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LIFTING BICATEGORIES INTO DOUBLE CATEGORIES: THE GLOBULARILY GENERATED CONDITION

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Abstract. This is the first part of a series of papers studying the problem of existence of double categories for which horizontal bicategory and object category are given. We refer to this problem as the problem of existence of internalizations for decorated bicategories. Motivated by this we introduce the condition of a double category being globularily generated. We prove that the problem of existence of internalizations for a decorated bicategory admits a solution if and only if it admits a globularily generated solution, and we prove that the condition of a double category being globularily generated is precisely the condition of a solution to the problem of existence of internalizations for a decorated bicategory being minimal in a sense which we will make precise. The study of the condition of a double category being globularily generated will thus be pivotal in our study of the problem of existence of internalizations.

1. Introduction

There exists, in the mathematical literature, a variety of competing definitions of what a higher order categorical structure should be [3,20,27]. Most common amongst which are those defined by the concepts of internalization and enrichment [22,30]. These two ideas reduce, in the case of categorical structures of order 2, to the concepts of double category and bicategory respectively, both types of structures introduced by Ehresmann, in [11] and [12]. These two notions are related in different ways. Every double category admits an underlying bicategory, its horizontal bicategory [28], and every bicategory can be considered as a double category through several different constructions. Examples of these are the trivial double category construction, the Ehresmann double category of quintets construction [8], the double category of adjoints construction [25] and the double category of spans construction [24] in the case of single 0-cell bicategories. All constructions mentioned are functorial. A double category is trivial if its vertical morphisms are all identities. This is expressed by the fact that the functor associated to the trivial double category construction is right adjoint to horizontalization.

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We are interested in situations analogous to the ones described above, only in which initial conditions in the form of a given collection of vertical morphisms is provided. Precisely, we are interested in the problem of existence of double categories having a given bicategory as horizontal bicategory and a category, not necessarily its category of horizontal cells, as category of objects. We call this problem the problem of existence of internalizations for decorated bicategories. Solutions to this problem are well known in the literature in the case in which the category of objects provided in the set of initial conditions is the category of horizontal cells of the given category [8,25]. Our main motivation for the study of the problem of existence of general internalizations of decorated bicategories stems from the work of Bartels, Douglas, and Henriques on the theory of correspondences between von Neumann algebras [4], from their theory of coordinate free conformal nets [5,6,7], and from the theory of regular and extended quantum field theories [1,2,21,26]. We are interested in particular in the problem of existence of a bicategory of not-necessarily semisimple coordinate free nets, internal to the 2-category of symmetric monoidal categories and equivalently in the problem of existence of a double category of not-necessarily semisimple von Neumann algebras with not-necessarily finite vertical morphisms.

This is the first installment of a series of papers studying the problem of existence of internalizations of decorated bicategories. We use the problem of existence of internalizations for decorated bicategories as motivation to introduce globularily generated double categories. Globularily generated double categories are those double categories which appear as minimal solutions to the problem of internalizations of decorated bicategories. Precisely, we prove that the problem of existence of internalizations of decorated bicategories is equivalent to the problem of existence of globularily generated internalizations, and we prove that every solution to the problem of existence of internalizations of decorated bicategories contains a globularily generated solution canonically in a sense that we will make precise. This will be our main motivation for the study of globularily generated double categories. We interpret the condition of a double category being globularily generated as a decorated analog of the condition of a double category being trivial. We introduce technical tools necessary to obtain results on the theory of globularily generated double categories, of which we make use in order to prove that the globularily generated condition is not trivial. We present functorial versions of structures involved in the theory of globularily generated double categories, e.g. vertical and horizontal filtrations, and we perform relevant computations in certain examples. Globularily generated double categories will be our main object of study.

The work presented in [4] suggests that an analytic solution to the problem of existence of functorial extensions of the Haagerup standard form construction and the Connes fusion operation, and thus of the problem of existence of internalizations of von Neumann algebras, would involve an understanding of a theory of conditional expectations of inclusions of von Neumann algebras. These inclusions not-necessarily being subfactors, and even when this is the case, the subfactors considered would not be assumed to have finite index. The techniques developed in this paper on the other hand suggest that the existence of a minimal e.g. globularily generated solution to the problem of existence of internalizations to any decorated bicategory is, from the perspective of bicategories, purely combinatorial. Computations of relevant pieces of structures e.g. globularily generated piece and vertical length, on double categories of algebras and semisimple von Neumann algebras provide clues on the conditions a combinatorial realization of a double category of general von Neumann algebras should satisfy. We pursue these ideas further in future papers. We now sketch the contents of this paper.

In section 2 we recall basic concepts related to the theory of categorical structures of second order, we introduce the concepts of decorated bicategory, decorated pseudofunctor, and decorated horizontalization; we introduce both the problem of existence of internalizations of decorated bicategories and the problem of existence of internalization functors which will serve as the main motivation for the material presented in the rest of the paper. We establish notational conventions and present examples relevant to our discussion. In section 3, motivated by problems presented in section 2 we define and study the concept of globularily generated double category. We define the globularily generated piece construction, which we furnish with the structure of a 2-functor. We use this to prove that the problem of existence of internalizations of decorated bicategories is equivalent to the problem of existence of globularily generated bicategories, and that the problem of existence of internalization functors is equivalent to the problem of existence of globularily generated internalization functors. We prove that the globularily generated piece 2-functor is a strict 2-coreflector which we interpret by saying that the condition of a double category being globularily generated is a decorated analog of the condition of a double category being trivial. In section 4 we introduce the technical framework needed in order to obtain results on the structure of globularily generated double categories. We introduce the vertical filtration of a globularily generated double category and the vertical length of a globularily generated 2-morphism. We use this technical framework to prove that the condition of a double category being globularily generated is not trivial. In section 5 we extend the definition of vertical filtration of a globularily generated double category introduced in section 4, to a filtration of the globularily generated piece functor. In section 6 we use results obtained in section 3 to perform computations of the globularily generated piece of double categories presented in section 2 thus providing non-trivial examples of globularily generated double categories. Precisely, we compute the globularily generated piece of double category of oriented cobordisms, double category of algebras, and of double category of semisimple von Neumann algebras.

2. Preliminaries

In this first section we recall basic concepts and set notational conventions regarding the theory of categorical structures of second order. Further, we introduce the concepts of decorated bicategory, decorated pseudofunctor, and decorated horizontalization. We use this to formally present the problem of existence of internalizations of decorated bicategories, which will serve as motivation for the rest of the work presented in this paper. We present examples relevant to our discussion.

2.1. Bicategories.
We refer the reader to [9,16] for concepts related to the theory of bicategories. Given a bicategory B, we will write B_0, B_1, and B_2 for the collections of 0-,1-, and 2-cells of B respectively. We will write i, $\circ$, and $*$ for the horizontal identity functions, and the vertical and horizontal compositions in B respectively. We will not make explicit use of left and right identity transformations or associator of bicategories and will thus omit notational conventions for these concepts. Nevertheless, bicategories will not be assumed to be strict unless explicitly stated.

Given a pseudofunctor F we will write F_0, F_1, and F_2 for the 0-,1-, and 2-cell components of F respectively. Pseudofunctors will not be assumed to be strict, unless explicitly stated. We will write **bCat** for the category of bicategories and pseudofunctors.

2.2. Double categories. We refer the reader to [22,28] for concepts related to the theory of double categories. Given a double category C we will write C_0 and C_1 for the category of objects and the category of morphisms of C respectively. We will denote by s, t, i, and $*$ the source, target, identity, and horizontal composition functors of a double category C. We will again omit notational conventions for identity transformations and associator of double categories. Double categories will not be assumed to be strict unless otherwise stated.

Given a double category C, a 2-morphism in C is said to be **globular** if its source and target are vertical identity endomorphisms. The definition of double category that we use requires for the components of the left and right identity transformations and of the associator of a double category to be globular.

Given a double functor F we will write F_0 and F_1 for the object functor and the morphism functor of F respectively. Double functors will not be assumed to be strict unless otherwise stated. Given a double natural transformation η we will write η_0 and η_1 for the object and morphism components of η respectively. We will denote the 2-category of double categories, double functors, and double natural transformations by **dCat**.

2.3. Trivial double categories. Given a bicategory B, we associate to B a double category $\overline{B}$. The category of objects $\overline{B}_0$ of $\overline{B}$ will be the discrete category dB_0 generated by the collection of 0-cells B_0 of B. Category of morphisms $\overline{B}_1$ of $\overline{B}$ will be the pair formed by the collections of 1- and 2-cells B_1 and B_2 of B, with vertical composition of 2-cells as composition operation. The obvious functors serve as source, target, and identity functors for $\overline{B}$. The horizontal composition functor of $\overline{B}$ will be the functor generated by the horizontal composition operation in B. The left and right identity transformations and the associator of $\overline{B}$ are given by the corresponding constraints in B. We call $\overline{B}$ the **trivial double category** associated to bicategory B. The construction of the trivial double category associated to a bicategory has an obvious extension to a functor from **bCat** to the category underlying **dCat**. We call this functor the **trivial double category**

functor. The trivial double category functor is an embedding, and thus makes category **bCat** a subcategory of **dCat**.

2.4. Horizontalization. Given a double category C, we now associate to C a bicategory HC whose collections of 0-,1-, and 2-cells are the collection of objects of C, the collection of horizontal morphisms of C, and the collection of globular 2-morphisms of C respectively. Vertical and horizontal composition of 2-cells in HC is the vertical and horizontal composition of 2-cells in C. The left and right identity transformations and associator of HC are induced by the left and right identity transformations and associator of C respectively. We call HC the **horizontal bicategory**, or simply the horizontalization of C [28]. Again the construction of the horizontal bicategory of a double category admits an obvious extension to a functor now from the category underlying **dCat** to **bCat**. We call this functor the **horizontalization functor** and we denote it by H. The horizontalization functor is right adjoint to the trivial double category functor and thus is a coreflector for the inclusion of **bCat** in **dCat**.

2.5. Decorated bicategories. Given a bicategory B we say that a category C is a decoration of B if the collection of objects of C is equal to the collection of 0-cells of B. We will usually indicate that a category C is a decoration of a bicategory B by writing C as B^*. We say that a pair B^*, B, formed by a category B^* and a bicategory B, is a **decorated bicategory** if category B^* is a decoration of bicategory B. In that case we will call B the underlying bicategory of the decorated bicategory B^*, B. Given decorated bicategories B^*, B and B'^*, B' we say that a pair F, G is a **decorated pseudofunctor** from B^*, B to B'^*, B' if F is a functor from B^* to B'^*, G is a pseudofunctor from B to B', and the object function of F and the 0-cell function of G coincide. Composition of decorated pseudofunctors is performed component-wise. We denote the category of decorated bicategories and decorated pseudofunctors by $\mathbf{bCat}^*$.

2.6. Decorated horizontalization. We extend the construction of the horizontalization functor H to a functor from category **dCat** to category $\mathbf{bCat}^*$. Given a double category C, we denote by H^*C the pair formed by the object category C_0 of C and by the horizontal bicategory HC of C. Thus defined, H^*C is a decorated bicategory. Given a double functor F between double categories C and C' we denote by H^*F the pair formed by the object functor F_0 of F and by the horizontalization HF of F. Thus defined, pair H^*F associated to the double functor F is a decorated pseudofunctor from the decorated horizontalization H^*C of C to the decorated horizontalization H^*C' of C'. The function associating, to every double category C the decorated horizontalization H^*C of C, together with the function associating, to every double functor F the decorated horizontalization H^*F of F, is a functor from **dCat** to $\mathbf{bCat}^*$. We denote this functor by H^* and we call it the **decorated horizontalization functor**.

2.7. Internalization. Given a decorated bicategory B^*, B we say that a double category C is an **internalization** of B^*, B if B^*, B is equal to decorated horizontalization

H^*C of C. Moreover, we say that a functor Ψ from $\mathbf{bCat}^*$ to $\mathbf{dCat}$ is an **internalization functor**, if Ψ is a section of decorated horizontalization functor H^*. We think of internalization functors as coherent ways of associating internalizations to decorated bicategories.

2.8. EXAMPLES.

1. **Trivially decorated bicategories:** Let B be a bicategory. In that case pair dB_0, B formed by the discrete category generated by the collection of 0-cells B_0 of B, and B, is a decorated bicategory. We call the pair dB_0, B the discretely decorated bicategory associated to B. The decorated horizontalization $H^*\overline{B}$ of the trivial double category $\overline{B}$ associated to bicategory B is equal to dB_0, B, that is, the trivial double category $\overline{B}$ associated to bicategory B is an internalization of the discretely decorated bicategory dB_0, B associated to B. Observe that a double category C is trivial if and only if equation $\overline{HC} = C$ holds.

2. **Cobordisms:** Let n be a positive integer. We denote by $\mathbf{Cob}(n)$ the double category of cobordisms between closed oriented manifolds of dimension n. Precisely, the category of objects $\mathbf{Cob}(n)_0$ of $\mathbf{Cob}(n)$ is the groupoid of closed oriented n-dimensional manifolds and their diffeomorphisms , the category of morphisms $\mathbf{Cob}(n)_1$ of $\mathbf{Cob}(n)$ is the groupoid of oriented cobordisms between closed oriented n-dimensional manifolds and equivariant diffeomorphisms between them. Source and target functors on $\mathbf{Cob}(n)$ associate domain and codomain to cobordisms and equivariant smooth maps, identity functor on $\mathbf{Cob}(n)$ is defined by associating cylinders to n-dimensional manifolds, and horizontal composition is defined by the gluing of cobordisms and smooth functions. Thus defined the decorated horizontalization $H^*\mathbf{Cob}(n)$ of $\mathbf{Cob}(n)$ is equal to bicategory $\underline{\mathbf{Cob}(n)}$ of n-dimensional cobordisms and diffeomorphisms presented in [19] decorated by the groupoid of diffeomorphisms of closed oriented n-dimensional manifolds, that is the double category $\mathbf{Cob}(n)$ is an internalization of bicategory $\underline{\mathbf{Cob}(n)}$ decorated by the groupoid of diffeomorphisms of closed oriented n-dimensional manifolds.

3. **Algebras:** We denote by $\mathbf{Alg}$ the double category of bimodules over algebras. Precisely, the category of objects $\mathbf{Alg}_0$ of $\mathbf{Alg}$ is the category of complex unital algebras and unital algebra morphisms, the category of morphisms $\mathbf{Alg}_1$ of $\mathbf{Alg}$ is the category of bimodules over algebras and equivariant bimodule morphisms. Source and target functors on $\mathbf{Alg}$ associate domain and codomain to bimodules and equivariant bimodule morphisms, identity functor on $\mathbf{Alg}$ is the functor that associates, to every algebra, the module defined by itself through left and right multiplication, and horizontal composition is defined by the relative tensor product bifunctors. The decorated horizontalization $H^*\mathbf{Alg}$ of $\mathbf{Alg}$ is equal to the bicategory $\underline{\mathbf{Alg}}$ of unital algebras, algebra bimodules, and bimodules morphisms presented in [19], decorated by category $\mathbf{Alg}_0$ of unital algebra morphisms. That is, the double

category **Alg** is an internalization of bicategory $\underline{\textbf{Alg}}$, decorated by the category $\textbf{Alg}_0$ of algebras, and unital algebra morphisms.

4. **von Neumann algebras:** We denote by $[W^*]^f$ the double category of Hilbert bimodules over semisimple von Neumann algebras. Precisely, the category of objects $[W^*]^f_0$ of $[W^*]^f$ is the category of semisimple von Neumann algebras and finite morphisms presented in [19], the category of morphisms $[W^*]^f_1$of $[W^*]^f$ is the category of Hilbert bimodules over semisimple von Neumann algebras and equivariant bimodule morphisms with finite domain and codomain. Source and target functors on $[W^*]^f$ associate domain and codomain to Hilbert bimodules and equivariant bimodules morphisms, identity functor on $[W^*]^f$ is the functor that associates, to every semisimple von Neumann algebra its Haagerup standard form presented in [14], and horizontal composition is defined by the Connes fusion operation bifunctor presented in [4]. Denote by $\underline{[W^*]}^f$ the sub-bicategory of bicategory $\underline{[W^*]}$ of von Neumann algebras, normal bimodules, and bimodule intertwiners defined in [19], generated by semisimple von Neumann algebras. In that case the decorated horizontalization $H^*[W^*]^f$ of double category $[W^*]^f$ is equal to bicategory $\underline{[W^*]}^f$, decorated by category $[W^*]^f_0$ of semisimple von Neumann algebras and finite morphisms, that is, double category $[W^*]^f$ is an internalization of bicategory $\underline{[W^*]}^f$ decorated by category $[W^*]^f_0$. An analogous construction provides the pair formed by the category of semisimple coordinate free conformal nets and finite natural transformations, and the category of sectors between semisimple coordinate free conformal nets and finite equivariant intertwiners $\mathbf{CN}^f$ presented in [6] with the structure of a double category.

2.9. The problem of existence of internalizations. It is not known if bicategory $\underline{[W^*]}$ of general i.e. not-necessarily semisimple von Neumann algebras, normal bimodules, and bimodule intertwiners appearing in [19], decorated by the category of general von Neumann algebras and general von Neumann algebra morphisms, admits an internalization. The existence of extensions of horizontal identity functor and horizontal composition bifunctor defining $[W^*]^f$, to the category of general von Neumann algebras and general von Neumann algebra morphisms and to the category of normal bimodules and general equivariant bimodule intertwiners would imply the existence of such an internalization. The existence of such extensions is conjectured in [4].

We are interested in the problem of finding internalizations for general decorated bicategories. We refer to this problem as the **problem of existence of internalizations for decorated bicategories**. The problem of existence of internalizations for decorated bicategories is the main motivation for the work at present. We will moreover be interested in the problem of finding internalization functors. We refer to this problem as **the problem of existence of internalization functors** and we regard it as a categorical version of the problem of existence of internalizations for decorated bicategories.

3. Globularily generated double categories

In this section we introduce globularily generated double categories. We prove that the condition of a double category being globularily generated is minimal with respect to having a given decorated bicategory as decorated horizontalization and that the problem of existence of internalizations for decorated bicategories is equivalent to the problem of existence of globularily generated internalizations. We do this by associating, to every double category, a globularily generated double category, its globularily generated piece. We furnish the globularily generated piece construction with the structure of a 2-functor. We extend results on the globularily generated piece construction to analogous results regarding internalization functors. Finally, we prove that globularily generated piece 2-functor is a 2-reflector. We interpret this by regarding the condition of a double category being globularily generated as a categorical analog of the condition of a double category being trivial.

Given a double category C, we will say that a sub-double category D of C is **complete**, if the collections of objects, vertical morphisms, and horizontal morphisms of D are equal to the collections of objects, vertical morphisms, and horizontal morphisms of C respectively. Given a collection of 2-morphisms Ω of double category C, we will call the intersection of all complete sub-double categories D, of C, such that Ω is contained in the collection of 2-morphisms of D, the **complete sub-double category** of C generated by Ω. We will say that a collection of 2-morphisms Ω of double a category C generates C if C is equal to the complete sub-double category of C generated by Ω. The following is the main definition of this section.

3.1. Definition. *Let C be a double category. We say that C is globularily generated if C is generated by its collection of globular 2-morphisms.*

Trivial double categories are examples of globularily generated double categories. Non-trivial examples will be provided in section 6.

Given a double category C, we write γC for the complete sub-double category of C generated by the collection of globular 2-morphisms of C. Thus defined, γC is a globularily generated complete sub-double category of C. We call double category γC associated to C the **globularily generated piece** of C. Thus defined, the globularily generated piece γC of double category C is equal to both the maximal globularily generated sub-double category of C and to the minimal complete sub-double category of C containing the collection of globular 2-morphisms of C.

3.2. Proposition. *Let C be a double category. The decorated horizontalization H^*C of C is equal to the decorated horizontalization $H^*\gamma C$ of the globularily generated piece γC of C. Moreover, the globularily generated piece γC of C is minimal among all sub-double categories D of C satisfying this property.*

Proof. Let C be a double category. The fact that the decorated horizontalization H^*C of C and the decorated horizontalization $H^*\gamma C$ of the globularily generated piece γC of C are equal follows from the fact that the globularily generated piece γC of C is complete in C and from the easy observation that the collection of globular 2-morphisms of both C and γC are equal. Now, suppose D is a sub-double category of C such that the decorated horizontalization H^*D of D is equal to the decorated horizontalization H^*C of C. We wish to prove that the globularily generated piece γC of C is a sub-double category of D. First observe that from the fact that H^*C and H^*D are equal it follows that D is complete in C. It follows that both the category of objects and the collection of horizontal morphisms of both D and γC are equal. Again from the equation $H^*C = H^*D$ it follows that the collection of globular 2-morphisms of C is contained in the collection of 2-morphisms of D. We conclude that the globularily generated piece γC is a sub-double category of D. This concludes the proof. ■

A consequence of proposition 3.2 is that the problem of existence of internalizations for a given decorated bicategory B^*, B is equivalent to the problem of existence of globularily generated internalizations for B^*, B. The following corollary says that the condition of a double category being globularily generated is precisely the condition of a double category being a minimal internalization.

3.3. COROLLARY. *Let C be a double category. C is globularily generated if and only if C is not properly contained in any internalization of decorated horizontalization H^*C of C.*

We interpret corollary 3.3 by saying that the condition of a double category being globularily generated is precisely the condition of a double category being minimal with respect to the decorated horizontalization functor. We regard this and proposition 3.2 as the main motivation for the study of globularily generated double categories.

We extend the construction of the globularily generated piece of a double category, to double functors as follows: Given a double functor $F : C \longrightarrow D$ from a double category C to a double category D, the image, under the morphism functor F_1 of F, of every globular 2-morphism in C, is a globular 2-morphism in D. It follows that the restriction of double functor F, to the globularily generated piece γC of C defines a double functor from the globularily generated piece γC of C to the globularily generated piece γD of D. We write γF for this double functor. We call γF the **globularily generated piece double functor of** F. We regard the following proposition as a functorial version of proposition 3.2.

3.4. PROPOSITION. *Let C, D be double categories. Let $F : C \longrightarrow D$ be a double functor. The decorated horizontalization H^*F of F is equal to the decorated horizontalization $H^*\gamma F$ of the globularily generated piece γF of F. Moreover, the globularily generated piece γF of F is minimal, in the sense of corollary 3.3, with respect to this property.*

PROOF. Let C, D be double categories. Let $F : C \longrightarrow D$ be a double functor. From the fact that the object functor and the object function of the morphism functor of the globularily generated piece γF of F are equal to the object functor and the object function of the morphism functor of F respectively, and from the fact that the globularily generated piece γF of F is equal to F on globular 2-morphisms of C it follows that the decorated horizontalization H^*F of F is equal to the decorated horizontalization $H^*\gamma F$

of the globularily generated piece γF of F. Now let C', D' be sub-double categories of C and D respectively. Let $F' : C' \longrightarrow D'$ be a double sub-functor of F such that H^*F and H^*F' are equal. We wish to prove, in that case, that the globularily generated piece γF of F is a sub-double functor of F'. The globularily generated piece γC is a sub-double category of C' and the globularily generated piece γD of D is a sub-double category of D' by proposition 3.2. By equation $H^*F' = H^*F$ it follows that the object functor and the object function of morphism functor of F' are equal to the object functor and the object function of morphism functor of γF respectively. From this and from the fact that the restriction of F' to the collection of globular 2-morphisms of C is equal to the restriction of F to the collection of globular 2-morphisms of C it follows that γF is a subfunctor of F'. This concludes the proof. ■

We write **gCat** for the subcategory of the underlying category of 2-category **dCat** of double categories, double functors, and double natural transformations, generated by globularily generated double categories. The function associating the globularily generated piece γC to every double category C and the globularily generated piece double functor γF to every double functor F, is a functor from the underlying category of 2-category **dCat** to **gCat**. We denote this functor by γ. We call functor γ the **globularily generated piece functor**. We consider the globularily generated piece functor γ as a functorial version of the globularily generated piece construction.

We will say that an internalization functor Ψ, as defined in the previous section, is a **globularily generated internalization functor**, if the image category of Ψ lies in category **gCat**. A consequence of proposition 3.4 is that the composition $\gamma\Phi$ of an internalization functor Φ and the globularily generated piece functor γ, is again an internalization functor, and thus the problem of existence of internalization functors is equivalent to the problem of existence of globularily generated internalization functors. The following corollary is a functorial analog of corollary 3.3.

3.5. Corollary. *Let Φ be an internalization functor. In that case Φ is globularily generated if and only if Φ does not properly contain internalization sub-functors.*

We extend the globularily generated piece functor to a 2-functor between appropriate 2-categories as follows: Given a double category C, we call 2-morphisms in C lying in globularily generated piece γC of C the **globularily generated 2-morphisms** of C. Given double functors $F, G : C \longrightarrow D$ from a double category C to a double category D we say that a double natural transformation $\eta : F \longrightarrow G$, from F to G, is a **globularily generated double natural transformation**, if every component of the morphism part η_1 of η is a globularily generated 2-morphism.

Identity double natural transformations are examples of globularily generated double natural transformations. Further, the collection of globularily generated double natural transformations is closed under the operations of taking vertical and horizontal compositions. It follows that the triple formed by collection of double categories, collection of double functors, and collection of globularily generated double natural transformations forms a sub 2-category of 2-category **dCat**. We denote this 2-category by $\mathbf{dCat}^{\mathrm{g}}$.

We keep denoting by **gCat** the full sub 2-category of $\mathbf{dCat}^{\mathrm{g}}$ generated by globularily generated double categories. 2-category **gCat** now has collection of globularily generated double categories, collection of double functors between globularily generated double categories, and collection of globularily generated double natural transformations as collections of 0-, 1-, and 2-cells respectively. The globularily generated piece functor γ admits a unique extension to a 2-functor from $\mathbf{dCat}^g$ to **gCat** such that this extension acts as the identity function on globularily generated double natural transformations. We keep denoting this extension by γ.

Given 2-categories B and B', such that B is a full sub 2-category of B', we will say that B is a **strictly 2-coreflective sub 2-category** of B' if the inclusion 2-functor of B in B' admits an right adjoint 2-functor with counit and unit being strict 2-natural transformations. In that case we will say that any 2-functor, right adjoint to the inclusion 2-functor of B in B', is a **strict 2-coreflector** of B' on B. The next proposition says that 2-category **gCat**, as a sub 2-category of 2-category $\mathbf{dCat}^g$ is strictly 2-coreflective, with the globularily generated piece 2-functor γ as 2-coreflector.

3.6. Proposition. *2-category **gCat** is a strictly 2-coreflective sub 2-category of 2-category $\mathbf{dCat}^g$ with globularily generated piece 2-functor γ as 2-coreflector.*

Proof. Let i denote the inclusion 2-functor of 2-category **gCat** in 2-category $\mathbf{dCat}^g$. We wish to provide pair (γ, i) formed by the globularily generated piece 2-functor γ and the inclusion 2-functor i with the structure of an adjoint pair. We associate, to pair (γ, i) a counit-unit pair (ϵ, η).

Let C be a double category. Write ϵ_C for the inclusion double functor of the globularily generated piece γC associated to C in C. We write ϵ for the collection of inclusions ϵ_C where C runs through collection of all double categories. We prove that thus defined ϵ is a 2-natural transformation [16] from the composition $i\gamma$ of the globularily generated piece 2-functor γ and inclusion i to the identity 2-endofunctor $id_{\mathbf{dCat}^g}$ of 2-category $\mathbf{dCat}^g$. Let C and D be double categories. Let $F : C \longrightarrow D$ be a double functor from C to D. Since double functor γ is defined by restriction on 1-cells, the following square

$$\begin{array}{ccc} i\gamma C & \xrightarrow{i\gamma F} & i\gamma D \\ {\scriptstyle \epsilon_C}\downarrow & & \downarrow{\scriptstyle \epsilon_D} \\ C & \xrightarrow[F]{} & D \end{array}$$

commutes. Now, let $F, G : C \longrightarrow D$ be double functors from double category C to double category D and let $\mu : F \longrightarrow G$ be a globularily generated natural transformation from F to G. We wish to prove that equation

$$\epsilon_D \mu = \mu \epsilon_C$$

holds. The above equation is equivalent to the following pair of equations

$$\epsilon_{D_0}\mu_0 = \mu_0\epsilon_{C_0} \text{ and } \epsilon_{D_1}\mu_1 = \mu_1\epsilon_{C_1}$$

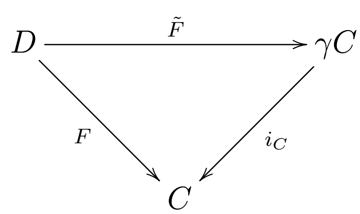
Observe that since the globularily generated piece 2-functor acts as the identity 2-functor on object categories, object functors, and object natural transformations, the first equation above is trivial. We thus need only to prove that the second equation holds. Let α be a horizontal morphism in C. In that case μ_α is a globularily generated 2-morphism and thus $\epsilon_D\mu_\alpha$ is equal to μ_α. Now, $\epsilon_C\alpha$ is equal to α and thus $\mu\epsilon_C\alpha$ is equal to μ_α. We conclude that both equations above hold and thus ϵ is a strict 2-natural transformation from the composition $i\gamma$ of the globularily generated piece 2-functor γ and inclusion i to the identity 2-endofunctor $id_{\mathbf{dCat}^g}$ of 2-category $\mathbf{dCat}^g$.

Now, since the globularily generated piece of a globularily generated double category is equal to the original globularily generated category and the globularily generated piece double functor γ acts by restriction on double functors and double natural transformations, the composition γi of the inclusion i of 2-category $\mathbf{gCat}$ in 2-category $\mathbf{dCat}^g$ and the globularily generated piece 2-functor γ is equal to the identity 2-endofunctor of 2-category $\mathbf{gCat}$. Denote by η the identity double natural transformation of the identity 2-endofunctor $id_{\mathbf{gCat}}$ of $\mathbf{gCat}$ as a natural transformation from $id_{\mathbf{gCat}}$ to the composition γi. Thus defined η is a strict natural transformation. Finally, observe that from the way η was defined, pair of natural transformations (ϵ, η) clearly satisfies the counit-unit triangle equations and it is thus a counit-unit pair for pair (γ, i). We conclude that 2-category $\mathbf{gCat}$ is a coreflective 2-subcategory of 2-category $\mathbf{dCat}^g$ and that 2-functor γ acts as a coreflector. ∎

We interpret proposition 3.6 by considering the globularily generated piece 2-functor γ as an analog of the horizontalization functor H and thus regarding the inclusion of $\mathbf{gCat}$ in $\mathbf{dCat}^g$ as an analogue of the inclusion of $\mathbf{bCat}$ in $\mathbf{dCat}$.

In the following corollary, for a given double category C, we keep denoting, as in the proof of proposition 3.6, the inclusion double functor from the globularily generated piece γC of C to C by ϵ_C

3.7. Corollary. *Let C be a double category. The globularily generated piece γC of C is characterized up to double isomorphisms by the following universal property: Let D be a globularily generated double category. Let $F : D \longrightarrow C$ be a double functor from D to C. In that case there exists a unique double functor $\tilde{F} : D \longrightarrow \gamma C$ from D to globularily generated piece γC of C such that the following triangle commutes:*



4. Structure

In this section we introduce technical tools in the theory of globularly generated double categories. We define the vertical length of a globularly generated 2-morphism and we use this to obtain general information on the structure of globularly generated double categories. We prove in particular that the condition of a double category being globularly generated is not trivial. We begin by recursively associating, to every globularly generated double category, a sequence of subcategories of its category of morphisms as follows:

Let C be a globularly generated double category. Denote by H_1^C the union of collection of globular 2-morphisms of C and collection of horizontal identities of vertical morphisms of C. Write V_1^C for the subcategory of the category of morphisms C_1 of C, generated by H_1^C, that is, V_1^C denotes the subcategory of C_1 whose morphisms are vertical compositions of globular 2-morphisms and horizontal identities of C.

Let n be an integer strictly greater than 1. Suppose that category V_{n-1}^C has been defined. We now define category V_n^C. First denote by H_n^C the collection of all possible horizontal compositions of 2-morphisms in category V_{n-1}^C. We make, in that case, category V_n^C to be the subcategory of the category of morphisms C_1 of C, generated by H_n^C. That is, category V_n^C is the subcategory of C_1 whose collection of morphisms is the collection of vertical compositions of elements of H_n^C.

We have thus associated, to every double category C, a sequence of subcategories $\{V_n^C\}$ of the category of morphisms C_1 of C. We call, for every n, category V_n^C the **n-th vertical category** associated to C. We have used, in the above construction, for every n, an auxiliary collection of 2-morphisms H_n^C of C. Observe that for each n, collection H_n^C both contains the horizontal identity of every vertical morphism in C and is closed under the operation of taking horizontal compositions. If double category C is strict, then, for every n, collection H_n^C is the collection of morphisms of a category whose collection of objects is the collection of vertical morphisms of C. In that case we call category H_n^C the **n-th horizontal category** associated to C.

By the way the sequence of vertical categories $\{V_n^C\}$ associated to a double category C was constructed it is easily seen that for every n, inclusions

$$\mathrm{Hom}V_n^C \subseteq H_{n+1}^C \subseteq \mathrm{Hom}V_{n+1}^C$$

hold. This implies that the n-th vertical category V_n^C associated to double category C is a subcategory of the $n+1$-th vertical category V_{n+1}^C associated to C for every n. Moreover, in the case in which double category C is strict, the n-th horizontal category H_n^C associated to C is a subcategory of the $n+1$-th horizontal category H_{n+1}^C associated to C for every n.

The following lemma says that the sequence of vertical categories of a globularly generated double category forms a filtration of its category of morphisms.

4.1. LEMMA. *Let C be a globularly generated double category. Morphism category C_1 of C is equal to the limit $\lim V_n^C$ in **Cat**, of sequence $\{V_n^C\}$ of vertical categories associated to C.*

Proof. Let C be a globularily generated double category. We wish to prove that morphism category C_1 of C is equal to the limit $limV_n^C$ of the sequence of vertical categories associated to C.

By the way it was defined it is easily seen that the category of morphisms γC_1 of the globularily generated piece γC of C is generated by collection H_1^C under vertical and horizontal composition operations in C. It follows that category γC_1 is equal to the union $\bigcup_{n=1}^{\infty} V_n^C$ of vertical categories V_n^C of C. The lemma follows from this and from the fact that category of morphisms C_1 of C equals category of morphisms γC_1 of γC. This concludes the proof. ■

Given a strict double category C the pair τC formed by collection of vertical morphisms of C and collection of 2-morphisms of C is a category. Composition operation in τC is horizontal composition in C. We call category τC associated to a strict double category C the **transversal category associated to** C.

4.2. Corollary. *Let C be a globularily generated double category. If C is a strict double category, then the transversal category τC associated to C is equal to the limit $limH_n^C$, in **Cat**, of the sequence of horizontal categories associated to C.*

Proof. Let C be a strict globularily generated double category. We wish to prove, in this case, that the transversal category τC associated to C is equal to the limit $limH_n^C$ of the sequence of horizontal categories associated to C.

By the way the sequence of horizontal categories associated to globularily generated double category C was constructed, it is easily seen that the collection of objects of the n-th horizontal category H_n^C associated to C is equal to the collection of vertical morphisms of C for every n. It follows, from this, that the collection of objects of the limit $limH_n^C$ of the sequence of horizontal categories associated to C is equal to the collection of vertical morphisms of C and is thus equal to the collection of objects of the transversal category τC of C. The collection of morphisms of the limit $limH_n^C$ is equal to the union $\bigcup_{n=1}^{\infty} \mathrm{Hom}H_n^C$ of the collections of morphisms of horizontal categories associated to C. This union is equal to the union $\bigcup_{n=1}^{\infty} \mathrm{Hom}V_n^C$ of the collections of morphisms of vertical categories associated to C, which by lemma 4.1 is equal to the collection of 2-morphisms of C. This concludes the proof. ■

4.3. Definition. *Let C be a globularily generated double category. Let Φ be a 2-morphism in C. We call the minimal integer n such that Φ is a morphism of the n-th vertical category V_n^C associated to C the vertical length of Φ.*

We now apply the concept of vertical length to the proof of results concerning the structure of globularily generated double categories. We first establish notational conventions.

Assuming a double category C is strict, the horizontal composition $\Phi_k * ... * \Phi_1$ of any composable sequence $\Phi_1, ..., \Phi_k$ of 2-morphisms in C, is unambiguously defined. This is not the case in general. If a double category C is not assumed to be strict, then the horizontal compositions of a composable sequence $\Phi_1, ..., \Phi_k$ of 2-morphisms in C, following different parentheses patterns, might yield different 2-morphisms. If a 2-morphism Φ in a double category C can be obtained as the horizontal composition, following a certain parentheses pattern, of composable sequence of 2-morphisms $\Phi_1, ..., \Phi_k$, we will write $\Phi \equiv \Phi_k * ... * \Phi_1$.

Given 2-morphisms Φ and Ψ in a double category C, we say that Φ and Ψ are **globularly equivalent** if there exist globular 2-isomorphisms Θ_1, Θ_2 in C such that the equation $\Phi = \Theta_1 \Psi \Theta_2^{-1}$ holds.

From the fact that associators in double categories satisfy the pentagon axiom and from the fact that the collection of globular 2-morphisms of any double category is closed under the operations of taking vertical and horizontal composition, it follows that if two 2-morphisms Φ and Ψ satisfy equation $\Phi, \Psi \equiv \Phi_k * ... * \Phi_1$ for a composable sequence of 2-morphisms $\Phi_1, ..., \Phi_k$ in C, then Φ and Ψ are globularily equivalent.

Finally, we will say that a 2-morphism Φ in a double category C is a **horizontal endomorphism** if the source and target $s\Phi, t\Phi$ of Φ, are equal. Horizontal identities are examples of horizontal endomorphisms. The next proposition says that a 2-morphism in a globular double category is either globular or a horizontal endomorphisms.

4.4. Proposition. *Let C be a globularily generated double category. Every non-globular 2-morphism in C is a horizontal endomorphism.*

Proof. Let C be a globularily generated double category. Let Φ be a non-globular 2-morphism in C. We wish to prove that Φ is a horizontal endomorphism.

We proceed by induction on the vertical length of Φ. Suppose first that Φ is an element of H_1^C. In that case, by the assumption that Φ is non-globular, Φ must be the horizontal identity of a vertical morphism in C and thus must be a horizontal endomorphism. Suppose now that Φ is a general element of the first vertical category V_1^C associated to C. Write Φ as a vertical composition $\Phi = \Phi_k \circ ... \circ \Phi_1$ where Φ_i is an element of H_1^C for every i. Moreover, assume that the length k of this decomposition is minimal. We prove by induction on k that Φ must be a horizontal endomorphism. Suppose first that $k = 1$. In that case Φ is an element of H_1^C and is thus a horizontal identity. Suppose now that k is strictly greater than 1 and that the result is true for every 2-morphism in the first vertical category V_1^C associated to C that can be written as a vertical composition of strictly less than k 2-morphisms in H_1^C. Write Ψ for composition $\Phi_k \circ ... \circ \Phi_2$. In this case equation $\Phi = \Psi \circ \Phi_1$ holds. Now, since the collection of globular 2-morphisms of C is closed under the operation of taking vertical composition, one of Ψ and Φ_1 is not globular. If both Ψ and Φ_1 are not globular, then by induction hypothesis both Ψ and Ψ_1 are horizontal endomorphisms and thus their vertical composition Φ is a horizontal endomorphism. Suppose now that Ψ is globular. In that case Φ_1 is a horizontal endomorphism. Now, from the fact that source and target of Ψ are in this case vertical identities and from the fact that source and target are functorial, equations $s\Phi = s\Phi_1$ and $t\Phi = t\Psi_1$ follow and thus Φ is a horizontal endomorphism. The case in which Φ_1 is globular is handled analogously. This concludes the base of the induction.

Let n be a positive integer strictly greater than 1. Assume now that every non-globular 2-morphism in C of vertical length strictly less than n is a horizontal endomorphism. Suppose first that Φ is an element of H_n^C. Let $\Phi_k * ... * \Phi_1$ represent a horizontal composition in C such that Φ_i is a morphism of V_{n-1}^C for each i and such that $\Phi \equiv \Phi_k * ... * \Phi_1$. Suppose that the length k of this decomposition is minimal. We proceed by induction over k. If $k = 1$ then Φ is an element of V_{n-1}^C and is thus a horizontal endomorphism by induction hypothesis. Suppose now that k is strictly greater than 1 and that the result is true for every non-globular 2-morphism in H_n^C that can be written as a horizontal composition of strictly less than k 2-morphisms in V_{n-1}^C. Choose Ψ such that $\Psi \equiv \Phi_k * ... * \Phi_2$. In this case Φ and $\Psi * \Phi_1$ are globularily equivalent and thus have the same source and target. Now, if both Ψ and Φ_1 are globular, then their horizontal composition, and every 2-morphism globularily equivalent to it, is globular. We thus assume that one of Ψ and Φ_1 is non-globular. If Ψ is globular, then the equation $t\Phi_1 = s\Psi$ together with induction hypothesis implies that Φ_1 is globular. An identical argument implies that if Φ_1 is globular then Ψ is globular. We conclude that both Ψ and Φ_1 are non-globular and thus by induction hypothesis are horizontal endomorphisms. This and equation $t\Phi_1 = s\Psi$ implies that $\Psi * \Phi_1$ and thus Φ is a horizontal endomorphism. Assume now that Φ is a general morphism of V_n^C. Write Φ as a vertical composition $\Phi_k \circ ... \circ \Phi_1$ where Φ is an element of H_n^C for every k. Moreover, assume again that the length k of this decomposition is minimal. An induction argument over k together with an argument analogous to that presented in the base of the induction proves that Φ is a horizontal endomorphism. This concludes the proof. ■

A direct consequence of proposition 4.4 is that neither double categories of the form $\mathbf{Cob}(n)$ nor double categories $\mathbf{Alg}$ or $[W^*]^f$, presented in section 2 are globularily generated. We interpret this by saying that the condition of a double category being globularily generated is not trivial. We will compute the globularily generated piece of these double categories in section 6. This will provide non-trivial examples of globularily generated double categories. The following corollary follows immediately from the previous proposition.

4.5. Corollary. *Let C be a globularily generated double category. Let Φ and Ψ be 2-morphisms in C. Suppose Φ and Ψ are composable. In that case horizontal composition $\Psi * \Phi$ is globular if and only if Φ and Ψ are both globular.*

We conclude this section with the following technical lemma.

4.6. Lemma. *Let C be a globularily generated double category. Let Φ be a 2-morphism in C. If the vertical length of Φ is equal to 1 then Φ can be written as a vertical composition of the form*

$$\Psi_k \circ \Phi_k \circ ... \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

where Φ_i is a horizontal identity for every $1 \leq i \leq k$ and Ψ_i is globular for every $0 \leq i \leq k$.

Proof. Let C be a globularily generated double category. Let Φ be a 2-morphism in C. Suppose that the vertical length of Φ is equal to 1. We wish to prove, in this case, that Φ admits a decomposition as described in the statement of the lemma.

Suppose first that Φ is an element of H_1^C. In that case Φ is either globular or Φ is the horizontal identity of a vertical morphism in C. Suppose first that Φ is globular. In that case make $k = 0$ and $\Psi_0 = \Phi$. Suppose now that Φ is the horizontal identity of a vertical morphism α in C, with domain and codomain x and y respectively. In that case make $k = 1$, make Ψ_0 to equal to the identity 2-morphism of the horizontal identity of x, make Φ_1 to be equal to Φ and make Ψ_1 to be equal to the identity 2-morphism of horizontal identity of y.

Suppose now that Φ is a general morphism of the first vertical category V_1^C associated to C. Write Φ as the vertical composition $\Phi = \Theta_m \circ ... \circ \Theta_1$, where Θ_i is an element of H_1^C for every i. Choose this decomposition in such a way that its length m is minimal. We proceed by induction on m. In the case in which m is equal to 1 Φ is an element of H_1^C. Suppose now that m is strictly greater than 1 and that the result is true for every 2-morphism in V_1^C that can be written as a vertical composition of strictly less than m elements of H_1^C. Write Ψ for vertical composition $\Theta_{m-1} \circ ... \circ \Theta_1$. In that case Ψ admits a decomposition as

$$\Psi = \Psi_k \circ \Phi_k \circ ... \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

for some k, where Φ_i is a horizontal identity for every $1 \leq i \leq k$ and Ψ_i is globular for every $0 \leq i \leq k$. Since Θ_m is an element of H_1^C then it is either globular or it is the horizontal identity of a vertical morphism in C. Suppose first that Θ_m is globular. In that case write Ψ'_k for vertical composition $\Theta_m \circ \Psi_k$. In that case the decomposition

$$\Phi = \Psi'_k \circ \Phi_k \circ ... \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

satisfies the conditions of the lemma. Suppose now that Θ_m is the vertical identity of a vertical morphism α, with domain and codomain x and y respectively. In that case write Φ_{k+1} for Θ_m and write Ψ_{k+1} for the identity 2-endomorphism of the horizontal identity of x. In that case the decomposition

$$\Phi = \Psi_{k+1} \circ \Phi_{k+1} \circ ... \circ \Psi_1 \circ \Phi_1 \circ \Psi_0$$

satisfies the conditions of the lemma. This concludes the proof. ■

5. Functoriality

In this section we present an extension of the definition of the vertical filtration of a globularily generated double category presented in the previous section, to a filtration on the globularily generated piece functor. We begin by establishing notational conventions.

Denote by π_0 and π_1 the 2-functors from **dCat** to **Cat** such that for every double category C, $\pi_0 C$ and $\pi_1 C$ are equal to the object category C_0 of C and to the morphism category C_1 of C respectively, such that for every double functor F, $\pi_0 F$ and $\pi_1 F$ are equal to the object functor F_0 of F and to the morphism functor F_1 of F respectively, and finally, such that for every double natural transformation η, $\pi_0\eta$ and $\pi_1\eta$ are equal to the object natural transformation η_0 associated to η and to the morphism natural transformation η_1 associated to η respectively. We call π_0 and π_1 the **object projection** and the **morphism projection** 2-functors of **dCat** respectively. We keep denoting by π_0 and π_1 restrictions of object and morphism projections of 2-category **dCat**, to sub 2-category **gCat**.

Denote by $\overline{\mathbf{dCat}}$ the sub 2-category of **dCat** generated by collection of strict double categories and collection of strict double functors between them, and denote by $\overline{\mathbf{gCat}}$ the sub 2-category of $\overline{\mathbf{dCat}}$ generated by collection of strict globularily generated double categories. Given a positive integer n, the pair of functions associating, for every strict globularily generated double category C the n-th horizontal category H_n^C associated to C, and to every strict double functor F, the n-th horizontal functor H_n^F associated to F, is a functor from the underlying category of 2-category $\overline{\mathbf{gCat}}$, to the underlying category of 2-category **Cat** of categories, functors, and natural transformations.

Given a strict double category C, we denoted, in section 4, by τC the category whose collection of objects is collection of vertical morphisms of C and whose collection of morphisms is collection of 2-morphisms of C. We called category τC the **transversal category associated to** C. Given a strict double functor $F : C \longrightarrow D$ from a strict double category C to a strict double category D, we denote by τF the functor from the transversal category τC associated to C to the transversal category τD associated to D, such that object and morphism functions of τF are the morphism function of object functor F_0 associated to F and the morphism function of morphism functor F_1 associated to F respectively. We call τF the **transversal functor associated to** F.

The pair of functions associating the transversal category τC to a double category C and the transversal functor τF to a double functor F forms a functor from the underlying category of 2-category $\overline{\mathbf{dCat}}$ to **Cat**. We denote this functor by τ. We call τ the **transversal category functor**. We keep denoting by τ the restriction of the transversal functor to the underlying category of 2-category $\overline{\mathbf{gCat}}$. We prove the following proposition.

5.1. Proposition. *The collection of functions associating to every globularily generated double category C its vertical filtration $\left\{V_n^C\right\}$ extends to a filtration of subfunctors $\{V_n\}$ of composition $\pi_1\gamma$. Likewise the collection of functions associating to every strict globularily generated double category C its horizontal filtration $\left\{H_n^C\right\}$ extends to a filtration of subfunctors $\{H_n\}$ of composition $\tau\gamma$.*

Proof. We wish to prove that the collection of functions associating to every globularily generated double category C its vertical filtration $\left\{V_n^C\right\}$ extends to a filtration of subfunctors $\{V_n\}$ of composition $\pi_1\gamma$ and that the collection of functions associating to every strict globularily generated double category C its horizontal filtration $\left\{H_n^C\right\}$ extends to a filtration of subfunctors $\{H_n\}$ of composition $\tau\gamma$.

Let C and D be globularily generated double categories. Let $F : C \longrightarrow D$ be a double functor from C to D. Let n be a positive integer. We first prove that the image of the n-th vertical category V_n^C associated to C, under morphism functor F_1 of F, is a subcategory of the n-th vertical category V_n^D associated to D. Moreover, we prove that if C, D, and F are all strict then the image of the n-th horizontal category H_n^C associated to C, under morphism functor F_1 of F, is a subcategory of the n-th horizontal category H_n^D associated to D.

We proceed by induction on n. Let Φ be a 2-morphism in the first vertical category V_1^C associated to C. We wish to prove, in this case that $F_1\Phi$ is a morphism in the first vertical category V_1^D associated to D. Suppose first that Φ is an element of H_1^C. In that case Φ is either globular or the horizontal identity of a vertical morphism in C. Suppose first that Φ is the horizontal identity of a vertical morphism α in C. In that case the image $F_1\Phi$ of Φ under functor F_1 is globularily conjugate to the horizontal identity of the image $F_0\alpha$ of α under functor F_0, and is thus a morphism in category V_1^D. Observe that in the case in which double functor F is strict $F_1\Phi$ is precisely the horizontal identity of vertical morphism $F_0\alpha$ and is thus an element of H_0^D. From this and from the fact that double functors preserve globular 2-morphisms it follows that the image of collection H_1^C, under morphism functor F_1, is contained in collection of morphisms of first vertical category V_1^D of D. Moreover, in the case in which F is strict, the image of H_1^C under F_1 is contained in H_1^D. Suppose now that Φ is a general element of the first vertical category V_1^C associated to C. Write Φ as a vertical composition

$$\Phi = \Phi_k \circ ... \circ \Phi_1$$

where Φ_i is an element of H_1^C for every $1 \leq i \leq k$. In that case the image of Φ under morphism functor F_1 of F is equal to vertical composition

$$F_1\Phi_k \circ ... \circ F_1\Phi_1$$

which is a morphism of the first vertical category V_1^D associated to D. Thus the image of the first vertical category V_1^C associated to C, under morphism functor F_1 of F is a subcategory of the first vertical category V_1^D associated to D. Moreover, if we assume that C, D, and F are strict, H_1^C and H_1^D are categories, and the image of H_1^C under morphism functor F_1 of F is a subcategory of H_1^D.

Let n now be strictly greater than 1. Suppose that the conclusions of the proposition are true for every $m < n$. Let Φ now be a morphism in the n-th vertical category V_n^C associated to C. We wish to prove in this case that the image $F_1\Phi$ of Φ under morphism functor F_1 of F is a morphism in the n-th vertical category V_n^D associated to D. Suppose first that Φ is a morphism in H_n^C. Write Φ, up to globular equivalences, as a horizontal composition of the form

$$\Phi \equiv \Phi_k * ... * \Phi_1$$

where Φ_i is an element of the $n-1$-th vertical category V_{n-1}^C associated to C for every $1 \leq i \leq k$. In that case the image $F_1\Phi$ under functor F_1 is globularily equivalent to any possible interpretation of horizontal composition

$$F_1\Phi_k * ... * F_1\Phi_1$$

in D. By induction hypothesis $F_1\Phi_i$ is a morphism of the $n-1$-th vertical category V^D_{n-1} associated to D for every $1 \leq i \leq k$ and thus any interpretation of the horizontal composition above is a morphism of the n-th horizontal category V^D_n associated to D. We conclude that the image $F_1\Phi$ of 2-morphism Φ under functor F_1 is a morphism in the n-th vertical category V^D_n associated to D. Moreover, if F is strict then image $F_1\Phi$ of Φ under functor F_1 is an element of H^D_n. Suppose now that Φ is a general morphism of the n-th vertical category V^C_n associated to C. Write Φ as a vertical composition of the form

$$\Phi = \Phi_k \circ ... \circ \Phi_1$$

where Φ_i is an element of H^C_n for every $1 \leq i \leq k$. In that case the image $F_1\Phi$ of Φ under morphism functor F_1 of F is equal to vertical composition

$$F_1\Phi_k \circ ... \circ F_i\Phi_1$$

in D and thus is an element of n-th vertical category V^D_n of D. We conclude that the image of the n-th vertical category V^C_n associated to C, under morphism functor of double functor F, is a subcategory of the n-th vertical category V^D_n associated to D and that if C, D, and F are strict then moreover the image of the n-th horizontal category H^C_n associated to C, under morphism functor F_1 of F, is a subcategory of the n-th horizontal category H^D_n associated to D.

Given a positive integer n and a functor F from globularily generated double category C to globularily generated double category D we write V^F_n for restriction, to the n-th vertical category V^C_n associated to C, of morphism functor F_1 of F. Thus defined V^F_n is, by lemma 5.1, a functor from the n-th vertical category V^C_n associated to C to the n-th vertical category V^D_n associated to D. We call V^F_n the **n-th vertical functor associated to** F.

Given a positive integer n we make V_n to be the pair formed by the function associating the n-th vertical category V^C_n associated to C to every double category C and the n-th vertical double functor V^F_n to every double functor F. The fact that composition $\pi_1\gamma$ is equal to the limit $limV_n$ of chain $\{V_n\}$ follows directly from lemma 4.1. The definition of chain of functors $\{H_n\}$ is performed analogously. The fact that composition $\pi_1\tau$ is equal to the limit $limH_n$ now follows directly from lemma 4.2. This concludes the proof

■

6. Computations

In this final section we present explicit computations of the globularily generated piece of double categories introduced in section 2. We begin by computing the globularily generated piece $\gamma\mathbf{Cob}(n)$ of double category $\mathbf{Cob}(n)$ of n-dimensional manifolds, diffeomorphisms, cobordisms, and equivariant diffeomorphisms, for every positive integer n

We will write a horizontal equivariant endomorphism (f,Φ,f) in double category $\mathbf{Cob}(n)$ simply as (f,Φ). If an equivariant morphism in $\mathbf{Cob}(n)$ is written in this way it will be assumed it is a horizontal endomorphism. We will say that cobordisms M and N from a closed manifold X to itself are **globularily diffeomorphic** if M and N are diffeomorphic relative to X.

In order to explicitly compute globularily generated piece $\gamma\mathbf{Cob}(n)$ of double category $\mathbf{Cob}(n)$, by proposition 4.4 we need only to compute collection of non-globular, globularily generated 2-morphisms between horizontal endomorphisms in $\mathbf{Cob}(n)$. We begin with the following lemma.

6.1. Lemma. *Let n be a positive integer. Let X and Y be closed n-dimensional manifolds. Let M be a cobordism from X to X and let N be a cobordism from Y to Y. If there exists a non-globular globularily generated diffeomorphism from M to N then M and N are globularily diffeomorphic to identity cobordisms i_X and i_Y respectively.*

Proof. Let n be a positive integer. Let X and Y be closed n-dimensional manifolds. Let M be a cobordism from X to X and let N be a cobordism from Y to Y. Suppose there exists a non-globular globularily generated diffeomorphism from M to N. In that case we wish to prove that M and N are globularily diffeomorphic to identity cobordisms i_X and i_Y respectively.

Let $(f,\Phi): M \longrightarrow N$ be a non-globular globularily generated diffeomorphism from M to N. We proceed by induction on the vertical length of (f,Φ) to prove that the existence of (f,Φ) implies that M and N are globularily diffeomorphic to horizontal identities i_X and i_Y respectively. Suppose first that the vertical length of (f,Φ) is equal to 1. By lemma 4.6 there exists a decomposition of (f,Φ) as a vertical composition of the form

$$(id_{X_k},\Psi_k)\circ(f_k, f_k\times id_{[0,1]})\circ ...\circ(id_{X_1},\Psi_1)\circ(f_1,f_1\times id_{[0,1]})\circ(id_{X_0},\Psi_0)$$

where $X_0,...,X_k$ are n-dimensional manifolds, X_0 and X_k are equal to X and Y respectively, $f_i: X_i \longrightarrow X_{i+1}$ is a diffeomorphism from X_i to X_{i+1} for all $i\leq k-1$, and where Ψ_i is a globular diffeomorphism from X_i to X_i for all i. Since we assume that (f,Φ) is not globular then the length k of this decomposition is greater than or equal to 1. The domain of Ψ_1 is equal to the horizontal identity i_X of manifold X and the codomain of Ψ_k is equal to the horizontal identity i_Y of manifold Y. Thus Ψ_0 is a globular diffeomorphism from M to i_{X_0} and Ψ_k is a globular diffeomorphism from i_{Y_0} to N.

Let m be a positive integer strictly greater than 1. Assume now that the result is true for every pair of cobordisms admitting a non-globular globularily generated diffeomorphism of vertical length strictly less than m. Assume first that non-globular globularily generated diffeomorphism (f,Φ) is an element of $H_m^{\mathbf{Cob}(n)}$. Write, in this case (f,Φ) as a horizontal composition

$$(f,\Phi)\equiv(f,\Phi_k)*...*(f,\Phi_1)$$

where (f, Φ_i) is a morphism in the $m-1$-th vertical category $V_{m-1}^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$ for every $i \leq k$. Moreover, assume that the length k of this decomposition is minimal. We proceed by induction on k to prove that in this case the existence of (f, Φ) implies that M and N satisfy the conditions of the lemma. If $k = 1$ then (f, Φ) is an element of the $m-1$-th vertical category $V_{m-1}^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$ and by induction hypothesis its existence implies that M and N satisfy the conditions of the lemma. Suppose now that k is strictly greater than 1. Write (f, Ψ) for any representative of $(f, \Phi_k) * ... * (f, \Phi_2)$. In that case the horizontal composition $(f, \Psi) * (f, \Phi_1)$ is equivalent to (f, Φ). From the assumption that (f, Φ) is not globular and from corollary 4.5 it follows that no globular 2-morphism is globularily equivalent to (f, Φ). Thus the horizontal composition $(f, \Psi) * (f, \Phi_1)$ is not globular and thus, again by corollary 3.6 neither (f, Ψ) nor (f, Φ_1) is globular. Both (f, Ψ) and (f, Φ_1) are globularily equivalent to the horizontal composition of strictly less than k morphisms in the $m-1$-th vertical category $V_{m-1}^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$. Let M_1 and N_1 be the domain and codomain of (f, Φ_1) and let M_2 and N_2 be the domain and codomain of (f, Ψ_1). By induction hypothesis M_1 and M_2 are both globularily diffeomorphic to the horizontal identity i_X of X and both N_1 and N_2 are globularily diffeomorphic to the horizontal identity i_Y of Y. It follows that $M_2 * M_1$ is globularily diffeomorphic to the horizontal identity i_X of X and that $N_2 * N_1$ is globularily diffeomorphic to the horizontal identity i_Y of Y. Finally, by the exchange property in $\mathbf{Cob}(n)$ we conclude that M and N are globularily diffeomorphic to horizontal identities i_X and i_Y of X and Y respectively.

Suppose now that (f, Φ) is a general element of the m-th vertical category $V_m^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$. In that case write (f, Φ) as a vertical composition

$$(f, \Phi) = (f_k, \Phi_k) \circ ... \circ (f_1, \Phi_1)$$

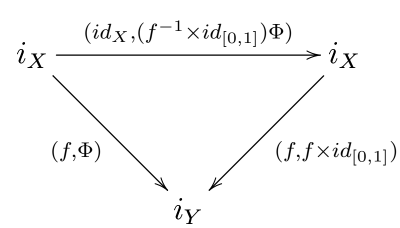
where (f_i, Φ_i) is an element of $H_m^{\mathbf{Cob}(n)}$ for every i. Moreover, assume that the length k of this decomposition is minimal. We again proceed by induction on k. If $k = 1$ then (f, Φ) is an element of $H_m^{\mathbf{Cob}(n)}$. Suppose now that k is strictly greater than 1 and that the existence of a non-globular globularily generated diffeomorphism in the m-th vertical category $V_m^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$, between manifolds X and Y, that can be written as a vertical composition of strictly less than k diffeomorphisms in $H_m^{\mathbf{Cob}(n)}$ implies the conclusion of the lemma for X and Y. Write (g, Ψ) for composition $(f_k, \Phi_k) \circ ... \circ (f_2, \Phi_2)$. In that case (f, Φ) is equal to vertical composition $(g, \Psi) \circ (f_1, \Phi_1)$. Moreover, from the assumption that (f, Φ) is not globular it follows that one of (g, Ψ) or (f_1, Φ_1) is non-globular. Assume first that (g, Ψ) is globular. In that case source and target of (f_1, Φ_1) are both equal to f. By induction hypothesis the domain and codomain of (f, Φ_1) are globularily diffeomorphic to the horizontal identity i_X of X and the horizontal identity i_Y of Y respectively. The domain of (f, Φ) is equal to the codomain of (f, Φ_1) and (g, Ψ) defines a globular diffeomorphism between the domain of (f, Φ) and the codomain of (f_1, Φ_1). We conclude that in this case, the existence of non-globular globularily generated diffeomorphism (f, Φ) implies the existence of a globular diffeomorphism between M and the horizontal identity i_X of X and between N and the horizontal identity i_Y of Y. The case in which it is assumed that (f_1, Φ_1) is globular is handled analogously. Suppose now that neither (g, Ψ) nor (f_1, Φ_1) are globular. In that case, the induction hypothesis implies that there exists a globular diffeomorphism between M, which is the domain of (f_1, Φ_1), and the horizontal identity i_X of X and that there exists a globular diffeomorphism between N, which is the codomain of (g, Ψ), and the horizontal identity i_Y of Y. This concludes the proof. ∎

As a consequence of lemma 6.1, in order to compute the globularily generated piece $\gamma\mathbf{Cob}(n)$ of double category $\mathbf{Cob}(n)$ it is enough to compute the collection of non-globular globularily generated diffeomorphisms between horizontal endomorphisms globularily diffeomorphic to horizontal identities of closed n-dimensional manifolds. This is achieved in the following proposition.

6.2. Proposition. *Let n be a positive integer. Let X and Y be closed n-dimensional manifolds. Let M be a cobordism from X to X and let N be a cobordism from Y to Y. Suppose that M is globularily diffeomorphic to the identity cobordism i_X associated to X and that N is globularily diffeomorphic to the identity cobordism i_Y associated to Y. In that case every horizontal 2-endomorphism from M to N, in double category $\boldsymbol{Cob}(n)$, is globularily generated and has vertical length equal to 1.*

Proof. Let n be a positive integer. Let X and Y be closed n-dimensional manifolds. Let M be a cobordism from X to X, globularily diffeomorphic to the horizontal identity i_X associated to manifold X and let N be a cobordism from Y to Y, globularily diffeomorphic to the horizontal identity i_Y associated to Y. We wish to prove, in this case, that every horizontal 2-endomorphism, in double category $\mathbf{Cob}(n)$, from M to N, is globularily generated and has vertical length equal to 1.

We first prove the proposition for the case in which M and N are equal to the horizontal identity cobordisms i_X and i_Y respectively. Let $(f, \Phi) : i_X \longrightarrow i_Y$ be a horizontal 2-endomorphism, in $\mathbf{Cob}(n)$, from i_X to i_Y. In that case the equivariant morphism $(id_X, (f^{-1} \times id_{[0,1]})\Phi)$ is a globular endomorphism of the horizontal identity i_X of X making the following triangle



commute. Since $(id_X, (f^{-1} \times id_{[0,1]})\Phi)$ is globular and $(f, f \times id_{[0,1]})$ is the horizontal identity i_f of diffeomorphism f of X, we conclude that (f, Φ) is globularily generated and that its vertical length is equal to 1.

Suppose now that M is a general cobordism from X to X globularly diffeomorphic to the horizontal identity i_X of X and that N is a general cobordism from Y to Y, globularly diffeomorphic to the horizontal identity i_Y of Y. Let $(f,\Phi) : M \longrightarrow N$ be a general 2-morphism, in $\mathbf{Cob}(n)$, from M to N. Let $(id_X,\varphi) : M \longrightarrow i_X$ be a globular diffeomorphism from M to the horizontal identity i_X of X and let $(id_Y,\phi) : N \longrightarrow i_Y$ be a globular diffeomorphism from N to the horizontal identity i_Y of Y. In that case composition $(f,\Psi) = (id_Y,\phi)(f,\Phi)(id_X,\varphi^{-1})$ is a 2-morphism from the horizontal identity i_X of X to the horizontal identity i_Y of Y and is thus a morphism in the first vertical category $V_1^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$. We conclude that $(f,\Phi) = (id_Y,\phi^{-1})(f,\Psi)(id_X,\varphi)$ is also a morphism in the first vertical category $V_1^{\mathbf{Cob}(n)}$ associated to $\mathbf{Cob}(n)$. This concludes the proof. ∎

By lemma 4.4 and lemma 6.1, proposition 6.2 provides an explicit description of the globularly generated piece of double categories of the form $\mathbf{Cob}(n)$. We now compute, using a procedure analogous to the one used to compute the globularly generated piece $\gamma\mathbf{Cob}(n)$ of double categories of the form $\mathbf{Cob}(n)$, the globularly generated piece $\gamma\mathbf{Alg}$ of double category $\mathbf{Alg}$ of complex algebras, unital algebra morphisms, bimodules, and equivariant bimodule morphisms. We make the same considerations regarding horizontal equivariant endomorphisms in $\mathbf{Alg}$ as we did with horizontal equivariant endomorphisms in $\mathbf{Cob}(n)$.

Given algebras A and B, a left-right A-bimodule M and a left-right B-bimodule N, we say that an equivariant morphism $(f,\varphi) : M \longrightarrow N$ from M to N, is 2-subcyclic if there exists a cyclic A-submodule L (we understand for cyclic A-module a quotient of A considered as an A-bimodule through left and right multiplication) of N, considering N as an A-bimodule via f, and a cyclic B-submodule K of N such that inclusions $\mathrm{Im}\varphi \subseteq L \subseteq K$ hold. Horizontal identities of algebra morphisms are examples of 2-subcyclic equivariant morphisms. Pair (i,i^2) formed by inclusion of $\mathbb{Z}$ in $\mathbb{Q}$ and inclusion of $\mathbb{Z}^2$ in $\mathbb{Q}^2$ is an example of an non-2-subcyclic equivariant morphism from $\mathbb{Z}$-bimodule $\mathbb{Z}^2$ to $\mathbb{Q}$-bimodule $\mathbb{Q}^2$. To see that this is the case it is enough to observe that pair L,K as above would, in this particular case, yield an inclusion of abelian groups $L \leq K$ where L is a rank 2 free abelian group and where K is a $\mathbb{Q}$ vector space of dimension 1 and thus a rank 1 free abelian group.

In order to explicitly compute the globularly generated piece $\gamma\mathbf{Alg}$ of double category $\mathbf{Alg}$, by proposition 4.4, we again need only to compute collection of non-globular, globularly generated 2-morphisms between horizontal endomorphisms in $\mathbf{Alg}$. We begin with the following lemma.

6.3. Lemma. *Let A and B be algebras. Let M and M' be left-right A-bimodules and let N and N' be left-right B-bimodules. Let $(f,\varphi) : M \longrightarrow N$ be an equivariant morphism from M to N and let $(f,\varphi') : M' \longrightarrow N'$ be equivariant morphism from M' to N'. If both (f,φ) and (f,φ') are 2-subcyclic then relative tensor product $(f,\varphi \otimes_f \varphi')$ is 2-subcyclic.*

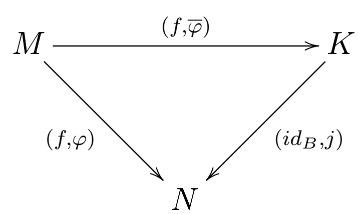
Proof. Let A and B be algebras. Let M and M' be left-right A-bimodules and let N and N' be left-right B-bimodules. Let $(f,\varphi) : M \longrightarrow N$ and $(f,\varphi') : M' \longrightarrow N'$ be equivariant morphisms from M to N and from M' to N' respectively. Suppose both (f, φ) and (f, φ') are 2-subcyclic. We wish to prove in this case that the relative tensor product $(f, \varphi \otimes_f \varphi')$ is 2-subcyclic.

Let L, L' and K, K' be bimodules such that L and L' are A-cyclic submodules of N and N' respectively and such that K and K' are B-cyclic submodules of N and N' respectively. Moreover, let L, L' and K, K' satisfy inclusions $\mathrm{Im}\varphi \subseteq L \subseteq K$ and $\mathrm{Im}\varphi' \subseteq L' \subseteq K'$ respectively. The relative tensor product $L \otimes_A L'$ is an A-cyclic submodule of $N \otimes_A N'$, the relative tensor product $K \otimes_B K'$ is a B-cyclic submodule of $N \otimes_B N'$, $L \otimes_A L'$ is contained in $K \otimes_{B'} K'$, and finally $\mathrm{Im}\varphi \otimes_f \varphi'$ is contained in $L \otimes_A L'$. This concludes the proof. ∎

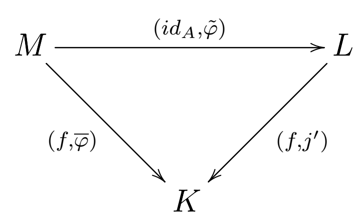
6.4. Proposition. *Let A and B be algebras. Let M be left-right A-bimodule and let N be a left-right B-bimodule. In that case collection of non-globular globularily generated equivariant morphisms from M to N is precisely the collection of non-globular 2-subcyclic equivariant 2-endomorphisms from M to N. Moreover, every globularily generated equivariant morphism from M to N has vertical length equal to 1.*

Proof. Let A and B be algebras. Let M be a left-right A-bimodule and let N be a left-right B-bimodule. We wish to prove that collection of non-globular globularily generated equivariant morphisms from M to N is precisely the collection of non-globular 2-subcyclic equivariant morphisms from M to N. Moreover, we wish to prove that every globularily generated equivariant morphism from M to N has vertical length equal to 1.

We first prove that every non-globular 2-subcyclic equivariant morphism from M to N is globularily generated. Let $(f, \varphi) : M \longrightarrow N$ be non-globular and 2-subcyclic. Let K be a B-cyclic submodule of N and let L be an A-cyclic submodule of K such that inclusions $\mathrm{Im}\varphi \subseteq L \subseteq K$ hold. Let j denote the inclusion of K in N. Let $\overline{\varphi}$ denote the codomain restriction of φ to K. Thus defined j is a globular and equivariant morphism $(f, \overline{\varphi})$ makes the following triangle:



commute. Denote now by j' the inclusion of L in K and denote by $\tilde{\varphi}$ the codomain restriction of φ to L. Considering K as an A-bimodule via f makes j' into a globular 2-morphism. Equivariant morphism $(f, \tilde{\varphi})$ makes the following triangle



commute. The following square:

$$\begin{array}{ccc} M & \xrightarrow{(f,\varphi)} & N \\ {\scriptstyle (id_A,\tilde{\varphi})}\downarrow & & \downarrow{\scriptstyle (id_B,j)} \\ L & \xrightarrow[(f,j')]{} & K \end{array}$$

is thus commutative. Commutativity of this square is clearly equivalent to commutativity of square:

$$\begin{array}{ccc} M & \xrightarrow{(f,\varphi)} & N \\ {\scriptstyle (f,\tilde{\varphi})}\downarrow & & \downarrow{\scriptstyle (id_B,j)} \\ L & \xrightarrow[(id_A,j')]{} & K \end{array}$$

Left and right hand sides of this last square are Globular. Finally, triangle:

$$\begin{array}{ccccc} L & & \xrightarrow{(id_B,j')} & & K \\ & {\scriptstyle (f,j')}\searrow & & \swarrow{\scriptstyle (f,id_K)} & \\ & & K & & \end{array}$$

commutes, which proves that equivariant morphism (f,j') is a morphism in the first vertical category $V_1^{\mathbf{Alg}}$ associated to **Alg**. We conclude that the 2-subcyclic equivariant morphism (f,φ) is globularily generated and has vertical length equal to 1.

We now prove that every non-globular globularily generated equivariant morphism from M to N is 2-subcyclic. Let $(f,\varphi): M \longrightarrow N$ be non-globular and globularily generated. Assume first that (f,φ) is an element of $H_1^{\mathbf{Alg}}$. From the assumption that (f,φ) is non-globular it follows that (f,φ) is the horizontal identity of an algebra morphism and thus is 2-subcyclic. Suppose now that (f,φ) is a general morphism in the first vertical category $V_1^{\mathbf{Alg}}$ associated to **Alg**. We wish to find, in this case, an A-cyclic submodule L of N and a B-cyclic submodule K of N such that inclusions $\mathrm{Im}\varphi \subseteq L \subseteq K$ hold. Write (f,φ) as a vertical composition of the form:

$$(id_B, \psi_{k+1}) \circ (f_k, \phi_k) \circ ... \circ (f_1, \phi_1) \circ (id_A, \psi_1)$$

as in lemma 4.6 where f_i is an algebra morphism and ϕ_i is a horizontal identity for every $i \leq k$. Write (f,Φ) for composition $(f_k,\phi_k) \circ ... \circ (f_1,\phi_1)$. Thus defined (f,Φ) is an equivariant morphism from left-right A-bimodule ${}_AA_A$ to left-right B-bimodule ${}_BB_B$.

Write K for the image $\mathrm{Im}\psi_{k+1}$ of ψ_{k+1}. The domain of ψ_{k+1} equals the codomain of Φ, which is equal to bimodule ${}_BB_B$. It follows that thus defined K is a cyclic left-right B-bimodule. Now make L to be equal to the image $\mathrm{Im}\psi_{k+1}\Phi$ of composition $\psi_{k+1}\Phi$. The domain of composition $\psi_{k+1}\Phi$ equals bimodule ${}_AA_A$. It now follows that thus defined L is a cyclic left-right A-module. Clearly L is contained in K. Moreover, the image $\mathrm{Im}\varphi$ of φ is contained in L. Pair L, K thus satisfies the conditions required to realize (f,φ) as a 2-subcyclic equivariant 2-morphism. We conclude that every equivariant morphism in the first vertical category $V_1^{\mathbf{Alg}}$ associated to **Alg** is 2-subcyclic. From this and from lemma 6.3 it follows that every globularily generated equivariant morphism between M and N is 2-subcyclic. The fact that every non-globular globularily generated 2-morphism in **Alg** has vertical length equal to 1 follows from this and from the first part of the proof. This concludes the proof. ∎

Proposition 6.4 provides an explicit description of the globularily generated piece $\gamma\mathbf{Alg}$ of double category **Alg**. A similar computation provides a complete description of the globularily generated piece $\gamma[W^*]^f$ of double category $[W^*]^f$ of semisimple von Neumann algebras, finite algebra morphisms, bimodules, and equivariant bimodule morphisms. Examples of globularily generated double categories having 2-morphisms of vertical length strictly greater than 1 will be studied in subsequent papers.

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