Theory and Applications of Categories, Vol. 34, No. 5, 2019, pp. 109–120.

DISTRIBUTORS AND THE COMPREHENSIVE FACTORIZATION SYSTEM FOR INTERNAL GROUPOIDS

GIUSEPPE METERE

ABSTRACT. In this note we prove that distributors between groupoids in a Barr-exact category $\mathcal{E}$ form the bicategory of relations relative to the comprehensive factorization system in $\mathbf{Gpd}(\mathcal{E})$. The case $\mathcal{E} = \mathbf{Set}$ is of special interest.

1. Introduction

Distributors (also called profunctors, or bimodules) were introduced by Bénabou in [2]. A fruitful approach is that of considering distributors as kind of *relations between categories* (see [3], and [4, §7.8]). In the set-theoretical case, relations can be presented as being *relative* to the epi/mono factorization system. As observed by Lawvere [10] such a factorization system can be obtained from a *comprehension schema*: for any set Y, one considers the *comprehension adjunction*

$$\mathbf{Set}/Y \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{2}^Y \ , \tag{1}$$

where the category $\mathbf{2}^Y$ is the partially ordered set of the subsets of Y. For any function $X \xrightarrow{f} Y$, the (epic) unit of the adjunction provides the factorization $f = m \cdot \eta_f$, where m is a mono:

$$\begin{tikzcd} X \arrow[r, "\eta_f"] \arrow[dr, "f"'] & Im(f) \arrow[d, "m"] \\ & Y \end{tikzcd}$$

As observed in [13], similar arguments can be used starting with the adjunction

$$\mathbf{Cat}/\mathbb{Y} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Set}^{\mathbb{Y}} \ ,$$

but climbing one dimension up produces two distinct factorizations of a given functor: (initial/discrete opfibration) and (final/discrete fibration). The first was named *comprehensive factorization of a functor* in [13], as arising from a *categorical comprehension schema.*

This work was partially supported by the Fonds de la Recherche Scientifique - F.N.R.S.: 2015/V 6/5/005 - IB/JN - 16385, and by the G.N.S.A.G.A. - Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni (I.N.D.A.M.).

Received by the editors 2018-05-26 and, in final form, 2019-02-22.

Transmitted by Tim Van der Linden. Published on 2019-02-26.

2010 Mathematics Subject Classification: 18A32,20L05.

Key words and phrases: distributor, profunctor, factorization system, internal groupoid.

A crucial point is that the two factorization systems coincide if we consider functors between groupoids. In this note we will show that, when restricted to the category of groupoids, distributors form a bicategory of relations relative to the comprehensive factorization system. More precisely, we will prove this result in the case of internal groupoids in a Barr-exact category $\mathcal{E}$; the category of groupoids is recovered for $\mathcal{E} = \mathbf{Set}$. The key fact is the elementary observation that, for groupoids, two-sided discrete fibrations are more simply described as usual discrete fibrations (Proposition 3.4). Then, since distributors can be formulated in terms of two-sided discrete fibrations, we can relate them to the comprehensive factorization system.

The internal case is of interest for some directions of research in categorical algebra and internal category theory. For instance, concerning internal non-abelian cohomology, Bourn has developed an intrinsic version of Schreier-Mac Lane Theorem of classification of extensions using internal distributors in [5], and the pointed version of a class of internal distributors, so-called *butterflies*, have been studied in the semi-abelian context by Abbad, Mantovani, Metere and Vitale in [1] and by Cigoli and Metere in [6]. On the other hand, the non-pointed version of butterflies, called *fractors* in [11], describe a notion of weak map between internal groupoids, where in the case of groupoids internal in groups, one recovers the notion of monoidal functor (see [14]).

Finally, a description of distributor composition in terms of the associated spans was missing. With this note we aim to fill this gap, and provide a useful tool for further investigations in the area.

2. Relations relative to a factorization system

Classically, a *relation* from a set A to a set B is a subset S of the cartesian product $A \times B$. For any two sets A and B, there is a (regular) epimorphic reflection between the preorder of relations from A to B and the category of spans from A to B:

$$\mathbf{Rel}(A,B) \underset{i_{A,B}}{\overset{r_{A,B}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Span}(A,B)\,.$$

The reflection is given by the (epi/mono) factorization: for a span

$$A \xleftarrow{e_1} E \xrightarrow{e_2} B$$

one obtains its associated relation by taking the image $r_{A,B}(E)$ of the function

$$E \xrightarrow{\langle e_1, e_2 \rangle} A \times B \ .$$

The (epi/mono) factorization system establishes also a connection between the composition of relations and the composition of spans. Indeed, given two relations, their usual composition is precisely the reflection of their composition as spans. Globally, this means that there is a lax biadjunction between the 2-category of relations and the bicategory of spans

$$\begin{tikzcd} \mathbf{Rel} \arrow[r, shift right, "i"'] \arrow[r, phantom, "\perp"] & \mathbf{Span} \arrow[l, shift right, "r"'] \end{tikzcd},$$

constant on objects, where only the 2-functor i is truly lax, since r is in fact a pseudo 2-functor.

More generally, one can start with any finitely complete category $\mathcal{C}$ endowed with an $(\mathcal{E}/\mathcal{M})$ factorization system (see Section 5.5 in [4]). Given two objects A and B, one defines the categories of $\mathcal{M}$-relations $\mathbf{Rel}(A, B)$ together with the local reflections $r_{A,B} \dashv i_{A,B}$. Hence, it is possible to define the composition of $\mathcal{M}$-relations as the reflection of their composition as spans, but such a composition need not be associative. As a consequence, we do not obtain a bicategory, in general. When we do get a bicategory $\mathbf{Rel}(\mathcal{C})$, then we call it:

the bicategory of relations in $\mathcal{C}$ *relative to the factorization system* $(\mathcal{E}/\mathcal{M})$.

This happens, for instance, when $\mathcal{C}$ is regular, or more generally, when $(\mathcal{E}/\mathcal{M})$ is a proper factorization system with the class $\mathcal{E}$ stable under pullbacks, but these conditions are not strictly necessary, as this article witnesses too. We will not provide further details on this general issue, but the literature on the subject is wide. The interested reader can consult [12] and the references therein.

3. Internal distributors and the comprehensive factorization

Distributors between internal categories have been introduced by Bénabou already in [2]. However, the cited reference is not as widely available as it would deserve, therefore we provide a secondary source [9].

Basic facts. For the notions of internal category and internal functor in a finitely complete category $\mathcal{E}$, the reader can consult [9, B2.3]. An internal functor F between internal categories $\mathbb{C}$ and $\mathbb{D}$ is represented by a diagram:

$$\begin{tikzcd} C_1 \arrow[r, "F_1"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & D_1 \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\ C_0 \arrow[r, "F_0"'] \arrow[u, "e" description] & D_0 \arrow[u, "e" description] \end{tikzcd}$$

The functor F is a *discrete fibration* if and only if $c \cdot F_1 = F_0 \cdot c$ is a pullback. It is a *discrete opfibration* if and only if $d \cdot F_1 = F_0 \cdot d$ is a pullback. Functors that are left orthogonal to the class of discrete fibrations are called final, those that are left orthogonal to the class of discrete opfibrations are called initial. Final functors and discrete fibrations give a factorization system for the category of internal categories in $\mathcal{E}$, and similarly so do initial functors and discrete opfibrations.

If $\mathbb{C}$ is an internal groupoid, the internal inverse map is denoted by $\tau\colon C_1 \to C_1$. $\mathbf{Gpd}(\mathcal{E})$ is the category of internal groupoids in $\mathcal{E}$. In the case of groupoids, discrete fibrations coincide with discrete opfibrations, and final functors with initial functors. Therefore, the two factorization systems mentioned above restrict to a single one that we denote by $(\mathcal{F}/\mathcal{D})$. This is called *comprehensive factorization system* ([13]).

Let us recall that the *connected components* functor

$$\Pi_0\colon \mathbf{Gpd}(\mathcal{E}) \longrightarrow \mathcal{E}$$

assigns to every internal groupoid, the coequalizer in $\mathcal{E}$ of its domain and codomain maps. Since $\mathcal{E}$ is Barr-exact, the joint factorization of d and c through the support of the groupoid (the image of the map $\langle d, c\rangle$) coincides with the kernel pair of such coequalizer.

Cigoli in [7] has characterized final functors between groupoids in a Barr-exact category $\mathcal{E}$.

3.1. Proposition. [7] *An internal functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids in a Barr-exact category $\mathcal{E}$ is final if and only if it is internally full and essentially surjective, i.e. if and only if*

- *the canonical comparison of C_1 with the joint pullback of d and c along F_0 is a regular epimorphism;*
- *$\Pi_0(F)$ is an isomorphism.*

Since we are dealing with groupoids, the first condition above can be rephrased.

3.2. Lemma. *For an internal functor $F\colon \mathbb{A} \to \mathbb{B}$ between groupoids as above, the following statements are equivalent:*

(i) the canonical comparison of C_1 with the joint pullback of d and c along F_0 is a regular epimorphism;

(ii) the canonical comparison of C_1 with the joint pullback of c and c along F_0 is a regular epimorphism;

(iii) the canonical comparison of C_1 with the joint pullback of d and d along F_0 is a regular epimorphism.

Distributors between groupoids are discrete fibrations. The definition of internal distributor closely follows the set-theoretical definition.

3.3. Definition. *([9]) Let $\mathbb{A}$ and $\mathbb{B}$ be internal groupoids in $\mathcal{E}$. A distributor*

$$\mathbb{B} \overset{S}{\leftarrow\!\!\!\!+\!\!\!-} \mathbb{A}$$

consists of the following data:

- *a span* $A_0 \xleftarrow{L} S_0 \xrightarrow{R} B_0$ *in* $\mathcal{E}$,
- *a left action* $A_1 \underset{A_0}{\times} S_0 \xrightarrow{\lambda_S} S_0$,
- *a right action* $S_0 \underset{B_0}{\times} B_1 \xrightarrow{\rho_S} S_0$,

which are associative, unital and compatible, where compatible means that the following diagram commutes:

$$\begin{tikzcd} A_1 \underset{A_0}{\times} S_0 \underset{B_0}{\times} B_1 \arrow[r, "1\times\rho_S"] \arrow[d, "\lambda_S\times 1"'] & A_1 \underset{A_0}{\times} S_0 \arrow[d, "\lambda_S"] \\ S_0 \underset{B_0}{\times} B_1 \arrow[r, "\rho_S"'] & S_0 \end{tikzcd}$$

Distributors between two given groupoids $\mathbb{A}$ and $\mathbb{B}$ form the category $\mathbf{Dist}(\mathbb{A},\mathbb{B})$, where an arrow between two distributors

$$\alpha\colon (L, S_0, R) \to (L', S_0', R')$$

is an arrow in the base category $\alpha\colon S_0 \to S_0'$ such that $L' \cdot \alpha = L$ and $R' \cdot \alpha = R$.

Like in the set-theoretical case, every internal distributor determines a span in $\mathbf{Gpd}(\mathcal{E})$. For instance, the distributor S above determines the span

$$\mathbb{A} \xleftarrow{L} \mathbb{S} \xrightarrow{R} \mathbb{C} \tag{2}$$

where the internal groupoid $\mathbb{S}$ has S_0 as the object of objects, and the object of arrows S_1 is obtained by the pullback

$$\begin{tikzcd} S_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S_0 \underset{B_0}{\times} B_1 \arrow[d, "\rho_S"] \\ A_1 \underset{A_0}{\times} S_0 \arrow[r, "\lambda_S"'] & S_0 \end{tikzcd}$$

with structure maps

$$d\colon S_1 \xrightarrow{\pi_1} A_1 \underset{A_0}{\times} S_0 \xrightarrow{\pi_2} S_0$$

$$c\colon S_1 \xrightarrow{\pi_2} S_0 \underset{B_0}{\times} B_1 \xrightarrow{\pi_1} S_0$$

and

$$e\colon S_0 \longrightarrow S_1$$

is the unique morphism such that $\pi \cdot e = \langle e, 1\rangle$ and $\pi_2 \cdot e = \langle 1, e\rangle$. Finally, the internal functors L and R are described below:

$$\begin{tikzcd} S_1 \arrow[r, "\pi_1"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & A_1 \times_{A_0} S_0 \arrow[r, "\pi_1"] & A_1 \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\ S_0 \arrow[u, "e" description] \arrow[rr, "L_0"'] & & A_0 \arrow[u, "e" description] \end{tikzcd} \qquad \begin{tikzcd} S_1 \arrow[r, "\pi_2"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & S_0 \times_{B_0} B_1 \arrow[r, "\pi_2"] & A_1 \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\ S_0 \arrow[u, "e" description] \arrow[rr, "R_0"'] & & B_0 \arrow[u, "e" description] \end{tikzcd}$$

The following result establishes the connection between the notion of distributor in groupoids and that of discrete fibration.

3.4. Proposition. *Giving a distributor*

$$\begin{tikzcd} \mathbb{B} & \mathbb{A} \arrow[l, "S"', "|" marking] \end{tikzcd}$$

between internal groupoids in a finitely complete category is equivalent to giving an internal discrete fibration

$$\mathbb{S} \xrightarrow{\langle L,R\rangle} \mathbb{A} \times \mathbb{B} \tag{3}$$

Proof. A span (L, R) as in (2) is determined by a distributor if and only if it is a two-sided discrete fibration. For groupoids in **Set**, it is easy to prove that such a span is a two-sided discrete fibration if and only if the induced functor into the product (3) is a discrete fibration. The result for internal groupoids follows by the usual Yoneda embedding argument. ∎

3.5. Remark. The notion of two-sided discrete fibration appears in the literature (although implicitly) as a discretization of so-called regular spans, introduced and studied by Yoneda in [15] (see also [8]). It is relevant to our discussion to recall that Yoneda, in Section 3.5 of the cited paper, introduces a (generalized) composition product of two composable regular spans as a suitable discretization of their composition as spans. This was the starting point for our investigations on the subject.

The last proposition allows us to describe the reflection of spans into distributors. Since $(\mathcal{F}/\mathcal{D})$ is a factorization system, we need not prove the following statement.

3.6. Proposition. *Let $\mathbb{A}$ and $\mathbb{B}$ be two groupoids in a finitely complete category $\mathcal{E}$. The comprehensive factorization defines the reflection R to the inclusion of distributors into spans, and so establishes an adjoint pair*

$$\begin{tikzcd} \mathbf{Dist}(\mathbb{A}, \mathbb{B}) \arrow[r, shift right=1.5, "I_{\mathbb{A},\mathbb{B}}"'] \arrow[r, phantom, "\perp"] & \mathbf{Span}(\mathbb{A}, \mathbb{B}) \arrow[l, shift right=1.5, "R_{\mathbb{A},\mathbb{B}}"'] \end{tikzcd}$$

with $R_{\mathbb{A},\mathbb{B}} \cdot I_{\mathbb{A},\mathbb{B}} \simeq id_{\mathbf{Dist}(\mathbb{A},\mathbb{B})}$.

For the span (P, Q), the reflection is obtained by the factorization

$$\begin{tikzcd} \mathbb{E} \arrow[r, "F"] \arrow[dr, "\langle P,Q\rangle"'] & R(\mathbb{E}) \arrow[d, "\langle L,R\rangle"] \\ & \mathbb{A} \times \mathbb{B} \end{tikzcd}$$

where F is final, and $\langle L, R\rangle$ a discrete fibration.

Composition of distributors. In this section we assume the base category $\mathcal{E}$ to be Barr-exact. Since in this case the category $\mathcal{E}$ admits coequalizers of reflexive pairs which are stable under pullback, distributors can be composed.

For two composable distributors T and S

$$\begin{tikzcd} \mathbb{C} & \mathbb{B} \arrow[l, "T"', "|" marking] & \mathbb{A} \arrow[l, "S"', "|" marking] \end{tikzcd}$$

their composition $A_0 \xleftarrow{L} (T \otimes S)_0 \xrightarrow{R} C_0$ is obtained by the universal property of the coequalizer in the first line in the diagram below

$$\begin{tikzcd} S_0 \underset{B_0}{\times} B_1 \underset{B_0}{\times} T_0 \arrow[r, shift left, "\rho_S \times id"] \arrow[r, shift right, "id \times \lambda_T"'] & S_0 \underset{B_0}{\times} T_0 \arrow[r, "Q_0"] \arrow[dr] & (T \otimes S)_0 \arrow[d, dashed, "\exists !"] \\ & & A_0 \times C_0 \end{tikzcd} \tag{4}$$

The action $\rho_{T\otimes S}$ is induced from ρ_T by pulling back along $c\colon C_1 \to C_0$; similarly, the action $\lambda_{T\otimes S}$ is induced from λ_S by pulling back along $d\colon A_1 \to A_0$.

The next statement is the key result of this note. It relates distributor composition to span composition.

3.7. Proposition. *Distributor composition agrees with (the reflection of) span composition, i.e for given groupoids $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, the following diagram commutes:*

$$\begin{tikzcd} \mathbf{Dist}(\mathbb{A},\mathbb{B}) \times \mathbf{Dist}(\mathbb{B},\mathbb{C}) \arrow[r, "\otimes"] \arrow[d, "I \times I"'] & \mathbf{Dist}(\mathbb{A},\mathbb{B}) \\ \mathbf{Span}(\mathbb{A},\mathbb{B}) \times \mathbf{Span}(\mathbb{B},\mathbb{C}) \arrow[r, "\diamond"'] & \mathbf{Span}(\mathbb{A},\mathbb{B}) \arrow[u, "R"'] \end{tikzcd}$$

where $\otimes$ is the composition of distributors and $\diamond$ is the composition of spans.

Proof. The way the composition $R \cdot \diamond \cdot I \times I$ acts on a pair of distributors S and T is shown in the following diagram

$$\begin{tikzcd} & & T \diamond S \arrow[r, "F"] \arrow[dl, "\bar{L}"'] \arrow[dr, "\bar{R}"] & R(T \diamond S) \arrow[dlll, "\hat{L}"' near start] \arrow[ddr, "\hat{R}"] & \\ & \mathbb{S} \arrow[dl, "L"'] \arrow[dr, "R"'] & & \mathbb{T} \arrow[dl, "L"] \arrow[dr, "R"] & \\ \mathbb{A} & & \mathbb{B} & & \mathbb{C} \end{tikzcd}$$

where the square $R \cdot \bar{L} = L \cdot \bar{R}$ is a pullback in $\mathbf{Gpd}(\mathcal{E})$. Hence, we consider two factorizations of the functor $\langle L \cdot \bar{L}, R \cdot \bar{R}\rangle$:

$$
\begin{tikzcd}
T \diamond S \arrow[r, "F"] \arrow[d, "Q"'] \arrow[dr, "\langle L\cdot\bar{L}, R\cdot\bar{R}\rangle" description] & R(T \diamond S) \arrow[d, "\langle \hat{L}, \hat{R}\rangle"] \\
T \otimes S \arrow[r, "\langle L, R\rangle"'] & \mathbb{A} \times \mathbb{C}
\end{tikzcd}
\tag{5}
$$

The first one is the comprehensive factorization, that is given by the final functor F followed by the discrete fibration $\langle \hat{L}, \hat{R}\rangle$. The second one extends at the level of arrows the factorization provided by the coequalizer diagram in (4). It consists of a functor Q (description below) followed by the discrete fibration $\langle L, R\rangle$ representing the distributor $T \otimes S$. By uniqueness of factorization, it suffices to prove that Q is final in order to prove that these two factorizations are isomorphic.

First we need to recall that the pullback groupoid $T \diamond S$ is computed level wise in $\mathcal{E}$, therefore $(T \diamond S)_i = S_i \underset{B_i}{\times} T_i$, for $i = 0, 1$. Moreover domain, codomain and unit maps are given by the universal properties of such pullbacks, namely $\langle d, d\rangle$, $\langle c, c\rangle$ and $\langle e, e\rangle$ respectively.

We are ready to describe the functor Q explicitly. For internal groupoids, the internal two-sided discrete fibration associated with $T \otimes S$ is a mere discrete fibration, and the cited factorization can be represented as follows:

$$
\begin{tikzcd}
(T \diamond S)_1 \arrow[r, dashed, "\exists ! Q_1"] \arrow[d, shift right=2, "\langle d,d\rangle"'] \arrow[d, shift left=2, "\langle c,c\rangle"] & (T \otimes S)_0 \underset{A_0 \times C_0}{\times} (A_1 \times C_1) \arrow[r, "\pi_2"] \arrow[d, shift right=2, "\bar{d}"'] \arrow[d, shift left=2, "\bar{c}"] & A_1 \times C_1 \arrow[d, shift right=2, "d \times d"'] \arrow[d, shift left=2, "c \times c"] \\
(T \diamond S)_0 \arrow[r, "Q_0"'] \arrow[u] & (T \otimes S)_0 \arrow[r, "\langle L_0, R_0\rangle"'] \arrow[u] & A_0 \times C_0 \arrow[u]
\end{tikzcd}
\tag{6}
$$

where the downward directed squares on the right are pullbacks. By Proposition 3.1, (Q_1, Q_0) is final if and only if it is internally full and essentially surjective. Therefore, the proof of the proposition will be achieved through the proof of the following two claims. ∎

3.8. Claim. *According to Lemma 3.2 (ii) above, the comparison map K with the joint pullback W in the diagram below is a regular epimorphism.*

$$
\begin{tikzcd}
(T \diamond S)_1 \arrow[dr, dashed, "K"] \arrow[drr, bend left, "Q_1"] \arrow[ddr, bend right, "\langle \langle c,c\rangle, \langle c,c\rangle\rangle"'] & & \\
& W \arrow[r, "K'"] \arrow[d, "J"] & (T \otimes S)_0 \underset{A_0 \times C_0}{\times} (A_1 \times C_1) \arrow[d, "\langle \bar{c}, \bar{c}\rangle"] \\
& (T \diamond S)_0 \times (T \diamond S)_0 \arrow[r, "Q_0 \times Q_0"'] & (T \otimes S)_0 \times (T \otimes S)_0
\end{tikzcd}
\tag{7}
$$

Proof of Claim 3.8. Let us consider the following diagram:

$$
\begin{tikzcd}
 & (T \diamond S)_1 \arrow[dl, "L"'] \arrow[dr, "K"] & & & \\
S_0 \underset{B_0}{\times} B_1 \underset{B_0}{\times} T_0 \arrow[dr, "H"'] & (i) & W \arrow[dl, "Y"'] \arrow[dr, "K'"] & & \\
 & \mathrm{Eq}(Q_0) \arrow[dl, "H'"'] \arrow[dr, "X"] & (ii) & (T \otimes S)_0 \underset{A_0 \times C_0}{\times} (A_1 \times C_1) \arrow[dl, "\bar{c}"'] \arrow[dr, "\pi_2"] & \\
(T \diamond S)_0 \times (T \diamond S)_0 \arrow[dr, "Q_0 \times Q_0"'] & (iii) & (T \otimes S)_0 \arrow[dl, "\Delta"'] \arrow[dr, "\langle L,R \rangle"'] & (iv) & A_1 \times C_1 \arrow[dl, "c \times c"] \\
 & (T \otimes S)_0 \times (T \otimes S)_0 & & A_0 \times C_0 &
\end{tikzcd}
$$

The regions labelled by (ii), (iii) and (iv) are pullbacks. Indeed, (iv) is the downward directed squares in diagram (6), (iii) defines $\mathrm{Eq}(Q_0)$ as the kernel pair of Q_0, and (ii)+(iii) is the pullback square in diagram (7). As we observed at the beginning of this section, $\mathrm{Eq}(Q_0)$ is the support of the groupoid $\mathbb{H}$, and the arrow H is the canonical comparison with such a support; therefore it is a regular epimorphism. Finally, the arrow L is the canonical projection

$$A_1 \underset{A_0}{\times} S_0 \underset{B_0}{\times} B_1 \underset{B_0}{\times} T_0 \underset{C_0}{\times} C_1 \longrightarrow S_0 \underset{B_0}{\times} B_1 \underset{B_0}{\times} T_0$$

Indeed, the proof of the claim now amounts to proving that the region (i) commutes: in this case, $(i) + (ii) + (iv)$ is precisely the definition of $(T \diamond S)_1$, it is a pullback, so that (i) is a pullback too and K is a regular epimorphism (pullback of the regular epimorphism H). The proof that (i) commutes is left to the reader, who can conveniently compose this diagram with the monomorphism H'. ■

3.9. Claim. *The arrow*

$$\Pi_0(Q_0, Q_1) \colon \Pi_0(T \diamond S) \longrightarrow \Pi_0(T \otimes S)$$

is an isomorphism.

Proof of Claim 3.9. Let us consider the following diagram of solid arrows, where horizontal and vertical forks are coequalizers.

$$
\begin{tikzcd}
 & (T \diamond S)_1 \arrow[r, "Q_1"] \arrow[d, shift right=2ex, "\langle d,d \rangle"'] \arrow[d, shift left=2ex, "\langle c,c \rangle"] & (T \otimes S)_0 \underset{A_0 \times C_0}{\times} (A_1 \times C_1) \arrow[d, shift right=2ex, "\bar{d}"'] \arrow[d, shift left=2ex, "\bar{c}"] \\
S_0 \underset{B_0}{\times} B_1 \underset{B_0}{\times} T_0 \arrow[ur, dashed, "\alpha"] \arrow[r, shift left=2ex, "\rho_S \times id"] \arrow[r, shift right=2ex, "id \times \lambda_T"'] & (T \diamond S)_0 \arrow[u] \arrow[l] \arrow[r, "Q_0"] \arrow[d, "P_\diamond"'] & (T \otimes S)_0 \arrow[u] \arrow[d, "P_\otimes"] \arrow[dl, dashed, "V"'] \\
 & \Pi_0(T \diamond S) \arrow[r, shift right=1ex, "\Pi_0(Q_0,Q_1)"'] & \Pi_0(T \otimes S) \arrow[l, dashed, shift right=1ex, "V'"']
\end{tikzcd}
\tag{8}
$$

Recall that $(i) + (ii) + (iv)$ is a pullback, and let α be the canonical section of L, i.e. the unique arrow such that

$$L \cdot \alpha = id$$

and

$$\pi_2 \cdot Q_1 \cdot \alpha = e \times e \cdot \langle L_0, R_0 \rangle \cdot Q_0 \cdot (id \times \lambda_T)$$

(or equivalently,

$$\pi_2 \cdot Q_1 \cdot \alpha = e \times e \cdot \langle L_0, R_0 \rangle \cdot Q_0 \cdot (\rho_S \times id)$$

since Q_0 coequalizes $id \times \lambda_T$ and $\rho_S \times id$).

We state the following facts.

1. $Q = K' \cdot K$ is a regular epimorphism, since K is a regular epimorphism by Claim 1, and K' is the pullback of the regular epimorphism $Q_0 \times Q_0$.

2. $P_\diamond$ coequalizes $id \times \lambda_T$ and $\rho_S \times id$, since they factor through $\langle d, d \rangle$ and $\langle c, c \rangle$ via α. Therefore, there exists a unique V as in the diagram, such that $V \cdot Q_0 = P_\diamond$. Eventually, one easily check $\Pi_0(Q_0, Q_1) \cdot V = P_\otimes$.

3. V coequalizes $\bar{d}$ and $\bar{c}$: just precompose with the regular epimorphism Q and follow the diagram. Therefore, there exists a unique V' as in the diagram, such that $V' \cdot P_\otimes = V$.

$\Pi_0(Q_0, Q_1)$ is a regular epimorphism (since precomposed with $P_\diamond$ is). Moreover,

$$V' \cdot \Pi_0(Q_0, Q_1) \cdot P_\diamond = V' \cdot P_\otimes \cdot Q_0 = V \cdot Q_0 = P_\diamond\,,$$

so that by canceling the regular epimorphism $P_\diamond$ one sees that $\Pi_0(Q_0, Q_1)$ is also a split monomorphism, and therefore an isomorphism. ∎

3.10. Remark. The arrow α described in the proof determines a natural transformation. One can prove that α is the a 2-limit, called *the identee* of the internal functor $\langle L\cdot\bar{L}, R\cdot\bar{R}\rangle$ of diagram (5). This gives a complementary viewpoint of the factorization of (5), where the internal functor (Q_0, Q_1) is actually the coidentifier of α, and as such, it is at the same time initial and final, even if we consider categories instead of groupoids. However, these aspects will not be further examined in the present paper.

We conclude the section with the expected result.

3.11. Theorem. *Let $\mathcal{E}$ be a Barr-exact category. Then*

$$\mathbf{DistGpd}(\mathcal{E}) = \mathbf{Rel}(\mathbf{Gpd}(\mathcal{E})) \text{ w.r.t. } (\mathcal{F}/\mathcal{D}),$$

i.e. the bicategory of distributors between internal groupoids in $\mathcal{E}$ is the bicategory of relations in $\mathbf{Gpd}(\mathcal{E})$ relative to the (final/discrete fibration) factorization system.

Proof. Proposition 3.6 identifies $\mathcal{D}$-relations, and Proposition 3.7 provides the bicategory structure. ■

Acknowledgments

I am grateful to Sandra Mantovani and Alan Cigoli for the many interesting conversations and suggestions, and to Tim Van der Linden for supporting my visit in Louvain-la-Neuve in 2016.

References

[1] O. Abbad, S. Mantovani, G. Metere, E. M. Vitale, Butterflies in a semi-abelian context, Adv. Math. **238** (2013) 140–183.

[2] J. Bénabou, Les distributeurs, Université catholique de Louvain, Institut de Mathématique Pure et Appliquée, rapport 33 (1973).

[3] J. Bénabou, Distributors at work, Notes from a course held by prof. Bénabou at TU Darmstadt, taken by T. Streicher (2000).

[4] F. Borceux, Handbook of Categorical Algebra. Vol 1. Cambridge University Press (1994).

[5] D. Bourn, Commutator theory, action groupoids, and an intrinsic Schreier-Mac Lane extension theorem, Adv. Math. **217** (2008), 2700–2735.

[6] A. Cigoli, G. Metere, Extension theory and the calculus of butterflies, J. Algebra **458** (2016) 87–119.

[7] A. Cigoli, A characterization of final functors between internal groupoids in exact categories, Theory and App. of Categories, **33** (11) (2018) 265–275.

[8] A. Cigoli, S. Mantovani, G. Metere, E. M. Vitale, Fibered aspects of Yoneda's regular span, *preprint*, arXiv:1806.02376 [math.CT] (2018).

[9] P. T. Johnstone, Sketches of an Elephant: A Topos Theory Compendium, Oxford Logic Guides, Clarendon Press (2002).

[10] F.W. Lawvere, Equality in hyperdoctrines and comprehension schema as an adjoint functor. In A. Heller, ed., Proc. New York Symp. on Applications of Categorical Algebra, AMS, (1970) 1–14.

[11] S. Mantovani, G. Metere and E. M. Vitale, Profunctors in Mal'tsev categories and fractions of functors, *J. Pure Appl. Algebra* **217** (7) (2013) 1173–1186.

[12] S. Milius, Relations in categories, *PhD Thesis* (2000).

[13] R. Street and R.F.C. Walters, The comprehensive factorization of a functor, Bull. Amer. Math. Soc. **79** (5) (1973).

[14] E. M. Vitale, Bipullbacks and calculus of fractions, Cahiers de Topol. et Géom. Diff. Catég. **51** (2010) 83–113.

[15] N. Yoneda, On Ext and exact sequences, J. Fac. Sci. Univ. Tokio **8** (1960) 507–576.

via Archirafi 34, Palermo, Italy
Email: giuseppe.metere@unipa.it

This article may be accessed at http://www.tac.mta.ca/tac/