*Theory and Applications of Categories,* Vol. 34, No. 39, 2019, pp. 1220–1245.

# A MODEL STRUCTURE ON PREDERIVATORS FOR $(\infty,1)$-CATEGORIES

DANIEL FUENTES-KEUTHAN, MAGDALENA KĘDZIOREK AND MARTINA ROVELLI

Abstract. By theorems of Carlson and Renaudin, the theory of $(\infty,1)$-categories embeds in that of prederivators. The purpose of this paper is to give a two-fold answer to the inverse problem: understanding which prederivators model $(\infty,1)$-categories, either strictly or in a homotopical sense. First, we characterize which prederivators arise on the nose as prederivators associated to quasicategories. Next, we put a model structure on the category of prederivators and strict natural transformations, and prove a Quillen equivalence with the Joyal model structure for quasicategories.

## Introduction

The notion of prederivator appeared independently and with different flavours in works by Grothendieck [Gro], Heller [Hel88] and Franke [Fra96]. A *prederivator* is a contravariant 2-functor $\mathbb{D}\colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$, which we regard as minimally recording the homotopical information of a given $(\infty,1)$-category.

The idea is that the value $\mathbb{D}(J)$ at a small category $J$ represents the homotopy category of $J$-shaped diagrams of the desired homotopy theory. For example: the prederivator $D_{\mathcal{C}}$ associated to any ordinary 1-category $\mathcal{C}$ records the functor categories $D_{\mathcal{C}}(J) := \mathcal{C}^J$, the prederivator $\mathbb{H}o(\mathcal{M})$ associated to any model category $\mathcal{M}$, defined by $\mathbb{H}o(\mathcal{M})(J) := \mathcal{M}^J[W^{-1}]$, is obtained by inverting the class $W$ of levelwise weak equivalences of $J$-shaped diagrams in $\mathcal{M}$, and the prederivator $\mathbb{H}o(X)$ associated to any quasi-category $X$ can be realized as $\mathbb{H}o(X)(J) := \mathrm{ho}(X^J)$, where $\mathrm{ho}\colon s\mathcal{S}et \to \mathcal{C}at$ denotes the left adjoint to the nerve functor.

The prederivator associated to an $(\infty,1)$-category of some flavour is suitably homotopy invariant. The initial intuition might suggest that passing to prederivators would cause a significant loss of information, given that it involves taking homotopy categories. However, as pointed out in Shulman's note [Shu], "derivators seem to suffice for all sorts of things that one might want to do in an $(\infty,1)$-category".

A heuristic explanation can be attributed to the fact that the data of the prederivator

---

The second author was supported by the NWO Veni grant 639.031.757. The third author was partially funded by the Swiss National Science Foundation, grant P2ELP2_172086.

Received by the editors 2018-10-19 and, in final form, 2019-11-06.

Transmitted by Kate Ponto. Published on 2019-11-13.

2010 Mathematics Subject Classification: 55U35, 18G30, 18A25.

Key words and phrases: prederivator, model structure, $(\infty,1)$-category, quasi-category.

associated to $X$ records the collection $\mathrm{ho}(X^J)$, where the parameter $J$ runs over $\mathcal{C}at^{\mathrm{op}}$, and is therefore a true enhancement of the bare homotopy category $\mathrm{ho}(X)$ of the $(\infty,1)$-category $X$. Additional evidence is given in the work of Riehl-Verity [RV15, RV], who show that much of $(\infty,1)$-category theory can be recovered by truncating each functor $(\infty,1)$-category $X^Y$ between two $(\infty,1)$-categories $X$ and $Y$ to their homotopy category $\mathrm{ho}(X^Y)$.

A more rigorous validation of the fact that prederivators carry all the relevant information of a given $(\infty,1)$-category was given by Renaudin in [Ren09]. He proves that the bi-localization of the 2-category of left-proper combinatorial model structures at the class of Quillen equivalences embeds in the 2-category of prederivators and pseudonatural transformations. Carlson shows in [Car16] an analogous result in a different framework. He proved that the functor $\mathbb{H}o$ actually gives a simplicial embedding $\mathbb{H}o\colon q\mathcal{C}at \to p\mathcal{D}er^{st}$ of the category of (small) quasicategories in that of (small) prederivators and strict natural transformations.

Inspired by the classical theorem of Brown representability, Carlson [Car16] raised the question of whether the essential image of $\mathbb{H}o$ could be characterized. In this paper, we provide such a characterization. More precisely, we recognise that all prederivators of the form $\mathbb{H}o(X)$ meet certain conditions, which we introduce in Section 2 under the terminology of "quasi-representability" (Section 2.8). Essentially, a prederivator is *quasi-representable* if and only if its value at the level of objects commutes with certain colimits, its value at the level of morphisms is suitably determined by that at the level of objects, and its underlying simplicial set is a quasi-category.

One of the main results of this paper is the following theorem, which appears in the paper as Section 2.16 and describes which prederivators arise from quasi-categories in a strict sense.

THEOREM A *A prederivator $\mathbb{D}$ is quasi-representable if and only if it is of the form $\mathbb{D} \cong \mathbb{H}o(X)$ for some quasi-category $X$.*

Next we concentrate on the homotopical analysis, identifying a suitable notion of weak equivalence of prederivators so that the homotopy category of $p\mathcal{D}er^{st}$ is equivalent to that of $q\mathcal{C}at$. We show in Section 3.6 that this class of weak equivalences is part of a model structure on $p\mathcal{D}er^{st}$ and that this model structure is equivalent to the model structure for quasi-categories. This is the main result of the paper, and it in particular validates prederivators as a model of $(\infty,1)$-categories.

THEOREM B *There exists a cofibrantly generated model structure on the category $p\mathcal{D}er^{st}$ of (small) prederivators and strict natural transformations that is Quillen equivalent to the Joyal model structure on the category $s\mathcal{S}et$ of simplicial sets.*

The desired model structure is transferred from the Joyal model structure on $s\mathcal{S}et$ using a certain functor $R\colon p\mathcal{D}er^{st} \to s\mathcal{S}et$, which will be defined in Construction 1.13 and was already used to prove [Car16, Proposition 2.9]. The methods that we use to transfer the model structure and prove the Quillen equivalence are formal, and rely on the fact that the functor $R$ fits in the middle of an adjoint triple and its left adjoint is fully faithful.

It is interesting to observe that, despite the functor $\mathbb{H}o\colon s\mathcal{S}et \to p\mathcal{D}er^{st}$ having been more extensively studied in the history of prederivators, it cannot be used to transfer a model structure in a standard fashion given that it does not admit an adjoint on either side.

Finally, we conclude the paper with an explicit description of the generating cofibrations, and a characterization of the fibrant objects and the acyclic fibrations in terms of suitable lifting properties. In particular, every quasi-representable prederivator is fibrant, and any fibrant prederivator is weakly equivalent to a quasi-representable one.

**Outline of the paper.** In Section 1 we introduce the category $p\mathcal{D}er^{st}$ of small prederivators and strict natural transformations, and the further structure that $p\mathcal{D}er^{st}$ possesses, which will be used later in the paper. In Section 2 we identify the image of

$$\mathbb{H}o\colon q\mathcal{C}at \to p\mathcal{D}er^{st}$$

as the class of quasi-representable prederivators. In Section 3 we right-transfer the Joyal model structure from $s\mathcal{S}et$ to $p\mathcal{D}er^{st}$ along the adjunction

$$L\colon s\mathcal{S}et \rightleftarrows p\mathcal{D}er^{st}\colon R,$$

study its properties, and prove the desired Quillen equivalence.

**Further directions.** The Quillen equivalence from Theorem B justifies that prederivators have the same *homotopy theory* as quasi-categories. In future work we aim to produce a rigorous comparison of the *category theory* of prederivators and that of quasi-categories[1].

The standard method to access the category theory of a model of $(\infty, 1)$-categories presented by a model category is to upgrade the model category to an $\infty$-cosmos, in the sense of Riehl-Verity. This is done by providing a model categorical enrichment over the Joyal model structure on simplicial sets. The model structure that we construct on the category of prederivators is unfortunately not enriched over the Joyal model structure, so we cannot conclude easily that the category of fibrant prederivators forms an $\infty$-cosmos.

However, given that Carlson proves that $\mathbb{H}o\colon q\mathcal{C}at \to s\mathcal{S}et$ is simplicially fully faithful, the functor $\mathbb{H}o$ induces an isomorphism of $\infty$-cosmoi onto its image. We plan to return to this topic in a future project and compare the 2-category of quasicategories (which has been developed e.g. in [Joy08, Lur09, RV]) with the 2-category of (quasi-representable) prederivators (which has been developed e.g. in [Gro13, Gro12]). Many independent results towards comparing the properties of certain 2-categorical constructions developed in terms of prederivators and in terms of quasi-categories have already been established; see e.g. [LV17, Lor18, Col19].

[1] Roughly speaking, the *homotopy theory* of a model of $(\infty, 1)$-categories is comprised of the information stored in the homotopy category of that model. It detects, for instance, when two given $(\infty, 1)$-categories are equivalent to one another. On the other hand, the *category theory* of a model of $(\infty, 1)$-categories focuses on the homotopy 2-category of the model. This focuses, for instance, on whether there is an adjunction between two given $(\infty, 1)$-categories, whether an $(\infty, 1)$-category is stable, or whether an element of an $(\infty, 1)$-category $Q$ is the limit of a given diagram in $Q$.

**Acknowledgements.** The authors are grateful to Emily Riehl for carefully reading an earlier draft of this paper and providing valuable feedback. We also would like to thank Kevin Carlson and Viktoriya Ozornova for many interesting discussions on the subject, and Bruno Vallette for helpful comments on the previous draft. The exposition benefited greatly from the referee's suggestions.

## 1. The category of prederivators

A prederivator is typically defined as a 2-functor $\mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$, where $\mathcal{C}at$ denotes the 2-category of small categories and $\mathcal{CAT}$ denotes the very large 2-category of large categories. Several authors (see e.g. [Ren09, Gro13, GPS14a, Car16]) have considered the (very large) category $p\mathcal{DER}^{st}$ of (large) prederivators and strict natural transformations, and the (very large) category $p\mathcal{DER}^{pseudo}$ of (large) prederivators and pseudonatural transformations. In this paper we are concerned with the former point of view.

We aim to study the homotopy theory of prederivators and compare it to the homotopy theory of quasi-categories, as presented by the Joyal model structure on the category $s\mathcal{S}et$ of small simplicial sets. In order to employ standard references for model category theory and enriched category theory, we will focus on a type of prederivators that assemble into a locally small category[2]. To this end, we replace the large 2-category $\mathcal{CAT}$ with the locally small 2-category $\mathcal{C}at$ as a target category of prederivators and we follow Grothendieck's approach from [Gro, §1.2] in replacing the (large) indexing 2-category $\mathcal{C}at^{op}$ with a smaller one: the 2-category *cat* of "homotopically finite categories", which was already considered e.g. by Carlson [Car16] and Muro-Raptis [MR11, MR16] (under the name of "finite direct categories"). With these arrangements, we show that the category of 2-functors $cat^{\mathrm{op}} \to \mathcal{C}at$ will turn out to be locally small.

1.1. Definition. [Car16, Definition 0.3] *A category $J$ is* homotopy finite*, also known as* finite direct*, if the nerve $NJ$ has finitely many nondegenerate simplices; equivalently, if $J$ is finite, skeletal, and admits no nontrivial endomorphisms. We denote by cat the full sub-2-category of $\mathcal{C}at$ consisting of homotopy finite categories.*

1.2. Example. Any category $[n]$ is homotopically finite, so *cat* includes $\Delta$ fully faithfully.

1.3. Example. The category containing a free isomorphism is not homotopically finite.

1.4. Example. The category associated to any infinite group is not homotopically finite.

1.5. Proposition. *The 2-category cat is essentially small, i.e., its class of isomorphism classes of objects is a set and all mapping categories between two objects are small categories.*

[2] While working with a locally small category of prederivators simplifies the exposition, this restriction is not necessary. An alternative approach would be to define prederivators to be 2-functors $\mathcal{C}at^{op} \to \mathcal{CAT}$, or 2-functors $\mathcal{C}at^{\mathrm{op}} \to \mathcal{C}at$, which assemble into a category that is not necessarily locally small, and to compare it with the very large category $s\mathcal{SET}$ of *large* simplicial sets. To see that all of our constructions go through unchanged upon ascending to a larger universe, see [Low13].

PROOF. We show that the class of isomorphism classes of objects of $cat$ is countable so in particular a set. If we denote by $C_{m,n}$ the class of isomorphism classes of homotopy finite categories which have precisely $m$ objects and $n$ arrows, then each $C_{m,n}$ is finite, and the set of objects

$$\mathrm{Ob}(cat) = \coprod_{m,n} C_{m,n},$$

is therefore countable.

Next, we observe that, for any homotopically finite categories $J$ and $K$, the class $\mathrm{Hom}_{cat}(J, K)$ of functors $J \to K$ is a set, given it is a subset of $\mathrm{Hom}_{\mathcal{S}et}(\mathrm{Mor}(J), \mathrm{Mor}(K))$.

Finally, we observe that for any homotopically finite categories $J$ and $K$ and functors $F, G\colon J \to K$, the class of strict natural transformations $F \Rightarrow G$ is a set, given that it is a subset of $\mathrm{Hom}_{\mathcal{S}et}(\mathrm{Ob}(J), \mathrm{Mor}(K))$. ∎

We mimic the usual definition of prederivator in our context.

1.6. DEFINITION. *A* (small) prederivator *is a 2-functor* $cat^{\mathrm{op}} \to \mathcal{C}at$. *We denote by* $p\mathcal{D}er^{st}$ *the category of (small) prederivators and (strict) natural transformations. We denote by* $\mathrm{Hom}_{p\mathcal{D}er^{st}}(\mathbb{D}, \mathbb{E})$ *the homset between two prederivators* $\mathbb{D}$ *and* $\mathbb{E}$.

1.7. PROPOSITION. *The category* $p\mathcal{D}er^{st}$ *is locally small.*

PROOF. Given that $\mathcal{C}at$ is a locally finitely presentable, and that $cat$ was shown to be an essentially small 2-category in Section 1.5, the category $p\mathcal{D}er^{st}$ is an instance of the functor category as described in [Kel82, §2.2], which is locally small. ∎

In this paper, all prederivators will be small and all natural transformations will be strict unless specified otherwise.

1.8. REMARK. Since $\mathcal{C}at$ is complete and cocomplete as a 2-category, as mentioned in [Kel82, §3.3] the category $p\mathcal{D}er^{st}$ of prederivators is complete and cocomplete, with limits and colimits computed pointwise in $\mathcal{C}at$.

The category of $p\mathcal{D}er^{st}$ is the underlying category of a 2-category. The hom-categories, which we denote by $\mathrm{Map}_{p\mathcal{D}er^{st}}(\mathbb{D}, \mathbb{E})$, are given by strict natural transformations $\mathbb{D} \to \mathbb{E}$ and modifications of such. This is discussed in more detail in [Gro13, §2.1].

As $p\mathcal{D}er^{st}$ is a category of enriched presheaves, standard constructions from enriched category theory apply.

1.9. NOTATION. [Kel82, §2.4] Let $D\colon \mathcal{C}at \to p\mathcal{D}er^{st}$ denote the Yoneda embedding, with the representable $D_K$ at a category $K$ being the 2-functor with values

$$D_K(J) \coloneqq K^J.$$

1.10. REMARK. The Yoneda embedding $D\colon \mathcal{C}at \to p\mathcal{D}er^{st}$ preserves binary products, i.e., there are isomorphisms of prederivators

$$D_{J\times K} \cong D_J \times D_K.$$

As proven in [Hel88, §4], the category $p\mathcal{D}er^{st}$ is cartesian closed.

1.11. PROPOSITION. *The category $p\mathcal{D}er^{st}$ is cartesian closed, with the internal hom $\mathbb{D}^{\mathbb{E}}$ given by*

$$\mathbb{D}^{\mathbb{E}}(J) \coloneqq \mathrm{Map}_{p\mathcal{D}er^{st}}(D_J \times \mathbb{E}, \mathbb{D}).$$

As proven in [MR16], the above cartesian closure can be used to enrich $p\mathcal{D}er^{st}$ over $s\mathcal{S}et$.

1.12. PROPOSITION. *The category $p\mathcal{D}er^{st}$ is the underlying category of a simplicial category whose hom-simplicial sets $\underline{\mathrm{Hom}}_{p\mathcal{D}er^{st}}(\mathbb{D}, \mathbb{E})_\bullet$ are given by*

$$\underline{\mathrm{Hom}}_{p\mathcal{D}er^{st}}(\mathbb{D}, \mathbb{E})_n \coloneqq \mathrm{Ob}(\mathbb{E}^{\mathbb{D}}([n])) = \mathrm{Hom}_{p\mathcal{D}er^{st}}(D_{[n]} \times \mathbb{D}, \mathbb{E}).$$

We remind the reader that at this point the category $p\mathcal{D}er^{st}$ has three relevant enrichments. We write $\mathrm{Hom}\, p\mathcal{D}er^{st}(\mathbb{D}, \mathbb{E})$ to denote the homset, $\underline{\mathrm{Hom}}_{p\mathcal{D}er^{st}}(\mathbb{D}, \mathbb{E})_\bullet$ to denote the simplicial enrichment, and $\mathrm{Map}_{p\mathcal{D}er^{st}}(\mathbb{D}, \mathbb{E})$ to denote the categorical enrichment. The following remark clarifies the relationship between them.

1.13. REMARK. For any prederivators $\mathbb{D}$ and $\mathbb{E}$, there is a canonical map

$$\alpha\colon \underline{\mathrm{Hom}}_{p\mathcal{D}er^{st}}(\mathbb{D}, \mathbb{E})_\bullet \to N\,\mathrm{Map}_{p\mathcal{D}er^{st}}(\mathbb{D}, \mathbb{E}).$$

The simplicial map $\alpha$ is induced on the set of $n$-simplices by postcomposition with the *underlying diagram* functors

$$dia_J^{[n]}\colon (\mathbb{E}^{D_{[n]}})(J) \to \mathbb{E}(J)^{[n]},$$

which are natural in $J$ and assemble into a map of prederivators $\mathbb{E}^{D_{[n]}} \to \mathbb{E}(\bullet)^{[n]}$. We refer the reader to [Gro13] for more details on the underlying diagram functors.

When $\mathbb{E} = D_K$ is represented by a category, the underlying diagram functors can be checked to be isomorphisms and the two enrichments agree, in the sense that $\alpha$ becomes an isomorphism

$$\alpha\colon \underline{\mathrm{Hom}}_{p\mathcal{D}er^{st}}(\mathbb{D}, D_K)_\bullet \cong N\,\mathrm{Map}_{p\mathcal{D}er^{st}}(\mathbb{D}, D_K).$$

This is not the case in general, even when $\mathbb{E} = \mathbb{H}o(X)$ is the prederivator associated to a quasi-category $X$. For instance, at the level of 1-simplices the underlying diagram functor can be identified with

$$\mathrm{ho}(X^{J\times[1]}) \to \mathrm{ho}(X^J)^{[1]},$$

and the corresponding map $\alpha$ is not bijective.

The enriched Yoneda Lemma from [Kel82, §2.4] specializes to the following.

1.14. Proposition. *There is a natural isomorphism of categories*

$$\mathrm{Map}_{p\mathcal{D}er^{st}}(D_J, \mathbb{E}) \cong \mathbb{E}(J).$$

*In particular, the isomorphisms induce natural bijections at the level of objects*

$$\mathrm{Hom}_{p\mathcal{D}er^{st}}(D_J, \mathbb{E}) \cong \mathrm{Ob}(\mathbb{E}(J)).$$

Given that $\Delta$ is a small category, that $s\mathcal{S}et$ is locally small and that $p\mathcal{D}er^{st}$ is cocomplete by Section 1.8, we use [Rie14, Construction 1.5.1], which is in turn a special case of the construction originally presented by Kan in [Kan58, §3], to obtain the following adjunction.

1.15. Construction. The restriction $D_{[\bullet]}\colon \Delta \subset \mathcal{C}at \to p\mathcal{D}er^{st}$ of the Yoneda embedding is a cosimplicial prederivator, and therefore induces an adjunction

$$L\colon s\mathcal{S}et \rightleftarrows p\mathcal{D}er^{st}\colon R.$$

The left adjoint $LX$ is the left Kan extension of $D_{[\bullet]}\colon \Delta \subset \mathcal{C}at \to p\mathcal{D}er^{st}$ along the Yoneda embedding $\Delta \subset s\mathcal{S}et$, explicitly

$$LX = \int^{[n]\in\Delta} \mathrm{Hom}_{s\mathcal{S}et}(\Delta[n], X)\cdot D_{[n]} = \mathrm{colim}_{\Delta[n_i]\to X} D_{[n_i]},$$

and the right adjoint $R\mathbb{D}$, which we call the *underlying simplicial set* of $\mathbb{D}$, is defined by

$$(R\mathbb{D})_n := \mathrm{Hom}_{p\mathcal{D}er^{st}}(D_{[n]}, \mathbb{D}) \cong \mathrm{Ob}(\mathbb{D}([n])).$$

We now collect three properties of the functors $R$ and $L$ that will be needed later.

1.16. Proposition. *The functor $R$ admits a right adjoint, and in particular it preserves colimits.*

Proof. We first observe that the functor $R$ can be expressed as the following composite

$$p\mathcal{D}er^{st} \to \mathcal{C}at^{\Delta^{\mathrm{op}}} \to \mathcal{S}et^{\Delta^{\mathrm{op}}} = s\mathcal{S}et,$$

where the first functor is the restriction along the inclusion of the discrete 2-category $\Delta^{\mathrm{op}}$ into the full 2-subcategory $cat^{\mathrm{op}}$ of $\mathcal{C}at$, and the second functor is induced by the functor $\mathrm{Ob}\colon \mathcal{C}at \to \mathcal{S}et$. In particular, since we are considering $\Delta^{\mathrm{op}}$ as a discrete 2-category, the category $\mathcal{C}at^{\Delta^{\mathrm{op}}}$ of ordinary functors coincides with the category of 2-functors.

Knowing from Section 1.5 that $cat$ is an essentially small 2-category, we can evoke [Kel82, Thorem 4.50] to say that the restriction along $\Delta^{\mathrm{op}} \to cat^{\mathrm{op}}$ admits a right 1-categorical adjoint, given by the enriched right Kan extension. The adjoint pair

$$\mathrm{Ob}\circ -\colon \mathcal{C}at^{\Delta^{\mathrm{op}}} \rightleftarrows s\mathcal{S}et\colon codisc \circ -$$

and the adjoint pair

$$(\Delta^{\mathrm{op}} \hookrightarrow cat^{\mathrm{op}})^*\colon p\mathcal{D}er^{st} \rightleftarrows \mathcal{C}at^{\Delta^{\mathrm{op}}}\colon Ran_{\Delta^{\mathrm{op}}\hookrightarrow cat^{\mathrm{op}}}$$

compose to an adjoint pair

$$R\colon p\mathcal{D}er^{st} \rightleftarrows \mathcal{C}at^{\Delta^{\mathrm{op}}} \rightleftarrows s\mathcal{S}et\colon G,$$

as desired. ∎

1.17. Remark. For any $K \in \mathcal{C}at$ there is a natural isomorphism of simplicial sets

$$RD_K \cong NK.$$

The functor $R$ is also a left inverse for $L$.

1.18. Proposition. *For any simplicial set $X$, the unit of the adjunction from Section 1.15 gives an isomorphism*

$$\eta_X \colon X \cong RL(X).$$

*In particular, the functor $L$ is fully faithful and the functor $R$ is a left inverse for $L$.*

Proof. We first prove that the unit of a representable simplicial set,

$$\eta_{\Delta[n]} \colon \Delta[n] \to RL(\Delta[n])$$

is an isomorphism. The component $m$ of the unit map,

$$\eta_{\Delta[n]_m} \colon \Delta[n]_m \to (RL\Delta[n])_m,$$

can be identified with the canonical isomorphism

$$\begin{array}{rcl} \Delta[n]_m & = & \mathrm{Hom}_{s\mathcal{S}et}(\Delta[m], \Delta[n]) \\ & \cong & \mathrm{Hom}_{\mathcal{C}at}([m],[n]) \\ & \cong & \mathrm{Hom}_{p\mathcal{D}er^{st}}(D_{[m]}, D_{[n]})) \\ & = & \mathrm{Hom}_{p\mathcal{D}er^{st}}(D_{[m]}, L(\Delta[n])) \\ & = & (RL\Delta[n])_m. \end{array}$$

As a consequence, the unit $\eta_{\Delta[n]}$ is an isomorphism.

We now show that the unit $\eta_X$ is an isomorphism for any simplicial set $X$. Given the canonical identification

$$\phi \colon \operatorname{colim}_{\Delta[n_i]\to X} \Delta[n_i] \cong X$$

and the fact that both $R$ and $L$ respect colimits, we obtain a further identification

$$\phi' \colon \operatorname{colim}_{\Delta[n_i]\to X}(RL\Delta[n_i]) \cong RL(\operatorname{colim}_{\Delta[n_i]\to X} \Delta[n_i]) \xrightarrow[\cong]{RL\phi} RLX\ .$$

Using the universal property of colimits and the naturality of $\eta$, a straightforward check shows that the following diagram commutes

$$\begin{tikzcd} \operatorname{colim}_{\Delta[n_i]\to X} \Delta[n_i] \arrow[r, "\operatorname{colim}\eta_{\Delta[n_i]}"] \arrow[d, "\cong"', "\phi"] & \operatorname{colim}_{\Delta[n_i]\to X} RL\Delta[n_i], \arrow[d, "\phi'"', "\cong"] \\ X \arrow[r, "\eta_X"'] & RLX \end{tikzcd}$$

and in particular there is an isomorphism

$$\eta_X \cong \operatorname{colim}_{\Delta[n_i]\to X} \eta_{\Delta[n_i]}.$$

The right hand map is an isomorphism, given that it is a colimit of isomorphisms, and so we conclude that $\eta_X$ is an isomorphism as well. ∎

The functor $L$ does not respect products, as shown by the following example. This obstruction will play an important role in a later discussion on the $(\infty, 2)$-categorical nature of $p\mathcal{D}er^{st}$. See Digression 3.18 for more details.

1.19. Example. We show that the canonical map

$$L(\Delta[1] \times \Delta[1]) \to L(\Delta[1]) \times L(\Delta[1])$$

is not an isomorphism of prederivators.

First, by construction of $L$ and by Section 1.10 we observe that

$$L(\Delta[1]) \times L(\Delta[1]) \cong D_{[1]} \times D_{[1]} \cong D_{[1]\times[1]},$$

while, using the fact that $\Delta[1] \times \Delta[1] \cong \Delta[2] \amalg_{\Delta[1]} \Delta[2]$, we obtain that

$$\begin{array}{rcl} L(\Delta[1] \times \Delta[1]) & \cong & L(\Delta[2] \amalg_{\Delta[1]} \Delta[2]) \\ & \cong & L(\Delta[2]) \amalg_{L(\Delta[1])} L(\Delta[2]) \\ & \cong & D_{[2]} \amalg_{D_{[1]}} D_{[2]}. \end{array}$$

Under these identifications, the comparison map can be written in the form

$$D_{[2]} \amalg_{D_{[1]}} D_{[2]} \to D_{[1]\times[1]}.$$

If we denote by $\Gamma$ the span shape category $\bullet \leftarrow \bullet \rightarrow \bullet$, it is enough to show that the induced map

$$f\colon \operatorname{Ob}((D_{[2]} \amalg_{D_{[1]}} D_{[2]})(\Gamma)) \to \operatorname{Ob}(D_{[1]\times[1]}(\Gamma))$$

which can be rewritten as

$$f\colon \operatorname{Ob}([2]^{\Gamma}) \amalg_{\operatorname{Ob}([1]^{\Gamma})} \operatorname{Ob}([2]^{\Gamma}) \to \operatorname{Ob}(([1] \times [1])^{\Gamma})$$

is not surjective. To this end, we observe that the span shape $\Gamma \to [1] \times [1]$, whose image is $(0, 1) \leftarrow (0, 0) \rightarrow (1, 0)$, defines an object of the category $([1] \times [1])^{\Gamma}$ that is not in the image of $f$. Indeed, by unravelling the definitions, one can see that any diagram $d'\colon \Gamma \to [1] \times [1]$ lying in the image of $f$ has to factor through one of the non degenerate 2-simplices of $[1] \times [1]$.

## 2. Quasi-representable prederivators

The Yoneda embedding $D\colon \mathcal{C}at \to p\mathcal{D}er^{st}$ from Section 1.9 provides a natural way to produce a prederivator from any category. There is in fact a canonical construction, which appears and is an object of study in several sources such as [Gro13, GPS14b, Car16, Len17, RV17], to extend the Yoneda embedding along the nerve inclusion $N\colon \mathcal{C}at \to s\mathcal{S}et$, and produce a prederivator from any quasi-category (and in fact from any simplicial set). This construction makes use of the "homotopy category" of a simplicial set.

2.1. Remark. We recall that the nerve functor admits a left adjoint $\mathrm{ho}\colon s\mathcal{S}et \to \mathcal{C}at$, which acts as a 1-truncation, see e.g. [RV, Definition 1.1.10] sending a simplicial set to its *homotopy category*. By [RV, Lemma 1.1.12], when $X$ is a quasi-category the homotopy category $\mathrm{ho}(X)$ has as its set of objects $\mathrm{Ob}(\mathrm{ho}(X)) \coloneqq X_0$, and as its set of morphisms the homotopy classes of 1-simplices of $X$. As defined in [RV, Definition 1.1.7], two 1-simplices $f$ and $g$ from $x$ to $y$ are homotopic if there exists a 2-simplex $\sigma$ such that

$$d_0(\sigma) = f, d_1(\sigma) = g \text{ and } d_2(\sigma) = s_0(x).$$

This set of morphisms can be described as the coequalizer

$$\mathrm{Mor}(\mathrm{ho}(X)) \coloneqq \mathrm{coeq}\left(X_0 \times_{X_1}^{d_2} X_2 \rightrightarrows X_1\right)$$

of the structure maps induced by the faces $d_0, d_1\colon X_2 \to X_1$.

2.2. Definition. *For any simplicial set $X$, its* homotopy prederivator *is the prederivator $\mathbb{H}o(X)$ that is defined by*

$$\mathbb{H}o(X)(J) \coloneqq \mathrm{ho}(X^{NJ}).$$

*Upon restricting the domain, this construction defines a functor $\mathbb{H}o\colon q\mathcal{C}at \to p\mathcal{D}er^{st}$.*

2.3. Remark. Using the isomorphism $\mathrm{ho}(NJ) \cong J$ for any $J \in \mathcal{C}at$, one can see that there is an isomorphism of prederivators

$$D_J \cong \mathbb{H}o(NJ).$$

In [Car16], Carlson observed that this functor is simplicial when the category $p\mathcal{D}er^{st}$ is endowed with the simplicial structure recalled in Section 1.12, and he proves that it is a simplicial embedding.

2.4. Theorem. [Car16, Theorem 2.1] *The functor $\mathbb{H}o\colon q\mathcal{C}at \to p\mathcal{D}er^{st}$ is simplicially fully faithful, i.e., it induces isomorphisms of simplicial sets*

$$\mathbb{H}o\colon\ \mathrm{Map}_{q\mathcal{C}at}(X, X')_\bullet \cong \underline{\mathrm{Hom}}_{p\mathcal{D}er^{st}}(\mathbb{H}o(X), \mathbb{H}o(X'))_\bullet.$$

The functor $R$ provides a left simplicial inverse for $\mathbb{H}o$.

2.5. Lemma. *The functor $R$ is a left inverse for $\mathbb{H}o$, i.e., for any simplicial set $X$ there is a natural isomorphism of simplicial sets*

$$R\mathbb{H}o(X) \cong X.$$

Proof. By definition, we obtain the natural identification

$$(R\mathbb{H}o(X))_n \coloneqq \mathrm{Ob}(\mathbb{H}o(X)[n]) = \mathrm{Ob}(\mathrm{ho}(X^{\Delta[n]})) = (X^{\Delta[n]})_0 = X_n,$$

which yields an isomorphism $R\mathbb{H}o(X) \cong X$, as desired. ∎

The following will be used later.

2.6. Proposition. *For any simplicial set $X$ and category $K$, there are isomorphisms of prederivators*

$$\mathbb{H}o(X)^K \cong \mathbb{H}o(X^{NK}),$$

*where $\mathbb{H}o(X)^K$ is defined by $\mathbb{H}o(X)^K(J) := \mathbb{H}o(X)(J \times K)$.*

Proof. By direct inspection and using Section 1.10, we see that there are isomorphisms of categories

$$\begin{array}{rcl} (\mathbb{H}o(X)^K)(J) & := & \mathbb{H}o(X)(J \times K) \\ & \cong & \mathrm{ho}(X^{NJ \times NK}) \\ & \cong & \mathrm{ho}((X^{NK})^{NJ}) \\ & \cong & \mathbb{H}o(X^{NK})(J), \end{array}$$

as desired. ∎

2.7. Remark. To clarify the role played by the different functors, we recap how the different functor between simplicial sets and prederivators interact with the inclusions given by the nerve construction and the Yoneda embedding:

$$N \colon \mathcal{C}at \to s\mathcal{S}et \quad \text{and} \quad D \colon \mathcal{C}at \to p\mathcal{D}er^{st}.$$

As a consequence of Sections 1.17 and 2.5, we see that the diagrams

$$\begin{tikzcd} \mathcal{C}at \arrow[r, "D"] \arrow[d, "N"'] & p\mathcal{D}er^{st} \\ s\mathcal{S}et \arrow[ur, "\mathbb{H}o"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C}at \arrow[r, "D"] \arrow[d, "N"'] & p\mathcal{D}er^{st}, \arrow[dl, "R"] \\ s\mathcal{S}et & \end{tikzcd}$$

commute up to isomorphism, whereas the diagram

$$\begin{tikzcd} \mathcal{C}at \arrow[r, "D"] \arrow[d, "N"'] & p\mathcal{D}er^{st} \\ s\mathcal{S}et \arrow[ur, "L"'] & \end{tikzcd}$$

does not. If it did, we would then obtain

$$L(\Delta[1] \times \Delta[1]) \cong L(N([1] \times [1])) \cong D_{[1]\times[1]} \cong D_{[1]} \times D_{[1]} \cong L\Delta[1] \times L\Delta[1],$$

contradicting Section 1.19.

We now address the question of identifying the essential image of the functor

$$\mathbb{H}o \colon q\mathcal{C}at \subset s\mathcal{S}et \to p\mathcal{D}er^{st}.$$

It follows from the definition of the functor $\mathbb{H}o$ that any prederivator of the form $\mathbb{D} = \mathbb{H}o(X)$ must send finite coproducts to finite products,

$$\mathbb{D}(J \amalg K) = \mathrm{ho}(X^{NJ \amalg NK}) \cong \mathrm{ho}(X^{NJ}) \times \mathrm{ho}(X^{NK}) = \mathbb{D}(J) \times \mathbb{D}(K),$$

but this condition is clearly not sufficient.

In this section, we will show that the prederivators of the form $\mathbb{D} = \mathbb{H}o(X)$ for some quasi-category $X$ are precisely the prederivators which satisfy the following three conditions, which we suggestively call the "quasi-representable prederivators".

2.8. DEFINITION. *A prederivator* $\mathbb{D}\colon cat^{\mathrm{op}} \to \mathcal{C}at$ *is* quasi-representable *if the following three conditions hold.*

1. *For any category $J \in cat$, the counit of the adjunction $(L, R)$*

$$LNJ \cong LR\mathbb{H}o(NJ) \xrightarrow{\epsilon_{NJ}} \mathbb{H}o(NJ) \cong D_J$$

*induces a bijection*

$$\mathrm{Ob}(\mathbb{D}(J)) \cong \mathrm{Hom}_{p\mathcal{D}er^{st}}(D_J, \mathbb{D}) \xrightarrow{\epsilon^*_{NJ}} \mathrm{Hom}_{p\mathcal{D}er}(LNJ, \mathbb{D}) \cong \mathrm{Hom}_{s\mathcal{S}et}(NJ, R\mathbb{D}).$$

2. *For any category $J \in cat$ the function induced by the underlying diagram functor*

$$dia_J^{[1]}\colon \mathbb{D}([1] \times J) \to \mathbb{D}(J)^{[1]}$$

*at the level of objects realizes a coequalizer diagram*

$$\mathrm{Ob}\left(\mathbb{D}([0] \times J) \times^{d_2}_{\mathbb{D}([1]\times J)} \mathbb{D}([2] \times J)\right) \rightrightarrows \mathrm{Ob}(\mathbb{D}([1] \times J)) \xrightarrow{\mathrm{Ob}(dia_J^{[1]})} \mathrm{Ob}\left(\mathbb{D}(J)^{[1]}\right)$$

*for the maps induced at the level of objects by*

$$\mathbb{D}([0] \times J) \times^{d_2}_{\mathbb{D}([1]\times J)} \mathbb{D}([2] \times J) \xrightarrow{pr_2} \mathbb{D}([2] \times J) \underset{\mathbb{D}(d^1\times J)}{\overset{\mathbb{D}(d^0\times J)}{\rightrightarrows}} \mathbb{D}([1] \times J).$$

3. *The underlying simplicial set $R\mathbb{D}$ is a quasi-category.*

The terminology is justified by the fact that we will prove in Section 2.16 that a prederivator is quasi-representable precisely when it is represented by a quasi-category. We start by showing the more direct implication.

2.9. PROPOSITION. *For any quasi-category $X$, the prederivator $\mathbb{H}o(X)$ is quasi-representable.*

PROOF. For Condition (1) of Section 2.8, we note that for any $J \in cat$ Section 2.5 yields a natural bijection

$$\mathrm{Hom}_{s\mathcal{S}et}(NJ, R\mathbb{H}o(X)) \cong \mathrm{Hom}_{s\mathcal{S}et}(NJ, X) \cong \mathrm{Ob}(\mathbb{H}o(X)(J)),$$

as desired.

In order to show that Condition (2) holds for $\mathbb{H}o(X)$ for any simplicial set $X$, we first prove it for $J = [0]$. In this case the diagram that we ought to show is a coequalizer is

$$\begin{array}{ccc} \mathrm{Ob}\left(\mathbb{H}o(X)([0]) \times^{d_2}_{\mathbb{H}o(X)([1])} \mathbb{H}o(X)([2])\right) & \rightrightarrows & \mathrm{Ob}(\mathbb{H}o(X)([1])) \\ & & \big\downarrow{\scriptstyle \mathrm{Ob}(dia^{[1]}_{[0]})} \\ & & \mathrm{Ob}\left(\mathbb{H}o(X)([0])^{[1]}\right). \end{array}$$

By the definition of $\mathbb{H}o(X)$, this diagram can be expressed as

$$X_0 \times^{d_2}_{X_1} X_2 \rightrightarrows X_1 \to \mathrm{Mor}(\mathrm{ho}(X)).$$

which is a coequalizer diagram by the description of the homotopy category $\mathrm{ho}(X)$ given in Section 2.1.

For the general case, the diagram that we need to show is a coequalizer is

$$\begin{array}{ccc} \mathrm{Ob}\left(\mathbb{H}o(X)([0]\times J) \times^{d_2}_{\mathbb{H}o(X)([1]\times J)} \mathbb{H}o(X)([2]\times J)\right) & \rightrightarrows & \mathrm{Ob}(\mathbb{H}o(X)([1]\times J)) \\ & & \big\downarrow{\scriptstyle \mathrm{Ob}(dia^{[1]}_{J})} \\ & & \mathrm{Ob}\left(\mathbb{H}o(X)(J)^{[1]}\right). \end{array}$$

By Section 2.6, this diagram can be rewritten as

$$\begin{array}{ccc} \mathrm{Ob}\left(\mathbb{H}o(X^{NJ})([0]) \times^{d_2}_{\mathbb{H}o(X^{NJ})([1])} \mathbb{H}o(X^{NJ})([2])\right) & \rightrightarrows & \mathrm{Ob}(\mathbb{H}o(X^{NJ})([1])) \\ & & \big\downarrow{\scriptstyle \mathrm{Ob}(dia^{[1]}_{[0]})} \\ & & \mathrm{Ob}\left(\mathbb{H}o(X^{J})([0])^{[1]}\right), \end{array}$$

and this was already observed to be a coequalizer because the prederivator $\mathbb{H}o(X^{NJ})$ satisfies condition (2) for $J = [0]$.

Finally, Condition (3) is a consequence of Section 2.5, which asserts that $R\mathbb{H}o(X) \cong X$. ∎

While Condition (3) in the definition above is self-explanatory, we elaborate on the meaning of conditions (1) and (2).

2.10. Remark. Given that for every prederivator $\mathbb{D}$ one finds the identification

$$\mathrm{Ob}\left(\mathbb{D}(J)^{[1]}\right) = \mathrm{Mor}(\mathbb{D}(J)),$$

Condition (2) essentially describes how the value of a quasi-representable derivator $\mathbb{D}$ on morphisms is completely determined by the $\mathcal{S}et$-valued functor $\mathrm{Ob} \circ \mathbb{D}$.

Condition (1), however, seems less transparent. We now explain how it can be interpreted as requiring compatibility of $\mathbb{D}$ with a certain class of colimits at the level of objects.

Recall that the nerve $N\colon \mathcal{C}at \to s\mathcal{S}et$ does not respect colimits in general. Therefore, when taking colimits of (nerves of) categories, we need to be careful. We focus on diagrams of the following form, for which the issue does not exist.

2.11. Definition. *Let $\{[n_i]\}_i\colon I \to \Delta \subset \mathcal{C}at$ be a diagram. We say that* the colimit of the diagram $\{[n_i]\}_i$ is created in $s\mathcal{S}et$ *if the colimit in $s\mathcal{S}et$ of the diagram $\{\Delta[n_i]\}_i\colon I \to s\mathcal{S}et$, obtained by postcomposing with the Yoneda embedding, is the nerve of some category $J \in cat$, i.e, if there is an isomorphism of simplicial sets*

$$\operatorname{colim}^{s\mathcal{S}et}_{i\in I} \Delta[n_i] \cong NJ.$$

*Since the nerve is fully faithful, by applying its left adjoint* $\mathrm{ho}\colon s\mathcal{S}et \to \mathcal{C}at$*, we in particular get an isomorphism of categories:*

$$\operatorname{colim}^{\mathcal{C}at}_{i\in I}[n_i] \cong J,$$

*which justifies the terminology*[3].

The following remark implies on the one hand that any category $J$ is a colimit of a diagram whose colimit is created in $s\mathcal{S}et$, and on the other hand that any quasi-representable derivator is determined on objects by its value on all $[n]$'s.

2.12. Remark. For every category $J$, the nerve $NJ$ is a presheaf and can therefore canonically be written as a colimit of representables $\Delta[n_i]$'s,

$$NJ \cong \operatorname{colim}^{s\mathcal{S}et}_{\Delta[n_i]\to NJ} \Delta[n_i],$$

indexed over the diagram $\Delta \downarrow X \to \Delta$ (see e.g. [Hov99, §3.1]). This presentation is natural in $J$. By definition, the colimit of this diagram is created in $s\mathcal{S}et$, and we obtain the isomorphism of categories

$$J \cong \operatorname{colim}^{\mathcal{C}at}_{[n_i]\to J}[n_i].$$

This essentially describes the fact that the category $J$ can be built by taking a copy of $[0]$ for any object of $J$, a copy of $[1]$ for any morphism of $J$, a copy of $[2]$ for any commutative triangle in $J$, a copy of $[3]$ for any triple of composable arrows in $J$, and so on.

2.13. Proposition. *A prederivator $\mathbb{D}\colon cat^{\mathrm{op}} \to \mathcal{C}at$ satisfies Condition (1) of Section 2.8 if and only if it satisfies the following condition.*

[3] In particular, the colimit of a diagram $\{[n_i]\}_i\colon I \to \Delta \subset \mathcal{C}at$ is created in $s\mathcal{S}et$ according to our definition if and only if it is created by the nerve functor $N\colon \mathcal{C}at \to s\mathcal{S}et$ in the sense of the more standard definition from [ML98, §V.1].

*(1') For any diagram of categories $\{[n_i]\}_i$ with colimit created in $s\mathcal{S}et$, the canonical map*

$$\mathbb{D}(\operatorname{colim}^{\mathcal{C}at}[n_i])) \to \lim{}^{\mathcal{C}at}\mathbb{D}([n_i]),$$

*induced by the map*

$$\operatorname{colim}^{p\mathcal{D}er} D_{[n_i]} \to D_{\operatorname{colim}^{cat}[n_i]},$$

*induces bijections at the level of objects*

$$\operatorname{Ob}(\mathbb{D}(\operatorname{colim}^{\mathcal{C}at}[n_i]))) \cong \operatorname{Ob}(\lim{}^{\mathcal{C}at}\mathbb{D}([n_i])) \cong \lim{}^{\mathcal{S}et}\operatorname{Ob}(\mathbb{D}([n_i])).$$

Proof. Because we can decompose any category $J$ as a colimit created in $s\mathcal{S}et$ by Section 2.12, Condition (1) becomes immediately equivalent to the assertion that for any colimit in $s\mathcal{S}et$ of the form $\operatorname{colim}\Delta[n_i] \cong NJ$, there is an isomorphism

$$\operatorname{Ob}(\mathbb{D}(\operatorname{colim}^{\mathcal{C}at}[n_i]))) \cong \operatorname{Hom}_{s\mathcal{S}et}(\operatorname{colim}^{s\mathcal{S}et}\Delta[n_i]), R\mathbb{D}).$$

Now observe that for any diagram $\{n_i\}_i$ whose colimit is created in $s\mathcal{S}et$ there are natural isomorphisms

$$\begin{array}{rcl}\operatorname{Hom}_{s\mathcal{S}et}(\operatorname{colim}^{s\mathcal{S}et}\Delta[n_i]), R\mathbb{D}) & \cong & \lim^{\mathcal{S}et}\operatorname{Hom}_{s\mathcal{S}et}([n_i], R\mathbb{D}) \\ & \cong & \lim^{\mathcal{S}et}(R\mathbb{D})_n \\ & \cong & \lim^{\mathcal{S}et}\operatorname{Ob}(\mathbb{D}([n_i])),\end{array}$$

and so if the isomorphism

$$\operatorname{Ob}(\mathbb{D}(\operatorname{colim}^{\mathcal{C}at}[n_i]))) \cong \operatorname{Hom}_{s\mathcal{S}et}(\operatorname{colim}^{s\mathcal{S}et}\Delta[n_i]), R\mathbb{D})$$

of Condition (1) holds, then so must the isomorphism

$$\operatorname{Ob}(\mathbb{D}(\operatorname{colim}^{\mathcal{C}at}[n_i]))) \cong \lim{}^{\mathcal{S}et}\operatorname{Ob}(\mathbb{D}([n_i]))$$

of Condition (1'), and vice versa. ∎

2.14. Example. For any $C \in \mathcal{C}at$, the representable prederivator $D_C$ (as described in Section 1.9) is quasi-representable.

2.15. Example. The following prederivators fail to be quasi-representable.

1. If $X$ is a non empty quasi-category, the prederivator $\mathbb{E} := \mathbb{H}o(X) \amalg D_{[0]}$ fails to satisfy condition (1) of Section 2.8. To see this, we observe for instance that this prederivator does not send coproducts to products, even at the level of objects:

$$\begin{array}{rcl}\operatorname{Ob}(\mathbb{E}([0]) \amalg [0])) & = & \operatorname{Ob}((\operatorname{ho}(X^{[0]\amalg[0]}) \amalg [0]) \\ & \cong & \operatorname{Ob}((\operatorname{ho}(X \times X)) \amalg [0]) \\ & \cong & \operatorname{Ob}((\operatorname{ho}(X) \times \operatorname{ho}(X)) \amalg [0]) \\ & \ncong & \operatorname{Ob}((\operatorname{ho}(X) \amalg [0]) \times (\operatorname{ho}(X) \amalg [0])) \\ & = & \operatorname{Ob}(\mathbb{E}([0])) \times \operatorname{Ob}(\mathbb{E}([0])).\end{array}$$

2. If $K$ is a non discrete category, the functor constant at $K$ fails to satisfy condition (2) of Section 2.8. To see this, we observe that the diagram
$$\mathrm{Ob}(K) \rightrightarrows \mathrm{Ob}(K) \xrightarrow{\mathrm{id}} \mathrm{Mor}(K),$$
where the parallel arrows are both identities on $\mathrm{Ob}(K)$, is not a coequalizer.

3. If $Y$ is a simplicial set that is not a quasi-category, the functor $\mathbb{H}o(Y)$ fails to satisfy condition (3) of Section 2.8, given that
$$R\mathbb{H}o(Y) \cong Y \notin q\mathcal{C}at$$
as follows from Section 2.5.

The terminology "quasi-representable" is justified by the following.

2.16. Theorem. *A prederivator $\mathbb{D}$ is quasi-representable if and only if it lies in the image of $\mathbb{H}o\colon q\mathcal{C}at \to p\mathcal{D}er^{st}$, i.e., if it is of the form*
$$\mathbb{D} \cong \mathbb{H}o(X)$$
*for a quasi-category $X$.*

The proof makes use of a the following result, which shows how the underlying simplicial set of a quasi-representable prederivator uniquely determines the prederivator.

2.17. Lemma. *The functor $R$ reflects the existence of an isomorphism between quasi-representable prederivators, i.e., given two quasi-representable prederivators $\mathbb{D}$ and $\mathbb{E}$,*
$$\text{if } R\mathbb{D} \cong R\mathbb{E} \text{ then } \mathbb{D} \cong \mathbb{E}.$$

Proof. As a consequence of Condition (1) of Section 2.8 we obtain that $\mathbb{E}$ and $\mathbb{D}$ agree at the level of objects, namely, for any $J \in cat$ there are natural bijections
$$\begin{aligned} \mathrm{Ob}(\mathbb{D}(J)) &\cong \mathrm{Hom}_{s\mathcal{S}et}(NJ, R\mathbb{D}) \\ &\cong \mathrm{Hom}_{s\mathcal{S}et}(NJ, R\mathbb{E}) \\ &\cong \mathrm{Ob}(\mathbb{E}(J)). \end{aligned}$$

As a consequence of Condition (2) of Section 2.8, we see that $\mathbb{E}$ and $\mathbb{D}$ agree at the level of morphisms, namely, for any $J \in cat$ there are natural bijections
$$\begin{gathered} \mathrm{Mor}(\mathbb{D}(J)) \\ \updownarrow{\scriptstyle\cong} \\ \mathrm{coeq}\left(\mathrm{Ob}(\mathbb{D}([0] \times J) \times^{d_2}_{\mathrm{Ob}(\mathbb{D}([1]\times J))} \mathrm{Ob}(\mathbb{D}([2] \times J))) \rightrightarrows \mathrm{Ob}(\mathbb{D}([1] \times J))\right) \\ \updownarrow{\scriptstyle\cong} \\ \mathrm{coeq}\left(\mathrm{Ob}(\mathbb{E}([0] \times J) \times^{d_2}_{\mathrm{Ob}(\mathbb{E}([1]\times J))} \mathrm{Ob}(\mathbb{E}([2] \times J))) \rightrightarrows \mathrm{Ob}(\mathbb{E}([1] \times J))\right) \\ \updownarrow{\scriptstyle\cong} \\ \mathrm{Mor}(\mathbb{E}(J)). \end{gathered}$$

Finally, a straightforward check shows that the natural bijections above are compatible with identities, source and target maps, and compositions, so that for any $J \in cat$ we get natural isomorphisms of categories

$$\mathbb{D}(J) \cong \mathbb{E}(J),$$

that assemble into an isomorphism of prederivators $\mathbb{D} \cong \mathbb{E}$, as desired. ∎

2.18. Remark. With a variation of the argument, one could also show that the functor $R$ is conservative on quasi-representable prederivators, i.e., given two quasi-representable prederivators $\mathbb{D}$ and $\mathbb{E}$ a map $f\colon \mathbb{D} \to \mathbb{E}$ is an isomorphism if and only if the simplicial map $Rf\colon R\mathbb{D} \to R\mathbb{E}$ is an isomorphism.

Notice also that $R$ does not reflect isomorphisms between non quasi-representable prederivators. Consider for instance the map

$$L(\Delta[1] \times \Delta[1]) \to L(\Delta[1]) \times L(\Delta[1])$$

from Section 1.19. It was proven not to be an isomorphism of prederivators, but it is sent by $R$ to an isomorphism of simplicial sets as a consequence of Section 1.18 and of the fact that $R$ commutes with products.

We now finish the proof of Section 2.16 by showing that any quasi-representable prederivator $\mathbb{D}$ is in the essential image of $\mathbb{H}o\colon q\mathcal{C}at \to p\mathcal{D}er^{st}$.

2.19. Proposition. *If a prederivator $\mathbb{D}$ is quasi-representable, then there is an isomorphism of prederivators*

$$\mathbb{D} \cong \mathbb{H}o(R\mathbb{D}).$$

Proof. By the definition of a quasi-representable prederivator, $R\mathbb{D}$ is a quasi-category. By Section 2.5, we have an isomorphism of simplicial sets

$$R\mathbb{H}o(R\mathbb{D}) \cong R\mathbb{D},$$

and by Section 2.17 we are able to construct an isomorphism of prederivators

$$\mathbb{H}o(R\mathbb{D}) \cong \mathbb{D},$$

as desired. ∎

## 3. The model category of prederivators

In this section, we put a model structure on $p\mathcal{D}er^{st}$ by transferring the Joyal model structure using the functor $R\colon p\mathcal{D}er^{st} \to s\mathcal{S}et$, and we prove that the induced Quillen pair is in fact a Quillen equivalence. In particular, the model category of prederivators provides a model for the homotopy theory of $(\infty,1)$-categories.

For further reference, we record here the main properties for the Joyal model structure. Denote by $\mathbb{I}$ the free living isomorphism category, i.e., the category containing two objects and two inverse isomorphisms between them.

3.1. Theorem. [Joyal] *There exists a cofibrantly generated model structure on the category* $s\mathcal{S}et$ *in which*

- *the cofibration are precisely the monomorphisms;*
- *the weak equivalences are precisely the categorical equivalences;*
- *the fibrant objects are precisely the quasi-categories;*
- *the fibrations between quasi-categories are precisely the maps between quasi-categories that have the right lifting properties with respect to the inner horn inclusions* $\Lambda^k[n] \hookrightarrow \Delta[n]$ *for* $n > 0$ *and* $0 < k < n$ *and with respect to either inclusion* $\Delta[0] \to N\mathbb{I}$*;*

This model structure is cofibrantly generated for formal reasons. A set of generating cofibrations is given by the boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ for $n > 0$, but there is no explicit description of the class of generating acyclic cofibrations.

The key ingredient to construct the desired Quillen equivalence between the Joyal model structure and the category of prederivators is a general fact about transferred model structures in presence of a triple of adjoint functors.

3.2. Theorem. *Let* $\mathcal{M}$ *be a cofibrantly generated model category,* $\mathcal{N}$ *a bicomplete category, and* $(L, R, G)$ *a triple of adjoint functors*

$$\begin{tikzcd} \mathcal{N} \arrow[r, "R" description] & \mathcal{M} \arrow[l, bend right=40, hook', "L"'] \arrow[l, bend left=40, "G"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"] \end{tikzcd}$$

*with* $L\colon \mathcal{M} \to \mathcal{N}$ *fully faithful. The category* $\mathcal{N}$ *admits the transferred model structure using the functor* $R\colon \mathcal{N} \to \mathcal{M}$*, where by definition fibrations and weak equivalences are created by* $R$*. Furthermore, with respect to this model category structure, the adjunctions*

$$L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R \quad \text{and} \quad R\colon \mathcal{N} \rightleftarrows \mathcal{M}\colon G$$

*are Quillen equivalences.*

3.3. Remark. Given adjoint triple of functors

$$\begin{tikzcd} \mathcal{N} \arrow[r, "R" description] & \mathcal{M} \arrow[l, bend right=40, "L"'] \arrow[l, bend left=40, "G"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"] \end{tikzcd}$$

if either $L$ or $G$ is fully faithful, so is the other outer adjoint (see e.g. [MLM94, §VII.4.1]). In particular, in Section 3.2 one can replace the assumption of full faithfulness of $L$ with that of $G$.

We give a proof of Section 3.2 that makes use of the following version of the classical transfer theorem (see e.g. [Hir03, Theorem 11.3.2]) and of a standard result about Quillen equivalences, which we also recall. Alternatively, the existence of the transferred model structure from Section 3.2 could also be obtained as a special case of [DCH19, Theorem 2.3].

3.4. Theorem. [Quillen] *Let $\mathcal{M}$ be a cofibrantly generated model category with a set of generating cofibrations $I$ and a set of generating acyclic cofibrations $J$, let $\mathcal{N}$ be a complete and cocomplete category, and let*

$$L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$$

*be an adjunction. Suppose that the following conditions hold.*

1. *The left adjoint $L$ preserves small objects; this is the case in particular when the right adjoint preserves filtered colimits.*

2. *The right adjoint $R$ takes maps that can be constructed as a transfinite compositions of pushouts of elements of $L(J)$ to weak equivalences.*

*Then, there is a cofibrantly generated model category structure on $\mathcal{N}$ in which the weak equivalences are the maps that $R$ takes to weak equivalences in $\mathcal{M}$ and the fibrations are the maps that $R$ takes to fibrations in $\mathcal{M}$. This model structure is cofibrantly generated, with $L(I)$ as a set of generating cofibrations, and $L(J)$ as a set of generating acyclic cofibrations. Furthermore, with respect to this model category structure, the adjunction*

$$L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$$

*is a Quillen pair.*

3.5. Proposition. *If in a Quillen pair $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ the right adjoint $R$ creates weak equivalences and the unit on any cofibrant object is a weak equivalence, then the Quillen pair is a Quillen equivalence.*

Proof. To show that $(L, R)$ is a Quillen equivalence it is enough to show that for a cofibrant $X$ in $\mathcal{M}$ and a fibrant $Y$ in $\mathcal{N}$, a map $f : L(X) \to Y$ is a weak equivalence in $\mathcal{N}$ if and only if its adjoint $f^\sharp : X \to R(Y)$ is a weak equivalence in $\mathcal{M}$.

We first observe that the derived unit at $X$ coincides with the unit $\eta_X$, since $R$ preserves all weak equivalences, and it is therefore a weak equivalence. Given that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\eta_X", "\simeq"'] \arrow[dr, "f^\sharp"'] & R(L(X)) \arrow[d, "R(f)"] \\ & R(Y), \end{tikzcd}$$

the adjoint map $f^\sharp$ is a weak equivalence if and only if $R(f)$ is a weak equivalence. Finally, since $R$ creates weak equivalences, this is true if and only if $f$ is a weak equivalence. ∎

We can now prove Section 3.2.

PROOF. We first check that the conditions of Section 3.4 hold for the adjunction $(L, R)$.

(1) The functor $R$ preserves all colimits since it has a right adjoint $G$, so in particular it preserves filtered colimits.

(2) Since $R$ preserves all colimits, to check the second condition it is enough to show that the image under $RL$ of any generating trivial cofibration of $\mathcal{M}$ is again a trivial cofibration. We conclude observing that by assumption $L$ is fully faithful, so the unit of the adjunction $(L, R)$ is a natural isomorphism $RL \cong \mathrm{id}_{\mathcal{M}}$.

So we obtain that the adjunction $(L, R)$ is a Quillen pair, and we now show that it is a Quillen equivalence. We already observed that the unit of the adjunction $(L, R)$ is an isomorphism. Since $R$ preserves all weak equivalences the unit on any cofibrant object is also the derived unit. Given that $R$ creates weak equivalences, it follows from Section 3.5 that the adjunction is a Quillen equivalence.

Finally, we notice that $R$ is also a left Quillen functor. Indeed, it preserves generating cofibrations, since $RL \cong \mathrm{id}_{\mathcal{M}}$, and it preserves all weak equivalences by definition. It follows that $(R, G)$ is a Quillen pair, and we now show that it is a Quillen equivalence. Since the left adjoint $R$ creates weak equivalences, by the dual of Section 3.5 it is enough to show that the counit of the adjunction $(R, G)$ is an isomorphism. This is true because by Section 3.3 the functor $G$ is fully faithful. ∎

By specializing the theorem to the triple of adjoints $(L, R, G)$ between the categories $s\mathcal{S}et$ and $p\mathcal{D}er^{st}$, we obtain the desired model structure on $p\mathcal{D}er^{st}$.

3.6. THEOREM. *The category $p\mathcal{D}er^{st}$ admits the transferred model structure using the functor $R\colon p\mathcal{D}er^{st} \to s\mathcal{S}et$, where by definition fibrations and weak equivalences are created by $R$. Furthermore, with respect to this model category structure, the adjunctions*

$$L\colon s\mathcal{S}et \rightleftarrows p\mathcal{D}er^{st}\colon R \quad \text{and} \quad R\colon p\mathcal{D}er^{st} \rightleftarrows s\mathcal{S}et\colon G$$

*are Quillen equivalences.*

PROOF. The Joyal model structure is cofibrantly generated, as mentioned in Section 3.1, and by Section 1.8, the category $p\mathcal{D}er^{st}$ is complete and cocomplete. We also know from Section 1.18 that the functor $L$ is fully faithful. We can then apply Section 3.2 to the triple of adjoint functors

$$\begin{tikzcd} p\mathcal{D}er^{st} \arrow[r, "R" description] & s\mathcal{S}et \arrow[l, bend right, "L"', "\perp"] \arrow[l, bend left, "G", "\perp"'] \end{tikzcd}$$

that was constructed in Section 1. ∎

3.7. Digression. Consider the adjunction

$$p_1^*\colon s\mathcal{S}et \rightleftarrows ss\mathcal{S}et\colon i_1^*$$

from [JT07, §4], where the right adjoint $i_1^*$ restricts a bisimplicial set to its 0-th row. This adjunction was proven in [JT07, Theorem 4.11] to be a Quillen equivalence between the Joyal model structure on $s\mathcal{S}et$ and the Rezk model structure for complete Segal spaces on $ss\mathcal{S}et$.

Since the functor $i_1^*$ also possesses a right adjoint and the functor $p_1^*$ is fully faithful, the adjunction above fits into an adjoint triple

$$\begin{tikzcd} ss\mathcal{S}et \arrow[r, "i_1^*" description] & s\mathcal{S}et \arrow[l, bend right=40, hook', "p_1^*"'] \arrow[l, bend left=40, hook] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd}$$

that satisfies the conditions of Section 3.2. In particular, one can transfer the Joyal model structure to the category of bisimplicial sets, using the functor $i_1^*$, obtaining a Quillen equivalence.

While the Rezk model structure and this transferred model structure are Quillen equivalent via the identity functor, the two model structures are not equal. Indeed, a bisimplicial set whose 0-th row is a quasi-category is not necessarily a complete Segal space, and therefore the fibrant objects do not coincide.

We now give a more explicit description of the model structure from Section 3.6 on the category $p\mathcal{D}er^{st}$, starting with the class of cofibrations.

Then we will characterize the fibrant prederivators, the fibrations between fibrant prederivators and the acyclic fibrations in terms of suitable lifting properties. The following results are straightforward consequences of Section 3.6 and the characterisations of corresponding classes of maps in the Joyal model structure on $s\mathcal{S}et$, which were recalled in Section 3.1.

We begin by introducing some notation.

3.8. Notation. For $n \geq 0$ and $0 \leq k \leq n$, the $k$-th horn of the representable prederivator $D_{[n]}$ is the prederivator

$$\mathbf{\Lambda}^k_{[n]} \coloneqq L(\Lambda^k[n]).$$

It comes with a canonical cofibration of prederivators

$$\mathbf{\Lambda}^k_{[n]} \to D_{[n]}$$

induced by the simplicial horn inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$.

3.9. Notation. For $n \geq 1$, the boundary of the representable prederivator $D_{[n]}$ is the prederivator

$$\boldsymbol{\partial} D_{[n]} \coloneqq L(\partial\Delta[n]).$$

It comes with a canonical cofibration of prederivators

$$\boldsymbol{\partial} D_{[n]} \to D_{[n]}$$

induced by the simplicial boundary inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$.

3.10. Corollary. *A map of prederivators $\varphi\colon \mathbb{D} \to \mathbb{E}$ is a cofibration if and only if it can be expressed as a retract of transfinite compositions of pushouts of boundary maps $\boldsymbol{\partial} D_{[n]} \to D_{[n]}$ for $n \geq 1$.*

3.11. Corollary. *A prederivator $\mathbb{D}$ is fibrant if and only if it has the right lifting property with respect to all inner horn cofibrations $\boldsymbol{\Lambda}^k_{[n]} \to D_{[n]}$ for $n \geq 2$ and $0 < k < n$.*

3.12. Corollary. *A map between fibrant prederivators $\varphi\colon \mathbb{D} \to \mathbb{E}$ is a fibration if and only if it has the right lifting property with respect to all inner horn cofibrations $\boldsymbol{\Lambda}^k_{[n]} \to D_{[n]}$ for $n \geq 2$ and $0 < k < n$ and with respect to either of the two canonical maps $D_{[0]} \to D_{\mathbb{I}}$.*

3.13. Corollary. *A map of prederivators $\varphi\colon \mathbb{D} \to \mathbb{E}$ is an acyclic fibration if and only if it has the right lifting property with respect to all boundary cofibrations $\boldsymbol{\partial} D_{[n]} \to D_{[n]}$ for $n \geq 1$.*

3.14. Proposition. *Any cofibration of prederivators $\varphi\colon \mathbb{D} \to \mathbb{E}$ induces an injective function*

$$\mathrm{Ob}(\varphi_J)\colon\ \mathrm{Ob}(\mathbb{D}(J)) \to \mathrm{Ob}(\mathbb{E}(J))$$

*for any $J \in cat$.*

Proof. The boundary cofibrations $\boldsymbol{\partial} D_{[n]} \to D_{[n]}$ induce injective functions

$$\mathrm{Ob}(\boldsymbol{\partial} D_{[n]}(J)) \to \mathrm{Ob}(D_{[n]}(J))$$

for any $J \in cat$. The result now follows from the fact that the functor Ob preserves retracts, transfinite compositions, and pushouts. ∎

3.15. Remark. By Condition (3) of Section 2.8, all quasi-representable prederivators are fibrant. Moreover, every prederivator is weakly equivalent to a quasi-representable one. Indeed, if $f\colon \mathbb{D} \to \widetilde{\mathbb{D}}$ is a fibrant replacement for a prederivator $\mathbb{D}$, we can consider the zig-zag

$$\mathbb{D} \xrightarrow{f} \widetilde{\mathbb{D}} \xleftarrow{\varepsilon_{\widetilde{\mathbb{D}}}} LR(\widetilde{\mathbb{D}}) \longrightarrow \mathbb{H}o(R(\widetilde{\mathbb{D}})),$$

where the last map is the adjoint of the isomorphism

$$R(\widetilde{\mathbb{D}}) \cong R\mathbb{H}o(R(\widetilde{\mathbb{D}}))$$

from Section 1.18. They are all weak equivalences, as a consequence of Sections 1.18 and 2.5.

Thus, the homotopy theory of (fibrant) prederivators is recovered by the homotopy theory of quasi-representable prederivators, which is isomorphic by Section 2.4 to the homotopy theory of quasi-categories.

Given that $\mathbb{H}o$ is fully faithful from Section 2.4, the following proposition gives a complete description of weak equivalences between prederivators that are in the image of the functor $\mathbb{H}o\colon s\mathcal{S}et \to p\mathcal{D}er^{st}$.

3.16. PROPOSITION. *The functor $\mathbb{H}o$ creates weak equivalences, i.e., a map of simplicial sets $f\colon X \to Y$ is a categorical equivalence if and only if the induced map of prederivators $\mathbb{H}o(f)\colon \mathbb{H}o(X) \to \mathbb{H}o(Y)$ is a weak equivalence.*

PROOF. Given the isomorphism $R\mathbb{H}o \cong \mathrm{id}_{s\mathcal{S}et}$ from Section 2.5, a map of simplicial sets $f\colon X \to Y$ is a categorical equivalence if and only if

$$R\mathbb{H}o(f)\colon R\mathbb{H}o(X) \to R\mathbb{H}o(Y)$$

is also one. Given that weak equivalences of prederivators are by definition created by the functor $R$, this is equivalent to saying that $\mathbb{H}o(f)\colon \mathbb{H}o(X) \to \mathbb{H}o(Y)$ is a weak equivalence of prederivators. ■

3.17. PROPOSITION. *If $f\colon X \to Y$ is a categorical equivalence between quasi-categories, the induced map $\mathbb{H}o(f)\colon \mathbb{H}o(X) \to \mathbb{H}o(Y)$ is levelwise an equivalence of categories.*

PROOF. Suppose that $f\colon X \to Y$ is a categorical equivalence between quasi-categories. For any category $J$, the induced map $f^{NJ}\colon X^{NJ} \to Y^{NJ}$ is a categorical equivalence, since the Joyal model structure is enriched over itself. By [Joy08, §1.12] it induces an equivalence of homotopy categories $\mathrm{ho}(f^{NJ})\colon \mathrm{ho}(X^{NJ}) \to \mathrm{ho}(Y^{NJ})$, and this is by definition precisely

$$\mathbb{H}o(f)(J)\colon \mathbb{H}o(X)(J) \to \mathbb{H}o(Y)(J),$$

as desired. ■

We conclude with a brief discussion on further directions.

3.18. DIGRESSION. With a model structure on $p\mathcal{D}er^{st}$ that is Quillen equivalent to the Joyal model structure on $s\mathcal{S}et$ the next natural goal is to attempt to endow the category of fibrant objects of $p\mathcal{D}er^{st}$ with the structure of an $\infty$-cosmos, in the sense of [RV15, Definition 2.1.1], and to attempt to show that this $\infty$-cosmos is biequivalent to the $\infty$-cosmos $q\mathcal{C}at$ of quasicategories, defined in [RV15, Example 2.1.4]. Having equivalent $\infty$-cosmoi guarantees that the 2-category theory in each $\infty$-cosmos is the same, so in particular the notions of limits and colimits, adjunctions, and cartesian fibrations coincide.

Mimicking what is done in similar scenarios (e.g. when starting with the model structure for complete Segal spaces, Segal categories, and 1-complicial sets, see [RV15, §2]), one would attempt to upgrade the model structure on $p\mathcal{D}er^{st}$ to one enriched over the Joyal model structure on $s\mathcal{S}et$, such that the functors $L$ and $R$ form a simplicial Quillen adjunction. The standard technique to do this is to use [RV15, Proposition 2.2.3] to transfer the simplicial enrichment, which works whenever the left adjoint $L$ is strong symmetric monoidal.

However, by Section 1.19, the functor $L$ does not preserve products. By [Rie14, Proposition 3.7.10], one can deduce that the functor $R$ is then not compatible with simplicial cotensors. These two equivalent conditions obstruct the simplicial enrichment on $p\mathcal{D}er^{st}$ discussed in Section 1.12 from giving an enrichment over the Joyal model structure, and indeed prevent the simplicial lifts of $L$ and $R$ from even being simplicially adjoint functors. This means that the conditions of [RV15, Proposition 2.2.3] are not satisfied.

At this moment, the authors have been unable to provide another method to achieve the desired $\infty$-cosmos structure on the category of fibrant prederivators using the functor $R$. Nonetheless, as a consequence of Carlson's Theorem (which was stated as Section 2.4) the simplicial category of quasi-representable prederivators is isomorphic via the functor $\mathbb{H}o$ to the simplicial category of quasicategories, which is indeed an $\infty$-cosmos. This in itself allows for several interesting results, and is the subject of a future project.

In order to reconnect with the classical 2-category theory of prederivators, we also plan to investigate Quillen equivalent variations of the model structure on prederivators, in which there are possibly more cofibrant objects and less fibrant objects. The aim is to obtain a different model structure on $p\mathcal{D}er^{st}$ for which the weak equivalences between bifibrant objects are related to the well-studied 2-categorical equivalences of prederivators.

## References


[Car16] Kevin Carlson. On the $\infty$-categorical whitehead theorem and the embedding of quasicategories in prederivators. arXiv:1612.06980, 2016.

[Col19] Ian Coley. Stabilization of derivators revisited. *Journal of Homotopy and Related Structures*, 14(2):525–577, 2019.

[DCH19] Gabriel C. Drummond-Cole and Philip Hackney. A criterion for existence of right-induced model structures. *Bulletin of the London Mathematical Society*, 51(2):309–326, 2019.

[Fra96] Jens Franke. Uniqueness theorems for certain triangulated categories possessing an adams spectral sequence. *$K$-theory Preprint Archives*, 139, 1996.

[GPS14a] Moritz Groth, Kate Ponto, and Michael Shulman. The additivity of traces in monoidal derivators. *Journal of $K$-theory*, 14(3):422–494, 2014.

[GPS14b] Moritz Groth, Kate Ponto, and Michael Shulman. Mayer-Vietoris sequences in stable derivators. *Homology, Homotopy and Applications*, 16(1):265–294, 2014.

[Gro] Alexander Grothendieck. Les dérivateurs. https://webusers.imj-prg.fr/~georges.maltsiniotis/groth/Derivateurs.html.

[Gro12] Moritz Groth. Monoidal derivators and additive derivators. arXiv:1203.5071, 2012.

[Gro13] Moritz Groth. Derivators, pointed derivators and stable derivators. *Algebraic & Geometric Topology*, 13(1):313–374, 2013.

[Hel88] Alex Heller. *Homotopy theories*, volume 383. American Mathematical Soc., 1988.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[Joy08] André Joyal. Notes on quasi-categories. https://www.math.uchicago.edu/~may/IMA/Joyal.pdf, 2008.

[JT07] André Joyal and Myles Tierney. Quasi-categories vs Segal spaces. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 277–326. Amer. Math. Soc., Providence, RI, 2007.

[Kan58] Daniel M. Kan. Functors involving c.s.s. complexes. *Transactions of the American Mathematical Society*, 87:330–346, 1958.

[Kel82] Max Kelly. *Basic concepts of enriched category theory*, volume 64. CUP Archive, 1982.

[Len17] Tobias Lenz. Homotopy (pre-)derivators of cofibration and quasi-categories. arXiv:1712.07845, 2017.

[Lor18] Fosco Loregian. Localization theory for derivators. arXiv:1802.08193, 2018.

[Low13] Zhen Lin Low. Universes for category theory. arXiv:1304.5227, 2013.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[LV17] Fosco Loregian and Simone Virili. Factorization systems on (stable) derivators. arXiv:1705.08565, 2017.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[MLM94] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[MR11] Fernando Muro and George Raptis. A note on $K$-theory and triangulated derivators. *Advances in Mathematics*, 227(5):1827–1845, 2011.

[MR16] Fernando Muro and Georgios Raptis. $K$-theory of derivators revisited. *Annals of $K$-theory*, 2(2):303–340, 2016.

[Ren09] Olivier Renaudin. Plongement de certaines théories homotopiques de Quillen dans les dérivateurs. *Journal of Pure and Applied Algebra*, 213(10):1916–1935, 2009.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.

[RV] Emily Riehl and Dominic Verity. Elements of $\infty$-category theory. http://www.math.jhu.edu/~eriehl/elements.pdf.

[RV15] Emily Riehl and Dominic Verity. Fibrations and Yoneda's lemma in an $\infty$-cosmos. arXiv:1506.05500, 2015.

[RV17] Emily Riehl and Dominic Verity. Kan extensions and the calculus of modules for $\infty$-categories. *Algebraic & Geometric Topology*, 17(1):189–271, 2017.

[Shu] Michael Shulman. Squeezing higher categories out of lower categories. https://golem.ph.utexas.edu/category/2010/05/squeezing_higher_categories_ou.html.

*Department of Mathematics,*
*Johns Hopkins University, Baltimore, USA*

*Mathematical Institute, Utrecht University*
*Utrecht, The Netherlands*

*Department of Mathematics,*
*Johns Hopkins University, Baltimore, USA*

Email: danielfk@jhu.edu
m.kedziorek@uu.nl
mrovelli@math.jhu.edu

This article may be accessed at http://www.tac.mta.ca/tac/