

## EXACT SEQUENCES IN THE ENCHILADA CATEGORY

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ABSTRACT. We define exact sequences in the enchilada category of  $C^*$ -algebras and correspondences, and prove that the reduced-crossed-product functor is not exact for the enchilada categories. Our motivation was to determine whether we can have a better understanding of the Baum-Connes conjecture by using enchilada categories. Along the way we prove numerous results showing that the enchilada category is rather strange.

### 1. Introduction

The Baum-Connes conjecture says (very roughly) that, given an action of a locally compact group on a  $C^*$ -algebra, the topological  $K$ -theory is naturally isomorphic to the  $K$ -theory of the reduced crossed product. Unfortunately, the conjecture is false in that form, essentially because the topological  $K$ -theory is an exact functor of actions, while the reduced crossed product is not. Some effort has been expended to “fix” the Baum-Connes conjecture (see, e.g., [2, 6, 4, 5, 10, 11, 12]). In this paper we investigate another possible strategy of fixing the conjecture: change the categories. All the work to date on the Baum-Connes conjecture has used categories of  $C^*$ -algebras, possibly with extra structure, where the morphisms are  $*$ -homomorphisms that preserve the structure. Here we change the morphisms to be isomorphism classes of  $C^*$ -correspondences — we call these “enchilada categories”. Perhaps we should explain the genesis of this unusual name: when the AMS Memoir [8] was being prepared, the authors decided to first introduce the general idea by writing a smaller paper [7], and we privately referred to these two papers as the “big enchilada” and the “little taco”, respectively. Since then, a few of us have been using the name “enchilada category” for the type of category studied in those two papers (see Section 3 for definitions).

More precisely, we investigate the following question: is the reduced crossed product functor exact between enchilada categories? In order to study this question rigorously, we first need to know: what are the exact sequences in the enchilada category of  $C^*$ -algebras? In this paper we give one answer to this latter question. We then apply this to answer the exactness question; unfortunately, the answer is no, the reduced crossed product is not exact for the enchilada categories.

Despite this failure of exactness, we believe that our investigation into exact sequences in the enchilada category will be useful. It turns out that the enchilada category is quite

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strange, in the sense that the morphisms are not mappings; additionally, the category is not abelian, or even preadditive, so the standard techniques of homological algebra are largely unavailable. As an illustration of our ignorance concerning the enchilada category, we have been unable to completely characterize either the monomorphisms or the epimorphisms.

This contributed to our most formidable hurdle: how to define the image of a morphism. Since a morphism in the enchilada category is (the isomorphism class of) a  $C^*$ -correspondence, it is fairly easy to guess that the image should involve the range of the inner product, and it is then a short step to imagine that the range of the correspondence should in fact be the closed span of the inner products. But how to put this on a rigorous footing? In abstract category theory, a common way to define image uses *subobjects*, which in turn use monomorphisms; this gave us trouble due to our inability to characterize monomorphisms. In some category-theory literature, the definition of subobject is modified by restricting the class of monomorphisms. We first tried the *strong monomorphisms* of [1]. This in turn led to another stumbling block: our limited understanding of epimorphisms in the enchilada category eventually defeated us because the definition of strong monomorphism uses epimorphisms. We then tried using *split monomorphisms* in the definition of image. This turned out to work very well, but it was unsatisfying since it seemed to depend upon the more-or-less arbitrary choice of split monomorphisms. Fortunately, we found in [15] an alternative notion of image, which we call *Schubert image*, based upon kernels and cokernels. Since we were able to prove that the enchilada category has kernels and cokernels, and even more importantly that in the enchilada category every kernel is a split monomorphism (see Proposition 3.6 and Corollary 3.10), we were happy to adopt Schubert's definition of image.

We begin in Section 2 with a brief review of the basic notions from category theory that we will need. Then in Section 3 we investigate these notions for the enchilada category, where our main objective is to define *kernel* and *Schubert image*. Once this is done, we characterize short exact sequences in the enchilada category (see Theorem 3.15). It is then easy to explain why the reduced crossed product functor is not exact for the enchilada categories (see Remark 3.16). Finally, in Section 4 we exhibit a few other ways in which the enchilada category is strange — it is not abelian, or even preadditive.

## 2. Preliminaries

This paper is written primarily for  $C^*$ -algebraists. We use a nontrivial portion of the concepts of category theory, so for the convenience of the reader we recall the definitions and basic results here. All abstract discussions of morphisms and objects will be tacitly in some category  $\mathcal{C}$ .

2.1. DEFINITION. A monomorphism is a morphism  $f$  such that  $f \circ g = f \circ h$  implies  $g = h$ . Dually,  $f$  is an epimorphism if  $g \circ f = h \circ f$  implies  $g = h$ .

2.2. DEFINITION. An object  $A$  is initial if for every object  $B$  there is exactly one morphism  $A \rightarrow B$ . Dually,  $A$  is terminal if for every object  $B$  there is exactly one morphism  $B \rightarrow A$ . Both of them are unique up to isomorphism.

2.3. DEFINITION. A zero is an object that is both initial and terminal. If it exists, a zero is unique up to isomorphism, and is denoted by  $0$ . For any two objects  $A, B$  the zero morphism  $0_{A,B} : A \rightarrow B$  is the unique morphism from  $A$  to  $B$  that factors through  $0$ . Frequently we just write  $0$  for  $0_{A,B}$ .

2.4. DEFINITION. Let  $f, g : A \rightarrow B$ . An equalizer of  $f, g$  is a morphism  $h : C \rightarrow A$  such that

- $f \circ h = g \circ h$ ;
- whenever  $k : D \rightarrow A$  satisfies  $f \circ k = g \circ k$  there exists a unique morphism  $p : D \rightarrow C$  such that  $h \circ p = k$ .

The situation is illustrated by the following commutative diagram:

$$\begin{array}{ccccc} D & & & & \\ \downarrow p & \searrow k & & & \\ C & \xrightarrow{h} & A & \xrightleftharpoons[g]{f} & B \end{array}$$

2.5. REMARK. As usual with universal properties, an equalizer, if it exists, is unique up to (unique) isomorphism. In this case this means not only that for any other equalizer  $k : L \rightarrow A$  the unique morphism  $u : L \rightarrow C$  making the diagram

$$\begin{array}{ccc} L & & \\ \downarrow u & \searrow k & \\ C & \xrightarrow{h} & A \end{array}$$

commute is an isomorphism, but that conversely for any isomorphism  $u : L \rightarrow C$  the morphism  $h \circ u$  is an equalizer of  $f, g$ . We will omit explicitly making similar remarks regarding other categorical gadgets.

2.6. DEFINITION. Coequalizer is the dual of equalizer, i.e., a coequalizer of  $f, g : A \rightarrow B$  is a morphism  $h : B \rightarrow C$  such that

- $h \circ f = h \circ g$ ;
- whenever  $k : B \rightarrow D$  satisfies  $k \circ f = k \circ g$  there exists a unique morphism  $p : C \rightarrow D$  such that  $k = p \circ h$ .

This is illustrated by the commutative diagram

$$\begin{array}{ccccc} & & & & D \\ & & & \nearrow k & \uparrow \\ A & \xrightleftharpoons[g]{f} & B & \xrightarrow{h} & C \\ & & & & \downarrow p \end{array}$$

2.7. **REMARK.** If  $f, g : A \rightarrow B$  and  $q : B \rightarrow E$  is a monomorphism, then  $f, g$  and  $q \circ f, q \circ g$  have the same equalizers. Dually, if  $q : E \rightarrow A$  is an epimorphism, then  $f, g$  and  $f \circ q, g \circ q$  have the same coequalizers.

2.8. **DEFINITION.** Let  $\mathcal{C}$  be a category with zero object  $0$ , and let  $f : A \rightarrow B$ . A kernel of  $f$  is an equalizer of the pair  $f, 0$  equivalently, a morphism  $h : C \rightarrow A$  such that

- $f \circ h = 0$ ;
- whenever  $k : D \rightarrow A$  satisfies  $f \circ k = 0$  there exists a unique morphism  $p : D \rightarrow C$  such that  $h \circ p = k$ .

2.9. **DEFINITION.** Cokernel is the dual of kernel, i.e., a cokernel of  $f : A \rightarrow B$  is a coequalizer of  $f, 0$ ; equivalently, a morphism  $h : B \rightarrow C$  such that

- $h \circ f = 0$
- whenever  $k : B \rightarrow D$  satisfies  $k \circ f = 0$  there exists a unique morphism  $p : C \rightarrow D$  such that  $p \circ h = k$ .

2.10. **REMARK.** Every equalizer, and hence every kernel, is a monomorphism, and by duality every coequalizer, and hence every cokernel, is an epimorphism. If  $f : A \rightarrow B$  and  $q : B \rightarrow E$  is a monomorphism, then  $f$  and  $q \circ f$  have the same kernels. Dually, if  $q : E \rightarrow A$  is an epimorphism, then  $f$  and  $f \circ q$  have the same cokernels.

2.11. **REMARK.** If  $0$  is a zero object, then for all objects  $A, B$  the morphism  $0 : A \rightarrow B$  has kernel  $1_A$  and cokernel  $0 : B \rightarrow 0$ .

2.12. **DEFINITION.** A split monomorphism is a morphism  $f : A \rightarrow B$  such that there exists a morphism  $g : B \rightarrow A$  with  $g \circ f = 1_A$ , and dually a split epimorphism is a morphism  $f : A \rightarrow B$  such that there exists a morphism  $g : B \rightarrow A$  with  $f \circ g = 1_B$ .

In the category-theory literature, one can find various definitions of image and coimage. A common definition of the image of a morphism uses subobjects, and we summarize this approach: if  $f : B \rightarrow A$  and  $g : C \rightarrow A$  are two monomorphisms with common codomain  $A$ , write  $f \leq g$  to mean that  $f = g \circ h$  for some  $h$ . When both  $f \leq g$  and  $g \leq f$  write  $f \sim g$ . This defines an equivalence relation among the monomorphisms with codomain  $A$ , and an equivalence class of these monomorphisms is called a *subobject* of  $A$ . The class (which could be proper) of all subobjects of  $A$  is partially ordered by the binary relation “ $\leq$ ”.

In practice, any monomorphism  $f : B \rightarrow A$  is referred to as a subobject, with the understanding that it is really just a representative of an equivalence class that is the actual subobject.

In some contexts the monomorphisms in the definition of subobject are required to satisfy some other conditions. For example, one could restrict to *strong monomorphisms* (see Remark 4.7 for the definition).

In [13, Section I.10] (for example) we find the following definition:

2.13. DEFINITION. *The image of a morphism  $f : A \rightarrow B$  is the “smallest” subobject of  $B$  through which  $f$  factors, equivalently a monomorphism  $u : I \rightarrow B$  such that*

- $f = u \circ f'$  for some  $f' : A \rightarrow I$
- if  $f = v \circ g$  for any other monomorphism  $v : J \rightarrow B$  and a morphism  $g : A \rightarrow J$ , then there is a unique morphism  $i : I \rightarrow J$  such that  $u = v \circ i$

In many categories this is a very useful definition, provided that it is not hard to determine what the subobjects are. For instance, in the category of sets subobjects are subsets, in the category of groups subobjects are subgroups, etc. However, as mentioned in the introduction, since we do not know what the monomorphisms — or the strong monomorphisms, for that matter — are in the enchilada category we were unable to determine what subobjects are. Therefore, we were unable to use this image definition. So, we use the following instead, which we took from [15, Definition 12.3.7].

2.14. DEFINITION. *In a category with kernels and cokernels, a Schubert image of a morphism  $f$  is a kernel of any cokernel of  $f$ , and dually a Schubert coimage of  $f$  is a cokernel of any kernel of  $f$ .*

We should note that Schubert defines an image of a morphism as above. However, in a category where subobjects can be fully identified, images (in the subobject sense) need not satisfy the condition of Definition 2.14 unless (for example) the category is abelian. For instance, in the category of groups the above definition is applicable if and only if the image is a normal subgroup of the codomain.

It is a trivial consequence of the definitions that the image of a morphism, if it exists, can be factored through the Schubert image.

It may be appropriate to mention briefly the duals of subobjects and images. If  $f : A \rightarrow B$  and  $g : A \rightarrow C$  are two epimorphisms with common domain  $A$ , write  $f \leq g$  to mean that  $f = h \circ g$  for some  $h$ . When both  $f \leq g$  and  $g \leq f$  write  $f \sim g$ . This defines an equivalence relation among the epimorphisms with domain  $A$ , and an equivalence class of these epimorphisms is called a *quotient object* of  $A$ . The *coimage* of a morphism  $f : A \rightarrow B$  is the smallest quotient object of  $A$  through which  $f$  factors.

### 3. The enchilada category

As we mentioned in the introduction, in the enchilada category our objects are  $C^*$ -algebras, and the morphisms from  $A$  to  $B$  are the isomorphism classes of nondegenerate  $A - B$  correspondences. The paper [7] and the memoir [8] (particularly Chapter 1 and 2, and Appendix A) contain a development of all the theory we will need, but for those not familiar with  $C^*$ -correspondences we give a quick review: a *Hilbert  $B$ -module* is a vector space  $X$  equipped with a right  $B$ -module structure and a  *$B$ -valued inner product*, i.e., a positive-definite  $B$ -valued sesquilinear form  $\langle \cdot, \cdot \rangle_B$  satisfying

$$\langle x, yb \rangle_B = \langle x, y \rangle_B b \quad \text{and} \quad \langle x, y \rangle_B^* = \langle y, x \rangle_B$$

for all  $x, y \in X, b \in B$ , and which is complete in the norm  $\|x\| = \|\langle x, x \rangle_B\|^{1/2}$ . The closed span of the inner products is an ideal  $B_X$  of  $B$ , and  $X$  is called *full* if  $B_X = B$ . The  $B$ -module operators  $T$  on  $X$  for which there is an operator  $T^*$  satisfying

$$\langle Tx, y \rangle_B = \langle x, T^*y \rangle_B \quad \text{for all } x, y \in X$$

(which is not automatic, even if  $T$  is bounded) form the  $C^*$ -algebra  $\mathcal{L}(X)$  of *adjointable operators* with the operator norm, and the closed linear span of the *rank-one operators*  $\theta_{x,y}$  given by

$$\theta_{x,y}z = x\langle y, z \rangle_B$$

is the closed ideal  $\mathcal{K}(X)$  of *compact operators*.

By an  $A$ - $B$  *correspondence*  $X$  we mean a Hilbert  $B$ -module  $X$  with a  $*$ -homomorphism<sup>1</sup>  $\phi_X : A \rightarrow \mathcal{L}(X)$ , and we say the correspondence is *nondegenerate* if  $AX = X$ .<sup>2</sup> *All our correspondences will be nondegenerate by standing hypothesis.* That is, from now on when we use the term correspondence we will tacitly assume the nondegeneracy condition. An *isomorphism*  $U : X \rightarrow Y$  of  $A$ - $B$  correspondences is a linear bijection such that

$$U(axb) = aU(x)b \quad \text{and} \quad \langle Ux, Uy \rangle_B = \langle x, y \rangle_B$$

for all  $a \in A, b \in B, x, y \in X$ .

The *balanced tensor product*  $X \otimes_B Y$  of an  $A$ - $B$  correspondence  $X$  and a  $B$ - $C$  correspondence  $Y$  is formed as follows: the algebraic tensor product  $X \odot Y$  is given the  $A$ - $C$  bimodule structure determined on the elementary tensors by

$$a(x \otimes y)c = ax \otimes yc \quad \text{for } a \in A, x \in X, y \in Y, c \in C,$$

and the unique  $C$ -valued sesquilinear form whose values on elementary tensors are given by

$$\langle x \otimes y, u \otimes v \rangle_C = \langle y, \langle x, u \rangle_B v \rangle_C \quad \text{for } x, u \in X, y, v \in Y.$$

The Hausdorff completion is an  $A$ - $C$  correspondence  $X \otimes_B Y$ . The term *balanced* refers to the property

$$xb \otimes y = x \otimes by \quad \text{for } x \in X, b \in B, y \in Y,$$

which is automatically satisfied. The *identity correspondence* on  $A$  is the vector space  $A$  with the  $A$ - $A$  bimodule structure given by multiplication and the inner product  $\langle a, b \rangle_A = a^*b$ . The *enchilada category* has  $C^*$ -algebras as objects, and the morphisms from  $A$  to  $B$  are the isomorphism classes of  $A$ - $B$  correspondences, with composition given by balanced tensor product and identity morphisms given by identity correspondences.

<sup>1</sup>and henceforth we will drop the  $*$ , so that all homomorphisms are assumed to be  $*$ -homomorphisms

<sup>2</sup>Note that we actually mean  $AX = \{ax : a \in A, x \in X\}$  — by the Cohen-Hewitt factorization theorem this coincides with the closed span.

We write “ ${}_A X_B$  is a correspondence” to mean that  $X$  is an  $A - B$  correspondence, and we write  $[X] = [{}_A X_B]$  for the associated morphism in the enchilada category. Unless otherwise specified,  ${}_A A_A$  will mean the identity correspondence over  $A$ . Note that composition is given by

$$[{}_B Y_C] \circ [{}_A X_B] = [{}_A (X \otimes_B Y)_C].$$

Actually, we will frequently drop the square brackets  $[\cdot]$ , since it will clean up the notation and no confusion will arise.

The *multiplier algebra* of  $A$  is the  $C^*$ -algebra  $M(A) = \mathcal{L}({}_A A_A)$ , and we identify  $A$  with its image under the left-module homomorphism  $\phi_A : A \rightarrow M(A)$ . In this way  $A$  becomes an ideal of  $M(A)$ . More generally, it is a standard fact that  $M(\mathcal{K}(X)) = \mathcal{L}(X)$ . A homomorphism  $\mu : A \rightarrow M(B)$  is *nondegenerate* if  $\mu(A)B = B$ , and nondegeneracy of a correspondence  ${}_A X_B$  is equivalent to nondegeneracy of the left-module homomorphism  $\phi_X : A \rightarrow M(\mathcal{K}(X))$ . Every nondegenerate homomorphism  $\mu : A \rightarrow M(B)$  extends uniquely to a homomorphism  $\bar{\mu} : M(A) \rightarrow M(B)$ , and we typically drop the bar, just writing  $\mu$  for the extension.

An  $A - B$  *Hilbert bimodule* is an  $A - B$  correspondence  $X$  that is also equipped with an  $A$ -valued inner product  ${}_A \langle \cdot, \cdot \rangle$ , which satisfies the “mirror image” of the properties of the  $B$ -valued inner product:

$${}_A \langle ax, y \rangle = a_A \langle x, y \rangle \quad \text{and} \quad {}_A \langle x, y \rangle^* = {}_A \langle y, x \rangle$$

for all  $a \in A, x, y \in X$ , as well as the compatibility property

$${}_A \langle x, y \rangle z = x \langle y, z \rangle_B \quad \text{for } x, y, z \in X.$$

A Hilbert bimodule  ${}_A X_B$  is *left-full* if the closed span  $A_X$  of  ${}_A \langle X, X \rangle$  is all of  $A$  (and to avoid confusion we sometimes refer to the property  $\overline{\text{span}} \langle X, X \rangle_B = B$  as *right-full*). In any event, if  $X$  is an  $A - B$  Hilbert bimodule then  $A_X$  is an ideal of  $A$  that is mapped isomorphically onto  $\mathcal{K}(X)$  via  $\phi_X$ . The *dual*  $B - A$  Hilbert bimodule  $\tilde{X}$  is formed as follows: write  $\tilde{x}$  when a vector  $x \in X$  is regarded as belonging to  $\tilde{X}$ , define the  $B - A$  bimodule structure by

$$b\tilde{x}a = \widetilde{a^*xb^*}$$

and the inner products by

$${}_B \langle \tilde{x}, \tilde{y} \rangle = \langle x, y \rangle_B \quad \text{and} \quad \langle \tilde{x}, \tilde{y} \rangle_A = {}_A \langle x, y \rangle$$

for  $b \in B, x, y \in X, a \in A$ . An  $A - B$  *imprimitivity bimodule* is an  $A - B$  Hilbert bimodule that is full on both the left and the right. Note that every Hilbert  $B$ -module may be regarded as a left-full  $\mathcal{K}(X) - B$  Hilbert bimodule with left inner product

$$\kappa(X) \langle x, y \rangle = \theta_{x,y}.$$

It is a fundamental fact about the enchilada category that the invertible morphisms are precisely the (isomorphism classes of) imprimitivity bimodules (see, for example, [7, Proposition 2.6], [8, Lemma 2.4], and [16, Proposition 2.3]).

In this section, we show the existence of the necessary ingredients, such as kernel and image, to construct an exact sequence in the enchilada category. As we mentioned in the introduction, we must be careful in defining the image of a morphism.

It is obvious that in the enchilada category a 0 object is any 0-dimensional  $C^*$ -algebra, and the 0 morphism from  $A$  to  $B$  is the 0 correspondence  ${}_A 0_B$ . If  $\mu : A \rightarrow M(B)$  is a homomorphism, then  $\mu(A)B$  is an  $A - B$  correspondence. When  $A$  is a closed ideal<sup>3</sup> of  $B$ , we get a correspondence  ${}_A A_B$ . For any ideal  $I$  of a  $C^*$ -algebra  $B$ , the quotient map  $B \rightarrow B/I$  gives rise to a correspondence  ${}_B (B/I)_{B/I}$ .

**3.1. PROPOSITION.** *Given correspondences  ${}_A X_B$  and  ${}_B Y_C$ , we have  $X \otimes_B Y = 0$  if and only if*

$$B_X \subset \text{Ker } \phi_Y.$$

**PROOF.** Assume that  $X \otimes_B Y = 0$ . Let  $x_1, x_2 \in X$  and  $y_1, y_2 \in Y$ . Then we have,

$$0 = \langle x_1 \otimes_B y_1, x_2 \otimes_B y_2 \rangle_C = \langle y_1, \langle x_1, x_2 \rangle_B y_2 \rangle_C.$$

This implies that  $\langle x_1, x_2 \rangle_B y = 0$  for all  $y \in Y$ , i.e.,  $\langle x_1, x_2 \rangle_B \in \text{Ker } \phi_Y$ . Since any element of  $B_X$  is a limit of linear combinations of elements  $\langle x_1, x_2 \rangle_B$  where  $x_i \in X$ , we conclude that  $B_X \subset \text{Ker } \phi_Y$ .

Now assume that  $B_X \subset \text{Ker } \phi_Y$ . Of course, since  $X \otimes_B Y$  is the closed span of elementary tensors, in order to show  $X \otimes_B Y = 0$  it suffices to show that  $x \otimes y = 0$  for any  $x \in X$  and  $y \in Y$ : by the Cohen-Hewitt factorization theorem we can write  $x$  as  $x_1 b$  for some  $x_1 \in X$  and  $b \in B_X$ , and then we have

$$x \otimes_B y = x_1 b \otimes_B y = x_1 \otimes_B b y = x_1 \otimes_B \phi_Y(b) y = 0.$$

■

**3.2. LEMMA.** *Let  ${}_A X_B$  be a correspondence and  $C$  be a  $C^*$ -subalgebra of  $B$  containing  $B_X$ . Then  $X$  becomes an  $A - C$  correspondence  ${}_A X_C$  by restricting the right-module structure to  $C$ , and the map*

$$x \otimes b \mapsto xb \quad \text{for } x \in X, b \in CB$$

*extends uniquely to an isomorphism*

$${}_A X_C \otimes_C (CB)_B \cong {}_A X_B.$$

Note that  $CB$  is a closed right ideal of  $B$ , by the Cohen-Hewitt factorization theorem.

**PROOF.**  $X$  already has a right  $B$ -module structure. Restricting this to  $C$ , we get a right  $C$ -module structure. Now, we need an inner product into  $C$ , which we get directly since  $B_X \subset C$ . Then the standard computation

$$\langle x \otimes b, x' \otimes b' \rangle_B = b^* \langle x, x' \rangle_B b' = \langle xb, x'b' \rangle_B$$

implies the assertion regarding the isomorphism. ■

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<sup>3</sup>and henceforth we will drop “closed”, so that all ideals are tacitly assumed to be closed



In Lemma 3.2,  $C$  will usually be an ideal of  $B$ , and then  ${}_A X_C \otimes_C {}_C X_B \cong {}_A X_B$ . A frequently used special case is when  $C = B$ , and then the main content is the isomorphism  $X \otimes_B B \cong X$ .

**3.3. LEMMA.** *Let  $X$  be a Hilbert  $B$ -module, and let  $\pi : A \rightarrow M(C)$  be a nondegenerate homomorphism.*

1. *Let  ${}_A X_B$  be a correspondence, and suppose that  $\text{Ker } \pi \subset \text{Ker } \phi_{A,X}$ . Let  $\phi_{C,X}$  be the unique homomorphism making the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi_{A,X}} & \mathcal{L}(X) \\ \pi \downarrow & \nearrow \phi_{C,X} & \\ C & & \end{array}$$

*commute, and let  ${}_C X_B$  be the associated correspondence. Then the map*

$$c \otimes x \mapsto \phi_{C,X}(c)x \quad \text{for } c \in C, x \in X$$

*extends uniquely to an isomorphism*

$${}_A C_C \otimes_C {}_C X_B \cong {}_A X_B$$

*of  $A - B$  correspondences.*

2. *Suppose that  $\pi$  is surjective. Let  ${}_C X_B$  and  ${}_C Y_B$  be correspondences, and define correspondences  ${}_A X_B$  and  ${}_A Y_B$  by*

$$\phi_{A,X} = \phi_{C,X} \circ \pi \quad \text{and} \quad \phi_{A,Y} = \phi_{C,Y} \circ \pi.$$

*Then  ${}_A X_B \cong {}_A Y_B$  if and only if  ${}_C X_B \cong {}_C Y_B$ .*

**PROOF.** (1) is folklore. For (2), first note that one direction follows quickly from part (1): if  ${}_C X_B \cong {}_C Y_B$  then

$${}_A X_B \cong C \otimes_C {}_C X_B \cong C \otimes_C {}_C Y_B \cong {}_A Y_B.$$

Conversely let

$$U : {}_A X_B \xrightarrow{\cong} {}_A Y_B$$

be an isomorphism. Then  $\text{Ad } U : \mathcal{L}_B(X) \rightarrow \mathcal{L}_B(Y)$  is also an isomorphism. For  $c \in C$  we can choose  $a \in A$  such that  $\pi(a) = c$ , and then

$$\begin{aligned} \text{Ad } U \circ \phi_{C,X}(c) &= \text{Ad } U \circ \phi_{C,X} \circ \pi(a) \\ &= \text{Ad } U \circ \phi_{A,X}(a) \\ &= \phi_{A,Y}(a) \\ &= \phi_{C,Y} \circ \pi(a) \\ &= \phi_{C,Y}(c), \end{aligned}$$

so that  $U$  also preserves the left  $C$ -module structures. ■

Frequently-used special cases of Lemma 3.3, for a given  $A - B$  correspondence  $X$ , are the isomorphisms

- $A \otimes_A X \cong X$  (where  $\pi = \text{id}_A$ ),
- $A/I \otimes_{A/I} X' \cong X$  (when  $I$  is an ideal of  $A$  contained in  $\text{Ker } \phi_X$ ), and
- $\mathcal{K}(X) \otimes_{\mathcal{K}(X)} X' \cong X$  [8, discussion preceding Proposition 2.27].

In connection with item (2) of Lemma 3.3, there is more to say:

3.4. PROPOSITION. *If  $\pi : A \rightarrow C$  is a surjective homomorphism, then  ${}_A C_C$  is an epimorphism in the enchilada category.*

PROOF. Given  $C - B$  correspondences  $X$  and  $Y$  such that

$${}_A C \otimes_C X_B \cong {}_A C \otimes_C Y_B,$$

we must show that  ${}_C X_B \cong {}_C Y_B$ . Using Lemma 3.3, we can regard  $X$  and  $Y$  as  $A - B$  correspondences  ${}_A X_B$  and  ${}_A Y_B$ , and then part (1) of the lemma and the hypothesis together tell us that  ${}_A X_B \cong {}_A Y_B$ , and so by part (2) of the lemma we also have  ${}_C X_B \cong {}_C Y_B$ . ■

3.5. PROPOSITION. *If  ${}_A X_B$  is a monomorphism in the enchilada category, then  $\phi_X : A \rightarrow \mathcal{L}_B(X)$  is injective.*

PROOF. Assume that  $\phi_X$  is not injective. Then,  $K = \text{Ker } \phi_X$  is a non-zero ideal of  $A$ . Consider the correspondence  ${}_K K_A$ . Since  $\langle K, K \rangle_K = K$ , by Proposition 3.1 we have

$$K \otimes_A X = 0 = 0 \otimes_A X,$$

but  $K \neq 0$ , so  $X$  is not a monomorphism. ■

We suspect that the converse of Proposition 3.5 is true, but have been unable to prove it. In any case, it seems that the property of being a monomorphism in the enchilada category is very weak.

3.6. PROPOSITION. *A correspondence  ${}_A X_B$  is a split monomorphism in the enchilada category if and only if it is a left-full Hilbert bimodule.*

PROOF. First assume that  $X$  is a split monomorphism. Let  ${}_B Y_A$  be a correspondence, and assume that we have an isomorphism

$$U : X \otimes_B Y \xrightarrow{\cong} A$$

of  $A - A$  correspondences. We must show that  $\phi_X : A \rightarrow \mathcal{L}(X)$  is an isomorphism onto  $\mathcal{K}(X)$ . We borrow an idea from [16, proof of Proposition 2.3]: note first that  $\text{Ad } U : \mathcal{L}(X \otimes_B Y) \xrightarrow{\cong} \mathcal{L}(A) = M(A)$  is an isomorphism, and define a homomorphism

$$\pi : \mathcal{L}(X) \rightarrow M(A)$$

by

$$\pi(T) = \text{Ad } U(T \otimes_B 1).$$

Then (the canonical extension to multipliers of)  $\pi \circ \text{id}_A$  is the identity on  $M(A)$ . Also, a routine argument using an approximate identity of  $A$  shows that  $\pi(\mathcal{K}(X)) \subset A$ . Thus (now deviating from Schweizer's proof)  $X$  gets an  $A$ -valued inner product:

$${}_A \langle x, y \rangle = \pi(\theta_{x,y}),$$

and so  $X$  is actually an  $A - B$  Hilbert bimodule. We have  $B - A$  correspondence isomorphisms

$$\begin{aligned} \tilde{X} \otimes_A X \otimes_B Y &\cong B_X \otimes_B Y && \text{(Lemma 3.7)} \\ &\cong B_X \otimes_{B_X} Y && \text{(Lemma 3.8)} \\ &\cong Y && \text{(Lemma 3.3)} \end{aligned}$$

while on the other hand

$$\begin{aligned} \tilde{X} \otimes_A X \otimes_B Y &\cong \tilde{X} \otimes_A A && \text{(hypothesis)} \\ &\cong \tilde{X} && \text{(Lemma 3.2)}. \end{aligned}$$

Thus

$$A \cong X \otimes_B Y \cong X \otimes_B \tilde{X} \cong A_X$$

as  $A - A$  correspondences. Therefore  $A = A_X$ , so the  $A - B$  Hilbert bimodule  $X$  is left-full.

Conversely, assume that  $X$  is a left-full Hilbert bimodule. Then we have a  $B - A$  Hilbert bimodule  $\tilde{X}$ , and

$$\begin{aligned} [\tilde{X}] \circ [X] &= [X \otimes_B \tilde{X}] \\ &= [X \otimes_{B_X} \tilde{X}] && \text{(by Lemma 3.8 again)} \\ &= [A], \end{aligned}$$

so  $[X]$  is a split monomorphism ■

**3.7. LEMMA.** *If  $X$  is an  $A - B$  Hilbert bimodule, then*

$$\begin{aligned} \tilde{X} \otimes_A X &\cong B_X && \text{as } B - B \text{ correspondences} \\ X \otimes_B \tilde{X} &\cong A_Y && \text{as } A - A \text{ correspondences.} \end{aligned}$$

**PROOF.** This is folklore. If  $X$  is an imprimitivity bimodule, then this is [14, Proposition 3.28] (for example), and the general case can be proved using the same techniques. ■

3.8. LEMMA. *If  ${}_A X_B$  and  ${}_B Y_C$  are correspondences, then*

$$X \otimes_B Y \cong X \otimes_{B_X} Y$$

*as  $A - C$  correspondences.*

PROOF. Just note that the balancing relations determined by  $\otimes_B$  and  $\otimes_{B_X}$  coincide: for every  $x \in X$ ,  $b \in B$ , and  $y \in Y$  we can choose  $x' \in X$ ,  $b' \in B_X$  such that  $x = x'b'$ , and then the following computation in  $X \otimes_{B_X} Y$  suffices:

$$\begin{aligned} xb \otimes y &= x'b'b \otimes y \\ &= x' \otimes b'by \quad (\text{since } b'b \in B_X) \\ &= x'b' \otimes by \\ &= x \otimes by. \end{aligned}$$

■

3.9. THEOREM. *Let  $X$  be an  $A - B$  correspondence and let  $K$  be the kernel of the associated homomorphism  $\phi_X : A \rightarrow \mathcal{L}_B(X)$ . Then the correspondence  ${}_K K_A$  is a kernel of  $X$ .*

PROOF. First,  $K \otimes_K X = 0$  by Lemma 3.1, because  $\langle K, K \rangle = K = \text{Ker } \phi_X$ .

Now suppose that  ${}_C Y_A$  is a correspondence such that  $Y \otimes_A X = 0$ . Then by Lemma 3.1 we have

$$A_Y \subset \text{Ker } \phi_X = K,$$

so by Lemma 3.2 we get a correspondence  ${}_C Y_K$  such that

$${}_C Y_K \otimes_K K_A \cong {}_C Y_A.$$

Moreover,  ${}_C Y_K$  is unique up to isomorphism because  ${}_K K_A$  is a left-full Hilbert bimodule, and hence is a split monomorphism by Proposition 3.6, so in particular is a monomorphism. ■

3.10. COROLLARY. *A correspondence  ${}_A X_B$  is a left-full Hilbert bimodule if and only if  $X$  is a kernel in the enchilada category, in which case it is a kernel of  ${}_B (B/B_X)_{B/B_X}$ .*

PROOF. First assume that  ${}_A X_B$  is a kernel of a correspondence  ${}_B Y_C$ . Then by Theorem 3.9  ${}_A X_B$  is isomorphic to the kernel  ${}_K K_B$ , where  $K = \text{Ker } \phi_Y$ , in the sense that there is an imprimitivity bimodule  $Y$  (equivalently,  $[Y] : A \rightarrow K$  is an isomorphism in the enchilada category) making the diagram

$$\begin{array}{ccc} A & \xrightarrow{X} & B \\ Y \downarrow & \nearrow K & \\ K & & \end{array}$$

commute. Since  ${}_K K_B$  is a left-full Hilbert bimodule, so is  ${}_A X_B$ .

Conversely, assume that  $X$  is a left-full Hilbert bimodule. By Lemma 3.2 we can regard  $X$  as an imprimitivity bimodule  ${}_A X_{B_X}$ , and

$${}_A X_B \cong {}_A X_{B_X} \otimes_{B_X} {}_{B_X} (B_X)_B.$$

By Theorem 3.9,  ${}_{B_X} (B_X)_B$  is a kernel of  ${}_B (B/B_X)_{B/B_X}$ , and hence so is  ${}_A X_B$  because  ${}_A X_{B_X}$  is an imprimitivity bimodule. ■

3.11. PROPOSITION. *A correspondence  ${}_A X_B$  has cokernel  ${}_B (B/B_X)_{B/B_X}$ .*

PROOF. First,  $X \otimes_B B/B_X = 0$ , because  $B_X$  is the kernel of the quotient map  $B \rightarrow B/B_X$ . Now suppose that  ${}_B Y_C$  is a correspondence such that  $X \otimes_B Y = 0$ . Then  $B_X \subset \text{Ker } \phi_Y$  by Lemma 3.1, so by Lemma 3.3 we may regard  $Y$  as a  $B/B_X - C$  correspondence, and we have

$${}_B (B/B_X)_{B/B_X} \otimes_{B/B_X} {}_{B/B_X} Y_C \cong {}_B Y_C. \tag{3.1}$$

Moreover, by Proposition 3.4  ${}_B (B/B_X)_{B/B_X}$  is an epimorphism in the enchilada category, so (3.1) determines  ${}_B Y_C$  up to isomorphism. ■

Finally, we are ready for images.

3.12. THEOREM. *A correspondence  ${}_A X_B$  has Schubert image  ${}_{B_X} (B_X)_B$  and Schubert coimage  ${}_A (A/\text{Ker } \phi)_{A/\text{Ker } \phi}$ , where  $\phi = \phi_X$  is the associated homomorphism.*

PROOF. By Proposition 3.11  ${}_B (B/B_X)_{B/B_X}$  is a cokernel of  $X$ , and by Theorem 3.9  ${}_{B_X} (B_X)_B$  is a kernel of  ${}_B (B/B_X)_{B/B_X}$ . Thus  ${}_{B_X} (B_X)_B$  is a cokernel of a kernel of  $X$ , and so is the Schubert image of  $X$  by definition.

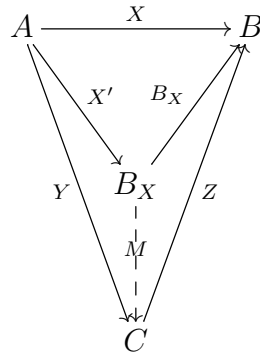
Very similarly,  ${}_{\text{Ker } \phi} (\text{Ker } \phi)_A$  is a kernel of  $X$ , and  ${}_A (A/\text{Ker } \phi)_{A/\text{Ker } \phi}$  is a cokernel of  ${}_{\text{Ker } \phi} (\text{Ker } \phi)_A$ . Thus, by definition, it is the Schubert coimage of  $X$ . ■

3.13. PROPOSITION. *Every Hilbert bimodule  ${}_A X_B$  has an image, and in fact it coincides with the Schubert image.*

PROOF. Assume that  ${}_A X_B$  is isomorphic to  ${}_A Y \otimes_C Z_B$  for a monomorphism  ${}_C Z_B$  and a morphism  ${}_A Y_C$ . Since  ${}_A X_{B_X}$  is a Hilbert bimodule, there exists a  $B_X - A$  Hilbert bimodule  $\tilde{X}$  such that  $\tilde{X} \otimes_A X \cong B_X$  as  $B_X - B_X$  correspondences (Lemma 3.7). Denote  $\tilde{X} \otimes_A Y$  by  $M$ . Then we have

$${}_{B_X} M \otimes_C Z_B \cong {}_{B_X} \tilde{X} \otimes_A Y \otimes_C Z_B \cong {}_{B_X} \tilde{X} \otimes_A X_B \cong {}_{B_X} (B_X)_B.$$

Thus  $B_X M_B$  is the unique monomorphism making the diagram



commute, which completes the proof. ■

And Theorem 3.12 in turn makes us ready for exact sequences:

3.14. DEFINITION. *A sequence*

$$\dots \longrightarrow A \xrightarrow{[X]} B \xrightarrow{[Y]} C \longrightarrow \dots$$

of morphisms in the enchilada category is exact at  $B$  if the image of  $X$  equals the kernel of  $Y$ , and is exact if it is exact at every node.

3.15. THEOREM. *Let  ${}_A X_B$  and  ${}_B Y_C$  be correspondences. Then the sequence*

$$0 \longrightarrow A \xrightarrow{X} B \xrightarrow{Y} C \longrightarrow 0$$

is exact in the enchilada category if and only if  $\phi_X$  is injective,  $B_X = \text{Ker } \phi_Y$ , and  $Y$  is full (i.e.,  $C_Y = C$ ).

PROOF. Since  $A_0 = \langle 0, 0 \rangle = 0$ , the morphism  $0 : 0 \rightarrow A$  is an image of  $0 : 0 \rightarrow A$ . On the other hand,  $\text{Ker } \phi_X : \text{Ker } \phi_X \rightarrow A$  is a kernel of  $X : A \rightarrow B$ . Thus the sequence is exact at  $A$  if and only if  $\phi_X$  is injective.

Next,  $C_Y : C_Y \rightarrow C$  is an image of  $Y : B \rightarrow C$ . On the other hand, the homomorphism  $\phi_0 : C \rightarrow \mathcal{L}(0)$  is 0, so has kernel  $C$ . Thus  $C : C \rightarrow C$  is a kernel of  $0 : C \rightarrow 0$ . Therefore the sequence is exact at  $C$  if and only if  $C_Y = C$ .

Finally,  $B_X : B_X \rightarrow C$  is an image of  $X : A \rightarrow B$ , and  $\text{Ker } \phi_Y : \text{Ker } \phi_Y \rightarrow B$  is a kernel of  $Y : B \rightarrow C$ , so the sequence is exact at  $B$  if and only if  $B_X = \text{Ker } \phi_Y$ . ■

3.16. REMARK. As we mentioned in the introduction, our primary motivation for investigating exact sequences in the enchilada category was to determine whether the reduced-crossed-product functor is exact in the enchilada categories, which would obviously be relevant for the Baum-Connes conjecture. But now we will show that it is not exact. Let

$$0 \longrightarrow A \xrightarrow{X} B \xrightarrow{Y} C \longrightarrow 0$$

be a short exact sequence of correspondences, and let  $G$  be a locally compact group. Further let  $\alpha : G \rightarrow \text{Aut } A$ ,  $\beta : G \rightarrow \text{Aut } B$ ,  $\gamma : G \rightarrow \text{Aut } C$ ,  $\zeta : G \rightarrow \text{Aut } X$ , and  $\eta : G \rightarrow \text{Aut } Y$  be actions as in [8, Section 3.1.1]. Then we can form the reduced-crossed-product correspondences

$$A \rtimes_{\alpha,r} G (X \rtimes_{\zeta,r} G)_{B \rtimes_{\beta,r} G} \quad \text{and} \quad B \rtimes_{\beta,r} G (Y \rtimes_{\eta,r} G)_{C \rtimes_{\gamma,r} G}.$$

Moreover, there are actions

$$\mu : G \rightarrow \text{Aut } \mathcal{K}(X) \quad \text{and} \quad \nu : G \rightarrow \text{Aut } \mathcal{K}(Y)$$

such that

$$\begin{aligned} \mathcal{K}(X \rtimes_{\zeta,r} G) &= \mathcal{K}(X) \rtimes_{\mu,r} G \\ \phi_{X \rtimes_{\zeta,r} G} &= \phi_X \rtimes_r G \\ \mathcal{K}(Y \rtimes_{\eta,r} G) &= \mathcal{K}(Y) \rtimes_{\nu,r} G \\ \phi_{Y \rtimes_{\eta,r} G} &= \phi_Y \rtimes_r G. \end{aligned}$$

Now consider the following sequence

$$0 \longrightarrow A \rtimes_{\alpha,r} G \xrightarrow{X \rtimes_{\zeta,r} G} B \rtimes_{\beta,r} G \xrightarrow{Y \rtimes_{\eta,r} G} C \rtimes_{\gamma,r} G \longrightarrow 0$$

One of the required conditions to make this sequence exact is

$$(B \rtimes_{\beta,r} G)_{X \rtimes_{\zeta,r} G} = \text{Ker } \phi_{Y \rtimes_{\eta,r} G}.$$

Now, by [8, Proposition 3.2] we have  $(B \rtimes_{\beta,r} G)_{X \rtimes_{\zeta,r} G} = B_X \rtimes_{\beta,r} G$ . Since  $\phi_{Y \rtimes_{\eta,r} G} = \phi_Y \rtimes_r G$  and  $B_X = \text{Ker } \phi_Y$ , for exactness we must have

$$(\text{Ker } \phi_X) \rtimes_{\alpha,r} G = \text{Ker}(\phi_X \rtimes_r G).$$

However this equality is not true in general if  $G$  is not exact. So, the reduced-crossed-product functor does not preserve exactness in the enchilada categories if  $G$  is a nonexact group.

**3.17. REMARK.** Although it is not directly relevant for our original investigation regarding the Baum-Connes conjecture, we will now point out that the full-crossed-product functor is also not exact in the enchilada categories.

Again let

$$0 \longrightarrow A \xrightarrow{X} B \xrightarrow{Y} C \longrightarrow 0$$

be a short exact sequence of correspondences, carrying compatible actions of  $G$ . For exactness, we would need the sequence

$$0 \longrightarrow A \rtimes G \xrightarrow{X \rtimes G} B \rtimes G \xrightarrow{Y \rtimes G} C \rtimes G \longrightarrow 0 \tag{3.2}$$

to be exact. To apply Theorem 3.15, we would need:

- $\phi_{X \rtimes G}$  to be injective,
- $(B \rtimes G)_{X \rtimes G} = \text{Ker } \phi_{Y \rtimes G}$ , and
- $(C \rtimes G)_{Y \rtimes G} = C \rtimes G$ .

This time, it may fail to be exact at  $A \rtimes G$ , i.e.,  $\phi_{X \rtimes G}$  need not be injective. To explain all this, note that, as for reduced crossed products,

$$\begin{aligned} \mathcal{K}(X \rtimes G) &= \mathcal{K}(X) \rtimes G \\ \mathcal{K}(Y \rtimes G) &= \mathcal{K}(Y) \rtimes G \\ \phi_{X \rtimes G} &= \phi_X \rtimes G \\ \phi_{Y \rtimes G} &= \phi_Y \rtimes G. \end{aligned}$$

Then we have

$$\begin{aligned} (B \rtimes G)_{X \rtimes G} &= B_X \rtimes G \\ &= (\text{Ker } \phi_Y) \rtimes G \quad (\text{since } B_X = \text{Ker } \phi_Y) \\ &= \text{Ker}(\phi_Y \rtimes G) \\ &= \text{Ker } \phi_{Y \rtimes G}, \end{aligned}$$

so (3.2) is exact in the middle.

Next,

$$\begin{aligned} (C \rtimes G)_{Y \rtimes G} &= C_Y \rtimes G \quad (\text{since } C_Y = C) \\ &= C \rtimes G, \end{aligned}$$

so (3.2) is exact at  $C \rtimes G$ .

To see how exactness at  $A \rtimes G$  might fail, we formulate a strategy for finding a counterexample: we will take a correspondence  $X$  arising from a homomorphism, and we will let all the actions be trivial. More precisely, we will have:

- an injection  $\pi$  of  $A$  into an ideal  $D$  of  $B$ , and
- the associated  $A - B$  correspondence  $X = AB$ .

Since all the actions are trivial, the crossed products are just the maximal tensor products with  $C^*(G)$ . In particular, the correspondence  ${}_{A \rtimes G}(X \rtimes G)_{B \rtimes G}$  is the tensor product

$${}_{A \otimes_{\max} C^*(G)}(X \otimes_{\max} C^*(G))_{B \otimes_{\max} C^*(G)}.$$

It follows from the standard theory of  $C^*$ -correspondences that this is isomorphic to

$${}_A X_B \otimes_{\max} C^*(G).$$



In particular, the left-module homomorphism  $\phi_{X \rtimes G}$  becomes the tensor product

$$\pi \otimes_{\max} \text{id} : A \otimes_{\max} C^*(G) \rightarrow B \otimes_{\max} C^*(G).$$

If we take any group  $G$  for which  $C^*(G)$  is nonnuclear (for example, the free group on 2 generators), then it is a subtle fact from  $C^*$ -algebra theory [3, Theorem IV.3.1.12] that there exist a  $C^*$ -algebra  $D$  and a  $C^*$ -subalgebra  $A$  such that the associated homomorphism  $\pi \otimes_{\max} \text{id}$  is noninjective. Then for any full correspondence  ${}_B Y_C$  with  $\text{Ker } \phi_Y = D$  we get a counterexample.

#### 4. Further properties of the enchilada category

In Corollary 3.10 we characterized kernels in the enchilada category as the left-full Hilbert bimodules. In view of other results that come in dual pairs, it is tempting to suspect that cokernels are precisely the right-full Hilbert bimodules. On the other hand, the enchilada category is decidedly left-challenged, so it is not surprising that it could have direction-related properties that are not satisfied when the directions are reversed. Indeed:

4.1. PROPOSITION. *In the enchilada category, a cokernel need not be a right-full Hilbert bimodule, and conversely a right-full Hilbert bimodule need not be a cokernel.*

PROOF. First, it follows from Proposition 3.11 that a quotient map  $B \rightarrow B/J$  is a cokernel in the enchilada category. However, it need not be a Hilbert bimodule, since the quotient  $B/J$  need not be isomorphic to an ideal of  $B$ .

Conversely, let  $A$  be a nonzero  $C^*$ -algebra. Then  ${}_A A_A$  is a right-full Hilbert bimodule, and we will show that it is not a cokernel. Arguing by contradiction, suppose  ${}_A A_A$  is a cokernel of  ${}_B X_A$ . Then

$$X \otimes_A A = 0,$$

so  $A_X \subset \text{Ker } \phi_A = 0$ , and hence  $X = 0$ . Now,  $0 : A \rightarrow 0$  is also a cokernel of  $0 : B \rightarrow A$ , so  $A$  and  $0$  are isomorphic in the enchilada category, and hence are Morita equivalent. Therefore  $A = 0$ , which is a contradiction. ■

4.2. PROPOSITION. *If a correspondence  ${}_A X_B$  is an epimorphism in the enchilada category, then  $X$  is full, i.e.,  $B_X = B$ .*

PROOF. Suppose that  $B_X \neq B$ . Then

$$X \otimes_B B(B/B_X)_{B/B_X} = 0 = X \otimes_B 0,$$

but  ${}_B(B/B_X)_{B/B_X} \not\cong 0$ . ■

Proposition 4.2 can be alternatively restated as follows:  ${}_A X_B$  if is an epimorphism then its image is the identity morphism  ${}_B B_B$ . In many categories, there is a converse: a morphism  $f : A \rightarrow B$  whose image is the identity morphism  $1_B$  must be an epimorphism — informally, surjections are epimorphisms. But not in the enchilada category:

4.3. PROPOSITION. *The enchilada category has a morphism  $X : A \rightarrow B$  that is not an epimorphism but whose image is  ${}_B B_B$ .*

PROOF. We must find correspondences  ${}_A X_B$ ,  ${}_B Y_C$ , and  ${}_B Z_C$  such that

1.  $B_X = B$ ,
2.  $X \otimes_B Y \cong X \otimes_B Z$ , and
3.  $Y \not\cong Z$ .

We let  $X$  be the  $\mathbb{C} - \mathbb{C}^2$  correspondence associated to the homomorphism  $a \mapsto (a, a)$ , and for  $Y$  and  $Z$  we take the  $\mathbb{C}^2 - \mathbb{C}$  correspondences associated to the homomorphisms  $(a, b) \mapsto a$  and  $(a, b) \mapsto b$ , respectively.

It follows from Lemma 3.3 that

$$X \otimes_B Y \cong {}_C \mathbb{C}_C \cong X \otimes_B Z.$$

On the other hand, the  $\mathbb{C}^2 - \mathbb{C}$  correspondences  $Y$  and  $Z$  are not isomorphic, since by [7, Proposition 2.3] two  $B - C$  correspondences coming from nondegenerate homomorphisms  $\pi, \rho : B \rightarrow M(C)$  are isomorphic if and only if there is a unitary  $u \in M(C)$  such that  $\text{Ad } u \circ \pi = \rho$ . ■

In spite of Proposition 4.3, there is a weaker result, namely Proposition 3.4.

4.4. COROLLARY. *The enchilada category does not have equalizers.*

PROOF. By Proposition 4.3, we can choose a correspondence  ${}_A X_B$  such that  $B_X = B$  but  $X$  is not an epimorphism in the enchilada category. Since  $B_X = B$ , the image of  $X$  is the correspondence  ${}_B B_B$ . Thus  $X$  factors through its image as follows:

$$X \cong X \otimes_B {}_B B_B.$$

Therefore, by [13, Proposition I.10.1], if the enchilada category had equalizers then  $X$  would have to be an epimorphism. ■

Despite having both kernels and cokernels, the enchilada category is not abelian. In fact:

4.5. COROLLARY. *The enchilada category is not additive.*

PROOF. Every additive category with kernels has equalizers. ■

Direct sum is a binary operation on isomorphism classes of  $A - B$  correspondences. However,  $([{}_A X_B], \oplus)$  does not have a group structure since direct sum is not cancellative (i.e., we can have  $X \oplus Y \cong X \oplus Z$  but  $Y \not\cong Z$ ). Therefore, the enchilada category is not even preadditive.

Proposition 3.6 implies a dual counterpart:

4.6. COROLLARY. *An  $A - B$  correspondence  $X$  is a right-full Hilbert bimodule if and only if  $X$  is a split epimorphism in the enchilada category.*

PROOF. First, if  $X$  is a split epimorphism, then we can choose a  $B - A$  correspondence  $Y$  such that

$$Y \otimes_A X \cong B$$

as  $B - B$  correspondences. It then follows from the proof of Proposition 3.6 that  $Y$  is a left-full Hilbert bimodule and  $X \cong \tilde{Y}$ . Thus  $X$  is a right-full Hilbert bimodule.

Conversely, if  $X$  is a right-full Hilbert bimodule, then, similarly to the proof of Proposition 3.6,  $\tilde{X}$  is a right inverse of  $X$ , so  $X$  is a split epimorphism. ■

4.7. REMARK. In [1], Arduini proposed a strengthening of the concepts of monomorphism and epimorphism, with an eye toward improving the concept of subobject. Here we only give his definition of monomorphism, which nowadays is called *strong monomorphism*: it is a morphism  $f : A \rightarrow B$  such that for every commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ h \downarrow & & \downarrow k \\ A & \xrightarrow{f} & B, \end{array}$$

where  $g$  is an epimorphism, there is a unique morphism  $\ell$  making the diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ h \downarrow & \begin{array}{c} \swarrow \ell \\ \downarrow \end{array} & \downarrow k \\ A & \xrightarrow{f} & B \end{array}$$

commute. Every split monomorphism is a strong monomorphism, and every strong monomorphism is a monomorphism. Thus, the uniqueness of  $\ell$  is automatic. Moreover, since  $g$  is an epimorphism, it is enough to know that the upper triangle commutes. As we mentioned in the introduction, for a time we thought we would be able to use Arduini's strong monomorphisms to define the image of a morphism in the enchilada category. But we had to abandon this approach, since we have an inadequate understanding of epimorphisms in the enchilada category.

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