A NOTE ON INTERNAL OBJECT ACTION REPRESENTABILITY
OF 1-CAT GROUPS AND CROSSED MODULES

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Abstract. The category of 1-cat groups, which is equivalent to the category of crossed modules, has internal object actions which are representable (by internal automorphism groups). Moreover, it is known that the crossed module, corresponding to the representing object \([X] = \text{Aut}(X)\) associated with a 1-cat group \(X\), must be isomorphic to the Norrie actor of the crossed module corresponding to \(X\). We recall the description of \(\text{Aut}(X)\) from the author’s PhD thesis, and construct that isomorphism explicitly.

1. Introduction

There are various contexts, unified in [BJK2005a] (see also [BJK2005b]) based on old ideas of G. M. Kelly, where one can consider the set \(\text{Act}(B, X)\) of actions of \(B\) on \(X\), and, moreover, make \(\text{Act}(\cdot, X)\) a functor \(\mathcal{C}^{\text{op}} \to \text{Sets}\), where \(\mathcal{C}\) is the suitable category of acting objects. We then might have

\[
\text{Act}(\cdot, X) \cong \text{hom}(\cdot, [X]),
\]

for some object \([X]\) in \(\mathcal{C}\), and if it is the case for each object \(X\) from the category of objects on which the objects of \(\mathcal{C}\) act, we say that the actions are representable. This notion of representable actions was also introduced in [BJK2005a], where the main example of internal object actions came from the general theory of semidirect products [BJ1998].

The context we need here is what is described in Subsection 4.4 of [BJK2005a], which is:

- Our ground category \(\mathcal{C}\) is the category of internal groups in a cartesian closed category \(\mathcal{E}\) with finite limits, and we assume \(\mathcal{C}\) to be semi-abelian in the sense of [JMT2002].

- An action of an object \(B\) of \(\mathcal{C}\) on an object \(X\) of \(\mathcal{C}\) can be identified with a morphism \(h : B \times X \to X\) satisfying the usual conditions, which, expressed in terms of generalized elements with \(h(b, x)\) written as \(bx\), are:

\[
0x = 0, \quad b_1(b_2x) = (b_1 + b_2)x, \quad b(x_1 + x_2) = bx_1 + bx_2;
\]
note that we are using additive notation, even though our groups are not necessarily abelian.

• In this case \([X] = Aut(X)\), the internal automorphism group of \(X\). As mentioned in [BJK2005a], it is constructed in a straightforward way; however, it involves long calculations, whose details can be found in [BCM2014].

More specifically, we are going to consider the case of \(E\) being the category of all (small) categories, which allows us to describe \(C\) either as:

• the category of crossed modules (see e.g. [BS1976]), or as:

• the category of 1-cat groups [L1982].

In my PhD Thesis [R2015], I was using, with help of George Janelidze, who was my PhD supervisor then, the second description to calculate \([X] = Aut(X)\) fully (and even to extend it to \(n\)-cat groups, for an arbitrary natural \(n\)). The result, recalled in Section 2 below, looks very different from what K. Norrie calls the *actor of a crossed module* in [N1990], and what is recalled in our Sections 3. Nevertheless, as follows from an observation in [BJK2005a], and in fact also from the results of [N1990], there should be no difference. More precisely, the crossed module corresponding to our calculated \([X] = Aut(X)\) should be isomorphic to the Norrie actor of crossed module corresponding to \(X\). Constructing this isomorphism explicitly (see Section 4) is the purpose of the present paper. What seems to make this construction interesting is that the two descriptions, of the internal automorphism group and of the Norrie actor, come from very different sources, namely from the theory of cartesian closed categories and from homotopical algebra (for the latter see references in [N1990], especially [L1979] and [W1948]).

2. The 1-cat presentation

Let \(M = \{1, s, t\}\) be the monoid in which

\[
st = t \quad \text{and} \quad ts = s,
\]

and which also implies that \(s\) and \(t\) are idempotents. A 1-cat group \(X\) can be described as an \(M\)-group \(X\) with

\[
(sx_1 = 0 = tx_2) \Rightarrow x_1 + x_2 = x_2 + x_1,
\]

for all \(x_1, x_2 \in X\), in the additive notation. Since the category of 1-cat groups can be identified with the category of internal groups in a cartesian closed category (of all small categories), each 1-cat group \(X\) has its internal automorphism group \(Aut(X)\), which is also a 1-cat group. Long but routine calculations made in [R2015] show that \(Aut(X)\) can
be presented as the 1-cat group of maps
\[ \alpha : M \times X \to X \] with
\[ m\alpha(m', x) = \alpha(mm', mx), \]
\[ \alpha(m, x_1 + x_2) = \alpha(m, x_1) + \alpha(m, x_2), \]
\[ \alpha(m, -) : X \to X \] is a bijection,
\[ \alpha(1, x) = \alpha(1, tx) - \alpha(s, tx) + \alpha(s, x), \]
\[ \alpha(1, x) = \alpha(t, x) - \alpha(t, sx) + \alpha(1, sx), \]
for all \( m, m' \in M \) and \( x_1, x_2 \in X \); the \( M \)-action on \( \text{Aut}(X) \) is defined by
\[ (m'\alpha)(m, x) = \alpha(mm', x), \]
while the addition on \( \text{Aut}(X) \) is defined by
\[ (\alpha + \beta)(m, x) = \alpha(m, \beta(m, x)), \]
making the second projection \( M \times X \to X \) the zero element of \( \text{Aut}(X) \).

3. The Norrie presentation

Recall that a crossed module is triple \((K, B, \partial)\), in which \( B \) and \( K \) are groups with \( B \) acting on \( K \), and \( \partial : K \to B \) is a group homomorphism with
\[ \partial(bk) = b + \partial(k) - b, \]
\[ \partial(k_1)k_2 = k_1 + k_2 - k_1, \]
for all \( b \in B \) and \( k_1, k_2 \in K \).

For example, a 1-cat group \( X \) presented as in Section 2 determines a crossed module \((K_X, B_X, \partial_X)\) in which:
\[ K_X = \{ x \in X | sx = 0 \}, \]
\[ B_X = \{ x \in X | sx = x \} = \{ x \in X | tx = x \}, \]
\[ B_X \text{ acts on } K_X \text{ via } bk = b + k - b, \]
\[ \partial_X : K_X \to B_X \text{ is defined by } \partial_X(k) = tk. \]

The Norrie actor \( A(K, B, \partial) = (D(B, K), \text{Aut}(K, B, \partial), \Delta) \) \[N1990\] of a crossed module \((K, B, \partial)\) is described as follows: •
Aut\((K, B, \partial)\) is the ordinary automorphism group of \((K, B, \partial)\), that is,
\[
\text{Aut}(K, B, \partial) = \{(\kappa, \beta) \in \text{Aut}(K) \times \text{Aut}(B) | \beta \partial = \partial \kappa \text{ and } \forall k \in K \forall b \in B \kappa(bk) = \beta(b)\kappa(k)\};
\]
(18)

a map \(d : B \to K\) is called a derivation (= a crossed homomorphism) if
\[
d(b_1 + b_2) = d(b_1) + b_1d(b_2),
\]
(19)
for all \(b_1, b_2 \in B\), and such derivations form an additive (not necessarily commutative) monoid \(\text{Der}(B, K)\), whose addition is defined by
\[
(d_1 + d_2)(b) = d_1(\partial d_2(b) + b) + d_2(b) = d_1\partial d_2(b) + d_2(b) + d_1(b)
\]
(20)
which makes the zero map \(B \to K\) the zero element of \(\text{Der}(B, K)\);

\(D(B, K)\) is defined as the group of invertible elements of the monoid \(\text{Der}(B, K)\);

the action of \(\text{Aut}(K, B, \partial)\) on \(D(B, K)\) is defined by
\[
(\kappa, \beta)d = \kappa d\beta^{-1};
\]
(21)
the homomorphism \(\Delta : D(B, K) \to \text{Aut}(K, B, \partial)\) is defined by
\[
\Delta(d) = (\Delta_1(d), \Delta_2(d)), \text{ where }
\]
\[
\Delta_1(d)(k) = d\partial(k) + k \text{ and } \Delta_2(d)(b) = \partial d(b) + b.
\]
(22)
Note: using the maps \(\Delta_1\) and \(\Delta_2\) one obtains nicer forms of (19), namely
\[
(d_1 + d_2)(b) = d_1\Delta_2(d_2)(b) + d_2(b) = \Delta_1(d_1)d_2(b) + d_1(b).
\]
(23)

4. The isomorphism

In this section we fix:

- a 1-cat group \(X\) presented as an \(M\)-group, as in Section 2;
- the corresponding crossed module \((K_X, B_X, \partial_X) = (K, B, \partial)\);
- the Norrie actor \(A(K, B, \partial) = (D(B, K), \text{Aut}(K, B, \partial), \Delta)\) of \((K, B, \partial)\);
- the crossed module corresponding to the 1-cat group \(\text{Aut}(X)\), which will be denoted by \((K, B, \partial)\).

And our aim is to construct an isomorphism
\[
(D(B, K), \text{Aut}(K, B, \partial), \Delta) \approx (K, B, \partial).
\]
(24)
4.1. Lemma.

(a) \( K = \{ \alpha \in Aut(X) | \forall x \in X \alpha(s, x) = x \} \);
(b) \( B = \{ \alpha \in Aut(X) | \forall x \in X \alpha(1, x) = \alpha(s, x) = \alpha(t, x) \} \);
(c) \( B \) acts on \( K \) via \( b_k(m, x) = b(m, k(m, b(m, -)^{-1}(x))) \);
(d) \( \partial : K \to B \) is defined by \( \partial(k)(m, x) = k(t, x) \).

Proof. (a) According to (14), \( K = \{ \alpha \in Aut(X) | s\alpha = 0 \} \), and we calculate, for any \( m \in M \):
\[
(s\alpha)(m, x) = \alpha(ms, x) \quad \text{by (10)}
\]
\[
= \alpha(s, x) \quad \text{(since } ms = s \text{ for every } m \in M).\]
Since 0 of \( Aut(X) \) is the second projection \( M \times X \to X \), as mentioned at the end of Section 2, this implies that \( s\alpha = 0 \) is equivalent to \( \forall x \in X \alpha(s, x) = x \).

(b) For \( \alpha \in Aut(X) \), we have:
\[
\alpha \in B \iff s\alpha = \alpha \quad \text{by (15)}
\]
\[
\iff \forall m \in M \forall x \in X \alpha(ms, x) = \alpha(m, x) \quad \text{by (10)}
\]
\[
\iff \forall m \in M \forall x \in X \alpha(s, x) = \alpha(m, x) \quad \text{(again since } ms = s \text{ for every } m \in M)
\]
\[
\iff \forall x \in X \alpha(1, x) = \alpha(s, x) = \alpha(t, x).
\]

(c) follows from (11) and (16).

(d) follows from (17), (10), and the fact that \( mt = t \) for every \( m \in M \).

4.2. Lemma. The formula
\[
f(\alpha)(x) = \alpha(1, x) - x
\]
(25)
defines a group homomorphism \( f : K \to D(B, K) \).

Proof. We need to prove the following:

(a) for every \( \alpha \in K \) and \( x \in B \), \( \alpha(1, x) - x \) belongs to \( K \).

(b) for every \( \alpha \in K \), the map \( f(\alpha) : B \to K \) defined by (25) is a derivation, that is, it belongs to \( Der(B, K) \);

(c) \( f(\alpha_1 + \alpha_2) = f(\alpha_1) + f(\alpha_2) \) for all \( \alpha_1, \alpha_2 \in K \);

(d) the derivations of the form \( f(\alpha) \) above are invertible, that is, they belong to \( D(B, K) \).

Proof of (a): We have
\[
s(\alpha(1, x) - x) = s\alpha(1, x) - sx \quad \text{(since } X \text{ is an } M\text{-group)}
\]
\[
= \alpha(s, sx) - sx \quad \text{(by (5))}
\]
\[
= sx - sx \quad \text{(by Lemma 4.1(a))}
\]
\[
= 0.
\]
Proof of (b): According to (19), we need to prove that
\[
\alpha(1, x_1 + x_2) - (x_1 + x_2) = \alpha(1, x_1) - x_1 + x_1(\alpha(1, x_2) - x_2),
\]
for all \(x_1, x_2 \in B\). We have:
\[
\alpha(1, x_1 + x_2) - (x_1 + x_2) = \alpha(1, x_1) + \alpha(1, x_2) - (x_1 + x_2) \quad \text{(by \(6\))}
\]
and so to prove (26) is to show that
\[
x_1 + \alpha(1, x_2) - x_2 - x_1 = x_1(\alpha(1, x_2) - x_2),
\]
which follows from (16).

Proof of (c): For \(\alpha_1, \alpha_2 \in K\) and \(x \in B\) we have:
\[
f(\alpha_1 + \alpha_2)(x) = (\alpha_1 + \alpha_2)(1, x) - x \quad \text{(by definition of \(f\))}
\]
\[
= \alpha_1(1, \alpha_2(1, x)) - x \quad \text{(by \(11\))}
\]
\[
= \alpha_1(1, t\alpha_2(1, x)) - \alpha_1(s, t\alpha_2(1, x)) + \alpha_1(s, \alpha_2(1, x)) - x \quad \text{(by \(8\) used for \(\alpha_2(1, x)\) instead of \(x\))}
\]
\[
= \alpha_1(1, t\alpha_2(1, x)) - t\alpha_2(1, x) + \alpha_2(1, x) - x \quad \text{(by Lemma 4.1(a))}
\]
and
\[
(f(\alpha_1) + f(\alpha_2))(x) = f(\alpha_1)(tf(\alpha_2)(x) + x) + f(\alpha_2)(x) \quad \text{(by the first equality of \(20\) and \(17\))}
\]
\[
= f(\alpha_1)(t(\alpha_2(1, x) - x) + x) + \alpha_2(1, x) - x \quad \text{(by the definition of \(f\))}
\]
\[
= f(\alpha_1)(t\alpha_2(1, x)) + \alpha_2(1, x) - x \quad \text{(since \(tx = x\), by \(15\))}
\]
\[
= \alpha_1(1, t\alpha_2(1, x)) - t\alpha_2(1, x) + \alpha_2(1, x) - x \quad \text{(by the definition of \(f\)).}
\]
That is, \(f(\alpha_1 + \alpha_2) = f(\alpha_1) + f(\alpha_2)\) for all \(\alpha_1, \alpha_2 \in K\).

Proof of (d): Since \(K\) is a group, having proved (c) we only need to prove \(f(0) = 0\).

For, just note:

- \(K\) is a subgroup of \(\text{Aut}(X)\);
- the 0 of \(\text{Aut}(X)\) the second projection \(M \times X \rightarrow X\), as mentioned at the end of Section 2;
- therefore \(f(0)(x) = x - x = 0\);
- the 0 of \(D(B, K)\) is the zero map \(B \rightarrow K\), as mentioned below (20).
Let us put \( f(\alpha) = d \) and try to recover \( \alpha \in K \) from \( d \). For, take an arbitrary each \( x \in X \), and observe:

- (a) since \( \alpha \) is in \( K \), \( \alpha(s,x) = x \) (by Lemma 4.1(a));
- (b) \( \alpha(1,sx) = d(sx) + sx \) and \( \alpha(1,tx) = d(tx) + tx \) (by (25));
- (c) \( \alpha(t,sx) = \alpha(t,tsx) = t\alpha(1,sx) \) (by (5));
- (d) \( \alpha(t,sx) = td(sx) + tsx = td(sx) + sx \) (by (c) and (b));
- (e) \( \alpha(1,x) = \alpha(1,tx) - \alpha(s,tx) + \alpha(s,x) = d(tx) + tx - tx + x = d(tx) + x \) (where the first equality is (8), while the second one follows from (b) and (a));
- (f) \( \alpha(t,x) = \alpha(1,x) - \alpha(1,sx) + \alpha(t,sx) = d(tx) + x - sx - dsx + td(sx) + sx = d(tx) - dsx + td(sx) + x \) (where: the first equality follows from (9); the second one follows from (e), (b), and (d); and the third one from (4), since \( s(x - sx) = 0 = t(-dsx + td(sx)) \)).

Denoting \( \alpha \) by \( g(d) \), this gives:

\[
g(d)(m,x) = \begin{cases} 
  d(tx) + x, & \text{if } m = 1; \\
  x & \text{if } m = s; \\
  d(tx) - dsx + td(sx) + x, & \text{if } m = t.
\end{cases}
\]  

(27)

We can also rewrite this as

\[
\begin{align*}
  g(d)(s,x) &= x, \\
  g(d)(m,x) &= d(tx) - dsx + md(sx) + x, & \text{if } m \neq s.
\end{align*}
\]  

(28)

4.3. **Lemma.** The pair (28) of formulas defines a map \( g : D(B, K) \to K \).

**Proof.** We need to prove that the map \( \alpha : M \times X \to X \), defined by

\[
\alpha(m,x) = \begin{cases} 
  d(tx) + x, & \text{if } m = 1; \\
  x & \text{if } m = s; \\
  d(tx) - dsx + td(sx) + x, & \text{if } m = t;
\end{cases}
\]  

(29)

belongs to \( Aut(X) \), that is, it satisfies (5)-(9); if so, then it will belong to \( K \) by Lemma 4.1(a). Note also, that verifying (5)-(7) we exclude the trivial case \( m = s \).

Verification of (5). Here the case \( m = 1 \) is also trivial, and so we only need to consider the case \( m = t \). We have:

\[
\begin{align*}
  t\alpha(1,x) &= t(d(tx) + x) = td(tx) + tx = d(tx) - d(tx) + td(tx) + tx \\
  &= d(ttx) - d(stx) + td(stx) + tx = \alpha(t,tx) = \alpha(t1,tx);
\end{align*}
\]
\[ t\alpha(s, x) = ts = \alpha(s, tx) = \alpha(ts, tx); \]

\[ t\alpha(t, x) = td(tx) - td(sx) + td(sx) + tx = td(tx) + tx = \alpha(t, tx) = \alpha(tt, tx), \]
(where the equality \( td(stx) + tx = \alpha(t, tx) \) already has appeared in the first calculation of this proof).

Verification of (6). First note that, for all \( b_1 \) and \( b_2 \) in 
\[ B = B_X = \{ x \in X \mid sx = x \} = \{ x \in X \mid tx = x \}, \]
we have
\[ d(b_1 + b_2) = d(b_1) + b_1 + d(b_2) - b_1 \tag{30} \]
in \( X \), as follows from (19) and (16). Then, suppose \( m = 1 \); we have:
\[ \alpha(1, x_1 + x_2) = d(tx_1 + tx_2) + x_1 + x_2 = d(tx_1) + tx_1 + d(tx_2) - tx_1 + x_1 + x_2 \ (\text{by (30)}) \]
\[ = d(tx_1) + tx_1 - tx_1 + x_1 + d(tx_2) + x_2 \ (\text{since } sd(tx_2) = 0 = t(-tx_1 + x_1)) \]
\[ = d(tx_1) + x_1 + d(tx_2) + x_2 = \alpha(1, x_1) + \alpha(1, x_2). \]

Now suppose \( m = t \); we have:
\[ \alpha(t, x_1 + x_2) = d(tx_1 + tx_2) - d(sx_1 + sx_2) + td(sx_1 + sx_2) + x_1 + x_2 \]
\[ = (d(tx_1) + tx_1 + d(tx_2) - tx_1) - (d(sx_1) + sx_1 + d(sx_2) - sx_1) + t(d(sx_1) + sx_1 + d(sx_2) - sx_1) + x_1 + x_2 \ (\text{by (30)}) \]
\[ = d(tx_1) + tx_1 + d(tx_2) - tx_1 + sx_1 - d(sx_2) - sx_1 + d(sx_1) + sx_1 + td(sx_2) - sx_1 + x_1 + x_2 \]
\[ \quad \text{(using } ts = s \text{ twice)} \]
\[ = d(tx_1) - d(sx_1) + td(sx_1) + tx_1 + d(tx_2) - tx_1 + sx_1 - d(sx_2) - sx_1 + sx_1 + td(sx_2) - sx_1 + x_1 + x_2 \]
\[ \quad \text{(since } s(tx_1 + d(tx_2) - tx_1 + sx_1 - d(sx_2) - sx_1) = 0 = t(-d(sx_1) + td(sx_1))) \]
\[ = d(tx_1) - d(sx_1) + td(sx_1) + tx_1 + d(tx_2) - tx_1 + sx_1 - d(sx_2) + td(sx_2) - sx_1 + x_1 + x_2 \]
\[ = d(tx_1) - d(sx_1) + td(sx_1) + tx_1 + d(tx_2) - tx_1 + sx_1 - sx_1 + x_1 - d(sx_2) + td(sx_2) + x_2 \]
\[ \quad \text{(since } s(-sx_1 + x_1) = 0 = t(-d(sx_2) + td(sx_2))) \]
\[ = d(tx_1) - d(sx_1) + td(sx_1) + tx_1 + d(tx_2) - tx_1 + x_1 + d(tx_2) - d(sx_2) + td(sx_2) + x_2 \]
\[ = d(tx_1) - d(sx_1) + td(sx_1) + x_1 + d(tx_2) - d(sx_2) + td(sx_2) + x_2 = \alpha(t, x_1) + \alpha(t, x_2). \]

Verification of (7). Let \( \beta : M \times X \rightarrow X \) be the map defined in the same way as \( \alpha \) but with \( d \) replaced with its inverse \( e \) in the group \( D(B, K) \). We have:
\[ \alpha(1, \beta(1, x)) = \alpha(1, e(tx) + x) = d(\alpha(e(tx) + x)) + e(tx) + x \]
\[ = d(te(tx) + tx) + e(tx) + x = (d + e)(tx) + x \ (\text{by the first equality of (20) and (17)}) \]
\[ = g(d + e)(1, x) \ (\text{by (27)}) \]
\[ = x \ (\text{which easily follows from } d + e = 0), \]
and similarly \( \beta(1, \alpha(1, x)) = x \), which proves that \( \alpha(1, -) : X \rightarrow X \) is a bijection.

Next, we have:
\[ \alpha(t, \beta(t, x)) = \alpha(t, e(tx) - e(sx) + te(sx) + x) \]
\[ = d(t(e(tx) - e(sx) + te(sx) + x)) - s(e(tx) - e(sx) + te(sx) + x) + td(s(e(tx) - e(sx) + te(sx) + x)) \]
te(sx) + x)) + e(tx) - e(sx) + te(sx) + x
= d(te(tx) - te(sx) + te(sx) + tx) - d(se(tx) - se(sx) + te(sx) + sx) + td(se(tx) - se(sx) + te(sx) + sx) + e(tx) - e(sx) + te(sx) + x
= d(te(tx) + tx) - d(te(sx) + sx) + td(te(sx) + sx) + e(tx) - e(sx) + te(sx) + x (using the fact that se(tx) = 0 = se(sx) twice)
= d(te(tx) + tx) + e(tx) - e(sx) - d(te(sx) + sx) + t(d(te(sx) + sx)) + te(sx) + x (since s(e(tx) - e(sx)) = 0 = t(-d(te(sx) + sx)) + t(d(te(sx) + sx)))
= (d + e)(tx) - (d + e)(sx) + t(d + e)(sx) + x (by the first equality of (20) and (17))
= g(d + e)(t, x) (by (27))
= x (since d + e = 0 again).
After that we conclude that \( \alpha(m, -) : X \to X \) is a bijection in the same way as we did for \( \alpha(1, -) \).

Verification of (8) and (9). We have:
\[
\alpha(1, tx) - \alpha(s, tx) + \alpha(s, x) = d(tx) + tx - tx + x = d(tx) + x = \alpha(1, x),
\]
and
\[
\alpha(t, x) - \alpha(t, sx) + \alpha(1, sx) = d(tx) - d(sx) + td(sx) + x - (d(tsx) - d(ssx) + td(ssx) + sx) + d(tsx) + sx
= d(tx) - d(sx) + td(sx) + x - sx - td(sx) + d(sx) + sx
= d(tx) - d(sx) + td(sx) - td(sx) + d(sx) + x - sx + sx (since s(x - sx) = 0
= t(-td(sx) + d(sx)))
= d(tx) + x = \alpha(1, x).
\]

4.4. Lemma. The maps \( f : K \to D(B, K) \) and \( g : D(B, K) \to \bar{K} \), of Lemmas 4.2 and 4.3, are inverse of each other.

Proof. \( gf = 1_K \) by the definition of \( g \). For \( fg \), any \( d \in D(B, K) \), and any \( x \in B \), we have:

\[
((fg))(d)(x) = (f(g)(d))(x) = g(d)(1, x) - x = d(tx) + x - x \quad \text{(by (27))}
= d(tx) = d(x) \quad \text{(since } x \text{ is in } B).\]

That is, \( fg(d) = d \) (for any \( d \in D(B, K) \)), and so \( fg = 1_{D(B, K)} \).

4.5. Lemma. The assignment
\[
p(\alpha) = (p_1(\alpha), p_2(\alpha)), \text{ where } p_1(\alpha)(k) = \alpha(1, k) \text{ and } p_2(\alpha)(b) = \alpha(1, b), \quad (31)
\]
defines a map \( p : B \to \text{Aut}(K, B, \partial) \).

Proof. Let us write \( p(\alpha) = (\kappa, \beta) \). First of all, for \( k \in K \), we have
\[ s(\kappa(k)) = s\alpha(1, k) = \alpha(s, sk) \quad \text{(by (5))} \]
\[ = \alpha(s, 0) \quad \text{(by (14))} \]
\[ = 0 \quad \text{(the fact that } \alpha(s, -) \text{ preserves zero follows from (6)),} \]

and, for \( b \in B \), we have

\[ s\beta(b) = s\alpha(1, b) = \alpha(s, sb) \quad \text{(as above)} \]
\[ = \alpha(s, b) \quad \text{(by (15))} \]
\[ = \beta(b); \]

that is, \( \kappa(k) \) and \( \beta(b) \) belong to \( K \) and \( B \), respectively. After that (6) (together with (11)) and (7) tell us that \( (\kappa, \beta) \) belongs to \( \text{Aut}(K) \times \text{Aut}(B) \), and we only need to prove that \( \beta\partial = \partial\kappa \) and \( \kappa(bk) = \beta(b)\kappa(k) \) for all \( k \in K \) and \( b \in B \). We have:

\[ \beta\partial(k) = \beta(tk) \quad \text{(by (17))} \]
\[ = \alpha(1, tk) \quad \text{(by (31))} \]
\[ = \alpha(t, tk) \quad \text{(by Lemma 4.1(b))} \]
\[ = t\alpha(1, k) \quad \text{(by (5))} \]

\[ = \partial\kappa(k), \]

\[ \kappa(bk) = \alpha(1, bk) = \alpha(1, b + k - b) \quad \text{(by (16))} \]
\[ = \alpha(1, b) + \alpha(1, k) - \alpha(1, b) \quad \text{(by (6))} \]
\[ = \beta(b) + \kappa(k) - \beta(b) = \beta(b)\kappa(k) \quad \text{(by (16)),} \]

and so \( (\kappa, \beta) \) belongs to \( \text{Aut}(K, B, \partial) \) as desired.

**4.6. Lemma.** The map \( p : B \to \text{Aut}(K, B, \partial) \) is a group homomorphism.

**Proof.** For \( \alpha_1 \) and \( \alpha_2 \) in \( B \), \( k \) in \( K \), and \( b \) in \( B \), we have to prove that \( p_1(\alpha_1 + \alpha_2)(k) = p_1(\alpha_1)(p_1(\alpha_2)(k)) \) and \( p_2(\alpha_1 + \alpha_2)(b) = p_2(\alpha_1)(p_2(\alpha_2)(b)) \), that is, to prove that

\[ (\alpha_1 + \alpha_2)(1, k) = \alpha_1(1, \alpha_2(1, k)) \]
\[ \text{and } (\alpha_1 + \alpha_2)(1, b) = \alpha_1(1, \alpha_2(1, b)), \]

but both of these equalities are special cases of (11).

**4.7. Lemma.** The formula

\[ q(\kappa, \beta)(m, x) = \kappa(x - sx) + \beta(sx) \quad \text{(32)} \]

defines a map \( q : \text{Aut}(K, B, \partial) \to B \).

**Proof.** First of all, the expression \( \kappa(x - sx) + \beta(sx) \) is well-defined, which follows from (14) and (15), since \( s(x - sx) = 0 \) and \( s(sx) = sx \). Next, it is independent of \( m \), as needed according to Lemma 4.1(b). After that we write \( \alpha(m, x) = \kappa(x - sx) + \beta(sx) \), and we have to prove that conditions (5)-(9) are satisfied.
Verification of (5). The case \( m = 1 \) is trivial. We have
\[
s\alpha(m', x) = s\kappa(x - sx) + s\beta(sx) = \beta(sx) \quad \text{(by (14) and (15), since \( \kappa(x - sx) \in K \) and \( \beta(sx) \in B \))}
\]
\[
= \kappa(sx - ssx) + \beta(ssx) = \alpha(smx', sx);
\]
\[
t\alpha(m', x) = t\kappa(x - sx) + t\beta(sx) = \partial\kappa(x - sx) + \beta(sx) \quad \text{(using (17) and the fact that \( \beta(sx) \) is in \( B \))}
\]
\[
= \beta(tx - sx) + \beta(sx) \quad \text{(by (17))}
\]
\[
= \beta(tx) - \beta(sx) + \beta(sx) \quad \text{(since \( \beta \) is in \( Aut(B) \), and both \( tx \) and \( sx \) are in \( B \))}
\]
\[
= \beta(tx) = \kappa(tx - stx) + \beta(stx) = \alpha(tm', tx).
\]
Verification of (6). We have
\[
\alpha(m, x_1 + x_2) = \kappa(x_1 + x_2 - s(x_1 + x_2)) + \beta(s(x_1 + x_2)) = \kappa(x_1 + x_2 - sx_2 - sx_1) + \beta(sx_1 + sx_2)
\]
\[
= \kappa(x_1 - sx_1 + sx_1 + x_2 - sx_2 - sx_1) + \beta(sx_1) + \beta(sx_2) \quad \text{(since both \( sx_1 \) and \( sx_2 \) are in \( B \))}
\]
\[
= \kappa(x_1 - sx_1 + (sx_1)(x_2 - sx_2)) + \beta(sx_1) + \beta(sx_2) \quad \text{(by (16))}
\]
\[
= \kappa(x_1 - sx_1) + \beta((sx_1)(x_2 - sx_2)) + \beta(sx_1) + \beta(sx_2) \quad \text{(since both \( x_1 - sx_1 \) and \( (sx_1)(x_2 - sx_2) \) are in \( K \))}
\]
\[
= \kappa(x_1 - sx_1) + \beta(sx_1) + \kappa((sx_1)(x_2 - sx_2)) + \beta(sx_1) + \beta(sx_2) \quad \text{(using (17))}
\]
\[
= \kappa(x_1 - sx_1) + \beta(sx_1) + \kappa(x_2 - sx_2) + \beta(sx_2)
\]
\[
= \alpha(m, x_1) + \alpha(m, x_2).
\]
Verification of (7). We define a map \( \gamma : X \to X \) by \( \gamma(x) = \kappa^{-1}(x - sx) + \beta^{-1}(sx) \), and calculate
\[
\gamma(\alpha(m, x)) = \kappa^{-1}(\alpha(m, x) - s\alpha(m, x)) + \beta^{-1}(s\alpha(m, x))
\]
\[
= \kappa^{-1}(k(x - sx) + \beta(sx) - s(\kappa(x - sx) + \beta(sx))) + \beta^{-1}(s(\kappa(x - sx) + \beta(sx)))
\]
\[
= \kappa^{-1}(k(x - sx)) + \beta^{-1}(\beta(sx)) \quad \text{(since \( s\kappa(x - sx) = 0 \) and \( s\beta(sx) = \beta(sx) \), which follows from (14) and (15), respectively)}
\]
\[
= x - sx + sx
\]
\[
= x,
\]
which tells us that the composite \( \gamma\alpha(m, -) \) is the identity map of \( X \); similarly, so is \( \alpha(m, -)\gamma \). This proves (7).

Verification of (8) and (9): just use the fact that \( \alpha(m, x) \) is independent of \( m \).

4.8. Lemma. The maps \( p : B \to Aut(K, B, \partial) \) and \( q : Aut(K, B, \partial) \to B \), of Lemmas 4.5 and 4.7, are inverse of each other.

Proof. To prove that \( pq \) is the identity map of \( Aut(K, B, \partial) \) is to prove that, for each \( (\kappa, \beta) \in Aut(K, B, \partial) \), we have
\[ p_1q(\kappa, \beta) = \kappa \text{ and } p_2q(\kappa, \beta) = \beta, \]

which means that, for each \( k \in K \) and each \( b \in B \), we have
\[ \kappa(k - sk) + \beta(sk) = \kappa(k) \text{ and } \kappa(b - sb) + \beta(sb) = \beta(b). \]
The first of these two formulas follows from (14), while the second one follows from (15).

To prove that \( qp \) is the identity map of \( B \) is to prove that, for each \( \alpha \in B \), we have
\[ qp(\alpha) = \alpha, \]
which means that, for each \( (m, x) \in M \times X \), we have
\[ \alpha(1, x - sx) + \alpha(1, sx) = \alpha(m, x). \]
Indeed,
\[ \alpha(1, x - sx) + \alpha(1, sx) = \alpha(1, x) \text{ (by (6))}, \text{ and } \alpha(1, x) = \alpha(m, x) \text{ (by Lemma 4.1(b))}. \]

4.9. Lemma. The diagram
\[ K \xrightarrow{\partial} B \xrightarrow{p} \]
\[ D(B, K) \xrightarrow{\Delta} Aut(K, B, \partial) \]
commutes.

Proof. To prove that diagram (33) commutes, is to prove that, for each \( \alpha \in K \), we have
\[ p_1(\partial(\alpha)) = \Delta_1(f(\alpha)) \text{ and } p_2(\partial(\alpha)) = \Delta_2(f(\alpha)), \]
which means that, for each \( k \in K \) and each \( b \in B \), we have
\[ \alpha(t, k) = f(\alpha)\partial(k) + k \text{ and } \alpha(t, b) = \partial f(\alpha)(b) + b, \]
or, equivalently (by (17) and our definition of \( f \)),
\[ \alpha(t, k) = \alpha(1, tk) - tk + k \text{ and } \alpha(t, b) = t\alpha(1, b) - tb + b. \]
The first of these two formulas follows from (8) and Lemma 4.1(a), while the second one follows from (5) and \( tb = b \) (by (15)).
4.10. Lemma. For every $k \in K$ and every $b \in B$, we have $f(bk) = p(b)f(k)$.

Proof. For any $x \in B$, we have

\[
(p(b)f(k))(x) = (p_1(b)f(k)p_2(b)^{-1})(x) \quad \text{(by (21))}
\]
\[
= p_1(b)(k_1,b(1,-)^{-1}(x) - b_1(1,-)^{-1}(x)) \quad \text{(by (25))}
\]
\[
= b_1(k_1,b(1,-)^{-1}(x)) - b_1(1,-)^{-1}(x) \quad \text{(by (31))}
\]
\[
= b_1(k_1,b_1(1,-)^{-1}(x)) - b_1(b_1(1,-)^{-1}(x)) \quad \text{(by (6))}
\]
\[
= b_1(k_1,b_1(1,-)^{-1}(x)) - x = bk_1(1,x) - x \quad \text{(by Lemma 4.1(c))}
\]
\[
= f(bk)(x) \quad \text{(by (25))}.
\]

Putting these lemmas together, we obtain the desired isomorphism (24). More precisely, we obtain

4.11. Theorem. Let $f, g, p, \text{ and } q$ be defined as in Lemmas 4.2, 4.3, 4.5, and 4.7, respectively. Then

\[
(f, p) : (K, B, \partial) \longrightarrow (D(B,K), \text{Aut}(K,B,\partial), \Delta)
\]

is an isomorphism of crossed modules, whose inverse is $(g, q)$.

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