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PRODUCTS AND COEQUALIZERS IN POINTED CATEGORIES

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Abstract. In this paper, we investigate the property (P) that binary products commute with arbitrary coequalizers in pointed categories. Examples of such categories include any regular unital or (pointed) majority category with coequalizers, as well as any pointed factor permutable category with coequalizers. We establish a Mal'tsev term condition characterizing pointed varieties of universal algebras satisfying (P). We then consider categories satisfying (P) locally, i.e., those categories for which every fibre $\mathrm{Pt}_{\mathbb{C}}(X)$ of the fibration of points $\pi : \mathrm{Pt}(\mathbb{C}) \to \mathbb{C}$ satisfies (P). Examples include any regular Mal'tsev or majority category with coequalizers, as well as any regular Gumm category with coequalizers. Varieties satisfying (P) locally are also characterized by a Mal'tsev term condition, which turns out to be equivalent to a variant of Gumm's shifting lemma. Furthermore, we show that the varieties satisfying (P) locally are precisely the varieties with normal local projections in the sense of Z. Janelidze.

1. Introduction

Let $\mathbb{C}$ be a category with binary products. Recall that products are said to commute with coequalizers in $\mathbb{C}$ when $\mathbb{C}$ satisfies the following property:

(P) For any two coequalizer diagrams

$$C_1 \overset{u_1}{\underset{v_1}{\rightrightarrows}} X_1 \xrightarrow{q_1} Q_1 \qquad C_2 \overset{u_2}{\underset{v_2}{\rightrightarrows}} X_2 \xrightarrow{q_2} Q_2,$$

in $\mathbb{C}$, the diagram

$$C_1 \times C_2 \overset{u_1\times u_2}{\underset{v_1\times v_2}{\rightrightarrows}} X_1 \times X_2 \xrightarrow{q_1\times q_2} Q_1 \times Q_2,$$

is a coequalizer diagram in $\mathbb{C}$.

For example, the dual categories $\mathbf{Set}^{\mathrm{op}}, \mathbf{Top}^{\mathrm{op}}, \mathbf{Ord}^{\mathrm{op}}$ of sets, topological spaces and ordered sets, as well as the dual of any topos, satisfy (P). More generally, any coextensive category [4] with coequalizers satisfies (P) (see Proposition 2.7 below). The dual category $\mathbb{E}_*^{\mathrm{op}}$ of pointed objects in any topos $\mathbb{E}$, or $\mathbf{Top}_*^{\mathrm{op}}$ the dual category of pointed topological spaces, are all pointed regular majority categories [10, 12], and therefore satisfy (P)(see

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Proposition 2.11 below). Given any category $\mathbb{C}$ with binary products which satisfies (P), the coslice category $(X \downarrow \mathbb{C})$ also satisfies (P) (since limits in $(X \downarrow \mathbb{C})$ are computed as in $\mathbb{C}$). If (P) is restricted to reflexive coequalizers, then (P) holds in any variety of algebras [1], as well any Cartesian closed category [17]. Also, if (P) is restricted to split coequalizers, then (P) holds in any category with binary products. Suppose that $\mathbb{C}$ is a variety of algebras satisfying (P). If the theory of $\mathbb{C}$ admits no constants, then the empty set would be the initial object 0 in $\mathbb{C}$, and applying (P) to the two diagrams

$$X \times X \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} X \longrightarrow 1 \qquad 0 \underset{0}{\overset{0}{\rightrightarrows}} X \xrightarrow{1_X} X$$

would imply that

$$0 \underset{0}{\overset{0}{\rightrightarrows}} X \times X \xrightarrow{\pi_2} X$$

is a coequalizer for any non-empty algebra X. But this would force $\mathbb{C}$ to be a preorder, since every algebra could have at most one element. Conversely, any preorder with binary products satisfies (P) trivially. Therefore, if $\mathbb{C}$ is a non-trivial variety which satisfies (P), then $\mathbb{C}$ must possess constants. In this paper, we are concerned with varieties which possess a unique constant, i.e., *pointed varieties.* We show that a pointed variety satisfies (P) if and only if there exists integers $0 \leqslant m$ and $1 \leqslant n$ such that its theory admits binary terms $b_i(x, y)$ and unary terms $c_i(x)$ for each $1 \leqslant i \leqslant m$ and $(m+2)$-ary terms $p_1, p_2, ..., p_n$ satisfying the equations:

$$\begin{aligned} p_1(x, y, b_1(x, y), b_2(x, y), ..., b_m(x, y)) &= x, \\ p_i(y, x, b_1(x, y), ..., b_m(x, y)) &= p_{i+1}(x, y, b_1(x, y), ..., b_m(x, y)), \\ p_n(y, x, b_1(x, y), b_2(x, y), ..., b_m(x, y)) &= y, \end{aligned}$$

and for each $i = 1, ..., n$ we have

$$p_i(0, 0, c_1(z), ..., c_m(z)) = z.$$

We then consider varieties which satisfy (P) *locally*, i.e., varieties for which each fibre $\mathrm{Pt}_{\mathbb{C}}(X)$ of the *fibration of points* $\pi : \mathrm{Pt}(\mathbb{C}) \to \mathbb{C}$ satisfies (P). Every pointed variety $\mathbb{C}$ satisfying (P) necessarily has *normal projections* in the sense of [13], i.e., every product projection in $\mathbb{C}$ is a cokernel, but the converse is not true. However, it turns out that a variety satisfies (P) locally if and only if it has *normal local projections* [14] (see Theorem 3.7). Furthermore, we show how both (P) and its local version may be seen as variants of Gumm's shifting lemma [9], so that in particular any congruence modular variety of algebras satisfies (P) locally.

2. Main Results

Recall from [13] that a pointed category $\mathbb{C}$ has normal projections if any product projection in $\mathbb{C}$ is a normal epimorphism. For example, every *subtractive* category [15], as well as any *unital* category [2], have normal projections.

2.1. PROPOSITION. *If $\mathbb{C}$ is a pointed category with binary products which satisfies (P), then the product of two normal epimorphisms is normal, and in particular $\mathbb{C}$ has normal projections.*

PROOF. The product of two normal epimorphisms being normal is a trivial consequence of (P), and any product projection $\pi_1 : X \times Y \to X$ may be obtained as the product of $X \xrightarrow{1_X} X$ and $Y \to 0$, which are normal epimorphisms. ■

The lemma below has a straightforward proof, and therefore we leave the proof of it to the reader.

2.2. LEMMA. *In a category, given morphisms*

$$\begin{tikzcd}
\bullet \arrow[dr, "i_1"] & & & & \\
& \bullet \arrow[r, shift left, "v"] \arrow[r, shift right, "u"'] & \bullet \arrow[r, "e_1"] & \bullet \arrow[r, "e_2"] & \bullet \\
\bullet \arrow[ur, "i_2"'] & & & &
\end{tikzcd}$$

such that $e_2 \circ e_1 \circ u = e_2 \circ e_1 \circ v$, where e_1 is a coequalizer of $u \circ i_1, v \circ i_1$ and e_2 is a coequalizer of $e_1 \circ u \circ i_2, e_1 \circ v \circ i_2$, then $e_2 \circ e_1$ is a coequalizer of v, u.

In what follows, the 0 symbol is used to denote an initial object in a category $\mathbb{C}$ (if it exists), and 0_X denotes the unique morphism $0_X : 0 \to X$.

2.3. LEMMA. *The following are equivalent for a category with binary products and an initial object.*

1. *$\mathbb{C}$ satisfies (P), i.e., the product of two coequalizers is a coequalizer.*
2. *The product of any coequalizer diagram $C \rightrightarrows X \to Y$ with the trivial coequalizer*

$$\begin{tikzcd} 0 \arrow[r, shift left, "0_Z"] \arrow[r, shift right, "0_Z"'] & Z \arrow[r, "1_Z"] & Z, \end{tikzcd}$$

is a coequalizer diagram for any object Z in $\mathbb{C}$.

PROOF. We show that (2) implies (1). Suppose that

$$\begin{tikzcd} C_1 \arrow[r, shift left, "u_1"] \arrow[r, shift right, "v_1"'] & X_1 \arrow[r, "q_1"] & Q_1 \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[r, shift left, "u_2"] \arrow[r, shift right, "v_2"'] & X_2 \arrow[r, "q_2"] & Q_2 \end{tikzcd}$$

are two coequalizer diagrams in $\mathbb{C}$. Then applying (2), we have that the diagrams

$$C_1 \times 0 \underset{v_1 \times 0_{X_2}}{\overset{u_1 \times 0_{X_2}}{\rightrightarrows}} X_1 \times X_2 \xrightarrow{q_1 \times 1_{X_2}} Q_1 \times X_2 \qquad 0 \times C_2 \underset{0_{Q_1} \times v_2}{\overset{0_{Q_1} \times u_2}{\rightrightarrows}} Q_1 \times X_2 \xrightarrow{1_{Q_1} \times q_2} Q_1 \times Q_2,$$

are coequalizer diagrams. We may then apply Lemma 2.2 to the diagram

$$\begin{tikzcd}
C_1 \times 0 \arrow[dr, "1_{C_1} \times 0_{C_2}"] & & & & \\
& C_1 \times C_2 \arrow[r, shift left, "u_1 \times u_2"] \arrow[r, shift right, "v_1 \times v_2"'] & X_1 \times X_2 \arrow[r, "q_1 \times 1_{X_2}"] & Q_1 \times X_2 \arrow[r, "1_{Q_1} \times q_2"] & Q_1 \times Q_2 \\
0 \times C_2 \arrow[ur, "0_{C_1} \times 1_{C_2}"'] & & & &
\end{tikzcd}$$

to obtain $q_1 \times 1_{X_2} \circ 1_{Q_1} \times q_2 = q_1 \times q_2$ as a coequalizer of $u_1 \times u_2$ and $v_1 \times v_2$. ∎

In what follows, we will be working with pointed categories. To simplify notation, we will always denote zero-morphisms between objects by 0, when there is no ambiguity.

The proof of following proposition is straightforward, and is left to the reader.

2.4. Proposition. *Given any reasonably commutative diagram*

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "u_1"] \arrow[r, shift right, "v_1"'] \arrow[d, "e"'] & X_1 \arrow[r, "q_1"] \arrow[d, "f"] & Q_1 \arrow[d, "g"] \\
C_2 \arrow[r, shift left, "u_2"] \arrow[r, shift right, "v_2"'] & X_2 \arrow[r, "q_2"'] & Q_2
\end{tikzcd}$$

in any category, where the top row is a coequalizer diagram and e is an epimorphism, the right-hand square is a pushout if and only if the bottom row is a coequalizer. In particular, any category with coequalizers admits pushouts along regular epimorphisms (take $e = 1_{C_1}$).

2.5. Corollary. *Let $\mathbb{C}$ be any category with coequalizers. For any two parallel pairs of morphisms $u_1, v_1 : C_1 \to X$ and $u_2, v_2 : C_2 \to X$ which have a common coequalizer, the pairs $f \circ u_1, f \circ v_1$ and $f \circ u_2, f \circ v_2$ have a common coequalizer, for any morphism $f : X \to Y$.*

Proof. A common coequalizer for u_1, v_1 and u_2, v_2 can be pushed out along f, by Proposition 2.4 to obtain a common coequalizer for $f \circ u_1, f \circ v_1$ and $f \circ u_2, f \circ v_2$. ∎

Recall that a category $\mathbb{C}$ with binary products is called *coextensive* if for any two objects X, Y in $\mathbb{C}$, the canonical 'product' functor $(X \downarrow \mathbb{C}) \times (Y \downarrow \mathbb{C}) \to (X \times Y \downarrow \mathbb{C})$ is an equivalence of categories. Equivalently, we may define a coextensive category according to the following theorem.

2.6. Theorem. [4] *A category with binary products is coextensive if and only if it has pushouts along product projections and in every commutative diagram*

$$\begin{tikzcd} A_1 \arrow[d, "f_1"'] & A_1 \times A_2 \arrow[l, "\pi_2"'] \arrow[r, "\pi_1"] \arrow[d, "f"'] & A_2 \arrow[d, "f_2"] \\ X_1 & X \arrow[l, "x_2"] \arrow[r, "x_1"'] & X_2 \end{tikzcd}$$

the bottom row is a product diagram if and only if the two squares are pushouts.

Note, that in every coextensive category, the product projections are automatically epimorphisms (see Proposition 2.6 in [4]). The following proposition is likely to be folklore, however, we could not find it in the literature, and we include it for completeness.

2.7. Proposition. *For any coextensive category $\mathbb{C}$ with coequalizers, the property (P) holds.*

Proof. Let $q_1 : X_1 \to Q_1$ be a coequalizer for $u_1, v_1 : C_1 \rightrightarrows X_1$, and $q_2 : X_2 \to Q_2$ a coequalizer for $u_2, v_2 : C_2 \rightrightarrows X_2$. Let $q : X_1 \times X_2 \to Q$ be a coequalizer for $u_1 \times u_2$ and $v_1 \times v_2$. Then we may apply Proposition 2.4 to the diagram

$$\begin{tikzcd} C_1 \arrow[d, shift right, "u_1"'] \arrow[d, shift left, "v_1"] & C_1 \times C_2 \arrow[l, two heads] \arrow[r, two heads] \arrow[d, shift right, "u_1 \times u_2"'] \arrow[d, shift left, "v_1 \times v_2"] & C_2 \arrow[d, shift right, "u_2"'] \arrow[d, shift left, "v_2"] \\ X_1 \arrow[d, "q_1"'] & X_1 \times X_2 \arrow[l] \arrow[r] \arrow[d, "q"'] & X_2 \arrow[d, "q_2"] \\ Q_1 & Q \arrow[l] \arrow[r] & Q_2 \end{tikzcd}$$

so that the bottom squares are pushouts, and hence the bottom row is a product by coextensivity. ∎

2.8. Remark. The notion of an $\mathcal{M}$-coextensive object in a category $\mathbb{C}$, where $\mathcal{M}$ is a distinguished class of morphisms from $\mathbb{C}$, has been recently introduced and studied in [12]. Let $\mathbb{C}$ be a category with binary products and coequalizers, and let $\mathcal{M} = \mathrm{Reg}(\mathbb{C})$ be the class of regular epimorphisms in $\mathbb{C}$. If every object in $\mathbb{C}$ is $\mathcal{M}$-coextensive, then the same argument above may be used to show that $\mathbb{C}$ satisfies (P). For example, every object in the category **Ring** of unitary rings, or the category **VonReg** of von Neumann regular rings, is $\mathcal{M}$-coextensive with $\mathcal{M} = \mathrm{Reg}(\mathbb{C})$. Moreover, every congruence distributive variety $\mathcal{V}$ of algebras which admits at least one constant, has every algebra $\mathcal{M}$-coextensive with $\mathcal{M} = \mathrm{Reg}(\mathbb{C})$.

2.9. Proposition. *Let $\mathbb{C}$ be a pointed category with finite limits and coequalizers, then the following are equivalent.*

1. *$\mathbb{C}$ satisfies (P).*

2. *For any regular epimorphism $q : X \to Y$ and any object Z in $\mathbb{C}$, the diagram*

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "{(1_X,0)}"'] & Y \arrow[d, "{(1_Y,0)}"] \\ X \times Z \arrow[r, "q\times 1_Z"'] & Y \times Z \end{tikzcd}$$

is a pushout.

3. *The product of a regular epimorphism in $\mathbb{C}$ with any identity morphism in $\mathbb{C}$ is a regular epimorphism, and for any effective equivalence relation θ on any product $X \times Z$ in $\mathbb{C}$, we have that $(x, 0)\theta(y, 0)$ implies $(x, z)\theta(y, z)$ for any generalized elements $x, y : C \to X$ and $z : C \to Z$. This property is illustrated by the diagram below, which can be seen as a variant of the "egg-box property" in the sense of [7].*

$$\begin{tikzcd} (x,0) \arrow[r, dash, "\mathrm{Eq}(\pi_2)"] \arrow[r, dash, bend left, "\theta"] \arrow[d, dash, "\mathrm{Eq}(\pi_1)"'] & (y,0) \arrow[d, dash, "\mathrm{Eq}(\pi_1)"] \\ (x,z) \arrow[r, dash, "\mathrm{Eq}(\pi_2)"'] \arrow[r, dash, dotted, bend right, "\theta"'] & (y,z) \end{tikzcd}$$

Proof. For (1) $\implies$ (2), let $q : X \to Y$ be a coequalizer of a parallel pair $x, y : C \to X$. By Lemma 2.3, the morphism $q \times 1_Z$ is a coequalizer for $(x, 0), (y, 0) : C \to X \times Z$. Applying Proposition 2.4, to the diagram

$$\begin{tikzcd} C \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] \arrow[d, "1_C"'] & X \arrow[r, "q"] \arrow[d, "{(1_X,0)}"] & Y \arrow[d, "{(1_Y,0)}"] \\ C \arrow[r, shift left, "{(x,0)}"] \arrow[r, shift right, "{(y,0)}"'] & X \times Z \arrow[r, "q\times 1_Z"'] & Y \times Z \end{tikzcd}$$

implies that the right-hand square is a pushout.

For (2) $\implies$ (3), let θ be any effective equivalence relation on a product $X \times Z$ in $\mathbb{C}$, and let $x, y : C \to X$ and $z : C \to Z$ be any morphisms such that $(x, 0)\theta(y, 0)$. Let $q : X \to Y$ be a coequalizer of x and y. Then the right-hand square in the above diagram being a pushout implies that $q \times 1_Z$ is a coequalizer of $(x, 0)$ and $(y, 0)$, by Proposition 2.4. Since θ is effective and $(x, 0)\theta(y, 0)$, it follows that $\mathrm{Eq}(q \times 1_Z) \leqslant \theta$ and hence $(x, z)\theta(y, z)$. Note that the product of a regular epimorphism in $\mathbb{C}$ with an identity morphism in $\mathbb{C}$ being regular is an immediate consequence of (2).

For (3) $\implies$ (1), suppose that we are given a coequalizer $q : X \to Y$ of $x, y : C \to X$. Let $k_1, k_2 : \mathrm{Eq}(q) \to X$ be the projections associated to the kernel equivalence $\mathrm{Eq}(q)$. Note that since x, y and k_1, k_2 have a common coequalizer, it follows that $(k_1, 0)$ and

$(k_2, 0)$ and $(u, 0), (v, 0)$ have a common coequalizer $q' : X \times Z \to Q'$ by Corollary 2.5. It suffices to show that $\mathrm{Eq}(q \times 1_Z) = \mathrm{Eq}(q')$, since $q \times 1_Z$ is a regular epimorphism by assumption. Note that we always have $\mathrm{Eq}(q') \leqslant \mathrm{Eq}(q \times 1_Z)$. Given any generalized elements $u, v : S \to X$ and $w : S \to Z$, if $(u, w)\mathrm{Eq}(q \times 1_Z)(v, w)$ then $u\mathrm{Eq}(q)v$. Since q' is a coequalizer of $(k_1, 0), (k_2, 0)$, and since $u\mathrm{Eq}(q)v$, it follows that $(u, 0)\mathrm{Eq}(q')(v, 0)$ and hence we may apply (3) to get $(u, w)\mathrm{Eq}(q')(v, w)$, which shows that $\mathrm{Eq}(q \times 1_Z) \leqslant \mathrm{Eq}(q')$. ∎

In what follows we will fix a finitely complete pointed category $\mathbb{C}$ which has coequalizers, and we will assume that regular epimorphisms in $\mathbb{C}$ are stable under binary products. Recall that $\mathbb{C}$ is *unital* if for any binary relation R in $\mathbb{C}$ between any two objects A and B in $\mathbb{C}$, we have $aR0$ and $0Rb$ implies aRb (see [2, 16]).

2.10. Proposition. *If $\mathbb{C}$ is unital, then $\mathbb{C}$ satisfies (P).*

Proof. We show that $\mathbb{C}$ fulfils condition (3) of Proposition 2.9. Let θ be any effective equivalence relation on any product $X \times Z$ in $\mathbb{C}$ such that $(x, 0)\theta(y, 0)$. Then we may consider the binary relation R between $A \times A$ and $B \times B$ defined by $(a, a')R(b, b')$ if and only if $(a, b)\theta(a', b')$. By assumption we have $(x, y)R(0, 0)$ and $(0, 0)R(z, z)$ (by reflexivity of θ) so that $(x, y)R(z, z)$ and hence $(x, z)\theta(y, z)$. ∎

Recall that $\mathbb{C}$ is a *majority category* [10, 12] if for any ternary relation R between objects X, Y, Z we have

$$(x, y, z') \in R \quad \text{and} \quad (x, y', z) \in R \quad \text{and} \quad (x', y, z) \in R \quad \implies (x, y, z) \in R, \qquad (*)$$

for any generalized elements $x, x' : S \to X$ and $y, y' : S \to Y$ and $z, z' : S \to Z$ in $\mathbb{C}$.

2.11. Proposition. *If $\mathbb{C}$ is a majority category, then $\mathbb{C}$ satisfies (P).*

Proof. Let θ be any effective equivalence relation on $X \times Z$ and suppose that x, y, z are as in (3) of Proposition 2.9. Then we consider the ternary relation R between A, B and $A \times B$ defined by $(a, b, (a', b')) \in R$ if and only if $(a, a')\theta(b, b')$. Then by assumption we have $(x, y, (0, 0)) \in R$, and by reflexivity we have $(x, x, (z, z)) \in R$ and $(y, y, (z, z)) \in R$. Applying the majority property $(*)$ above to these three elements yields $(x, y, (z, z)) \in R$, so that $(x, z)\theta(y, z)$. ∎

The notion of a *Gumm* category [3] is the categorical analogue of varieties in which Gumm's shifting lemma holds [9], i.e., congruence modular varieties. A finitely complete category $\mathbb{C}$ is a Gumm category if for any three equivalence relations R, S, T on any object X in $\mathbb{C}$ such that $R \cap S \leqslant T$, if $(x, y), (w, z) \in R$ and $(y, z), (x, w) \in S$ and $(y, z) \in T$ then we get $(x, w) \in T$. This implication of relations between the elements above is usually depicted with a diagram

$$\begin{tikzcd} y \arrow[r, dash, "S"] \arrow[r, dash, bend left=60, "T"] \arrow[d, dash, "R"'] & z \arrow[d, dash, "R"] \\ x \arrow[r, dash, "S"'] \arrow[r, dash, dotted, bend right=60, "T"'] & w \end{tikzcd}$$

where the dotted curve represents the relation induced from the relations indicated by the solid curves.

2.12. Proposition. *If $\mathbb{C}$ is a Gumm category, then $\mathbb{C}$ satisfies (P).*

Proof. The diagrammatic condition characterizing (P) in (3) of Proposition 2.9 is a restriction of the shifting lemma, where $R = \mathrm{Eq}(\pi_1)$, $S = \mathrm{Eq}(\pi_2)$ and $T = \theta$. Since we always have $\mathrm{Eq}(\pi_1) \cap \mathrm{Eq}(\pi_2) = \Delta_X$ for any two complementary product projections π_1 and π_2 of an object X in $\mathbb{C}$, we may apply the shifting lemma to the diagram in (3) of Proposition 2.9. ■

Recall that an equivalence relation F on an object X is called a *factor relation* on X if there exists a product projection $p : X \to A$ of X such that $F = \mathrm{Eq}(p)$.

2.13. Definition. [8] *A regular category $\mathbb{C}$ is said to be* factor permutable *if for any factor relation F and any equivalence relation E on any object X in $\mathbb{C}$, we have $F \circ E = E \circ F$.*

2.14. Lemma. [Lemma 2.5 in [8]] *In any factor permutable category $\mathbb{C}$ the* weak shifting lemma holds*: for any equivalence relations R and S on $A \times B$ in $\mathbb{C}$ such that $\mathrm{Eq}(\pi_1) \cap R \leqslant S$, if $(a,b), (a,c), (d,e), (d,f)$ are related via the solid arrows as in the diagram*

$$\begin{tikzcd} (a,c) \arrow[r, dash, "R"] \arrow[r, dash, bend left=60, "S"] \arrow[d, dash, "\mathrm{Eq}(\pi_1)"'] & (d,f) \arrow[d, dash, "\mathrm{Eq}(\pi_1)"] \\ (a,b) \arrow[r, dash, "R"'] \arrow[r, dash, dotted, bend right=60, "S"'] & (d,e) \end{tikzcd}$$

then we have $(a,b)S(d,e)$.

2.15. Proposition. *Any pointed regular factor permutable category with coequalizers satisfies (P).*

Proof. Similar to the proof of Proposition 2.12, we may apply the weak shifting property of Lemma 2.14 to the diagram in (2) of Proposition 2.9. ■

The theorem below is a Mal'tsev type characterization of pointed varieties of universal algebras satisfying (P). In the proof, we will use 2×2 matrices to represent elements of a congruence C on a product $A \times B$ in the following way:

$$\begin{pmatrix} a & a' \\ b & b' \end{pmatrix} \in C \iff (a,b)C(a',b').$$

2.16. THEOREM. *A pointed variety $\mathcal{V}$ of algebras satisfies (P) if and only if there exists integers $n \geqslant 1$ and $m \geqslant 0$ such that $\mathcal{V}$ admits binary terms $b_i(x,y)$ and unary terms $c_i(x)$ for each $1 \leqslant i \leqslant m$ and $(m+2)$-ary terms $p_1, p_2, ..., p_n$ satisfying the equations:*

$$\begin{aligned} p_1(x,y,b_1(x,y),b_2(x,y),...,b_m(x,y)) &= x, \\ p_i(y,x,b_1(x,y),...,b_m(x,y)) &= p_{i+1}(x,y,b_1(x,y),...,b_m(x,y)), \\ p_n(y,x,b_1(x,y),b_2(x,y),...,b_m(x,y)) &= y, \end{aligned}$$

and for each $i = 1,...,n$ we have $p_i(0,0,c_1(z),...,c_m(z)) = z$.

PROOF. We recall that $\mathcal{V}$ being a regular category, the product of two regular epimorphisms is automatically a regular epimorphism. Consider the principal congruence C on $F_{\mathcal{V}}(x,y) \times F_{\mathcal{V}}(z)$ generated by the single relation $(x,0)C(y,0)$, where $F_{\mathcal{V}}(x,y)$ and $F_{\mathcal{V}}(z)$ are the free algebras over $\{x,y\}$ and $\{z\}$ respectively. Thus, by (3) of Proposition 2.9 it follows that $(x,z)C(y,z)$. The congruence C may be obtained by closing the relation

$$\left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix} \right\}$$

first under reflexivity, then under all operations in $\mathcal{V}$, and then under transitivity. Therefore, there exists $w_1, ..., w_{n+1} \in F_{\mathcal{V}}(x,y) \times F_{\mathcal{V}}(z)$ such that $w_1 = (x,z)$ and $w_{n+1} = (y,z)$, where:

$$(w_i, w_{i+1}) = p_i(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b_{1,i}(x,y) & b_{1,i}(x,y) \\ c_{1,i}(z) & c_{1,i}(z) \end{pmatrix}, \dots, \begin{pmatrix} b_{m_i,i}(x,y) & b_{m_i,i}(x,y) \\ c_{m_i,i}(z) & c_{m_i,i}(z) \end{pmatrix}).$$

for certain terms $p_1, \dots, p_n$ and elements $(b_{ij}(x,y), c_{ij}(z)) \in F_{\mathcal{V}}(x,y) \times F_{\mathcal{V}}(z)$. Without loss of generality, we may assume that $m_1 = \dots = m_n = m$, $b_{1j} = \dots = b_{nj} = b_j$ and $c_{1j} = \dots = c_{nj} = c_j$ for all $j \in \{1, \dots, m\}$. Then, writing out the identities above coordinate-wise and noting that since $(b_j, c_j) \in F_{\mathcal{V}}(x,y) \times F_{\mathcal{V}}(z)$, each b_j is a binary term $b_j(x,y)$ and each c_j is a unary term $c_j(z)$, we get the identities in the statement of the theorem.

For the converse, suppose that C is any congruence on $X \times Z$ in $\mathcal{V}$, and that $(x,0)C(y,0)$. Consider the elements of C defined by:

$$(w_{1,i}, w_{2,i}) = p_i(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b_1(x,y) & b_1(x,y) \\ c_1(z) & c_1(z) \end{pmatrix}, \dots, \begin{pmatrix} b_m(x,y) & b_m(x,y) \\ c_m(z) & c_m(z) \end{pmatrix}).$$

Then the equations in the statement of the theorem imply that $w_{2,i} = w_{1,i+1}$ as well as $(x, z) = w_{1,1}$ and $(y, z) = z_{2,n}$, so that by the transitivity of C we get $(x, z)C(y, z)$. ■

2.17. Example. If $\mathcal{V}$ is a unital variety of algebras, then the theory of $\mathcal{V}$ admits a Jónsson-Tarski operation $+$, i.e., a binary operation satisfying $x + 0 = x = 0 + x$ where 0 is the unique constant. We may take $n = 1 = m$ in the theorem above, and define $p_1(x, y, z) = x + z$ and $b_1(x, y) = 0$ and $c_1(z) = z$. Then the equations in the statement of the theorem hold with respect to these terms.

2.18. Example. If $\mathcal{V}$ is a pointed variety which admits a majority term $m(x, y, z)$, then we may take $n = 1$ and $m = 2$ where $p_1(x, y, z, w) = m(x, z, w)$ and $b_1(x, y) = x$ and $b_2(x, y) = y$, and $c_1(z) = z = c_2(z)$, then the equations in the theorem above hold with respect to these terms.

3. Normal local projections and products of coequalizers

Recall that the *category of points* $\mathrm{Pt}_{\mathbb{C}}(X)$ of an object X in a category $\mathbb{C}$ consists of triples (A, p, s) where $p : A \to X$ is a split epimorphism and s is a splitting for p. A morphism $f : (A, p, s) \to (B, q, t)$ in $\mathrm{Pt}_{\mathbb{C}}(X)$ is a morphism $f : A \to B$ such that $q \circ f = p$ and $f \circ s = t$. The category $\mathrm{Pt}_{\mathbb{C}}(X)$ is pointed, where the zero-object is $(X, 1_X, 1_X)$. The zero-morphism from (A, p, s) to (B, q, t) is given by $t \circ p$. When $\mathbb{C}$ has coequalizers and pullbacks, then $\mathrm{Pt}_{\mathbb{C}}(X)$ has products and coequalizers. Moreover, if $\mathbb{C}$ is a regular category, then so is $\mathrm{Pt}_{\mathbb{C}}(X)$.

In what follows, we will say that a category $\mathbb{C}$ *satisfies (P) locally* if for every object X in $\mathbb{C}$ the category $\mathrm{Pt}_{\mathbb{C}}(X)$ satisfies (P).

3.1. Example. Every regular Mal'tsev category [6, 5] with coequalizers satisfies (P) locally. This is because a finitely complete category $\mathbb{C}$ is Mal'tsev if and only if for any object X the category $\mathrm{Pt}_{\mathbb{C}}(X)$ is unital (see [2]). Moreover, $\mathbb{C}$ being regular with coequalizers implies that $\mathrm{Pt}_{\mathbb{C}}(X)$ is pointed regular with coequalizers. Then by Proposition 2.10 it follows that $\mathrm{Pt}_{\mathbb{C}}(X)$ satisfies (P), for any object X in $\mathbb{C}$.

3.2. Example. For essentially the same reasons as the above example every regular majority category with coequalizers satisfies (P) locally: if $\mathbb{C}$ is a regular majority category with coequalizers, then so is $\mathrm{Pt}_{\mathbb{C}}(X)$ for any object X, which is moreover pointed (see Example 2.15 in [10]). Hence by Proposition 2.11 we have that $\mathbb{C}$ satisfies (P) locally.

3.3. Example. Every regular Gumm category $\mathbb{C}$ with coequalizers satisfies (P) locally: if $\mathbb{C}$ is a Gumm category with coequalizers, then so is $\mathrm{Pt}_{\mathbb{C}}(X)$ (see Lemma 2.5 in [3] and the discussion proceeding it), and hence by Proposition 2.12 it follows that $\mathrm{Pt}_{\mathbb{C}}(X)$ satisfies (P) for any object X in $\mathbb{C}$.

Recall from the introduction that a category $\mathbb{C}$ is said to have normal local projections if for any object X in $\mathbb{C}$, the category $\mathrm{Pt}_{\mathbb{C}}(X)$ has normal projections.

3.4. THEOREM. [14] *The following are equivalent for a variety $\mathcal{V}$ of universal algebras.*

- *$\mathcal{V}$ has normal local projections.*
- *There exists integers $n \geqslant 1$ and $m \geqslant 0$ such that $\mathcal{V}$ admits binary terms $b_1, ..., b_m$, $c_1, ..., c_m$ and $(m+2)$-ary terms $p_1, p_2, ..., p_n$ satisfying*
  - $p_1(x, y, b_1(x, y), ..., b_m(x, y)) = x$.
  - $p_n(y, x, b_1(x, y), ..., b_m(x, y)) = y$.
  - *For any $i \in \{1, 2, ..., n-1\}$ we have*

$$p_i(y, x, b_1(x, y), ..., b_m(x, y)) = p_{i+1}(x, y, b_1(x, y), ..., b_m(x, y)).$$

  - *For any $i \in \{1, 2, ..., n\}$ we have $p_i(u, u, c_1(u, v), ..., c_m(u, v)) = v$ and $b_i(z, z) = c_i(z, z)$.*

In what follows we will see that a variety of algebras satisfies (P) locally if and only if it has normal local projections.

3.5. CHARACTERIZATION OF VARIETIES SATISFYING (P) LOCALLY. Given two objects (A, p, s) and (B, q, t) in $\mathrm{Pt}_{\mathbb{C}}(X)$ consider the diagram below where the square is a pullback

$$\begin{tikzcd}
X \arrow[dr, "{(s,t)}"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\
& A \times_X B \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "d"] & B \arrow[d, "q"] \\
& A \arrow[r, "p"'] & X
\end{tikzcd}$$

Then $(A \times_X B, d, (s, t))$ together with p_1, p_2 form a product for (A, p, s) and (B, q, t) in $\mathrm{Pt}_{\mathbb{C}}(X)$. Then we may adapt Lemma 2.3 to the local situation, and obtain the following:

3.6. PROPOSITION. *For a category $\mathbb{C}$ with finite limits, the following are equivalent.*

- *$\mathbb{C}$ satisfies (P) locally.*
- *For any object X in $\mathbb{C}$, and for any coequalizer diagram*

$$(C, r, n) \underset{u}{\overset{v}{\rightrightarrows}} (A, p, s) \xrightarrow{f} (B, q, t)$$

in $\mathrm{Pt}_{\mathbb{C}}(X)$, the diagram

$$C \underset{(u, j \circ r)}{\overset{(v, j \circ r)}{\rightrightarrows}} A \times_X D \xrightarrow{f \times_X 1_D} B \times_X D$$

is a coequalizer in $\mathbb{C}$ for any object (D, m, j) in $\mathrm{Pt}_{\mathbb{C}}(X)$.

3.7. Theorem. *For a variety $\mathcal{V}$ the following are equivalent.*

1. *For any object X in $\mathcal{V}$ the product of two coequalizer diagrams in $\mathrm{Pt}_{\mathcal{V}}(X)$ is a coequalizer diagram.*

2. *For any object X in $\mathcal{V}$ the product of two normal-epimorphisms in $\mathrm{Pt}_{\mathcal{V}}(X)$ is a normal epimorphism in $\mathrm{Pt}_{\mathcal{V}}(X)$.*

3. *$\mathcal{V}$ has normal local projections.*

4. *There exists $n \geqslant 1$ and $m \geqslant 0$ such that $\mathcal{V}$ admits binary terms $b_1, ...b_m, c_1, ..., c_m$ and $(m+2)$-ary terms $p_1, p_2, ..., p_n$ such that*

   - $p_1(x, y, b_1(x, y), ..., b_m(x, y)) = x$.
   - $p_n(y, x, b_1(x, y), ..., b_m(x, y)) = y$.
   - *For any $i \in \{1, 2, ..., n-1\}$ we have*

   $$p_i(y, x, b_1(x, y), ..., b_m(x, y)) = p_{i+1}(x, y, b_1(x, y), ..., b_m(x, y)).$$

   - *For any $i \in \{1, 2, ..., n\}$ we have $p_i(u, u, c_1(u, v), ..., c_m(u, v)) = v$ and $b_i(z, z) = c_i(z, z)$.*

5. *For any two morphisms $f : A \to X$ and $g : B \to X$, and any congruence C on the pullback $A \times_X B$ of f along g, if $(x, u)C(y, u)$ then $(x, v)C(y, v)$ where $(x, v), (y, v) \in A \times_X B$. This can be visualized as follows*

$$\begin{tikzcd}
(x,u) \arrow[r, dash, "\mathrm{Eq}(p_2)"] \arrow[r, dash, bend left, "C"] \arrow[d, dash, "\mathrm{Eq}(p_1)"'] & (y,u) \arrow[d, dash, "\mathrm{Eq}(p_1)"] \\
(x,v) \arrow[r, dash, "\mathrm{Eq}(p_2)"'] \arrow[r, dash, dotted, bend right, "C"'] & (y,v)
\end{tikzcd}$$

In the proof below, we make use of the same notation as described in the paragraph preceding Theorem 2.16.

Proof. The implications (1) $\implies$ (2) $\implies$ (3) are the content of Proposition 2.1, and (3) $\implies$ (4) is the content of Theorem 3.4. We show (4) $\implies$ (5) $\implies$ (1). Suppose that C is a congruence on the pullback $A \times_X B$ in (5), and that we are given $(x, u)C(y, u)$, and suppose that $(x, v), (y, v) \in A \times_X B$ are any two elements. Note that since $(x, u), (y, u), (x, v), (y, v)$ are all elements of $A \times_X B$, it follows that $f(x) = f(y) = g(u) = g(v)$, and since $b_i(z, z) = c_i(z, z)$ we have $f(b_i(x, y)) = b_i(f(x), f(y)) = c_i(g(u), g(v)) = g(c_i(u, v))$, so that $(b_i(x,y), c_i(u,v)) \in A \times_X B$. Consider the elements $(z_{1,i}, z_{2,i})$ of C given by:

$$(z_{1,i}, z_{2,i}) = p_i\left(\begin{pmatrix} x & y \\ u & u \end{pmatrix}, \begin{pmatrix} y & x \\ u & u \end{pmatrix}, \begin{pmatrix} b_1(x,y) & b_1(x,y) \\ c_1(u,v) & c_1(u,v) \end{pmatrix}, \dots, \begin{pmatrix} b_m(x,y) & b_m(x,y) \\ c_m(u,v) & c_m(u,v) \end{pmatrix} \right).$$

Then the equations in the statement of the theorem imply that $z_{1,1} = (x,v)$ and $z_{2,n} = (y,v)$, and moreover that $z_{2,i} = z_{1,i+1}$, so that by the transitivity of C we have $(x,v)C(y,v)$. For (5) $\implies$ (1), we note that condition (5) implies that $\mathrm{Pt}_{\mathcal{V}}(X)$ satisfies (3) of Proposition 2.9, since products in $\mathrm{Pt}_{\mathcal{V}}(X)$ are pullbacks in $\mathcal{V}$, and the domain functor $\mathrm{Pt}_{\mathcal{V}}(X) \to \mathcal{V}$ sends equivalence relations in $\mathrm{Pt}_{\mathcal{V}}(X)$ to equivalence relations in $\mathcal{V}$ (since it preserves pullbacks and equalizers), and reflects isomorphisms. ■

In light of the theorem above, it is natural to ask if every variety with normal projections satisfies (P). We answer this question in the negative: every subtractive category has normal product projections, but not every subtractive variety satisfies (P). To see this, consider the subtraction algebra X which has underlying set $\{0, x, y\}$, and whose subtraction is defined as $a - 0 = a$ and $a - b = 0$ if $b \neq 0$, where a and b are any elements of X. Similarly, let Z be the subtraction algebra with underlying set $\{0, z\}$, where the subtraction is defined as X's is. Let θ be the congruence on $X \times Z$ generated by the relation $(x,0)\theta(y,0)$. By (3) of Proposition 2.9 it is enough to show that $((x,z),(y,z)) \notin \theta$. Using the same notation as we described in the paragraph preceding Theorem 2.16, we claim that $\theta = R \cup \Delta_{X\times Z}$ where

$$R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Note that $R \cup \Delta_{X\times Z}$ is an equivalence relation, so that it remains only to show that it is a subalgebra of $(X \times Z) \times (X \times Z)$. Since both R and $\Delta_{X\times Z}$ are subalgebras of $(X \times Z) \times (X \times Z)$, it suffices to show that the subtraction of any element of R with any element of $\Delta_{X\times Z}$, and visa-versa, results with an element in $R \cup \Delta_{X\times Z}$. The following table, together with the table that results from swapping x and y, shows that this is the case. Therefore $((x,z),(y,z)) \notin \theta$, which shows that the variety of subtraction algebras does not satisfy (P).

.

4. Concluding remarks

We have been unable to establish a categorical counterpart of Theorem 3.7, and leave the investigation of this question for a future work. Moreover, for varieties which are not necessarily pointed we have been unable to determine whether or not the property (P) can be characterized by a Mal'tsev condition as we did in Theorem 2.16. We leave it as an open question of whether or not (P) is a Mal'tsev property for not-necessarily pointed varieties, and conjecture that it is not. As mentioned earlier in Remark 2.8, every category

A	B	$A - B$	$B - A$
$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} y & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ z & z \end{pmatrix}$	$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ z & z \end{pmatrix}$
$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & x \\ z & z \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ z & z \end{pmatrix}$
$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} y & y \\ z & z \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ z & z \end{pmatrix}$

with binary products in which every object is $\mathcal{M}$-coextensive in the sense of [11], where $\mathcal{M} = \mathrm{Reg}(\mathbb{C})$ is the class of regular epimorphisms in $\mathbb{C}$, satisfies (P). Thus for example, the category **Ring** of unitary rings satisfies (P), but **Ring** is not pointed. Moreover, in any congruence distributive $\mathcal{V}$ which admits constants every object is $\mathcal{M}$-coextensive with $\mathcal{M} = \mathrm{Reg}(\mathcal{V})$, and thus $\mathcal{V}$ satisfies (P).

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