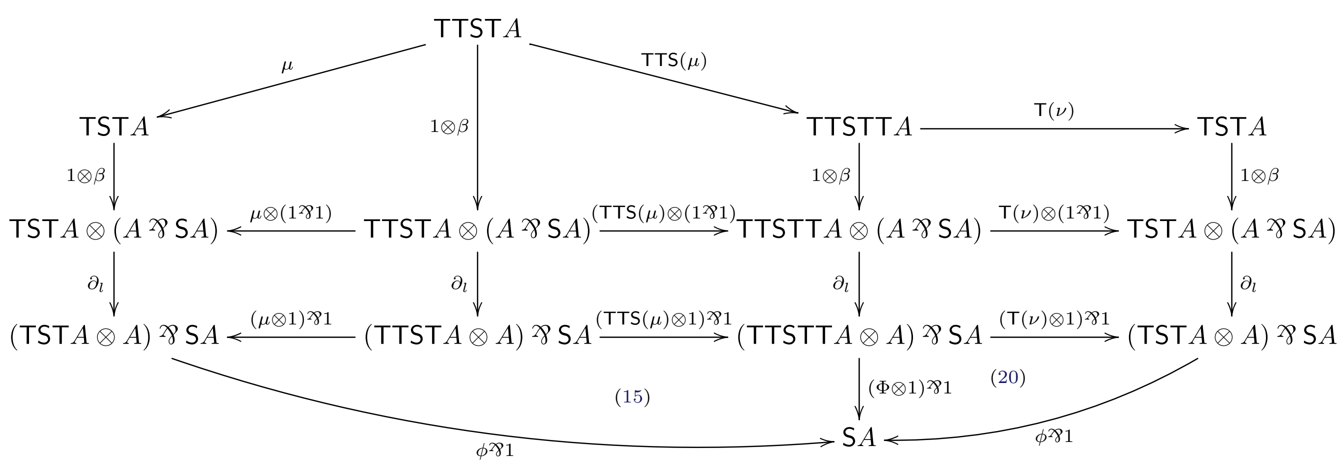
Proof. We show that ν satisfies both diagrams of (2). First note that $\mathsf{S}(\eta):\mathsf{ST}A\to\mathsf{S}A$ is equal to the following composite:

$$\mathsf{S}(\eta)=\left(\mathsf{ST}A\xrightarrow{1\otimes\beta}\mathsf{ST}A\otimes(A\parr\mathsf{S}A)\xrightarrow{\partial_l}(\mathsf{ST}A\otimes A)\parr\mathsf{S}A\xrightarrow{(1\otimes\eta)\parr 1}(\mathsf{ST}A\otimes\mathsf{T}A)\parr\mathsf{S}A\xrightarrow{\alpha\parr 1}\mathsf{S}A\right)$$

Then that ν satisfies the left diagram of (2) follows from commutativity of the following diagram:



That ν satisfies the right diagram of (2) follows from commutativity of the following diagram:



Similar arguments can be used to show that ν' satisfies both diagrams of (2) as well. ■

We now have that S and S' lift to the Eilenberg-Moore category $\mathbb{X}^{\mathsf{T}}$. The next step is to show that $\parr$ lifts to $\mathbb{X}^{\mathsf{T}}$ and that we have a second comonoidal structure on T. First observe the following general results on lifting isomorphisms:

5.6. LEMMA. *Let T be a monad and (A, a) a T-algebra. If $f : A \to B$ is an isomorphism, then the following map:*

$$\mathsf{T}B \xrightarrow{\mathsf{T}(f^{-1})} \mathsf{T}A \xrightarrow{\mathsf{a}} A \xrightarrow{f} B \tag{21}$$

provides a T-algebra structure on B and also that f is a T-algebra morphism.

As an immediate consequence of Lemma 5.6, the canonical isomorphisms $A \cong \mathsf{S}\mathsf{S}'A \cong \mathsf{S}'\mathsf{S}A$ lift to $\mathbb{X}^{\mathsf{T}}$. Note that while this isomorphism is defined using the distributors and the evaluation and coevaluation maps, we have yet to show that the latter are indeed T-algebra morphisms. At this point however, we can define a comonoidal structure on the monad T, which in turn determines $\parr$ for $\mathbb{X}^{\mathsf{T}}$.

Define the natural transformation $\mathsf{n}_2 : \mathsf{T}(A \parr B) \to \mathsf{T}A \parr \mathsf{T}B$ as follows:

$$\begin{gathered}\mathsf{T}(A \parr B) \xrightarrow{\cong} \mathsf{T}\mathsf{S}(\mathsf{S}'B \otimes \mathsf{S}'A) \xrightarrow{\mathsf{T}\mathsf{S}(\nu' \otimes \nu')} \\ \mathsf{T}\mathsf{S}\left(\mathsf{T}\mathsf{S}'\mathsf{T}B \otimes \mathsf{T}\mathsf{S}'\mathsf{T}A\right) \xrightarrow{\mathsf{T}\mathsf{S}(\mathsf{m}_2)} \mathsf{T}\mathsf{S}\mathsf{T}\left(\mathsf{S}'\mathsf{T}B \otimes \mathsf{S}'\mathsf{T}A\right) \xrightarrow{\nu} \\ \mathsf{S}\left(\mathsf{S}'\mathsf{T}B \otimes \mathsf{S}'\mathsf{T}A\right) \xrightarrow{\cong} \mathsf{T}A \parr \mathsf{T}B\end{gathered} \tag{22}$$

5.7. PROPOSITION. $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ *is a comonoidal monad on* $(\mathbb{X}, \parr, \bot)$.

PROOF. Rather than proving this directly, we will define a monoidal structure on $\mathbb{X}^{\mathsf{T}}$ which is strictly preserved by the forgetful functor. Let (A, a) and (B, b) be T-algebras. Since $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1)$ is a comonoidal monad and both S and S' lifts to $\mathbb{X}^{\mathsf{T}}$, we can build the T-algebra $\overline{\mathsf{S}}\left(\overline{\mathsf{S}'}(B, \mathsf{b}) \otimes \overline{\mathsf{S}'}(A, \mathsf{a})\right)$ whose underlying object is $\mathsf{S}(\mathsf{S}'B \otimes \mathsf{S}'A)$. Since $A \parr B \cong \mathsf{S}(\mathsf{S}'B \otimes \mathsf{S}'A)$, we can apply Lemma 5.6 to obtain a T-algebra structure on $A \parr B$ which one can easily check ends up being:

$$\mathsf{T}(A \parr B) \xrightarrow{\mathsf{n}_2} \mathsf{T}A \parr \mathsf{T}B \xrightarrow{\mathsf{a} \parr \mathsf{b}} A \parr B \tag{23}$$

We define $(A, \mathsf{a}) \parr (B, \mathsf{b})$ as this new T-algebra. Furthermore, the canonical isomorphisms $A \parr B \cong \mathsf{S}(\mathsf{S}'B \otimes \mathsf{S}'A) \cong \mathsf{S}'(\mathsf{S}B \otimes \mathsf{S}A)$ are all T-algebra morphisms. It follows that $(\mathbb{X}^{\mathsf{T}}, \parr, (\bot, \mathsf{n}_1))$ is a monoidal category and the forgetful functor $\mathsf{U} : (\mathbb{X}^{\mathsf{T}}, \parr, (\bot, \mathsf{n}_1)) \to (\mathbb{X}, \parr, \bot)$ preserves the monoidal structure strictly. This induces a comonoidal monad structure on (T, μ, η), which is precisely $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$. ∎

To show that we obtain a linearly distributive monad with negation, it remains to show that the distributors (∂_l and ∂_r) and the four evaluation and coevaluation maps (α, β, α', and β') are all T-algebra morphisms (which recall is equivalent to checking the remaining coherence axioms for a linearly distributive monad with negation). We will use the same trick that Pastro uses in his paper [9].

5.8. Theorem. *Let* $(\mathbb{X}, \otimes, \top, \parr, \bot, \partial_l, \partial_r)$ *be a linearly distributive category with negation* $(\mathsf{S}, \mathsf{S}', \alpha, \beta, \alpha', \beta')$*, and* $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1)$ *be a Hopf monad on* $(\mathbb{X}, \otimes, \top)$ *with a* T*-algebra structure* $\mathsf{n}_1 : \mathsf{T}\bot \to \bot$ *on* $\bot$*. Then, with natural transformations* ν, ν' *and* n_2 *defined as above,* $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1, \mathsf{n}_2, \mathsf{n}_1, \nu, \nu')$ *is a linearly distributive monad with negation on* $\mathbb{X}$*.*

Proof. It suffices to show that the distributors, the evaluation and coevaluation maps are T-algebra morphisms. First recall that every linearly distributive category with negation admits a closed monoidal structure with respect to $\otimes$. In particular the left and right internal homs are respectively $\mathsf{S}(A \otimes \mathsf{S}'B)$ and $\mathsf{S}'(\mathsf{S}B \otimes A)$, we have an evaluation map $\mathsf{e} : \mathsf{S}(A \otimes \mathsf{S}'B) \otimes A \to B$ and a coevaluation map $\mathsf{e}' : A \otimes \mathsf{S}'(\mathsf{S}B \otimes A) \to B$. Now since $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1)$ is a Hopf monad, it follows that both e and e' are T-algebra morphisms (when A and B are T-algebras).

Now note that the following diagrams all commute:

$$\begin{array}{ccccc}
A \otimes (B \parr C) & \xrightarrow{\cong} & A \otimes \mathsf{S}'(\mathsf{S}C \otimes \mathsf{S}B) & \xrightarrow{1 \otimes \mathsf{S}'(1 \otimes \mathsf{e})} & A \otimes \mathsf{S}'(\mathsf{S}C \otimes \mathsf{S}(A \otimes \mathsf{S}'\mathsf{S}B) \otimes A) \\
 & & & & \downarrow{\scriptstyle \cong} \\
 & & & & A \otimes \mathsf{S}'(\mathsf{S}\mathsf{S}'(\mathsf{S}C \otimes \mathsf{S}(A \otimes \mathsf{S}'\mathsf{S}B)) \otimes A) \\
{\scriptstyle \partial_l}\downarrow & & & & \downarrow{\scriptstyle \mathsf{e}'} \\
 & & & & \mathsf{S}'(\mathsf{S}C \otimes \mathsf{S}(A \otimes \mathsf{S}'\mathsf{S}B)) \\
 & & & & \downarrow{\scriptstyle \cong} \\
(A \otimes B) \parr C & & \xleftarrow{\qquad\cong\qquad} & & (A \otimes \mathsf{S}'\mathsf{S}B) \parr C
\end{array}$$

$$\begin{array}{ccccc}
(A \parr B) \otimes C & \xrightarrow{\cong} & \mathsf{S}(\mathsf{S}'B \otimes \mathsf{S}'A) \otimes C & \xrightarrow{\mathsf{S}(\mathsf{e}' \otimes 1) \otimes 1} & \mathsf{S}(C \otimes \mathsf{S}'(\mathsf{S}\mathsf{S}'B \otimes C) \otimes \mathsf{S}'A) \otimes C \\
 & & & & \downarrow{\scriptstyle \cong} \\
 & & & & \mathsf{S}(C \otimes \mathsf{S}'\mathsf{S}(\mathsf{S}'(\mathsf{S}\mathsf{S}'B \otimes C) \otimes \mathsf{S}'A)) \otimes C \\
{\scriptstyle \partial_l}\downarrow & & & & \downarrow{\scriptstyle \mathsf{e}} \\
 & & & & \mathsf{S}(\mathsf{S}'(\mathsf{S}\mathsf{S}'B \otimes C) \otimes \mathsf{S}'A) \\
 & & & & \downarrow{\scriptstyle \cong} \\
A \parr (B \otimes C) & & \xleftarrow{\qquad\cong\qquad} & & A \parr (\mathsf{S}\mathsf{S}'B \otimes C)
\end{array}$$

$$\begin{array}{ccccc} \mathsf{S}A \otimes A & \xrightarrow{\cong} & \mathsf{S}(A \otimes \mathsf{S}'\mathsf{S}\top) \otimes A & \xrightarrow{\mathsf{e}} & \mathsf{S}\top \\ & \searrow^{\alpha} & & & \downarrow{\cong} \\ & & & & \bot \end{array} \qquad \begin{array}{ccccc} A \otimes \mathsf{S}'A & \xrightarrow{\cong} & A \otimes \mathsf{S}'(\mathsf{S}\mathsf{S}'\top \otimes A) & \xrightarrow{\mathsf{e}'} & \mathsf{S}'\top \\ & \searrow^{\alpha'} & & & \downarrow{\cong} \\ & & & & \bot \end{array}$$

$$\begin{array}{ccccc} \top & \xrightarrow{\cong} & \mathsf{S}\mathsf{S}'\top & \xrightarrow{\mathsf{S}(\mathsf{e}')} & \mathsf{S}\left(\mathsf{S}'\mathsf{S}A \otimes \mathsf{S}'(\mathsf{S}\mathsf{S}'\top \otimes \mathsf{S}'\mathsf{S}A)\right) \\ & & & & \downarrow{\cong} \\ & \searrow^{\beta} & & & \mathsf{S}\left(\mathsf{S}'\mathsf{S}A \otimes \mathsf{S}'A\right) \\ & & & & \downarrow{\cong} \\ & & & & A \parr \mathsf{S}A \end{array}$$

$$\begin{array}{ccccc} \top & \xrightarrow{\cong} & \mathsf{S}'\mathsf{S}\top & \xrightarrow{\mathsf{S}'(\mathsf{e})} & \mathsf{S}'\left(\mathsf{S}(\mathsf{S}\mathsf{S}'A \otimes \mathsf{S}'\mathsf{S}\top) \otimes \mathsf{S}\mathsf{S}'A\right) \\ & & & & \downarrow{\cong} \\ & \searrow^{\beta'} & & & \mathsf{S}'\left(\mathsf{S}A \otimes \mathsf{S}\mathsf{S}'A\right) \\ & & & & \downarrow{\cong} \\ & & & & \mathsf{S}'A \parr A \end{array}$$

This implies that ∂_l, ∂_r, α, β, α', β' are all composites of T-algebra morphisms, and are therefore T-algebras morphisms themselves. And hence, we indeed have a linearly distributive monad with negation. ∎

In terms of $*$-autonomous categories and $*$-autonomous monads, our theorem can be stated as follows.

5.9. THEOREM. *Let* $(\mathbb{X}, \otimes, \top, \mathsf{S}, \mathsf{S}')$ *be a* $*$*-autonomous category, and* $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1)$ *be a Hopf monad on* $(\mathbb{X}, \otimes, \top)$ *with a* T*-algebra structure* $\mathsf{n}_1 : \mathsf{T}\bot \to \bot$ *on* $\bot = \mathsf{S}\top$*. Then, with natural transformations* ν, ν' *defined as above,* $(\mathsf{T}, \mu, \eta, \mathsf{m}_2, \mathsf{m}_1, \nu, \nu')$ *is a* $*$*-autonomous monad on* $(\mathbb{X}, \otimes, \top, \mathsf{S}, \mathsf{S}')$*.*

Let us conclude this paper with a few examples.

5.10. EXAMPLE. An autonomous category can be seen as a linearly distributive category with negation with $\otimes = \parr$ and $\top = \bot$. In this "compact" case, as Pastro observed [9], the notions of Hopf monads and linearly distributive monads with negation coincide: there is an algebra structure on $\bot = \top$ given by the comonoidality $\mathsf{T}\top \to \top$.

5.11. EXAMPLE. More generally, when $\top \cong \bot$, the notions of Hopf monads and linearly distributive monads with negation coincide (apply Lemma 5.6 to the comonoidality $\mathsf{T}\top \to \top$ to get an algebra structure on $\bot$). Therefore, on isoMIX categories (in the sense of Cockett and Seely [7]), Hopf monads are the same as linearly distributive monads with negation.

5.12. EXAMPLE. Suppose that H is a Hopf algebra with invertible antipode in a symmetric linearly distributive category with negation (or symmetric $*$-autonomous category). The monad $\mathsf{T} = H \otimes (-)$ is a Hopf monad [5], and every object A has a trivial T-algebra structure (H-module structure) $H \otimes A \to A$ induced by the counit $H \to \top$ of the Hopf algebra. In particular, the dualizing object $\bot$ has a T-algebra structure. It follows that T is a linearly distributive monad with negation, and the linearly distributive structure with negation is lifted to category of T-algebras (or H-modules).

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