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MODELS OF LINEAR LOGIC BASED ON THE SCHWARTZ ε-PRODUCT.

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ABSTRACT. From the interpretation of Linear Logic multiplicative disjunction as the epsilon product defined by Laurent Schwartz, we construct several models of Differential Linear Logic based on the usual mathematical notions of smooth maps. This improves on previous results by Blute, Ehrhard and Tasson based on convenient smoothness where only intuitionist models were built. We isolate a completeness condition, called k-quasi-completeness, and an associated notion which is stable under duality called k-reflexivity, allowing for a star-autonomous category of k-reflexive spaces in which the dual of the tensor product is the reflexive version of the epsilon-product. We adapt Meise's definition of smooth maps into a first model of Differential Linear Logic, made of k-reflexive spaces. We also build two new models of Linear Logic with conveniently smooth maps, on categories made respectively of Mackey-complete Schwartz spaces and Mackey-complete Nuclear Spaces (with extra reflexivity conditions). Varying slightly the notion of smoothness, one also recovers models of DiLL on the same star-autonomous categories. Throughout the article, we work within the setting of Dialogue categories where the tensor product is exactly the epsilon-product (without reflexivization).

Contents

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1. Introduction

Smooth models of classical Linear Logic. Since the discovery of linear logic by Girard [Gir87], thirty years ago, many attempts have been made to obtain denotational models of linear logic in the context of categories of vector spaces with linear proofs interpreted as linear maps [Blu96, Ehr02, Gir04, Ehr05, BET]. Models of linear logic are often inspired by coherent spaces, or by the relational model of linear logic. Coherent Banach spaces [Gir99], coherent probabilistic or coherent quantum spaces [Gir04] are Girard's attempts to extend the first model, as finiteness spaces [Ehr05] or Köthe spaces [Ehr02] were designed by Ehrhard as a vectorial version of the relational model. Following the construction of Differential linear logic [ER06], one would want moreover to find natural models of it where non-linear proofs are interpreted by some classes of smooth maps. This requires the use of more general objects of functional analysis which were not directly constructed from coherent spaces. We see this as a strong point, as it paves the way towards new computational interpretations of functional analytic constructions, and a denotational interpretation of continuous or infinite data objects.

A consequent categorical analysis of the theory of differentiation was tackled by Blute, Cockett and Seely [BCS06, BCS09]. They gave several structures in which a differentiation operator is well-behaved. Their definition then restricts to models of Intuitionistic Differential Linear Logic. Our paper takes another point of view as we look for models of classical DiLL, in which spaces equal some double dual. We want to emphasize the classical computational nature of Differential Linear Logic.

Three difficulties appear in this semantical study of linear logic. The equivalence between a formula and its double negation in linear logic asks for the considered vector spaces to be isomorphic to their double duals. This is constraining in infinite dimension. This infinite dimensionality is strongly needed to interpret exponential connectives. Then one needs to find a good category with smooth functions as morphisms, which should give a cartesian closed category. This is not at all trivial, and was solved by using a quantative setting, i.e. power series as the interpretation for non-linear proofs, in most of the previous works [Gir99, Gir04, Ehr05, Ehr02]. Finally, imposing a reflexivity condition to respect the first requirement usually implies issues of stability by natural tensor products of this condition, needed to model multiplicative connectives. This corresponds to the hard task of finding $*$-autonomous categories [Ba79]. As pointed out in [Ehr16], the only model of differential linear logic using smooth maps [BET] fails to satisfy the $*$-autonomous property for classical linear logic.

Our paper solves all these issues simultaneously and produces several denotational models of classical linear logic with some classes of smooth maps as morphism in the Kleisli category of the monad. We will show that the constraint of finding a $*$-autonomous category in a compatible way with a cartesian closed category of smooth maps is even relevant to find better mathematical notions of smooth maps in locally convex spaces. Let us explain this mathematical motivation first.

A framework for differential calculus. It seems that, historically, the development of differential calculus beyond normed spaces suffered from the lack of interplay between analytic considerations and categorical, synthetic or logical ones. As a consequence, analysts often gave up looking for good properties stable under duality and focused on one side of the topological or bornological viewpoint.

An analytic summary of the early theory can be found in Keller's book [Kel]. It already gives a unified and simplified approach based on continuity conditions of derivatives in various senses. But it is well-known that in order to look for strong categorical properties such as cartesian closedness, the category of continuous maps is not a good starting point, the category of maps continuous on compact sets would be better. This appears strongly in all the developments made to recover continuity of evaluation on the topological product (instead of considering the product of a cartesian closed category), which is unavoidable for full continuity of composition of derivatives in the chain rule. This leads to considering convergence notions beyond topological spaces on spaces of linear maps, but then, no abstract duality theory of those vector convergence spaces or abstract tensor product theory is developed. In the end, everything goes well only on restricted classes of spaces that lack almost any categorical stability properties, and nobody understands half of the notions introduced. The situation became slightly better when [Me] considered k-space conditions and obtained what analysts call kernel representation theorems (Seely isomorphisms for linear logicians), but still the only classes stable by products were Fréchet spaces and (DFM)-spaces, which are by their very nature not stable under duality.

The general lesson here is that, if one wants to stay within better studied and commonly used locally convex spaces, one had better not stick to functions continuous on products, and the corresponding projective topological tensor product, but always take tensor products that come from a $*$-autonomous category, since one also needs duality, or at least a closed category, to control the spaces of linear maps in which the derivatives take values. $*$-autonomous categories are the better behaved categories having all those data. Ideally, following developments inspired by game semantics [MT], we will be able to get more flexibility and allow larger dialogue categories containing such $*$-autonomous categories as their category of continuation. We will get slightly better categorical properties on those larger categories.

A better categorical framework was later found and summarized in [FK, KM] the so-called convenient smoothness. A posteriori, as seen in [Ko], the notion is closely related to synthetic differential geometry as diffeological spaces are. It chooses a very liberal notion of smoothness, that does not imply continuity except on very special compact sets, the images of finite dimensional compact sets by smooth maps. It gives a nice cartesian closed category and this enabled [BET] to obtain a model of intuitionistic differential linear logic. As we will see, this may give the wrong idea that this very liberal notion of smoothness is the only way of getting cartesian closedness and it also takes the viewpoint of focusing on bornological properties. This is the main reason why, in our view, they don't obtain $*$-autonomous categories since bornological locally convex spaces have complete duals which gives an asymmetric requirement on duals since they only need a much weaker Mackey-completeness on their spaces to work with their notion of smooth maps. We will obtain in this paper several models of linear logic using conveniently smooth maps, and we will explain logically this Mackey-completeness condition in section 4.1. It is exactly a compatibility condition on F enabling one to force our models to satisfy $!E \multimap F = (!E \multimap 1) \parr F$. Of course, as usual for vector spaces, our models will satisfy the mix rule making the unit for multiplicative connectives self-dual. This formula in the form $!E \multimap F = (F^*) \multimap (!E \multimap 1)$ is interpreted mathematically as saying that smooth maps with value in some sufficiently complete space are reduced by duality to the scalar case in the sense that composed with a map in the dual of the target space, one must recover a scalar valued smooth map and one gets exactly the classes of vector valued smooth maps if one imposes the right continuity on F^* which is captured by $\multimap$. It requires to identify the right completeness notion for each notion of smoothness.

A smooth interpretation for the $\parr$. Another insight in our work is that the setting of models of Linear logic with smooth maps offers a decisive interpretation for the multiplicative disjunction. In the setting of smooth functions, the epsilon product introduced by Laurent Schwartz is well studied and behaves exactly as wanted: under a certain completeness condition, one indeed has $\mathscr{C}^\infty(E, \mathbb{R})\varepsilon F \simeq \mathscr{C}^\infty(E, F)$. This required for instance in [Me] some restrictive conditions. We reduce these conditions to the definition 3.2 of k-complete spaces, which is also enough to get associativity and commutativity of ε. The interpretation of the tensor product follows as the negation of the ε product. We would like to point out that many possibilities exists for defining a topological tensor product (see subsection 2.6 for reminders), and that choosing to build our models from the ε-product offers a simplifying and intuitive guideline.

With this background in mind, we can describe in more detail our results and our strategy.

Organisation of the first part. The first part of the paper will focus on building several $*$-autonomous categories. This work started with a negative lesson the first author learned from the second author's results in [Ker]. Combining strong properties on concrete spaces as for instance in [BD, D] will never be sufficient since it makes stability of these properties by tensor product and duality too difficult. The only way out is to get a duality functor that makes spaces reflexive for this duality in order to correct tensor products by double dualization. The lesson is that identifying a proper notion of duality is therefore crucial if one wants to get an interesting analytic tensor product. From an analytic viewpoint, the inductive tensor product is too weak to deal with extensions to completions and therefore the weak dual or the Mackey dual, shown to work well with this tensor product in [Ker], and which are the first duality functors implying easy reflexivity properties, is not strong enough for our purposes. The insight is given by a result of [S] that implies that another slightly different dual, the Arens dual always satisfies the algebraic equality $((E'_c)'_c)'_c = E'_c$ hence one gets a functor enabling one to get reflexive spaces, in some weakened sense of reflexivity. Moreover, Laurent Schwartz also developed there a related tensor product, the so called ε-product which is intimately related. This tensor product is a dual tensor product, a generalization of the (dual) injective tensor product of (dual) Banach spaces and linear logicians would say it is a negative connective (as in polarized linear logic [Gir91], as seen for instance from its commutation with categorical projective limits) suitable for interpreting $\parr$. Moreover, it is strongly related with Seely-like isomorphisms for various classes of non-linear maps, from continuous maps (see e.g. [T]) to smooth maps [Me]. It is also strongly related with nuclearity and Grothendieck's approximation property. This is thus a well-established analytic tool desirable as a connective for a natural model of linear logic. We actually realize that most of the general properties for the Arens dual and the ε-product in [S] are nicely deduced from a very general $*$-autonomous category we will explain at the end of the preliminary section 3. This first model of MALL that we will obtain takes seriously the lack of self-duality of the notion of locally convex space and notices that adjoining a bornology with weak compatibility conditions enables one to get a framework where building a $*$-autonomous category is almost tautological. This is probably related to some kind of Chu construction (cf. [Ba96] and appendix to [Ba79]), but we won't investigate this idea here. This is opposite to the consideration of bornological locally convex vector spaces where bornology and topology are linked to determine one another, here they can be almost independently chosen and correspond to encapsulating on the same space the topology of the space and of its dual (given by the bornology).

Then, the work necessary to obtain a $*$-autonomous category of locally convex spaces is twofold, it requires one to impose some completeness condition required to get associativity maps for the ε-product and we must then make the Arens dual compatible with some completion process to keep a reflexivity condition and get another duality functor with duals isomorphic to triple duals. We repeat this general plan twice in sections 3 and 4 to obtain two extreme cases where this plan can be carried out. The first version uses the notion of completeness used in [S], or rather a slight variant we will call k-quasi-completeness and builds a model of MALL with the only requirement being k-quasi-completeness and being the Arens dual of a k-quasi-complete space. This notion is equivalent to a reflexivity property that we call k-reflexivity. This first $*$-autonomous category is important because its positive tensor product is a completed variant of an algebraic tensor product $\otimes_\gamma$ having universal properties for bilinear maps which have a hypocontinuity condition implying continuity on product of compact sets (see section 2.6 for more preliminary background). This suggested to us a relation to the well-known notion of cartesian closed category (equivalent to k-spaces) of topological spaces with maps all maps continuous on compact sets. Using the fact that we have obtained a $*$-autonomous category, this enables us to provide the strongest notion of smoothness (on locally convex spaces) that we can imagine having a cartesian closedness property. Contrary to convenient smoothness, it satisfies a much stronger continuity condition of all derivatives on compacts sets. Here, we thus combine the $*$-autonomous category with a cartesian closed category in taking inspiration of the former to define the latter. This is developed in subsection 3.21.

Then in section 5, we turn to the complementary goal of finding a $*$-autonomous framework that will be well-suited for the already known and more liberal notion of smoothness, namely convenient smoothness. Here, we need to combine Mackey-completeness with a Schwartz space property to reach our goals. This is strongly based on preliminary work in section 4 that identifies an intermediate notion of $\mathscr{C}$-smooth maps between convenient smoothness and our stronger notion of smoothness in subsection 3.21. We can then associate a corresponding notion of $\mathscr{C}$-completeness and a notion of $\mathscr{C}$-space property that generalizes the notion of Schwartz space. To complement our identification of a logical meaning of Mackey-completeness (as a special case of $\mathscr{C}$-completeness), we also relate the extra Schwartz property condition with the logical interpretation of the transpose of the dereliction $\mathsf{d}^t : E^* \multimap (!E)^*$. This asks for the topology on E^* to be finer than the one induced by $(!E)^*$. If moreover one wants to recover later a model of differential linear logic, we need a morphism: $\overline{\mathsf{d}} : !E \longrightarrow E$ such that $\mathsf{d} \circ \overline{\mathsf{d}} = \mathrm{Id}_E$.

This enforces the fact that the topology on E^* must equal the one induced by $(!E)^*$. In this way, various natural topologies on conveniently smooth maps suggest various topologies on duals. We investigate in more detail in section 5 the two extreme cases again, corresponding to well-known functional analytic conditions, both invented by Grothendieck, namely Schwartz topologies and the subclass of nuclear topologies.

Combining those two notions : the $\mathscr{C}$-completeness and the $\mathscr{C}$-space property, one gets a dialogue category in section 4.23 and an associated $*$-autonomous category of $\mathscr{C}$-reflexive spaces. It is quite noticeable that we recover the associativity of the ε-product on $\mathscr{C}$-complete $\mathscr{C}$-spaces in using strongly the cartesian closedness for $\mathscr{C}$-smooth maps. We develop this generally in section 4, and give more concrete examples in specializing to the Mackey-complete setting in section 5.

Technically, it is convenient to decompose our search for a $*$-autonomous category in two steps. Once we have identified the right duality notion and the corresponding reflexivity, we produce first a Dialogue category [MT] that deduces its structure from a kind of intertwining with the $*$-autonomous category obtained in section 2. Then we use [MT] to recover a $*$-autonomous category in a standard way. This gives us the notion of ρ-dual and the $*$-autonomous category of ρ-reflexive spaces. As before, those spaces can be described in saying that they are Mackey-complete with Mackey-complete Mackey dual (coinciding with Arens dual here) and they have the Schwartz topology associated to their Mackey topology. We give these structures the name ρ-dual since this was the first and more fruitful way (as seen its relation developed later with convenient smoothness) of obtaining a reflexive space by duality, hence the letter ρ for reflexive, while staying close to the letter σ that would have remembered the key Schwartz space property, but which was already taken by weak duals.

At the end of the first part of the paper, we have a kind of generic methodology enabling to produce $*$-autonomous categories of locally convex spaces from a kind of universal one from section 2. We also have obtained two examples that we want to extend to denotational models of full (differential) Linear logic in the second part.

Organisation of the second part about LL and DiLL. In the second part of the paper, we develop a theory for variants of conveniently smooth maps, which we restrict to allow for continuous, and not only bounded, differentials. We start with the convenient smoothness setting in section 6. Actually we work with several topological variants of this setting (all having the same bornologification). We obtain in that way in section 6 two denotational models of LL on the same $*$-autonomous category (of ρ-reflexive spaces), with the same cartesian closed category of conveniently smooth maps, but with two different comonads. We actually show this difference in remark 6.10 using Banach spaces without the approximation property. This also gives an insight of the functional analytic significance of the two structures. Technically, we use dialogue categories again, but not via the models of tensor logic from [MT], but rather with a variant we introduce to retain cartesian closedness of the category equipped with non-linear maps as morphisms.

Finally, in section 7, we extend our models to models of (full) differential linear logic. In the k-reflexive space case, we have already identified the right notion of smooth maps for that in section 4, but in the ρ-reflexive case, which generalizes convenient vector spaces, we need to slightly change our notion of smoothness and introduce a corresponding notion of ρ-smoothness. Indeed, for the new ρ-reflexive spaces which are not bornological, the derivative of conveniently smooth maps are only bounded and need not be in spaces of continuous linear maps which are the maps of our $*$-autonomous categories. Taking inspiration of our use of dialogue categories and its interplay with cartesian closed categories in section 6, we introduce in section 7.1 a notion merging dialogue categories with differential λ-categories of [BEM]. It turns out that the properties of ρ-smooth maps that we need to check in order to get a DiLL model can essentially be reduced to properties of conveniently smooth maps in imposing the extra continuity conditions on derivatives in a general categorical way. We explain in section 7.4 this categorical reduction step which enables us to turn our specific type of LL models from section 6 to what we needed in section 7.1 to construct DiLL models. This enables us to get a class of models of DiLL with at least 3 new different models in that way, one on k-reflexive spaces (section 7.13) and two being on the same category of ρ-reflexive spaces with ρ-smooth maps (section 7.7). This is done concretely by considering only smooth maps whose derivatives are smooth in their non-linear variable with value in (iterated) spaces of continuous linear maps.

A first look at the interpretation of Linear Logic constructions For the reader familiar with other denotational models of Linear Logic, we would like to point out some of the constructions involved in the first model **k-Ref**. Our two other main models make use of similar constructions, with a touch of Mackey-completeness.

First, we define a *k-quasi-complete* space as a space in which the closed absolutely convex cover of a compact subset is still compact. We detail a procedure of k-quasi-completion, which is done inductively.

We take as the interpretation $E^{\perp}$ of the negation the k-quasi completion of E'_c, the dual of E endowed with the compact-open topology, at least when E is k-quasi-complete. We define $!E$ as $\mathscr{C}^{\infty}_{co}(E, \mathbb{K})^{\perp}$, the k-quasicompletion of the dual of the space of scalar smooth functions. This definition is in fact forced on us as soon as we have a $*$-autonomous category with a co-Kleisli category of smooth maps. Here we define the space of smooth functions as the space of infinitely many times Gâteaux-differentiable functions with derivatives continuous on compacts, with a good topology (see subsection 3.21). This definition, adapted from the one of Meise, allows for cartesian closedeness.

We then interpret the $⅋$ as the (double dual of) the ε product: $E\varepsilon F = \mathcal{L}_{\epsilon}(E'_c, F)$, the space of all linear continuous functions from E'_c to F endowed with the topology of uniform convergence on equicontinuous subsets of E'. The interpretation of $\otimes$ is the dual of ε, and can be seen as the k-quasi-completion of a certain topological tensor product $\otimes_{\gamma}$.

The additive connectives $\times$ and $\oplus$ are easily interpreted as the product and the co-product. In our vectorial setting, they coincide in finite arity.

In the differential setting, codereliction $\overline{\mathsf{d}}$ is interpreted as usual by the transpose of differentiation at 0 of scalar smooth maps.

I. Models of MALL

2. Preliminaries

We will be working with *locally convex separated* topological vector spaces. We will write in short lcs for such spaces, following [K] in that respect. We refer to the book by Jarchow [Ja] for basic definitions. We will recall the definitions from Schwartz [S] concerning the ε product. We write $|E| = |F|$ when two lcs E and F are equal algebraically and $E \simeq F$ or $E = F$ when they are isomorphic or equal topologically as well. We will define sets and functions using the notation $E := \cdots$ or $f := \cdots$.

2.1. Remark. As usual we will use the term *embedding* for a continuous linear map $E \longrightarrow F$ which is one-to-one and such that the topology of E agrees with that induced from this inclusion.

2.2. Remark. We will use projective kernels as in [K]. They are more general than categorical limits, which are more general than the projective limits of [K], which coincide with those categorical limits indexed by directed sets.

2.3. Reminder on topological vector spaces.

2.4. Definition. *Consider E a vector space. A* bornology *on E is a collection of sets (the bounded sets of E) such that the union of all those sets covers E, and such that the collection is stable under inclusion and finite unions.*

In the lcs setting, convexity plays a crucial role to identify relevant bornologies and is expressed in terms of bipolars. For a set $A \subset E$, its polar and bipolar are defined as follows:

$$A^\circ := \{f \in E' : \forall x \in A, |f(x)| \leq 1\},$$

$$A^{\circ\circ} := \{x \in E : \forall f \in A^\circ, |f(x)| \leq 1\}.$$

The bipolar theorem [Ja, 8.2.2] then states that the bipolar $A^{\circ\circ}$ is the same as the weakly closed absolutly convex hull of A (see Definition 2.8 for a reminder on the weak topology) and equivalently the closed absolutly convex hull $\overline{\Gamma(A)}$ of A [Ja, 8.2.5].

When E is a topological vector space, one defines the Von-Neumann bornology β as those sets which are absorbed by any neighbourhood of 0. Without any further qualification, the name bounded set will refer to a bounded set for the Von-Neumann bornology. Other examples of bornology are the collections γ of all absolutely convex compact subsets of E, and σ of all bipolars of finite sets. When E is a space of continuous linear maps, one can also consider on E the bornology ε of all equicontinuous parts of E. When E is a lcs, we only consider saturated bornologies, namely those which contain the subsets of the bipolars of each of its members.

2.5. Definition. *Consider E, F, G topological vector spaces and $h : E \times F \mapsto G$ a bilinear map.*

- *h is* continuous *if it is continuous from $E \times F$ endowed with the product topology to G.*

- *h is* separately continuous *if for any $x \in E$ and $y \in F$, $h(x,.)$ is continuous from F to G and $h(.,y)$ is continuous from E to G.*

- *Consider $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) a bornology on E (resp. F). Then h is said to be* $\mathcal{B}_1$,$\mathcal{B}_2$ hypocontinuous *[S2] if for every 0-neighbourhood W in G, every bounded set A_E in E, and every bounded set A_F in F, there are 0-neighbourhoods $V_F \subset F$ and $V_E \subset E$ such that $h(A_E \times V_F) \subset W$ and $h(V_E \times A_F) \subset W$. When no qualification is given, a hypocontinuous bilinear map is a map hypocontinuous for both Von-Neumann bornologies.*

Consider A an absolutely convex and bounded subset of a lcs E. We write E_A for the linear span of A in E. It is a normed space when endowed with the Minkowski functional

$$\|x\|_A \equiv p_A(x) := \inf \{\lambda \in \mathbb{R}^+ \mid x \in \lambda A\}.$$

A lcs E is said to be *Mackey-complete* (or locally complete [Ja, 10.2]) when for every bounded closed and absolutely convex subset A, E_A is a Banach space. A sequence is *Mackey-convergent* if it is convergent in some E_B. This notion can be generalized for any bornology $\mathcal{B}$ on E: a sequence is said to be *$\mathcal{B}$-convergent* if it is convergent is some E_B for $B \in \mathcal{B}$.

Consider E a lcs and τ its topology. Recall that a filter in E' is said to be equicontinuously convergent if it is ε-convergent. E is a *Schwartz space* if it is endowed with a Schwartz topology, that is a space such that every continuously convergent filter in E' converges equicontinuously. We refer to [HNM, chapter 1] and [Ja, sections 10.4, 21.1] for an overview on Schwartz topologies. We recall some facts below.

The finest Schwartz locally convex topology coarser than τ is the topology τ_0 of uniform convergence on sequences of E' converging equicontinuously to 0. We write $\mathscr{S}(E) := \mathscr{S}(E,\tau) = (E,\tau_0)$. We have $|(\mathscr{S}(E))'| = |E'|$, and $\mathscr{S}(E)$ is always separated. A lcs E is a Schwartz space if and only if we have the topological equality $\mathscr{S}(E) = E$, if and only if the completion $\tilde{E}$ is a Schwartz space. The *Mackey-Arens theorem* [Ja, 8.5.5] states that whenever E' is endowed with a topology finer that the weak topology, and coarser than the Mackey topology, then $E = E''$ algebraically. A usual consequence of its combination with [Ja, 8.4.4] is that every topology with the same dual has the same bounded sets. Hence, we do know also that $\mathscr{S}(E)$ is Mackey-complete as soon as E is (as both spaces have the same dual, they have the same bounded sets). Moreover, any subspace of a Schwartz space is a Schwartz space.

2.6. Reminder on tensor products and duals of locally convex spaces. Several topologies can be associated with the tensor product of two topological vector space. Our formulation follows the characterization [S2, 3° p 11].

2.7. Definition. *Consider E and F two lcs.*

- *The* projective tensor product *$E \otimes_\pi F$ is the finest locally convex topology on $E \otimes F$ making $E \times F \to E \otimes_\pi F$ continuous. (cf. [K2, §41.2], [S2, 1° p 12], or [PC, 0.6.3])*

- *The* inductive tensor product *$E \otimes_i F$ is the finest locally convex topology on $E \otimes F$ making $E \times F \to E \otimes_i F$ separately continuous. (cf. [K2, p 266] or[S2, 4° p 12])*

- *The* hypocontinous tensor product $E \otimes_\beta F$ *is the finest locally convex topology on* $E \otimes F$ *making* $E \times F \to E \otimes_\beta F$ *hypocontinuous. (cf. [S2,* 2° *p 12] or [PC, 11.3.3])*
- *The* γ-tensor product $E \otimes_\gamma F$ *is the finest locally convex topology on* $E \otimes F$ *making* $E \times F \to E \otimes_\gamma F$ γ*-hypocontinuous. (cf. [S2,* 3° *p 12])*
- *Suppose that E and F are duals. The* ε-hypocontinous tensor product $E \otimes_{\beta e} F$ *is the finest locally convex topology on* $E \otimes F$ *making* $E \times F \to E \otimes_{\beta e} F$ ε*-hypocontinuous in the sense of [S, p 18].*
- *Consider* $\mathcal{B}_1$ *(resp.* $\mathcal{B}_2$*) a bornology on E (resp. F).The* $\mathcal{B}_1 - \mathcal{B}_2$-hypocontinous tensor product $E \otimes_{\mathcal{B}_1,\mathcal{B}_2} F$ *is the finest locally convex topology on* $E \otimes F$ *making* $E \times F \to E \otimes_{\mathcal{B}_1,\mathcal{B}_2} F$ $\mathcal{B}_1, \mathcal{B}_2$*-hypocontinuous. (cf. [S2, p 10])*

All the above tensor products, except the last one, are commutative and the $\otimes_\pi$ product is associative. With the last generic notation one gets $\otimes_i := \otimes_{\sigma,\sigma}, \otimes_\beta := \otimes_{\beta,\beta}, \otimes_\gamma := \otimes_{\gamma,\gamma}, \otimes_{\beta e} := \otimes_{\varepsilon,\varepsilon}$ and we will sometimes consider during proofs non-symmetric variants such as: $\otimes_{\varepsilon,\gamma}, \otimes_{\sigma,\gamma}$ etc. Note that the injective tensor product $\otimes_\varepsilon \neq \otimes_{\varepsilon,\varepsilon}$ is a dual version we will discuss later (see e.g. [PC, 0.6.3] or [K2, §44.2]). It does not have the above type of universal property.

One can define several topologies on the dual E' of a lcs E, all of them are studied in [Ho], but we usually refer to the more encyclopedic books by G. Köthe [K, K2]. Those authors may use a different notation. We will make use of :

2.8. Definition. *Consider a lcs E.*

- *The* strong dual E'_β*, endowed with the strong topology* $\beta(E', E)$ *of uniform convergence on bounded subsets of E. (cf. [Ho, p 201] or [K,* §21.2*])*
- *The* Arens dual E'_c *endowed with the topology* $\gamma(E', E)$ *of uniform convergence on absolutely convex compact subsets of E. (cf. [Ho, p 235] or [K2,* §*43.1])*
- *The* Mackey dual E'_μ*, endowed with the Mackey topology of uniform convergence on absolutely convex weakly compact subsets of E. (cf. [Ho, p 206] or [K,* §*21.4])*
- *The* weak dual E'_σ *endowed with the weak topology* $\sigma(E', E)$ *of simple convergence on points of E. (cf. [Ho, p 185] or [K,* §*21.1])*
- *The* ε-dual E'_ε *of a dual* $|E| = F'$ *is the dual* E' *endowed with the topology of uniform convergence on equicontinuous sets in* F'*. (cf. [Ho, p 229] or [K,* §*21.3])*

Remember that when it is considered as a set of linear forms acting on E'*, E is always endowed with the topology of uniform convergence on equicontinuous parts of* E'*, equivalent to the original topology of E, hence* $(E'_\mu)'_\epsilon \simeq (E'_c)'_\epsilon \simeq (E'_\sigma)'_\epsilon \simeq E$*. A lcs is said to be reflexive when it is topologically equal to its strong double dual* $(E'_\beta)'_\beta$*.*

By the Mackey-Arens Theorem, one has

$$|E| = |(E'_c)'|. \tag{1}$$

As explained by Laurent Schwartz [S, section 1], the equality $E \simeq (E'_c)'_c$ holds as soon as E is endowed with its γ topology, i.e. with the topology of uniform convergence on absolutely convex compact subsets of E'_c. He proves moreover that an Arens dual is always endowed with its γ-topology, that is: $E'_c \simeq ((E'_c)'_c)'_c$. This fact is the starting point of the construction of a $*$-autonomous category in section 5.

The *ε-product* has been extensively used and studied by Laurent Schwartz [S, section 1]. By definition $E\varepsilon F := (E'_c \otimes_{\beta e} F'_c)'$ is the set of ε-hypocontinuous bilinear forms on the duals E'_c and F'_c. When E, F have their γ topologies we have $E\varepsilon F \simeq (E'_c \otimes_\gamma F'_c)'$.

The topology on $E\varepsilon F$ is the topology of uniform convergence on products of equicontinuous sets in E', F'. If E, F are quasi-complete spaces (resp. complete spaces , resp. complete spaces with the approximation property) so is $E\varepsilon F$ (see [S, Prop 3 p29, Corol 1 p 47]). The ε tensor product $E \otimes_\varepsilon F$ coincides with the topology on $E \otimes F$ induced by $E\varepsilon F$ (see [S, Prop 11 p46]), $\otimes_\varepsilon$ is associative, and $E\hat{\otimes}_\varepsilon F \simeq E\varepsilon F$ if E, F are complete and E has the approximation property.

The ε-product is also defined on any finite number of space as $\varepsilon_i E_i$, the space of ε-equicontinuous multilinear forms on $\prod_i (E_i)'_c$,, endowed the the topology of uniform convergence on equicontinuous sets. Schwartz proves the associativity of the ε-product when the spaces are quasi-complete. We do so when the spaces are Mackey-complete and Schwartz, see proposition 5.17.

2.9. DIALOGUE AND $*$-AUTONOMOUS CATEGORIES. It is well-known that models of (classical) linear logic require the construction of $*$-autonomous categories as introduced in [Ba79]. If we add categorical completeness, they give models of MALL. We need some background about them, as well as a generalization introduced in [MT]: the notion of dialogue category that will serve us as an intermediate in between a general $*$-autonomous category we will introduce in the next subsection and more specific ones requiring a kind of reflexivity of locally convex spaces that we will obtain by double dualization, hence in moving to the so-called continuation category of the dialogue category.

Recall the definition (cf. [Ba79]):

2.10. DEFINITION. *A* $*$-autonomous category *is a symmetric monoidal closed category* $(C, \otimes_C, 1_C, [\cdot,\cdot]_C)$ *with an object* $\perp$ *giving an equivalence of categories* $(\cdot)^* := [\cdot, \perp]_C : C^{\mathrm{op}} \longrightarrow C$ *and with the canonical map* $d_A : A \longrightarrow (A^*)^*$ *being a natural isomorphism.*

Since our primary data will be functional, based on spaces of linear maps (and tensorial structure will be deduced since it requires various completions), we will need a consequence of the discussion in [Ba79, (4.4) (4.5) p 14-15]. We outline the proof for the reader's convenience. We refer to [DeS, p 25] (see also [DL]) for the definition of symmetric closed category.

2.11. LEMMA. *Let* $(C, 1_C, [\cdot,\cdot]_C)$ *a symmetric closed category, which in particular implies there is a natural isomorphism* $s_{X,Y,Z} : [X, [Y, Z]_C]_C \longrightarrow [Y, [X, Z]_C]_C$ *and let* $\perp := [1_C, 1_C]_C$. *Assume moreover that there is a natural isomorphism,* $d_X : X \longrightarrow [[X, \perp]_C, \perp]_C$. *Define* $X^* := [X, \perp]_C$ *and* $(X \otimes_C Y) := ([X, Y^*]_C)^*$. *Then* $(C, \otimes_C, 1_C, [\cdot,\cdot]_C, (\cdot)^*)$ *is a* $*$*-autonomous category.*

Proof. Recall for instance that $i_X : X \longrightarrow [1_C, X]_C$ is an available natural isomorphism. Note first that there is a natural isomorphism defined by:

$$d_{X,Y} : [X, Y]_C \xrightarrow{[X,d_Y]_C} [X, Y^{**}]_C \xrightarrow{s_{X,Y^*,\perp}} [Y^*, [X, \perp]_C]_C.$$

The assumptions give a natural isomorphism:

$$\begin{aligned} C(X, [Y, Z^*]_C) &\simeq C(1, [X, [Y, Z^*]_C]_C) \simeq C(1, [X, [Z, Y^*]_C]_C) \\ &\simeq C(1, [Z, [X, Y^*]_C]_C) \simeq C(Z, [Y, X^*]_C). \end{aligned}$$

Moreover, we have a bijection $C(X^*, Y^*) \simeq C(1, [X^*, Y^*]_C) \simeq C(1, [Y, X]_C) \simeq C(Y, X)$ so that the assumptions in [Ba79, (4.4)] are satisfied. His discussion in (4.5) gives a natural isomorphism: $\pi_{XYZ} : C(X \otimes_C Y, Z) \longrightarrow C(X, [Y, Z]_C)$.

We are thus in the third basic situation of [DeS, IV .4] which gives (from s) a natural transformation $p_{XYZ} : [X \otimes_C Y, Z]_C \longrightarrow [X, [Y, Z]_C]_C$. Then the proof of his Prop VI.4.2 proves his compatibility condition MSCC1 from SCC3, hence we have a monoidal symmetric closed category in the sense of [DeS, Def IV.3.1].

Then [DeS, Thm VI.6.2 p 136] gives us a symmetric monoidal closed category in the sense of [EK]. This concludes the proof. ■

We finally recall the more general definition in [MT]:

2.12. Definition. *A* dialogue category *is a symmetric monoidal category* $(C, \otimes_C, 1_C)$ *with a functor, called tensorial negation:* $\neg : C \longrightarrow C^{\mathrm{op}}$ *which is associated to a natural bijection* $\varphi_{A,B,C} : C(A \otimes_C B, \neg C) \simeq C(A, \neg(B \otimes_C C))$ *and satisfying the commutative diagram with associators* $\mathrm{Ass}^{\otimes_C}_{A,B,C} : A \otimes_C (B \otimes_C C) \longrightarrow (A \otimes_C B) \otimes_C C$*:*

$$\begin{array}{ccccc} C((A \otimes_C B) \otimes_C C, \neg D) & \xrightarrow{\varphi_{A\otimes_C B,C,D}} & C(A \otimes_C B, \neg(C \otimes_C D)) & \xrightarrow{\varphi_{A,B,C\otimes_C D}} & C\big(A, \neg[B \otimes_C (C \otimes_C D)]\big) \\ \downarrow{\scriptstyle C(\mathrm{Ass}^{\otimes_C}_{A,B,C}, \neg D)} & & & & \uparrow{\scriptstyle C(A, \neg \mathrm{Ass}^{\otimes_C}_{B,C,D})} \\ C(A \otimes_C (B \otimes_C C), \neg D) & & \xrightarrow{\varphi_{A,B\otimes C,D}} & & C\big(A, \neg[(B \otimes_C C) \otimes_C D]\big) \end{array} \quad (2)$$

Tabareau shows in his theses [T, Prop 2.9] that if the continuation monad $\neg\neg$ of a dialogue category is commutative and idempotent then, the continuation category is $*$-autonomous. Actually, according to a result attributed to Hasegawa [MT], for which we don't have a published reference, it seems that idempotency and commutativity are equivalent in the above situation. This would simplify our developments since, as discussed in the introduction, we will choose our duality functor to ensure idempotency, but we don't use this second result in the sequel.

Hence, dialogue categories are a source of $*$-autonomous categories by the following result:

2.13. Lemma. *Let* $(C^{\mathrm{op}}, \parr_C, I, \neg)$ *be a dialogue category with a commutative and idempotent continuation monad and* $\mathcal{D} \subset C$ *the full subcategory of objects of the form* $\neg C, C \in C$*. Then* $\mathcal{D}$ *is equivalent to the Kleisli category of the comonad* $T := \neg\neg$ *in* C*. If we define* $\cdot \otimes_{\mathcal{D}} \cdot := \neg(\neg(\cdot) \parr_C \neg(\cdot))$*, then* $(\mathcal{D}, \otimes_{\mathcal{D}}, I, \neg)$ *is a* $*$*-autonomous category and* $\neg : C^{\mathrm{op}} \longrightarrow \mathcal{D}$ *is strongly monoidal.*

Proof. From the already quoted unpublished result of Hasegawa [T, Prop 2.9], the cited Kleisli category (or Continuation category $C^{\neg}$) is a $*$-autonomous category since we start from a dialogue category with commutative and idempotent continuation monad. Consider $\neg : C^{\neg} \longrightarrow \mathcal{D}$. We have $\mathcal{D}(\neg A, \neg B) = C^{op}(\neg B, \neg A) = C^{\neg}(A, B)$ which gives that $\neg$ is fully faithful on the continuation category. The map $\neg : \mathcal{D} \longrightarrow C^{\neg}$ is the strong inverse of the equivalence since $\neg \circ \neg \simeq \mathrm{Id}_{\mathcal{D}}$ by choice of $\mathcal{D}$, and idempotency of the continuation and the canonical map $J_{\neg A} \in C^{\neg}(\neg\neg(A), \mathrm{Id}_{C^{\neg}}(A)) = C^{\mathrm{op}}(\neg A, \neg(T(a))$ is indeed natural in A and it is an isomorphism in $C^{\neg}$. Therefore we have a strong equivalence. Recall that the commutative strength $t_{A,B} : A \parr_C T(B) \longrightarrow T(A \parr_C B), t'_{A,B} : T(A) \parr_C B \longrightarrow T(A \otimes_C B)$ in C^{op}, implies that we have isomorphisms

$$I_{A,B} := \neg\Big(T(t'_{A,B}) \circ t_{T(A),B}\Big) \circ J^{2op}_{\neg(A \parr_C B)} : \neg(A \parr_C B) \simeq \neg(\neg\neg A \parr_C B) \simeq \neg(\neg\neg A \parr_C \neg\neg B)$$

with commutation relations $I_{A,B} := \neg\Big(T(t_{A,B}) \circ t'_{A,T(B)}\Big) \circ J^{2op}_{\neg(A \parr_C B)}$. This gives in $\mathcal{D}$ the compatibility map for the strong monad: $\mu_{A,B} := I^{op-1}_{A,B} : \neg(A \parr_C B) \simeq \neg A \otimes_{\mathcal{D}} \neg B$. Checking the associativity and unitarity for this map is a tedious computation left to the reader using axioms of strengths, commutativity, functoriality. This concludes. ∎

2.14. A model of MALL related to the Arens dual and the Schwartz ε-product. We introduce first a $*$-autonomous category that captures categorically the part of [S] that does not use quasi-completeness. Since bornological and topological concepts are dual to one another, it is natural to fix a saturated bornology on E in order to create a self-dual concept. Then, if one wants every object to be a dual object as in a $*$-autonomous category, one must consider only bornologies that can arise as the natural bornology on the dual, namely, the equicontinuous bornology. We could take a precompactness condition to ensure that, but to be able to use the Arens dual and ε-product (and not the polar topology and Meise's variant of the ε-product), we use instead a compactness condition. A weak-compactness condition would work for the self-duality requirement by Mackey Theorem but not for dealing with tensor products.

We will thus use a (saturated, topological) variant of the notion of compactology used in [Ja, p 157]. We say that a saturated bornology B_E on a lcs E is a *compactology* if it consists of relatively compact sets. Hence, the bipolar of each bounded set for this bornology is an absolutely convex compact set in E, and it is bounded for this bornology. A separated locally convex space with a compactology will be called a *compactological locally convex space*.

2.15. Definition. *Let* **LCS** *be the category of separated locally convex spaces with continuous linear maps and* **CLCS** *the category of compactological locally convex spaces, with maps given by bounded continuous linear maps. For $E, F \in$* **CLCS** *the internal Hom $L_b(E, F)$ is the above set of maps given the topology of uniform convergence on the bornology of E and the bornology of equibounded equicontinuous sets. Let $E'_b := L_b(E, \mathbb{K})$ (its bornology is merely the equicontinuous bornology, see step 1 of next proof). The algebraic tensor product $E \otimes_H F$ is $E \otimes_{B_E, B_F} F$ as lcs with the notation of definition 2.7, and with the bornology generated by bipolars of sets $A \otimes C$ for $A \in B_E, C \in B_F$.*

2.16. Remark. Recall that we only consider saturated bornologies as explained after definition 2.4. In order to make the tensor product map bounded, we want $A \otimes C$ bounded if A is bounded and C is bounded. Hence, we have considered on the tensor product the saturated bornology generated by this type of tensor products of bounded sets. Equivalently, we consider the bornology generated by bipolars of those sets, as in the previous definition. Note that we don't claim that $E \otimes_H F$ is in **CLCS**, it may not be. Recall that the tensor product we took $E \otimes_{B_E,B_F} F$ as a universal property for B_E, B_F-hypocontinuous bilinear maps. Hence, instead of fixing a functorial bornological space associated to the topology as in the five first points of definition 2.7, we consider here an hypocontinuity condition depending on the extra bornology we assumed given for our spaces in **CLCS**.

Note that compositions of bounded continuous linear maps are of the same type, hence **CLCS** is indeed a category.

Recall also that **LCS** is complete and cocomplete since it has small products and coproducts, kernels and cokernels (given by the quotient by the closure $\overline{Im[f-g]}$) [K, §18.3.(1,2,5), 18.5.(1)].

2.17. Theorem. **CLCS** *is a complete and cocomplete $*$-autonomous category with dualizing object $\mathbb{K}$ and internal Hom $L_b(E,F)$.*

1. *The functor $(.)'_c : \mathbf{LCS} \longrightarrow \mathbf{CLCS}^{op}$ giving the Arens dual the equicontinuous bornology, is right adjoint to $U((.)'_b)$, with U the underlying lcs and $U((.)'_b) \circ (.)'_c = \mathrm{Id}_{\mathbf{LCS}}$. The functor $(.)'_\sigma : \mathbf{LCS} \longrightarrow \mathbf{CLCS}^{op}$ giving the weak dual the equicontinuous bornology, is left adjoint to $U((.)'_b)$ and $U((.)'_b) \circ (.)'_\sigma = \mathrm{Id}_{\mathbf{LCS}}$.*
2. *The functor $U : \mathbf{CLCS} \longrightarrow \mathbf{LCS}$ is left adjoint and also left inverse to $(.)_c$, the functor $E \mapsto E_c$ the space with the same topology and the absolutely convex compact bornology. U is right adjoint to $(.)_\sigma$, the functor $E \mapsto E_\sigma$ the space with the same topology and the saturated bornology generated by finite sets. $U, (.)_c, (.)_\sigma$ are faithful.*
3. *The ε-product in **LCS** is given by $E\varepsilon F := U(E_c \parr_b F_c)$ with $G \parr_b H = L_b(G'_b, H)$ and of course the Arens dual by $U((E_c)'_b)$, and more generally $L_c(E,F) = U(L_b(E_c, F_c))$. The inductive tensor product $E \otimes_i F = U(E_\sigma \otimes_b F_\sigma)$ with $G \otimes_b H = (G'_b \parr_b H'_b)'_b$ and of course the weak dual is $U((E_\sigma)'_b)$.*

Proof. **Step 1:** Internal Hom functor L_b.

We first need to check that the equibounded equicontinuous bornology on $L_b(E,F)$ is made of relatively compact sets when $E, F \in \mathbf{CLCS}$. In the case $F = \mathbb{K}$, the bornology is the equicontinuous bornology since an equicontinuous set is equibounded for von Neumann bornologies [K2, §39.3.(1)]. Our claimed statement is then explained in [S, note 4 p 16] since it is proved there that every equicontinuous closed absolutely convex set is compact in $(U(E))'_c$ and our assumption that the saturated bornology is made of relatively compact sets implies there is a continuous map $E'_c \longrightarrow E'_b$. This proves the case $F = \mathbb{K}$.

Note that by definition, $G = L_b(E, F)$ identifies with the dual $H := (E \otimes_H F'_b)'_b$. Indeed, the choice of bornologies implies the topology of H is the topology of uniform convergence on equicontinuous sets of F' and on bounded sets of E which is the topology of G. An equicontinuous set in H is known to be an equihypocontinuous set [S2, p 10], i.e. a set taking a bounded set in E and giving an equicontinuous set in $(F'_b)'$, namely a bounded set in F, hence the equibounded condition, and taking symmetrically a bounded set in F'_b i.e. an equicontinuous set and sending it to an equicontinuous set in E', hence the equicontinuity condition [K2, §39.3.(4)].

Let $E\widehat{\otimes}_H F'_b \subset E\widetilde{\otimes}_H F'_b$ the subset of the completion obtained by taking the union of bipolars of bounded sets. It is easy to see that this is a vector subspace on which we put the induced topology. One deduces that $H \simeq (E\widehat{\otimes}_H F'_b)'_b$ where the $E\widehat{\otimes} F'_b$ is given the bornology generated by bipolars of bounded sets (which covers it by our choice of subspace). Indeed the completion does not change the dual and the equicontinuous sets herein [K, §21.4.(5)] and the extension to bipolars does not change the topology on the dual either. But in $E\widehat{\otimes}_H F'_b$, bounded sets for the above bornology are included into bipolars of tensor product of bounded sets. Let us recall why tensor products $A \otimes B$ of such bounded sets are precompact in $E \otimes_H F'_b$ (hence also in $E\widehat{\otimes}_H F'_b$ by [K, §15.6.(7)]) if $E, F \in$ **CLCS**. Take U' (resp. U) a neighbourhood of 0 in it (resp. such that $U + U \subset U'$), by definition there is a neighbourhood V (resp. W) of 0 in E (resp. F'_b) such that $V \otimes B \subset U$ (resp. $A \otimes W \subset U$). Since A, B are relatively compact hence precompact, cover $A \subset \cup_i x_i + V$, $x_i \in A$ (resp. $B \subset \cup_j y_j + W$, $y_j \in B$) so that one obtains the finite cover, thus giving total boundedness:

$$A \otimes B \subset \cup_i x_i \otimes B + V \otimes B \subset \cup_{i,j} x_i \otimes y_j + x_i \otimes W + V \otimes B \subset \cup_{i,j} x_i \otimes y_j + U + U \subset \cup_{i,j} x_i \otimes y_j + U'.$$

Note that we used strong compactness here in order to exploit hypocontinuity, and weak compactness and the definition of Jarchow for compactologies wouldn't work with our argument.

Thus from hypocontinuity, we deduced the canonical map $E \times F'_b \longrightarrow E\widehat{\otimes}_H F'_b$ send $A \times B$ to a precompact (using [K, §5.6.(2)]), hence its bipolar is complete (since we took the bipolar in the completion which is closed there) and precompact [K, §20.6.(2)] hence compact (by definition [K, §5.6]). Thus $E\widehat{\otimes}_H F'_b \in$ **CLCS**, if $E, F \in$ **CLCS**. From the first case for the dual, one deduces $L_b(E, F) \in$ **CLCS** in this case. Moreover, once the next step obtained, we will know $E\widehat{\otimes}_H F'_b \simeq E \otimes_b F'_b$.

Step 2: CLCS as closed category.

It is well known that Vect, the category of vector spaces, is a symmetric monoidal category and furthermore a closed category in the sense of [EK]. **CLCS** $\subset$ Vect is a (far from being full) subcategory, but we see that we can induce maps on our smaller internal Hom. Indeed, the linear map $L^E_{FG} : L_b(F, G) \longrightarrow L_b(L_b(E, F), L_b(E, G))$ is well defined since a bounded family in $L_b(F, G)$ is equibounded, hence it sends an equibounded set in $L_b(E, F)$ to an equibounded set in $L_b(E,G)$, and also equicontinuous, hence its transpose sends an equicontinuous set in $(L_b(E,G))'$ (described as bipolars of bounded sets in E tensored with equicontinuous sets in G') to an equicontinuous set in $(L_b(E,F))'$. This reasoning implies L^E_{FG} is indeed valued in continuous equibounded maps and even bounded with our choice of bornologies. Moreover we claim L^E_{FG} is continuous. Indeed, an equicontinuous set in $(L_b(L_b(E,F),L_b(E,G)))'$ is generated by the bipolar of an equicontinuous set C in G', a bounded set B in E and an equibounded set A in $(L_b(E,F))$ and the transpose consider $A(B) \subset F$ and C to generate a bipolar which is indeed equicontinuous in $(L_b(F,G))'$. Hence, L^E_{FG} is a map of our category. Similarly, the morphism giving identity maps $j_E : \mathbb{K} \longrightarrow L_b(E,E)$ is indeed valued in the smaller space and the canonical $i_E : E \longrightarrow L_b(\mathbb{K},E)$ indeed sends a bounded set to an equibounded equicontinuous set and is tautologically equicontinuous. Now all the relations for a closed category are induced from those in Vect by restriction. The naturality conditions are easy.

Step 3: The $*$-autonomous property.

First note that $L_b(E,F) \simeq L_b(F'_b,E'_b)$ by transposition. Indeed, the space of maps and their bornologies are the same since equicontinuity (resp. equiboundedness) $E \longrightarrow F$ is equivalent to equiboundedness (resp. equicontinuity) of the transpose $F'_b \longrightarrow E'_b$ for equicontinuous bornologies (resp. for topologies of uniform convergence of corresponding bounded sets). Moreover the topology is the same since it is the topology of uniform convergence on bounded sets of E (identical to equicontinuous sets of $(E'_b)'$) and equicontinuous sets of F' (identical to bounded sets for F'_b). Similarly $\mathscr{S}L_b(E,F) \simeq \mathscr{S}L_b(F'_b,E'_b)$ since on both sides one considers the bornology generated by Mackey-null sequences for the same bornology.

It remains to check $L_b(E,L_b(F,G)) \simeq L_b(F,L_b(E,G))$. The map is of course the canonical map. Equiboundedness in the first space means sending a bounded set in E and a bounded set in F to a bounded set in G and also a bounded set in E and an equicontinuous set in G' to an equicontinuous set in F'. This second condition is exactly equicontinuity $F \longrightarrow L_b(E,G)$. Finally, analogously, equicontinuity of the map $E \longrightarrow L_b(F,G)$ implies that it sends a bounded set in F and an equicontinuous set in G' to an equicontinuous set in E' which was the missing part of equiboundedness in $L_b(F,L_b(E,G))$. The identification of spaces and bornologies follows. Finally, the topology on both spaces is the topology of uniform convergence on products of bounded sets of E,F.

Again, the naturality conditions of the above two isomorphisms are easy, and the last one induces from Vect again the structure of a symmetric closed category, hence lemma 2.11 concludes to **CLCS** $*$-autonomous.

Step 4: Completeness and cocompleteness.

Let us describe first coproducts and cokernels. This is easy in **CLCS**, it is given by the colimit of separated locally convex spaces, given the corresponding final bornology. Explicitly, the coproduct is the direct sum of vector spaces with coproduct topology and the bornology is the one generated by finite sum of bounded sets, hence included in finite sums of compact sets which are compact [K, §15.6.(8)]. Hence the direct sum is in **CLCS** and clearly has the universal property from those of topological/bornological direct sums. For the cokernel of $f, g : E \longrightarrow F$, we take the coproduct in **LCS**, $\mathrm{Coker}(f, g) := F/\overline{(f-g)(E)}$ with the final bornology, i.e. the bornology generated by images of bounded sets. Since the quotient map is continuous between Hausdorff spaces, the image of a compact containing a bounded set is compact, hence $\mathrm{Coker}(f, g) \in$ **CLCS**. Again the universal property comes from the one in locally convex and bornological spaces. Completeness then follows from the $*$-autonomous property since one can see $\lim_i E_i := (\mathrm{colim}_i (E_i)'_b)'_b$ gives a limit.

Step 5: Adjunctions and consequences.

The fact that the stated maps are functors is easy. We start with the adjunction for U in (2): $\mathbf{LCS}(U(F), E) = L_b(F, E_c) = \mathbf{CLCS}(F, E_c)$ since the extra condition of boundedness beyond continuity is implied by the fact that a bounded set in F is contained in an absolutely convex compact set which is sent to the same type of set by a continuous linear map. Similarly, $\mathbf{LCS}(E, U(F)) = L_b(E_\sigma, F) = \mathbf{CLCS}(E_\sigma, F)$ since the image of a finite set is always in any bornology (which must cover E and is stable under union), hence the equiboundedness is also automatic.

Moreover, for the adjunction in (1), we have the equality as set (using involutivity and functoriality of $(.)'_b$ and the previous adjunction):

$$\mathbf{CLCS}^{\mathrm{op}}(F, E'_c) = L_b((E_c)'_b, F) \simeq L_b(F'_b, E_c) = \mathbf{LCS}(U(F'_b), E),$$

$$\mathbf{CLCS}^{\mathrm{op}}(E'_\sigma, F) = L_b(F, (E_\sigma)'_b) \simeq L_b(E_\sigma, F'_b) = \mathbf{LCS}(E, U(F'_b)).$$

The other claimed identities are obvious by definition. ∎

The second named author explored in [Ker] models of linear logic using the positive tensor product $\otimes_i$ and $(.)'_\sigma$. We will use in this work the negative multiplicative disjunction ε and the Arens dual $(.)'_c$ which appeared with a dual role in the previous result. Let us summarize the properties obtained in [S] that are consequences of our categorical framework.

2.18. Corollary.

1. *Let $E_i \in$ **LCS**$, i \in I$. The iterated ϵ-product is $\varepsilon_{i\in I} E_i := U(\parr_{b,i\in I}(E_i)_c)$, it is symmetric in its arguments and commutes with limits.*
2. *There is a continuous injection $(E_1 \varepsilon E_2 \varepsilon E_3) \longrightarrow E_1 \varepsilon (E_2 \varepsilon E_3)$.*
3. *For any continuous linear map $f : F_1 \longrightarrow E_1$ (resp. continuous injection, closed embedding), so is $f \varepsilon Id : F_1 \varepsilon E_2 \longrightarrow E_1 \varepsilon E_2$.*

Note that (3) is also valid for non-closed embeddings and (2) is also an embedding [S], but this is not a categorical consequence of our setting.

Proof. The equality in (1) is a reformulation of definitions, symmetry is an obvious consequence. Commutation with limits come from the fact that $U, ()_c$ are right adjoints and $\parr_b$ commutes with limits from universal properties.

Using associativity of $\parr_b$: $E_1\varepsilon(E_2\varepsilon E_3) = U((E_1)_c \parr_b [U((E_2)_c \parr_b (E_3)_c)]_c)$ hence functoriality and the natural transformation coming from adjunction $Id \longrightarrow (U(\cdot))_c$ concludes to the continuous map in (2). It is moreover a monomorphism since $E \longrightarrow (U(E))_c$ is one since $U(E) \longrightarrow U((U(E))_c)$ is identity and U reflects monomorphisms and one can use the argument for (3).

For (3) functoriality gives definition of the map, and recall that closed embeddings in **LCS** are merely regular monomorphisms, hence a limit, explaining its commutation by (1). If f is a monomorphism, in the categorical sense, so is $U(f)$ using a right inverse for U and so is $(f)_c$ since $U((f)_c) = f$ and U reflects monomorphisms as a faithful functor. Hence it suffices to see $\parr_b$ preserves monomorphisms but $g_1, g_2 : X \longrightarrow E \parr_b F$ correspond by cartesian closedness to maps $X \otimes_b F'_b \longrightarrow E$ that are equal when composed with $f : E \longrightarrow G$ if f monomorphism, hence so is $f \parr_b \mathrm{Id}_F$. ■

In general, we have just seen that ε has features for a negative connective as $\parr$, but it lacks associativity. We will have to work to recover a monoidal category, and then models of LL. In that respect, we want to make our fix of associativity compatible with a class of smooth maps, this will be the second leitmotiv. We don't know if there is an extension of the model of MALL given by **CLCS** into a model of LL using a type of smooth maps.

3. The original setting for the Schwartz ε-product and smooth maps.

In his original paper [S], Schwartz used quasi-completeness as his basic assumption to ensure associativity, instead of restricting to Schwartz spaces and assuming only Mackey-completeness as we will do soon, or more generally to $\mathscr{C}$-complete $\mathscr{C}$-spaces. Actually, what is really needed is that the absolutely convex cover of a compact set is still compact. Indeed, as soon as one takes the image (even of an absolutely convex) compact set by a continuous bilinear map, one gets only what we know from continuity, namely compactness and the need to recover absolutely convex sets, for compatibility with the vector space structure, thus makes the above assumption natural. Since this notion is related to compactness and continuity, we call it *k-quasi-completeness*.

This small remark reveals that this notion is also relevant for differentiability since it is necessarily based on some notion of continuity, at least at some level, even if this is only on $\mathbb{R}^n$, as in convenient smoothness. Avoiding the technical use of Schwartz spaces for now and benefiting from [S], we find a $*$-autonomous category and an adapted notion of smooth maps.

We will see that this will give us a strong notion of differentiability with cartesian closedness. We will come back to convenient smoothness in the next sections starting from what we will learn in this basic example with a stronger notion of smoothness.

3.1. k-quasi-complete and k-reflexive spaces.

3.2. Definition. *A (separated) locally convex space E is said to be* k-quasi-complete*, if for any compact set $K \subset E$, its closed absolutely convex cover $\overline{\Gamma(K)}$ is complete (equivalently compact [K, §20.6.(3)]). We denote by* **Kc** *the category of k-quasi-complete spaces and linear continuous maps.*

3.3. Remark. There is a k-quasi-complete space which is not quasi-complete, hence our new notion of k-quasi-completeness does not reduce to the usual notion. Indeed in [V], one finds a completely regular topological space W such that $C^0(W)$ with the compact-open topology is bornological and such that it is a hyperplane in its completion but this completion is not bornological. If $C^0(W)$ were quasi-complete, it would be complete by [Ja, Corol 3.6.5] and this is not the case. $C^0(W)$ is k-quasi-complete since by the Ascoli Theorem [Bo, X.17 Thm 2] a compact set for the compact open topology is pointwise bounded and equicontinuous, hence so is the absolutely closed convex cover of such a set, which is thus compact too.

3.4. Lemma. *The intersection $\widehat{E}^K$ of all k-quasi-complete spaces containing E and contained in the completion $\tilde{E}$ of E, is k-quasi-complete and called the k-quasi-completion of E.*

We define $E_0 := E$, and for any ordinal λ, the subspace $E_{\lambda+1} := \cup_{K \in C(E_\lambda)} \overline{\Gamma(K)} \subset \tilde{E}$ where the union runs over all compact subsets $C(E_\lambda)$ of E_λ with the induced topology, and the closure is taken in the completion. We also let for any limit ordinal $E_\lambda := \cup_{\mu<\lambda} E_\mu$. Then for any ordinal λ, $E_\lambda \subset \widehat{E}^K$ and eventually for λ large enough, we have equality.

Proof. The first statement comes from stability of k-quasi-completeness under intersection. Indeed, in any k-quasi-complete subspace $F \subset \widetilde{E}$ the closed absolutely convex cover $\overline{\Gamma(K)}^F$ is compact hence closed in $\widetilde{E}$ and dense in the corresponding closure of the completion, and thus it coincides with $\overline{\Gamma(K)}^{\widetilde{E}}$. Hence, in case of an intersection, if K is in the intersection, $\overline{\Gamma(K)}$ can be computed in any space of the intersection, and hence this common compact set is also compact in the intersection.

It is easy to see that E_λ is a subspace. Moreover if at some λ, $E_{\lambda+1} := E_\lambda$, then by definition, E_λ is k-quasi-complete and then the ordinal sequence is eventually constant. Since for cardinal reasons, the ordinal sequence is necessarily eventually constant (for instance starting at a regular cardinal λ larger than the cardinal of $\widetilde{E}$), there is necessarily such a λ_0 and then $\widehat{E}^K \subset E_{\lambda_0}$.

Let us see that we have the converse $E_\lambda \subset \widehat{E}^K$, for any λ. We proceed by transfinite induction. This is clear for $\lambda = 0$ and unions. But if true at stage λ, a compact $K \subset E_\lambda \subset \widehat{E}^K$ must have $\overline{\Gamma(K)}$ computed in $\widehat{E}^K$ coincide with the closure in the completion, hence this closure that we add in $E_{\lambda+1}$ is also in $\widehat{E}^K$.

■

3.5. Definition. *For a (separated) locally convex space E, the topology $k(E', E)$ on E' is the topology of uniform convergence on absolutely convex compact sets of $\widehat{E}^K$. The dual*

$$(E', k(E', E)) := (\widehat{E}^K)'_c$$

is nothing but the Arens dual of the k-quasi-completion and is written E'_k. We let $E^*_k := \widehat{E'_k}^K$. A (separated) locally convex space E is said k-reflexive *if E is k-quasi-complete and if $E = (E'_k)'_k$ topologically. Their category is written* **k**-**Ref**.

From the Mackey theorem, we know that $|(E'_k)'| = |(E^*_k)'| = |\widehat{E}^k|$.

We first want to check that **k-Ref** is logically relevant. We show that $(E'_k)'_k$ and E^*_k are always k-reflexive. Hence we will get a k-reflexivization functor. This is the first extension of the relation $E'_c \simeq ((E'_c)'_c)'_c$ that we need.

We start by proving a general lemma we will reuse several times. Of course to get a $*$-autonomous category, we will need some stability of our notions of completion by dual. The following lemma says that if a completion can be decomposed by an increasing ordinal decomposition as above and that for each step the duality we consider is sufficiently compatible in terms of its equicontinuous sets, then the process of completion in the dual does not alter any type of completeness in the original space.

3.6. Lemma. *Let D a contravariant duality functor on* **LCS**, *in the sense that algebraically $|D(E)| = |E'|$. We assume it is compatible with duality ($|(D(E))'| = |E|$). Let $E_0 \subset E_\lambda \subset \widetilde{E_0}$ an increasing family of subspaces of the completion $\widetilde{E_0}$ indexed by ordinals $\lambda \leq \lambda_0$. We assume that for limit ordinals $E_\lambda = \cup_{\mu<\lambda} E_\mu$ and, at successor ordinals that every point $x \in E_{\lambda+1}$ lies in $\overline{\Gamma(L)}$, for a set $L \subset E_\lambda$, equicontinuous in $[D(E_{\lambda_0})]'$.*

Then any complete set K in $D(E_0)$ is also complete for the stronger topology of $D(E_{\lambda_0})$.

Proof. Let $E := E_0$. Note that since $|D(E)| = |D(\widetilde{E})|$ we have $|D(E)| = |D(E_\lambda)|$ algebraically.

Take a net $x_n \in K$ which is a Cauchy net in $D(E_{\lambda_0})$. Thus $x_n \longrightarrow x \in K$ in $D(E_0)$. We show by transfinite induction on λ that $x_n \longrightarrow x$ in $D(E_\lambda)$.

First take λ a limit ordinal. The continuous embeddings $E_\mu \longrightarrow E_\lambda$ gives by functoriality a continuous identity map $D(E_\lambda) \longrightarrow D(E_\mu)$ for any $\mu < \lambda$. Therefore since we know $x_n \longrightarrow x$ in any $D(E_\mu)$ the convergence takes place in the projective limit $D_\lambda := \operatorname{proj\,lim}_{\mu<\lambda} D(E_\mu)$.

But we have a continuous identity map $D(E_\lambda) \longrightarrow D_\lambda$ and both spaces have the same dual $E_\lambda = \cup_{\mu<\lambda} E_\mu$. For any equicontinuous set L in $(D(E_\lambda))'$ x_n is Cauchy and thus converges uniformly in $C^0(L)$ on the Banach space of weakly continuous maps. It moreover converges pointwise to x, thus we have uniform convergence to x on any equicontinuous set i.e. $x_n \longrightarrow x$ in $D(E_\lambda)$.

Let us prove convergence in $D(E_{\lambda+1})$ assuming it in $D(E_\lambda)$. Take an absolutely convex closed equicontinuous set L in $(D(E_{\lambda+1}))' = E_{\lambda+1}$, we have to show uniform convergence on any such equicontinuous set. Since L is weakly compact, one can look at the Banach space of weakly continuous functions $C^0(L)$. Let $\iota_L : D(E_{\lambda+1}) \longrightarrow C^0(L)$. $\iota_L(x_n)$ is Cauchy by assumption and therefore converges uniformly to some y_L. We want to show $y_L(z) = \iota_L(x)(z)$ for any $z \in L$. Since $z \in E_{\lambda+1}$ there is by assumption a set $M \subset E_\lambda$ equicontinuous in $[D(E_{\lambda_0})]'$ such that $z \in \overline{\Gamma(M)}$ computed in $E_{\lambda+1}$. Let $N = \overline{\Gamma(M)}$ computed in E_{λ_0}, so that $z \in N$. Since M is equicontinuous in $(D(E_{\lambda_0}))'$ we conclude that so is N and it is also weakly compact there. One can apply the previous reasoning to N instead of L (since x_n Cauchy in $D(E_{\lambda_0})$, not only in $D(E_{\lambda+1})$). $\iota_N(x_n) \longrightarrow y_N$ and since $z \in L \cap N$ and using pointwise convergence $y_L(z) = y_N(z)$. Note also $\iota_N(x)(z) = \iota_L(x)(z)$. Moreover, for $m \in M \subset E_\lambda$, $\iota_N(x_n)(m) \longrightarrow \iota_N(x)(m)$ since $\{m\}$ is always equicontinuous in $(D(E_\lambda))'$ so that $\iota_N(x)(m) = y_N(m)$. Since both sides are affine on the convex N and weakly continuous (for $\iota_N(x)$ since $x \in D(E_{\lambda_0}) = E'_{\lambda_0}$), we extend the relation to any $m \in N$ and thus $\iota_N(x)(z) = y_N(z)$. Altogether, this gives the expected $y_L(z) = \iota_L(x)(z)$. Thus K is complete as expected. ∎

3.7. Lemma. *For any separated locally convex space, $E_k^* = ((E_k')_k')_k'$ is k-reflexive. A space is k-reflexive if and only if $E \simeq (E_c')_c'$ and both E and E_c' are k-quasi-complete. More generally, if E is k-quasi-complete, so are $(E_k')_c' \simeq (E_c')_c'$ and $(E_k^*)_c'$ and $\gamma(E) = \gamma((E_c')_c') = \gamma((E_k^*)_k^*)$.*

3.8. Remark. The example $E = C^0(W)$ in Remark 3.3, which is not quasi-complete, is even k-reflexive. Indeed it remains to see that E_c' is k-quasi-complete. But from [Ja, Thm 13.6.1], it is not only bornological but ultrabornological, hence by [Ja, Corol 13.2.6], E_μ' is complete (and so is $F = \mathscr{S}(E_\mu')$.) But for a compact set in E_c', the closed absolutely convex cover is closed in E_c', hence E_μ', hence complete there. Thus, by Krein's Theorem [K, §24.5.(4)], it is compact in E_c', making E_c' k-quasi-complete.

Proof. One can assume E is k-quasi-complete (all functors start by this completion) thus so is $(E_c')_c'$ by [Bo2, IV.5 Rmq 2] since $(E_c')_c' \longrightarrow E$ continuous with same dual (see [S]). There is a continuous map $(E_k^*)_c' \longrightarrow (E_c')_c'$ we apply lemma 3.6 to $E_0 = E_c'$, E_λ the λ-th step of the completion in lemma 3.4. Any $\overline{\Gamma(K)}$ in the union defining $E_{\lambda+1}$ is equicontinuous in $((E_{\lambda+1})_c')'$ so a fortiori in $((E_{\lambda_0})_c')'$ for λ_0 large enough. We apply the lemma to another K closed absolutely convex cover of a compact set of $(E_k^*)_c'$ computed in $(E_c')_c'$ therefore compact there by assumption. The lemma gives K is complete and therefore contains the bipolar of the compact computed in $(E_k^*)_c'$ which must also be compact as a closed subset of a compact. In this case we have deduced that $(E_k^*)_c' = (E_k')_k'$ is k-quasi-complete.

Clearly $((E_k')_k')_k' = ((E_k^*)_k')_k' \longrightarrow E_k^*$ continuous. Dualizing the continuous $(E_k')_k' \longrightarrow E$, one gets $E_k' \longrightarrow ((E_k')_k')_k' = ((E_k^*)_k')_k' \longrightarrow E_k^*$ and since the space in the middle is already k-quasi-complete inside E_k^* which is the k-quasi-completion, it must be equal to E_k^* and thus E_k^* is k-reflexive and we have the stated equality.

For the next-to-last statement, sufficiency is clear, we have already noted in the k-quasi-complete case the continuous map $(E_k')_k' = (E_k^*)_c' \longrightarrow (E_c')_c' \longrightarrow E$ which implies $(E_c')_c' \simeq E$ under our assumption of k-reflexivity $(E_k')_k' \simeq E$ and $E_k^* = ((E_k')_k')_k' = ((E_k')_k')_c' = E_c'$ implies this space is also k-quasi-complete. For the comparison of absolutely convex compact sets, note that $(E_k^*)_k^* \longrightarrow (E_c')_c'$ ensures one implication and if $K \in \gamma((E_c')_c')$ we know it is equicontinuous in $(E_c')'$ hence [K, §21.4.(5)] equicontinuous in $(\widehat{E_c'}^K)'$ and as a consequence included in an absolutely convex compact in $(\widehat{E_c'}^K)_c' = (E_k^*)_k^*$, i.e. $K \in \gamma((E_k^*)_k^*)$. $\gamma(E) = \gamma((E_c')_c')$ is a reformulation of $E_c' \simeq ((E_c')_c')_c'$. ∎

We consider γ-**Kc** the full subcategory of **Kc** with their γ-topology, and γ-**Kb** the full subcategory of **LCS** made of spaces of the form E_c' with E k-quasi-complete.

We first summarize the results of [S]. We call γ-**LCS** $\subset$ **LCS** the full subcategory of spaces having their γ-topology, namely $E = (E_c')_c'$. This is equivalent to saying that subsets of absolutely convex compact sets in E_c' are (or equivalently are exactly the) equicontinuous subsets in E'. With the notation of Theorem 2.17, this can be reformulated by an intertwining relation $(\cdot)_c \circ (\cdot)_c' = (\cdot)_b' \circ (\cdot)_c$ in **CLCS** which explains the usefulness of these spaces:

$$E \in \gamma\text{-}\mathbf{LCS} \Leftrightarrow (E_c')_c = (E_c)_b' \Leftrightarrow (E_c)_b' = [U((E_c)_b')]_c \tag{3}$$

3.9. Proposition. *k-quasi-complete spaces are stable by ε-product, and* $(\mathbf{Kc}, \varepsilon, \mathbb{K})$ *form a symmetric monoidal category . Moreover, if E, F are k-quasi-complete, a set in $E\varepsilon F$ is relatively compact if and only if it is ε-equihypocontinuous. Therefore we have canonical embeddings:*

$$E'_c \otimes_{\beta e} F'_c \longrightarrow (E\varepsilon F)'_c \longrightarrow E'_c \widehat{\otimes}^K_{\beta e} F'_c.$$

The reader should note that $(\mathbf{Kc}, \varepsilon, \mathbb{K})$ is not a closed category. Duals, a specific case of Hom spaces are usually not k-quasi-complete (see 5.8 for an example of dual having almost no completeness at all). This is the reason of introducing k-reflexive spaces. We will soon see that this is the only correction needed : k-reflexive spaces form a $*$-autonomous category.

Proof. The characterization of relatively compact sets is [S, Prop 2], where it is noted that the direction proving relative compactness does not use any quasi-completeness. It gives $(E\varepsilon F)'_c = (E\varepsilon F)'_\epsilon$ with the epsilon topology as a bidual of $E'_c \otimes_{\beta e} F'_c$ and in general anyway a continuous linear map:

$$(E\varepsilon F)'_c \longrightarrow (E\varepsilon F)'_\epsilon \tag{4}$$

For a compact part in $E\varepsilon F$, hence equicontinuous in $(E'_c \otimes_{\beta e} F'_c)'$, its bipolar is still ε-equihypocontinuous hence compact by the characterization, as we have just explained. This gives stability of k-quasi-completeness.

Associativity of ε is Schwartz' Prop 7 but we give a reformulation giving a more detailed proof that $(\mathbf{Kc}, \varepsilon)$ is symmetric monoidal. The restriction to $\mathbf{Kc}$ of the functor $(\cdot)_c$ of Theorem 2.17 gives a functor we still call $(\cdot)_c : \mathbf{Kc} \longrightarrow \mathbf{CLCS}$. It has left adjoint $\hat{\cdot}^K \circ U$. Note that for $E, F \in \mathbf{Kc}$, $E\varepsilon F = \hat{\cdot}^K \circ U(E_c \parr_b F_c)$ from our previous stability of $\mathbf{Kc}$. Moreover, note that

$$\forall E, F \in \mathbf{Kc}, \quad (E\varepsilon F)_c = E_c \parr_b F_c \tag{5}$$

thanks to the characterization of relatively compact sets, since the two spaces were already known to have same topology and the bornology on the right was defined as the equicontinuous bornology of $(E'_c \otimes_{\beta e} F'_c)'$ and on the left the one generated by absolutely convex compact sets or equivalently the saturated bornology generated by compact sets (using $E\varepsilon F \in \mathbf{Kc}$). Lemma 3.11 concludes to $(\mathbf{Kc}, \varepsilon, \mathbb{K})$ symmetric monoidal. They also make $(\cdot)_c$ a strong monoidal functor.

We could deduce from [S] the embeddings, but we prefer seeing them as coming from **CLCS**.

Let us apply the next lemma to the embedding of our statement. Note that by definition $E'_c \otimes_{\beta e} F'_c = U((E_c)'_b \otimes_H (F_c)'_b)$, and $(E\varepsilon F)'_\varepsilon = U((E_c \parr_b F_c)'_b) = U((E_c)'_b \otimes_b (F_c)'_b)$ so that we obtain the embeddings for $E, F \in \mathbf{LCS}$:

$$E'_c \otimes_{\beta e} F'_c \longrightarrow (E\varepsilon F)'_\varepsilon \longrightarrow E'_c \widehat{\otimes}^K_{\beta e} F'_c \tag{6}$$

which specializes to the statement in the k-quasi-complete case by the beginning of the proof to identify the middle terms. ∎

We have used and are going to reuse several times the following:

3.10. Lemma. *Let $E, F \in$ **CLCS**we have the topological embedding (for U the map giving the underlying lcs):*

$$U(E \otimes_H F'_b) \longrightarrow U(E \otimes_b F'_b) \longrightarrow [\hat{\cdot}^K \circ U](E \otimes_H F'_b). \tag{7}$$

Proof. Recall that for $E, F \in$ **CLCS**, $E \otimes_H F$ has been defined before the proof of Theorem 2.17 and is the algebraic tensor product. By construction we saw $(E \otimes_b F'_b)'_b = E'_b \parr_b F = L_b(E, F) = (E \otimes F'_b)'_b$, hence by $*$-autonomy $E \otimes_b F'_b = ((E \otimes_H F'_b)'_b)'_b = (L_b(E, F))'_b$ and it has been described as a subspace $E\widehat{\otimes}_H F'_b$ inside the completion (in step 1 of this proof) with induced topology, obtained as union of bipolars of $A \otimes B$ or $\overline{A} \otimes \overline{B}$ (image of the product), for A bounded in E, B bounded in F'_b. Hence the embeddings follow from the fact we checked $\overline{A} \otimes \overline{B}$ is precompact, and of course closed in the completion hence compact and the bipolar is one of those appearing in the first step of the inductive description of the k-quasicompletion. ■

We have also used the elementary categorical lemma:

3.11. Lemma. *Let $(C, \otimes_C, I)$ be a symmetric monoidal category and $\mathcal{D}$ a category. Consider a functor $R : \mathcal{D} \longrightarrow C$ with left adjoint $L : C \longrightarrow \mathcal{D}$ and define $J = L(I)$, and $E \otimes_{\mathcal{D}} F = L(R(E) \otimes_C R(F))$. Assume that for any $E, F \in \mathcal{D}$, $L(R(E)) = E$, $R(J) = I$ and*

$$R(E \otimes_{\mathcal{D}} F) = R(E) \otimes_C R(F).$$

Then, $(\mathcal{D}, \otimes_{\mathcal{D}}, J)$ is a symmetric monoidal category.

Proof. The associator is obtained as $\mathrm{Ass}^{\otimes_{\mathcal{D}}}_{E,F,G} = L(\mathrm{Ass}^{\otimes_C}_{R(E),R(F),R(G)})$ and the same intertwining defines the braiding and units and hence transports the relations which concludes the result. For instance in the pentagon we used the relation $L(\mathrm{Ass}^{\otimes_C}_{R(E),R(F)\otimes_C R(G),R(H)}) = \mathrm{Ass}^{\otimes_{\mathcal{D}}}_{E,F\otimes_{\mathcal{D}}G,H}$. ■

We deduce a description of internal hom-sets in these categories: we write $L_{co}(E, F)$, the space of all continuous linear maps from E to F endowed with the topology of uniform convergence on compact subsets of E. When E is a k-quasi-complete space, note this is the same lcs as $L_c(E, F)$, endowed with the topology of uniform convergence on absolutely convex compacts of E.

3.12. Corollary. *For $E \in \gamma$-**Kc** and $F \in$ **Kc** (resp. $F \in$ **Mc**), one has $L_c(E'_c, F) \simeq E\varepsilon F$, which is k-quasi-complete (resp. Mackey-complete).*

Proof. Algebraically, $E\varepsilon F = L(E'_c, F)$ and the first space is endowed with the topology of uniform convergence on equicontinuous sets in E'_c which coincides with subsets of absolutely convex compact sets since E has its γ-topology. ■

Using that for $E \in \gamma$-**Kb**, $E = F'_c$ for $F \in$ **Kc**, hence $E'_c = (F'_c)'_c \in$ **Kc** by lemma 3.7.

3.13. Corollary. *Consider $E \in \gamma$-**Kb**, $F \in$ **Kc** (resp. $F \in$ **Mc**) then $L_c(E,F)$ is k-quasi-complete (resp. Mackey-complete).*

3.14. Proposition. *γ-**Kc** $\subset$ **Kc** is a coreflective subcategory with coreflector$((\cdot)'_c)'_c$, which commutes with $\widehat{\cdot}^K$ on γ-**Kb**. For $F \in \gamma$-**Kc**, $\cdot \widehat{\otimes}^K_\gamma F'_c$: **LCS** $\longrightarrow$ **Kc** (resp. **Kc** $\longrightarrow$ **Kc**, γ-**Kc** $\longrightarrow \gamma$-**Kc**) is left adjoint to $F\varepsilon \cdot$ (resp. $F\varepsilon \cdot, ((F\varepsilon \cdot)'_c)'_c$). More generally, for $F \in$ **Kc**,$\cdot \widehat{\otimes}^K_{\gamma,\varepsilon} F'_c$: **LCS** $\longrightarrow$ **Kc** is left adjoint to $F\varepsilon\ \cdot$. Finally, γ-**Kb** is stable by $\widehat{\otimes}^K_\gamma$.*

Proof. (1) We start by proving the properties of the inclusion γ-**Kc** $\subset$ **Kc**. Let $E \in$ **Kc**. We know the continuous map $(E'_c)'_c \longrightarrow E$ and both spaces have the same dual, therefore for K compact in $(E'_c)'_c$ its closed absolutely convex cover is the same computed in both by the bipolar Thm [K, §20.7.(6) and 8.(5)] and it is complete in E by assumption so that by [Bo2, IV.5 Rmq 2] again also in $(E'_c)'_c$ which is thus k-quasi-complete too. Hence, by functoriality of Arens dual, we got a functor: $((\cdot)'_c)'_c$: **Kc** $\longrightarrow \gamma$-**Kc**. Then we deduce from functoriality the continuous inverse maps $L(F,E) \longrightarrow L((F'_c)'_c, (E'_c)'_c) = L(F, (E'_c)'_c) \longrightarrow L(F,E)$ (for $F \in \gamma$-**Kc**, $E \in$ **Kc**) which gives the first adjunction. The unit is $\eta = id$ and counit given by the continuous identity maps: $\varepsilon_E : ((E)'_c)'_c \longrightarrow E$.

(2) Let us turn to proving commutation with completion. For $H \in \gamma$-**Kb**, $H = G'_c = ((G'_c)'_c)'_c, G \in$ **Kc** we thus have to note that the canonical map $((\widehat{H}^K)'_c)'_c \longrightarrow \widehat{H}^K$ is inverse of the map obtained from canonical map $H \longrightarrow \widehat{H}^K$ by applying functoriality: $H \longrightarrow (\widehat{H}^K)'_c)'_c$ and then k-quasi-completion (since we saw the target is in γ-**Kc**:) $\widehat{H}^K \longrightarrow ((\widehat{H}^K)'_c)'_c$.

(3) For the adjunctions of tensor products, let us start with a heuristic computation. Fix $F \in \gamma - Kc, E \in$ **LCS**, $G \in$ **Kc**. From the discussion before (3), $L_\gamma(F'_c, G) \simeq F\varepsilon G$ thus, there is a canonical injection

$$\mathbf{Kc}(E\widehat{\otimes}^K_\gamma F'_c, G) = L(E \otimes_\gamma F'_c, G) \longrightarrow L(E, L_\gamma(F'_c, G)) = L(E, F\varepsilon G).$$

But an element in $L(E, F\varepsilon G)$ sends a compact set in E to a compact set in $F\varepsilon G$ therefore an ϵ-equihypocontinuous set by proposition 3.9 which is a fortiori an equicontinuous set in $L(F'_c, G)$. This gives the missing hypocontinuity to check the injection is onto.

Let us now give a more abstract alternative proof of the first adjunction. Fix $F \in \gamma$-**Kc**. Let us define $\cdot \widehat{\otimes}^K_\gamma F'_c$: **LCS** $\longrightarrow$ **Kc** as the composition $\widehat{\cdot}^M \circ U \circ (\cdot \otimes_b (F_c)'_b) \circ (.)_c$ so that we will be able to describe the unique adjunction by composing known adjunctions. (Similarly, for $F \in$ **Kc** one can define $\cdot \widehat{\otimes}^K_{\gamma,\varepsilon} F'_c$: **LCS** $\longrightarrow$ **Kc** as the same composition $\widehat{\cdot}^M \circ U \circ (\cdot \otimes_b (F_c)'_b) \circ (.)_c$). We have to check this is possible by agreement on objects. This reads for $E \in$ **LCS** as application of (7), (3) and reformulation of the definition $\cdot \otimes_\gamma \cdot = U((\cdot)_c \otimes_H (\cdot)_c)$:

$$\widehat{\cdot}^K \circ U(E_c \otimes_b (F_c)'_b) = \widehat{\cdot}^K \circ U(E_c \otimes_H (F_c)'_b) = \widehat{\cdot}^K \circ U(E_c \otimes_H (F'_c)_c) = E\, \widehat{\otimes}^K_\gamma F'_c.$$

The case $F \in$ **Kc** is similar since by definition $\cdot \otimes_{\gamma,\varepsilon} (\cdot)'_c = U((\cdot)_c \otimes_H ((\cdot)_c)'_b)$.

Then, to compute the adjunction, one needs to know the adjoints of the composed functors, which are from Theorem 2.17 and the proof of proposition 3.9. This gives as adjoint $U \circ (\cdot \parr_b F_c) \circ (.)_c = \cdot\varepsilon F$.

(4) The second adjunction and the last are consequences if we see $\cdot\widehat{\otimes}^K_\gamma F'_c : \gamma\text{-}\mathbf{Kc} \longrightarrow \gamma\text{-}\mathbf{Kc}$ as composition of $i : \gamma\text{-}\mathbf{Kc} \longrightarrow \mathbf{Kc}$, $\cdot\widehat{\otimes}^K_\gamma F'_c : \mathbf{Kc} \longrightarrow \mathbf{Kc}$ and the right adjoint of i (which we will see is not needed here). Indeed, by proposition 3.9, for $E \in \gamma\text{-}\mathbf{Kc}$, $E'_c\widehat{\otimes}^K_\gamma F'_c = E'_c\widehat{\otimes}^K_{\beta e} F'_c$ is the k-quasi-completion of $(E\varepsilon F)'_c \in \gamma\text{-}\mathbf{Kb}$, and therefore from the commutation of γ-topology and k-quasi-completion in that case, that we have just established in (2), it is also in $\gamma\text{-}\mathbf{Kc}$. Hence, the adjunction follows by composition of previous adjunctions and we have also just proved that $\gamma\text{-}\mathbf{Kb}$ is stable by $\widehat{\otimes}^K_\gamma$. ∎

We emphasize expected consequences from the $*$-autonomous structure we will soon demonstrate since we will use them in slightly more general form.

3.15. Corollary. *For any* $Y \in \mathbf{Kc}, X, Z_1, ..., Z_m, Y_1, ..., Y_n \in \gamma\text{-}\mathbf{Kc}, T \in \mathbf{k}\text{-}\mathbf{Ref}$ *the following canonical linear maps are continuous*

$$\mathrm{ev}_{X'_c} : (Y\varepsilon X)\widehat{\otimes}^K_\gamma X'_c \longrightarrow Y, \quad \mathrm{Comp}^*_{T'_c} : (Y\varepsilon T)\widehat{\otimes}^K_\gamma ((T'_c\varepsilon Z_1 \cdots \varepsilon Z_m)^*_k)^*_k \longrightarrow (Y\varepsilon Z_1 \cdots \varepsilon Z_m),$$
$$\mathrm{Comp}_{T'_c} : (Y_1\varepsilon \cdots \varepsilon Y_n \varepsilon T) \otimes_\gamma (T'_c\varepsilon Z_1 \cdots \varepsilon Z_m) \longrightarrow (Y\varepsilon Y_1 \cdots \varepsilon Y_n\varepsilon Z_1 \cdots \varepsilon Z_m),$$
$$\mathrm{Comp}^\sigma_{T'_c} : (Y\varepsilon Y_1 \cdots \varepsilon Y_n \varepsilon T) \otimes_{\sigma,\gamma} (T'_c\varepsilon Z_1 \cdots \varepsilon Z_m) \longrightarrow (Y\varepsilon Y_1 \cdots \varepsilon Y_n\varepsilon Z_1 \cdots \varepsilon Z_m),$$

Moreover for any $F, G \in \mathbf{Kc}$, $V, W \in \gamma\text{-}\mathbf{Kb}$ *and* U, E *any separated lcs, there are continuous associativity maps*

$$\mathrm{Ass}_\varepsilon : E\varepsilon(F\varepsilon G) \longrightarrow (E\varepsilon F)\varepsilon G, \quad \mathrm{Ass}_\gamma : (U\widehat{\otimes}^K_\gamma V)\widehat{\otimes}^K_\gamma W \longrightarrow U\widehat{\otimes}^K_\gamma (V\widehat{\otimes}^K_\gamma W),$$
$$\mathrm{Ass}_{\gamma,\varepsilon} : V\widehat{\otimes}^K_\gamma (T\varepsilon X) \longrightarrow (V\widehat{\otimes}^K_\gamma T)\varepsilon X.$$

Proof. (1) From the adjunction, the first evaluation map is given by the symmetry map in $L((Y\varepsilon X), (X\varepsilon Y)) = L((Y\varepsilon X)\widehat{\otimes}^K_\gamma X'_c, Y)$.

(2) For the associativity Ass_ε, recall that using definitions and (5) (using $F, G \in \mathbf{Kc}$):

$$\begin{aligned} E\varepsilon(F\varepsilon G) &= U(E_c \parr_b [F\varepsilon G]_c) = U(E_c \parr_b [F_c \parr_b G_c]) \\ &\longrightarrow U([E_c \parr_b F_c] \parr_b G_c) \longrightarrow U([U(E_c \parr_b F_c)]_c \parr_b G_c) = (E\varepsilon F)\varepsilon G, \end{aligned}$$

where the first map is $U(\mathrm{Ass}^{\parr_b}_{E_c,F_c,G_c})$ and the second obtained by functoriality from the unit $\eta_{E_c\parr_b F_c} : E_c \parr_b F_c \longrightarrow [U(E_c \parr_b F_c)]_c$.

(3) For the associativity Ass_γ, we know from the adjunction again, since $V'_c, W'_c \in \gamma\text{-}\mathbf{Kc}$, $V = (V'_c)'_c$, $W = (W'_c)'_c$:

$$\begin{aligned} L((U\widehat{\otimes}^K_\gamma V)\widehat{\otimes}^K_\gamma W, U\widehat{\otimes}^K_\gamma (V\widehat{\otimes}^K_\gamma W)) &= L((U\widehat{\otimes}^K_\gamma V), W'_c\varepsilon\big(U\widehat{\otimes}^K_\gamma (V\widehat{\otimes}^K_\gamma W)\big)) \\ &= L(U, V'_c\varepsilon\Big(W'_c\varepsilon\big(U\widehat{\otimes}^K_\gamma (V\widehat{\otimes}^K_\gamma W)\big)\Big)). \end{aligned}$$

Then composing with Ass_ε (note the γ tensor product term is the term requiring nothing but k-quasi-completeness for the adjunction to apply) gives a map:

$$\begin{aligned} &L(U\widehat{\otimes}^K_{\gamma,\varepsilon}\big(V'_c\varepsilon W'_c\big)'_c, \big(U\widehat{\otimes}^K_\gamma (V\widehat{\otimes}^K_\gamma W)\big)) \\ &\qquad \simeq L(U, \big(V'_c\varepsilon W'_c\big)\varepsilon\big(U\widehat{\otimes}^K_\gamma (V\widehat{\otimes}^K_\gamma W)\big)) \longrightarrow L(U, V'_c\varepsilon\Big(W'_c\varepsilon\big(U\widehat{\otimes}^K_\gamma (V\widehat{\otimes}^K_\gamma W)\big)\Big)) \end{aligned}$$

Since an equicontinuous set in $\left(V'_c\varepsilon W'_c\right)'_c$ is contained in an absolutely convex compact set, one gets by universal properties a continuous linear map: $U\widehat{\otimes}^K_{\gamma,\varepsilon}\left(V'_c\varepsilon W'_c\right)'_c \longrightarrow U\widehat{\otimes}^K_{\gamma}\left(V'_c\varepsilon W'_c\right)'_c$.

Finally by functoriality and the embedding of proposition 3.9 there is a canonical continuous linear map: $U\widehat{\otimes}^K_{\gamma}\left(V'_c\varepsilon W'_c\right)'_c \longrightarrow U\widehat{\otimes}^K_{\gamma}(V\widehat{\otimes}^K_{\gamma} W)$. Dualizing, we also have a map which we can evaluate at the identity map composed with all our previous maps to get Ass_γ:

$$L(U\widehat{\otimes}^K_{\gamma}(V\widehat{\otimes}^K_{\gamma} W), \left(U\widehat{\otimes}^K_{\gamma}(V\widehat{\otimes}^K_{\gamma} W)\right)) \longrightarrow L(U\widehat{\otimes}^K_{\gamma,\varepsilon}\left(V'_c\varepsilon W'_c\right)'_c, \left(U\widehat{\otimes}^K_{\gamma}(V\widehat{\otimes}^K_{\gamma} W)\right))$$

(4) We treat similarly the map $\mathrm{Comp}^*_{T'_c}$ in the case $m = 2$, for notational convenience. It is associated to $\mathrm{ev}_{T'_c}\circ(\mathrm{Id}\otimes\mathrm{ev}_{(Z_1)'_c})\circ(\mathrm{Id}\otimes\mathrm{ev}_{(Z_2)'_c}\otimes\mathrm{Id})$ via the following identifications. One obtains first a map between Hom-sets using the previous adjunction:

$$L\left(\left[((Y\varepsilon T)\widehat{\otimes}^K_{\gamma}(((T'_c\varepsilon Z_1)\varepsilon Z_2)^*_k)^*_k)\widehat{\otimes}^K_{\gamma}(Z_2)'_c\right]\widehat{\otimes}^K_{\gamma}(Z_1)'_c, Y\right) = L((Y\varepsilon T)\widehat{\otimes}^K_{\gamma}(((T'_c\varepsilon Z_1)\varepsilon Z_2)^*_k)^*_k, (Y\varepsilon Z_1)\varepsilon Z_2).$$

We compose this twice with Ass_γ and the canonical map $(E^*_k)^*_k \longrightarrow E$ for E k-quasi-complete:

$$\begin{aligned}
&L\left((Y\varepsilon T)\widehat{\otimes}^K_{\gamma}\left[\left(((T'_c\varepsilon Z_1)\varepsilon Z_2)\widehat{\otimes}^K_{\gamma}(Z_2)'_c\right)\widehat{\otimes}^K_{\gamma}(Z_1)'_c\right], Y\right)\\
&\quad\longrightarrow L\left((Y\varepsilon T)\widehat{\otimes}^K_{\gamma}\left[\left((((T'_c\varepsilon Z_1)\varepsilon Z_2)^*_k)^*_k\widehat{\otimes}^K_{\gamma}(Z_2)'_c\right)\widehat{\otimes}^K_{\gamma}(Z_1)'_c\right], Y\right)\\
&\quad\longrightarrow L\left(\left[(Y\varepsilon T)\widehat{\otimes}^K_{\gamma}\left((((T'_c\varepsilon Z_1)\varepsilon Z_2)^*_k)^*_k\widehat{\otimes}^K_{\gamma}(Z_2)'_c\right)\right]\widehat{\otimes}^K_{\gamma}(Z_1)'_c, Y\right)\\
&\quad\longrightarrow L\left(\left[\left((Y\varepsilon T)\widehat{\otimes}^K_{\gamma}(((T'_c\varepsilon Z_1)\varepsilon Z_2)^*_k)^*_k\right)\widehat{\otimes}^K_{\gamma}(Z_2)'_c\right]\widehat{\otimes}^K_{\gamma}(Z_1)'_c, Y\right).
\end{aligned}$$

Note that the first associativity uses the added $((\cdot)^*_k)^*_k$ making the Arens dual of the space k-quasi-complete as it should to use Ass_γ and the second since $(((T'_c\varepsilon Z_1)\varepsilon Z_2)^*_k)^*_k\widehat{\otimes}^K_{\gamma}(Z_2)'_c \in \gamma$-**Kb** from Proposition 3.14.

Note that $T'_c \in$ **Kc** is required for definition of $\mathrm{ev}_{(Z_i)'_c}$ hence the supplementary assumption $T \in$ **k**-**Ref** and not only $T \in \gamma$-**Kc**.

(5) By the last statement in lemma 3.7, we already know that the same absolutely convex compact sets are shared by $((T'_c\varepsilon Z_1\cdots\varepsilon Z_m)^*_k)^*_k$ and $T'_c\varepsilon Z_1\cdots\varepsilon Z_m$. Hence for any absolutely compact set in this set, $\mathrm{Comp}^*_{T'_c}$ induces an equicontinuous family in:

$$L(Y_1\varepsilon\cdots\varepsilon Y_n\varepsilon T, Y\varepsilon Y_1\cdots\varepsilon Y_n\varepsilon Z_1\cdots\varepsilon Z_m).$$

But now by symmetry on ε product and of the assumption on Y_i, Z_j one gets the second hypocontinuity to define $\mathrm{Comp}_{T'_c}$ by a symmetric argument.

(6) One uses $\mathrm{Comp}_{T'_c}$ on $|((Y'_c)'_c\varepsilon Y_1\cdots\varepsilon Y_n\varepsilon T)| = |(Y\varepsilon Y_1\cdots\varepsilon Y_n\varepsilon T)|$, since $(Y'_c)'_c \in \gamma$-**Kc**. This gives the separate continuity needed to define $\mathrm{Comp}^\sigma_{T'_c}$, the one sided γ-hypocontinuity follows from $\mathrm{Comp}^*_{T'_c}$ as in (5).

(7) We finish by $\mathrm{Ass}_{\gamma,\varepsilon}$. We know from the adjunction again composed with Ass_ε and symmetry of ε that we have a map:

$$L(T\varepsilon X, (V\widehat{\otimes}^K_{\gamma}T)\varepsilon\left(V'_c\varepsilon X\right)) \longrightarrow L(T\varepsilon X, V'_c\varepsilon\left((V\widehat{\otimes}^K_{\gamma}T)\varepsilon X\right)) = L(V\widehat{\otimes}^K_{\gamma}(T\varepsilon X), (V\widehat{\otimes}^K_{\gamma}T)\varepsilon X)$$

Similarly, we have canonical maps:

$$L((T\varepsilon X)\widehat{\otimes}^K_\gamma(V\widehat{\otimes}^K_\gamma T)'_c, \big(V'_c\varepsilon X\big)) \simeq L(T\varepsilon X, ((V\widehat{\otimes}^K_\gamma T)'_c)'_c\varepsilon\big(V'_c\varepsilon X\big)) \longrightarrow L(T\varepsilon X, (V\widehat{\otimes}^K_\gamma T)\varepsilon\big(V'_c\varepsilon X\big))$$

$$L((X\varepsilon T)\widehat{\otimes}^K_\gamma(T'_c\varepsilon V'_c), X\varepsilon V'_c) \longrightarrow L((T\varepsilon X)\widehat{\otimes}^K_\gamma((V'_c\varepsilon T'_c)'_c)'_c, V'_c\varepsilon X) \longrightarrow L((T\varepsilon X)\widehat{\otimes}^K_\gamma(V\widehat{\otimes}^K_\gamma T)'_c, V'_c\varepsilon X)).$$

The image of $\mathrm{Comp}_{T'_c} \in L((X\varepsilon T)\widehat{\otimes}^K_\gamma(T'_c\varepsilon V'_c), \big(X\varepsilon V'_c\big))$ gives $\mathrm{Ass}_{\gamma,\varepsilon}$ since $X, V'_c \in \gamma$-**Kc**. ∎

3.16. The dialogue category of k-quasi-complete spaces and the $*$-autonomous category of k-reflexive spaces. We refer to [MT, T] for the theory of dialogue categories and remind the reader that we have already given the definition in subsection 2.9. Note that $*$-autonomous categories are a special case.

We state first a transport lemma for dialogue categories along monoidal functors, which we will use several times.

3.17. Lemma. *Consider $(C, \otimes_C, 1_C)$, and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$, two symmetric monoidal categories, $R : C \longrightarrow \mathcal{D}$ a functor, and $L : \mathcal{D} \longrightarrow C$ the left adjoint to R which is assumed strictly monoidal. If $\neg$ is a tensorial negation on C, then $E \mapsto R(\neg L(E)) = E'_{\mathcal{D}}$ is a tensorial negation on $\mathcal{D}$.*

Proof. Let φ^C the natural isomorphism making $\neg$ a tensorial negation. Denote the natural bijections given by the adjunction as follows

$$\psi_{A,B} : \mathcal{D}(A, R(B)) \simeq C(L(A), B).$$

Define

$$\varphi^{\mathcal{D}}_{A,B,C} = \psi^{-1}_{A,\neg(L(B\otimes_{\mathcal{D}}C))} \circ \varphi^C_{L(A),L(B),L(C)} \circ \psi_{A\otimes_{\mathcal{D}}B,\neg(L(C))} : \\ \mathcal{D}(A \otimes_{\mathcal{D}} B, F(\neg(L(C)))) \longrightarrow \mathcal{D}(A, F(\neg(L(B \otimes_{\mathcal{D}} C)))).$$

It gives the expected natural bijection:

$$\mathcal{D}(A \otimes_{\mathcal{D}} B, F(\neg(L(C)))) \simeq C(L(A \otimes_{\mathcal{D}} B), \neg(L(C))) = C(L(A) \otimes_C L(B)), \neg(L(C))) \\ \simeq C(L(A), \neg(L(B \otimes_{\mathcal{D}} C)))) \simeq \mathcal{D}(A, F(\neg(L(B \otimes_{\mathcal{D}} C))))$$

where we have used strict monoidality of L: $L(B \otimes_{\mathcal{D}} C) = L(B) \otimes_C L(C)$, and the structure of dialogue category on C.

It remains to check the compatibility relation (2). It suffices to note that by naturality of the adjunction, one has for instance:

$$\mathcal{D}(\mathrm{Ass}^{\otimes_{\mathcal{D}}}_{A,B,C}, F(\neg(L(D)))) = \psi^{-1}_{A\otimes_{\mathcal{D}}(B\otimes_{\mathcal{D}}C),\neg(L(D))} \circ C(L(\mathrm{Ass}^{\otimes_{\mathcal{D}}}_{A,B,C}), \neg(L(D))) \circ \psi_{(A\otimes_{\mathcal{D}}B)\otimes_{\mathcal{D}}C,\neg(L(D))}.$$

and since $L(\mathrm{Ass}^{\otimes_{\mathcal{D}}}_{A,B,C}) = \mathrm{Ass}^{\otimes_C}_{L(A),L(B),L(C)}$ from compatibility of a strong monoidal functor, the new commutative diagram in $\mathcal{D}$ reduces to the one in C by intertwining. ∎

3.18. Remark. Note that we have seen or will see several examples of such monoidal adjunctions:

- between $(\mathbf{Kc}^{\rm op}, \varepsilon)$ and $(\mathbf{CLCS}^{\rm op}, \mathbin{\rotatebox[origin=c]{180}{$\&$}}_b)$ through the functors $L = (\cdot)_c$ and $R = (\hat{\cdot}^K \circ U)$ (proof of proposition 3.9 and (5)),
- between $(\mathscr{C}\text{-}\mathbf{CLCS}^{\rm op}, \mathbin{\rotatebox[origin=c]{180}{$\&$}}_{\mathscr{C}})$ and $(\mathscr{C}c\mathscr{C}\text{-}\mathbf{LCS}^{\rm op}, \varepsilon)$ through the functors $L = (\cdot)_{\mathscr{C}}$ and $R = (\hat{\cdot}^{\mathscr{C}} \circ U)$ (proof of lemma 4.17).

We complement this result in the direction of getting a $*$-autonomous category.

3.19. Lemma. *We assume the setting of lemma 3.17. We define $[E, F]_C = \neg(E \otimes_C \neg(F))$. Assume that $(C, \otimes_C, 1_C, [\cdot,\cdot]_C, \neg)$ is a $*$-autonomous category with $RL = id$. Also assume that the counit $\epsilon_A : LR(A) \longrightarrow A$ is a monomorphism for every $A \in C$ and an isomorphism if $A = \neg L(B'_{\mathcal{D}})$ for $B \in \mathcal{D}$. Then $(\cdot)'_{\mathcal{D}}$ has a commutative continuation monad.*

Proof. We already know that $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}}, (\cdot)'_{\mathcal{D}})$ is a dialogue category. From [T, Prop 2.4], we know that the continuation monad $(T = (\cdot)'_{\mathcal{D}}) \circ (\cdot)'_{\mathcal{D}}), \mu, \eta^T)$ admits a right tensor strength $t_{A,B} : A \otimes T(B) \longrightarrow T(A \otimes B)$ and a left tensor strength $t'_{A,B} : T(A) \otimes B \longrightarrow T(A \otimes B)$.

We have to check the commutation relation $\mu_{A,B} \circ T(t'_{A,B}) \circ t_{TA,B} = \mu_{A,B} \circ T(t_{A,B}) \circ t'_{A,TB}$.

First note that by the basic relation [T, (1.12)] for strength (applied twice), naturality of η^T, functoriality of T and unitality of the monad:

$$\begin{aligned}
\mu_{A,B} \circ T(t'_{A,B}) \circ t_{TA,B} \circ (\eta^T_A \otimes_{\mathcal{D}} \eta^T_B) &= \mu_{A,B} \circ T(t'_{A,B}) \circ \eta^T_{TA \otimes_{\mathcal{D}} B} \circ \eta^T_A \otimes_{\mathcal{D}} \mathrm{Id}_B \\
&= \mu_{A,B} \circ T(t'_{A,B}) \circ T(\eta^T_A \otimes_{\mathcal{D}} \mathrm{Id}_B) \circ \eta^T_{A \otimes_{\mathcal{D}} B} \\
&= \mu_{A,B} \circ T(\eta^T_{A \otimes_{\mathcal{D}} B}) \circ \eta^T_{A \otimes_{\mathcal{D}} B} \\
&= \eta^T_{A \otimes_{\mathcal{D}} B}
\end{aligned}$$

and similarly $\mu_{A,B} \circ T(t_{A,B}) \circ t'_{A,TB} \circ (\eta^T_A \otimes_{\mathcal{D}} \eta^T_B) = \eta^T_{A \otimes_{\mathcal{D}} B}$. Hence, it suffices to see that $(\eta^T_A \otimes_{\mathcal{D}} \eta^T_B)$ is an epimorphism.

In terms of the unit $\eta_A = \mathrm{Id}_A : A \longrightarrow RL(A) = A$, counit $\epsilon_B : LR(B) \longrightarrow B$ of the adjunction R, L and isomorphisms $d_A : A \longrightarrow \neg\neg(A)$ coming from $*$-autonomy, it is easy to see that:

$$\eta^T_B = R(\neg(\epsilon_{\neg L(B)}) \circ d_{L(B)}) \circ \eta_B.$$

By our assumption, $\epsilon_{\neg L(B)}$ is a monomorphism, hence $\neg(\epsilon_{\neg L(B)}) \circ d_{L(B)}$ an epimorphism. By the adjunction relation, we have:

$$\neg(\epsilon_{\neg L(B)}) \circ d_{L(B)} = \epsilon_{\neg L(B)'_{\mathcal{D}}} \circ L(\eta^T_B).$$

By assumption, $\epsilon_{\neg L(B)'_{\mathcal{D}}}$ is an isomorphism, hence $L(\eta^T_B)$ is an epimorphism and since L reflects epimorphisms as any faithful functor (it is even full and faithful since $RL = id$), so is η^T_B. Finally, for the same reason, it suffices to see $L(\eta^T_A \otimes_{\mathcal{D}} \eta^T_B) = L(\eta^T_A) \otimes_C L(\eta^T_B)$ is an epimorphism. But L also preserves epimorphisms as any left adjoint, hence $L(\eta^T_A), L(\eta^T_B)$ are epimorphism, and so is their tensor product as in any $*$-autonomous category (cf. proof of corollary 2.18). ■

3.20. Theorem. *$\mathbf{Kc}^{\mathrm{op}}$ is a dialogue category with tensor product ε and tensorial negation $(\cdot)^*_k$ which has a commutative and idempotent continuation monad $((\cdot)^*_k)^*_k$.*

Its continuation category is equivalent to the $*$-autonomous category **k**-**Ref** *with tensor product $E \otimes_k F = (E^*_k \varepsilon F^*_k)^*_k$, dual $(.)^*_k$ and dualizing object $\mathbb{K}$. It is stable under arbitrary products and direct sums.*

Proof. The structure of a dialogue category follows from the first case of the previous remark since $(\mathbf{CLCS}^{\mathrm{op}}, \parr_b, (\cdot)'_b)$ is a $*$-autonomous category, hence a dialogue category by Theorem 2.17 and then the new tensorial negation is $R(\neg L(\cdot)) = \hat{\cdot}^K \circ (\cdot)'_c$ which coincides with $(\cdot)^*_k$ on **Kc**. The idempotency of the continuation monad comes from lemma 3.7.

In order to check that the monad is commutative, one uses lemma 3.19. There remains 3 assumptions to check:

- $RL(A) = \widehat{U((A)_c)}^K = \widehat{A}^K = A$ for any $A \in \mathbf{Kc}$.
- $\epsilon_A \in \mathbf{CLCS}^{\mathrm{op}}(LR(A), A)$ correspond to the completion map $j_{U(A)} \in \mathbf{LCS}(U(A), \widehat{U(A)}^K)$ which is known to be an epimorphism, via the adjunction in theorem 2.17.(2) $\eta^{U,L}_A : A \longrightarrow (U(A))_c$:

  $$\mathbf{LCS}(U(A), \widehat{U(A)}^K) \simeq \mathbf{CLCS}(A, (\widehat{U(A)}^K)_c) = \mathbf{CLCS}(A, LR(A)),$$

  so that $\epsilon^{\mathrm{op}}_A = (j_{U(A)})_c \circ \eta^{U,L}_A$ is an epimorphism. Indeed, $\eta^{U,L}_A$ is a split epimorphism since U is full and $(.)_c$ preserves epimorphisms since its right inverse U reflects them as any faithful functor.
- In the case $A = ((B^*_k)_c)'_b$ for $B \in \mathbf{Kc}$, we have $U(A) = (B^*_k)'_c$ which is k-quasi-complete by lemma 3.7 so that $j_{U(A)} = \mathrm{Id}_{U(A)}$. Hence $\epsilon^{\mathrm{op}}_A = \eta^{U,L}_A$ is an isomorphism since U is full and faithful.

The $*$-autonomous property follows from lemma 2.13. ∎

3.21. A strong notion of smooth maps. During this subsection, $\mathbb{K} = \mathbb{R}$ so that we deal with smooth maps and not holomorphic ones while we explore the consequence of our $*$-autonomy results for the definition of a nice notion of smoothness.

We recall the definition of (conveniently) smooth maps as used by Frolicher, Kriegl and Michor: a map $f : E \longrightarrow F$ is smooth if and only if for every smooth curve $c : \mathbb{R} \longrightarrow E$, $f \circ c$ is a smooth curve. See [KM]. They define on a space of smooth curves the usual topology of uniform convergence on compact subsets of each derivative. Then they define on the space of smooth functions between Mackey-complete spaces E and F the projective topology with respect to all smooth curves in E (see also section 6.1 below).

As this definition fits well in the setting of bounded linear maps and bounded duals, but not in our setting using continuous linear maps, we make use of a slightly different approach by Meise [Me]. Meise works with k-spaces, that is spaces E in which continuity on E is equivalent to continuity on compacts subsets of E. We change his definition and rather use a continuity condition on compact sets in the definition of smooth functions.

3.22. Definition. *For X, F separated lcs, denote by $C^\infty_{co}(X, F)$ the space of infinitely Gâteaux-differentiable functions with derivatives continuous on compacts with values in the space*

$$L^{n+1}_{co}(E, F) = L_{co}(E, L^n_{co}(E, F)),$$

where, for any n, the space $L^n_{co}(E, F)$ is endowed with the topology of uniform convergence on compact sets. We put on this space the topology of uniform convergence on compact sets of all derivatives in the space $L^n_{co}(E, F)$.

We denote by $C^\infty_{co}(X)$ the space $C^\infty_{co}(X, \mathbb{K})$.

One could treat similarly the case of an open set $\Omega \subset X$. We always assume X k-quasi-complete.

Our definition is almost the same as in [Me], except for the continuity condition restricted to compact sets. Meise works with k-spaces, that is spaces E in which continuity on E is equivalent to continuity on compacts subsets of E. Thus for X a k-space, one recovers exactly Meise's definition. Since a (DFM) space X is a k-space ([KM, Th 4.11 p 39]) his corollary 7 gives us that for such an X, $C^\infty_{co}(X, F)$ is a Fréchet space as soon as F is. Similarly for any (F)-space or any (DFS)-space X then his corollary 13 gives $C^\infty_{co}(X, \mathbb{R})$ is a Schwartz space.

As in his lemma 3 p 271, if X k-quasi-complete, the Gâteaux differentiability condition is automatically uniform on compact sets (continuity on absolutely convex closure of compacts of the derivative is enough for that), and as a consequence, this smoothness implies convenient smoothness. We will therefore use the differential calculus from [KM].

One are now ready to obtain a category.

3.23. Proposition. *The k-reflexive spaces form a category with $C^\infty_{co}(X, F)$ as spaces of maps, that we denote* **k-Ref**$_\infty$*. Moreover, for any $g \in C^\infty_{co}(X, Y)$, $Y, X \in$* **k-Ref***, any F Mackey-complete, $\cdot \circ g : C^\infty_{co}(Y, F) \longrightarrow C^\infty_{co}(X, F)$ is linear and continuous.*

Proof. For stability by composition, we in fact show more, consider $g \in C^\infty_{co}(X, Y)$, $f \in C^\infty_{co}(Y, F)$ with $X, Y \in$ **k-Ref** and $F \in$ **Mc** we aim at showing $f \circ g \in C^\infty_{co}(X, F)$. We use stability of composition of conveniently smooth maps, we can use the chain rule [KM, Thm 3.18]. This enables us to make the derivatives valued in F if F is Mackey-complete so that, up to passing to the completion, we can assume $F \in$ **Kc** since the continuity conditions are the same when the topology of the target is induced. This means that we must show continuity on compact sets of expressions of the form

$$(x, h) \mapsto df^l(g(x))(d^{k_1}g(x), ..., d^{k_l}g(x))(h_1, ..., h_m), m = \sum_{i=1}^{l} k_i, h \in Q^m.$$

First note that $L_{co}(X, F) \simeq X'_c \varepsilon F$, $L^n_{co}(X, F) \simeq (X'_c)^{\varepsilon n} \varepsilon F$ fully associative for the spaces above.

Of course for K compact in X, $g(K) \subset Y$ is compact, so $df^l \circ g$ is continuous on compacts with value in $(Y'_c)^{\varepsilon l} \varepsilon F$ so that continuity comes from continuity of the map obtained by composing various $\mathrm{Comp}^*_Y, \mathrm{Ass}_\varepsilon$ from Corollary 3.15 (note Ass_γ is not needed with chosen parentheses):

$$\Big(\Big(...\Big(((Y'_c)^{\varepsilon l} \varepsilon F) \otimes_\gamma (((X'_c)^{\varepsilon k_1} \varepsilon Y)^*_k)^*_k\Big) \otimes_\gamma ...\Big) \otimes_\gamma (((X'_c)^{\varepsilon k_l} \varepsilon Y)^*_k)^*_k\Big) \longrightarrow ((X'_c)^{\varepsilon m} \varepsilon F)$$

and this implies continuity on products of absolutely convex compact sets of the corresponding multilinear map even without $((\cdot)^*_k)^*_k$ since from lemma 3.7 absolutely convex compact sets are the same in both spaces (of course with same induced compact topology). We can compose it with the continuous function on compacts with value in a compact set (on compacts in x):$x \mapsto (df^l(g(x)), d^{k_1}g(x), ..., d^{k_l}g(x))$. The continuity in f is similar and uses hypocontinuity of the above composition (and not only its continuity on products of compacts). ∎

We now prove the *cartesian closedeness* of the category **k-Ref**, the proofs being a slight adaptation of the work of Meise [Me]

3.24. Proposition. *For any $X \in$ **k-Ref**, $C^\infty_{co}(X, F)$ is k-quasi-complete (resp. Mackey-complete) as soon as F is.*

Proof. This follows from the projective kernel topology on $C^\infty_{co}(X, F)$, Corollary 3.13 and the corresponding statement for $C^0(K, F)$ for K compact. In the Mackey-complete case we use the remark at the beginning of step 2 of the proof of Theorem 5.12 that a space is Mackey-complete if and only if the bipolar of any Mackey-Cauchy sequence is complete. We treat the two cases in parallel, if F is k-quasi-complete (resp Mackey-complete), take L a compact set (resp. a Mackey-Cauchy sequence) in $C^0(K, F)$, M its bipolar, its image by evaluations L_x are compact (resp. a Mackey-null sequence) in F and the image of M is in the bipolar of L_x which is complete in F hence a Cauchy net in M converges pointwise in F. But the Cauchy property of the net then implies as usual uniform convergence of the net to the pointwise limit. This limit is therefore continuous, hence the result. ∎

The following two propositions are an adaptation of a result by Meise [Me, Thm 1 p 280]. Remember though that his ε product and E'_c are different from ours, his correspond to replacing absolutely convex compact sets by precompact sets. This different setting forces him to assume quasi-completeness to obtain a symmetric ε-product in his sense.

3.25. Proposition. *For any k-reflexive space X, any compact K, and any separated k-quasi-complete space F one has $C^\infty_{co}(X, F) \simeq C^\infty_{co}(X)\varepsilon F, C^0(K, F) \simeq C^0(K)\varepsilon F$. Moreover, if F is any lcs, we still have a canonical embedding $J_X : C^\infty_{co}(X)\varepsilon F \longrightarrow C^\infty_{co}(X, F)$.*

Proof. We build a map $\mathrm{ev}_X \in C^\infty_{co}(X, (C^\infty_{co}(X))'_c)$ defined by $\mathrm{ev}_X(x)(f) = f(x)$ and show that $\cdot \circ \mathrm{ev}_X : C^\infty_{co}(X)\varepsilon F = L_\epsilon((C^\infty_{co}(X))'_c, F) \longrightarrow C^\infty_{co}(X, F)$ is a topological isomorphism and an embedding if F is only Mackey-complete. The case with a compact K is embedded in our proof and left to the reader.

(a) We first show that the expected j-th differential $\mathrm{ev}^j_X(x)(h)(f) = d^j f(x).h$ indeed gives a map:

$$\mathrm{ev}^j_X \in C^0_{co}(X, L^j_{co}(X, (C^\infty_{co}(X))'_c)).$$

First note that for each $x \in X$, $\mathrm{ev}^j_X(x)$ is in the expected space. Indeed, by definition of the topology $f \mapsto d^j f(x)$ is linear continuous in $L(C^\infty_{co}(X), L^j_{co}(X, \mathbb{R})) \subset L(((C^\infty_{co}(X))'_c)'_c, L^j_{co}(X, \mathbb{R})) = (C^\infty_{co}(X))'_c\varepsilon(X'_c)^{\varepsilon j}$. Using successively Ass^ϵ from Corollary 3.15 (note no completeness assumption on $(C^\infty_{co}(X))'_c$ is needed for that) hence

$$\mathrm{ev}^j_X(x) \in (\cdots((C^\infty_{co}(X))'_c\varepsilon X'_c)\cdots\varepsilon X'_c) = L^j_{co}(X, (C^\infty_{co}(X))'_c).$$

Then, once the map is well-defined, we must check its continuity on compacts sets in variable $x \in K \subset X$, uniformly on compacts sets for $h \in Q$, one must check convergence in $(C^\infty_{co}(X))'_c$. But everything takes place in a product of compact sets and from the definition of the topology on $C^\infty_{co}(X)$, $\mathrm{ev}^j_X(K)(Q)$ is equicontinuous in $(C^\infty_{co}(X))'$. But from [K, §21.6.(2)] the topology $(C^\infty_{co}(X))'_c$ coincides with $(C^\infty_{co}(X))'_\sigma$ on these sets. Hence we only need to prove for any f continuity of $d^j f$ and this follows by assumption on f. This concludes the proof of (a).

(b) Let us note that for $f \in L_\varepsilon((C^\infty_{co}(X))'_c, F)$, $f \circ \mathrm{ev}_X \in C^\infty_{co}(X, F)$. We first note that $f \circ \mathrm{ev}^j_X(x) = d^j(f \circ \mathrm{ev}_X)(x)$ as in step (c) in the proof of [Me, Thm 1]. This shows for $F = (C^\infty_{co}(X))'_c$ that the Gâteaux derivative is $d^j\mathrm{ev}_X = \mathrm{ev}^j_X$ and therefore the claimed $\mathrm{ev}_X \in C^\infty_{co}(X, (C^\infty_{co}(X))'_c)$.

(c)$f \mapsto f \circ \mathrm{ev}_X$ is the stated isomorphism. The monomorphism property is the same as (d) in Meise's proof. Finding a right inverse j proving surjectivity is the same as his (e) . Let us detail this since we only assume k-quasi-completeness on F. We want $j : C^\infty_{co}(X, F) \longrightarrow C^\infty_{co}(X)\varepsilon F = L(F'_c, C^\infty_{co}(X))$ for $y' \in F'$, $f \in C^\infty_{co}(X, F)$ we define $j(f)(y') = y' \circ f$. Note that from convenient smoothness we know that the derivatives are $y' \circ d^j f(x)$ and $d^j f(x) \in |(X'_c)^{\varepsilon n}\varepsilon F| = |(X'_c)^{\varepsilon n}\varepsilon(F'_c)'_c|$ algebraically so that, since $(F'_c)'_c$ k-quasi-complete, one can use $\mathrm{ev}_{F'_c}$ from Corollary 3.15 to see $y' \circ d^j f(x) \in (X'_c)^{\varepsilon n}$ and one even deduces (using only separate continuity of $\mathrm{ev}_{F'_c}$) its continuity in y'. Hence $j(f)(y')$ is valued in $C^\infty_{co}(X)$ and from the projective kernel topology, $j(f)$ is indeed continuous. The simple identity showing that j is indeed the expected right inverse proving surjectivity is the same as in Meise's proof. ∎

3.26. Proposition. *For any space $X_1, X_2 \in$ **k-Ref** and any Mackey-complete lcs F we have:*

$$C^\infty_{co}(X_1 \times X_2, F) \simeq C^\infty_{co}(X_1, C^\infty_{co}(X_2, F)).$$

Proof. Construction of the curry map Λ is analogous to [Me, Prop 3 p 296]. Since all spaces are Mackey-complete, we already know from [KM, Th 3.12] that there is a Curry map valued in $C^\infty(X_1, C^\infty(X_2, F))$, it suffices to see that the derivatives $d^j\Lambda(f)(x_1)$ are continuous on compacts with value $C^\infty_{co}(X_2, F)$. But this derivative coincides with a partial derivative of f, hence it is valued pointwise in $C^\infty_{co}(X_2, F) \subset C^\infty_{co}(X_2, \widehat{F}^K)$. Since we already know all the derivatives are pointwise valued in F, we can assume F k-quasi-complete. But the topology for which we must prove continuity is a projective kernel, hence we only need to see that $d^k(d^j\Lambda(f)(x_1))(x_2)$ continuous on compacts in x_1 with value in $L^j_c(X_1, C^0(K_2, L^j_c(X_2, F)))$. But we are in the case where the ε product is associative, hence the above space is merely $C^0(K_2)\varepsilon L^j_c(X_1, L^j_c(X_2, F)) = C^0(K_2, L^j_c(X_1, L^j_c(X_2, F)))$. We already know the stated continuity in this space from the choice of f. The reasoning for the inverse map is similar using again the convenient smoothness setting (and cartesian closedness $C^0(K_1, C^0(K_2)) = C^0(K_1 \times K_2)$). ∎

Thus **k-Ref** is cartesian closed.

4. Models of MALL coming from classes of smooth maps

Any denotational model of linear logic has a morphism interpreting dereliction on any space E: $d_E : E \longrightarrow ?E$. In our context of smooth functions and reflexive spaces, it means that the topology on E must be finer than the one induced by $\mathscr{C}^\infty(E^*, \mathbb{K})$, and in this section we continue with the assumption $\mathbb{K} = \mathbb{R}$. From the model of k-reflexive spaces, we introduce a variety of new classes of smooth functions, each one inducing a different topology and a new smooth model of classical Linear Logic. These notions of smoothness are indexed by a cartesian subcategory $\mathscr{C}$ of **k-Ref**. We show in particular that each time the $\parr$ is interpreted as the ε-product. As explained in the introduction, we then need a notion of $\mathscr{C}$-complete space in order to recover the largest possible cartesian closed setting for this notion of $\mathscr{C}$-smoothness. One also gets a notion of $\mathscr{C}$-completion with an inductive construction in replacing arbitrary compact sets by images of compact subsets of spaces in $\mathscr{C}$ by smooth maps.

We want to start with cartesian closedness [KM, Th 3.12] and its corollary, but we want an exponential law in the topological setting, and not in the bornological setting. We thus change slightly the topology on (conveniently)-smooth maps $C^\infty(E, F)$ between two locally convex spaces. We follow the simple idea to consider spaces of smooth curves on a family of base spaces stable by product, thus at least on any $\mathbb{R}^n$. Since we choose at this stage a topology, it seems reasonable to look at the induced topology on linear maps.We think the Seely isomorphism is crucial to select such a topology in transforming a desirable stability by ε-product into stability by product. For $\mathscr{C}$-smooth maps, we therefore introduce a notion of $\mathscr{C}$-space. We will see in the next section that this generalizes Schwartz spaces and nuclear spaces when we take the most basic examples of $\mathscr{C}$.

We then follow the same strategy as before to recover a dialogue and then a $*$-autonomous category. We restrict the model of MALL from Theorem 2.17 in section 4.23 to $\mathscr{C}$-spaces. Here, we use crucially cartesian closedness of $\mathscr{C}$-smooth maps. We then get an intertwining with $\mathscr{C}$-complete $\mathscr{C}$-spaces to recover a symmetric monoidal category in section 4.25, reusing again cartesian closedness. We study a reflexivization functor in section 4.18 and we conclude in the two last sections using the same categorical preliminaries as in section 3.16.

4.1. $\mathscr{C}$-Smooth maps and $\mathscr{C}$-completeness. We first fix a small cartesian category $\mathscr{C}$ that will replace the category of finite dimensional spaces $\mathbb{R}^n$ as parameter space of curves.

We will soon restrict to the full category $\mathbf{F} \times \mathbf{DFS} \subset \mathbf{LCS}$ consisting of (finite) products of Fréchet spaces and strong duals of Fréchet-Schwartz spaces, but we first explain the most general context in which we know our formalism works. We assume $\mathscr{C}$ is a full cartesian small subcategory of **k-Ref** containing $\mathbb{R}$, with smooth maps as morphisms.

Proposition 3.26 and the convenient smoothness case suggests the following space and topology. For any $X \in \mathscr{C}$, for any $c \in C^\infty_{co}(X, E)$ a (**k-Ref** space parametrized) curve we define $C^\infty_{\mathscr{C}}(E, F)$ as the set of maps f such that $f \circ c \in C^\infty_{co}(X, F)$ for any such curve c. We call them *$\mathscr{C}$-smooth maps*. Note that $. \circ c$ is in general not surjective, but valued in the closed subspace:

$$[C^\infty_{co}(X, F)]_c = \{g \in C^\infty_{co}(X, F) : \forall x \neq y : c(x) = c(y) \Rightarrow g(x) = g(y)\}.$$

One gets a linear map $. \circ c : C^\infty_{\mathscr{C}}(E, F) \longrightarrow C^\infty_{co}(X, F)$. We equip the target space of the topology of uniform convergence of all differentials on compact subsets as before. We equip $C^\infty_{\mathscr{C}}(E, F)$ with the projective kernel topology of those maps for all $X \in \mathscr{C}$ and c smooth maps as above, with connecting maps all smooth maps $C^\infty_{co}(X, Y)$ inducing reparametrizations. Note that this projective kernel can be identified with a projective limit (indexed by a directed set). Indeed, we put an order on the set of curves $C^\infty_{co}(\mathscr{C}, E) := \sqcup_{X\in\mathscr{C}} C^\infty_{co}(X, E)/\sim$ (where two curves are identified with the equivalence relation making the preorder we define into an order). This is an ordered set with $c_1 \leq c_2$ if $c_1 \in C^\infty_{co}(X, E), c_2 \in C^\infty_{co}(Y, E)$ and there is $f \in C^\infty_{co}(X, Y)$ such that $c_2 \circ f = c_1$. This is moreover a directed set. Indeed given $c_i \in C^\infty_{co}(X_i, E)$, one considers $c_i' \in C^\infty_{co}(X_i \times \mathbb{R}, E)$, $c_i'(x, t) = tc_i(x)$ so that $c_i' \circ (., 1) = c_i$ giving $c_i \leq c_i'$. Then one can define $c \in C^\infty_{co}(X_1 \times \mathbb{R} \times X_2 \times \mathbb{R}, E)$ given by $c(x, y) = c_1'(x) + c_2'(y)$. This satisfies $c \circ (., 0) = c_1', c \circ (0, .) = c_2'$, hence $c_i \leq c_i' \leq c$. We claim that $C^\infty_{\mathscr{C}}(E, F)$ identifies with the projective limit along this directed set (we fix one c in each equivalence class) of $[C^\infty_{co}(X, F)]_c$ on the curves $c \in C^\infty_{co}(X, E)$ with connecting maps for $c_1 \leq c_2$, $. \circ f$ for one fixed f such that $c_2 \circ f = c_1$. This is well-defined since if g is another curve with $c_2 \circ g = c_1$, then for $u \in [C^\infty_{co}(X_2, F)]_{c_2}$ for any $x \in X_1$, $u \circ g(x) = u \circ f(x)$ since $c_2(g(x)) = c_1(x) = c_2(f(x))$ hence $\cdot \circ g = \cdot \circ f : [C^\infty_{co}(X_2, F)]_{c_2} \longrightarrow [C^\infty_{co}(X_1, F)]_{c_1}$ does not depend on the choice of f.

For a compatible sequence of such maps in $[C^\infty_{co}(X, F)]_c$, one associates the map $u : E \longrightarrow F$ such that $u(x)$ is the value at the constant curve c_x equal to x in $C^\infty_{co}(\{0\}, E) = E$. For, the curve $c \in C^\infty_{co}(X, E)$ satisfies for $x \in X$, $c \circ c_x = c_{c(x)}$, hence $u \circ c$ is the element of the sequence associated to c, hence $u \circ c \in [C^\infty_{co}(X, F)]_c$. Since this holds for any curve c, this implies $u \in C^\infty_{\mathscr{C}}(E, F)$ and the canonical map from this space to the projective limit is therefore surjective. The topological identity is easy.

We summarize this with the formula:

$$C^\infty_{\mathscr{C}}(E, F) = \mathrm{proj} \lim_{c\in C^\infty_{co}(X,E)} [C^\infty_{co}(X, F)]_c \tag{8}$$

For $\mathscr{C} = \mathrm{Fin}$ the category of finite dimensional spaces, $C^\infty_{\mathrm{Fin}}(E, F) = C^\infty(E, F)$ is the space of conveniently smooth maps considered by Kriegl and Michor. We merely call them smooth maps. Note that our topology on this space is slightly stronger than theirs (before they bornologify) and that any $\mathscr{C}$-smooth map is smooth, since all our $\mathscr{C} \supset \mathrm{Fin}$. Another important case for us is $\mathscr{C} = \mathrm{Ban}$ a small category of Banach spaces. (In order to apply our projective limit constructions, we need to stick to a small category $\mathscr{C}$. Hence we need a little set theoretic care at this point. The reader can either assume Ban is the category of Banach spaces in a given Grothendieck universe, or assuming ZFC and existence of a large cardinal as is usual in functional analysis, Ban can then be the category of Banach spaces of density character smaller than some fixed inaccessible cardinal. This way of writing will be easier but we could also take Ban to be the more concrete category of separable Banach spaces, with minor modifications).

4.2. Lemma. *We fix $\mathscr{C}$ any cartesian small and full subcategory of* **k**-**Ref** *containing $\mathbb{R}$ and the above projective limit topology on $C^\infty_{\mathscr{C}}$. For any E, F, G lcs, with G k-quasi-complete, there is a topological isomorphism:*

$$C^\infty_{\mathscr{C}}(E, C^\infty_{\mathscr{C}}(F, G)) \simeq C^\infty_{\mathscr{C}}(E \times F, G) \simeq C^\infty_{\mathscr{C}}(E \times F)\varepsilon G.$$

Moreover, the first isomorphism also holds for G Mackey-complete, and $C^\infty_{\mathscr{C}}(F, G)$ is Mackey-complete (resp. k-quasi-complete) as soon as G is. If $X \in \mathscr{C}$ then $C^\infty_{\mathscr{C}}(X, G) \simeq C^\infty_{co}(X, G)$ and if only $X \in$ **k**-**Ref** *there is a continuous inclusion: $C^\infty_{co}(X, G) \longrightarrow C^\infty_{\mathscr{C}}(X, G)$.*

Proof. The first algebraic isomorphism comes from [KM, Th 3.12] in the case $\mathscr{C}$ = Fin (since maps smooth on smooth curves are automatically smooth when composed by "smooth varieties" by their Corollary 3.13). More generally, for any $\mathscr{C}$, the algebraic isomorphism works with the same proof in using Proposition 3.26 instead of their Proposition 3.10. We also use their notation $f^\vee, f^\wedge$ for the maps given by the algebraic cartesian closedness isomorphism.

Concerning the topological identification we take the viewpoint of projective kernels, for any curve $c = (c_1, c_2) : X \longrightarrow E \times F$, one can associate a curve $(c_1 \times c_2) : (X \times X) \longrightarrow E \times F$, $(c_1 \times c_2)(x, y) = (c_1(x), c_2(y))$ and for $f \in C^\infty(E, C^\infty(F, G))$, one gets $(\cdot \circ c_2)(f \circ c_1) = f^\wedge \circ (c_1 \times c_2)$ composed with the diagonal embedding gives $f^\wedge \circ (c_1, c_2)$ and thus uniform convergence of the latter is controlled by uniform convergence of the former. This gives by taking projective kernels, continuity of the direct map.

Conversely, for $f \in C^\infty(E \times F, G)$, $(\cdot \circ c_2)(f^\vee \circ c_1) = (f \circ (c_1 \times c_2))^\vee$ with c_1 on X_1, c_2 on X_2 is controlled by a map $f \circ (c_1 \times c_2)$ with $(c_1 \times c_2) : X_1 \times X_2 \longrightarrow E \times F$ and this gives the converse continuous linear map (using proposition 3.26).

The topological isomorphism with the ε product comes from its commutation with projective limits ([K2, §44.5.(4)]) as soon as we note that $[C^\infty_{\mathrm{co}}(X, G)]_c = [C^\infty_{\mathrm{co}}(X, \mathbb{R})]_c \varepsilon G$ but these are also projective limits as intersections and kernels of evaluation maps. Therefore, this comes from proposition 3.25.

Finally, $C^\infty_{\mathscr{C}}(F, G)$ is a closed subspace of a product of $C^\infty_{\mathrm{co}}(X, G)$ which are Mackey-complete or k-quasi-complete if so is G by proposition 3.24.

For the last statement, since $id : X \longrightarrow X$ is smooth, we have a continuous map $I : C^\infty_{\mathscr{C}}(X, G) \longrightarrow C^\infty_{\mathrm{co}}(X, G)$ in case $X \in \mathscr{C}$. Conversely, it suffices to note that for any $Y \in \mathscr{C}$, $c \in C^\infty_{\mathrm{co}}(Y, X)$, $f \in C^\infty_{\mathrm{co}}(X, G)$, then $f \circ c \in C^\infty_{\mathrm{co}}(Y, G)$ by the chain rule from proposition 3.23 and that this map is continuous linear in f for c fixed. This shows I is the identity map and gives continuity of its inverse by the universal property of the projective limit. ■

We now want to extend this result beyond the case G k-quasi-complete in finding the appropriate notion of completeness depending on $\mathscr{C}$.

Recall we introduced a map J_X in proposition 3.25.

4.3. Lemma. *Consider the statements:*

1. *F is Mackey-complete.*
2. *For any $X \in \mathscr{C}$, $J_X : C^\infty_{\mathrm{co}}(X) \varepsilon F \longrightarrow C^\infty_{\mathrm{co}}(X, F)$ is a topological isomorphism*
3. *For any lcs E, $J^{\mathscr{C}}_E : C^\infty_{\mathscr{C}}(E) \varepsilon F \longrightarrow C^\infty_{\mathscr{C}}(E, F)$ is a topological isomorphism.*
4. *For any $X \in \mathscr{C}$, $f \in (C^\infty_{\mathrm{co}}(X))'_c$, any $c \in C^\infty_{\mathrm{co}}(X, F) \subset C^\infty_{\mathrm{co}}(X, \tilde{F}) = C^\infty_{\mathrm{co}}(X) \varepsilon \tilde{F}$, we have $(f \varepsilon \mathrm{Id})(c) \in F$ instead of its completion (equivalently with its k-quasi-completion).*
5. *For any $X \in \mathscr{C}$, $c \in C^\infty_{\mathrm{co}}(X, F)$, and compact $K \subset X$, the closed absolutely convex hull of $c(K)$ is compact.*

We have equivalence of (2),(3),(4) and (5) for any $\mathscr{C}$ cartesian small and full subcategory of **k-Ref** *containing $\mathbb{R}$. They always imply (1) and when $\mathscr{C} \subset$* **F** $\times$ **DFS**, *(1) is also equivalent to them.*

This suggests the following condition weaker than k-quasi-completeness:

4.4. Definition. *A locally convex space E is said $\mathscr{C}$-*complete *(for a $\mathscr{C}$ as above) if one of the equivalent conditions (2),(3),(4),(5) is satisfied. We write $K_{\mathscr{C}}(E)$ the images of compact sets by $c \in C^\infty_{\text{co}}(X, E)$ for $X \in \mathscr{C}$.*

Proof. (2) implies (3) by the commutation of ε product with projective limits as in lemma 4.2 and (3) implies (2) using $C^\infty_{\mathscr{C}}(X, F) = C^\infty_{\text{co}}(X, F)$, for $X \in \mathscr{C}$. (2) implies (4) is obvious since the map $(f\varepsilon\text{Id})$ gives the same value when applied in $C^\infty_{\text{co}}(X)\varepsilon F$. Conversely, looking at $u \in C^\infty_{\text{co}}(X, F) \subset C^\infty_{\text{co}}(X, \tilde{F}) = L((C^\infty_{\text{co}}(X))'_c, \tilde{F})$, (4) says that the image of the linear map u is valued in F instead of $\tilde{F}$, so that since continuity is induced, one gets $u \in L((C^\infty_{\text{co}}(X))'_c, F)$ which gives the missing surjectivity hence (2) (using some compatibility of J_X for a space and its completion).

Let us assume (4) and prove (1). We use a characterization of Mackey-completeness in [KM, Thm 2.14 (2)], we check that any smooth curve has an anti-derivative. As in their proof of (1) implies (2) we only need to check any smooth curve has a weak integral in E (instead of the completion, in which it always exists uniquely by their lemma 2.5). But take $\text{Leb}_{[0,x]} \in (C^\infty_{\text{co}}(\mathbb{R}))'$, for a curve $c \in C^\infty(\mathbb{R}, \tilde{F})$ it is easy to see that $(\text{Leb}_{[0,x]}\epsilon\text{Id})(c) = \int_0^x c(s)ds$ is this integral (by commutation of both operations with application of elements of F'). Hence (4) gives exactly that this integral is in F instead of its completion, as we wanted.

Let us show that (1) implies (2) first in the case $\mathscr{C} = \text{Fin}$ and take $X = \mathbb{R}^n$. One uses [FK, Thm 5.1.7] which shows that $S = \text{Vect}(\text{ev}_{\mathbb{R}^n}(\mathbb{R}^n))$ is Mackey-dense in $C^\infty(\mathbb{R}^n)'_c$. But for any map $c \in C^\infty(\mathbb{R}^n, F)$, there is a unique possible value of $f \in L(C^\infty(\mathbb{R}^n)'_c, F)$ such that $J_X(f) = c$ once restricted to $\text{Vect}(\text{ev}_{\mathbb{R}^n}(\mathbb{R}^n))$. Moreover $f \in L(C^\infty(\mathbb{R}^n)'_c, \tilde{F})$ exists and Mackey-continuity implies that the value on the Mackey-closure of S lies in the Mackey closure of F in the completion, which is F. This gives surjectivity of J_X.

In the case $X \in \mathscr{C} \subset$ **F** $\times$ **DFS**, it suffices to show that $S = \text{Vect}(\cup_{k\in\mathbb{N}}\text{ev}^{(k)}_X(X^{k+1}))$ is Mackey dense in $C^\infty(X)'_c$. Indeed, one can then reason similarly since for $c \in C^\infty_{\text{co}}(X, F)$ and $f \in L(C^\infty_{\text{co}}(X)'_c, \tilde{F})$ with $J_X(f) = c$ satisfies $f \circ \text{ev}^{(k)}_X = c^{(k)}$ which takes value in F by convenient smoothness and Mackey-completeness, hence also Mackey limits so that f will be valued in F. Let us prove the claimed density. First recall that $C^\infty_{\text{co}}(X)$ is a projective kernel of spaces $C^0(K, (X'_c)^{\epsilon k})$ via maps induced by differentials and this space is itself a projective kernel of $C^0(K \times L^k)$ for absolutely convex compact sets $K, L \subset X$. Hence by [K, §22.6.(3)], $(C^\infty_{\text{co}}(X))'$ is a locally convex hull (at least a quotient of a sum) of the space of signed measures $(C^0(K \times L^k))'$. As recalled in the proof of [Me, Corol 13 p 279], every compact set K in $X \in$ **F** $\times$ **DFS** is a compact subset of a Banach space, hence metrizable. Hence the space of signed measures $(C^0(K\times L^k))'$ is metrizable too for the weak-* topology (see e.g. [DM]), and by Krein-Millman's Theorem [K, §25.1.(3)] every point in the (compact) unit ball is a weak-* limit of an absolutely convex combination of extreme points, namely Dirac masses [K, §25.2.(2)], and by metrizability one can take a sequence of such combinations, which is bounded in $(C^0(K \times L^k))'$. Hence its image in $E = (C^\infty_{co}(X))'$ is bounded in some Banach subspace, with equicontinuous ball B (by image of an equicontinuous sets, a ball in a Banach space by the transpose of a continuous map) and converges weakly. But from [Me, Prop 11 p 276], $C^\infty_{co}(X)$ is a Schwartz space, hence there is an other equicontinuous set C such that B is compact in E_C hence the weakly convergent sequence admitting only at most one limit point must converge normwise in E_C. Finally, we have obtained Mackey convergence of this sequence in $E = (C^\infty_{co}(X))'$ and looking at its form, this gives exactly Mackey-density of S.

Let us prove that (4) implies (5). We first claim that any $y \in \overline{\Gamma(c(K))}$ can be written $y = (I(t)\varepsilon\mathrm{Id})(c)$ for some $t \in \overline{\Gamma(\delta(K))}$ for $\delta : K \longrightarrow (C^0(K))'_c$ the Dirac mass map and $I : (C^0(K))'_c \longrightarrow (C^\infty_{co}(X))'_c$ the dual of the restriction map. For, take a net (y_n) with $y_n = \sum_m \lambda_{n,m} c(x_{n,m})$ an absolutely convex combination of elements in $c(K)$, i.e. $x_{n,m} \in K$. Consider the net (T_n) with $T_n = \sum_m \lambda_{n,m}\delta(x_{n,m})$. One can consider this net in $(C^0(K))'_c = \mathscr{S}((C^0(K))'_\mu)$ which is complete since $C^0(K)$ ultrabornological as any Banach space (and using [Ja, Prop 13.2.(5)]). Hence, since $\delta : K \longrightarrow (C^0(K))'_c$ is continuous, (T_n) belongs to the closed absolutely convex hull of $\delta(K)$ which is compact, hence there is a subnet (T_ν) converging to some t. Hence $y_\nu = (I(T_\nu)\varepsilon\mathrm{Id})(c) \longrightarrow (I(t)\varepsilon\mathrm{Id})(c)$. Hence assumption (4) implies $(I(t)\varepsilon\mathrm{Id})(c) = y \in F$ (and the convergence takes place there as well as in its completion). This proves the claim. Take now any net $y_n \in \overline{\Gamma(c(K))}$, and choose some t_n as above with $y_n = (I(t_n)\varepsilon\mathrm{Id})(c)$. By compactness of $\overline{\Gamma(\delta(K))}$, get a converging subnet $t_\nu \longrightarrow t$ so that $y_\nu \longrightarrow (I(t)\varepsilon\mathrm{Id})(c)$. As before, (4) concludes to the limit in F. Hence, $\overline{\Gamma(c(K))} \subset F$ is compact.

Let us finally prove (4) starting from (5). As before, it suffices to note that any point f is in a fundamental equicontinuous set, which is nothing but the closed absolutely convex hull of the image of compact by $d^k c$. This differential being smooth, (5) applies to get the conclusion. ∎

This can be the basis to define a $\mathscr{C}$-completion similar to Mackey completion with a projective definition (as intersection in the completion) based on (2) and an inductive construction (as union of a chain in the completion) based on (5). We conclude with a variant of lemma 3.4, left to the reader.

4.5. Lemma. *The intersection $\widehat{E}^{\mathscr{C}}$ of all $\mathscr{C}$-complete spaces containing E and contained in the completion $\tilde{E}$ of E, is $\mathscr{C}$-complete and called the $\mathscr{C}$-completion of E.*

We define $E_{\mathscr{C};0} = E$, and for any ordinal λ, the subspace $E_{\mathscr{C};\lambda+1} = \cup_{K \in K_{\mathscr{C}}(E_\lambda)} \overline{\Gamma(K)} \subset \tilde{E}$ where the union runs over all compact subsets $K_{\mathscr{C}}(E_\lambda)$ of E_λ of the form defined in definition 4.4, and the closure is taken in the completion. We also let for any limit ordinal $E_{\mathscr{C};\lambda} = \cup_{\mu<\lambda} E_{\mathscr{C};\mu}$. Then for any ordinal λ, $E_{\mathscr{C};\lambda} \subset \widehat{E}^{\mathscr{C}}$ and eventually for λ large enough, we have equality.

4.6. Induced topologies on spaces of linear maps. In the setting of the previous subsection, we have $E' \subset C^\infty_{\mathscr{C}}(E, \mathbb{R})$. From Mackey-completeness, this extends to an inclusion of the Mackey completion, on which one obtains an induced topology which coincides with the topology of uniform convergence on images by smooth curves with source $X \in \mathscr{C}$ of compacts in this space. Indeed, the differentials of the smooth curve is also smooth on a product and the condition on derivatives therefore reduces to this one. This can be described functorially in the spirit of the associated Schwartz topology functor $\mathscr{S}$.

We still consider $\mathscr{C} \subset$ **k-Ref** a full cartesian small subcategory.

Let $\mathscr{C}^\infty$ be the smallest class of locally convex spaces containing $C^\infty_{co}(X, \mathbb{K})$ for $X \in \mathscr{C}$ ($X = \{0\}$ included) and stable by products and subspaces. Let $\mathscr{S}_\mathscr{C}$ the functor on **LCS** of associated topology in this class described by [Ju, 2.6.4]. The functor endows a lcs E with the finest topology coarser than the original one and making $\mathscr{S}_\mathscr{C}(E)$ an object of $\mathscr{C}^\infty$. As $\mathscr{C}^\infty$ contains the field of scalars and is stable by product, E endowed with its weak topology is in particular an element of this space. The functor $\mathscr{S}_\mathscr{C}$ commutes with products.

4.7. Example. If $\mathscr{C} = \{0\}$ then $\mathscr{C}^\infty =$ **Weak** the category of spaces with their weak topology, since $\mathbb{K}$ is a universal generator for spaces with their weak topology. Thus the weak topology functor is $\mathscr{S}_{\{0\}}(E)$.

4.8. Example. If $\mathscr{C}^\infty \subset \mathscr{D}^\infty$ (e.g. if $\mathscr{C} \subset \mathscr{D}$) then, from the very definition, there is a natural transformation $id \longrightarrow \mathscr{S}_\mathscr{D} \longrightarrow \mathscr{S}_\mathscr{C}$ with each map $E \longrightarrow \mathscr{S}_\mathscr{D}(E) \longrightarrow \mathscr{S}_\mathscr{C}(E)$ is a continuous identity map.

4.9. Lemma. *For any lcs E,* $|(\mathscr{S}_\mathscr{C}(E))'| = |E'|$ *algebraically.*

Proof. Since $\{0\} \subset \mathscr{C}$, there is a continuous identity map $E \longrightarrow \mathscr{S}_\mathscr{C}(E) \longrightarrow \mathscr{S}_{\{0\}}(E) = (E'_\sigma)'_\sigma$. The Mackey-Arens theorem concludes. ∎

As a consequence, E and $\mathscr{S}_\mathscr{C}(E)$ have the same bounded sets and therefore are simultaneously Mackey-complete. Hence $\mathscr{S}_\mathscr{C}$ commutes with Mackey-completion. Moreover, the class $\mathscr{C}^\infty$ is also stable by ε-product, since this product commutes with projective kernels and $C^\infty_{co}(X, \mathbb{K})\varepsilon C^\infty_{co}(Y, \mathbb{K}) = C^\infty_{co}(X \times Y, \mathbb{K})$ and we assumed $X \times Y \in \mathscr{C}$.

We now consider the setting of the previous subsection, namely we also assume $\mathbb{R} \in \mathscr{C}$, $\mathscr{C}$ small and identify the induced topology $E'_{\infty\mathscr{C}} \subset C^\infty_\mathscr{C}(E, \mathbb{R})$.

4.10. Lemma. *For any lcs E, there is a continuous identity map:* $E'_{\infty\mathscr{C}} \longrightarrow \mathscr{S}_\mathscr{C}(E'_c)$.
If moreover E is $\mathscr{C}$-complete, this is a topological isomorphism.

Proof. For the direct map we use the universal property of projective kernels. Consider a continuous linear map $f \in L(E'_c, C^\infty_{co}(X, \mathbb{K})) = C^\infty_{co}(X, \mathbb{K})\varepsilon E$ and the corresponding $J_X(f) \in C^\infty_{co}(X, E)$, then by definition of the topology $\cdot \circ J_X(f) : C^\infty_\mathscr{C}(X, E) \longrightarrow C^\infty_{co}(X, \mathbb{R})$ is continuous and by definition, its restriction to E' agrees with f, hence $f : E'_{\infty\mathscr{C}} \longrightarrow C^\infty_{co}(X, \mathbb{K})$ is also continuous. Taking a projective kernel over all those maps gives the expected continuity.

Conversely, if E is $\mathscr{C}$-complete, note that $E'_c \longrightarrow E'_{\infty\mathscr{C}}$ is continuous using again the universal property of a kernel, it suffices to see that for any $X \in \mathscr{C}$, $c \in C^\infty_\mathscr{C}(X, E)$ then $\cdot \circ c : E'_c \longrightarrow C^\infty_\mathscr{C}(X, K)$ is continuous, and this is the content of the surjectivity of J_X in lemma 4.3 (2) since $\cdot \circ c = J_X^{-1}(c)$. Hence since $E'_\mathscr{C} \in \mathscr{C}^\infty$ by definition as projective limit, one gets by functoriality the continuity of $\mathscr{S}_\mathscr{C}(E'_c) \longrightarrow E'_{\infty\mathscr{C}}$. ∎

4.11. First model of MALL in the $\mathscr{C}$-space setting. We still fix $\mathscr{C}$ any cartesian small and full subcategory of **k-Ref** containing $\mathbb{K} = \mathbb{R}$. This will remain our current assumption in this section. We revisit our first model of MALL from Theorem 2.17 in the context of $\mathscr{C}$-spaces, and we will deduce our new result by restricting this result to our new situation. Of course, we need dually a notion of $\mathscr{C}$-bornology we introduce first.

4.12. Definition. *Let $\mathscr{C}$-**LCS** $\subset$ **LCS** be the full subcategory of $\mathscr{C}$-*spaces*, i.e. those of the form $\mathscr{S}_{\mathscr{C}}(E) = E$ and $\mathscr{C}$-**CLCS** $\subset$ **CLCS** the full subcategory of $\mathscr{C}$-*compactological *lcs, namely those spaces which are $\mathscr{C}$-spaces as locally convex spaces and for which E'_b is a $\mathscr{C}$-space too.*

For a compactological space E, we call $E_{\mathscr{C}} = [\mathscr{S}_{\mathscr{C}}(E'_b)]'_b$ the associated *$\mathscr{C}$-bornology*. We first give a more concrete description. We denote $C^{\infty}_{\mathrm{co},b}(X, E)$ for $X \in \mathscr{C}$, the set of maps $\varphi \in C^{\infty}_{\mathrm{co}}(X, E)$ such that differentials of φ send products of compacts sets to bounded set in the bornology of E.

4.13. Lemma. *For $E, F \in$ **CLCS**, the bornology $E_{\mathscr{C}}$ is generated by bipolars of images of compact sets by smooth maps $\varphi \in C^{\infty}_{\mathrm{co},b}(X, E)$ for $X \in \mathscr{C}$. Moreover, if $g \in L_b(E, F), \varphi \in C^{\infty}_{\mathrm{co},b}(X, E)$ then $g \circ \varphi \in C^{\infty}_{\mathrm{co},b}(X, F)$ and we have a vector space equality $|C^{\infty}_{\mathrm{co},b}(X, E)| = |C^{\infty}_{\mathrm{co},b}(X, E_{\mathscr{C}})|$.*

Proof. Let us call $E_{\mathscr{C}-}$ the associated bornology. Recall the map introduced in the proof of proposition 3.25 : $\mathrm{ev}_X \in C^{\infty}_{\mathrm{co}}(X, (C^{\infty}_{\mathrm{co}}(X))'_c)$ defined by $\mathrm{ev}_X(x)(f) = f(x)$. Also note that the next-to-last statement is obvious since differentials of $g \circ \varphi$ are differentials of φ composed with g, which keeps bounded sets in the associated bornologies.

Since $(.)'_b$ exchange limits and colimits, $E_{\mathscr{C}}$ is described as an inductive limit of dual maps f' of maps $f : E'_b \longrightarrow C^{\infty}_{\mathrm{co}}(X, \mathbb{K})$ which enable to describe the projective limit topology $\mathscr{S}_{\mathscr{C}}(E'_b)$. But since $i : E'_c \longrightarrow E'_b$, $j : (E'_c)'_c \longrightarrow E$ continuous, a map f is associated to an element $\varphi = J_X(f \circ i) = j \circ (f \circ i)'_c \circ \mathrm{ev}_X \in C^{\infty}_{\mathrm{co}}(X, E)$. Let us see it has the claimed property if and only if $f : E'_b \longrightarrow C^{\infty}_{\mathrm{co}}(X, \mathbb{K})$ is continuous. Note that we could even produce such a φ from $f : (\widetilde{E})'_c \longrightarrow C^{\infty}_{\mathrm{co}}(X, \mathbb{K})$ and then we are about to show that the supplementary stated boundedness for φ is equivalent to the continuity of f starting from E'_b. Indeed, differentials of φ are compositions of $j \circ (f \circ i)'_c$ with differentials of ev_X which by definition send products of compact sets to equicontinuous sets in $(C^{\infty}_{\mathrm{co}}(X, \mathbb{K}))'$ and actually to a generating familly of such equicontinuous sets (when all orders of differentiation are considered).

Hence φ has all differentials sending products of compact sets to bounded elements of E if and only if $(f)'_c = j \circ (f \circ i)'_c : (C^{\infty}_{\mathrm{co}}(X, \mathbb{K}))' \longrightarrow E$ (as vector space map) send equicontinuous sets to bounded sets. This is the same as $f : E'_b \longrightarrow C^{\infty}_{\mathrm{co}}(X, \mathbb{K})$ continuous.

By definition, $E_{\mathscr{C}}$ is generated by images by $(f)'_c$ of equicontinuous sets in $(C^{\infty}_{\mathrm{co}}(X))'_c$ and as said above, this are the same as (subsets of) bipolars of images of compacts by differentials of φ above. Such differentials are still smooth maps on products of X, which are still elements of $\mathscr{C}$, by assumption. Hence every bounded set in $E_{\mathscr{C}}$ is bounded in $E_{\mathscr{C}-}$. And the converse is similar.

Since the identity map is in $L_b(E_{\mathscr{C}}, E)$, one deduces: $C^{\infty}_{\mathrm{co},b}(X, E) \supset C^{\infty}_{\mathrm{co},b}(X, E_{\mathscr{C}})$. Conversely, differentials of $\varphi \in C^{\infty}_{\mathrm{co},b}(X, E)$ send compact sets to generators of the bornology $E_{\mathscr{C}}$, hence one automatically gets $\varphi \in C^{\infty}_{\mathrm{co},b}(X, E_{\mathscr{C}})$. ∎

All our future results on $\mathscr{C}$-spaces will be based on the following consequence of cartesian closedness in the form of proposition 3.26.

4.14. Lemma. *Let $E, F \in$ **CLCS** and $\varphi_1 \in C^{\infty}_{\mathrm{co},b}(X_1, E), \varphi_2 \in C^{\infty}_{\mathrm{co},b}(X_1, L_b(F, E)), \psi \in C^{\infty}_{\mathrm{co},b}(X_2, F)$ for $X_i \in \mathscr{C}$ (resp. $\varphi_1 \in C^{\infty}_{\mathrm{co}}(X_1, E), \varphi_2 \in C^{\infty}_{\mathrm{co}}(X_1, L_b(F, E)), \psi \in C^{\infty}_{\mathrm{co}}(X_2, F)$). Then $(\varphi_1 \otimes \psi)(x_1, x_2) = \varphi_1(x_1) \otimes \psi(x_2)$ and $(\varphi_2(\psi))(x_1, x_2) = [\varphi_2(x_1)](\psi(x_2))$ define smooth maps:*

$$(\varphi_1 \otimes \psi) \in C^{\infty}_{\mathrm{co},b}(X_1 \times X_2, E\widehat{\otimes}_H F), \quad (resp.\ (\varphi_1 \otimes \psi) \in C^{\infty}_{\mathrm{co}}(X_1 \times X_2, E\widehat{\otimes}_H F))$$

$$\varphi_2(\psi) \in C^\infty_{co,b}(X_1 \times X_2, E), \quad (resp.\ \varphi_2(\psi) \in C^\infty_{co}(X_1 \times X_2, E)).$$

Proof. Let us first note that

$$(\varphi_1 \otimes \psi) \in C^\infty_{co}(X_1, C^\infty_{co}(X_2, E\widehat{\otimes}_H F)).$$

Indeed, $(\varphi_1 \otimes \psi)(x_1) = \varphi_1(x_1) \otimes \psi$ is clearly a smooth function on X_2. For its smoothness in X_1, one needs to consider differentiability in each seminorm, hence for each expression $\varphi_1(x_1) \otimes d^k\psi$ one notices that images of compact sets by $d^k\psi$ are valued in a set bounded for F. Hence hypocontinuity of $E \times F \longrightarrow E\widehat{\otimes}_H F$ implies all derivatives exists, are of the form $d^l\varphi_1(x_1) \otimes d^k\psi$ and are continuous on compact sets from the continuity of $d^l\varphi_1$ and the same hypocontinuity.

Similarly one gets $(\varphi_1 \otimes \psi) \in C^\infty_{co}(X_2, C^\infty_{co}(X_1, E\widehat{\otimes}_H F))$. This is crucial since the target space need not be Mackey-complete and cartesian closedness does not apply directly, one needs to apply the Mackey-completion functor C_M. Hence, one deduces from proposition 3.26:

$$(\varphi_1 \otimes \psi) \in C^\infty_{co}(X_1, C^\infty_{co}(X_2, C_M(E\widehat{\otimes}_H F))) = C^\infty_{co}(X_1 \times X_2, C_M(E\widehat{\otimes}_H F))).$$

In ordeer to conclude, one needs to see that all partial derivatives are valued in $E\widehat{\otimes}_H F$, since then the continuity is induced (and the Gâteaux differentiability deduced as usual). But by induction, as before, one can check that all partial derivatives are among

$$d^l\varphi_1 \otimes d^k\psi \in C^\infty_{co}(X_2, C^\infty_{co}(X_1, E\widehat{\otimes}_H F)) \cap C^\infty_{co}(X_1, C^\infty_{co}(X_2, E\widehat{\otimes}_H F)).$$

This concludes to the statement for $(\varphi_1 \otimes \psi)$ since, as seen the form of partial derivatives, images by derivatives of compact sets are valued in finite sums of tensor products of bounded sets, hence bounded in $E\widehat{\otimes}_H F$. From this, one gets a map

$$(\varphi_2 \otimes \psi) \in C^\infty_{co,b}(X_1 \times X_2, L_b(F, E)\widehat{\otimes}_H F).$$

And one can compose it with the canonical map from the $*$-autonomous category **CLCS**. We have an evaluation map $e : L_b(L_b(F, E)\widehat{\otimes}_H F, E)$ and, as noticed in lemma 4.13, we deduce:

$$\varphi_2(\psi) \equiv e \circ (\varphi_2 \otimes \psi) \in C^\infty_{co,b}(X_1 \times X_2, E).$$

■

We call $\mathscr{C}$-$L_b(E, F)$ the same lcs as $L_b(E, F)$ but given the bornology $(B_{L_b(E,F)})_{\mathscr{C}}$ namely the associated $\mathscr{C}$-bornology. Note that $\mathscr{C}$-$L_b(E, \mathbb{K}) = E'_b$ as compactological lcs for $E \in \mathscr{C}$-**CLCS**. We also include in our next result a variant of Proposition 3.10.

4.15. Theorem. *$\mathscr{C}$-**CLCS** is a complete and cocomplete $*$-autonomous category with dualizing object $\mathbb{K}$ and internal Hom $\mathscr{C}$-$L_b(E, F)$*

*The functor $U : \mathscr{C}$-**CLCS**$\longrightarrow\mathscr{C}$-**LCS** is left adjoint and also left inverse to $(.)_{\mathscr{C}}$, the functor $E \mapsto E_{\mathscr{C}}$ the space with the same topology and the $\mathscr{C}$-bornology associated to the absolutely convex compact bornology. U is again right adjoint to $(.)_\sigma$ (restriction of the functor from Theorem 2.17). $U, (.)_{\mathscr{C}}, (.)_\sigma$ are faithful. Finally, for any $E, F \in$ **CLCS** such that E, F'_b have $\mathscr{C}$-bornologies, we have the topological embedding :*

$$U(E \otimes_H F'_b) \longrightarrow U(E \otimes_b F'_b) \longrightarrow [\,\widehat{\cdot}^{\mathscr{C}} \circ U](E \otimes_H F'_b). \tag{9}$$

Proof. **Step 1:** Internal Hom functor $\mathscr{C}$-L_b.

Let us explain why $\mathscr{C}$-$CLCS$ is stable by the above internal Hom functor. First, for $E, F \in \mathscr{C}$-$CLCS$, we must see that $(L_b(E,F))'_b = E\widehat{\otimes_H}F'_b$ has a $\mathscr{C}$-bornology. Hence take $A \subset E, B \subset F'_b$ bounded. By definition, F, E'_b are $\mathscr{C}$-spaces, hence by lemma 4.13, one can assume there are smooth maps $\varphi : X_1 \longrightarrow E, \psi : X_2 \longrightarrow F'_b$, with $X_i \in \mathscr{C}$ and the boundedness property stated there such that $A = \varphi(K_1), B = \psi(K_2)$ and K_i compact sets. From lemma 4.14, $A \otimes B = (\varphi \otimes \psi)(K_1 \times K_2)$ is also an image of a compact by the same kind of smooth maps so that $L_b(E,F)$ is a $\mathscr{C}$-space. Note also that bipolars of such $A \otimes B$ are therefore in the first step of the $\mathscr{C}$-completion, hence we also deduce the only missing inclusion to deduce (9) from Proposition 3.10.

From the choice of bornology which is a $\mathscr{C}$-bornology, $\mathscr{C}$-$L_b(E,F) \in \mathscr{C}$-**CLCS**.

Step 2: $\mathscr{C}$-**CLCS** as closed category.

First, let us see that for $E \in \mathscr{C}$-**CLCS**,

$$\mathscr{C}\text{-}L_b(E, \mathscr{C}\text{-}L_b(F,G)) = \mathscr{C}\text{-}L_b(E, L_b(F,G)) \tag{10}$$

Let us first see the equality as vector spaces $|L_b(E, \mathscr{C}\text{-}L_b(F,G))| = |L_b(E, L_b(F,G))|$. $\subset$ is obvious, thus take $f \in L_b(E, L_b(F,G))$, continuity and equicontinuity of bounded sets are the same, thus we have to see equiboundedness. From lemma 4.13, it suffices to consider $\varphi \in C^\infty_{\text{co},b}(X,E)$, generating a bounded set of E. Note that $f \circ \varphi \in C^\infty_{\text{co},b}(X, L_b(F,G))$. Hence the bipolar of the image of a compact set generates, by lemma 4.13 again, a bounded set for $\mathscr{C}$-$L_b(F,G)$. This gives the equiboundedness.

The topology of $L_b(E,H)$ only depends on the topology of H, hence we have the topological equality since both target spaces have the same topology. It remains to compare the bornologies. But from the equal target topologies, again, the equicontinuity condition is the same on both spaces hence boundedness of the map $L_b(E, \mathscr{C}\text{-}L_b(F,G)) \longrightarrow L_b(E, L_b(F,G))$ is obvious.

Take a smooth map $\varphi \in C^\infty_{\text{co},b}(X, L_b(E, L_b(F,G)))$ as in lemma 4.13. In order to conclude to the stated bornological equality, it suffices to see that $\varphi \in C^\infty_{\text{co},b}(X, L_b(E, \mathscr{C}\text{-}L_b(F,G)))$. Since the target topology has not changed, it suffices to see the extra boundedness, hence the image A of a product of compact sets by a higher order differential must be equibounded. Take a generating bounded set $B = \psi(K_2) \subset E$ and see $A = \phi(K_1)$, with $\phi : X_1 \longrightarrow L_b(E, L_b(F,G)), \psi : X_2 \longrightarrow E$. We deduce from lemma 4.14 that $A(B) = (\varphi(\psi))(K_1 \times K_2)$ generates a bounded set of $\mathscr{C}$-$L_b(F,G)$ as expected. This concludes the proof of (10).

As a consequence, for $E, F, G \in \mathscr{C}$-**CLCS**, the previous map L^E_{FG} induces a map

$$\begin{aligned} L^E_{FG} : \mathscr{C}\text{-}L_b(F,G) \longrightarrow \mathscr{C}\text{-}L_b(L_b(E,F), L_b(E,G)) &\longrightarrow \mathscr{C}\text{-}L_b(\mathscr{C}\text{-}L_b(E,F), L_b(E,G)) \\ &= \mathscr{C}\text{-}L_b(\mathscr{C}\text{-}L_b(E,F), \mathscr{C}\text{-}L_b(E,G)) \end{aligned}$$

coinciding with the one from **CLCS** as map. Note that we have used the canonical continuous equibounded map $L_b(L_b(E,F),G) \longrightarrow L_b(\mathscr{C}\text{-}L_b(E,F),G)$ obviously given by the definition of associated $\mathscr{C}$-bornologies which is a smaller bornology.

Step 3: The $*$-autonomous property.

Note that (10) implies the compactological isomorphism

$$\mathscr{C}\text{-}L_b(E,\mathscr{C}\text{-}L_b(F,G)) \simeq \mathscr{C}\text{-}L_b(E,L_b(F,G)) \simeq \mathscr{C}\text{-}L_b(F,L_b(E,G)) \simeq \mathscr{C}\text{-}L_b(F,\mathscr{C}\text{-}L_b(E,G)).$$

Hence, application of lemma 2.11 gives the desired result in the same way as in the corresponding step in the proof of Theorem 2.17, after inducing from Vect the structure of a symmetric closed category.

Step 4: Completeness and cocompleteness.

Let us describe first coproducts and cokernels. The colimit of $\mathscr{C}$-bornologies is still of the same type since the dual is a projective limit of $\mathscr{C}$-spaces hence a $\mathscr{C}$-space. We therefore claim that the colimit is the $\mathscr{C}$-topological space associated to the colimit in **CLCS** with same bornology. Indeed this gives a space in $\mathscr{C}$-**CLCS** since there are more compact sets hence the compatibility condition in **CLCS** is still satisfied and functoriality of $\mathscr{S}_{\mathscr{C}}$ implies the universal property. Completeness follows by duality.

Step 5: Adjunctions and consequences.

Since $(E_\sigma)'_b = E'_\sigma$ is always a $\mathscr{C}$-space, since $\mathbb{K} \in \mathscr{C}$, the functor $(.)_\sigma$ is valued in $\mathscr{C}$-**CLCS** $\subset$ **CLCS** hence the adjunction. Moreover $U((E_{\mathscr{C}})'_b) = \mathscr{S}_{\mathscr{C}}(E'_c)$ by construction. The key vector space (even lcs) identity

$$\mathscr{C}\text{-}\mathbf{LCS}(U(F),E) = \mathbf{LCS}(U(F),E) = L_b(F,E_c) = \mathscr{C}\text{-}\mathbf{CLCS}(F,E_{\mathscr{C}})$$

comes from (10) in seeing $E_c = L_b((E_c)'_b,\mathbb{K})$ and $E_{\mathscr{C}} = \mathscr{C}\text{-}L_b((E_c)'_b,\mathbb{K})$. All naturality conditions are easy.

■

4.16. Preliminaries in the $\mathscr{C}$-complete $\mathscr{C}$-space setting. Let $\mathscr{C}c\mathscr{C}$-**LCS** the full subcategory of **LCS** of $\mathscr{C}$-complete $\mathscr{C}$-spaces. Our first work is to recover a monoidal category. We first deduce from Theorem 4.15 and a variant of [S, Prop 2] a useful:

4.17. Lemma. *Let $\invamp_{\mathscr{C}b}$ be the $\invamp$ of the complete $*$-autonomous category $\mathscr{C}$-**CLCS** given by $A \invamp_{\mathscr{C}b} B = \mathscr{C}\text{-}L_b(A'_b, B)$. Then, we have the equality in $\mathscr{C}$-**CLCS**:*

$$\forall E, F \in \mathscr{C}c\mathscr{C}\text{-}\mathbf{LCS}, \quad E_{\mathscr{C}} \invamp_{\mathscr{C}b} F_{\mathscr{C}} = (E\varepsilon F)_{\mathscr{C}}. \tag{11}$$

As a consequence, $(\mathscr{C}c\mathscr{C}\text{-}\mathbf{LCS}, \varepsilon, \mathbb{K})$ is a symmetric monoidal category. Moreover, $\mathscr{C}$-spaces are stable by ε.

Contrary to the k-quasi-complete setting, we don't know if $\mathscr{C}$-complete spaces are stable by ε, we only prove that $\mathscr{C}$-complete $\mathscr{C}$-spaces are.

Proof. First assume only E, F are $\mathscr{C}$-spaces. Note that $(E_{\mathscr{C}})'_b$ is $\mathscr{S}_{\mathscr{C}}(E'_c)$ with equicontinuous bornology, which is a $\mathscr{C}$-bornology. Moreover, by definition $F_{\mathscr{C}} = \mathscr{C}\text{-}L_b((F_c)'_b, \mathbb{K}), F_c = L_b((F_c)'_b, \mathbb{K})$, hence (10) and the fact that $(\cdot)_c$ is right adjoint to the underlying functor (theorem 2.17 (2)) gives:

$$U(E_{\mathscr{C}} \invamp_{\mathscr{C}b} F_{\mathscr{C}}) = U(L_b((E_{\mathscr{C}})'_b, F_{\mathscr{C}})) = U(L_b((E_{\mathscr{C}})'_b, F_c) = L_\epsilon(\mathscr{S}_{\mathscr{C}}(E'_c), F) = L_\epsilon(E'_c, F) = E\varepsilon F$$

using in the next-to-last equality F is a $\mathscr{C}$-space. Especially, this imples that $E\varepsilon F$ is a $\mathscr{C}$-space.

From now on, all spaces E, F are in $\mathscr{C}c\mathscr{C}$-**LCS**. Now in $\mathscr{C}$-**CLCS** we even have using again (10):

$$\begin{aligned} E_{\mathscr{C}} \invamp_{\mathscr{C}b} F_{\mathscr{C}} &= \mathscr{C}\text{-}L_b((E_{\mathscr{C}})'_b, \mathscr{C}\text{-}L_b((F_c)'_b, \mathbb{K})) \\ &= \mathscr{C}\text{-}L_b(\mathscr{S}_{\mathscr{C}}((E_c)'_b), L_b((F_c)'_b, \mathbb{K})) = \mathscr{C}\text{-}L_b((E_c)'_b, F_c) \end{aligned}$$

so that the bornology is the $\mathscr{C}$-bornology associated to the ϵ-equicontinuous bornology of $E\varepsilon F$ (the one of $E_c \invamp_b F_c$).

It remains to identify this bornology with the one of $[E\varepsilon F]_{\mathscr{C}}$. Of course from this description, and the boundedness of $E_c \invamp_b F_c \longrightarrow [E\varepsilon F]_c$ recalled in the proof of lemma 3.9, the identity map $E_{\mathscr{C}} \invamp_{\mathscr{C}b} F_{\mathscr{C}} \longrightarrow [E\varepsilon F]_{\mathscr{C}}$ is bounded. One must check the converse.

Using lemma 4.13, it suffices to see that any $\varphi : C^\infty_{\mathrm{co}}(X_1, [E\varepsilon F]_c)$ is also in $C^\infty_{\mathrm{co},b}(X_1, E_c \invamp_b F_c)$. We must check that $\varphi_1 = d^k\varphi \in C^\infty_{\mathrm{co}}(X_1^{k+1}, L_b((E_c)'_b, F))$ send a compact set $K_1 \subset X_1^{k+1}$ to an ε-equihypocontinuous set $A = d^k\varphi(K_1)$. Up to symmetry $E - F$, it suffices to see that given an equicontinuous set $B \subset (E_c)'$, the image $A(B) \subset F$ generates an absolutely convex compact set. But since E is a $\mathscr{C}$-space, it suffices to consider $B = \psi(K_2)$ for $\psi \in C^\infty_{\mathrm{co},b}(X_2, (E_c)'_b)$. But $A(B) = \varphi_1(\psi)(K_1 \times K_2)$ and, from lemma 4.14, $\varphi_1(\psi) \in C^\infty_{\mathrm{co}}(X_1^{k+1} \times X_2, F)$. Hence, since F is $\mathscr{C}$-complete, using lemma 4.3 (5), the closed absolutely convex hull of $A(B)$ is still compact, hence bounded in F_c as expected. Moreover, it also follows in that way that the absolutely closed convex hull of $\varphi(K)$ is ε-equihypocontinuous in $E\varepsilon F$ hence relatively compact. Therefore, $E\varepsilon F$ is also $\mathscr{C}$-complete, by the characterization from lemma 4.3 (5).

Let us now prove that $(\mathscr{C}c\mathscr{C}\text{-}\mathbf{LCS}, \varepsilon, \mathbb{K})$ is symmetric monoidal using lemma 3.11 starting from $(\mathscr{C}\text{-}\mathbf{CLCS}, \mathfrak{F}_{\mathscr{C}}, \mathbb{K})$. We apply it to the functor $(\cdot)_{\mathscr{C}} : \mathscr{C}c\mathscr{C}\text{-}\mathbf{LCS} \longrightarrow \mathscr{C}\text{-}\mathbf{CLCS}$ with left adjoint $\widehat{\cdot}^{\mathscr{C}} \circ U$ using U from Theorem 4.15. The lemma concludes since the assumptions are easily satisfied, especially $E\varepsilon F = U([E\varepsilon F]_{\mathscr{C}}) = \widehat{\cdot}^{\mathscr{C}} \circ U([E\varepsilon F]_{\mathscr{C}})$ from stability of $\mathscr{C}$-completeness and using the key relation (11). ∎

4.18. $\mathscr{C}$-REFLEXIVE SPACES. As before, $\mathscr{C}$ is a full small cartesian subcategory of **k-Ref** containing $\mathbb{K} = \mathbb{R}$. We define a new notion for the dual of E. It consists in taking the Arens-dual of the $\mathscr{C}$-completion of the $\mathscr{C}$-space $\mathscr{S}_{\mathscr{C}}(E)$, which is once again transformed into a $\mathscr{C}$-complete $\mathscr{C}$-space $E^*_{\mathscr{C}}$.

4.19. DEFINITION. *For a lcs E we write* $E'_{\mathscr{C}} := \mathscr{S}_{\mathscr{C}}(E'_c), E'_{\mathscr{C}\rho} = (\widehat{\mathscr{S}_{\mathscr{C}}(E)}^{\mathscr{C}})'_{\mathscr{C}}$ *and* $E^*_{\mathscr{C}} := \widehat{E'_{\mathscr{C}\rho}}^{\mathscr{C}}$.

First note functoriality:

4.20. LEMMA. $(\cdot)^*_{\mathscr{C}}$, $(\cdot)'_{\mathscr{C}}$ *and* $(\cdot)'_{\mathscr{C}\rho}$ *are contravariant endofunctors on* **LCS**.

PROOF. They are obtained by composing the already recalled functors $\mathscr{S}_{\mathscr{C}}$, $(\cdot)'_c$ and $\widehat{\cdot}^{\mathscr{C}}$. ∎

From the Mackey theorem and the fact that completion of a space does not change its dual, we can deduce immediately that we have the following algebraic identities $|(E'_{\mathscr{C}})'| = |E|$ and $|(E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho}| = |\widehat{\mathscr{S}_{\mathscr{C}}(E)}^{\mathscr{C}}|$.

From these we deduce the fundamental algebraic equality:

$$\left|(E^*_{\mathscr{C}})^*_{\mathscr{C}}\right| = \left|\widehat{\mathscr{S}_{\mathscr{C}}(E)}^{\mathscr{C}}\right| \tag{12}$$

4.21. DEFINITION. *A lcs E is said* $\mathscr{C}$-reflexive *if the canonical map* $E \longrightarrow \widehat{\mathscr{S}_{\mathscr{C}}(E)}^{\mathscr{C}} \simeq (E^*_{\mathscr{C}})^*_{\mathscr{C}}$ *gives a topological isomorphism* $E \simeq (E^*_{\mathscr{C}})^*_{\mathscr{C}}$.

We write $\mathscr{C}$-**Ref** $\subset$ **LCS** the full subcategory of $\mathscr{C}$-reflexive spaces.

We are looking for a condition necessary to make the above equality a topologically one. The following theorem demonstrates an analogous to $E'_c \simeq ((E'_c)'_c)'_c$ for our new dual. We now make use of lemma 3.6.

4.22. THEOREM. *For any separated locally convex space,* $((E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} \simeq E^*_{\mathscr{C}}$ *is* $\mathscr{C}$*-reflexive. A space is* $\mathscr{C}$*-reflexive if and only if* $E \simeq (E'_{\mathscr{C}})'_{\mathscr{C}}$ *and both E and* $E'_{\mathscr{C}}$ *are* $\mathscr{C}$*-complete.*

PROOF. One can assume E is a $\mathscr{C}$-complete $\mathscr{C}$-space (all functors start by $\mathscr{S}_{\mathscr{C}}$ followed by $\mathscr{C}$-completion and this $\mathscr{C}$-completion is a $\mathscr{C}$-space by [Ju, Prop 7.2.1-3] as any subspace of the completion of a $\mathscr{C}$-space). Let us see that $(E'_{\mathscr{C}})'_{\mathscr{C}}$ is also $\mathscr{C}$-complete. Indeed $(E'_{\mathscr{C}})'_c \longrightarrow (E'_c)'_c \longrightarrow E$ continuous with same dual hence so is $(E'_{\mathscr{C}})'_{\mathscr{C}} \longrightarrow E$ by functoriality of $\mathscr{S}_{\mathscr{C}}$. But any smooth $\varphi \in C^\infty_{co}(X, (E'_{\mathscr{C}})'_{\mathscr{C}})$ give the same kind of map with value in E, the closed absolutely convex hull agrees (with the common bipolar) and hence is complete in E and also in $(E'_{\mathscr{C}})'_{\mathscr{C}}$ by [Bo2, IV.5 Rmq 2].

Let us write $C_{\mathscr{C}}(.) = \widehat{\cdot}^{\mathscr{C}}$ for the $\mathscr{C}$-completion functor and for an ordinal λ, $C^\lambda_{\mathscr{C}}(E) = E_{\mathscr{C},\lambda}$ from lemma 4.5.

There is a continuous map $(E^*_{\mathscr{C}})'_{\mathscr{C}} \longrightarrow (E'_{\mathscr{C}})'_{\mathscr{C}}$. We apply lemma 3.6 to the duality functor $D = (\cdot)'_{\mathscr{C}}$ to $E_0 = E'_{\mathscr{C}}$, $E_\lambda = C^\lambda_{\mathscr{C}}(E_0)$. Any $\overline{\Gamma(K)}$ in the union defining $E_{\lambda+1}$ is equicontinuous in $((E_{\lambda+1})'_{\mathscr{C}})' = (E_{\lambda+1})_{\mathscr{C}}$ since K is the image of a compact by $\varphi \in C^\infty_{\mathrm{co}}(X, E_\lambda)) \subset C^\infty_{\mathrm{co},b}(X, (E_{\lambda+1})_c)$. So a fortiori $\overline{\Gamma(K)}$ is equicontinuous in $((E_{\lambda_0})'_{\mathscr{C}})'$ for λ_0 large enough.

We apply the lemma to another K closed absolutely convex cover of a compact set of $K_{\mathscr{C}}((E^*_{\mathscr{C}})'_{\mathscr{C}})$ computed in $(E'_{\mathscr{C}})'_{\mathscr{C}}$ therefore compact there by assumption. The lemma gives K is complete in $(E^*_{\mathscr{C}})'_{\mathscr{C}}$ and therefore contains the bipolar of the compact computed in $(E^*_{\mathscr{C}})'_{\mathscr{C}}$ which must also be compact as a closed subset of a compact. In this case, we have deduced that $(E^*_{\mathscr{C}})'_{\mathscr{C}} = (E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho}$ is $\mathscr{C}$-complete.

Clearly $((E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} = ((E^*_{\mathscr{C}})'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} \longrightarrow E^*_{\mathscr{C}}$ continuous. Dualizing the continuous linear map $(E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} \longrightarrow E$, one gets $E'_{\mathscr{C}\rho} \longrightarrow ((E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} = ((E^*_{\mathscr{C}})'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} \longrightarrow E^*_{\mathscr{C}}$ and since the space in the middle is already $\mathscr{C}$-complete inside $E^*_{\mathscr{C}}$ which is the $\mathscr{C}$-completion, it must be equal to $E^*_{\mathscr{C}}$ and thus $E^*_{\mathscr{C}}$ is $\mathscr{C}$-reflexive (since any odd number larger than 2 of applications of $(\cdot)'_{\mathscr{C}\rho}$ lead to isomorphic spaces[1]) and we have the stated equality.

For the last statement, sufficiency is clear, since if $E = (E'_{\mathscr{C}})'_{\mathscr{C}}$, E is a $\mathscr{C}$-space and since we assume it is $\mathscr{C}$-complete $E'_{\mathscr{C}\rho} = E'_{\mathscr{C}}$ and since we assume it is $\mathscr{C}$-complete too it is equal to $E^*_{\mathscr{C}}$ too. The same applies to $E'_{\mathscr{C}}$ which satisfies the same assumptions hence $(E^*_{\mathscr{C}})^*_{\mathscr{C}} = (E'_{\mathscr{C}})^*_{\mathscr{C}} = (E'_{\mathscr{C}})'_{\mathscr{C}} = E$.

Conversely, obviously a $\mathscr{C}$-reflexive space is a $\mathscr{C}$-complete $\mathscr{C}$-space and we already noted $(E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} = (E^*_{\mathscr{C}})'_{\mathscr{C}} \longrightarrow (E'_{\mathscr{C}})'_{\mathscr{C}} \longrightarrow E$ in the $\mathscr{C}$-complete $\mathscr{C}$-space case which implies $(E'_{\mathscr{C}})'_{\mathscr{C}} \simeq E$ if $(E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} \simeq E$ (which is the case if E is $\mathscr{C}$-reflexive). Finally $E^*_{\mathscr{C}} = ((E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} = ((E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho})'_{\mathscr{C}} = E'_{\mathscr{C}}$ implies this space is also k-quasi-complete. ∎

4.23. The dialogue category $\mathscr{C}c\mathscr{C}$-**LCS**. We will now use lemma 3.17 to obtain a dialogue category.

4.24. Proposition. *The negation $(.)^*_{\mathscr{C}}$ gives $\mathscr{C}c\mathscr{C}$-$\mathbf{LCS}^{\mathrm{op}}$ the structure of a dialogue category with tensor product ε.*

Proof. Lemma 4.17 gives $\mathscr{C}c\mathscr{C}$-$\mathbf{LCS}^{\mathrm{op}}$ the structure of a symmetric monoidal category. We have to check that $(.)^*_{\mathscr{C}} : \mathscr{C}c\mathscr{C}\text{-}\mathbf{LCS}^{\mathrm{op}} \longrightarrow \mathscr{C}c\mathscr{C}\text{-}\mathbf{LCS}$ is a tensorial negation on $\mathscr{C}c\mathscr{C}$-$\mathbf{LCS}^{\mathrm{op}}$.

For, we write it as a composition of functors involving $\mathscr{C}$-**CLCS**.

Note that $(\cdot)_{\mathscr{C}} : \mathscr{C}c\mathscr{C}\text{-}\mathbf{LCS} \longrightarrow \mathscr{C}\text{-}\mathbf{CLCS}$ (the composition of inclusion and the functor of the same name in Theorem 4.15) is right adjoint to $L := \hat{\cdot}^{\mathscr{C}} \circ U$ in combining this result with the completion functor, which is clearly left adjoint to $\mathscr{C}c\mathscr{C}\text{-}\mathbf{LCS} \subset \mathscr{C}\text{-}\mathbf{LCS}$. Then on $\mathscr{C}c\mathscr{C}$-**LCS**,

$$(.)^*_{\mathscr{C}} = \hat{\cdot}^{\mathscr{C}} \circ \mathscr{S}_{\mathscr{C}} \circ (\cdot)'_c = L \circ \mathscr{S}_{\mathscr{C}} \circ (\cdot)'_b \circ (\cdot)_c = L \circ (\cdot)'_b \circ (\cdot)_{\mathscr{C}}.$$

Lemma 3.17 and the following remark concludes. ∎

[1]The restriction on the number larger than two of applications comes from the fact that we need $\mathscr{C}$-completeness of E to get a continuous map $(E'_{\mathscr{C}\rho})'_{\mathscr{C}\rho} \longrightarrow E$. We don't need to get any completeness of $E'_{\mathscr{C}\rho}$ but we have just showed that any larger number of iteration of this dual leads to $\mathscr{C}$-complete spaces.

4.25. The $*$-autonomous category $\mathscr{C}$-**Ref**. We conclude, mutatis mutandis, as in Theorem 3.20.

4.26. Theorem. *The category $\mathscr{C}$-**Ref**, endowed with the tensor product $E \otimes_{\mathscr{C}} F = (E^*_{\mathscr{C}} \epsilon F^*_{\mathscr{C}})^*_{\mathscr{C}}$, dual $(\cdot)^*_{\mathscr{C}}$ and dualizing object $\mathbb{K}$, is a $*$-autonomous category. It is equivalent to the Kleisli category of the comonad $T = ((\cdot)^*_{\mathscr{C}})^*_{\mathscr{C}}$ in $\mathscr{C}c\mathscr{C}$-**LCS**.*

Proof. ($\mathscr{C}c\mathscr{C}$-**LCS**$, \epsilon, \mathbb{K}$) is a dialogue category by proposition 4.24 with idempotent continuation monad T by Theorem 4.22. We conclude as in Theorem 3.20 in using theorem 4.15 instead of theorem 2.17 and theorem 4.22 instead of lemma 3.7. ∎

5. Schwartz locally convex spaces, Mackey-completeness and the ρ-dual.

In order to obtain a $*$-autonomous category adapted to convenient smoothness, we want to replace k-quasi-completeness by the weaker Mackey-completeness and specialize our previous section. In order to ensure associativity of the dual of the ε-product, Mackey-completeness is not enough we have to restrict simultaneously to Schwartz topologies. Subsection 5.1 identifies various cases of $\mathscr{C}$ we can use in the Mackey-complete setting. This preliminary work will enable us to recover several models of linear logic on the same $*$-autonomous category later in the next part. We thus define our appropriate weakened reflexivity (ρ-reflexivity) in subsection 5.6 using for $\mathscr{C}$ a sufficiently large category of Banach spaces. We investigate categorical completeness, which is easier in the Schwartz space setting, in 5.18. The two last sections detail the equivalence of $\mathscr{C}$-**Ref** for various $\mathscr{C}$. We still assume $\mathbb{K} = \mathbb{R}$ in this section.

5.1. More details in the case $\mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$. We are going to give more examples in a more restricted context. We now fix Fin $\subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$. But the reader may assume $\mathscr{C} \subset$ Ban if he or she wants, our case is not such more general. Recall that $\mathbf{F} \times \mathbf{DFS} \subset \mathbf{LCS}$ is the full subcategry consisting of (finite) products of Fréchet spaces and strong duals of Fréchet-Schwartz spaces

Note that we already know from lemma 4.3, that $\mathscr{C}$-completeness is equivalent to Mackey-completeness in that case. Note that then $|C^\infty_{\mathscr{C}}(E, F)| = |C^\infty(E, F)|$ algebraically. For it suffices to see $|C^\infty_{\mathrm{co}}(X, F)| = |C^\infty(X, F)|$ for any $X \in \mathbf{F} \times \mathbf{DFS}$ (since then the extra smoothness condition will be implied by convenient smoothness). Note that any such X is ultrabornological (using [Ja, Corol 13.2.4], [Ja, Corol 13.4.4,5] since a DFS space is reflexive hence its strong dual is barrelled [Ja, Prop 11.4.1] and for a dual of a Fréchet space, the quoted result implies it is also ultrabornological, for products this is [Ja, Thm 13.5.3]). By cartesian closedness of both sides this reduces to two cases. For any Fréchet space X, Fréchet smooth maps are included in $C^\infty_{\mathrm{co}}(X, F)$ which is included in $C^\infty(X, F)$ which coincides with the first space of Fréchet smooth maps by [KM, Th 4.11.(1)] (which ensures the continuity of Gateaux derivatives with value in bounded linear maps with strong topology for derivatives, those maps being the same as continuous linear maps as seen the bornological property). The case of strong duals of Fréchet-Schwartz spaces is similar using [KM, Th 4.11.(2)]. The index $\mathscr{C}$ in $C^\infty_{\mathscr{C}}(E, F)$ remains to point out the different topologies.

5.2. Example. If $\mathscr{C} = \mathbf{F} \times \mathbf{DFS}$ (say with objects of density character smaller than some inaccessible cardinal) then $\mathscr{C}^\infty \subset \mathbf{Sch}$, from [Me, Corol 13 p 279]. Let us see equality. Indeed, $(\ell^1(\mathbb{N}))'_c \subset C^\infty_{\rm co}(\ell^1(\mathbb{N}), \mathbb{K})$ and $(\ell^1(\mathbb{N}))'_c = (\ell^1(\mathbb{N}))'_\mu$ (since on $\ell^1(\mathbb{N})$ compact and weakly compact sets coincide [HNM, p 37]), and $(\ell^1(\mathbb{N}))'_\mu$ is a universal generator of Schwartz spaces [HNM, Corol p 36], therefore $C^\infty_{\rm co}(\ell^1(\mathbb{N}), \mathbb{K})$ is also such a universal generator. Hence, we even have $\mathscr{C}^\infty = \mathrm{Ban}^\infty = \mathbf{Sch}$. Let us deduce even more of such type of equalities.

Note also that $\mathrm{Sym}(E'_c \varepsilon E'_c) \subset C^\infty_{\rm co}(E, \mathbb{K})$ is a complemented subspace given by quadratic forms. In case $E = H$ is an infinite dimensional Hilbert space, by Buchwalter's theorem $H'_c \varepsilon H'_c = (H \widehat{\otimes}_\pi H)'_c$ and it is well-known that $\ell^1(\mathbb{N}) \simeq D$ is a complemented subspace (therefore a quotient) of $H \widehat{\otimes}_\pi H$ as diagonal copy (see e.g. [Ry, ex 2.10]) with the projection a symmetric map. Thus $D'_c \subset H'_c \varepsilon H'_c$ and it is easy to see it is included in the symmetric part $\mathrm{Sym}(E'_c \varepsilon E'_c)$. As a consequence, $C^\infty_{\rm co}(H, \mathbb{K})$ is also such a universal generator of Schwartz spaces.

Finally, consider $E = \ell^m(\mathbb{N}, \mathbb{C})$ $m \in \mathbb{N}, m \geq 1$. The canonical multiplication map, given by Hölder inequality, $\ell^m(\mathbb{N}, \mathbb{C})^{\otimes_\pi m} \longrightarrow \ell^1(\mathbb{N}, \mathbb{C})$ is a metric surjection realizing the target as a quotient of the symmetric subspace generated by tensor powers (indeed $\sum a_k e_k$ is the image of $(\sum a_k^{1/m} e_k)^{\otimes m}$ so that $(\ell^1(\mathbb{N}, \mathbb{C}))'_c \subset \mathrm{Sym}([(\ell^m(\mathbb{N}, \mathbb{C}))'_c]^{\varepsilon m})$. Thus $C^\infty_{\rm co}(\ell^m(\mathbb{N}, \mathbb{C}), \mathbb{K})$ is also such a universal generator of Schwartz spaces.

We actually checked that for any $\mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$ with $\ell^1(\mathbb{N}) \in \mathscr{C}$ or $\ell^2(\mathbb{N}) \in \mathscr{C}$ or $\ell^m(\mathbb{N}, \mathbb{C}) \in \mathscr{C}$ then $\mathscr{C}^\infty = \mathbf{Sch}$ so that

$$\mathscr{S} = \mathscr{S}_{\rm Ban} = \mathscr{S}_{Hilb} = \mathscr{S}_{\mathscr{C}} = \mathscr{S}_{\mathbf{F} \times \mathbf{DFS}}.$$

As a consequence, we can improve slightly our previous results in this context:

5.3. Lemma. *Let $\mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$ as above. For any lcs E, there is a continuous identity map: $E'_{\mathscr{C}} \longrightarrow \mathscr{S}_{\mathscr{C}}(E'_\mu) \longrightarrow \mathscr{S}_{\mathscr{C}}(E'_c)$. If moreover E is Mackey-complete, this is a topological isomorphism.*

Proof. Indeed, by definition, $\mathscr{S}_{\mathscr{C}}(E'_\mu)$ is described by a projective limit over maps in the spaces:

$$L(E'_\mu, C^\infty_{\rm co}(X, \mathbb{K})) = C^\infty_{\rm co}(X, \mathbb{K}) \varepsilon E \subset C^\infty_{\rm co}(X, E)$$

with equality by the Schwartz property (used via lemma 5.11). As in lemma 4.10, the identity map $E'_{\mathscr{C}} \longrightarrow \mathscr{S}_{\mathscr{C}}(E'_\mu)$ is continuous. But by functoriality, one has also a continuous identity map $\mathscr{S}_{\mathscr{C}}(E'_\mu) \longrightarrow \mathscr{S}_{\mathscr{C}}(E'_c)$ and in the Mackey-complete case $\mathscr{S}_{\mathscr{C}}(E'_c) \longrightarrow E'_{\mathscr{C}}$ by lemma 4.10. (This uses that Mackey-complete implies $\mathscr{C}$-complete in our case by the last statement in lemma 4.3). ■

5.4. Example. Note also that if D is a quotient with quotient topology of a Fréchet space C with respect to a closed subspace, then $C^\infty_{\rm co}(D, \mathbb{K})$ is a subspace of $C^\infty_{\rm co}(C, \mathbb{K})$ with induced topology. Indeed, the injection is obvious and derivatives agree, and since from [K, §22.3.(7)], compacts are quotients of compacts, the topology is indeed induced. Therefore if $\mathscr{D}$ is obtained from $\mathscr{C} \subset \mathbf{F}$, the category of Fréchet spaces, by taking all quotients by closed subspaces, then $\mathscr{C}^\infty = \mathscr{D}^\infty$.

5.5. Example. If $\mathscr{C}$ = Fin then Fin^∞ = **Nuc**, since $C^\infty_{co}(\mathbb{R}^n, \mathbb{K}) \simeq \mathfrak{s}^{\mathbb{N}}$ [V82, (7) p 383], a countable direct product of classical sequence space $\mathfrak{s}$, which is a universal generator for nuclear spaces. Thus, the associated nuclear topology functor is $\mathscr{N}(E) = \mathscr{S}_{\mathrm{Fin}}(E)$.

5.6. ρ-reflexive spaces and their Arens-Mackey duals. We now specialize even further and consider the case $\mathscr{C}$ = Ban in the following subsections.

We define a new notion for the dual of E, which consists in taking the Arens-dual of the Mackey-completion of the Schwartz space $\mathscr{S}(E)$, which is once again transformed into a Mackey-complete Schwartz space E^*_ρ.

5.7. Definition. *For a lcs E, the topology $\mathscr{S}\rho(E', E)$ on E' is the topology of uniform convergence on absolutely convex compact sets of $\widehat{\mathscr{S}(E)}^M$. We write $E'_{\mathscr{S}\rho} = (E', \mathscr{S}\rho(E', E)) = (\widehat{\mathscr{S}(E)}^M)'_c$. We write $E^*_\rho = \widehat{\mathscr{S}(E'_{\mathscr{S}\rho})}^M$ and $E'_{\mathscr{R}} = \mathscr{S}(E'_{\mathscr{S}\rho})$.*

Note that, with our previous notation, $E^*_\rho = E^*_{\mathrm{Ban}}$, $E'_{\mathscr{R}} = E'_{\mathscr{C}\rho}$.

5.8. Remark. Note that $E'_{\mathscr{S}\rho}$ is in general not Mackey-complete: there is an Arens dual of a Mackey-complete space (even of a nuclear complete space with its Mackey topology) which is not Mackey-complete using [BD, thm 34, step 6]. Indeed take Γ a closed cone in the cotangent bundle (with 0 section removed) $\dot{T}^*R^n$. Consider Hörmander's space $E = \mathcal{D}'_\Gamma(\mathbb{R}^n)$ of distributions with wave front set included in Γ with its normal topology in the terminology of [BD, Prop 12,29]. It is shown there that E is nuclear complete. Therefore the strong dual is $E'_\beta = E'_c$. Moreover, [BD, Lemma 10] shows that this strong dual if $\mathcal{E}'_\Lambda$, the space of compactly supported distributions with a wave front set in the open cone $\Lambda = -\Gamma^c$ with a standard inductive limit topology. This dual is shown to be nuclear in [BD, Prop 28]. Therefore we have $E'_c = E'_{\mathscr{S}\rho}$. Finally, as explained in the step 6 of the proof of [BD, Thm 34] where it is stated it is not complete, as soon as Λ is not closed (namely by connectedness when $\Gamma \notin \{\emptyset, \dot{T}^*R^n\}$), then E'_c is not even Mackey-complete. This gives our claimed counter-example. The fact that E above has its Mackey topology is explained in [D].

5.9. Definition. *A lcs E is said ρ-*reflexive* if the canonical map $E \longrightarrow \widehat{\mathscr{S}(E)}^M = (E^*_\rho)^*_\rho$ gives a topological isomorphism $E \simeq (E^*_\rho)^*_\rho$.*

We also need a technical definition and a well-known fact. We first define a variant of the Schwartz ε-product:

5.10. Definition. *For two separated lcs E and F, we define $E\eta F = L(E'_\mu, F)$ the space of continuous linear maps on the Mackey dual with the topology of uniform convergence on equicontinuous sets of E'. We write $\eta(E, F) = (E'_\mu \otimes_{\beta e} F'_\mu)'$ with the topology of uniform convergence on products of equicontinuous sets.*

This space $E\eta F$ has already been studied in [K2]. For us it will be only used once in lemma 5.26 to quote a technical result from [K2] in section 5.25.

5.11. Lemma. *If E is a Schwartz lcs, $E'_c \simeq E'_\mu$. so that for any lcs F, $E\eta F \simeq E\varepsilon F$ topologically. Thus for any lcs E, $E'_\mu \simeq (\mathscr{S}(E))'_c$ and $(E'_{\mathrm{Ban}})'_{\mathrm{Ban}} = \mathscr{S}((E'_\mu)'_\mu)$ has the Schwartz topology associated to the Mackey topology of its dual $\mu_{(s)}(E, E')$.*

Proof. Take K an absolutely convex $\sigma(E', E)$-weakly compact in E, it is an absolutely convex closed set in E and precompact as any bounded set in a Schwartz space [Ho, 3,§15 Prop 4]. [Bo2, IV.5 Rmq 2] concludes to K complete since $E \longrightarrow (E'_\mu)'_\sigma$ continuous with same dual and K complete in $(E'_\mu)'_\sigma$, and since K precompact, it is therefore compact in E. As a consequence E'_c is the Mackey topology. Hence, $|E\eta F| = |L(E'_\mu, F)| = |L(E'_c, F)| = |E\varepsilon F|$ algebraically and the topologies are defined in the same way. The newt-to-last statement comes from the first and $E'_\mu \simeq (\mathscr{S}(E))'_\mu$. The last statement follows since $(E'_{\mathrm{Ban}})'_{\mathrm{Ban}} \simeq \mathscr{S}([\mathscr{S}(E'_c)]'_c) \simeq \mathscr{S}([E'_c]'_\mu) \simeq \mathscr{S}([E'_\mu]'_\mu)$ (the last equality being obvious since the Mackey-topology only depends on the dual pair). ■

If we specialize Theorem 4.22 to our current context, we obtain:

5.12. Theorem. *Let E be a separated locally convex space, then E^*_ρ is ρ-reflexive. A lcs is ρ-reflexive, if and only if it is Mackey-complete, has its Schwartz topology associated to the Mackey topology of its dual $\mu_{(s)}(E, E')$ and its dual is also Mackey-complete with its Mackey topology. As a consequence, Mackey duals of ρ-reflexive spaces coincide with their Arens duals, and are exactly Mackey-complete locally convex spaces with their Mackey topology such that their Mackey dual is Mackey-complete.*

5.13. Remark. A k-quasi-complete space is Mackey-complete hence for a k-reflexive space E, $\mathscr{S}((E'_\mu)'_\mu)$ is ρ-reflexive (since E'_c k-quasi-complete implies that so is E'_μ which is a stronger topology). Our new setting is a priori more general than the one of section 4. We will pay the price of a weaker notion of smooth maps. Note that a Mackey-complete space need not be k-quasi-complete (see lemma 5.16 below).

Proof. For the last and only unseen statement, we have already seen that the condition is necessary, it is sufficient since for F Mackey-complete with its Mackey topology with Mackey-complete Mackey-dual, $\mathscr{S}(F)$ is ρ-reflexive by what we just saw and so that $(\mathscr{S}(F))'_c$ is the Mackey topology on F', by symmetry $[\mathscr{S}([\mathscr{S}(F)]'_c)]'_c = F$ and therefore F is both Mackey and Arens dual of the ρ-reflexive space $\mathscr{S}([\mathscr{S}(F)]'_c)$. ■

Several relevant categories have appeared in the course of the proof. μ**LCS** $\subset$ **LCS** the full subcategory of spaces having their Mackey topology. μ**Sch** $\subset$ **LCS** the full subcategory of spaces having the Schwartz topology associated to its Mackey topology. **Mb** $\subset$ **LCS** the full subcategory of spaces with a Mackey-complete Mackey dual. And then by intersection always considered as full subcategories, one obtains:

$$\mathbf{Mc}\mu\mathbf{Sch} = \mathbf{Mc} \cap \mu\mathbf{Sch}, \quad \mathbf{Mb}\mu\mathbf{Sch} = \mathbf{Mb} \cap \mu\mathbf{Sch},, \quad \mathbf{McMb} = \mathbf{Mb} \cap \mathbf{Mc},$$

$$\mu\mathbf{McMb} = \mathbf{McMb} \cap \mu\mathbf{LCS}, \quad \rho\text{-}\mathbf{Ref} = \mathbf{McMb} \cap \mu\mathbf{Sch}.$$

We can summarize the situation as follows: There are two functors $(.)'_c$ and μ the associated Mackey topology (contravariant and covariant respectively) from the category ρ-**Ref** to μ**McMb** the category of Mackey duals of ρ-Reflexive spaces (according to the previous proposition). There are two other functors $(.)^*_\rho, \mathscr{S}$ and they are the (weak) inverses of the two previous ones.

In order to give a counterexample, we recall two lemmas that will turn out to be useful later:

5.14. Lemma. *The associated Schwartz topology functor $\mathscr{S}$ commutes with arbitrary products, quotients and embeddings (and as a consequence with arbitrary projective kernels or categorical limits).*

Proof. For products and (topological) quotients, this is [Ju, Prop 7.4.2]. For embeddings (that he calls topological injections), this is [Ju, Prop 7.4.8] based on the previous ex 7.4.7. The consequence comes from the fact that any projective kernel is a subspace of a product, as a categorical limit is a kernel of a map between products. ∎

5.15. Lemma. *For any separated lcs F, we have a topological isomorphism:*

$$\widehat{((F'_\mu)'_\mu)}^M \simeq ((\widehat{F}^M)'_\mu)'_\mu.$$

Proof. Recall also from [K, §21.4.(5)] the completion of the Mackey topology has its Mackey topology $\widetilde{((F'_\mu)'_\mu)} = ((\widetilde{F})'_\mu)'_\mu$. This implies, as for any completion that $(F'_\mu)'_\mu) \subset ((\widehat{F}^M)'_\mu)'_\mu$ has the induced topology. A fortiori, the continuous inclusions $((F'_\mu)'_\mu) \longrightarrow (\hat{F}^M)'_\mu)'_\mu \longrightarrow ((\widetilde{F})'_\mu)'_\mu$ always have the induced topology. In the transfinite description of the Mackey completion, the Cauchy sequences and the closures are the same in $((\widetilde{F})'_\mu)'_\mu$ and $\widetilde{F}$ (since they have same dual hence same bounded sets), therefore one finds the stated topological isomorphism. ∎

Finally, the following lemma explains that our new setting is more general than the k-quasi-complete setting of section 3:

5.16. Lemma. *There is a space $E \in$* **McμSch** *which is not k-quasi-complete.*

Proof. We take $\mathbb{K} = \mathbb{R}$ (the complex case is similar). Let $F = C^0([0,1])$ the Banach space with the topology of uniform convergence. We take $G = \mathscr{S}(F'_\mu) = F'_c$ which is complete since F ultrabornological [Ja, Corol 13.2.6]. Consider $H = \mathrm{Vect}\{\delta_x, x \in [0,1]\}$ the vector space generated by Dirac measures and $E = \widehat{H}^M$ the Mackey completion with induced topology (since we will see E identifies as a subspace of G). Let K be the unit ball of F', the space of measures on $[0,1]$. It is absolutely convex, closed for any topology compatible with duality, for instance in G, and since G is a Schwartz space, it is precompact, and complete by completeness of G, hence compact. By Krein-Millman's theorem [K, §25.1.4] it is the closed convex cover of its extreme points. Those are known to be $\delta_x, -\delta_x, x \in [0,1]$ [K, §25.2.(2)]. Especially, E is dense in G, which is therefore its completion. By the proof of lemma 5.15, the Mackey-topology of E is induced by G and thus by lemma 5.14, $\mathscr{S}(E_\mu)$ is also the induced topology from G. Hence $E \in$ **McμSch**. But by Maharam decomposition of measures, it is known that F' has the following decomposition (see e.g. [Ha, p 22]) as an ℓ^1-direct sum:

$$F' = L^1(\{0,1\}^\omega)^{\oplus_1 2^\omega} \oplus_1 \ell^1([0,1])$$

and the Dirac masses generate part of the second component, so that $H \subset \ell^1([0,1])$ in the previous decomposition. But the bounded sets in G are the same as in F'_β (by principle of uniform boundedness), hence Mackey-convergence in G implies norm convergence in F'_β, so that by completeness of $\ell^1([0,1])$, $E \subset \ell^1([0,1])$. Hence Lebesgue measure (which gives one of the summands $L^1(\{0,1\}^\omega)$) gives $\lambda \notin E$. Finally, consider $\delta : [0,1] \longrightarrow K \subset G$ the dirac mass map. It is continuous since a compact set in F is equicontinuous by the Ascoli theorem, which gives exactly uniform continuity of δ on compact sets in F. Hence $\delta([0,1])$ is compact in E while its absolutely convex cover in G contains λ so that the intersection with E cannot be complete, hence E is not k-quasi-complete. ■

A specialization of proposition 4.24 and theorem 4.26 gives:

5.17. Proposition. *The negation* $(.)^*_\rho$ *gives* **McSch**$^{\mathrm{op}}$ *the structure of a dialogue category with tensor product* ε*, with commutative continuation monad*

5.18. Relation to projective limits and direct sums. We now deduce the following stability properties from Theorem 5.12.

5.19. Corollary. *The class of ρ-reflexive spaces is stable by countable locally convex direct sums and arbitrary products.*

Proof. Let $(E_i)_{i\in I}$ a countable family of ρ-reflexive spaces, and $E = \oplus_{i\in I} E_i$. Using Theorem 5.12, we aim at proving that E is Mackey-complete, has its Schwartz topology associated to the Mackey topology of its dual $\mu_{(s)}(E, E')$ and its dual is also Mackey-complete with its Mackey topology.

From the Theorem 5.12, E_i itself has the Schwartz topology associated to its Mackey topology. From [K, §22. 5.(4)], the Mackey topology on E is the direct sum of Mackey topologies. Moreover the maps $E_i \longrightarrow \mathscr{S}(E_i)$ give a direct sum map $E \longrightarrow \oplus_{i\in I} \mathscr{S}(E_i)$ and thus a continuous map $\mathscr{S}(E) \longrightarrow \oplus_{i\in I} \mathscr{S}(E_i)$ since a countable direct sum of Schwartz spaces is a Schwartz space. Conversely the maps $E_i \longrightarrow E$ give maps $\mathscr{S}(E_i) \longrightarrow \mathscr{S}(E)$ and by the universal property this gives $\mathscr{S}(E) \simeq \oplus_{i\in I} \mathscr{S}(E_i)$. Therefore, if all spaces E_i are ρ-reflexive, E carries the Schwartz topology associated to its Mackey topology. From [KM, Th 2.14, 2.15], Mackey-complete spaces are stable by arbitrary projective limits and direct sums, thus the Mackey-completeness condition on the space and its dual (using the computation of dual Mackey topology from [K, §22. 5.(3)]) are also satisfied.

For an arbitrary product, [K, §22. 5.(3)] again gives the Mackey topology, universal properties and stability of Schwartz spaces by arbitrary products give the commutation of $\mathscr{S}$ with arbitrary products and the stability of Mackey-completeness can be safely used (even for the dual, uncountable direct sum).

■

5.20. Lemma. *For $(E_i, i \in I)$ a (projective) directed system of Mackey-complete Schwartz locally convex space if $E = \mathrm{proj\ lim}_{i\in I} E_i$, then:*

$$((E)^*_\rho)^*_\rho \simeq \Big[\Big[\mathrm{proj}\ \lim_{i\in I}((E_i)^*_\rho)^*_\rho)\Big]^*_\rho\Big]^*_\rho.$$

The same holds for general locally convex kernels and categorical limits.

Proof. The bidualization functor give maps $((E)^*_\rho)^*_\rho \longrightarrow ((E_i)^*_\rho)^*_\rho$ and then universal property of projective limits gathers those maps into $(E^*_\rho)^*_\rho \longrightarrow \text{proj } \lim_{i\in I}((E_i)^*_\rho)^*_\rho$, (see [K, §19.6.(6)] for l.c. kernels) and bidualization and ρ-reflexivity concludes to the first map. Conversely, the canonical continuous linear map in the Mackey-complete Schwartz case $((E_i)^*_\rho)^*_\rho = ((E_i)'_c)^*_\rho \longrightarrow E_i$ gives the reverse map after passing to the projective limit and double ρ-dual. The localy convex kernel case and the categorical limit case are identical. ■

5.21. Proposition. *The category ρ-**Ref** is complete and cocomplete, with products and countable direct sums agreeing with those in **LCS** and limits given in lemma 5.20*

Proof. Bidualazing after application of **LCS**-(co)limits clearly gives (co)limits. Corollary 5.19 gives the product and sum case. ■

5.22. The $*$-autonomous category ρ-**Ref**.

5.23. Definition. *We thus consider ρ-**Ref**, the category of ρ-reflexive spaces, with tensor product $E \otimes_\rho F = ((E^*_\rho \varepsilon F^*_\rho)^*_\rho)$ and internal hom $E \multimap_\rho F = (((E^*_\rho)\varepsilon F)^*_\rho)^*_\rho$.*

5.24. Theorem. *The category ρ-**Ref** endowed with the tensor product $\otimes_\rho$, and internal Hom $\multimap_\rho$ is a complete and cocomplete $*$-autonomous category with dualizing object $\mathbb{K}$. It is equivalent to the Kleisli category of the comonad $T = ((\cdot)^*_\rho)^*_\rho$ in **McSch**.*

Proof. Corollary 5.21 has already dealt with categorical (co)completeness. The rest is a special case of Theorem 4.26. ■

5.25. $\mathscr{C}$-**Ref** in the case $\mathscr{C} \subset$ **F** $\times$ **DFS**. We now fix Fin $\subset \mathscr{C} \subset$ **F** $\times$ **DFS**.

We saw that $\mathscr{S}_\mathscr{C}\mathscr{S} = \mathscr{S}_\mathscr{C}$ so that from lemma 5.11, $(\mathscr{S}_\mathscr{C}(E))'_c \longrightarrow (\mathscr{S}(E))'_c = E'_\mu$ is continuous, and since they have the same dual we have topological equality $(\mathscr{S}_\mathscr{C}(E))'_c = E'_\mu$. As for $\mathscr{S}$, $\left(\widehat{\mathscr{S}_\mathscr{C}(E)}\right)^M = \mathscr{S}_\mathscr{C}(\widehat{E}^M)$. Hence we have: $E'_{\mathscr{C}\rho} = \mathscr{S}_\mathscr{C}(\widehat{E}^M)'_\mu$.

We defined $E^*_\mathscr{C}$ for $E \in$ **McSch** as the Mackey completion of $\mathscr{S}_\mathscr{C}((\widehat{E}^M)'_\mu)$ and thus:

$$E^*_\mathscr{C} = \mathscr{S}_\mathscr{C}(E^*_\rho).$$

Moreover, since $E \longrightarrow \mathscr{S}_\mathscr{C}(E)$ is the continuous identity map, the full subcategory $\mathscr{C}$-**Mc** $\subset$ **McSch** of objects satisfying $E = \mathscr{S}_\mathscr{C}(E)$ is reflective of reflector $\mathscr{S}_\mathscr{C}$.

We need a general lemma deduced from (9).

5.26. Lemma. *Let $X, Y \in \mathscr{C}$-**LCS** and define $G = (X\varepsilon Y)'_\varepsilon$ the dual with the topology of convergence on equicontinuous sets from the duality with $H = X'_c \otimes_{\beta e} Y'_c$. Then we have the embedding $H \subset G \subset \widehat{H}^M$, and the topological isomorphisms: $((H)'_\mu)'_\mu \simeq X'_\mu \otimes_i Y'_\mu$ and $((\widehat{H}^M)'_\mu)'_\mu \simeq X'_\mu \widehat{\otimes}_i^M Y'_\mu$.*

Proof. We apply (9) to $(X_c)'_b, (Y_c)'_b$ which have $\mathscr{C}$-bornologies since $X, Y \in \mathscr{C}$-**LCS**. Note that $H = U((X_c)'_b \otimes_H (Y_c)'_b)$ and that

$$U((X_c)'_b \otimes_b (Y_c)'_b) = U\Big(\Big[((X_c)'_b)'_b \parr_b ((X_c)'_b)'_b\Big]'_b\Big) = U\Big(\Big[X_c \parr_b Y_c\Big]'_b\Big) = G.$$

In order to prove the second statement, we use the space introduced in definition 5.10. Let us check $\eta(X,Y) = X\varepsilon Y$ (lemma 5.11) is also the set of bilinear forms on $E'_\mu \times F'_\mu$ which are separately continuous. Of course $\eta(X,Y) = X\varepsilon Y$ is included in the space of separately continuous forms. Conversely, if $f : E'_\mu \times F'_\mu \longrightarrow \mathbb{K}$ is separately continuous, from [Ja, Corol 8.6.5], it is also separately continuous on $E'_\sigma \times F'_\sigma$ and the fact it coincides with $\eta(X,Y)$ follows from [K2, §40.4.(5)].

Then, we need to see that the Mackey topology $((X'_c \otimes_{\beta e} Y'_c)'_\mu)'_\mu = X'_\mu \otimes_i Y'_\mu$ is the inductive tensor product. The fact that both algebraic tensor products have the same dual implies there is, by Arens-Mackey Theorem, a continuous identity map $((X'_c \otimes_{\beta e} Y'_c)'_\mu)'_\mu \longrightarrow X'_\mu \otimes_i Y'_\mu$. Conversely, one uses the universal property of the inductive tensor product which gives a separately continuous map $X'_\mu \times Y'_\mu \longrightarrow X'_\mu \otimes_{\beta e} Y'_\mu = X'_c \otimes_{\beta e} Y'_c$. But applying functoriality of Mackey duals on each side gives for each $x \in X'_\mu$ a continuous map $Y'_\mu \longrightarrow ((X'_c \otimes_{\beta e} Y'_c)'_\mu)'_\mu$ and by symmetry, a separately continuous map $X'_\mu \times Y'_\mu \longrightarrow ((X'_c \otimes_{\beta e} Y'_c)'_\mu)'_\mu$. The universal property of the inductive tensor product again concludes. The last isomorphisms for Mackey-completions follows by lemma 5.15. ∎

We can thus improve our Theorem 4.26 and get:

5.27. Theorem. *Let* Fin $\subset \mathscr{C} \subset$ **F** $\times$ **DFS** *a full cartesian small subcategory.* $\mathscr{C}$-**Ref** *is a complete and cocomplete* $*$*-autonomous category with tensor product* $E \otimes_{\mathscr{C}} F = (E^*_{\mathscr{C}} \varepsilon F^*_{\mathscr{C}})^*_{\mathscr{C}}$ *and dual* $(.)^*_{\mathscr{C}}$ *and dualizing object* $\mathbb{K}$. *It is stable by arbitrary products. It is equivalent to the Kleisli category of* $\mathscr{C}$-**Mc** *and to* ρ-**Ref** *as a* $*$*-autonomous category via the inverse functors:* $\mathscr{S}_{\mathscr{C}} : \rho$-**Ref** $\longrightarrow \mathscr{C}$-***Ref*** *and* $\mathscr{S}([.]_\mu) : \mathscr{C}$-***Ref*** $\longrightarrow \rho$-***Ref***.

Proof. Once the last statement proven, everything will follow from Theorems 4.26 and 5.24. Recall from the previous proof that $(\mathscr{S}_{\mathscr{C}}(E))'_\mu = E'_\mu$ and $((E)^*_{\mathscr{C}})^*_{\mathscr{C}} = \mathscr{S}_{\mathscr{C}}((E^*_\rho)^*_\rho)$. This implies the two functors are inverse of each other as stated.

We show they intertwine the other structure. We already noticed $E^*_{\mathscr{C}} = \mathscr{S}_{\mathscr{C}}(E^*_\rho)$.

We computed in lemma 5.26 the Mackey-completions:

$$C_M((E^*_\rho \varepsilon F^*_\rho)'_\mu) \simeq (E^*_\rho)'_\mu \widehat{\otimes}_i^M (F^*_\rho)'_\mu \simeq C_M((E^*_{\mathscr{C}} \varepsilon F^*_{\mathscr{C}})'_\mu).$$

Since ε product keeps Mackey-completeness, one can compute $(\cdot)^*_{\mathscr{C}}$ and $(\cdot)^*_\rho$ by applying respectively $\mathscr{S}_{\mathscr{C}}(\hat{\cdot}^M)$ and $\mathscr{S}(\hat{\cdot}^M)$, which gives the missing topological identity:

$$\mathscr{S}_{\mathscr{C}}\Big((E^*_\rho \varepsilon F^*_\rho)^*_\rho\Big) \simeq (E^*_{\mathscr{C}} \varepsilon F^*_{\mathscr{C}})^*_{\mathscr{C}}.$$

∎

We now provide several more advanced examples which will enable us to prove that we obtain different comonads in several of our models of *LL*. They are all based on the important approximation property of Grothendieck.

5.28. Example. If E a Fréchet space without the approximation property (in short AP, for instance $E = B(H)$ the space of bounded operators on a Hilbert space), then from [Me, Thm 7 p 293], $C^\infty_{co}(E)$ does not have the approximation property. Actually, $E'_c \subset C^\infty_{co}(E)$ is a continuously complemented subspace so that so is $((E'_c)^*_\rho)^*_\rho \subset ((C^\infty_{co}(E))^*_\rho)^*_\rho$. But for any Banach space $E'_c = \mathscr{S}(E'_\mu)$ is Mackey-complete so that $(E'_c)^*_\rho = \mathscr{S}(E)$, $((E'_c)^*_\rho)^*_\rho = E^*_\rho = E'_c = \mathscr{S}(E'_\mu)$. Thus since for a Banach space E has the approximation property if and only if $\mathscr{S}(E'_\mu)$ has it [Ja, Thm 18.3.1], one deduces that $((C^\infty_{co}(E))^*_\rho)^*_\rho$ does not have the approximation property [Ja, Prop 18.2.3].

5.29. Remark. We will see in the appendix in lemma 9.2 that for any lcs E, $((C^\infty_{\mathrm{Fin}}(E))^*_\rho)^*_\rho$ is Hilbertianisable, hence it has the approximation property. This implies that $\mathscr{N}(E'_\mu) \subset C^\infty_{\mathrm{Fin}}(E)$ with induced topology is not complemented, as soon as E is Banach space without AP, since otherwise $((\mathscr{N}(E'_\mu))^*_\rho)^*_\rho \subset ((C^\infty_{\mathrm{Fin}}(E))^*_\rho)^*_\rho$ would be complemented and $((\mathscr{N}(E'_\mu))^*_\rho)^*_\rho = ((E'_c)^*_\rho)^*_\rho = E'_c$ would have the approximation property, and this may not be the case. This points out that the change to a different class of smooth function in the next section is necessary to obtain certain models of DiLL. Otherwise, the differential that would give such a complementation cannot be continuous.

II. Models of LL and DiLL

From now on, since we only wish to deal with smooth maps, we assume $\mathbb{K} = \mathbb{R}$.

6. Smooth maps and new models of LL

6.1. A general construction for LL models. We have used intensively dialogue categories from [MT, T] to obtain $*$-autonomous categories, but their notion of models of tensor logic is less fit for our purposes since the cartesian category they use need not be cartesian closed. For us trying to check their conditions involving an adjunction at the level of the dialogue category would imply introducing a non-natural category of smooth maps while we have already a good cartesian closed category. Therefore we propose a variant of their definition using relative adjunctions [U].

6.2. Definition. *Consider categories C, C' and $\mathcal{D}$, and a functor $J : C' \longrightarrow C$. A functor $F : \mathcal{D} \longrightarrow C$ admits $G : C' \longrightarrow \mathcal{D}$ as* right J-adjoint *if there is a binatural isomorphism:*

$$\mathcal{D}(\_, G(\_)) \simeq C(F(\_), J(\_)).$$

6.3. Definition. *A* linear (resp. and commutative) categorical model of λ-tensor logic *is a complete and cocomplete dialogue category $(C^{\mathrm{op}}, \parr_C, I, \neg)$ with a (resp. commutative and idempotent) continuation monad $T = \neg\neg$, jointly with a cartesian category $(\mathcal{M}, \times, 0)$, a symmetric strongly monoidal functor* NL $: \mathcal{M} \longrightarrow C^{\mathrm{op}}$ *having a right $\neg$-relative adjoint U. The model is said to be a Seely categorical model of λ-tensor logic if U is moreover bijective on objects.*

This last terminology is inspired from a well-known result [PAM, Prop 25] which states that being bijective on objects is the only missing information for a linear-non-linear adjunction to give rise to a *Seely category*. We will use this result in Theorem 6.6.

This definition is convenient for its concision, but it does not emphasize that $\mathcal{M}$ must be cartesian closed. Since our primitive objects are functional, we will prefer an equivalent alternative based on the two relations we started to show in lemma 4.2, namely an enriched adjointness of cartesian closedness and a compatibility with $\parr$.

6.4. Definition. *A* (resp. commutative) λ-categorical model of λ-tensor logic *is a complete and cocomplete dialogue category* $(\mathcal{C}^{op}, \parr_{\mathcal{C}}, 1_{\mathcal{C}} = \mathbb{K}, \neg)$ *with a (resp. commutative and idempotent) continuation monad* $T = \neg\neg$*, jointly with a cartesian closed category* $(\mathcal{M}, \times, 0, [\cdot,\cdot])$*, and a functor* NL : $\mathcal{M} \longrightarrow \mathcal{C}^{op}$ *having a right* $\neg$*-relative adjoint* U*, which is assumed faithful, and compatibility natural isomorphisms in* $\mathcal{M}, \mathcal{C}$ *respectively:*

$$\Xi_{E,F} : U(\mathrm{NL}(E) \parr_{\mathcal{C}} F) \longrightarrow [E, U(F)], \Lambda^{-1}_{E,F,G} : \mathrm{NL}(E) \parr_{\mathcal{C}} \Big(\mathrm{NL}(F) \parr_{\mathcal{C}} G\Big) \longrightarrow \mathrm{NL}(E \times F) \parr_{\mathcal{C}} G$$

satisfying the following six commutative diagrams (where $\mathrm{Ass}^{\parr}, \rho, \lambda, \sigma^{\parr}$ *are associator, right and left unitors and braiding in* $\mathcal{C}^{op}$ *and* $\Lambda^{\mathcal{M}}, \sigma^{\times}, \ell, r$ *are the curry map, braiding and unitors in the cartesian closed category* $\mathcal{M}$*) expressing an intertwining between curry maps:*

$$\begin{array}{ccccc} U\Big(\mathrm{NL}(E) \parr_{\mathcal{C}} \big(\mathrm{NL}(F) \parr_{\mathcal{C}} G\big)\Big) & \xrightarrow{\Xi_{E,\mathrm{NL}(F)\parr_{\mathcal{C}} G}} & [E, U(\mathrm{NL}(F) \parr_{\mathcal{C}} G)] & \xrightarrow{[\mathrm{Id}_E, \Xi_{F,G}]} & [E,[F,U(G)]] \\ \uparrow{\scriptstyle U(\Lambda_{E,F,G})} & & & & \uparrow{\scriptstyle \Lambda^{\mathcal{M}}} \\ U\Big(\mathrm{NL}(E \times F) \parr_{\mathcal{C}} G\Big) & & \xrightarrow{\Xi_{E\times F,G}} & & [E \times F, U(G)] \end{array}$$

compatibility of Ξ *with the (relative) adjunctions (written* $\simeq$ *and* φ*, the characteristic isomorphism of the dialogue category* $\mathcal{C}^{op}$*):*

$$\begin{array}{ccccc} \mathcal{M}(0, U\Big(\mathrm{NL}(E) \parr_{\mathcal{C}} F\Big)) & \xrightarrow{\simeq} & \mathcal{C}(\neg(\mathrm{NL}(0)), \mathrm{NL}(E) \parr_{\mathcal{C}} F) & \xrightarrow{\varphi^{op}_{\mathrm{NL}(E),F,\mathrm{NL}(0)}} & \mathcal{C}(\neg(F \parr_{\mathcal{C}} \mathrm{NL}(0)), \mathrm{NL}(E)) \\ \uparrow{\scriptstyle M(I_{\mathcal{M}})} & & & & \uparrow{\scriptstyle \simeq} \\ M(U\Big(\mathrm{NL}(E) \parr_{\mathcal{C}} F\Big)) & & \xrightarrow{M(\Xi_{E,F})} & & M([E, U(F)]) = \mathcal{M}(E, U(F)) \end{array}$$

compatibility with associativity:

$$\begin{array}{ccccc} \mathrm{NL}(E) \parr_{\mathcal{C}} \Big(\mathrm{NL}(F) \parr_{\mathcal{C}} G\Big) & \xrightarrow{\Lambda^{-1}_{E,F,G}} & \mathrm{NL}(E \times F) \parr_{\mathcal{C}} G & \xrightarrow{\rho_{\mathrm{NL}(E\times F)\parr_{\mathcal{C}} G}} & \Big(\mathrm{NL}(E \times F) \parr_{\mathcal{C}} \mathbb{K}\Big) \parr_{\mathcal{C}} G \\ \downarrow{\scriptstyle \mathrm{Ass}^{\parr}_{\mathrm{NL}(E),\mathrm{NL}(F),G}} & & & & \uparrow{\scriptstyle \Lambda^{-1}_{E,F,\mathbb{K}} \parr_{\mathcal{C}} G} \\ \Big(\mathrm{NL}(E) \parr_{\mathcal{C}} \mathrm{NL}(F)\Big) \parr_{\mathcal{C}} G & & \xrightarrow{(\mathrm{NL}(E)\parr\, \rho_{\mathrm{NL}(F)})\parr\, G} & & \Big(\mathrm{NL}(E) \parr_{\mathcal{C}} (\mathrm{NL}(F) \parr_{\mathcal{C}} \mathbb{K})\Big) \parr_{\mathcal{C}} G \end{array}$$

compatibility with symmetry,

$$\begin{array}{ccccc}
\mathrm{NL}(E) \parr_C \mathrm{NL}(F) & \xrightarrow{\mathrm{NL}(E)\parr\, \rho_{\mathrm{NL}(F)}} & \mathrm{NL}(E) \parr_C \left(\mathrm{NL}(F) \parr_C \mathbb{K}\right) & \xrightarrow{\Lambda^{-1}_{E,F,\mathbb{K}}} & \mathrm{NL}(E \times F) \parr_C \mathbb{K} \\
\Big\downarrow{\scriptstyle \sigma^{\parr}_{\mathrm{NL}(E),\mathrm{NL}(F)}} & & & & \Big\downarrow{\scriptstyle \mathrm{NL}(\sigma^{\times}_{E,F})\parr_C\mathbb{K}} \\
\mathrm{NL}(F) \parr_C \mathrm{NL}(E) & \xrightarrow{\mathrm{NL}(F)\parr\, \rho_{\mathrm{NL}(E)}} & \mathrm{NL}(F) \parr_C \left(\mathrm{NL}(E) \parr_C \mathbb{K}\right) & \xrightarrow{\Lambda^{-1}_{F,E,\mathbb{K}}} & \mathrm{NL}(F \times E) \parr_C \mathbb{K}
\end{array}$$

and compatibility with unitors for a given canonical isomorphism $\epsilon : \mathbb{K} \longrightarrow \mathrm{NL}(0_{\mathcal{M}})$*:*

$$\begin{array}{ccccc}
\mathrm{NL}(0_{\mathcal{M}}) \parr_C \mathrm{NL}(F) & \xrightarrow{\mathrm{NL}(0_{\mathcal{M}})\parr\, \rho_{\mathrm{NL}(F)}} & \mathrm{NL}(0_{\mathcal{M}}) \parr_C \left(\mathrm{NL}(F) \parr_C \mathbb{K}\right) & \xrightarrow{\Lambda^{-1}_{0_{\mathcal{M}},F,\mathbb{K}}} & \mathrm{NL}(0_{\mathcal{M}} \times F) \parr_C \mathbb{K} \\
\Big\uparrow{\scriptstyle \epsilon\parr_C\mathrm{NL}(F)} & & & & \Big\downarrow{\scriptstyle \mathrm{NL}(\ell_F)\parr_C\mathbb{K}} \\
\mathbb{K} \parr_C \mathrm{NL}(F) & \xrightarrow{\lambda^{-1}_{\mathrm{NL}(F)}} & \mathrm{NL}(F) & \xrightarrow{\rho_{\mathrm{NL}(F)}} & \mathrm{NL}(F) \parr_C \mathbb{K}
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{NL}(E) \parr_C \mathrm{NL}(0_{\mathcal{M}}) & \xrightarrow{\mathrm{NL}(E)\parr\, \rho_{\mathrm{NL}(0_{\mathcal{M}})}} & \mathrm{NL}(E) \parr_C \left(\mathrm{NL}(0_{\mathcal{M}}) \parr_C \mathbb{K}\right) & \xrightarrow{\Lambda^{-1}_{E,0_{\mathcal{M}},\mathbb{K}}} & \mathrm{NL}(E \times 0_{\mathcal{M}}) \parr_C \mathbb{K} \\
\Big\uparrow{\scriptstyle \mathrm{NL}(E)\parr_C\epsilon} & & & & \Big\downarrow{\scriptstyle \mathrm{NL}(r_E)\parr_C\mathbb{K}} \\
\mathrm{NL}(E) \parr_C \mathbb{K} & & \xrightarrow{\qquad id \qquad} & & \mathrm{NL}(E) \parr_C \mathbb{K}
\end{array}$$

The model is said to be a Seely model if U is bijective on objects.

In our examples, U must be thought of as an underlying functor that forgets the linear structure of $\mathcal{C}$ and sees it as a special smooth structure in $\mathcal{M}$. Hence we could safely assume it faithful and bijective on objects.

6.5. Proposition. *A Seely λ-model of λ-tensor logic is a Seely linear model of λ-tensor logic.*

Proof. Start with a λ-model. Let

$$\mu^{-1}_{E,F} = \rho^{-1}_{\mathrm{NL}(E\times F)} \circ \Lambda^{-1}_{E,F,\mathbb{K}} \circ (\mathrm{Id}_{\mathrm{NL}(E)} \parr \rho_{\mathrm{NL}(F)}) : \mathrm{NL}(E) \parr_C \mathrm{NL}(F) \longrightarrow \mathrm{NL}(E \times F)$$

using the right unitor ρ of $\mathcal{C}^{\mathrm{op}}$, and composition in $\mathcal{C}$. The identity isomorphism ϵ is also assumed given. Since μ is an isomorphism it suffices to see it makes NL a lax symmetric monoidal functor. The symmetry condition is exactly the diagram of compatibility with symmetry that we assumed and similarly for the unitality conditions. The first assumed diagram with Λ used in conjunction with U faithful enables us to transport any diagram valid in the cartesian closed category to an enriched version, and the second diagram concerning compatibility with associativity is then the only missing part needed so that μ satisfies the relation with associators of $\parr, \times$. ∎

These models enable us to recover models of linear logic. We get a linear-non-linear adjunction in the sense of [Ben] (see also [PAM, def 21 p 140]).

6.6. Theorem. *Assume that* $(\mathcal{C}^{\mathrm{op}}, \parr_{\mathcal{C}}, I, \neg, \mathcal{M}, \times, 0, \mathrm{NL}, U)$ *is a Seely linear model of* λ*-tensor logic. Let* $\mathcal{D} \subset \mathcal{C}$ *the full subcategory of objects of the form* $\neg C, C \in \mathcal{C}$*. Then,* $\mathcal{N} = U(\mathcal{D})$ *is equivalent to* $\mathcal{M}$*.* $\neg \circ \mathrm{NL} : \mathcal{N} \longrightarrow \mathcal{D}$ *is left adjoint to* $U : \mathcal{D} \longrightarrow \mathcal{N}$ *and forms a linear-non-linear adjunction. Finally* $! = \neg \circ \mathrm{NL} \circ U$ *gives a comonad on* $\mathcal{D}$ *making it a* $*$*-autonomous complete and cocomplete Seely category with Kleisli category for* ! *isomorphic to* $\mathcal{N}$*.*

Proof. This is a variant of [T, Thm 2.13]. We already saw in lemma 2.13 that $\mathcal{D}$ is $*$-autonomous with the structure defined there. Composing the natural isomorphisms for $F \in \mathcal{D}, E \in \mathcal{M}$

$$\mathcal{M}(E, U(F)) \simeq \mathcal{C}^{\mathrm{op}}(\mathrm{NL}(E), \neg F) \simeq \mathcal{D}(\neg(\mathrm{NL}(E)), F),$$

one gets the stated adjunction. The equivalence is the inclusion with inverse $\neg\neg : \mathcal{M} \longrightarrow \mathcal{N}$ which is based on the canonical map in $\mathcal{C}$, $\eta_E : \neg\neg E \longrightarrow E$ which is mapped via U to a corresponding natural transformation in $\mathcal{M}$. It is an isomorphism in $\mathcal{N}$ since any element is image of U enabling to use the $\neg$-relative adjunction for $E \in \mathcal{C}$:

$$\mathcal{M}(U(E), U(\neg\neg E)) \simeq \mathcal{C}^{\mathrm{op}}(\mathrm{NL}(U(E)), \neg\neg\neg E) \simeq \mathcal{C}^{\mathrm{op}}(\mathrm{NL}(U(E)), \neg E) \simeq \mathcal{M}(U(E), U(E)).$$

Hence the element corresponding to identity gives the inverse of η_E. Since $\mathcal{D}$ is coreflective in $\mathcal{C}$, the coreflector preserves limits enabling to compute them in $\mathcal{D}$, and by $*$-autonomy, it therefore has colimits (which must coincide with those in $\mathcal{C}$). By [PAM, Prop 25 p 149], since $U : \mathcal{D} \longrightarrow \mathcal{N}$ is still a bijection on objects, the fact that $\mathcal{D}$ is a Seely category follows and the computation of its Kleisli category too. The co-unit and co-multiplication of the co-monad ! come from the relative adjunction $U \dashv_{\neg} \mathrm{NL}$, and correspond respectively to the identity on E in M, and to the composition of the unit of the adjunction by ! on the left and U on the right. ■

6.7. Remark. In the previous situation, we checked that $U(E) \simeq U(\neg\neg E)$ in $\mathcal{M}$ and we even obtained a natural isomorphism $U \circ \neg\neg \simeq U$ and this has several consequences we will reuse. First $\neg$ is necessarily faithful on $\mathcal{C}$ since if $\neg(f) = \neg(g)$ then $U \circ \neg\neg(f) = U \circ \neg\neg(g)$ hence $U(f) = U(g)$ and U is assumed faithful hence $f = g$. Let us see that as a consequence, as for ε, $\parr_{\mathcal{C}}$ preserves monomorphisms. Indeed if $f : E \longrightarrow F$ is a monomorphism, $\neg\neg(f \parr_{\mathcal{C}} \mathrm{Id}_G)$ is the application of the $\neg\neg(\cdot) \parr G$ for the $*$-autonomous continuation category, hence a right adjoint functor, hence $\neg\neg(f \parr_{\mathcal{C}} \mathrm{Id}_G)$ is a monomorphism since right adjoints preserve monomorphisms. Since $\neg\neg$ is faithful one deduces $f \parr_{\mathcal{C}} \mathrm{Id}_G$ is a monomorphism too.

6.8. A class of examples of LL models. We now fix $\mathrm{Fin} \subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$. Recall that then $|C^{\infty}_{\mathscr{C}}(E, F)| = |C^{\infty}(E, F)|$ algebraically for any lcs E, F. The index $\mathscr{C}$ remains to point out the different topologies.

Let $\mathscr{C}$-$\mathbf{Ref}_{\infty}$, $\mathscr{C}$-$\mathbf{Mc}_{\infty}$ be the cartesian categories with the same spaces as $\mathscr{C}$-$\mathbf{Ref}$, $\mathscr{C}$-$\mathbf{Mc}$ but with $\mathscr{C}$-smooth maps, namely conveniently smooth maps. Let $U : \mathscr{C}\text{-}\mathbf{Ref} \longrightarrow \mathscr{C}\text{-}\mathbf{Ref}_{\infty}$ the inclusion functor (forgetting linearity and continuity of the maps). Note that, for $\mathscr{C} \subset \mathscr{D}$, $\mathscr{C}$-$\mathbf{Mc}_{\infty} \subset \mathscr{D}$-$\mathbf{Mc}_{\infty}$ is a full subcategory.

6.9. Theorem. *Let* Fin $\subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$ *as above.* $\mathscr{C}$-**Ref** *is also a Seely category with structure extended by the comonad* $!_{\mathscr{C}}(\cdot) := (C^\infty_{\mathscr{C}}(\cdot))^*_{\mathscr{C}}$ *associated to the adjunction with left adjoint* $!_{\mathscr{C}} : \mathscr{C}\text{-}\mathbf{Ref}_\infty \longrightarrow \mathscr{C}\text{-}\mathbf{Ref}$ *and right adjoint* U.

Proof. We apply Theorem 6.6 to $C = \mathscr{C}\text{-}\mathbf{Mc}$ so that $\mathcal{D} := \mathscr{C}\text{-}\mathbf{Ref}$ and $\mathcal{N} := \mathscr{C}\text{-}\mathbf{Ref}_\infty$. For that we must check the assumptions of a λ-categorical model for $\mathcal{M} = \mathscr{C}\text{-}\mathbf{Mc}_\infty$. Lemma 4.2 shows that $\mathcal{M}$ is a cartesian closed category since the internal hom functor $C^\infty_{\mathscr{C}}(E, F)$ is almost by definition in $\mathscr{C}$-**Mc**. Indeed it is a projective limit of $\mathscr{C}^\infty_{\text{co}}(X)\varepsilon F$ which is a projective kernel of $\mathscr{C}^\infty_{\text{co}}(X)\varepsilon\mathscr{C}^\infty_{\text{co}}(Y) = \mathscr{C}^\infty_{\text{co}}(X\times Y)$ with $X, Y \in \mathscr{C}$ as soon as $F \in \mathscr{C}\text{-}\mathbf{Mc}$. The identity in lemma 4.3 gives the natural isomorphisms for the $(\cdot)^*_{\mathscr{C}}$-relative adjunction (the last one algebraically using $C^\infty_{\mathscr{C}}(E) \in \mathscr{C}\text{-}\mathbf{Mc}$):

$$C^\infty_{\mathscr{C}}(E, F) \simeq C^\infty_{\mathscr{C}}(E)\varepsilon F \simeq L(F'_c, C^\infty_{\mathscr{C}}(E)) \simeq L(F^*_{\mathscr{C}}, C^\infty_{\mathscr{C}}(E)),\quad |L(F^*_{\mathscr{C}}, C^\infty_{\mathscr{C}}(E))| = |C^{\text{op}}(C^\infty_{\mathscr{C}}(E), F^*_{\mathscr{C}})|.$$

It remains to see that $C^\infty_{\mathscr{C}} : \mathcal{M} \longrightarrow C$ is a symmetric unital functor satisfying the extra assumptions needed for a λ-categorical model. Note that Lemmas 4.2 and 4.3 also provide the definitions of the map Λ, Ξ respectively, the second diagram for Ξ. The diagram for Ξ comparing the internal hom functors is satisfied by definition of the map Λ which is given by a topological version of this diagram. Note that unitality and functoriality of $C^\infty_{\mathscr{C}}$ are obvious and that $\Lambda_{E,F,G}$ is even defined for any $G \in \mathbf{Mc}$. It remains to prove symmetry and the second diagram for Λ. We first reduce it to $\mathscr{C}$ replaced by Fin. For, note that, by their definition as projective limit, there is a continuous identity map $C^\infty_{\mathscr{C}}(E) \longrightarrow \mathscr{C}^\infty_{\text{Fin}}(E)$ for any lcs E, and since smooth curves only depend on the bornology, $\mathscr{C}^\infty_{\text{Fin}}(E) \simeq \mathscr{C}^\infty_{\text{Fin}}(\mathscr{N}(E))$ topologically (recall $\mathscr{N} = \mathscr{S}_{\text{Fin}}$ is the reflector of I : Fin-**Mc** $\subset \mathscr{C}$-**Mc**, which is a cartesian functor [Ju], and thus also of I_∞ : Fin-$\mathbf{Mc}_\infty \subset \mathscr{C}\text{-}\mathbf{Mc}_\infty$ by this very remark.) Composing both, one gets easily a natural transformation $J_{\mathscr{C},\text{Fin}} : C^\infty_{\mathscr{C}} \longrightarrow I \circ \mathscr{C}^\infty_{\text{Fin}} \circ \mathscr{N}$. It intertwines the Curry maps Λ as follows for $G \in \mathbf{Mc}$:

$$\begin{array}{ccc}
C^\infty_{\mathscr{C}}(E)\varepsilon\Big(C^\infty_{\mathscr{C}}(F)\varepsilon G\Big) & \xrightarrow{J_{\mathscr{C},\text{Fin}}(E)\varepsilon(J_{\mathscr{C},\text{Fin}}(F)\varepsilon G)} & \mathscr{C}^\infty_{\text{Fin}}(\mathscr{N}(E))\varepsilon\Big(\mathscr{C}^\infty_{\text{Fin}}(\mathscr{N}(F))\varepsilon G\Big) \\
\uparrow \Lambda_{E,F,G} & & \uparrow \Lambda_{\mathscr{N}(E),\mathscr{N}(F),G} \\
C^\infty_{\mathscr{C}}(E\times F)\varepsilon G & \xrightarrow{J_{\mathscr{C},\text{Fin}}(E\times F)\varepsilon G} & \mathscr{C}^\infty_{\text{Fin}}(\mathscr{N}(E)\times\mathscr{N}(F))\varepsilon G
\end{array}$$

Now, the associativity, symmetry and unitor maps are all induced from **McSch**, hence, it suffices to prove the compatibility diagrams for Λ in the case of $\mathscr{C}^\infty_{\text{Fin}}$ with $G \in \mathbf{McSch}$. In this case, we can further reduce it using that from naturality of associator, unitor and braiding, they commute with projective limits as ε does, and from its construction in lemma 4.2 $\Lambda_{E,F,G}$ is also a projective limit of maps, hence the projective limit description of $\mathscr{C}^\infty_{\text{Fin}}$ reduces those diagrams to E, F finite dimensional. Note that for the terms with products $E \times F$ the cofinality of product maps used in the proof of lemma 4.2) enables to rewrite the projective limit for $E \times F$ with the product of projective limits for E, F separately. The key to check the relations is to note that the target space of the diagrams is a set of multilinear maps on $(C^\infty(\mathbb{R}^n\times\mathbb{R}^m))', G'$ and to prove equality of the evaluation of both composition on an element in the source space, by linearity continuity and since $\overline{\text{Vect}\,\delta_{\mathbb{R}^{n+m}}(\mathbb{R}^{n+m})} = (C^\infty(\mathbb{R}^n \times \mathbb{R}^m))'$ ([FK, Thm 5.1.7] with $\delta_{\mathbb{R}^{n+m}}$ the Dirac mass map as in lemma 6.12 bellow), it suffices to evaluate the argument in $(C^\infty(\mathbb{R}^n \times \mathbb{R}^m))'$ on Dirac masses which have a product form. Then when reduced to a tensor product argument, the associativity and braiding maps are canonical and the relation is obvious to check. ∎

6.10. Remark. In $\mathscr{C}$-**Ref** we defined $!_{\mathscr{C}}E = ((C^\infty_{\mathscr{C}}(E))^*_{\mathscr{C}})$. In order to compare the various exponential structures for different $\mathscr{C}$, we compare them in ρ-**Ref** after applying the isomorphism of $*$-autonomous category of Theorem 5.27, one gets $\mathscr{S}([!_{\mathscr{C}}E]_\mu) = ((C^\infty_{\mathscr{C}}(E))^*_\rho)$. Recall the applied isomorphism is identity when $\mathscr{C}$ = Ban. By lemma 9.2, for $\mathscr{C}$ = Fin one gets a space with its ρ-dual having the approximation property. However, for $\mathscr{C}$ = Ban, we claim that $(!_{\mathrm{Ban}}\mathscr{S}(E))^*_\rho$ does not have the approximation property as soon as E is Banach space in Ban without the approximation property. Indeed, example 5.28 concludes to this statement as soon as we notice that we have $(!_{\mathscr{C}}\mathscr{S}(E))^*_\rho = ((C^\infty_{co}(E))^*_\rho)^*_\rho$ if E is a Banach space. This is the case since we have the topological identity $C^\infty_{\mathscr{C}}(E, \mathbb{K}) \simeq C^\infty_{\mathscr{C}}(\mathscr{S}(E), \mathbb{K})$ coming from the identical indexing set of curves coming from the algebraic equality $|C^\infty_{co}(X, E)| = |C^\infty_{\mathrm{Ban}}(X, E)| = |C^\infty(X, E)| = |C^\infty_{co}(X, \mathscr{S}(E))|$. Therefore, if E is the Schwartz space associated to a Banach space in Ban without the approximation property:

$$\mathscr{S}([!_{\mathrm{Fin}}E]_\mu) \subsetneq !_{\mathrm{Ban}}E$$

(since both duals are algebraically equal to $C^\infty(E, \mathbb{K})$, the difference of topology implies different duals algebraically). It is natural to wonder if there are infinitely many different exponentials obtained in that way for different categories $\mathscr{C}$. It is also natural to wonder if one can characterize ρ-reflexive spaces (or even Banach spaces) for which there is equality $\mathscr{S}([!_{\mathrm{Fin}}E]_\mu) = !_{\mathrm{Ban}}E$.

6.11. Exponential structures from biadditive models of linear logic. A model of LL: a Seely category. This section describes the construction of a model of differential linear logic from the material presented in the previous section and categorical axiomatisation of Biadditive Intuitionistic Linear Logic. We referred to [PAM] in order to produce a Seely category. For extensions to DiLL models, it is better to make more explicit the structure we obtained. First recall the various functors. When $f : E \longrightarrow F$ is a continuous linear map with $E, F \in \mathscr{C}$-**Mc**, we used $!_{\mathscr{C}}f : !_{\mathscr{C}}E \longrightarrow !_{\mathscr{C}}F$ defined as $(\cdot \circ f)^*_{\mathscr{C}}$. Hence $!_{\mathscr{C}}$ is indeed a functor from $\mathscr{C}$-**Mc** to $\mathscr{C}$-**Ref**.

Since $C^\infty_{\mathscr{C}}$ is a functor too on $\mathscr{C}$-$\mathbf{Mc}_\infty$, the above functor is decomposed in an adjunction as follows. For $F : E \longrightarrow F$ $\mathscr{C}$-smooth, $C^\infty_{\mathscr{C}}(F)(g) = g \circ F, g \in C^\infty_{\mathscr{C}}(F, \mathbb{R})$ and for a linear map f as above, $U(f)$ is the associated smooth map, underlying the linear map. Hence we also noted $!_{\mathscr{C}}F = (C^\infty_{\mathscr{C}}(F))^*_{\mathscr{C}}$ gives the functor, left adjoint to $U : \mathscr{C}\text{-}\mathbf{Mc} \longrightarrow \mathscr{C}\text{-}\mathbf{Mc}_\infty$ and our previous $!_{\mathscr{C}}$ is merely the new $!_{\mathscr{C}} \circ U$.

For any $E \in \mathscr{C}$-**Ref**, we recall the continuous isomorphism from E to $(E^*_{\mathscr{C}})^*_{\mathscr{C}} = \mathscr{S}((E'_\mu)'_\mu)$

$$\mathrm{ev}_E : \begin{cases} E \to (E^*_{\mathscr{C}})^*_{\mathscr{C}} = E \\ x \mapsto (l \in E^*_{\mathscr{C}} \mapsto l(x)) \end{cases}$$

Note that if E is only Mackey-complete, the linear isomorphism above is still defined, in the sense that we take the extension to the Mackey-completion of $l \mapsto l(x)$, but it is only a bounded/smooth algebraic isomorphism but it is not continuous by Theorem 5.12. However, ev_E^{-1} is always linear continuous in this case too.

We may still use the notation e_E for any separated locally convex space E as the bounded linear injective map, obtained by composition of the canonical map $E \longrightarrow \widehat{E}^M$ and $\mathrm{ev}_{\widehat{E}^M}$. We also consider the similar canonical maps:

6.12. Lemma. *For any space $E \in \mathscr{C}$-**Mc**, there is a smooth map (the Dirac mass map):*

$$\delta_E : \begin{cases} E \to (C^\infty_{\mathscr{C}}(E))' \subset !_{\mathscr{C}} E \\ x \mapsto (f \in C^\infty_{\mathscr{C}}(E, \mathbb{K}) \mapsto f(x) = \delta_E(x)(f)), \end{cases}$$

Proof. We could see this directly using convenient smoothness, but it is better to see how it comes from our λ-categorical model structure. We have an adjunction:

$$C^\infty_{\mathscr{C}}(E, !_{\mathscr{C}} E) \simeq \mathscr{C}\text{-}\mathbf{Mc}^{\mathrm{op}}(C^\infty_{\mathscr{C}}(E), (!_{\mathscr{C}} E)^*_{\mathscr{C}}) = \mathscr{C}\text{-}\mathbf{Mc}((C^\infty_{\mathscr{C}}(E)^*_{\mathscr{C}})^*_{\mathscr{C}}, C^\infty_{\mathscr{C}}(E))$$

and δ_E is the map in the first space, associated to $\mathrm{ev}^{-1}_{C^\infty_{\mathscr{C}}(E)}$ in the last. ∎

Hence, δ_E is the unit of the adjunction giving rise to $!_{\mathscr{C}}$, considered on the opposite of the continuation category.

As usual, see e.g. [PAM, section 6.7], the adjunction giving rise to $!_{\mathscr{C}}$ produces a comonad structure on this functor. The counit implementing the dereliction rule is the continuous linear map $\mathsf{d}_E : !_{\mathscr{C}}(E) \longrightarrow E$ obtained in looking at the map corresponding to identity in the adjunction:

$$C^\infty_{\mathscr{C}}(E, E) \simeq \mathscr{C}\text{-}\mathbf{Mc}^{\mathrm{op}}(C^\infty_{\mathscr{C}}(E), (E)^*_{\mathscr{C}}) = \mathscr{C}\text{-}\mathbf{Mc}(E^*_{\mathscr{C}}, C^\infty_{\mathscr{C}}(E)) \simeq \mathscr{C}\text{-}\mathbf{Mc}((C^\infty_{\mathscr{C}}(E))^*_{\mathscr{C}}, E)$$

The middle map $\epsilon_E^{C^\infty_{\mathscr{C}}} \in \mathscr{C}\text{-}\mathbf{Mc}(E^*_{\mathscr{C}}, C^\infty_{\mathscr{C}}(E))$ is the counit of the $(\cdot)^*_{\mathscr{C}}$-relative adjunction and it gives $\mathsf{d}_E = \mathrm{ev}_E^{-1} \circ (\epsilon_E^{C^\infty_{\mathscr{C}}})^*_{\mathscr{C}}$ when $E \in \mathscr{C}$-**Ref**. The comultiplication map implementing the promotion rule is obtained as $\mathsf{p}_E = !_{\mathscr{C}}(\delta_E) = (C^\infty_{\mathscr{C}}(\delta_E))^*_{\mathscr{C}}$.

We can now summarize the structure. Note, that we write the usual $\top$, unit for $\times$ as 0, for the $\{0\}$ vector space.

6.13. Proposition. *The functor $!_{\mathscr{C}}$ is an exponential modality for the Seely category of Theorem 6.9 in the following way:*

- $(!_{\mathscr{C}}, \mathsf{p}, \mathsf{d})$ *is a comonad, with* $\mathsf{d}_E = \mathrm{ev}_E^{-1} \circ (\epsilon_E^{C^\infty_{\mathscr{C}}})^*_{\mathscr{C}}$ *and* $\mathsf{p}_E = !_{\mathscr{C}}(\delta_E) = (C^\infty_{\mathscr{C}}(\delta_E))^*_{\mathscr{C}}$.
- $!_{\mathscr{C}} : (\mathscr{C}\text{-}\mathbf{Ref}, \times, 0) \to (\mathscr{C}\text{-}\mathbf{Ref}, \otimes, \mathbb{K})$ *is a strong and symmetric monoidal functor, thanks to the isomorphisms $m^0 : \mathbb{K} \simeq !_{\mathscr{C}}(0)$ and (the map composing tensor strengths and adjoints of Ξ, Λ of λ-tensor models):*

$$\begin{aligned} m^2_{E,F} : !_{\mathscr{C}} E \otimes !_{\mathscr{C}} !F &= \left((C^\infty_{\mathscr{C}}(E)^*_{\mathscr{C}})^*_{\mathscr{C}} \parr_{\mathscr{C}} (C^\infty_{\mathscr{C}}(F)^*_{\mathscr{C}})^*_{\mathscr{C}}\right)^*_{\mathscr{C}} \simeq \left(C^\infty_{\mathscr{C}}(E, \mathbb{K}) \parr_{\mathscr{C}} C^\infty_{\mathscr{C}}(F, \mathbb{K})\right)^*_{\mathscr{C}} \\ &\simeq \left(C^\infty_{\mathscr{C}}(E, C^\infty_{\mathscr{C}}(F, \mathbb{K}))\right)^*_{\mathscr{C}} \simeq (C^\infty(E \times F, \mathbb{K}))^*_{\mathscr{C}} \simeq !_{\mathscr{C}}(E \times F) \end{aligned}$$

- *the following diagram commutes:*

$$\begin{array}{ccccc} !_{\mathscr{C}} E \otimes_{\mathscr{C}} !_{\mathscr{C}} F & \xrightarrow{m^2_{E,F}} & !(E \times F) & \xrightarrow{\mathsf{p}_{E \times F}} & !_{\mathscr{C}} !_{\mathscr{C}}(E \times F) \\ \downarrow{\scriptstyle \mathsf{p}_E \otimes_{\mathscr{C}} \mathsf{p}_F} & & & & \downarrow{\scriptstyle !_{\mathscr{C}}\langle !_{\mathscr{C}}\pi_1, !_{\mathscr{C}}\pi_2 \rangle} \\ !_{\mathscr{C}} !_{\mathscr{C}} E \otimes_{\mathscr{C}} !_{\mathscr{C}} !_{\mathscr{C}} F & & \xrightarrow[m^2_{!_{\mathscr{C}} E, !_{\mathscr{C}} F}]{} & & !_{\mathscr{C}}(!_{\mathscr{C}} E \times !_{\mathscr{C}} F) \end{array}$$

Moreover, the comonad induces a structure of bialgebra on every space $!_{\mathscr{C}}E$ and this will be crucial to obtain models of DiLL [Ehr16]. We conclude this review section in recalling how all the diagrams there not involving the codereliction map are satisfied. In general, we have maps giving a commutative comonoid structure (this is the coalgebra part of the bialgebra, but it must not be confused with the coalgebra structure from the comonad viewpoint):

- $\mathsf{c}_E : !_{\mathscr{C}}E \to !_{\mathscr{C}}(E \times E) \simeq !_{\mathscr{C}}E \otimes !_{\mathscr{C}}E$ given by $\mathsf{c}_E = (m^2_{E,E})^{-1} \circ !_{\mathscr{C}}(\Delta_E)$ with $\Delta_E(x) = (x, x)$ the canonical diagonal map of the cartesian category.
- $\mathsf{w}_E = (m^0)^{-1} \circ !_{\mathscr{C}}(n_E) : !_{\mathscr{C}}E \to !_{\mathscr{C}}0 \simeq \mathbb{R}$ with $n_E : E \longrightarrow 0$ the constant map, hence more explicitly $\mathsf{w}_E(h) = h(1)$ for $h \in !_{\mathscr{C}}E$ and $1 \in C^\infty_{\mathscr{C}}(E)$ the constant function equal to 1.

This is exactly the structure considered in [Bie93, Chap 4 §6] giving a Seely category (in his terminology a new-Seely category) the structure of a Linear category (called $\mathcal{L}^!_\otimes$-model in [F]) from his Definition 35 in his Thm 25. See also [PAM, 7.4] for a recent presentation. This especially also contains the compatibility diagrams of [Ehr16, 2.6.1]. Especially, $\mathsf{p}_E : (!_{\mathscr{C}}E, \mathsf{w}_E, \mathsf{c}_E) \longrightarrow (!_{\mathscr{C}}!_{\mathscr{C}}E, \mathsf{w}_{!_{\mathscr{C}}E}, \mathsf{c}_{!_{\mathscr{C}}E})$ is a comonoid morphism as in [Ehr16, 2.6.3]. Also $!_{\mathscr{C}}$ is given the structure of a symmetric monoidal endofunctor on $\mathscr{C}$-**Ref**, $(!_{\mathscr{C}}, \mu^0, \mu^2)$ making $\mathsf{w}_E, \mathsf{c}_E$ coalgebra morphisms. For instance, $\mu^0 : \mathbb{R} \longrightarrow !_{\mathscr{C}}(\mathbb{R})$ (the space of distributions) is given by [Bie93, Chap 4 Prop 20] as $!_{\mathscr{C}}(v_{\mathbb{R}}) \circ m^0$, i.e. $\mu^0(1) = \delta_1$ with $v_{\mathbb{R}} : 0 \longrightarrow \mathbb{R}$ the map with $u_R(0) = 1$. By [Bie93], a Linear category with products is actually the same thing as a Seely category. This is what is called in [F] a $\mathcal{L}^!_{\otimes,\times}$-model. So far, this structure is available in the setting of Theorem 6.6, and we will use it in this setting later.

As explained in [F], the only missing piece of structure to get a bicomonoid structure on every $!_{\mathscr{C}}E$ is a biproduct compatible with the symmetric monoidal structure, or equivalently a **Mon**-enriched symmetric monoidal category, where **Mon** is the category of monoids. This is what he calls a $\mathcal{L}^!_{\otimes,*}$-model.

His Theorem 3.1 then provides us with the two first compatibility diagrams in [Ehr16, 2.6.2] and the second diagram in [Ehr16, 2.6.4].

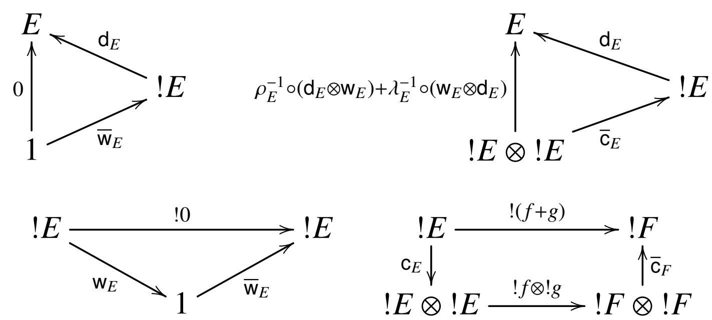
In our case $\nabla_E : E \times E \longrightarrow E$ is the sum when seeing $E \times E = E \oplus E$ as coproduct and its unit $u : 0 \longrightarrow E$ is of course the 0 map. Hence $(\mathscr{C}\text{-}Mc, 0, \times, u, \nabla; n, \Delta)$ is indeed a biproduct structure. And compatibility with the monoidal structure, which boils down to biadditivity of tensor product, is obvious. One gets cocontraction and coweakening maps:

- $\overline{\mathsf{c}}_E : !_{\mathscr{C}}E \otimes !_{\mathscr{C}}E \simeq !_{\mathscr{C}}(E \times E) \to !_{\mathscr{C}}E$ is the convolution product, namely it corresponds to $!_{\mathscr{C}}(\nabla_E)$.
- $\overline{\mathsf{w}}_E : \mathbb{R} \simeq !_{\mathscr{C}}(0) \to !_{\mathscr{C}}E$ is given by $\overline{\mathsf{w}}_E(1) = (\mathrm{ev}_0)^*_{\mathscr{C}}$ with $\mathrm{ev}_0 = C^\infty_{\mathscr{C}}(u_E)$ i.e. $\mathrm{ev}_0(f) = f(0)$.

From [F, Prop 3.2] $(!_{\mathscr{C}}E, \mathsf{c}_E, \mathsf{w}_E, \overline{\mathsf{c}}_E, \overline{\mathsf{w}}_E)$ is a commutative bialgebra. The remaining first diagram in [Ehr16, 2.6.4] is easy and comes in our case for $f \in C^\infty_{\mathscr{C}}(!E)$ from

$$[!_{\mathscr{C}}(\delta_E \circ u_E)](f) = \delta_{\delta_0}(f) = \delta_1(\lambda \mapsto f(\lambda(\delta_0)) = [!_{\mathscr{C}}\overline{\mathsf{w}}_E(\delta_1)](f) = [!_{\mathscr{C}}\overline{\mathsf{w}}_E(\mu^0(1))](f).$$

To finish checking the assumptions in [Ehr16], it remains to check the assumptions in 2.5 and 2.6.5. As [F] is a conference paper, they were not explicitly written there.



(13)

(14)

The first is $\mathrm{ev}_E^{-1}\circ(\epsilon_E^{C^\infty_\mathscr{C}})^*_\mathscr{C}\circ(C^\infty_\mathscr{C}(u_E))^*_\mathscr{C}=\mathrm{ev}_E^{-1}\circ(C^\infty_\mathscr{C}(u_E)\circ\epsilon_E^{C^\infty_\mathscr{C}})^*_\mathscr{C}=\mathrm{ev}_E^{-1}\circ((u_E)^*_\mathscr{C})^*_\mathscr{C}=u_E=0$ as expected. The second is $\mathrm{ev}_E^{-1}\circ(\epsilon_E^{C^\infty_\mathscr{C}})^*_\mathscr{C}\circ(C^\infty_\mathscr{C}(\nabla_E))^*_\mathscr{C}=\mathrm{ev}_E^{-1}\circ(C^\infty_\mathscr{C}(\nabla_E)\circ\epsilon_E^{C^\infty_\mathscr{C}})^*_\mathscr{C}=\mathrm{ev}_E^{-1}\circ((\nabla_E)^*_\mathscr{C})^*_\mathscr{C}\circ(\epsilon_E^{C^\infty_\mathscr{C}})^*_\mathscr{C}\otimes(\epsilon_E^{C^\infty_\mathscr{C}})^*_\mathscr{C}$ which gives the right value since $\nabla_E=r_E^{-1}\circ(\mathrm{Id}_E\times n_E)+\ell_E^{-1}\circ(n_E\times\mathrm{Id}_E)$.

The third diagram comes from $n_Eu_E=0$ and the last diagram from $\nabla_Y\circ(f\times g)\circ\Delta_X=f+g$ which is the definition of the additive structure on maps.

6.14. Comparison with Convenient Analysis and Blute-Ehrhard-Tasson. In [BET], the authors use the convenient setting of global analysis [FK, KM] in order to produce a model of Intuitionistic differential Linear logic. They work on the category **Conv** of convenient vector spaces, i.e. bornological Mackey-complete (separated) lcs, with continuous (equivalently bounded), linear maps as morphisms. Thus, apart for the bornological requirement, the setting seems similar to ours. It is time to compare them.

First any bornological space has its Mackey topology, let us explain why

$$\mathscr{S}:\mathbf{Conv}\longrightarrow\rho\text{-}\mathbf{Ref}$$

is an embedding giving an isomorphic full subcategory (of course with inverse $(\cdot)_\mu$ on its image). Indeed, for $E\in\mathbf{Conv}$ we use Theorem 5.12 in order to see that $\mathscr{S}(E)\in\rho\text{-}\mathbf{Ref}$ and it only remains to note that E'_μ is Mackey-complete.

As in Remark 3.8, E bornological Mackey-complete, thus ultrabornological, implies E'_μ and even $\mathcal{S}(E'_\mu)$ complete hence Mackey-complete (and E'_c k-quasi-complete).

Said otherwise, the bornological requirement ensures a stronger completeness property of the dual than Mackey-completeness, the completeness of the space, our functor $(\widehat{(\cdot)'_\mu})'_\mu$ should thus be thought of as a replacement of the bornologification functor in [FK] and $((\cdot)^*_\rho)^*_\rho$ is our analogue of their Mackey-completion functor in [KM] (recall that their Mackey completion is what we would call Mackey-completion of the bornologification). Of course, we already noticed that we took the same smooth maps and $\mathscr{S}:\mathbf{Conv}_\infty\longrightarrow\rho\text{-}\mathbf{Ref}_\infty$ is even an equivalence of categories. Indeed, $E\longrightarrow E_{\mathrm{born}}$ is smooth and gives the inverse for this equivalence.

Finally note that $|E^*_\rho\varepsilon F|=|L_\beta(E,F)|$ algebraically if $E\in\mathbf{Conv}$ since $E^*_\rho\varepsilon F\simeq L_\epsilon((E^*_\rho)'_c,F)\simeq L_\mathscr{R}(E,F)$ topologically and the space of continuous and bounded linear maps are the same in the bornological case. $L_\beta(E,F)\longrightarrow L_\mathscr{R}(E,F)$ is clearly continuous hence so is

$$\mathscr{S}(L_\beta(E,F))\longrightarrow\mathscr{S}(L_\mathscr{R}(E,F))=L_\mathscr{R}(E,F).$$

But the closed structure in **Conv** is given by $(L_\beta(E,F))_{\mathrm{born}}$ which uses a completion of the dual and hence we only have a lax closed functor property for $\mathscr{S}$, in form (after applying $((\cdot)^*_\rho)^*_\rho$) of a continuous map:

$$\mathscr{S}((L_\beta(E,F))_{\mathrm{born}}) \longrightarrow ((E^*_\rho \varepsilon F)^*_\rho)^*_\rho. \tag{15}$$

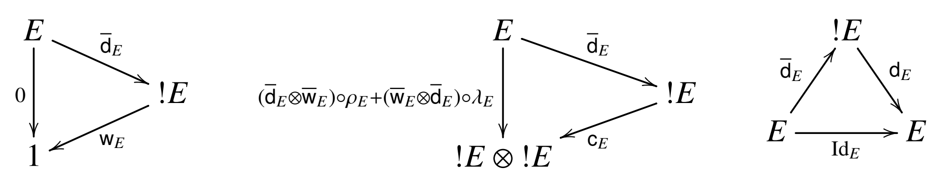
Similarly, most of the linear logical structure is not preserved by the functor $\mathscr{S}$.

7. Models of DiLL

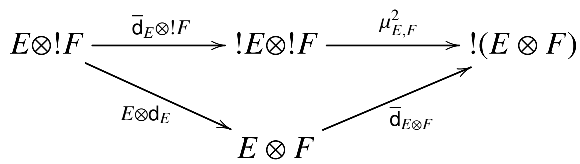
Smooth linear maps in the sense of Frölicher are bounded but not necessarily continuous. Taking the differential at 0 of functions in $\mathscr{C}^\infty(E,F)$ thus would not give us a morphisms in **k**-**Ref**, thus we have no interpretation for the codereliction $\bar{\mathsf{d}}$ of DiLL. We first introduce a general differential framework fitting dialogue categories, and show that the variant of smooth maps introduced in section 3.21 allows for a model of DiLL.

7.1. An intermediate notion: models of differential λ-Tensor logic. We refer to [Ehr11, Ehr16] for surveys on differential linear logic.

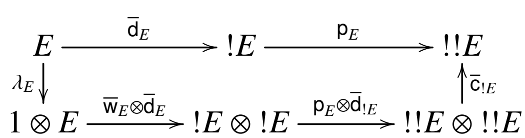
According to Fiore and Ehrhard [F, Ehr16], models of differential linear logic are given by Seely $*$-autonomous complete categories C with a biproduct structure and either a creation operator natural transformation $\partial_E : !E \otimes E \longrightarrow !E$ or a creation map/codereliction natural transformation $\bar{\mathsf{d}}_E : E \longrightarrow !E$ satisfying proper conditions. We recalled in subsection 6.11 the structure available without codereliction. Moreover, in the codereliction picture, one requires the following diagrams to commute [Ehr16, 2.5, 2.6.2,2.6.4]:



(16)



(17)



(18)

Then from [F, Thm 4.1] (see also [Ehr16, section 3]) the creation operator is $\partial_E = \bar{\mathsf{c}}_E \circ (!E \otimes \bar{\mathsf{d}}_E)$

We again need to extend this structure to a dialogue category context. In order to get a natural differential extension of cartesian closed category, we use differential λ-categories from [BEM]. This notion gathers the maybe very general cartesian differential categories of Blute-Cockett-Seely to cartesian closedness, via the key axiom (D-curry), relating applications of the differential operator D and the curry map Λ for $f : C\times A \longrightarrow B$ (we don't mention the symmetry of cartesian closed category $(C \times C \times A) \times A \simeq (C \times A) \times (C \times A)$):

$$D(\Lambda(f)) = \Lambda\Big(D(f) \circ \langle(\pi_1 \times 0_A), \pi_2\rangle\Big) : (C \times C) \longrightarrow [A, B].$$

We also use $\mathrm{Diag}(E) = E \times E$ the obvious functor. We also suppose that the cartesian structure is a biproduct, a supposition that is equivalent to supposing a **Mon**-enriched category as shown by Fiore [F].

The idea is that while D encodes the usual rules needed for differential calculus, d encodes the fact that we want the derivatives to be smooth, that is compatible with the linear duality structure we had before.

7.2. Definition. A (commutative) model of differential λ-tensor logic *is a (commutative) λ-categorical model of λ-tensor logic with dialogue category $(\mathcal{C}^{\mathrm{op}}, \parr_{\mathcal{C}}, 1_{\mathcal{C}} = \mathbb{K}, \neg)$ with a biproduct structure compatible with the symmetric monoidal structure, a cartesian closed category $(\mathcal{M}, \times, 0, [\cdot,\cdot])$, which is a differential λ-category with operator D internalized as a natural transformation $D_{E,F} : [E, F] \longrightarrow [\mathrm{Diag}(E), F]$ (so that D in the definition of those categories is given by $M(D_{E,F}) : \mathcal{M}(E, F) \longrightarrow \mathcal{M}(E \times E, F)$ with M the basic functor to sets of the closed category $\mathcal{M}$). We assume $U : \mathcal{C} \longrightarrow \mathcal{M}$ and $\neg : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}$ are* **Mon***-enriched functors. We also assume given an internalized differential operator, given by a natural transformation*

$$d_{E,F} : \mathrm{NL}(U(E)) \parr_{\mathcal{C}} F \longrightarrow \mathrm{NL}(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} F)$$

satisfying the following commutative diagrams (with the opposite of the counit of the relative adjointness relation, giving a map in $\mathcal{C}$ written: $\epsilon^{\mathrm{NL}}_E : \neg E \longrightarrow \mathrm{NL}(U(E)) \equiv \mathrm{NL}_E$) expressing compatibility of the two differentials. We have a first diagram in $\mathcal{M}$:

$$\begin{array}{ccccc}
U\Big(\mathrm{NL}_E \parr_{\mathcal{C}} F\Big) & \xrightarrow{U(d_{E,F})} & U\Big(\mathrm{NL}_E \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} F)\Big) & \xrightarrow{U(\mathrm{NL}_E)\parr_{\mathcal{C}}(\epsilon^{\mathrm{NL}}_E \parr_{\mathcal{C}} F))} & U\Big(\mathrm{NL}_E \parr_{\mathcal{C}} (\mathrm{NL}_E \parr_{\mathcal{C}} F)\Big) \\
\downarrow{\scriptstyle \Xi_{E,F}} & & & & \downarrow{\scriptstyle [\mathrm{Id}_{U(E)},\Xi_{E,F}]\circ\Xi_{E,\mathrm{NL}(E)\parr_{\mathcal{C}} F}} \\
[U(E), U(F)] & \xrightarrow{D_{U(E),U(F)}} & [U(E \times E), U(F)] & \xrightarrow{\Lambda^{\mathcal{M}}_{U(E),U(E),U(F)}} & [U(E), [U(E), U(F)]]
\end{array}$$

and weak differentiation property diagram in $\mathcal{C}$:

$$\begin{array}{ccccc}
\mathrm{NL}(U(E)) \parr_{\mathcal{C}} F & \xrightarrow{d_{E,F}} & \mathrm{NL}(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} F) & \xrightarrow{\mathrm{Ass}^{\parr_{\mathcal{C}}}_{\mathrm{NL}(U(E)),\neg E,F}} & (\mathrm{NL}(U(E)) \parr_{\mathcal{C}} \neg E) \parr_{\mathcal{C}} F \\
\downarrow{\scriptstyle \rho_{\mathrm{NL}(U(E))}\parr_{\mathcal{C}} F} & & & & \downarrow{\scriptstyle \mathrm{NL}(U(E))\parr_{\mathcal{C}}\rho_{\neg E}\parr_{\mathcal{C}} F} \\
(\mathrm{NL}(U(E)) \parr_{\mathcal{C}} \mathbb{K}) \parr_{\mathcal{C}} F & & \xrightarrow{d_{E,\mathbb{K}}\parr_{\mathcal{C}} F} & & (\mathrm{NL}(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} \mathbb{K})) \parr_{\mathcal{C}} F
\end{array}$$

We now produce a model of DiLL from this data.

7.3. Theorem. *Assume that $(\mathcal{C}^{\mathrm{op}}, \parr_{\mathcal{C}}, I, \neg, \mathcal{M}, \times, 0, \mathrm{NL}, U, D, d)$ is a Seely model of differential λ-tensor logic. Let $\mathcal{D} \subset \mathcal{C}$ the full subcategory of objects of the form $\neg C, C \in \mathcal{C}$, equipped with $! = \neg \circ \mathrm{NL} \circ U$ as comonad on $\mathcal{D}$ making it a $*$-autonomous complete and cocomplete Seely category with Kleisli category for $!$ isomorphic to $\mathcal{N} = U(\mathcal{D})$. With the dereliction d_E as in subsection 6.11 and the codereliction interpreted by: $\overline{\mathsf{d}}_E = \neg\Big((\mathrm{NL}(E) \parr_{\mathcal{C}} \rho^{-1}_{\neg E}) \circ d_{E,\mathbb{K}} \circ \rho_{\mathrm{NL}(E)}\Big) \circ (\overline{\mathsf{w}}_E \otimes \mathrm{Id}_E) \circ \lambda_E$, this makes $\mathcal{D}$ a model of differential Linear Logic.*

Proof. The setting comes from Theorem 6.6 giving already a model of Linear logic. Recall from subsection 6.11 that we have already checked all diagrams not involving codereliction. We can and do fix $E = \neg C$ so that $\epsilon_E^{\neg\neg} : \neg\neg E \longrightarrow E$ is an isomorphism that we will ignore safely in what follows.

Step 1: Internalization of D-curry from [BEM]

Let us check:

$$\begin{array}{ccccc}
\mathrm{NL}(E) \parr \mathrm{NL}(E) \parr (\neg E \parr \mathbb{K}) & \xleftarrow{\mathrm{NL}(E)\parr d_{E,\mathbb{K}}} & \mathrm{NL}(E) \parr (\mathrm{NL}(E) \parr \mathbb{K}) & \xleftarrow{\Lambda_{E,E,\mathbb{K}}} & \mathrm{NL}(E \times E) \parr \mathbb{K} \\
\uparrow{\scriptstyle \Lambda_{E,E,\neg E\parr\mathbb{K}}} & & & \swarrow{\scriptstyle d_{E\times E,\mathbb{K}}} & \\
\mathrm{NL}(E \times E) \parr (\neg E \parr \mathbb{K}) & \xleftarrow{\mathrm{NL}(E\times E)\parr(\pi_2\parr\mathbb{K})} & \mathrm{NL}(E \times E) \parr ((\neg E)^{\oplus 2}) \parr \mathbb{K}) & &
\end{array}$$

Indeed, using compatibility with symmetry from definition of λ-categorical models, it suffices to check a flipped version with the derivation acting on the first term. Then, applying the faithful U, intertwining with Ξ and using the compatibility with $D_{E,F}$ the commutativity then follows easily from D-curry.

Step 2: Internalization of chain rule D5 from [BEM]

Note that

$$\begin{aligned}
g \in \mathcal{M}(U(E), U(F)) = \mathcal{M}(0, [U(E), U(F)]) &\simeq \mathcal{M}(0, U(\mathrm{NL}(E) \parr_C F) \\
&\simeq C(\neg(\mathrm{NL}(0)), \mathrm{NL}(E) \parr_C F \parr_C K),
\end{aligned}$$

gives a map $h : \neg(\mathbb{K}) \longrightarrow \mathrm{NL}(E) \parr_C F \parr_C \mathbb{K}$. One gets $d_{E,F} \circ h : \neg(\mathbb{K}) \longrightarrow \mathrm{NL}(E) \parr_C \neg E \parr_C F \parr_C \mathbb{K}$ giving by characteristic diagram of dialogue categories (for C^{op}, recall our maps are in the opposite of this dialogue category) a map $dH : \neg F \longrightarrow \mathrm{NL}(E) \parr_C \neg E \parr_C \mathbb{K}$. We leave as an exercise to the reader to check that D5 can be rewritten as before:

$$\begin{array}{ccccc}
\mathrm{NL}(U(E)) \parr_C (\neg E \parr_C \mathbb{K}) & \xleftarrow{d_{E,\mathbb{K}}} & \mathrm{NL}(U(E)) \parr_C \mathbb{K} & \xleftarrow{\mathrm{NL}(g)\parr_C\mathbb{K}} & \mathrm{NL}(U(F)) \parr_C \mathbb{K} \\
\uparrow{\scriptstyle \left(\mathrm{NL}(\Delta_{U(E)})\Lambda_{E,E}^{-1}\right)\parr_C(\neg E\parr_C\mathbb{K})} & & & & \downarrow{\scriptstyle d_{F,\mathbb{K}}} \\
\mathrm{NL}(U(E)) \parr_C \mathrm{NL}(U(E)) \parr_C (\neg E \parr_C \mathbb{K}) & & \xleftarrow[\mathrm{NL}(g)\parr_C dH]{} & & \mathrm{NL}(U(F)) \parr_C (\neg(F) \parr_C \mathbb{K})
\end{array}$$

Note that we can see dH in an alternative way using our weak differentiation property. Composing with a minor isomorphism, if we see $h : \neg(\mathbb{K}) \longrightarrow (\mathrm{NL}(E) \parr_C \mathbb{K}) \parr_C F$ then one can consider $(d_{E,\mathbb{K}} \parr F) \circ h$ and it gives $d_{E,F} \circ h$ after composition by a canonical map. But if $H : \neg(F) \longrightarrow (\mathrm{NL}(E) \parr_C \mathbb{K})$ is the map associated to h by the map φ of dialogue categories, the naturality of this map gives exactly $dH = d_{E,\mathbb{K}} \circ H$. Note that if $g = U(g')$, by the naturality of the isomorphisms giving H, it is not hard to see that $H = \epsilon_E^{\mathrm{NL}} \circ \neg(g')$.

Step 3: Two first diagrams in (16).

For the first diagram,by functoriality, it suffices to see $d_{E,\mathbb{K}} \circ (\mathrm{NL}(n_E) \parr \mathbb{K}) = 0$. Applying step 2 to $g = n_E$, one gets $H = u_{\mathrm{NL}(E)\parr\mathbb{K}}$ hence $dH = d_{E,\mathbb{K}} \circ H = 0$ as expected thanks to axiom D1 of [BEM] giving $D(0) = 0$.

For the second diagram, we compute $\mathsf{c}_E\overline{\mathsf{d}}_E = (m^2_{E,E})^{-1} \circ \neg(\mathrm{NL}(\Delta_{U(E)})) \circ \neg\Big((\mathrm{NL}(u_E) \parr_C \rho^{-1}_{\neg E}) \circ d_{E,\mathbb{K}} \circ \rho_{\mathrm{NL}(E)}\Big) \circ \lambda_E$. We must compute $[\mathrm{NL}(u_E) \parr_C (\neg E \parr_C \mathbb{K})] \circ d_{E,\mathbb{K}} \circ (\mathrm{NL}(\Delta_{U(E)}) \parr \mathbb{K})$ using step 2 again with $g = \Delta_{U(E)} = U(\Delta_E)$, hence $H = \epsilon^{\mathrm{NL}}_E \circ \neg(\Delta_E) = \epsilon^{\mathrm{NL}}_E \circ \bigtriangledown_{\neg E}$.

Using (19) below, one gets $dH = (\mathrm{NL}(n_E) \parr \bigtriangledown_{\neg E} \parr \mathbb{K}) \circ Isom$, so that, using $\mathrm{Isom} \circ (\Delta_{U(E)} \times n_E) \circ \Delta_{U(E)} = \Delta_{U(E)}$, one obtains

$$\mathrm{CD}_E := [\mathrm{NL}(u_E) \parr_C (\neg E \parr_C \mathbb{K})] \circ d_{E,\mathbb{K}} \circ (\mathrm{NL}(\Delta_{U(E)}) \parr \mathbb{K}) = (\mathrm{NL}(\Delta_{U(E)} \circ u_E) \parr \nabla_{\neg E} \parr \mathbb{K}) \circ d_{E^2,\mathbb{K}}.$$

Hence, noting that by naturality $\mathrm{NL}(0) \parr \nabla_{\neg E} \parr \mathbb{K} = \nabla_{\mathrm{NL}(0)\parr\neg E\parr\mathbb{K}}$ and using the formula in step 1,

$$\mathrm{CD}_E\Lambda^{-1}_{E,E,\mathbb{K}} = \nabla_{\mathrm{NL}(0)\parr\neg E\parr\mathbb{K}} \bigoplus_{i=1,2} (\mathrm{NL}(u_{E^2}) \parr \pi_i \parr \mathbb{K}) \circ d_{E^2,\mathbb{K}}\Lambda^{-1}_{E,E,\mathbb{K}} =$$
$$\nabla_{\mathrm{NL}(0)\parr\neg E\parr\mathbb{K}}\Big[\mathrm{NL}(u_E) \parr_C \big[(\mathrm{NL}(u_E) \parr \neg E \parr \mathbb{K}))d_{E,\mathbb{K}}\big], \big[(\mathrm{NL}(u_E) \parr \neg E \parr \mathbb{K}))d_{E,\mathbb{K}}\big] \parr_C \mathrm{NL}(u_E)\Big].$$

On the other hand, we can compute

$$(\overline{\mathsf{w}}_E \otimes \overline{\mathsf{d}}_E) \circ \lambda_E = \neg\Big((\mathrm{NL}(u_E) \parr_C \rho^{-1}_{\neg E}) \circ \lambda^{-1} \circ (\mathrm{NL}(u_E) \parr_C d_{E,\mathbb{K}}) \circ (\mathrm{NL}(E) \parr_C \rho_{\mathrm{NL}(E)})\Big) \circ \lambda_E.$$

From the symmetric computation, one sees (in using $\neg$ is additive) that our expected equation reduces to proving the formula which reformulates our previous result:

$$\begin{aligned}\mathrm{CD}_E \circ \Lambda^{-1}_{E,E,\mathbb{K}} &= \lambda^{-1} \circ (\mathrm{NL}(u_E) \parr_C \big[(\mathrm{NL}(u_E) \parr \neg E \parr \mathbb{K}))d_{E,\mathbb{K}}\big]) \\ &\quad + \rho^{-1} \circ (\big[(\mathrm{NL}(u_E) \parr \neg E \parr \mathbb{K}))d_{E,\mathbb{K}}\big] \parr_C \mathrm{NL}(u_E)).\end{aligned}$$

Step 4: Final Diagrams for codereliction.

To prove (18),(17), one can use [F, Thm 4.1] (and the note added in proof making (14) redundant, but we could also check it in the same vein as below using step 2) and only check (16) and the second part of his diagram (15) on $\partial_E = \neg\Big((\mathrm{NL}(E) \parr_C \rho^{-1}_{\neg E}) \circ d_{E,\mathbb{K}} \circ \rho_{\mathrm{NL}(E)}\Big)$. Indeed, our choice $\overline{\mathsf{d}}_E = (\partial_E) \circ (\overline{\mathsf{w}}_E \otimes \mathrm{Id}_E) \circ \lambda_E$ is exactly the direction of this bijection producing the codereliction.

One must check:

$$\begin{array}{ccccc} !E \otimes E & \xrightarrow{\partial_E} & !E & \xrightarrow{\mathsf{p}_E} & !!E \\ \downarrow{\scriptstyle \mathsf{c}_E\otimes E} & & & & \uparrow{\scriptstyle \partial_{!E}} \\ !E \otimes !E \otimes E & & \xrightarrow{\mathsf{p}_E\otimes\partial_E} & & !!E \otimes !E \end{array}$$

and recall $\mathsf{c}_E = (m^2_{E,E})^{-1} \circ \neg(\mathrm{NL}(\Delta_E))$, $\mathsf{p}_E = \neg(\mathrm{NL}(\delta_E))$, and $(m^2_{E,E})^{-1} = \neg(\Lambda^{-1}_{E,E}(\epsilon^\neg \parr \epsilon^\neg))$,with $\Lambda^{-1}_{E,E} = \rho_{\mathrm{NL}(E\times E)}\Lambda^{-1}_{E,E,\mathbb{K}}(\mathrm{NL}(E) \parr \rho_{\mathrm{NL}(E)})$, $\epsilon^\neg : \neg\neg E \longrightarrow E$ the counit of self-adjunction.

Hence our diagram will be obtained by application of $\neg$ (after intertwining with ρ) if we prove:

$$\begin{array}{ccccc}
\mathrm{NL}(U(E)) \parr_C (\neg E \parr_C \mathbb{K}) & \xleftarrow{d_{E,\mathbb{K}}} & \mathrm{NL}(U(E)) \parr_C \mathbb{K} & \xleftarrow{\mathrm{NL}(\delta_E)\parr_C\mathbb{K}} & \mathrm{NL}(U(!E)) \parr_C \mathbb{K} \\
\uparrow{\scriptstyle (\mathrm{NL}(\Delta_E)\Lambda^{-1}_{E,E})\parr_C(\neg E\parr_C\mathbb{K})} & & & & \downarrow{\scriptstyle d_{!E,\mathbb{K}}} \\
\mathrm{NL}(U(E)) \parr_C \mathrm{NL}(U(E)) \parr_C (\neg E \parr_C \mathbb{K}) & & \xleftarrow[\mathrm{NL}(\delta_E)\parr_C(d_{E,\mathbb{K}}\circ\epsilon^\neg)]{} & & \mathrm{NL}(U(!E)) \parr_C (\neg(!E) \parr_C \mathbb{K})
\end{array}$$

This is the diagram in step 2 for $g = \delta_E$ if we see that $dH = (d_{E,\mathbb{K}} \circ \epsilon^\neg)$. For, it suffices to see $H = \epsilon^\neg$, which is essentially the way δ_E is defined as in proposition 6.12.

We also need to check the diagram [F, (16)]which will follow if we check the (pre)dual diagram:

$$\begin{array}{ccccc}
\mathrm{NL}(E\times E) \parr (\neg E \parr \mathbb{K}) & \xleftarrow{\mathrm{NL}(E)\parr d_{E,\mathbb{K}}} & \mathrm{NL}(E) \parr (\mathrm{NL}(E) \parr \mathbb{K}) & \xleftarrow{\Lambda_{E,E,\mathbb{K}}} & \mathrm{NL}(E\times E) \parr \mathbb{K} \\
& \nwarrow{\scriptstyle \Lambda_{E,E,\neg E\parr\mathbb{K}}\circ(\mathrm{NL}(U(\nabla_E))\parr(\neg E\parr\mathbb{K}))} & & & \uparrow{\scriptstyle \mathrm{NL}(U(\nabla_E))\parr\mathbb{K}} \\
& & \mathrm{NL}(E) \parr (\neg E \parr \mathbb{K}) & \xleftarrow{d_{E,\mathbb{K}}} & \mathrm{NL}(E) \parr \mathbb{K}
\end{array}$$

Using step 1 and step 2 with $g = U(\nabla_E)$, it reduces to:

$$\begin{array}{ccc}
\mathrm{NL}(E^2) \parr (\neg E \parr \mathbb{K}) & \xleftarrow{\mathrm{NL}(E^2)\parr(\pi_2\parr\mathbb{K})} & \mathrm{NL}(E^2) \parr ((\neg E)^{\oplus 2}) \parr \mathbb{K}) \\
\uparrow{\scriptstyle (\mathrm{NL}(U(\nabla_E))\parr(\neg E\parr\mathbb{K}))} & & \uparrow{\scriptstyle (\mathrm{NL}(\Delta_{E^2})\Lambda^{-1}_{E^2,E^2})\parr_C((\neg E)^{\oplus 2}\parr_C\mathbb{K})} \\
\mathrm{NL}(E) \parr (\neg E \parr \mathbb{K}) & \xrightarrow{\mathrm{NL}(U(\nabla_E))\parr dH} & \mathrm{NL}(E^2) \parr \mathrm{NL}(E^2) \parr ((\neg E)^{\oplus 2}) \parr \mathbb{K})
\end{array}$$

Recall that here, from step 2, $dH = d_{E^2,\mathbb{K}} \circ H$. In our current case, we noticed that $H = \epsilon^{\mathrm{NL}}_{E^2} \circ \neg(\nabla_E)$. Using (19) with E^2 instead of E, and $\mathrm{Isom} \circ (\mathrm{Id}_{E^2} \times n_{E^2})\Delta_{E^2} = \mathrm{Id}_{E^2}$, the right hand side of the diagram we must check reduces to the map $\mathrm{NL}(U(\nabla_E)) \parr (\pi_2 \circ \neg(\nabla_E)) \parr \mathbb{K} = \mathrm{NL}(U(\nabla_E)) \parr \neg E \parr \mathbb{K}$ as expected, using only the defining property of ∇_E from the coproduct.

Let us turn to proving the first diagram in [F, (15)], which will give at the end $\overline{\mathsf{d}}_E \circ \mathsf{d}_E = \mathrm{Id}_E$. Modulo applying $\neg$ and intertwining with canonical isomorphisms, it suffices to see:

$$\begin{array}{ccccc}
& & \mathrm{NL}(U(E)) \parr \mathbb{K} & & \\
& \swarrow{\scriptstyle d_{E,\mathbb{K}}} & & \nwarrow{\scriptstyle \epsilon^{\mathrm{NL}}_E\parr\mathbb{K}} & \\
\mathrm{NL}(U(E)) \parr \neg E \parr \mathbb{K} & \xleftarrow[\mathrm{NL}(n_E)\parr\neg E\parr\mathbb{K}]{} & \mathrm{NL}(U(0)) \parr \neg E \parr \mathbb{K} & \xleftarrow{\simeq} & \neg E \parr \mathbb{K}
\end{array} \tag{19}$$

For it suffices to get the diagram after precomposition by any $h : \neg\mathbb{K} \longrightarrow \neg E$ (using the $\mathcal{D}$ is closed with unit $\neg\mathbb{K}$ for the closed structure). Since $E \in \mathcal{D}$ this is the same thing as $g = \neg h : E \simeq \neg\neg E \longrightarrow \mathbb{K}$ so that one can apply naturality in E of all the maps in the above diagram. This reduces the diagram to the case $E = \mathbb{K}$.

But from axiom D3 of [BEM], we have $D(\mathrm{Id}_{U(E)}) = \pi_2$, projection on the second element of a pair, for $E \in C$. When we apply the compatibility diagram between D and $d_{E,E}$ to $\mathsf{d}_{\neg E}$, which corresponds through Ξ to $\mathrm{Id}_U(E)$, we have (for $\pi_2 \in M(E \times E, E)$ the second projection):

$$M\Big[\mathrm{Isom} \circ U(\mathrm{NL}_E \parr_C (\epsilon_E^{\mathrm{NL}} \parr_C E)) \circ U(d_{E,E}) \circ U((\epsilon_E^{\mathrm{NL}} \parr \mathbb{K}) \circ (I_E))\Big] = \pi_2$$

Here we used $I_E : \neg\mathbb{K} \longrightarrow \neg E \parr E$ used from the axiom of dialogue categories corresponding via φ to $\mathrm{Id}_{\neg E}$ and where we use $M(\Xi_{E,E} \circ (U((\epsilon_E^{\mathrm{NL}} \parr \mathbb{K}) \circ (I_E)))) = \mathrm{Id}_{U(E)}$. This comes via naturality for φ from the association via φ of $(\epsilon_E^{\mathrm{NL}} \parr \mathbb{K}) \circ (I_E)$ to the map $\epsilon_E^{\mathrm{NL}} : \neg E \longrightarrow \mathrm{NL}(E)$, and then from the use of the compatibility of Ξ with adjunctions in definition 6.4 jointly with the definition of ϵ^{NL} as counit of adjunction, associating it to $\mathrm{Id}_{U(E)}$. Thus applying this to $E = \mathbb{K}$ and since we can always apply the faithful functors U, M to our relation and compose it with the monomorphism applied above after $U(d_{E,E})$ and on the other side to $U(I_{\mathbb{K}}) \simeq Id$, it is easy to see that the second composition is also π_2. ■

7.4. A general construction for DiLL models. Assume given the situation of Theorem 6.6, with C having a biproduct structure with $U, \neg$ **Mon**-enriched and assume that $\mathcal{M}$ is actually given the structure of a differential λ-category with operator internalized as a natural transformation $D_{E,F} : [E, F] \longrightarrow [\mathrm{Diag}(E), F]$ (so that D in the definition of those categories is given by $M(D_{E,F}) : \mathcal{M}(E, F) \longrightarrow \mathcal{M}(E \times E, F)$ with M the basic functor to sets of the closed category $\mathcal{M}$) and U bijective on objects. Assume also that there is a map $D'_{E,F}$: $\mathrm{NL}(E) \parr_C F \longrightarrow \mathrm{NL}(E \times E) \parr_C F$ in C, natural in E such that

$$\Xi_{E \times E, F} \circ U(D'_{E,F}) = D_{U(E),U(F)} \circ \Xi_{E,F}.$$

and

$$D'_{E,F} = \left(\rho^{-1}_{\mathrm{NL}(E^2)} \circ D'_{E,\mathbb{K}} \circ \rho_{\mathrm{NL}(E)}\right) \parr_C F. \tag{20}$$

Our non-linear variables are the first one after differentiation.

We assume $\parr_C$ commutes with limits and finite coproducts in C and recall from remark 6.7 that it preserves monomorphisms and that $\neg$ is faithful. Note that since C is assumed complete and cocomplete, it has coproducts $\oplus = \times$, by the biproduct assumption, and that $\neg(E \times F) = \neg(E) \oplus \neg(F)$ since $\neg : C^{\mathrm{op}} \longrightarrow C$ is left adjoint to its opposite functor $\neg : C \longrightarrow C^{\mathrm{op}}$ which therefore preserves limits.

Finally, we need the following:

$$\begin{array}{ccccc}
\neg(E \times F) \parr_C G & \xrightarrow{\epsilon^{\mathrm{NL}}_{E\times F} \parr_C G} & \mathrm{NL}(E \times F) \parr_C G & \xrightarrow{\mathrm{NL}(U((\mathrm{Id}_E \times 0_F)\circ r)) \parr_C G} & \mathrm{NL}(E) \parr_C G \\
\Big\downarrow \simeq & & & & \Big\uparrow \epsilon^{\mathrm{NL}}_E \parr_C G \\
\left(\neg E \parr_C G\right) \times \left(\neg F \parr_C G\right) & & \xrightarrow{\qquad\qquad \pi_1 \qquad\qquad} & & \neg E \parr_C G
\end{array} \tag{21}$$

This reduces to the case $G = \mathbb{K}$ by functoriality and then, this is a consequence of naturality of ϵ^{NL} since the main diagonal of the diagram taking the map via the lower left corner is nothing but $\neg((\mathrm{Id}_E \times 0_F) \circ r)$ with $r : E \longrightarrow E \times 0$ the right unitor for the cartesian structure on C.

We want to build from that data a new category $\mathcal{M}_C$ giving jointly with C the structure of a model of differential λ-tensor logic.

$\mathcal{M}_C$ has the same objects as $\mathcal{M}$ (and thus as C too) but new morphisms that will have as derivatives maps from C, or rather from its continuation category. Consider the category $\mathrm{Diff}_{\mathbb{N}}$ with objects $\{0\}\times\mathbb{N}\cup\{1\}\times\mathbb{N}^*$ generated by the following family of morphisms without relations: one morphism $d = d_i : (0,i) \longrightarrow (0,i+1)$ for all $i \in \mathbb{N}$ which will be mapped to a differential and one morphism $j = j_i : (1,i+1) \longrightarrow (0,i+1)$ for all $i \in \mathbb{N}$ which will give an inclusion. Hence all the morphism are given by $d^k : (0,i) \longrightarrow (0,i+k)$, $d^k \circ j : (1,i+1) \longrightarrow (0,i+k+1)$.

We must define the new Hom set. We actually define an internal Hom. Consider, for $E, F \in C$ the functor $\mathrm{Diff}_{E,F}, \mathrm{Diff}_E : \mathrm{Diff}_{\mathbb{N}} \longrightarrow C$ on objects by

$$\mathrm{Diff}_E((0,i)) = \mathrm{NL}(U(E)^{i+1}) \parr_C \mathbb{K}, \quad \mathrm{Diff}_E((1,i+1)) = (\mathrm{NL}(U(E)) \parr_C ((\neg E)^{\parr_C i+1} \parr_C \mathbb{K}))$$

with the obvious inductive definition $(\neg E)^{\parr_C i+1} \parr_C \mathbb{K} = \neg E \parr_C \left[(\neg E)^{\parr_C i} \parr_C \mathbb{K}\right]$. Then we define $\mathrm{Diff}_{E,F} = \mathrm{Diff}_E \parr_C F$.

The values of the functor on the generating morphisms are defined as follows:

$$\mathrm{Diff}_E(d_i) = \Lambda^{-1}_{U(E)^2,U(E)^i,\mathbb{K}} \circ (D'_{E,\mathrm{NL}(U(E)^i)\parr_C\mathbb{K}}) \circ \Lambda_{U(E),U(E)^i,\mathbb{K}},$$

$$\mathrm{Diff}_E(j_{i+1}) = \left[\rho_{\mathrm{NL}(U(E)^{i+1})} \circ \Lambda^{-1}_{U(E),U(E)^{i+1},\mathbb{K}} \circ \left(\mathrm{NL}(U(E)) \parr_C \left[\Lambda^{-1}_{U(E);i+1,\mathbb{K}} \circ ((\epsilon_E^{\mathrm{NL}})^{\parr_C i+1} \parr_C \mathbb{K})\right]\right)\right]$$

where we wrote

$$\Lambda^{-1}_{U(E);i+1,F} = \Lambda^{-1}_{U(E),U(E^i),F} \circ \cdots \circ (\mathrm{NL}(U(E))^{\parr_C i-1} \parr_C \Lambda^{-1}_{U(E),U(E),F}).$$

Since $\mathcal{M}$ has all small limits, one can consider the limit of the functor $U \circ \mathrm{Diff}_{E,F}$ and write it $[U(E), U(F)]_C$. Since U bijective on objects, this induces a Hom set:

$$\mathcal{M}_C(U(E), U(F)) = M([U(E), U(F)]_C).$$

We define $\mathrm{NL}_C(U(E))$ as the limit in C of Diff_E. Note that, since $\parr_C$ commutes with limits in C, $\mathrm{NL}_C(U(E)) \parr_C F$ is the limit of $\mathrm{Diff}_{E,F} = \mathrm{Diff}_E \parr_C F$.

From the universal property of the limit, it comes with canonical maps

$$D^k_{E,F} : \mathrm{NL}_C(U(E)) \parr_C F \longrightarrow \mathrm{Diff}_{E,F}((1,k)), \quad j = j_{E,F} : \mathrm{NL}_C(U(E)) \parr_C F \longrightarrow \mathrm{Diff}_{E,F}((0,0)).$$

Note that $j_{E,F} = j_{E,\mathbb{K}} \parr_C F$ is a monomorphism since for a pair of maps f, g with target $\mathrm{NL}_C(U(E)) \parr_C F$, using that lemma 7.5 below implies that all $\mathrm{Diff}_E(j_{i+1})$ are monomorphisms, one deduces that all the compositions with all maps of the diagram are equal, hence, by the uniqueness in the universal property of the projective limit, f, g must be equal.

Moreover, since $U : C \longrightarrow \mathcal{M}$ is right adjoint to $\neg \circ \mathrm{NL}$, it preserves limits, so that one gets an isomorphism $\Xi^{\mathcal{M}_C}_{U(E),F} : U(\mathrm{NL}_C(U(E)) \parr_C F) \simeq U(\lim \mathrm{Diff}_{E,F}) \simeq [U(E), U(F)]_C$. It will remain to build $\Lambda^{\mathcal{M}_C}$ but we can already obtain $d_{E,F}$.

We build it by the universal property of limits, consider the maps (obtained using canonical maps for the monoidal category C^{op})

$$D^{(1,k)}_{E,F} : \ \mathrm{NL}_C(U(E)) \parr_C F \xrightarrow{D^{k+1}_{E,F}} (\mathrm{NL}(U(E)) \parr_C ((\neg E)^{\parr_C k+1} \parr_C \mathbb{K})) \parr_C F \xrightarrow{\simeq} \mathrm{Diff}_{E,\neg E\parr_C F}((1,k))$$

$$J^1 : \mathrm{NL}_C(U(E)) \parr_C F \xrightarrow{D^1_{E,F}} (\mathrm{NL}(U(E)) \parr_C ((\neg E) \parr_C \mathbb{K})) \parr_C F \xrightarrow{\simeq} \mathrm{Diff}_{E,\neg E \parr_C F}((0,0))$$

Those maps extends uniquely to a cone enabling us to obtain by the universal properties of limits our expected map $d_{E,F}$. This required checking the identities

$$\mathrm{Diff}_{E,\neg E \parr_C F}(d^k) \circ J^1 = \mathrm{Diff}_{E,\neg E \parr_C F}(j_k) \circ D^{(1,k)}_{E,F}$$

that comes from $\mathrm{Diff}_{E,F}(d^k \circ j_1) \circ D^1_{E,F} = \mathrm{Diff}_{E,F}(j_{k+1}) \circ D^{1+k}_{E,F}$ (by definition of $D^{1+k}_{E,F}$ as map coming from a limit) which is exactly the previous identity after composition with structural isomorphisms and $\mathrm{NL}(E^{k+1}) \parr_C \epsilon^{\mathrm{NL}}_E \parr_C F$ which is a monomorphism, hence the expected identity, thanks to:

7.5. Lemma. *In the previous situation, ϵ^{NL}_E is a monomorphism.*

Proof. Since $\neg : C^{\mathrm{op}} \longrightarrow C$ is faithful, it suffices to see $\neg(\epsilon^{\mathrm{NL}}_E) : \neg(\mathrm{NL}(U(E)) \longrightarrow \neg\neg E$ is an epimorphism. But its composition with the epimorphism $\neg\neg E \longrightarrow E$, as counit of an adjunction with faithful functors $\neg$, is also the counit of $\neg \circ \mathrm{NL}$ with right adjoint U which is faithful too, hence the composition is an epimorphism too. But $U(\neg\neg E) \simeq U(E)$ by the proof of Theorem 6.6, thus $U(\neg(\epsilon^{\mathrm{NL}}_E))$ is an epimorphism and U is also faithful so reflects epimorphisms. ■

7.6. Theorem. *In the above situation, $(C^{\mathrm{op}}, \parr_C, I, \neg, \mathcal{M}_C, \times, 0, [.,.]_C, \mathrm{NL}_C, U, D, d)$ has the structure of Seely model of differential λ-tensor logic.*

Proof. For brevity, we call $A_k = (1,k), k > 0, A_0 = (0,0) = B_0, B_k = (0,k)$

Step 1: $\mathcal{M}_C$ is a cartesian (not full) subcategory of $\mathcal{M}$ and $U : C \longrightarrow \mathcal{M}_C, \mathrm{NL}_C : \mathcal{M}_C \longrightarrow C$ are again functors, the latter being right $\neg$-relative adjoint of the former.

Fix

$$g \in \mathcal{M}_C(U(E), U(F)) = C(\neg K, \mathrm{NL}_C(U(E)) \parr_C F)$$
$$\longrightarrow C(\neg K, \mathrm{NL}(U(E)) \parr_C F) \simeq C(\neg F, \mathrm{NL}(U(E))) \ni d^0 g.$$

Similarly, composing with $D^k_{E,F}$ one obtains:

$$d^k g \in C(\neg K, \mathrm{Diff}_{E,\mathbb{K}}(A_k) \parr_C F) \simeq C(\neg F, \mathrm{Diff}_{E,\mathbb{K}}(A_k)).$$

We first show that $\mathrm{NL}(g) = \cdot \circ g : \mathrm{NL}(U(F)) \longrightarrow \mathrm{NL}(U(E))$ induces via the monomorphisms j a map $\mathrm{NL}_C(g) : \mathrm{NL}_C(F) \longrightarrow \mathrm{NL}_C(E)$ such that $j_{E,\mathbb{K}}\mathrm{NL}_C(g) = \mathrm{NL}(g) j_{F,\mathbb{K}}$. This relation already determines at most one $\mathrm{NL}_C(g)$, one must check such a map exists in using the universal property for $\mathrm{NL}_C(E)$. Said in words, this map is supposed to show that the previous notion of composition $f \circ g$ is differentiable in the new stronger sense with derivatives with value in C. Since this differentiation must be compatible with D that satisfies the chain rule, and hence Faà di Bruno's formula, we will produce a lifting in C of this formula in $\mathcal{M}$. Concretely, we must build maps:

$$\mathrm{NL}^k_C(g) : \mathrm{NL}_C(F) \longrightarrow \mathrm{Diff}_{E,\mathbb{K}}(A_k)$$

with $\mathrm{NL}^0_C(g) = \mathrm{NL}(g) j_{F,\mathbb{K}}$ satisfying the relations for $k \geq 0$ ($j_0 = id$):

$$\mathrm{Diff}_{E,\mathbb{K}}(d_k \circ j_k) \circ \mathrm{NL}^k_C(g) = \mathrm{Diff}_{E,\mathbb{K}}(j_{k+1}) \circ \mathrm{NL}^{k+1}_C(g). \tag{22}$$

This map $\mathrm{NL}^k_C(g)$ is supposed to give $d^k(f \circ g)$ and the expected relations are consistency relations for this expectation. The form of $\mathrm{NL}^k_C(g)$ will be an abstract version of the combinatorial form of Faà di Bruno's formula (expressed as a sum over partitions) in the sense of being a lift expressed in terms of the structure of C of this formula valid in the cartesian differential category $\mathcal{M}$. We will obtain it as a sum of $\mathrm{NL}^{k,\pi}_C(g) : \mathrm{Diff}_{F,\mathbb{K}}(A_{|\pi|}) \longrightarrow \mathrm{Diff}_{E,\mathbb{K}}(A_k)$ for $\pi = \{\pi_1, ..., \pi_{|\pi|}\} \in P_k$ the set of partitions of $[\![1, k]\!]$. This map is supposed to be the composition map we need to apply to the term $d^{|\pi|} f$ in the term indexed by the partition π in Faà di Bruno's formula for $d^k(f \circ g)$. We define it as

$$\mathrm{NL}^{k,\pi}_C(g) = (\mathrm{NL}(\Delta^{|\pi|+1}_E) \parr_C \mathrm{Id}) \circ \mathrm{IsomAss}_{|\pi|} \circ [d^0 g \parr_C d^{|\pi_1|} g \parr_C \cdots \parr_C d^{|\pi_{|\pi|}|} g \parr_C \mathrm{Id}_{\mathbb{K}}]$$

with $\mathrm{IsomAss}_k : \mathrm{NL}(E) \parr_C (\mathrm{NL}(E) \parr_C E_1) \parr_C \cdots \parr_C (\mathrm{NL}(E) \parr_C E_k) \simeq \mathrm{NL}(E^{k+1}) \parr_C (E_1 \parr_C \cdots \parr_C E_k)$ and $\Delta_k : E \longrightarrow E^k$ the diagonal of the cartesian category C. We will compose it with d^{P_k} : $\mathrm{NL}_C(F) \longrightarrow \prod_{\pi \in P_k} \mathrm{Diff}_{F,\mathbb{K}}(A_{|\pi|})$ given by the universal property of product composing to $d^{|\pi|}$ in each projection.

Then using the canonical sum map $\Sigma^k_E : \prod_{i=1}^k E \simeq \oplus_{i=1}^k E \longrightarrow E$ obtained by universal property of coproduct corresponding to identity maps, one can finally define the map inspired by Faà di Bruno's Formula:

$$\mathrm{NL}^k_C(g) = \Sigma^{|P_k|}_{\mathrm{Diff}_{E,\mathbb{K}}(A_k)} \circ \Big(\prod_{\pi \in P_k} \mathrm{NL}^{k,\pi}_C(g) \Big) \circ d^{P_k}.$$

Applying U and composing with Ξ, (22) is then obtained in using the chain rule D5 on the inductive proof of Faà di Bruno's Formula, using also that U is additive.

Considering $\mathrm{NL}_C(g) \parr_C G : \mathrm{NL}_C(F) \parr_C G \longrightarrow \mathrm{NL}_C(E) \parr_C G$ which induces a composition on $\mathcal{M}_C$, one gets that $\mathcal{M}_C$ is a subcategory of $\mathcal{M}$ (from the agreement with previous composition based on intertwining with j) as soon as we see $\mathrm{Id}_{U(E)} \in \mathcal{M}_C(U(E), U(E))$. This boils down to building a map in C, $I_{\mathcal{M}_C} : \neg(\mathbb{K}) \longrightarrow \mathrm{NL}_C(U(E)) \parr_C E$ using the universal property such that $j_{E,E} \circ I_{\mathcal{M}_C} = I_{\mathcal{M}} : \neg(\mathbb{K}) \longrightarrow \mathrm{NL}(U(E)) \parr_C E$ corresponds to identity map. We define it in imposing $D^k_{E,E} \circ I_{\mathcal{M}_C} = 0$ if $k \geq 2$ and

$$D^1_{E,E} \circ I = \mathrm{NL}(0_E) \parr_C i_C : \neg(\mathbb{K}) \simeq \mathrm{NL}(0) \parr_C \neg(\mathbb{K}) \longrightarrow \mathrm{NL}(E) \parr_C \neg E \parr_C E$$

with $i_C \in C(\neg(\mathbb{K}), \neg E \parr_C E) \simeq C(\neg E, \neg E)$ corresponding to identity via the compatibility for the dialogue category $(C^{\mathrm{op}}, \parr_C, \mathbb{K}, \neg)$. This satisfies the compatibility condition enabling us to define a map by the universal property of limits because of axiom D3 in [BEM] implying (recall our linear variables are in the right contrary to theirs) $D(\mathrm{Id}_{U(E)}) = \pi_2, D(\pi_2) = \pi_2\pi_2$ (giving vanishing starting at second derivative via D-curry) and of course $(\epsilon^{\mathrm{NL}}_E \parr_C E) \circ i_C = I_{\mathcal{M}}$ from the adjunction defining ϵ^{NL}.

As above we can use known adjunctions to get the isomorphism

$$\begin{aligned}\mathcal{M}_C(U(E),U(F)) = \mathcal{M}(0,[U(E),U(F)]_C) \simeq C^{\mathrm{op}}(\mathrm{NL}(0),\neg(\mathrm{NL}_C(U(E))\parr_C F)) \\ \simeq C(\neg(\mathbb{K}),\mathrm{NL}_C(U(E))\parr_C F) \simeq C^{\mathrm{op}}(\mathrm{NL}_C(U(E)),\neg F)\end{aligned} \tag{23}$$

where the last isomorphism is the compatibility for the dialogue category $(C^{\mathrm{op}},\parr_C,\mathbb{K},\neg)$. Hence the map $\mathrm{Id}_{U(E)}$ we have just shown to be in the first space gives $\epsilon_E^{\mathrm{NL}_C}:\neg E\longrightarrow \mathrm{NL}_C(U(E))$ with $j_{E,\mathbb{K}}\circ\rho_{\mathrm{NL}_C(U(E))}\circ\epsilon_E^{\mathrm{NL}_C}=\rho_{\mathrm{NL}(U(E))}\circ\epsilon_E^{\mathrm{NL}}$.

Let us see that U is a functor too. Indeed $\epsilon_E^{\mathrm{NL}_C}\parr_C F:\neg E\parr_C F\longrightarrow \mathrm{NL}_C(U(E))\parr_C F$ can be composed with the adjunctions and compatibility for the dialogue category again to get:

$$C(E,F)\longrightarrow C(\neg\neg E,F)\simeq C(\neg\mathbb{K},\neg E\parr_C F)\longrightarrow C(\neg\mathrm{NL}(0),\mathrm{NL}_C(U(E))\parr_C F),$$

the last space being nothing but $\mathcal{M}_C(U(E),U(F)) = \mathcal{M}(0,[U(E),U(F)]_C)$ giving the wanted $U(g)$ for $g\in C(E,F)$ which is intertwined via j with the $\mathcal{M}$ valued one, hence U is indeed a functor too. The previous equality is natural in F via the intertwining with j and the corresponding result for $\mathcal{M}$.

Now one can see that (23) is natural in $U(E),F$. For it suffices to note that the first equality is natural by definition and all the following ones are already known. Hence the stated ¬-relative adjointeness.

This implies U preserve products as right adjoint of $\neg\circ\mathrm{NL}_C$, hence the previous products $U(E)\times U(F)=U(E\times F)$ are still products in the new category, and the category $\mathcal{M}_C$ is indeed cartesian.

Step 2: The Curry map

There remain some structure to define, most notably the internalized Curry map: $\Lambda^C_{E,F,G}$: $\mathrm{NL}_C(E\times F)\parr_C G\longrightarrow \mathrm{NL}_C(E)\parr_C\left(\mathrm{NL}_C(F)\parr_C G\right)$. We use freely the structure isomorphisms of the monoidal category C.

For we use the universal property of limits as before, we need to define:

$$\Lambda^k_{E,F,G}:\mathrm{NL}_C(E\times F)\parr_C G\longrightarrow \mathrm{Diff}_{E,(\mathrm{NL}_C(F)\parr_C G)}(A_k)$$

satisfying the relations for $k\geq 0$ ($j_0=id$):

$$\mathrm{Diff}_{E,(\mathrm{NL}_C(F)\parr_C G)}(d_k\circ j_k)\circ\Lambda^k_{E,F,G}=\mathrm{Diff}_{E,(\mathrm{NL}_C(F)\parr_C G)}(j_{k+1})\circ\Lambda^{k+1}_{E,F,G}. \tag{24}$$

Since $\mathrm{Diff}_{E,(\mathrm{NL}_C(F)\parr_C G)}(A_k)\simeq \mathrm{NL}_C(F)\parr_C\left(\mathrm{NL}(E)\parr_C(\neg E)^{\parr_C k}\parr_C\mathbb{K}\right)\parr_C G$ we use again the same universal property to define the map $\Lambda^k_{E,F,G}$ and we need to define:

$$\Lambda^{k,l}_{E,F,G}:\mathrm{NL}_C(E\times F)\parr_C G\longrightarrow \mathrm{Diff}_{F,\left(\mathrm{NL}(E)\parr_C(\neg E)^{\parr_C k}\parr_C\mathbb{K}\right)\parr_C G}(A_l).$$

satisfying the relations:

$$\mathrm{Diff}_{F,\left(\mathrm{NL}(E)\parr_C(\neg E)^{\parr_C k}\parr_C\mathbb{K}\right)\parr_C G}(d_l\circ j_l)\circ\Lambda^{k,l}_{E,F,G}=\mathrm{Diff}_{F,\left(\mathrm{NL}(E)\parr_C(\neg E)^{\parr_C k}\parr_C\mathbb{K}\right)\parr_C G}(j_{l+1})\circ\Lambda^{k,l+1}_{E,F,G}. \quad (25)$$

But we can consider the map:

$$D^{k+l}_{E\times F,G}:\mathrm{NL}_C(E\times F)\parr_C G\longrightarrow(\mathrm{NL}(U(E\times F))\parr_C((\neg(E\times F))^{\parr_C k+l}\parr_C\mathbb{K}))\parr_C G$$

Let us describe an obvious isomorphism of the space of value to extract the component we need. First, using the assumptions on $\parr_C$ and $\neg$:

$$\begin{aligned}((\neg(E_1\times E_2))^{\parr_C k+l}\parr_C\mathbb{K}))\parr_C G&\simeq\bigoplus_{i:[\![1,k+l]\!]\longrightarrow\{1,2\}}(\neg E_{i_1}\parr_C\neg E_{i_2}\parr_C\cdots\parr_C\neg E_{i_{k+l}})\parr_C G\\&\simeq\bigoplus_{i:[\![1,k+l]\!]\longrightarrow\{1,2\}}(\neg(E_1)^{\parr_C(\#f^{-1}(\{1\}))}\parr_C\neg(E_2)^{\parr_C(\#i^{-1}(\{2\}))})\parr_C G.\end{aligned}$$

Hence using also $\Lambda_{E,F,.}$ one gets:

$$\begin{aligned}&\Lambda:(\mathrm{NL}(U(E\times F))\parr_C((\neg(E\times F))^{\parr_C k+l}\parr_C\mathbb{K}))\parr_C G\\&\simeq\bigoplus_{i:[\![1,k+l]\!]\longrightarrow\{1,2\}}\mathrm{Diff}_F(A_{\#i^{-1}(\{2\}))})\parr_C\left(\mathrm{Diff}_E(A_{\#i^{-1}(\{1\}))})\parr_C G\right).\end{aligned}$$

Composing with $P_{k,l}$ a projection on a term with $\#i^{-1}(\{1\})=k$, one gets the map $P_{k,l}\circ\Lambda\circ D^{k+l}_{E\times F,G}=\Lambda^{kl}_{E,F,G}$ we wanted. One could check this does not depend on the choice of term using axiom (D7) of Differential cartesian categories giving an abstract Schwarz lemma, but for simplicity we choose $i(1)=\cdots=i(l)=2$ which corresponds to differentiating all variables in E first and then all variables in F. The relations we want to check will follow from axiom (D-curry) of differential λ-categories.

Then to prove the relation (25) we can prove it after composition by a Λ (hence the left hand side ends with application of $D'_{F,\mathrm{NL}(U(F)^l)\parr_C\mathbb{K}}\parr_C\left(\mathrm{Diff}_E(A_k)\parr_C G\right)$). We can then apply $\mathrm{Diff}_F(B_l)\parr_C\left(\mathrm{Diff}_E(j_k)\parr_C G\right)$ which is a monomorphism and obtain, after decurryfying and applying U and various Ξ, maps in $[U(F)^2\times U(F)^l\times U(E)^{k+1},U(G)]$, and finally only prove equality there, the first variable F being a non-linear one.

Of course, we start from $\mathrm{Diff}_{E\times F,G}(d_{k+l}\circ j_{k+l})\circ D^{k+l}_{E\times F,G}=\mathrm{Diff}_{E\times F,G}(j_{k+l+1})\circ D^{k+l+1}_{E\times F,G}$ and use an application of (21):

$$\begin{aligned}&\left[\mathrm{Diff}_F(B_l)\parr_C\left(\mathrm{Diff}_E(j_k)\parr_C G\right)\right]\circ\mathrm{Diff}_{F,\mathrm{Diff}_E(A_k)\parr_C G}(j_l)\circ P_{k,l}\circ\Lambda\\&=\mathrm{Isom}\circ\Lambda_{E,F,\mathrm{Diff}_F(B_l)\parr_C\left(\mathrm{Diff}_E(B_k)\parr_C G\right)}\circ\mathrm{NL}(0_{l,k})\circ\mathrm{Diff}_{E\times F,G}(j_{k+l})\end{aligned}$$

with $0_{l,k}:U(E\times F)\times U(F)^l\times U(E)^k\simeq U(E\times F)\times U(0\times F)^l\times U(E\times 0)^k\longrightarrow U(E\times F)^{k+l+1}$ the map corresponding to $\mathrm{Id}_{E\times F}\times(0_E\times\mathrm{Id}_F)^l\times(\mathrm{Id}_E\times 0_F)^k$. We thus need the following commutation relation:

$$\mathrm{NL}(0_{l+1,k})\circ\mathrm{Diff}_{E\times F,G}(d_{k+l})=\mathrm{Isom}\circ\mathrm{NL}(0_{1,0})\circ(D'_{E\times F,\left(\mathrm{Diff}_F(B_l)\parr_C\mathrm{Diff}_E(B_k)\parr_C G\right)})\circ\mathrm{NL}(0_{l,k})$$

This composition $\mathrm{NL}(0_{1,0}) \circ D'_{E\times F,.}$ gives exactly after composition with some Ξ the right hand side of (D-curry), hence composing all our identities, and using canonical isomorphisms of λ-models of λ-tensor logic, and this relation gives the expected (25) at the level of $[U(F)^2 \times U(F)^l \times U(E)^{k+1}, U(G)]$.

Let us turn to checking (24). It suffices to check it after composition with the monomorphism $\mathrm{Diff}_E(B_k) \mathbin{⅋_C} j_{F,G}$. Then the argument is the same as for (25) in the case $k = 0$ and with E and F exchanged. The inverse of the Curry map is obtained similarly.

Step 3: $\mathcal{M}_C$ is a differential λ-category.

We first need to check that $\mathcal{M}_C$ is cartesian closed, and we already know it is cartesian. Since we defined the internalized curry map and Ξ one can use the first compatibility diagram in the definition 6.4 to define $\Lambda^{\mathcal{M}_C}$. To prove the defining adjunction of exponential objects for cartesian closed categories, it suffices to see naturality after applying the basic functor to sets M. From the defining diagram, naturality in E, F of $\Lambda^{\mathcal{M}_C} : [E \times F, U(G)]_C \longrightarrow [E, [F, U(G)]_C]_C$ will follow if one checks the naturality of $\Xi^C_{E,F}$ and $\Lambda^C_{E,F,G}$ that we must check anyway while naturality in $U(G)$ and not only G will have to be considered separately.

For $\Lambda^{C-1}_{E,F,G}$, take $e : E \longrightarrow E'$, $f : F \longrightarrow F'$, $g : G' \longrightarrow G$ the first two in $\mathcal{M}_C$ the last one in C. We must see $\Lambda^{C-1}_{E,F,G} \circ [\mathrm{NL}_C(e) ⅋_C (\mathrm{NL}_C(f) ⅋_C g)] = [\mathrm{NL}_C(e \times f) ⅋_C g] \circ \Lambda^{C-1}_{E',F',G'}$ and it suffices to see equality after composition with the monomorphism $j_{U^{-1}(E\times F),G} : \mathrm{NL}_C(E \times F) ⅋_C G \longrightarrow \mathrm{NL}(E \times F) ⅋_C G$. But by definition, $j_{U^{-1}(E\times F),G} \Lambda^{C-1}_{E,F,G} = \Lambda^{-1}_{E,F,G}(\mathrm{NL}(E) ⅋ j_{U^{-1}(F),G}) j_{U^{-1}(E),\mathrm{NL}_C(F)⅋G}$ and similarly for NL_C functors which are also induced from NL, hence the relation comes from the one for Λ of the original model of λ-tensor logic we started with. The reasoning is similar with Ξ. Let us finally see that $M(\Lambda^{\mathcal{M}_C})$ is natural in $U(G)$, but again from step 1 composition with a map $g \in \mathcal{M}_C(U(G), U(G')) \subset \mathcal{M}(U(G), U(G'))$ is induced by the one from $\mathcal{M}$ and so is $\Lambda^{\mathcal{M}_C}$ from $\Lambda^{\mathcal{M}}$ in using the corresponding diagram for the original model of λ-tensor logic we started with and all the previous induced maps for Ξ, Λ^C. Hence also this final naturality in $U(G)$ is induced.

Having obtained the adjunction for a cartesian closed category, we finally see that all the axioms D1–D7 of cartesian differential categories in [BEM] and D-Curry are also induced. Indeed, our new operator D is also obtained by restriction as well as the left additive structure. Note that as a consequence the new U is still a **Mon**-enriched functor.

Step 4: $(\mathcal{M}_C, C)$ forms a λ-categorical model of λ-tensor logic.

We have already built all the data for definition 6.4, and shown $\neg$-relative adjointness in step 1. It remains to see the four last compatibility diagrams.

But from all the naturality conditions for canonical maps of the monoidal category, one can see them after composing with monomorphisms $\mathrm{NL}_C \longrightarrow \mathrm{NL}$ and induce them from the diagrams for NL.

Among all the data needed in definition 7.2, it remains to build the internalized differential $D^C_{E,F}$ for D in $\mathcal{M}^\lrcorner$ and see the two compatibility diagrams there. From the various invertible maps, one can take the first diagram as definition of $D^C_{U(E),U(F)}$ and must see that, then $M(D^C_{U(E),U(F)})$ is indeed the expected restriction of D. Let $j^{E,F}_{\mathcal{M}} : [U(E), U(F)]_C \longrightarrow [U(E), U(F)]$ the monomorphism. It suffices to see $j^{E\times E,F}_{\mathcal{M}} \circ D^C_{U(E),U(F)} \circ \Xi^C_{E,F} = \Xi_{E\times E,F} \circ U(D'_{E,F} \circ j_{E,F})$ (note that this also gives the naturality in E, F of d from the one of D'). Hence from the definition of D^C, it suffices to see the following diagram:

$$\begin{array}{ccc} U\Big(\mathrm{NL}_C(U(E)) \parr_C F\Big) & \xrightarrow{U(\mathrm{NL}_C(U(E)))\parr_C(\epsilon_E^{\mathrm{NL}_C}\parr_C F))\circ U(d_{E,F})} & U\Big(\mathrm{NL}_C(U(E)) \parr_C (\mathrm{NL}_C(U(E)) \parr_C F)\Big) \\ {\scriptstyle \Xi_{E\times E,F}\circ U(D'_{E,F}\circ j_{E,F})}\downarrow & & \downarrow{\scriptstyle [\mathrm{Id}_{U(E)},\Xi^C_{E,F}]_C\circ\Xi^C_{E,\mathrm{NL}_C(U(E))\parr_C F}} \\ [U(E\times E), U(F)] \xleftarrow{j^{E\times E,F}_{\mathcal{M}}} [U(E\times E), U(F)]_C & \xleftarrow{(\Lambda^{\mathcal{M}_C}_{U(E),U(E),U(F)})^{-1}} & [U(E), [U(E), U(F)]_C]_C. \end{array}$$

First we saw from induction of our various maps that the right hand side of the diagram can be written without maps with index C:

$$(\Lambda^{\mathcal{M}}_{U(E),U(E),U(F)})^{-1} \circ [\mathrm{Id}_{U(E)}, \Xi_{E,F}] \circ \Xi_{E,\mathrm{NL}(E)\parr_C F} \circ U(\mathrm{NL}(U(E))) \parr_C (\epsilon_E^{\mathrm{NL}} \parr_C F)) \circ U(j_{E,\neg E\parr_C F} \circ d_{E,F}).$$

The expected diagram now comes the definition of $d_{E,F}$ by universal property which gives $j_{E,\neg E\parr_C F} \circ d_{E,F} = J^1 = \mathrm{Isom} \circ D^1_{E,F}$ and similarly $\mathrm{Diff}_{E,F}(j_1) \circ D^1_{E,F} = D'_{E,F} \circ j_{E,F}$ so that composing the above diagrams (and an obvious commutation of the map involving ϵ^{NL} through various natural isomorphisms) gives the result.

For the last diagram in definition 7.2, since $j = j_{E,\neg E\parr_C F}$ is a monomorphism, it suffices to compose $d_{E,F}$ and the equivalent map stated in the diagram by j and see equality, and from the recalled formula above reducing it to $D'_{E,F}$, this reduces to (20). ∎

7.7. ρ-smooth maps as model of DiLL. The previous categories that were described in Theorem 6.9 cannot give a model of DiLL with $\mathscr{C}$-$\mathbf{Ref}_\infty$ as smooth maps. The differential map would not have values in $E \multimap_{\mathscr{C}} F$, but in spaces of bounded linear maps $L_{\mathrm{bd}}(E, F)$. We will have to restrict to maps with iterated differentials valued in $E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F := E \multimap_{\mathscr{C}} (\cdots (E \multimap_{\mathscr{C}} F) \cdots)$. This is what we did abstractly in the previous subsection that will enable us to obtain efficiently a model.

7.8. Lemma. *The categories of Theorem 6.9 satisfy the assumptions of subsection 7.4.*

Proof. We already saw in Theorem 6.9 that the situation of Theorem 6.6 is satisfied with dialogue category $C = (\mathscr{C}\text{-}\mathbf{Mc}^{\mathrm{op}}, \epsilon, \mathbb{K}, (\cdot)^*_{\mathscr{C}})$, and $\mathcal{M} = \mathscr{C}\text{-}\mathbf{Mc}_\infty$. We already know that the ε-product commutes with limits and monomorphisms and the biproduct property is straigtforward. The key is to check that we have an internalized derivative. From [KM] we know that we have a derivative $d : C^\infty_{\mathscr{C}}(E, F) \longrightarrow C^\infty_{\mathscr{C}}(E, L_b(E, F))$ and the space of bounded linear maps $L_{\mathrm{bd}}(E, F) \subset C^\infty_{\mathscr{C}}(E, F)$ the set of conveniently smooth maps. Clearly, the inclusion is continuous since all the images of compact sets under appearing in the projective kernel definition of $C^\infty_{\mathscr{C}}(E, F)$ are bounded. Thus one gets $d : C^\infty_{\mathscr{C}}(E, F) \longrightarrow C^\infty_{\mathscr{C}}(E, C^\infty_{\mathscr{C}}(E, F)) \simeq C^\infty_{\mathscr{C}}(E \times E, F)$. It remains to demonstrate continuity. By the projective kernel definition, one must check that for $c = (c_1, c_2) \in \mathscr{C}^\infty_{co}(X, E \times E)$, $f \mapsto df \circ (c_1, c_2)$ is continuous $C^\infty_{\mathscr{C}}(E, F) \longrightarrow \mathscr{C}^\infty_{co}(X, F)$. But consider the curve $c_3 : X\times X\times\mathbb{R} \longrightarrow E$ given by $c_3(x, y, t) = c_1(x)+c_2(y)t$, since $X\times X\times\mathbb{R} \in \mathscr{C}$, we know that $f \circ c_3$ is smooth and $\partial_t(f \circ c_3)(x, y, 0) = df(c_1(x))(c_2(y))$ and its derivatives in x, y are controlled by the seminorms for $C^\infty_{\mathscr{C}}(E, F)$, hence the stated continuity. It remains to note that $\mathcal{M}$ is a differential λ-category since we already know it is cartesian closed and all the properties of derivatives are well-known for conveniently smooth maps. For instance, the chain rule D7 is [KM, Thm 3.18]. ■

Concretely, one can make explicit the stronger notion of smooth maps considered in this case. We thus consider d^k the iterated (convenient) differential giving

$$d^k : C^\infty_{\mathscr{C}}(E, F) \longrightarrow C^\infty_{\mathscr{C}}(E, L_{bd}(E^{\otimes_\beta k}, F)).$$

Since $E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F$ is a subspace of $L_{bd}(E^{\otimes_\beta k}, F)$ (unfortunately this does not seem to be in general boundedly embedded), we can consider:

$$C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F) := \Big\{u \in C^\infty_{\mathscr{C}}(E, F) : \forall k \geq 1 : d^k(u) \in C^\infty_{\mathscr{C}}(E, E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F)\Big\}.$$

7.9. Remark. A map $f \in C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F)$ will be called $\mathscr{C}$-**Ref**-*smooth*. In the case $\mathscr{C}$ = Ban, we say ρ-smooth maps, associated to the category ρ-Ref, and write $C^\infty_\rho = C^\infty_{\mathrm{Ban}\text{-}\mathbf{Ref}}$. Actually, for Fin $\subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$, from the equivalence of $*$-autonomous categories in Theorem 5.27, and since the inverse functors keep the bornology of objects, hence don't change the notion of conveniently smooth maps, we have algebraically

$$|C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F)| = |C^\infty_\rho(\mathscr{S}(E_\mu), \mathscr{S}(F_\mu))|.$$

Hence, we only really introduced one new notion of smooth maps, namely, ρ-smooth maps. Of course, the topologies of the different spaces differ.

Thus d^k induces a map $C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F) \longrightarrow C^\infty_{\mathscr{C}}(E, E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F)$ ($d^0 = id$) and we equip $C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F)$ with the corresponding locally convex kernel topology $\mathrm{K}_{n,\geq 0}(d^n)^{-1}(C^\infty(E, E^{\otimes n} \multimap F))$ with the notation of [K] and the previous topology given on any $C^\infty(E, E^{\otimes k} \multimap F)$.[2]

We call $\mathscr{C}$-$\mathbf{Ref}_{\infty\text{-}\mathscr{C}\text{-}\mathbf{Ref}}$ the category of $\mathscr{C}$-reflexive spaces with $C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}$ as spaces of maps. Then from section 7.4 we even have an induced $d : C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F) \longrightarrow C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F)$.

Let us call $d_0(f) := df(0)$ so that $d_0 : C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F) \longrightarrow C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F)$ is continuous. Recall also that we introduced $\partial_E := \bar{\mathsf{c}}_E \circ (!E \otimes \bar{\mathsf{d}}_E)$ and dually $\bar{\partial}_E := (!E \otimes \mathsf{d}_E)\mathsf{c}_E : !E \longrightarrow !E \otimes E$. We conclude to our model:

7.10. Theorem. *Let* Fin $\subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$ *as above.* $\mathscr{C}$-**Ref** *is also a Seely category with biproducts with structure extended by the comonad* $!_{\mathscr{C}\text{-}\mathbf{Ref}}(\cdot) := (C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(\cdot))^*_{\mathscr{C}}$ *associated to the adjunction with left adjoint* $!_{\mathscr{C}\text{-}\mathbf{Ref}} : \mathscr{C}\text{-}\mathbf{Ref}_{\infty\text{-}\mathscr{C}\text{-}\mathbf{Ref}} \longrightarrow \mathscr{C}\text{-}\mathbf{Ref}$ *and right adjoint* U. *It gives a model of DiLL with codereliction* $(d_0)^*_{\mathscr{C}}$.

Proof. This is a combination of Theorem 7.6, 7.3 and the previous lemma. ■

[2] This definition is quite similar to one definition (for the corresponding space of value $E^{\otimes k} \multimap F$ which can be interpreted as a space of hypocontinuous multilinear maps for an appropriate bornology) in [Me] except that instead of requiring continuity of all derivatives, we require their smoothness in the sense of Kriegl-Michor.

7.11. Remark. One can check that

1. for any $E \in \mathscr{C}$-**Ref**, $\partial_E \overline{\partial}_E + \mathrm{Id}_E$ is invertible,
2. The model is Taylor in the sense of [Ehr16, 3.1], i.e. for any $f_1, f_2 :!_{\mathscr{C}\text{-}\mathbf{Ref}} E \longrightarrow F$ if $f_1 \partial_E = f_2 \partial_E$ then $f_1 + f_2 \overline{\mathsf{w}}_E \mathsf{w}_E = f_2 + f_1 \overline{\mathsf{w}}_E \mathsf{w}_E$.

Indeed, the Taylor property is obvious since $df_1 = df_2$ in the convenient setting implies the same Gâteaux derivatives, hence $f_1 + f_2(0) = f_2 + f_1(0)$ on each line hence everywhere.

For (1), we define the inverse by $(I_E)^*_{\mathscr{C}}$ with $I_E : C^\infty_{\mathscr{C}}(E) \longrightarrow C^\infty_{\mathscr{C}}(E)$ as in [Ehr16, 3.2.1] by $I_E(f)(x) = \int_0^1 f(tx)dt$, which is a well-defined weak Riemann integral by Mackey-completeness of the space [KM].

By [Ehr16], the two conditions reformulate the two fundamental theorems of calculus. See also [CL] for a further developments on the two conditions above.

7.12. Remark. Let us continue our comparison of subsection 6.14. Note that if $E, F \in$ **Conv**, we have $C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F) = C^\infty_{\mathscr{C}}(E, F)$ so that we did not need to introduce a new class of smooth maps for convenient vector spaces. Recall also that the functor $\mathscr{S} :$ **Conv** $\longrightarrow \rho$-**Ref** is not essentially surjective, there are ρ-Reflexive spaces not having the Schwarz topology associated to a bornological topology (those specific spaces have complete Mackey duals). Our notion of smoothness turning our model into a model of DiLL is only crucial on these spaces in ρ-**Ref** $\cap$ $(\mathcal{S}(\mathbf{Conv}))^c$.Let us explain why, for $E, F \in$ **Conv**, we have $C^\infty_{\mathscr{C}\text{-}\mathbf{Ref}}(E, F) = C^\infty_{\mathscr{C}}(E, F)$. It suffices to see that $f \in C^\infty_{\mathscr{C}}(E, F)$ is ρ-smooth. But (15) gives that the derivative automatically smooth with value $L_\beta(E, F)$ by convenient smoothness is also smooth by composition with value $E^*_\rho \varepsilon F$ as expected. Since this equation only depends on the source space E to be bornological, it extends to spaces for higher derivatives, hence the conclusion.

Hence we have a functor $\mathscr{S} : \mathbf{Conv}_\infty \longrightarrow \mathscr{C}\text{-}\mathbf{Ref}_{\infty\text{-}\mathscr{C}\text{-}\mathbf{Ref}}$ for any $\mathscr{C}$ as above. We don't think this is an equivalence of category any more, as was the corresponding functor in 6.14. But finding a counterexample to essential surjectivity may be difficult, even thought we didn't really try.

7.13. *k*-smooth maps as model of DiLL. We now turn to improve the $*$-autonomous category **k-Ref** of section 3 into a model of DiLL using the much stronger notion of k-smooth map considered in subsection 3.21. For $X, Y \in$ **k-Ref**, $C^\infty_{\mathrm{co}}(X, Y) \subset \mathscr{C}^\infty(X, Y)$, hence there is a differential map $d : C^\infty_{\mathrm{co}}(X, Y) \longrightarrow C^\infty_{\mathrm{co}}(X, L_\beta(X, Y))$ but it is by definition valued in $C^0_{\mathrm{co}}(X, L_{\mathrm{co}}(X, Y))$. But actually since the derivatives of these map are also known, it is easy to use the universal property of projective limits to induce a continuous map: $d : C^\infty_{\mathrm{co}}(X, Y) \longrightarrow C^\infty_{\mathrm{co}}(X, L_{\mathrm{co}}(X, Y))$. Finally, note that $L_{\mathrm{co}}(X, Y) = X^*_k \varepsilon Y$, hence the space of value is the one expected for the dialogue category Kc^{op} from Theorem 3.20.

For simplicity, in this section we slightly change **k-Ref** to be the category of k-reflexive spaces of density character smaller than a fixed inaccessible cardinal κ, in order to have a small category $\mathscr{C}$ = **k-Ref** and in order to define without change $C^\infty_{\mathscr{C}}(X, Y)$

We call $\mathbf{k}\text{-}\mathbf{Ref}_\infty$ the category of k-reflexive spaces with maps $C^0_{\mathrm{co}}(X, Y)$ as obtained in subsection 3.21. We call $\mathbf{Kc}_\infty$ the category of k-quasi-complete spaces (with density character smaller than the same κ) with maps $C^\infty_{\mathscr{C}}(X,Y)$. This is easy to see that this forms a category by definition of $C^\infty_{\mathscr{C}}$. We first check our assumptions to produce models of LL. We call $C^\infty_{\mathscr{C}} : \mathbf{Kc}_\infty \longrightarrow \mathbf{Kc}^{\mathrm{op}}$ the functor associating $C^\infty_{\mathscr{C}}(X) = C^\infty_{\mathscr{C}}(X,\mathbb{R})$ to a space X.

7.14. Lemma. $(\mathbf{Kc}^{\mathrm{op}}, \varepsilon, \mathbb{K}, (\cdot)^*_\rho, \mathbf{Kc}_\infty, \times, 0, C^\infty_{\mathscr{C}}, U)$ *is a Seely linear model of λ-tensor logic.*

Proof. We checked in Theorem 3.20 that $\mathcal{C} = (\mathbf{Kc}^{\mathrm{op}}, \varepsilon, \mathbb{K}, (\cdot)^*_\rho)$ is a dialogue category. κ-Completeness and κ-cocompleteness are obvious using the k-quasicompletion functor to complete colimits in **LCS**. Lemma 4.2 gives the maps Ξ, Λ and taking the first diagram as definition of $\Lambda^{\mathcal{M}}$ one gets cartesian closedness of $\mathcal{M} = (\mathbf{Kc}_\infty, \times, 0, C^\infty_{\mathscr{C}}(\cdot,\cdot))$, and this result also gives the relative adjunction. The other compatibility diagrams are reduced to conveniently smooth maps $\mathscr{C}^\infty_{\mathrm{Fin}}$ as in the proof of Theorem 6.9. ∎

Note that since $\mathbf{k}$-$\mathbf{Ref}^{\mathrm{op}}$ is already $*$-autonomous and isomorphic to its continuation category

7.15. Lemma. *The categories of the previous lemma satisfy the assumptions of subsection 7.4.*

Proof. The differential λ-category part reduces to convenient smoothness case. The above construction of d make everything else easy. ∎

7.16. Theorem. $\mathbf{k}$-$\mathbf{Ref}$ *is also a complete Seely category with biproducts with $*$-autonomous structure extended by the comonad $!_{\mathrm{co}}(\cdot) = (C^\infty_{\mathrm{co}}(\cdot))^*_k$ associated to the adjunction with left adjoint $!_{\mathrm{co}} : \mathbf{k}\text{-}\mathbf{Ref}_{\mathrm{co}} \longrightarrow \mathbf{k}\text{-}\mathbf{Ref}$ and right adjoint U. It gives a model of DiLL with codereliction $(d_0)^*_k$.*

Proof. Note that on $\mathbf{k}$-$\mathbf{Ref}$ which corresponds to $\mathcal{D}$ in the setting of subsection 7.4, we know that $C^\infty_{\mathrm{co}} = C^\infty_{\mathscr{C}}$ by the last statement in lemma 4.2. But our previous construction of d implies that the new class of smooth maps obtained by the construction of subsection 7.4 is again C^∞_{co}. The result is a combination of Theorem 7.6, 7.3 and the previous lemmas. ∎

8. Conclusion

This work provides strong evidence for the validity of the classical setting of Differential Linear Logic. Indeed, we present here the first smooth models of Differential Linear Logic which model the classical structure. Our axiomatization of the rules for differential categories within the setting of dialogue categories can be seen as a first step towards a computational classical understanding of Differential Linear Logic. We plan to explore the categorical content of our construction for new models of Smooth Linear Logic, and the diversity of models which can be constructed this way. Our results also argue for an exploration of a classical differential term calculus, as initiated by Vaux [Vaux], and inspired by works on the computational signification of classical logic [CH] and involutive linear negation [Mu].

The clarification of a natural way to obtain $*$-autonomous categories in an analytic setting suggests one should reconsider known models such as [Gir99] from a more analytic viewpoint, and should lead the way to exploit the flourishing operator space theory in logic, following the inspiration of the tract [Gir04]. An obvious notion of coherent operator space should enable this.

This interplay between functional analysis, physics and logic is also strongly needed as seen the more and more extensive use of convenient analysis in some algebraic quantum field theory approaches to quantum gravity [BFR]. Here the main need would be to improve the infinite dimensional manifold theory of diffeomorphism groups on non-compact manifolds. From that geometric viewpoint, differential linear logic went only part way in considering smooth maps on linear spaces, rather that smooth maps on some type of smooth manifold. By providing well-behaved ?-monads, our work suggests to try using ?-algebras for instance in k-reflexive or ρ-reflexive spaces as a starting point (giving a base site of a Grothendieck topos) to capture better infinite dimensional features than the usual Cahier topos. Logically, this probably means obtaining a better interplay between intuitionist dependent type theory and linear logic. Physically, this would be useful to compare recent homotopic approaches [BSS] with applications of the BV formalism [FR, FR2]. Mathematically this probably means merging recent advances in derived geometry (see e.g. [To]) with infinite dimensional analysis. Since we tried to advocate the way linear logic nicely captures (for instance with two different tensor products) infinite dimensional features, this finally strongly suggests for an interplay of parametrized analysis in homotopy theory and parametrized versions of linear logic [CFM].

9. Appendix

We conclude with two technical lemmas only used to show we have built two different examples of ! on the same category ρ-Ref.

9.1. Lemma. *For any ultrabornological spaces E_i, any topological locally convex hull $E = \Sigma_{i\in I}A_i(E_i)$, then we have the topological identity:*

$$\mathscr{S}(E'_\mu) = \mathrm{K}_{i\in I}(A_i^t)^{-1}(\mathscr{S}((E_i)'_\mu).$$

Proof. We start with case where E_i are Banach spaces. By functoriality one gets a map between two topologies on the same space (see for Mackey duals [K, p 293]):

$$\mathscr{S}(E'_\mu) \longrightarrow \mathrm{K}_{i\in I}(A_i^t)^{-1}(\mathscr{S}((E_i)'_\mu)) =: F.$$

In order to identify the topologies, it suffices to identify the duals and the equicontinuous sets on them. From [K, §22.6.(3)], the dual of the right hand side is $F' = \Sigma_{i\in I}(A_i^{tt})(\mathscr{S}((E_i)'_\mu)' = \Sigma_{i\in I}(B_i)(E_i) \longrightarrow E$ where the injective continuous map to E is obtained by duality of the previous surjective map (and the maps called B_i again are in fact compositions of A_i^{tt} and the isomorphism between $[\mathscr{S}((E_i)'_\mu)]' = E_i$). From the description of E the map above is surjective and thus we must have $F' = E$ as vector spaces.

Let us now identify equicontinuous sets. From continuity of $\mathscr{S}(E'_\mu) \longrightarrow F$ every equicontinuous set in F' is also equicontinuous in $E = (\mathscr{S}(E'_\mu))'$. Conversely an equicontinuous set in $E = (\mathscr{S}(E'_\mu))'$ is contained in the absolutely convex cover of a null-sequence $(x_n)_{n\geq 0}$ for the bornology of absolutely convex weakly-compact sets, (thus also for the bornology of Banach disks [Ja, Th 8.4.4 b]). By a standard argument, there is $(y_n)_{n\geq 0}$ null sequence of the same type such that $(x_n)_{n\geq 0}$ is a null sequence for the bornology of absolutely convex compact sets in a Banach space E_B with B the closed absolutely convex cover of $(y_n)_{n\geq 0}$.

Of course $(y_n)_{n\leq m}$ can be seen inside a minimal finite sum $G_m = \Sigma_{i\in I_m}(B_i)(E_i)'$ and G_m is increasing in F so that one gets a continuous map $I : \text{ind lim}_{m\in\mathbb{N}} G_m \longrightarrow F'$. Moreover each G_m being a finite hull of Banach space, it is again a Banach space thus one gets a linear map $j : E_B \longrightarrow \text{ind lim}_{m\in\mathbb{N}} G_m = G$. Since $I\circ j$ is continuous, j is a sequentially closed map, E_B is Banach space, G a (LB) space therefore a webbed space, by De Wilde's closed graph theorem [K2, §35.2.(1)], one deduces j is continuous. Therefore by Grothendieck's Theorem [K, §19.6.(4)], there is a G_m such that j is valued in G_m and continuous again with value in G_m. Therefore $(j(x_n))_{n\geq 0}$ is a null sequence for the bornology of absolutely convex compact sets in G_m. We want to note it is equicontinuous there, which means it is contained in a sum of equicontinuous sets.
By [K, §19.2.(3)], G_m is topologically a quotient by a closed linear subspace $\bigoplus_{i\in I_m}(B_i)(E_i)'/H$. By [K, §22.2.(7)] every compact subset of the quotient space $\bigoplus_{i\in I_m}(B_i)(E_i)'/H$ of a Banach space by a closed subspace H is a canonical image of a compact subset of the direct sum, which can be taken a product of absolutely convex covers of null sequences.Therefore our sequence $(j(x_n))_{n\geq 0}$ is contained in such a product which is exactly an equicontinuous set in $G_m = \left(\mathrm{K}_{i\in I_m}(A_i^t)^{-1}(\mathscr{S}((E_i)'_\mu)\right)'$ [K, §22.7.(5)] (recall also that for a Banach space $\mathscr{S}((E_i)'_\mu) = (E_i)'_c$). Therefore it is also equicontinuous in F' (by continuity of $F\longrightarrow G_m'$). This concludes to the Banach space case.

For the ultrabornological case decompose E_i as an inductive limit of Banach spaces. Get in this way a three terms sequence of continuous maps with middle term $\mathrm{K}_{i\in I}(A_i^t)^{-1}(\mathscr{S}((E_i)'_\mu)$ and end point the corresponding iterated kernel coming from duals of Banach spaces by transitivity of Kernels/hulls. Conclude by the previous case of equality of topologies between the first and third term of the sequence, and this concludes to the topological equality with the middle term too. ∎

To prove the approximation property we use an intermediate class between Schwartz spaces and nuclear spaces: hilbertianisable spaces [Ja, p 243] (see also [H] where they are called (gH)-spaces).

9.2. Lemma. *For any lcs E, $((C^\infty_{\mathrm{Fin}}(E))^*_\rho)^*_\rho$ is Hilbertianisable, hence it has the approximation property.*

Proof. We actually show that $F = ((C^\infty_{\mathrm{Fin}}(E))^*_\rho)^*_\rho = \mathscr{S}[\left(\mathscr{C}_M\left[(C^\infty_{\mathrm{Fin}}(E))'_\mu\right]\right)'_\mu]$ is Hilbertianisable. This is enough thanks to [H, Rmk 1.5.(4)].

Note that $G = C^\infty_{\mathrm{Fin}}(E)$ is a complete nuclear space. It suffices to show that for any complete nuclear space G, $\mathscr{S}[\left(\mathscr{C}_M\left[G'_\mu\right]\right)'_\mu]$ is a complete (gH) space. Of course, we use lemma 3.6 but we need another description of the Mackey completion $\mathscr{C}^\lambda_M(G'_\mu)$. We let $E_0 = G'_\mu, E_{\lambda+1} = \cup_{\{x_n\}\in\mathrm{RMC}(E_\lambda)}\overline{\gamma(x_n, n\in\mathbb{N})}E_\lambda = \cup_{\mu<\lambda}E_\mu$ for limit ordinals.

Here $\mathrm{RMC}(E_\lambda)$ is the set of sequences $(x_n)\in E_\lambda^{\mathbb{N}}$ which are rapidly Mackey-Cauchy in the sense that if x is their limit in the completion there is a bounded disk $B\subset E_{\lambda+1}$ such that for all k, $(x_n - x)\in n^{-k}B$ for n large enough. For λ_0 large enough, $E_{\lambda_0+1} = E_{\lambda_0}$ and any Mackey-Cauchy sequence x_n in E_{λ_0}, let us take its limit x in the completion and B a closed bounded disk in E_{λ_0} such that $||x_n - x||_B \longrightarrow 0$ one can extract x_{n_k} such that $||x_{n_k} - x||_B \leq k^{-k}$ so that $(x_{n_k} - x) \in k^{-l}B$ for k large enough (for any l) thus $(x_{n_k}) \in \mathrm{RMC}(E_{\lambda_0})$ thus its limit is in $E_{\lambda_0+1} = E_{\lambda_0}$ which is thus Mackey-complete. To apply lemma 3.6 with $D = \mathscr{N}((\cdot)'_\mu))$ one needs to see that $\{x_n, n \in \mathbb{N}\}$ is equicontinuous in $D(E_{\lambda_0})'$. But since E_{λ_0} is Mackey-complete, one can assume the bounded disk B is a Banach disk and $||x_n - x||_B = O(n^{-k})$ so that x_n is rapidly convergent. From [Ja, Prop 21.9.1] $\{(x_n - x), n \in \mathbb{N}\}$ is equicontinuous for the strongly nuclear topology associated to the topology of convergence on Banach disks and a fortiori equicontinuous for $D(E_{\lambda_0})'$. By translation, so is $\{x_n, n \in \mathbb{N}\}$ as expected. From application of lemma 3.6, $H^{\lambda_0} := \mathscr{N}[(\mathscr{C}_M(G'_\mu))'_\mu]$ is complete since $\mathscr{N}[(G'_\mu)'_\mu]$ is already complete (G is complete nuclear so that $\mathscr{N}[(G'_\mu)'_\mu] \longrightarrow G$ continuous and use again [Bo2, IV.5 Rmq 2]).

H^{λ_0} is nuclear thus a (gH)-space. Since H^{λ_0} is a complete (gH) space, it is a reduced projective limit of Hilbert spaces [H, Prop 1.4] and semi-reflexive [H, Rmk 1.5 (5)]. Therefore its Mackey=strong dual [K, §22.7.(9)] is an inductive limit of the Mackey duals, thus Hilbert spaces.

One can apply lemma 9.1 to get $\mathscr{S}([H^{\lambda_0}]_\mu)$ as a projective kernel of $\mathscr{S}(H)$ with H Hilbert spaces. But from [Bel, Thm 4.2] this is the universal generator of Schwartz (gH) spaces, therefore the projective kernel is still of (gH) space.

For λ_0 as above, this concludes to $\mathscr{S}[(\mathscr{C}_M((C^\infty_{\mathrm{Fin}}(E))'_\mu))'_\mu]$ (gH) space, as expected. ■

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