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COLIMITS OF MONOIDS

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ABSTRACT. If $\mathcal{C}$ is a monoidal category with reflexive coequalizers which are preserved by tensoring from both sides, then the category **Mon**$\mathcal{C}$ of monoids over $\mathcal{C}$ has all coequalizers and these are regular epimorphisms in $\mathcal{C}$. This implies that **Mon**$\mathcal{C}$ has all colimits which exist in $\mathcal{C}$, provided that $\mathcal{C}$ in addition has (regular epi, jointly monomorphic)-factorizations of discrete cones and admits arbitrary free monoids. A further application is a lifting theorem for adjunctions with a monoidal right adjoint whose left adjoint is not necessarily strong to adjunctions between the respective categories of monoids.

Not much can be found in the literature about the existence of colimits in **Mon**$\mathcal{C}$. Assuming that $\mathcal{C}$ is cocomplete and its tensor product $\otimes$ preserves colimits, it is shown in [16] that certain pushouts exist in **Mon**$\mathcal{C}$, while [14] deals with the situation that $\mathcal{C}$ is locally finitely presentable and $\otimes$ preserves directed colimits: Then **Mon**$\mathcal{C}$ is locally presentable and, hence, cocomplete. We believe, though, that more is known in the categorical folklore in the case where $\mathcal{C}$ is cocomplete and, in addition, **Mon**$\mathcal{C}$ is monadic over $\mathcal{C}$: this is summed up as Fact 3.4 below, including some arguments and references.

The main topic of this note is the existence of colimits in **Mon**$\mathcal{C}$ if the category $\mathcal{C}$ fails to have *all* colimits. In particular we will consider the case where the functor $|-|$ is regularly monadic in the sense of [1]. Then by a well known result on such functors the category **Mon**$\mathcal{C}$ has all colimits which exist in $\mathcal{C}$. The application of this criterion however requires more information about coequalizers and regular epimorphisms in **Mon**$\mathcal{C}$ than seems to be available up to now.

We therefore first prove, as Theorem 2.3 below, that **Mon**$\mathcal{C}$ has *all* coequalizers and that regular epimorphisms in **Mon**$\mathcal{C}$ are regular epimorphisms in $\mathcal{C}$ as well, provided that $\mathcal{C}$ has reflexive coequalizers and these are preserved by $\otimes$. This strengthens the unpublished result [2, Proposition 2.1.6], where existence of all coequalizers and their preservation by $\otimes$ is assumed. In fact, this result motivated the author to compile this note, since he felt it very unlikely it would be found there (neither the publication [3] with a similar title nor the actual arXiv-version contains this result) and the authors of that preprint have no intention to publish it [7]. We would like to stress the fact that our proof uses essentially the techniques of [2], while we add a short remark explaining the naturality of this construction.

As a further application we prove as Theorem 3.6 below a lifting theorem for an

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adjunction with a (lax) monoidal right adjoint $\mathcal{C} \xrightarrow{R} \mathcal{D}$, whose left adjoint is not necessarily strong, to an adjunction between the respective categories of monoids. Again, the crucial assumption on $\mathcal{C}$ is — besides admitting free monoids — that this category has reflexive coequalizers preserved by $\otimes$. This theorem generalizes a similar one (see [2, Theorem 2.2.8]) in the preprint mentioned above, while its proof uses crucial elements of that proof. We make clear, however, that what appears to be an ad hoc construction in [2], in fact is based on general principles for the lifting of adjunctions.

Finally, the following comments might be in place. The standard assumption used in this note, that the monoidal category $\mathcal{C}$ has reflexive coequalizers and these are preserved by $\otimes$, is considerably weaker then the standard assumption in [2], where $\mathcal{C}$ is assumed to have all coequalizers and these are preserved by $\otimes$. In fact it is well known that the tensor product of a monoidal category often preserves reflexive coequalizers but not arbitrary ones. In view of a known argument (see [6, page 18] — with $\times$ replaced by $\otimes$) existence of reflexive coequalizers and preservation of these by $\otimes$ in $\mathcal{C}$ guarantees existence of reflexive coequalizers in $\mathbf{Mon}\mathcal{C}$; thus, the novelty in Theorem 3.6 is the construction of *arbitrary* coequalizers.

1. Preliminaries and notations

1.1. Some notations. A monoidal category will be denoted as $(\mathcal{C}, \otimes, I)$ or simply as $\mathcal{C}$. $\mathbf{Mon}\mathcal{C}$ then denotes the category of monoids in $\mathcal{C}$ and $|-|$ its forgetful functor; whenever this functor has a left adjoint it will be denoted by T. We will often omit the functor $|-|$ when no confusion is possible; in particular monoids will usually be denoted as $\mathsf{A} = (A, A \otimes A \xrightarrow{m_\mathsf{A}} A, I \xrightarrow{e_\mathsf{A}} A)$.

By a *monoidal functor* is meant what also is known as a lax monoidal functor. Consequently, we call a monoidal functor *strong* if its multiplications and units are isomorphisms.

By the phrase $\otimes$ *preserves colimits (of some type)* in the monoidal category $\mathcal{C}$ we mean that, for each C in $\mathcal{C}$, the functors $C \otimes -$ and $- \otimes C$ preserve these colimits. Similarly, $\otimes$ *preserves regular epimorphisms* means that all functors $C \otimes -$ and $- \otimes C$ preserve those.

1.2. Regular factorizations. With respect to regular factorizations we use the terminology of [1]. In particular a category $\mathcal{C}$ is said to *have regular factorizations* or to be a *regular category* if it has (regular epi, jointly monomorphic)-factorizations of (possibly class indexed) discrete cones. Every regular category has coequalizers.

A functor $\mathcal{D} \xrightarrow{U} \mathcal{C}$ is *regularly monadic* if it is monadic, $\mathcal{C}$ is regular, and U or, equivalently, the respective monad functor preserves regular epimorphisms. Every regularly monadic functor U *detects colimits*, that is, a diagram $D\colon \mathcal{I} \to \mathcal{D}$ has a colimit, provided the diagram $U \circ D$ has a colimit in $\mathcal{C}$ (see [1, Chapter 23]).

1.3. Bimodules. Given a monoidal category $\mathcal{C}$ and a monoid A in $\mathcal{C}$ there are the categories ${}_\mathsf{A}\mathcal{C}$ of *left* A*-modules* $(C, A \otimes C \xrightarrow{l} C)$, $\mathcal{C}_\mathsf{A}$ of *right* A*-modules* $(C, C \otimes A \xrightarrow{r} C)$(called left and right A-actions in [10]) and ${}_{\mathsf{A}}\mathcal{C}_{\mathsf{A}}$ of A-*bimodules* $(C, A \otimes C \xrightarrow{l} C, C \otimes A \xrightarrow{r} C)$. The monoid A is a left (right, bi)module by means of its multiplication.

The obvious forgetful functors from these categories into $\mathcal{C}$ have left adjoints. In particular, the left adjoint $\mathcal{C} \to {}_{\mathsf{A}}\mathcal{C}_{\mathsf{A}}$ assigns to a $\mathcal{C}$-object X the bimodule

$$(A \otimes X \otimes A, A \otimes A \otimes X \otimes A \xrightarrow{m_{\mathsf{A}} \otimes X \otimes A} A \otimes X \otimes A, A \otimes X \otimes A \otimes A \xrightarrow{A \otimes X \otimes m_{\mathsf{A}}} A \otimes X \otimes A)$$

and the bimodule structure on A described above is the free A-bimodule on I.

The unit of this adjunction is the $\mathcal{C}$-morphism $X \simeq I \otimes X \otimes I \xrightarrow{e_{\mathsf{A}} \otimes X \otimes e_{\mathsf{A}}} A \otimes X \otimes A$ and the homomorphic extension of $\mathcal{C}$-morphism $X \xrightarrow{\alpha} A$ is the morphism

$$\Lambda_\alpha := A \otimes X \otimes A \xrightarrow{A \otimes \alpha \otimes A} A \otimes A \otimes A \xrightarrow{m_{\mathsf{A}} \otimes A} A \otimes A \xrightarrow{m_{\mathsf{A}}} A. \tag{1}$$

The following observations will be of use later, where the first two are immediate consequences of Λ_α being the homomorphic extension of α.

1. For every $\mathcal{C}$-morphism $X \xrightarrow{\beta} A$
$$\tau \circ \Lambda_\alpha = \tau \circ \Lambda_\beta \iff \tau \circ \alpha = \tau \circ \beta \tag{2}$$
2. For every monoid morphism $A \xrightarrow{\gamma} B$
$$\gamma \circ \Lambda_\alpha = \Lambda_{\gamma \circ \alpha} \circ (\gamma \otimes X \otimes \gamma)$$
3. The $\mathcal{C}$-morphism Λ_α has the $\mathcal{C}$-morphism $s = A \simeq I \otimes I \otimes A \xrightarrow{e_{\mathsf{A}} \otimes e_{\mathsf{X}} \otimes A} A \otimes X \otimes A$ as a section, provided that $\mathsf{X} \xrightarrow{\alpha} \mathsf{A}$ is monoid morphism.

Being a ${}_{\mathsf{A}}\mathcal{C}_{\mathsf{A}}$-morphism it makes the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes X \otimes A \arrow[r, "A \otimes \Lambda_\alpha"] \arrow[d, "m_{\mathsf{A}} \otimes X \otimes A"'] & A \otimes A \arrow[d, "m_{\mathsf{A}}"] \\ A \otimes X \otimes A \arrow[r, "\Lambda_\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes X \otimes A \otimes A \arrow[r, "\Lambda_\alpha \otimes A"] \arrow[d, "A \otimes X \otimes m_{\mathsf{A}}"'] & A \otimes A \arrow[d, "m_{\mathsf{A}}"] \\ A \otimes X \otimes A \arrow[r, "\Lambda_\alpha"] & A \end{tikzcd} \tag{3}$$

Note that for $\mathcal{C} = \mathsf{Ab}$, the monoidal category of abelian groups, monoids A are rings and (two-sided) ideals in A are A-bimodules. If $X \xrightarrow{\iota} A$ is the embedding of a subgroup X of the additive group of the ring A, then the ideal generated by X is $\mathrm{im}\Lambda_\iota$, the image of the free A-bimodule over X under the map Λ_ι, considered as a subgroup of A,

2. Coequalizers in $\mathbf{Mon}\mathcal{C}$

Somewhat surprisingly, coequalizers in $\mathbf{Mon}\mathcal{C}$ for an arbitrary monoidal category $\mathcal{C}$ can be constructed the same way as in the special case of the category of unital rings, that is, as in $\mathsf{Ring} = \mathbf{Mon}\mathsf{Ab}$ where Ab denotes the monoidal category of abelian groups, provided that $\mathcal{C}$ shares with Ab the property that $\otimes$ preserves reflexive coequalizers. We therefore recall this simple construction as follows: If

$$\begin{tikzcd} \mathsf{B} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \mathsf{A} \arrow[r, "\pi"] & \mathsf{Q} \end{tikzcd}$$

is a coequalizer diagram in Ring, then $\mathsf{Q} = \mathsf{A}/I_{\alpha,\beta}$, where $I_{\alpha,\beta}$ is the two-sided ideal generated by the image $\mathrm{im}(\alpha - \beta)$ in Ab. In other words, $I_{\alpha,\beta}$ is, considered as a group, the subgroup $X := \mathrm{im}\Lambda_{\alpha-\beta} = \mathrm{im}(\Lambda_\alpha - \Lambda_\beta)$ of A, such that

$$\begin{tikzcd} A \otimes X \otimes A \arrow[r, shift left, "\Lambda_\alpha"] \arrow[r, shift right, "\Lambda_\beta"'] & A \arrow[r, "\pi"] & Q \end{tikzcd}$$

is a coequalizer diagram in Ab.

We show next how this can be generalized to an arbitrary monoidal category $\mathcal{C}$.

2.1. Lemma. [2] *Let* A *be a monoid in* $\mathcal{C}$ *and* $\alpha, \beta \colon X \to A$ *be* $\mathcal{C}$*-morphisms. Assume that*

$$\begin{tikzcd} A \otimes X \otimes A \arrow[r, shift left, "\Lambda_\alpha"] \arrow[r, shift right, "\Lambda_\beta"'] & A \arrow[r, "\pi"] & Q \end{tikzcd} \tag{4}$$

is a coequalizer diagram in $\mathcal{C}$ *which is preserved by* $\otimes$*. Then* Q *carries a unique monoid structure such that* π *is a monoid morphism.*

If $\alpha', \beta' \colon X' \to A$ *is another such pair, then the coequalizers of* $(\Lambda_\alpha, \Lambda_\beta)$ *and* $(\Lambda_{\alpha'}, \Lambda_{\beta'})$ *coincide provided that, for every monoid morphism* $\mathsf{A} \xrightarrow{\tau} \mathsf{C}$*,*

$$\tau \circ \alpha = \tau \circ \beta \iff \tau \circ \alpha' = \tau \circ \beta'. \tag{5}$$

Proof. Consider the diagram

$$\begin{tikzcd} A \otimes X \otimes A \otimes A \arrow[r, shift left, "\Lambda_\alpha \otimes A"] \arrow[r, shift right, "\Lambda_\beta \otimes A"'] \arrow[d, "A \otimes X \otimes A \otimes \pi"'] & A \otimes A \arrow[r, "m_{\mathsf{A}}"] \arrow[d, "A \otimes \pi"] & A \arrow[d, "\pi"] \\ A \otimes X \otimes A \otimes Q \arrow[r, shift left, "\Lambda_\alpha \otimes Q"] \arrow[r, shift right, "\Lambda_\beta \otimes Q"'] & A \otimes Q \arrow[r, dotted, "m"'] \arrow[dr, "\pi \otimes Q"'] & Q \\ & & Q \otimes Q \arrow[u, dashed, "m_Q"'] \end{tikzcd}$$

The left hand rectangle commutes serially since π coequalizes $(\Lambda_\alpha, \Lambda_\beta)$. By assumption $A \otimes A \xrightarrow{A \otimes \pi} A \otimes Q$ is a coequalizer of $A \otimes \Lambda_\alpha$ and $A \otimes \Lambda_\beta$, such that one obtains, using the first diagram of (3), a unique $\mathcal{C}$-morphism $A \otimes Q \xrightarrow{m} Q$ making the right hand square commute.

Since $A \otimes X \otimes A \otimes \pi$ is a (regular) epimorphism and $A \otimes Q \xrightarrow{\pi \otimes Q} Q \otimes Q$ is the coequalizer of $\Lambda_\alpha \otimes Q$ and $\Lambda_\beta \otimes Q$ by assumption, there exists a unique $\mathcal{C}$-morphism $Q \otimes Q \xrightarrow{m_Q} Q$ satisfying $m_Q \circ (\pi \otimes Q) = m$.

Using the facts that A is a monoid and π is a regular epimorphism (hence $\otimes^n \pi$ an epimorphism for each n) one shows easily that $\mathsf{Q} = (Q, m_{\mathsf{Q}}, \pi \circ e_{\mathsf{Q}})$ is a monoid and π a monoid morphism.

Concerning the second statement assume that $A \xrightarrow{\pi} E$ and $A \xrightarrow{\pi'} E'$ are coequalizers of $(\Lambda_\alpha, \Lambda_\beta)$ and $(\Lambda_{\alpha'}, \Lambda_{\beta'})$, respectively. Since π and π' are monoid morphisms by the above, one obtains by the equivalences (2) and (5) the equivalences

$$\pi \circ \Lambda_\alpha = \pi \circ \Lambda_\beta \iff \pi \circ \alpha = \pi \circ \beta \iff \pi' \circ \alpha' = \pi' \circ \beta' \iff \pi' \circ \Lambda_{\alpha'} = \pi' \circ \Lambda_{\beta'},$$

which implies the claim. ■

2.2. Remark. If the forgetful functor of $\mathbf{Mon}\mathcal{C}$ has a left adjoint T, the equivalence (5) holds in particular, if α' and β' are the homomorphic extensions $TX \to A$ of α and β.

The following strengthens a result of [2], where preservation of all coequalizers is required.

2.3. Theorem. *Let $\mathcal{C}$ be a monoidal category with reflexive coequalizers preserved by $\otimes$. Consider, for any pair of monoid morphisms $\alpha, \beta \colon \mathsf{X} \to \mathsf{A}$, the following coequalizer diagram in $\mathcal{C}$.*

$$A \otimes X \otimes A \underset{\Lambda_\beta}{\overset{\Lambda_\alpha}{\rightrightarrows}} A \xrightarrow{\pi} Q \tag{6}$$

Then Q carries a (unique) monoid structure such that π becomes a monoid morphism and

$$\mathsf{X} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathsf{A} \xrightarrow{\pi} \mathsf{Q}$$

is a coequalizer diagram in $\mathbf{Mon}\mathcal{C}$.

In particular the category $\mathbf{Mon}\mathcal{C}$ has coequalizers and the forgetful functor $\mathbf{Mon}\mathcal{C} \to \mathcal{C}$ preserves regular epimorphisms.

Proof. Let $\alpha, \beta \colon \mathsf{D} \to \mathsf{A}$ be monoid morphisms and $A \xrightarrow{\pi} Q$ the coequalizer of Λ_α and Λ_β in $\mathcal{C}$. By Lemma 2.1 Q carries a unique monoid structure such that π is a monoid morphism, since Diagram (6) displays a reflexive coequalizer by item 3 of Section 1.3. If $\mathsf{A} \xrightarrow{\tau} \mathsf{C}$ is a monoid morphism such that $\tau \circ \alpha = \tau \circ \beta$, then $\tau \circ \Lambda_\alpha = \tau \circ \Lambda_\beta$ by the equivalence (2), such that there exists a unique $\mathcal{C}$-morphism $Q \xrightarrow{\sigma} C$ with $\sigma \circ \pi = \tau$. It remains to prove that σ is a monoid morphism. But this is clear since π is a regular epimorphism and, hence, $\pi \otimes \pi$ is an epimorphism. ■

2.4. Remark. Given two parallel pairs of morphisms in a category with coequalizers

$$X_1 \underset{g_1}{\overset{f_1}{\rightrightarrows}} A \qquad X_2 \underset{g_2}{\overset{f_2}{\rightrightarrows}} A$$

one obtains a multiple coequalizer of the morphisms f_1, g_1, f_2, g_2 as follows: Form the coequalizer $A \xrightarrow{q_1} Q_1$ of f_1, g_1 and then the coequalizer $Q_1 \xrightarrow{q_2} Q$ of $q_1 \circ f_2, q_1 \circ g_2$; then $A \xrightarrow{q_1} Q_1 \xrightarrow{q_2} Q$ is the required multiple coequalizer.

In particular, every category with coequalizers has such multiple coequalizers and any such is a composite of ordinary ones.

3. Applications

Monadicity. Applying the result above we first provide two similar criteria for the forgetful functor $|-|\colon \mathbf{Mon}\mathcal{C} \to \mathcal{C}$ to be regularly monadic. For this we use the following fact which follows by the Beck-Paré-Theorem (the argument given in [11] for the case of semigroups applies by replacing $\times$ by $\otimes$).

3.1. Fact. *$|-|\colon \mathbf{Mon}\mathcal{C} \to \mathcal{C}$ is monadic provided that it has a left adjoint.*

3.2. Proposition. *Let $\mathcal{C}$ be a monoidal category with regular factorizations. Then the forgetful functor $|-|\colon \mathbf{Mon}\mathcal{C} \to \mathcal{C}$ is regularly monadic, provided that either of the following conditions is satisfied.*

1. *$\mathcal{C}$ has denumerable coproducts and these as well as regular epimorphisms are preserved by $\otimes$.*
2. *$|-|\colon \mathbf{Mon}\mathcal{C} \to \mathcal{C}$ has a left adjoint and $\otimes$ preserves reflexive coequalizers.*

Proof. Monadicity is clear both cases.

In the first case the respective monad $\mathbb{T}$ acts on a morphism f by $\mathbb{T}f = \coprod_n \otimes^n f$, hence, maps regular epimorphisms to regular epimorphisms by assumption, which proves this claim. In the second case $|-|$ preserves regular epimorphisms by Theorem 2.3. ∎

3.3. Remarks. There are monoidal categories with regular factorizations where $\otimes$ preserves regular epimorphisms but not coequalizers and which admit free monoids, which cannot be constructed canonically. The monoidal categories Unif and $\mathsf{Unif}_\star$ of uniform spaces with its cartesian structure and the (non symmetric monoidal right closed) structure given by the semi-uniform product, respectively, are examples.

General Colimits in $\mathbf{Mon}\mathcal{C}$. As far as we know, not much about the existence of colimits in $\mathbf{Mon}\mathcal{C}$ appears in the published literature, though probably quite a bit is known in categorical folklore. The only published results we are aware of are

- If $\mathcal{C}$ is cocomplete and $\otimes$ preserves colimits, then the category $\mathbf{Mon}\mathcal{C}$ has all pushouts of the form

$$\begin{tikzcd} FC \arrow[r, "Ff"] \arrow[d] & FD \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

  where $C \xrightarrow{f} D$ is a morphism in $\mathcal{C}$ and $F\colon \mathcal{C} \to \mathbf{Mon}\mathcal{C}$ is the free monoid functor (see [16]).

- If $\mathcal{C}$ admits a monoidal topological functor $U\colon \mathcal{C} \to \mathsf{Set}$, then the induced functor $\bar{U}\colon \mathbf{Mon}\mathcal{C} \to \mathsf{Mon}$ into the category of (algebraic) monoids is topological again (the argument used in [12] for a strict monoidal functor generalizes). In particular $\mathbf{Mon}\mathcal{C}$ is cocomplete (see [1, Proposition 21.15]).

- If $\mathcal{C}$ is locally λ-presentable and $\otimes$ preserves λ-directed colimits, then $\mathbf{Mon}\mathcal{C}$ is locally presentable and, hence, cocomplete in particular (see [14]).

The first of these results is a special instance of the following fact which be believe to be part of the categorical folklore; it is a consequence of the familiar fact that a monadic category over a cocomplete category is cocomplete if it has reflexive coequalizers (see e.g. [4, Theorem 4.3.4]), where existence of those follows from Johnstone's observation [6, page 18] (with $\times$ replaced by $\otimes$) — but clearly from Theorem 2.3 as well.

3.4. Fact. *If the forgetful functor* $|-|\colon \mathbf{Mon}\mathcal{C} \to \mathcal{C}$ *has a left adjoint and the category* $\mathcal{C}$ *is cocomplete with reflexive coequalizers preserved by* $\otimes$ *then* $\mathbf{Mon}\mathcal{C}$ *is cocomplete.*

Without the requirement of cocompleteness of $\mathcal{C}$ we obtain the following.

3.5. Theorem. *In the situation of Proposition 3.2* $\mathbf{Mon}\mathcal{C}$ *has coequalizers and all other colimits which exist in* $\mathcal{C}$*.*

Proof. Every category with regular factorizations has coequalizers (see [1, Proposition 20.33]); hence, the result follows from [1, Theorem 23.11], since every regularly monadic functor is essentially algebraic. ∎

Lifting adjunctions. Let $R\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor. It is well known that R induces a functor $\bar{R}\colon \mathbf{Mon}\mathcal{C} \to \mathbf{Mon}\mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathbf{Mon}\mathcal{C} \arrow[r, "\bar{R}"] \arrow[d] & \mathbf{Mon}\mathcal{D} \arrow[d] \\ \mathcal{C} \arrow[r, "R"'] & \mathcal{D} \end{tikzcd}$$

commutes, where the vertical arrows denote the respective forgetful functors (denoted by $|-|$ if necessary). It is of quite some interest (see e.g. [8], [15]) to know under which conditions the functor $\bar{R}$ has a left adjoint if R has one.

The answer to this question is rather trivial and well known in case that the left adjoint L of R is strong: then L also lifts to a functor $\bar{L}\colon \mathbf{Mon}\mathcal{D} \to \mathbf{Mon}\mathcal{C}$, and this is a left adjoint of $\bar{R}$ (see e.g. [14]); this applies in particular for monoidal adjunctions $L \dashv R$.

A standard approach to the general case of this problem would be to apply Dubuc's Adjoint Triangle Theorem, which would require both forgetful functors to have left adjoints and $\mathbf{Mon}\mathcal{C}$ to have coequalizers of reflexive pairs. Tambara [17] claimed without a proof that it suffices to assume that $\mathcal{C}$ is cocomplete and that $\otimes$ preserves all colimits. A proof of this claim is contained in [2]. The following is a generalization of this result in that we do not assume the free monoids over $\mathcal{C}$ to be given by MacLane's standard construction and only require preservation of reflexive coequalizers by $\otimes$.

3.6. Theorem. *Let $R\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor with left adjoint L. Assume that $\mathcal{C}$ has reflexive coequalizers which are preserved by $\otimes$ and that the forgetful functor $\mathbf{Mon}\mathcal{C} \xrightarrow{|-|_{\mathcal{C}}} \mathcal{C}$ has left adjoint T. Then the functor $\bar{R}\colon \mathbf{Mon}\mathcal{C} \to \mathbf{Mon}\mathcal{D}$ has a left adjoint.*

Proof. We use the following notations.

1. Unit and counit of the adjunction $T \dashv |-|_{\mathcal{C}}$ are denoted by ξ and ζ, respectively. For every C in BC m_{TC} and e_{TC} are the multiplication and unit, respectively, of the free monoid TC.

2. Φ denotes the multiplication and ϕ the unit of the monoidal structure of R; Ψ and ψ denote the opmonoidal structure of L. $\kappa\colon id \Rightarrow RL$ and $\lambda\colon LR \Rightarrow id$ denote the unit and counit, respectively, of the adjunction $L \dashv R$.

3. Unit and counit of the adjunction $F_1 := TL \dashv R|-|_{\mathcal{C}} =: P_1$ are denoted by η and ϵ, respectively. In particular, for any $\mathcal{D}$-object D,

$$\eta_D = D \xrightarrow{\kappa_D} RLD \xrightarrow{R\xi_{LD}} R|TLD|_{\mathcal{C}} \tag{7}$$

Following the analysis of adjoint triangles in [18] one should try to obtain the left adjoint of $\bar{R}$ as follows:

1. Find in $\mathbf{Mon}\mathcal{C}$, for each $\mathcal{D}$-monoid $\mathsf{D} := (D, m_{\mathsf{D}}, e_{\mathsf{D}})$, a suitable morphism

$$F_1 D = TLD \xrightarrow{\pi_{\mathsf{D}}} L_{\mathsf{D}}.$$

2. Find in $\mathbf{Mon}\mathcal{D}$ a morphism $\mathsf{D} \xrightarrow{\gamma_{\mathsf{D}}} \bar{R}L_{\mathsf{D}}$ with

$$|\gamma_{\mathsf{D}}|_{\mathcal{D}} = D \xrightarrow{\eta_D} P_1 F_1 D \xrightarrow{R|\pi_{\mathsf{D}}|} |\bar{R}L_{\mathsf{D}}|_{\mathcal{D}}.$$

γ_{D} so defined has the potential of being $\bar{R}$-universal for D, hence the family $\gamma = (\gamma_{\mathsf{D}})_{\mathsf{D}}$ to be the unit of the desired adjunction.

Now π_{D} has to be an epimorphism since the forgetful functor $\mathbf{Mon}\mathcal{D} \xrightarrow{|-|_{\mathcal{D}}} \mathcal{D}$ is faithful (see [18]). A natural choice of π_{D}, thus, would be to consider a (multiple) coequalizer of $\mathbf{Mon}\mathcal{C}$-morphisms which in some way reflect the monoid structure of $\mathsf{D} = (D, m_{\mathsf{D}}, e_{\mathsf{D}})$ and the monoidal structure of R (equivalently, the opmonoidal structure of L). We therefore use the following morphisms.

1. $\alpha_1^{\mathsf{D}} = LI_{\mathcal{D}} \xrightarrow{\psi} I_{\mathcal{C}} \xrightarrow{e_{TLD}} |TLD|$
   $\beta_1^{\mathsf{D}} = LI_{\mathcal{D}} \xrightarrow{Le_{\mathsf{D}}} LD \xrightarrow{\xi_{LD}} |TLD|$
2. $\alpha_2^{\mathsf{D}} = L(D \otimes D) \xrightarrow{\Psi_{D,D}} LD \otimes LD \xrightarrow{\xi_{LD\otimes LD}} |TLD| \otimes |TLD| \xrightarrow{m_{TLD}} |TLD|$
   $\beta_2^{\mathsf{D}} = L(D \otimes D) \xrightarrow{Lm_{\mathsf{D}}} LD \xrightarrow{\xi_{LD}} |TLD|$

and consider the homomorphic extensions of these maps, that is, the monoid morphisms

3. $\bar{\alpha}_1^{\mathsf{D}}, \bar{\beta}_1^{\mathsf{D}} \colon TLI_{\mathcal{D}} \to TLD$ with $\bar{\alpha}_1^{\mathsf{D}} \circ \xi_{LI_{\mathcal{D}}} = \alpha_1^{\mathsf{D}}$ and $\bar{\beta}_1^{\mathsf{D}} \circ \xi_{LI_D} = \beta_1^D$
4. $\bar{\alpha}_2^{\mathsf{D}}, \bar{\beta}_2^{\mathsf{D}} \colon TL(D \otimes D) \to TLD$ with $\bar{\alpha}_2^{\mathsf{D}} \circ \xi_{L(D\otimes D)} = \alpha_2^{\mathsf{D}}$ and $\bar{\beta}_2^{\mathsf{D}} \circ \xi_{L(D\otimes D)} = \beta_2^D$.

Now let $TLD \xrightarrow{\pi_{\mathsf{D}}} L_{\mathsf{D}}$ be the multiple coequalizer (which exists by assumption — see Remark 2.4) of the morphisms $\bar{\alpha}_1^{\mathsf{D}}$, $\bar{\beta}_1^{\mathsf{D}}$, $\bar{\alpha}_2^{\mathsf{D}}$, $\bar{\beta}_2^{\mathsf{D}}$ in $\mathbf{Mon}\mathcal{C}$.

According to step 2. we check that the following $\mathcal{D}$-morphism is a morphism of $\mathcal{D}$-monoids $\mathsf{D} \xrightarrow{\gamma_{\mathsf{D}}} \bar{R}L_{\mathsf{D}}$.

$$\gamma_{\mathsf{D}} := D \xrightarrow{\eta_D} |\bar{R}TLD|_{\mathcal{D}} \xrightarrow{R\pi_{\mathsf{D}}} |\bar{R}L_{\mathsf{D}}|_{\mathcal{D}} = D \xrightarrow{\eta_D} R|TLD|_{\mathcal{C}} \xrightarrow{R\pi_{\mathsf{D}}} R|L_{\mathsf{D}}|_{\mathcal{C}}$$

Compatibility with the multiplications is equivalent to commutativity of the outer frame of the following diagram (where we omit in the right hand cells the underlying functors $|-|_{\mathcal{C}}$ and $|-|_{\mathcal{D}}$), which is clear by standard arguments for monoidal functors and the fact that π_{D} coequalizes $\bar{\alpha}_2^{\mathsf{D}}$ and $\bar{\beta}_2^{\mathsf{D}}$, i.e., $|\pi_{\mathsf{D}}|$ coequalizes α_2^{D} and β_2^{D} (see Remark 2.2) so that $R|\pi_{\mathsf{D}}|$ coequalizes the two paths around the pentagon appearing in the diagram.

$$\begin{tikzcd}
D \otimes D \arrow[r, "\kappa_D\otimes\kappa_D"'] \arrow[rr, bend left, "\eta_D\otimes\eta_D"] \arrow[ddd, "m_{\mathsf{D}}"'] \arrow[ddr, "\kappa_{D\otimes D}"'] & RLD \otimes RLD \arrow[r, "R\xi_{LD}\otimes R\xi_{LD}"'] \arrow[d, "\Phi_{LD,LD}"] & RTLD \otimes RTLD \arrow[r, "R\pi_{\mathsf{D}}\otimes R\pi_{\mathsf{D}}"'] \arrow[d, "\Phi_{TLD,TLD}"] & RL_{\mathsf{D}} \otimes RL_{\mathsf{D}} \arrow[d, "\Phi_{L_{\mathsf{D}},L_{\mathsf{D}}}"] \\
 & R(LD \otimes LD) \arrow[r, "R(\xi_{LD}\otimes\xi_{LD})"'] & R(TLD \otimes TLD) \arrow[r, "R(\pi_{\mathsf{D}}\otimes\pi_{\mathsf{D}})"'] \arrow[dd, "Rm_{TLD}"] & R(L_{\mathsf{D}} \otimes L_{\mathsf{D}}) \arrow[dd, "Rm_{L_{\mathsf{D}}}"] \\
 & RL(D \otimes D) \arrow[u, "R\Psi_{D,D}"'] \arrow[d, "RLm_{\mathsf{D}}"] & & \\
D \arrow[r, "\kappa_D"] \arrow[rr, bend right, "\eta_D"'] & RLD \arrow[r, "R\xi_{LD}"] & RTLD \arrow[r, "R\pi_{\mathsf{D}}"] & RL_{\mathsf{D}}
\end{tikzcd}$$

Preservation of units follows by a similar argument.

It remains to prove that γ_{D} is $\bar{R}$-universal for D. Here we again use elements of the proof given in [2]. First define, for a $\mathcal{C}$-monoid A, a morphism $L_{\bar{R}\mathsf{A}} \xrightarrow{\sigma_{\mathsf{A}}} \mathsf{A}$ in $\mathbf{Mon}\mathcal{C}$. This will in fact be the counit of the desired adjunction. By the easily checked identities

$$\epsilon_A \circ \alpha_1^{\bar{R}\mathsf{A}} = \epsilon_A \circ \beta_1^{\bar{R}\mathsf{A}} \text{ and } \epsilon_A \circ \alpha_2^{\bar{R}\mathsf{A}} = \epsilon_A \circ \beta_2^{\bar{R}\mathsf{A}}.$$

one obtains, using the equivalence (2) and the universal property of $\pi_{\bar{R}\mathsf{A}}$, a unique monoid morphism $L_{\bar{R}\mathsf{A}} \xrightarrow{\sigma_{\mathsf{A}}} \mathsf{A}$ making the following diagram commute

$$\begin{tikzcd} L_{\bar{R}\mathsf{A}} \arrow[r, "\sigma_{\mathsf{A}}"] & \mathsf{A} \\ TLRA \arrow[u, "\pi_{\bar{R}\mathsf{A}}"] \arrow[ur, "\epsilon_A"'] & \end{tikzcd} \tag{8}$$

Next one checks, for any morphism $\mathsf{D} \xrightarrow{h} \mathsf{C}$ in $\mathbf{Mon}\mathcal{D}$, the identities

$$\begin{array}{cc} TLh \circ \alpha_1^{\mathsf{D}} = \alpha_1^{\mathsf{C}} & TLh \circ \beta_1^{\mathsf{D}} = \beta_1^{\mathsf{C}} \\ TLh \circ \alpha_2^{\mathsf{D}} = \alpha_2^{\mathsf{C}} \circ L(h \otimes h) & TLh \circ \beta_2^{\mathsf{D}} = \beta_2^{\mathsf{C}} \circ L(h \otimes h) \end{array}$$

These imply

$$\begin{array}{rcl} (\pi_{\mathsf{C}} \circ TLh) \circ \alpha_1^{\mathsf{D}} & = & \pi_{\mathsf{C}} \circ \alpha_1^{\mathsf{C}} = \pi_{\mathsf{C}} \circ \beta_1^{\mathsf{C}} \\ & = & (\pi_{\mathsf{C}} \circ TLh) \circ \beta_1^{\mathsf{D}} \\ (\pi_{\mathsf{C}} \circ TLh) \circ \alpha_2^{\mathsf{D}} & = & \pi_{\mathsf{C}} \circ \alpha_2^{\mathsf{C}} \circ L(h \otimes h) = \pi_{\mathsf{C}} \circ \beta_2^{\mathsf{C}} \circ L(h \otimes h) \\ & = & (\pi_{\mathsf{C}} \circ TLh) \circ \beta_2^{\mathsf{D}} \end{array}$$

Hence, $\pi_{\mathsf{C}} \circ TLh$ coequalizes simultaneously the $\mathcal{C}$-morphisms $\alpha_1, \alpha_2, \beta_1, \beta_2$ and, thus, by Remark 2.2 the monoid morphisms $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_1, \bar{\beta}_2$. Consequently there exist a unique monoid morphism $L_{\mathsf{D}} \xrightarrow{L_h} L_{\mathsf{C}}$ making the following diagram commute

$$\begin{tikzcd} L_{\mathsf{D}} \arrow[r, "L_h"] & L_{\mathsf{C}} \\ TLD \arrow[u, "\pi_D"] \arrow[r, "TLh"'] & TLC \arrow[u, "\pi_C"'] \end{tikzcd} \tag{9}$$

Then, for every morphism $\mathsf{D} \xrightarrow{d} \bar{R}\mathsf{A}$ in $\mathbf{Mon}\mathcal{D}$, the following diagram obviously commutes (in $\mathcal{D}$) and illustrates the required one-to-one correspondence between morphisms $\mathsf{D} \xrightarrow{d} \bar{R}\mathsf{A}$ in $\mathbf{Mon}\mathcal{D}$ and morphisms $L_{\mathsf{D}} \xrightarrow{\sigma_{\mathsf{A}} \circ L_d} \mathsf{A}$ in $\mathbf{Mon}\mathcal{C}$.

$$
\begin{tikzcd}
D \arrow[r, "\kappa_D"'] \arrow[rr, bend left, "\eta_D"] \arrow[rrr, bend left=30, "\gamma_{\mathsf{D}}"] \arrow[ddrr, bend right, "d"'] & RLD \arrow[r, "R\xi_{LD}"'] \arrow[ddr, "RLd"'] & RTLD \arrow[r, "R\pi_{\mathsf{D}}"'] \arrow[d, "RTLd"'] & RL_{\mathsf{D}} \arrow[d, "RL_d"] \\
 & & RTLRA \arrow[r, "R\pi_{\bar{R}\mathsf{A}}"] \arrow[d, "R\epsilon_A"'] & RL_{\bar{R}\mathsf{A}} \arrow[dl, "R\sigma_{\mathsf{A}}"] \\
 & & RA &
\end{tikzcd}
$$

■

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References

[1] J. Adámek, H. Herrlich and G.E. Strecker, *Abstract and Concrete Categories*, John Wiley, New York, 1990.

[2] A. Ardizzoni, L. El Katoutit and C. Menini, Coendomorphism bialgebroid and chain complexes, `arXiv:0044572v1` (2010).

[3] A. Ardizzoni, L. El Katoutit and C. Menini, Coendomorphism left bialgebroids, *J. Algebra Appl.* **12**(3) (2013) 1250181 (21 pages).

[4] F. Borceux, *Handbook of Categorical Algebra 2*, Cambridge University Press (1994).

[5] E. Dubuc, Adjoint triangles, in *Lecture Notes in Mathematics* **61**, 69–81, Springer, Heidelberg (1968).

[6] P.T. Johnstone, *Topos Theory*, Academic Press (1977).

[7] L. El Katoutit, Private communication (2018).

[8] I. Goyvaerts and J. Vercruysse, On the duality of generalized Lie and Hopf algebras, *Advances in Mathematics* **258** (2014), 154–190.

[9] S. Lack, Note on the Construction of Free Monoids, *Appl. Categor. Struct.* **18** (2010), 17–29.

[10] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Springer, New York (1998).

[11] R. Paré, Absolute coequalizers, in *Lecture Notes in Mathematics* **61**, 132–145, Springer, Heidelberg (1968).

[12] H.–E. Porst, How Topological are Topological Groups?, *Quaest. Math.* **13** (1990), 291–299.

[13] H.–E. Porst, Towards a new construction of free topological groups, *Quaest. Math.* **16** (1993), 209–220.

[14] H.–E. Porst, On Categories of Monoids, Comonoids, and Bimonoids, *Quaest. Math.* **31** (2008), 127–139.

[15] H.–E. Porst and R. Street, Generalizations of the Sweedler Dual, *Appl. Categor. Struct.*, **24** (2016), 619–647.

[16] S. Schwede and B. Shipley, Algebras and modules in monoidal model categories *Proc. London Math. Soc.* **80 (2)** (2000), 491-511.

[17] D. Tambara, The coendomorphism bialgebra of an algebra *J. Fac. Sci. Univ. Tokyo, Sect. IA Math* **37** (1990), 425–456.

[18] W. Tholen, Adjungierte Dreiecke, Colimites und Kan-Erweiterungen, *Math. Ann.* **217** (1975), 121–129.

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