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A NOTE ON THE CATEGORICAL CONGRUENCE DISTRIBUTIVITY

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ABSTRACT. Having given a characterization of the categorical congruence modularity getting rid of the assumption that the ground category is regular in [10], we give now a characterization of the categorical congruence distributivity. We have a look as well at the case where the congruence distributivity is only involved, in some sense, for a subclass Γ of equivalence relations.

Introduction

In the varietal context of Universal Algebra, the notions of congruence modularity and congruence distributivity are quite classical, see among others [20], [24], [25], [17]. Similar categorical notions have been investigated in the context of Mal'tsev and Goursat regular categories in [14], [13], [23], [3], [4], [19] and [6] where there is a natural notion of suprema of pairs of equivalence relations. Later on was introduced in [10] a condition which guarantees the existence of suprema of pairs of equivalence relations without any regularity assumption on the ground category $\mathbb{E}$. This condition led to a characterization of the categorical congruence modular formula in a non-regular context.

The purpose of this article is twofold. First: congruence distributivity being a particular case of congruence modularity, we shall think afresh about a characterization of the categorical congruence distributive formula in a non-regular context. It will be given by the stability of binary infima under *cocartesian maps* in the category $Equ\mathbb{E}$ (of equivalence relations in $\mathbb{E}$) with respect to the forgetful functor $Equ\mathbb{E} \to \mathbb{E}$ to the ground category. We shall show that, in this case, *any internal groupoid is necessarily an equivalence relation*, as it is the case in the varietal context.

Secondly, the existence of suprema of equivalence relations $R \vee S$ keeps a full meaning when it is restricted to some subclass Γ of equivalence relations R, giving rise to Γ-modular and Γ-distributive formula indexed by this class Γ. So that we shall investigate what is remaining in this partial context of the global characterization of the modular and distributive formula.

Regular Σ-Mal'tsev categories in the sense of [9] will be shown to be examples of such Σ-modular categories. Among them, there are the categories Mon of monoids and SRg

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of semi-rings, when Σ is chosen to be the class of Schreier split epimorphisms as defined in [11], see Section 3.8. The variety of Quandles, [18] [22], will be another interesting example of the validity of such a Σ-modularity, when the class Σ in question is the class of acupuncturing split epimorphisms as defined in [7], again see Section 3.8.

The category $BoSrg$ of boolean semi-rings, again when Σ is chosen to be the class of Schreier split epimorphisms, gives rise to an example of Σ-distributive formula. In this restricted Σ-context, we only have: *any Σ-groupoid is necessarily a Σ-equivalence relation.*

A very interesting situation will emerge with the category Rg of commutative rings. On the one hand, as any regular Mal'tsev category, it is congruence modular.

Now consider the reflection $I : Rg \to BoRg$ towards the boolean rings and the class Σ of I-cartesian split epimorphisms $(f, s) : X \rightleftarrows Y$ in Rg, namely those which are such that the following square is a pullback:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & I(X') \arrow[d, "I(f)"] \\ Y \arrow[r, "\eta_X"'] & I(Y) \end{tikzcd}$$

Then, on the other hand, the category Rg will be shown to produce a congruence distributive formula when it is restricted to the only subclass of Σ-equivalence relations.

1. Suprema of pairs of equivalence relations

1.1. Congruence modular and congruence distributive varieties. A variety of Universal Algebra $\mathbb{V}$ is congruence modular when any of its algebras is such that its lattice of congruences is modular, namely such that:

$$(R \vee S) \wedge T) = R \vee (S \wedge T);\ \text{provided that : R} \subset \text{T}$$

it is congruence distributive when this same lattice is distributive, namely such that any of the two equivalent conditions holds:

$$T \vee (R \wedge S) = (T \vee R) \wedge (T \vee S)\ ;\ T \wedge (R \vee S) = (T \wedge R) \vee (T \wedge S)$$

It is well known that the distributive formula implies de modular one.

1.2. Existence of suprema of pairs of equivalence relations. We shall be interested in the categorical aspect of these conditions. Let $\mathbb{E}$ be a finitely complete category. We shall use the simplicial notations to describe any equivalence relation R on an object X in $\mathbb{E}$, as on the left hand side, or any internal groupoid $\underline{X}_1$ in $\mathbb{E}$, as on the right hand side,:

$$\begin{tikzcd} R \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] & X \arrow[l, "s_0^R" description] \end{tikzcd} \quad ; \quad \begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description] \end{tikzcd}$$

We shall denote by $Equ\mathbb{E}$ the category of internal equivalence relations in $\mathbb{E}$ and by $(\ )_0 : Equ\mathbb{E} \to \mathbb{E}$ the forgetful functor associating with any equivalence relation R on X its ground object X. It is a fibration whose cartesian maps are produced by the inverse images of equivalence relations. Any of its fibers is a preorder. As usual, we shall denote by $R[h]$ the kernel equivalence relation of a map h. Let us recall from [10] the following:

1.3. Proposition. *Let $\mathbb{E}$ be a finitely complete category. The following conditions are equivalent:*
1) any pair of equivalence relations (R, S) has a supremum $R \bigvee S$ in $Equ\mathbb{E}$
2) above any split epimorphism $(f, s) : X \rightleftarrows Y$ in $\mathbb{E}$ there is a cocartesian map (and hence a regular epimorphism) $(f, \hat{f}) : W \to T$ in $Equ\mathbb{E}$ whichever is its domain W above X.

Under these conditions, the map $(f, \hat{f}) : R \to T$ is cocartesian above (f, s) if and only if $f^{-1}(T) = R[f] \vee R$, while $R \vee S$ is obtained in the following way: first take the inverse image $(d_0^R)^{-1}(S)$, then take the cocartesian map above the split epimorphism (d_1^R, s_0^R) having this domain; the codomain of this cocartesian map gives you $R \vee S$:

$$\begin{tikzcd}
 & & S \arrow[ddl, shift right=2, "d_0^S"'] \arrow[ddl, shift left=2, "d_1^S"] \\
(d_0^R)^{-1}(S) \arrow[urr, bend left, "\check{d}_0^R"] \arrow[r, dotted, "\bar{d}_1"] \arrow[d, shift right=2, "\bar{\delta}_0^R"'] \arrow[d, shift left=2, "\bar{\delta}_1^R"] & R \vee S \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2] & \\
R \arrow[u] \arrow[r, "d_1^R"] & X \arrow[l, dotted, bend left, "s_0^R"] \arrow[u] \arrow[uur] &
\end{tikzcd}$$

From now on, we shall denote by $f_!(R)$ *this cocartesian image $T = R \vee S$ of R along the split epimorphism* (f, s), and by $f_!^R : R \twoheadrightarrow f_!(R)$ the induced cocartesian map in $Equ\mathbb{E}$.

1.4. Corollary. *Given any category $\mathbb{E}$ and a morphism of equivalence relations:*

$$\begin{tikzcd}
R \arrow[r, two heads, "\hat{f}"] \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] & T \arrow[d, shift right=2, "d_0^T"'] \arrow[d, shift left=2, "d_1^T"] \\
X \arrow[u] \arrow[r, two heads, "f"] & Y \arrow[u] \arrow[l, dotted, bend left, "s"]
\end{tikzcd}$$

where f is a split epimorphism in $\mathbb{E}$ and $\hat{f}$ an extremal epimorphism in $\mathbb{E}$, then it is a regular epimorphism in $Equ\mathbb{E}$ and we necessarily have $f^{-1}(T) = R[f] \vee R$.

Recall now that a Mal'tsev category is a finitely complete category in which any reflexive relation is an equivalence relation [14] and [15].

1.5. Proposition. *Suppose $\mathbb{E}$ is a Mal'tsev category with suprema of pairs of equivalence relations. Then a map $(f, \hat{f})$ in $Equ\mathbb{E}$ is cocartesian above the split epimorphism (f, s) if and only if the map $\hat{f}$ is an extremal epimorphism in $\mathbb{E}$.*

Proof. Consider such a cocartesian map in $Equ\mathbb{E}$, with a factorization of $\hat{f}$ through a monomorphism m in $\mathbb{E}$:

$$\begin{tikzcd}
 & U \arrow[dr, "m"] \arrow[dd, shift right] \arrow[dd, shift left] & \\
R \arrow[ur, "\phi"] \arrow[rr, "\hat{f}"'] \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] & & T \arrow[d, shift right=2, "d_0^T"'] \arrow[d, shift left=2, "d_1^T"] \\
X \arrow[u] \arrow[r, two heads, "f"'] & Y \arrow[uu, dotted] \arrow[r, equal] & Y \arrow[u]
\end{tikzcd}$$

Since f is a split epimorphism and m a monomorphism, the commutative upward square with maps (s_0^R, s_0^T) produces a factorization of s_0^T through U; this gives U a structure of reflexive relation on Y. The Mal'tsev condition makes it an equivalence relation. Since $(f, \hat{f})$ is cocartesian in $Equ\mathbb{E}$, the monomorphism m of equivalence relations is an isomorphism. ∎

1.6. Stability under slices and coslices. Suppose that $\mathbb{E}$ has suprema of pairs of equivalence relations. Then any slice category $\mathbb{E}/Y$ or coslice category $Y/\mathbb{E}$ have them.

Proof. Given any object $h : W \to Y$ in $\mathbb{E}/Y$, an equivalence relation R on W lies in $\mathbb{E}/Y$ if and only if $R \subset R[h]$. So, when R and S are in $\mathbb{E}/Y$, we get $R \vee S \subset R[h]$ and $R \vee S$ is in $\mathbb{E}/Y$.

Given any object $h : W \to Y$ in $W/\mathbb{E}$, any equivalence relation R on Y lies in $W/\mathbb{E}$ since its domain in $W/\mathbb{E}$ is necessarily $s_0^R.h$. ∎

1.7. Categorical modularity. The existence of suprema of pairs of equivalence relations being guaranteed, we can investigate the validity of the modular formula in full generality. Again from [10], recall:

1.8. Proposition. *Let $\mathbb{E}$ be a category with suprema of pairs of equivalence relations. The following conditions are equivalent:*
1) the modular formula holds
2) the cocartesian maps in $Equ\mathbb{E}$ above split epimorphisms in $\mathbb{E}$ are stable under pullbacks along maps in the fibres of the functor $(\ )_0$.

A category satisfying all the conditions of the previous proposition is said to be *cc-modular* (i.e. categorically congruence modular).

1.9. Proposition. *Let $\mathbb{E}$ be a Mal'tsev category with suprema of pairs of equivalence relations. It is cc-modular as soon as extremal epimorphisms in $\mathbb{E}$ are stable under pullbacks along monomorphisms.*

Proof. Under this assumption and according to Proposition 1.5, the cocartesian morphisms in $Equ\mathbb{E}$ above split epimorphisms in $\mathbb{E}$ are clearly stable under pullbacks along maps in the fibers of $(\ )_0$. ∎

Whence the immediate:

1.10. Corollary. *Any regular Mal'tsev category is cc-modular.*

This result is well-known from the introduction of regular Mal'tsev categories in [14], but, here, we get to it only with purely categorical proofs and without any use of the regular embedding theorem [1].

1.11. Categorical modularity and shifting property. In [17], Gumm characterized the congruence modular varieties by the validity of the *Shifting Lemma*: given any triple (R, S, T) of equivalence relations such that $S \cap T \subset R$ the following left hand side situation implies the right hand side one:

$$\begin{tikzcd}
x \arrow[r, dotted, "S"] \arrow[d, dotted, "T"'] \arrow[d, dotted, bend right=60, "R"'] & y \arrow[d, dotted, "T"] & & y \arrow[d, dotted, bend left=60, "R"] \\
x' \arrow[r, dotted, "S"'] & y' & & y'
\end{tikzcd}$$

The categorical description of this shifting property was given in [12] and is the following one: given any triple (R, S, T) of equivalence relations on X such that $S \cap T \subset R \subset T$, the following morphism of equivalence relations is fibrant, i.e. any commutative square is a pullback:

$$\begin{tikzcd}
S\square R \arrow[r, shift left=2, "\delta_1^R"] \arrow[r, shift right=2, "\delta_0^R"'] \arrow[d, tail, "S\square i"'] & R \arrow[l] \arrow[d, tail, "i"] \\
S\square T \arrow[r, shift left=2, "\delta_1^T"] \arrow[r, shift right=2, "\delta_0^T"'] & T \arrow[l]
\end{tikzcd}$$

A category satisfying the shifting property for any such triple of equivalence relations was called a Gumm category in [5]. If $\mathbb{T}$ is any algebraic theory in the sense of Universal algebra, let us denote by $\mathbb{V}(\mathbb{T})$ the corresponding variety of $\mathbb{T}$-algebras and by $\mathbb{T}(\mathbb{E})$ the category of internal $\mathbb{T}$-algebras in the finitely complete category $\mathbb{E}$. There is a natural dotted factorization of the Yoneda embedding Y making the following square a pullback and the functor $Y_{\mathbb{E}}$ a left exact conservative fully faithful functor (which consequently reflects pullbacks).

$$\begin{tikzcd}
\mathbb{T}(\mathbb{E}) \arrow[r, dotted, "Y_{\mathbb{E}}"] \arrow[d, "\mathcal{U}_{\mathbb{T}}"'] & \mathcal{F}(\mathbb{E}^{op}, \mathbb{V}(\mathbb{T})) \arrow[d, "\mathcal{F}(\mathbb{E}^{op}, \mathcal{U})"] \\
\mathbb{E} \arrow[r, "Y"'] & \mathcal{F}(\mathbb{E}^{op}, Set)
\end{tikzcd}$$

By this factorization $Y_{\mathbb{E}}$, the Gumm characterization guarantees that *the category* $\mathbb{T}(\mathbb{E})$ *is a Gumm category as soon as* $\mathbb{V}(\mathbb{T})$ *is a congruence modular variety.* In the categorical context, we only get the implication in one direction, see Proposition 2.11 in [10]:

1.12. Proposition. *Any cc-modular category $\mathbb{E}$ is a Gumm category.*

The converse is not true in general: indeed, any category $\mathbb{T}(\mathbb{E})$ of the previous kind is not necessarily provided with suprema of pairs of equivalence relations, and therefore is not in a position to satisfy the modular formula.

1.13. Internal groupoids. Let us briefly recall our presentation of internal groupoids which comes from the observation that the category $Grd\mathbb{E}$ of internal groupoids in $\mathbb{E}$ is monadic above the category $Pt\mathbb{E}$ of split epimorphisms in $\mathbb{E}$ [2]. An internal groupoid $\underline{X}_1$ is a internal reflexive graph, as on the right hand side:

$$\begin{tikzcd}
R[d_0] \arrow[r, shift left=3, "d_0^R"] \arrow[r, shift right=3, "d_1^R"'] \arrow[r, bend right=50, "d_2"'] & X_1 \arrow[l, "s_0^R"' description] \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

such that $R[d_0]$ is endowed with a map d_2, satisfying all the simplicial identities up to level 3, when the diagram is completed by $R[d_0^R]$. Then necessarily all the commutative squares of the induced 3-truncated simplicial object diagram are *pullbacks*. In set-theoretical terms, d_2 is defined by $d_2(\phi, \psi) = \psi.\phi^{-1}$ for any pair (ϕ, ψ) of morphisms in X_1 having same domain.

2. Categorical congruence distributivity

In a similar way, let us introduce the following:

2.1. Definition. *Let $\mathbb{E}$ be a finitely complete category. We shall say it is cc-distributive ("categorically congruence distributive") when it has suprema of pairs of equivalence relations and when the equivalent conditions of Section 1.1 holds for any triple (R, S, T) of equivalence relations. We shall say that it is weakly cc-distributive when only the following implication holds:*

$$(T \wedge R = \Delta_X \text{ and } T \wedge S = \Delta_X) \Rightarrow T \wedge (R \vee S) = \Delta_X$$

where Δ_X denotes the discrete equivalence relation.

The following categories are examples of cc-distributive categories: $BoRg$ of boolean rings, $VNRg$ of von Neumann regular rings, $Heyt$ of Heyting Algebras, the dual $\mathbb{E}^{op}$ of any topos, and in particular the dual Set^{op} of the category of sets. According to the equivalent conditions of Section 1.1, any cc-distributive category is weakly cc-distributive.

2.2. Weakly cc-distributive categories. The pertinence of the second definition is based upon the three following strong results:

2.3. Theorem. *In a weakly cc-distributive category, any internal groupoid is an equivalence relation.*

Proof. Let us show first that any internal group structure in $\mathbb{E}/Y$ on the object $f : X \to Y$ is trivial, namely such that f is an isomorphism. Such a structure is actually a special case of groupoid structure:

$$\begin{tikzcd} X \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y \arrow[l, "s_0" description] \end{tikzcd}$$

with $d_0 = f = d_1$; we shall set $s_0 = s$ for the splitting. Accordingly the following square is a pullback of split epimorphisms:

$$\begin{tikzcd} R[f] \arrow[r, shift left, "d_2"] \arrow[d, shift right, "d_0^f"'] & X \arrow[l, shift left, "s_1"] \arrow[d, shift right, "f"'] \\ X \arrow[u, shift right, "s_0^f"'] \arrow[r, shift left, "d_1^f"] & Y \arrow[l, shift left, "s"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

where d_2 is the division map of this group. Consequently the pair (d_2, d_0^f) is jointly monic and we get $R[d_2] \wedge R[d_0^f] = \Delta_{(R[f])}$ (1). We get $R[d_2] \wedge R[d_1^f] = \Delta_{(R[f])}$ (2) by duality. Finally the following commutative square:

$$\begin{tikzcd} R[f] \arrow[r, "d_1^f"] \arrow[d, shift right, "d_0^f"'] & X \arrow[d, shift right, "f"'] \\ X \arrow[u, shift right, "s_0^f"'] \arrow[r, "f"'] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

gives us $R[d_0^f] \vee R[d_1^f] = R[f.d_0^f]$ by Corollary 2.2 in [10]; so, since we have $f.d_2 = f.d_0^f$, we get $R[d_2] \wedge (R[d_0^f] \vee R[d_1^f]) = R[d_2]$ (3), the map d_2 being a binary operation in $\mathbb{E}/Y$. By (1), (2) and (3), the weak cc-distributivity makes $R(d_2) = \Delta_{(R[f])}$, and d_2 a monomorphism; being split it is an isomorphism. Now the first pullback above makes d_1^f an isomorphism. Accordingly the split epimorphism f is a monomorphism as well, and consequently an isomorphism.

Now, let us consider any internal groupoid structure:

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description] \end{tikzcd}$$

and the kernel equivalence relation of the morphism (d_0, d_1):

$$\begin{tikzcd} R[(d_0,d_1)] \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] & X_1 \arrow[l, "s_0^R" description] \arrow[r, "{(d_0,d_1)}"] & X_0 \times X_0 \end{tikzcd}$$

We are going to show that there is an internal group structure on the object d_0^R in $\mathbb{E}/X_1$; this will prove that $(d_0, d_1) : X_1 \to X_0 \times X_0$ is a monomorphism, and consequently that the groupoid in question is actually an equivalence relation.

In set-theoretic terms, $R[(d_0, d_1)]$ is the set of parallel pairs of the groupoid. Given any pair $(\phi,\chi),(\phi,\psi)$ in $R[(d_0, d_1)]$ with same image by d_0^R, the multiplication of the group structure is given by:

$$(\phi,\chi) * (\phi,\psi) = (\phi, \psi\phi^{-1}\chi)$$

Clearly its unit element is (ϕ,ϕ), while the inverse of (ϕ,ψ) is $(\phi,\phi\psi^{-1}\phi)$. Notice that the definition of the unit map is precisely s_0^R. ■

The two other results are dealing with the Mal'tsev context:

2.4. Proposition. *Given any regular Mal'tsev category $\mathbb{E}$, the following conditions are equivalent:*
1) $\mathbb{E}$ is weakly cc-distributive
2) any unital fibre $Pt_Y\mathbb{E}$ has no non-trivial abelian object
3) any internal groupoid in $\mathbb{E}$ is an equivalence relation
4) given any regular epimorphism $f : X \twoheadrightarrow Y$, and any pair (R,S) of equivalence relations on X, we get:

$$(R[f] \wedge R = \Delta_X \text{ and } R[f] \wedge S = \Delta_X) \Rightarrow f(R \wedge S) = f(R) \wedge f(S)$$

Proof. It is just Theorem 3.9 in [3], enriched with Theorem 3.1 and Theorem 3.4 in [6]. ■

2.5. Proposition. *Given any exact Mal'tsev category $\mathbb{E}$, the following conditions are equivalent:*
1) $\mathbb{E}$ is cc-distributive
2) $\mathbb{E}$ is weakly cc-distributive.

Proof. It is Theorem 3.2 in [3], according to the previous proposition. ■

This last equivalence is valid in particular for any Mal'tsev variety $\mathbb{V}$.

2.6. Characterization of the cc-distributive categories. The cc-distributivity can be very easily characterized by the following:

2.7. Theorem. *Given any category $\mathbb{E}$ with suprema of pairs of equivalence relations, the following conditions are equivalent:*
1) $\mathbb{E}$ is cc-distributive
2) given any split epimorphism $(f,s) : X \rightleftarrows Y$ in $\mathbb{E}$, the cocartesian images above it preserves binary infima, namely: given any pair (S,T) of equivalence relations on X, we get:

$$f_!(S \wedge T) = f_!(S) \wedge f_!(T)$$

Proof. On the one hand, we noticed that $R \vee S = (d_1^R)_!((d_0^R)^{-1}(S))$. So that condition 2) applies to the split epimorphism (d_1^R, s_0^R) and $S \wedge T$ produces the distributive condition since the inverse image $(d_0^R)^{-1}$ preserves intersections as well.

Conversely, when f is split, the inverse image $f^{-1} : Equ_Y\mathbb{E} \to Equ_X\mathbb{E}$ between the fibers is monomorphic; so that $f_!(S \wedge T) = f_!(S) \wedge f_!(T)$ is equivalent to $f^{-1}(f_!(S \wedge T)) = f^{-1}(f_!(S) \wedge f_!(T)) = f^{-1}(f_!(S)) \wedge f^{-1}(f_!(T))$ which is just: $R[f] \vee (S \wedge T) = (R[f] \vee S) \wedge (R[f] \vee T)$. ■

2.8. Main consequences of the cc-distributivity. By Theorem 2.3, the first consequence is that *in any cc-distributive category* $\mathbb{E}$ *the only groupoids are the equivalence relations.*

On the other hand, we recalled that the distributive formula implies the modular one. From that we immediately get a second important consequence:

2.9. Proposition. *Suppose* $\mathbb{E}$ *is cc-distributive. Then it is cc-modular, and consequently the cocartesian maps in* $Equ\mathbb{E}$ *above split epimorphisms in* $\mathbb{E}$ *are stable under pullbacks along maps in the fibers of the fibration* $(\ )_0$.

Proof. It is a straightforward consequence of Proposition 1.8. ∎

3. Partial cc-modularity and cc-distributivity

In this section, we shall investigate the situation where there are only cocartesian maps above a certain class Σ of split epimorphisms.

3.1. Σ-modularity.

3.2. Definition. *Given any class* Σ *of split epimorphisms in* $\mathbb{E}$*, a graph* X_1 *on an object* X *will be said to be a* Σ*-graph when it is reflexive*

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X \quad (s_0 : X \to X_1)$$

and such that the split epimorphism (d_0, s_0) *belongs to the class* Σ*. Similar definitions can be given for the notions of* Σ*-relations,* Σ*-categories and* Σ*-groupoids.*

A morphism $f : X \to Y$ *is called* Σ*-special when its kernel relation* $R[f]$ *is a* Σ*-equivalence relation. An object* X *is said to be* Σ*-special when the terminal map* $\tau_X : X \to 1$ *is* Σ*-special.*

3.3. Proposition. *Suppose* $\mathbb{E}$ *is endowed with a class* Σ *of split epimorphisms and is such that, given any* Σ*-equivalence relation* R*, there is a cocartesian map in* $Equ\mathbb{E}$ *above the split epimorphism* (d_1^R, s_0^R)*. Then* $R \vee S$ *does exist for any pair* (R, S) *of equivalence relations, provided that* R *is a* Σ*-equivalence relation. Moreover the modular formula holds for* R*:*

$$(R \vee S) \wedge T) = R \vee (S \wedge T);\ \text{provided that : R} \subset \text{T}$$

as soon as these cocartesian maps are stable under pullbacks along maps in the fibers of $(\ )_0$.

Proof. The first point is a consequence of Proposition 2.4 in [10], while the second one is a consequence of Proposition 2.7. ∎

We shall need the following specifications where $\Sigma\mathbb{E}$ denotes the full subcategory of $Pt\mathbb{E}$ whose objects are the split epimorphisms in Σ:

3.4. Definition. *The class Σ is said to be:*
1) fibrational *when Σ is stable under pullbacks and contains the isomorphisms;*
2) point-congruous *when, in addition, $\Sigma\mathbb{E}$ is stable under finite limits in $Pt\mathbb{E}$.*

Suppose the class Σ is fibrational. Then any Σ-special split epimorphism is in Σ. For that consider the following pullback:

$$\begin{tikzcd}
R[f] \arrow[d, shift right, "d_0^f"'] & X \arrow[l, "s_1"'] \arrow[d, shift right, "f"'] \\
X \arrow[u, shift right, "s_0^f"'] & Y \arrow[l, "s"] \arrow[u, shift right, "s"']
\end{tikzcd}$$

The converse is not true in general. However any Σ-groupoid $\underline{X}_1$ is such that (d_0, s_0) is actually a Σ-special split epimorphism. For that consider the following diagram which is a pullback of split epimorphisms:

$$\begin{tikzcd}
R[d_0^R] \arrow[r, shift left, "d_2"] \arrow[d, shift right, "d_0^R"'] & X_1 \arrow[l, shift left, "s_1^R"] \arrow[d, shift right, "d_0"'] \\
X_1 \arrow[r, shift left, "d_1"] \arrow[u, shift right, "s_0^R"'] & X_0 \arrow[l, shift left, "s_0"] \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

3.5. Theorem. *Suppose $\mathbb{E}$ is endowed with a fibrational class Σ of split epimorphisms. The following conditions are equivalent:*
1) any pair (R, S) of equivalence relations admits a supremum in $Equ\mathbb{E}$, provided that R is a Σ-equivalence relation;
2) above any Σ-special split epimorphism $(f, s) : X \rightleftarrows Y$ in $\mathbb{E}$, there is a cocartesian map (and hence a regular epimorphism) $(f, \hat{f}) : W \to T$ in $Equ\mathbb{E}$ above f whichever is its domain W above X.

The modular formula holds for any Σ-equivalence relation R if and only if these cocartesian maps are stable under pullbacks along maps in the fibers of the fibration $(\ )_0$.

Proof. Suppose 1) and consider any Σ-special split epimorphism $(f, s) : X \rightleftarrows Y$ in $\mathbb{E}$. Accordingly $R[f]$ is a Σ-equivalence relation. Given any equivalence relation S on X, the suprema $R[f] \vee S$ does exist according to the previous proposition and the following construction produces the desired cocartesian map above f:

$$\begin{tikzcd}
S \arrow[r, tail] \arrow[drr, shift right, "d_0^S"'] \arrow[drr, shift left] \arrow[rrr, bend left, "\hat{f}"] & R[f] \bigvee S \arrow[rr] \arrow[dr, shift right=2, "d_0"'] \arrow[dr, shift left=2, "d_1"] & & V \arrow[ll, shift left, dotted] \arrow[d, shift right=2, "d_0^V"'] \arrow[d, shift left=2, "d_1^V"] \\
& & X \arrow[ull] \arrow[ul] \arrow[r, "f"] & Y \arrow[l, shift left, dotted, "s"] \arrow[u]
\end{tikzcd}$$

where we have $f^{-1}(V) = R[f] \bigvee S$ or equivalently $V = s^{-1}(R[f] \bigvee S)$.

Conversely suppose 2). Starting with a Σ-equivalence relation R, we noticed that (d_1^R, s_0^R) is actually a Σ-special split epimorphism. So, there are cocartesian maps above it, and we get $R \vee S$ for any equivalence relation S by the construction described below Proposition 1.3.

The last point of the theorem is given by the proof of Proposition 2.7 in [10], since $R[f]$ is a Σ-equivalence relation and consequently the modular formula holds for $R[f]$. ∎

3.6. Definition. *When all the conditions (including the last one) of this theorem hold, we shall say that* $\mathbb{E}$ *is* Σ-modular.

Let us recall from [9] that a Σ-Mal'tsev category is a category $\mathbb{E}$ in which any square of split epimorphisms:

$$\begin{tikzcd}
X' \arrow[d, shift right, "f'"'] \arrow[r, shift right, "\bar{g}"'] & X \arrow[l, shift right, "\bar{t}"'] \arrow[d, shift right, "f"'] \\
Y' \arrow[u, shift right, "s'"'] \arrow[r, shift right, "g"'] & Y \arrow[l, shift right, "t"'] \arrow[u, shift right, "s"']
\end{tikzcd}$$

is such that the factorization ϕ towards the pullback of f along g is an extremal epimorphism as soon as the split epimorphism (f, s) is in Σ (an extremal epimorphism being a morphism $\phi : U \to V$ such that any decomposition $\phi = m\psi$ with a monomorphic m makes m an isomorphism). When $\mathbb{E}$ is regular, it is equivalent to saying that the rightward and downward square is a regular pushout.

A category $\mathbb{E}$ is Σ-protomodular category [8] when any pullback of split epimorphisms:

$$\begin{tikzcd}
X' \arrow[d, shift right, "f'"'] \arrow[r, "\bar{g}"] & X \arrow[d, shift right, "f"'] \\
Y' \arrow[u, shift right, "s'"'] \arrow[r, "g"'] & Y \arrow[u, shift right, "s"']
\end{tikzcd}$$

makes the pair $(s, \bar{g})$ jointly strongly epic as soon as the split epimorphism (f, s) is in Σ. Any Σ-protomodular category is a Σ-Mal'tsev one.

3.7. Proposition. *Any regular* Σ*-Mal'tsev category* $\mathbb{E}$ *with a fibrational class* Σ *of split epimorphisms is* Σ*-modular.*

Proof. In this case, the direct image along a Σ-special regular map f of any equivalence relation is an equivalence relation as well [9]. So this regular direct image coincides with the cocartesian image and produces the cocartesian map above f in $Equ\mathbb{E}$. Then:
1) there are cocartesian maps above Σ-special split epimorphisms and, according to the previous theorem, there are suprema of pairs (R, S), provided that R is a Σ-equivalence relation.
2) this same observation on direct images means that, when f is a Σ-special regular epimorphism, the cocartesian map above it is a level wise regular epimorphisms in $\mathbb{E}$. As such, the ground category $\mathbb{E}$ being regular, the cocartesian maps in $Equ\mathbb{E}$ above Σ-special

regular epimorphisms in $\mathbb{E}$ are stable under pullbacks along any map in $Equ\mathbb{E}$. So, in particular, $\mathbb{E}$ is Σ-modular. ■

Accordingly the following categories are examples of Σ-modular categories.

3.8. Example. 1) The category Mon of monoids and the category SRg of semi-rings are Σ-protomodular categories (and thus Σ-Mal'tsev categories) with respect to the class of Schreier split epimorphisms [11], namely those split epimorphisms (f, s) which are such that, for every $y \in Y$, the map $\mu_Y : \mathrm{Ker} f \to f^{-1}(y)$ defined by $\mu_Y(k) = k.s(y)$ (resp. by $\mu_Y(k) = k + s(y)$) is bijective.
2) More generally consider any Jónsson-Tarski variety $\mathbb{V}$, i.e. a variety whose corresponding theory has a unique constant 0 and a binary term + satisfying $x + 0 = x = 0 + x$. The same definition of Schreier split epimorphisms as above makes sense in $\mathbb{V}$. Then $\mathbb{V}$ is Σ-protomodular for this class Σ of split epimorphisms, see [21].
3) A quandle is a set X endowed with a binary operation $\triangleright : X \times X \to X$ which is idempotent and such that for any object x the translation $- \triangleright x : X \to X$ is an automorphism with respect to the binary operation $\triangleright$; its inverse is denoted by $- \triangleright^{-1} x$. A homomorphism of quandles is an application $f : (X, \triangleright) \to (Y, \triangleright)$ which respects the binary operations. This defines the variety Qnd of quandles. The notion was independently introduced in [18] and [22] in strong relationship with Knot Theory.

Any group G produces a quandle with the binary operation $x \triangleright y = yxy^{-1}$. A set endowed with the binary operation given by the first projection is a quandle, so that Set appears as subvariety of Qnd.

In [7] a split epimorphism $(f, s) : X \rightleftarrows Y$ in Qnd was called an *acupuncturing* split epimorphism when, for any element $y \in Y$, the application $s(y) \triangleleft - : f^{-1}(y) \to f^{-1}(y)$ is bijective. The class Σ of acupuncturing split epimorphisms was shown to be point-congruous and the category Qnd to be a Σ-Mal'tsev category (but not a Σ-protomodular one).
4) Define a *prequandle* as a set X endowed with an idempotent binary operation $\triangleright$ such that the application $- \triangleright x$ is bijective for any object $x \in X$ (the inverse operation being again denoted by $- \triangleright^{-1} x$), and denote $PrQnd$ the associated variety. The same definition of acupuncturing split epimorphisms obviously holds in $PrQnd$. It is clear that we have $Qnd \subset PrQnd$. Actually the same proofs as in [7] show that this class Σ of acupuncturing split epimorphisms is point-congruous and that the category $PrQnd$ is a Σ-Mal'tsev category, since these proofs do not use the "automorphism axiom".

3.9. Σ-modularity and Σ-shifting property. Mimicking what was done in [10] and recalled in Proposition 1.12, we are going to show that the Σ-modularity implies some kind of shifting property.

3.10. Definition. *Let Σ be a fibrational class of split epimorphisms in $\mathbb{E}$. We shall say that $\mathbb{E}$ is a Σ-Gumm category when, given any triple (R, S, T) of equivalence relations on*

X such that $S \cap T \subset R \subset T$, the following morphism of equivalence relations is fibrant:

$$\begin{tikzcd}
S\Box R \arrow[r, shift left=2, "\delta_1^R"] \arrow[r, shift right=2, "\delta_0^R"'] \arrow[d, tail, "S\Box i"'] & R \arrow[l] \arrow[d, tail, "i"] \\
S\Box T \arrow[r, shift left=2, "\delta_1^T"] \arrow[r, shift right=2, "\delta_0^T"'] & T \arrow[l]
\end{tikzcd}$$

provided that R is a Σ-equivalence relation.

3.11. Proposition. *Let Σ be a fibrational class of split epimorphisms in $\mathbb{E}$. Any Σ-modular category $\mathbb{E}$ is a Σ-Gumm category.*

Proof. Suppose that $\mathbb{E}$ is Σ-modular, that R is a Σ-equivalence relation and that we have $S \cap T \subset R \subset T$; then consider the following commutative diagram:

$$\begin{tikzcd}
S\Box R \arrow[rr, shift left=2, "\delta_1^R"] \arrow[rr, shift right=2, "\delta_0^R"'] \arrow[dr] \arrow[dd, tail] & & R \arrow[ll] \arrow[dr] \arrow[dd, tail] & \\
& S\Box(R\bigvee S) \arrow[rr, shift left=2] \arrow[rr, shift right=2] \arrow[dd, tail] & & R\bigvee S \arrow[ll] \arrow[dd, tail, "i\bigvee S"] \\
S\Box T \arrow[rr, shift left=2, "\delta_1^T"] \arrow[rr, shift right=2, "\delta_0^T"'] \arrow[dr] & & T \arrow[ll] \arrow[dr] & \\
& S\Box\nabla_X \arrow[rr, shift left=2] \arrow[rr, shift right=2] & & \nabla_X \arrow[ll]
\end{tikzcd}$$

The right hand side vertical square is a pullback by the Σ-modular law, and so is the left hand side vertical one since inverse image preserves intersections (remind that $S\Box R = (d_0^S)^{-1}(R) \cap (d_1^S)^{-1}(R)$ in the fiber $Equ_S\mathbb{E}$). The front morphism of equivalence relations is fibrant since both $R\bigvee S$ and ∇_X contain S, see Lemma 1.7 in [10]. Accordingly the back morphism of equivalence relations is fibrant as well. ∎

3.12. Σ-distributivity. We are now able to specify the following:

3.13. Proposition. *Suppose $\mathbb{E}$ is endowed with a fibrational class Σ of split epimorphisms and satisfies any of the equivalent conditions 1) or 2) of the previous theorem. Then the following conditions are equivalent:*
1) the following distributive formula holds:

$$R \vee (S \wedge T) = (R \vee S) \wedge (R \vee T)$$

provided that R is Σ-equivalence relation;
2) when (f, s) is a Σ-special split epimorphism, the cocartesian images above it preserve binary infima.
Such a category $\mathbb{E}$ is necessarily Σ-modular.

Proof. The proofs are exactly the same as in Theorem 2.7, once carefully checked that all the used ingredients do exist in the present partial context.

The last point can be checked in the same way. Suppose R is any Σ-equivalence relation satisfying $R \subset T$. Then we easily get:

$$R \vee (S \wedge T) = (R \vee S) \wedge (R \vee T) = (R \vee S) \wedge T$$

■

3.14. Definition. *A category* $\mathbb{E}$ *satisfying all the conditions of the previous theorem will be called* Σ-distributive.

3.15. Proposition. *The category BoSRg of boolean semi-rings is* Σ*-distributive with respect to the (point-congruous) class* Σ *of Schreier split epimorphisms.*

Proof. Recall that $(f, s) : X \rightleftarrows Y$ is a Schreier split epimorphism when for any $x \in X$, there is a unique $k \in Kerf$ such that $x = sf(x) + k$. So, a map $g : X \to Z$ is Σ-special if and only if $Kerg$ is a (boolean) ring. The category $BoSRg$ being Σ-Mal'tsev according to Proposition 3.7, the direct image of any equivalence relation along a surjective Σ-special morphism g is actually an equivalence relation. So, let (T, S) be a pair on equivalence relation on X. We have to show that $f(T) \wedge f(S) \subset f(T \wedge S)$. Suppose $y(f(T) \wedge f(S))y'$ in Y, so that there is a quadruple (x, x', t, t') in X such that xSx', tTt', $f(x) = f(t) = y$ and $f(x') = f(t') = y'$. Accordingly there is a $k \in Kerf$ such that $t = k + x$; so we get $(x + k)S(x' + k)$, namely $tS(x' + k)$.

Let us set $x" = x' + k$. We get $tSx"$ and tTt' with $f(x") = y' = f(t')$. Now there is another $\alpha \in Kerf$ such that $t' = x" + \alpha$. Consequently, on the one hand, we get $(t+\alpha t+\alpha t)S(x"+\alpha t+\alpha x'')$, namely $tS(x"+\alpha t+\alpha x")$, since $\alpha t+\alpha t = 0$ in the boolean ring $Kerf$. Moreover we get $f(x" + \alpha t + \alpha x'') = f(x") = y'$.

On the other hand we get $(t + \alpha t + \alpha t)T(x" + \alpha + \alpha(x" + \alpha) + \alpha t)$, namely $tT(x" + \alpha x" + \alpha t)$. In this way, we get $y(f(T \wedge S))y'$. ■

3.16. Proposition. *Let* $\mathbb{V}$ *be any Mal'tsev variety with a fixed Mal'tsev term* p *(i.e. satisfying* $p(x, y, y) = x = p(y, y, x)$*) and* $\mathbb{V}'$ *be any Birkhoff subvariety in which this same* p *satisfies the Pixley identity* $p(x, y, x) = x$ *[24]. Let* $I : \mathbb{V} \to \mathbb{V}'$ *be the induced reflection and* Σ *the class of* I*-cartesian split epimorphisms in* $\mathbb{V}$*. Then the class* Σ *is point-congruous and* $\mathbb{V}$ *is* Σ*-distributive.*

Proof. Since $\mathbb{V}$ is a Mal'tsev variety, the cocartesian maps above regular epimorphisms (=surjections) in $Equ\mathbb{V}$ are levelwise regular epimorphisms in $\mathbb{V}$. A map g in $\mathbb{V}$ is I-cartesian when the following square is a pullback:

$$\begin{tikzcd} X' \arrow[r, "\eta_{X'}"] \arrow[d, "g"'] & I(X') \arrow[d, "I(g)"] \\ X \arrow[r, "\eta_X"'] & I(X) \end{tikzcd}$$

Let us show that the class of I-cartesian split epimorphisms is fibrational.

First recall that $\mathbb{V}'$ being a Birkhoff subcategory, any projection $\eta_X : X \twoheadrightarrow I(X)$ is a regular epimorphism. Then consider the following diagram where the left hand side square is a pullback and (f, s) is I-cartesian:

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, shift right, "f'"'] & X \arrow[r, two heads, "\eta_X"] \arrow[d, shift right, "f"'] & I(X) \arrow[d, shift right, "I(f)"'] \\
Y' \arrow[r, "k"'] \arrow[u, shift right, "s'"'] & Y \arrow[r, two heads, "\eta_Y"'] \arrow[u, shift right, "s"'] & I(Y) \arrow[u, shift right, "I(s)"']
\end{tikzcd}$$

Accordingly, the right hand side square is a pullback, so that the whole rectangle is a pullback; now this rectangle is the following one as well:

$$\begin{tikzcd}
X' \arrow[r, two heads, "\eta_{X'}"] \arrow[d, shift right, "f'"'] & I(X') \arrow[r, "I(h)"] \arrow[d, shift right, "I(f')"'] & I(X) \arrow[d, shift right, "I(f)"'] \\
Y' \arrow[r, two heads, "\eta_{Y'}"'] \arrow[u, shift right, "s'"'] & I(Y') \arrow[r, "I(k)"'] \arrow[u, shift right, "I(s')"'] & I(Y) \arrow[u, shift right, "I(s)"']
\end{tikzcd}$$

Observe that the left hand square is a regular pushout since it is a levelwise regular epimorphism of split epimorphisms in a Mal'tsev category. According to Lemma 1.1 in [16], both squares are then pullbacks. So that (f', s') is I-cartesian and the pullback in question in $\mathbb{D}$ is preserved by I. The fact that the class Σ is point-congruous follows from the following observation: a split epimorphism (f, s) is in Σ if and only if it is the pullback of some map in $\mathbb{V}'$, which is a straightforward consequence of the fact that any map in $\mathbb{C}$ is I-cartesian and that the class Σ of I-cartesian split epimorphisms is fibrational.

Now let (T, S) be a pair of equivalence relations on the domain X of a I-cartesian split epimorphism (f, s). Let us show that $f(T) \wedge f(S) \subset f(T \wedge S)$. Suppose $y(f(T) \wedge f(S))y'$ in Y, so that there is a quadruple (x, x', t, t') in X such that xSx', tTt', $f(x) = f(t) = y$ and $f(x') = f(t') = y'$. Let us consider $\alpha = p(t, p(t, x, x'), t')$. We have $tSp(t, x, x')$, whence $\alpha St'$. We have also $f(p(t, x, x')) = f(t') = y'$ and $f(\alpha) = f(t) = y$.

We get also $\alpha Tp(t', p(t, x, x'), t')$, with $f(p(t', p(t, x, x'), t')) = p(y', y', y') = y'$. To conclude, it remains to show that $p(t', p(t, x, x'), t') = t'$. We know that their images by f are equal to y'. Since f is I-cartesian, its pullback definition implies now that we get $p(t', p(t, x, x'), t') = t'$ if and only if their images by η_X are equal as well. But the codomain of η_X is in $\mathbb{V}'$ which satisfies the Pixley identity for p. This guarantees the equality of these images. ∎

So $\mathbb{V}$ happens to be, on the one hand, a cc-modular category since it is a regular Mal'tsev one, which, on the other hand, appears to be Σ-distributive with respect to some specific class of Σ-equivalence relations.

This situation is the source of many examples. Take for instance the variety $\mathbb{V} = Rg$ of commutative rings with Mal'tsev term defined by $p(x, y, z) = (x-y)(z-y)+x-y+z$, and

$\mathbb{V}' = BoRg$ the subvariety of boolean rings where the previous term becomes $p(x, y, z) = x + xy + yz + zx + z$, which satisfies the Pixley identity in $BoRg$.

Similarly, you can choose $\mathbb{V}$ as a variety of R-algebras for a given ring R, while $\mathbb{V}'$ is the subvariety of its idempotent R-algebras.

3.17. Σ-distributivity and Σ-groupoids. The equivalence between the two conditions given at the beginning of Section 1.1 is broken if the first one is only valid when R is supposed to be a Σ-equivalence relation. However we can get the following:

3.18. Proposition. *Given any Σ-distributive category $\mathbb{E}$, then we get:*

$$T \wedge (R \vee S) = (T \wedge R) \vee (T \wedge S)$$

provided that $T \wedge R$ and S are Σ-equivalence relations.

Proof. We get: $(T \wedge R) \vee (T \wedge S) = ((T \wedge R) \vee T)) \wedge ((T \wedge R) \vee S)) = T \wedge ((T \wedge R) \vee S))$ since $T \wedge R$ is a Σ-equivalence relation. Moreover we get:
$T \wedge ((T \wedge R) \vee S)) = T \wedge ((T \vee S) \wedge (R \vee S)) = T \wedge (R \vee S)$ since S is a Σ-equivalence relation. ∎

Whence the following:

3.19. Definition. *Suppose Σ is a class of split epimorphisms and $\mathbb{E}$ is such that any pair (R, S) of equivalence relations admits a supremum in $Equ\mathbb{E}$, provided that S is a Σ-equivalence relation. We shall say that $\mathbb{E}$ is weakly Σ-distributive when the following implication holds:*

$$(T \wedge R = \Delta_X \ \text{ and } \ T \wedge S = \Delta_X) \Rightarrow T \wedge (R \vee S) = \Delta_X$$

provided that S is a Σ-equivalence relation.

The previous proposition shows that any Σ-distributive category $\mathbb{E}$ is a weakly Σ-distributive one.

3.20. Proposition. *Suppose that the class Σ is point-congruous and $\mathbb{E}$ is weakly Σ-distributive. Then the only Σ-groupoids are the Σ-equivalence relations.*

Proof. We shall re-read the proof of Theorem 2.3. As for the groups in $\mathbb{E}/Y$, we need that $R[d_0^f]$ is a Σ-equivalence relation, which means that f is Σ-special. This is the case as soon as (f, s) is in Σ and is underlying a group structure. So, any group structure on a split epimorphism $(f, s) : X \rightleftarrows Y$ in Σ is trivial.

Now starting with a groupoid $\underline{X}_1$, we need to show that $R[(d_0, d_1)]$ is a Σ-equivalence relation. For that consider the following commutative factorization diagram where the right hand side square is a pullback:

$$\begin{tikzcd}
X_1 \arrow[r, "{(d_0,d_1)}"] \arrow[rr, bend left, "s_0^{X_1}"] \arrow[d, shift right, "d_0"'] & R[d_0] \arrow[r, tail] \arrow[d, shift right, "f.d_0^f"'] & X_1 \times X_1 \arrow[d, shift right, "d_0 \times d_0"'] \\
X_0 \arrow[r, equal] \arrow[u, shift right, "s_0"'] & X_0 \arrow[r, tail, "s_0^X"'] \arrow[u, shift right, "s_0^f.s_0"'] & X_0 \times X_0 \arrow[u, shift right, "s_0 \times s_0"']
\end{tikzcd}$$

Since Σ is point-congruous, the right hand side vertical split epimorphism is Σ-special, since so is (d_0, s_0); accordingly so is the middle one as well. The factorization (d_0, d_1) is then a morphism in $\mathbb{E}/X_0$ between Σ-special maps. According to [9], it is Σ-special. ■

Whence the following:

3.21. Corollary. *In any Σ-distributive category $\mathbb{E}$, where Σ is point-congruous, the only Σ-groupoids are the Σ-equivalence relations.*

Accordingly, all the examples produced by the results following Section 3.12 satisfy this property.

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