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SYMMETRIC MONOIDAL CATEGORIES AND Γ-CATEGORIES

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ABSTRACT. In this paper we construct a symmetric monoidal closed model category of coherently commutative monoidal categories. The main aim of this paper is to establish a Quillen equivalence between a model category of coherently commutative monoidal categories and a natural model category of Permutative (or strict symmetric monoidal) categories, **Perm**, which is not a symmetric monoidal closed model category. The right adjoint of this Quillen equivalence is the classical Segal's Nerve functor.

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1. Introduction

In the paper [BF78] Bousfield and Friedlander constructed a model category of Γ-spaces and proved that its homotopy category is equivalent to a homotopy category of *connective spectra*. Their research was taken further by Schwede [Sch99] who constructed a Quillen

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equivalent model structure on Γ-spaces whose fibrant objects can be described as (pointed) *spaces* having a *coherently commutative group* structure. Schwede's model category is a symmetric monoidal closed model category under the *smash product* defined by Lydakis in [Lyd99] which is just a version of the *Day convolution product* [Day70] for normalized functors. This paper is the first in a series of papers in which we study *coherently commutative monoidal* objects in cartesian closed model categories. Our long term objective is to understand coherently commutative monoidal objects in suitable model categories of (∞, n)-categories such as [Rez10], [Ara14]. The current paper deals with the case of ordinary categories which is an intermediate step towards achieving the aforementioned goal. A Γ- category is a functor from the (skeletal) category of finite based sets Γ^{op} into the category of all (small) categories **Cat**. We denote the category of all Γ- categories and natural transformations between them by Γ**Cat**. Along the lines of the construction of the stable Q-model category in [Sch99] we construct a symmetric monoidal closed model category structure on Γ**Cat** which we refer to as the model category structure of *coherently commutative monoidal categories*. A Γ- category is called a *coherently commutative monoidal category* if it satisfies the Segal condition, see [Seg74] or equivalently it is a *homotopy monoid* in **Cat** in the sense of Leinster [Lei00]. These Γ- categories are fibrant objects in our model category of coherently commutative monoidal categories. The main objective of this paper is to compare the category of all (small) symmetric monoidal categories with our model category of coherently commutative monoidal categories. There are many variants of the category of symmetric monoidal categories all of which have equivalent homotopy categories, see [Man10, Theorem 3.9]. All of these variant categories are *fibration categories* but they do not have a model category structure. Due to this shortcoming, in this paper we will work in a subcategory **Perm** which inherits a model category structure from **Cat**. The objects of **Perm** are *permutative categories* (also called strict symmetric monoidal categories) and maps are strict symmetric monoidal functors. We recall that a *permutative category* is a symmetric monoidal category whose tensor product is strictly associative and unital. It was shown by May [May72] that permutative categories are algebras over the categorical Barrat-Eccles operad, in **Cat**. We will construct a model category structure on **Perm** by transferring along the functor F which assigns to each category, the free permutative category generated by it. This functor is a right adjoint of an adjunction $U : \mathbf{Perm} \rightleftharpoons \mathbf{Cat} : F$ where U is the forgetful functor. This model category structure also follows from results in [BM07] and [Lac07]. We will refer to this model structure on **Perm** as the *natural model category structure* of permutative categories. The weak equivalences and fibrations in this model category structure are inherited from the natural model category structure on **Cat**, namely they are equivalence of categories and isofibrations respectively. The homotopy category of **Perm** is equivalent to the homotopy categories of all the variant categories of symmetric monoidal categories mentioned above. The model category of all (small) permutative categories is a **Cat**-model category. However the shortcoming of the natural model category structure is that it is not a symmetric monoidal closed model category structure. In the paper [Sch08] a tensor product of symmetric monoidal categories has been defined but this tensor product does not endow the category of symmetric monoidal categories with a symmetric monoidal closed structure. However it follows from [Bou17, Prop. 6.4] that there is a symmetric *skew* monoidal structure on **Perm** which induces a symmetric monoidal closed structure on the homotopy category of **Perm**.

The model category structure of coherently commutative monoidal categories on $\Gamma\mathbf{Cat}$ is obtained by localizing the projective (or strict) model category structure on $\Gamma\mathbf{Cat}$. The guiding principle of this construction is to introduce a *semi-additive* structure on the homotopy category. We achieve this by inverting all canonical maps

$$X \sqcup Y \longrightarrow X \times Y$$

in the homotopy category of the projective model category structure on $\Gamma\mathbf{Cat}$. The fibrant objects in this model category structure are coherently commutative monoidal categories. We show that $\Gamma\mathbf{Cat}$ is a symmetric monoidal closed model category with respect to the Day convolution product. In the paper [KS15] the authors construct a model category of E_∞-*quasicategories* whose underlying category is the category of (honest) commutative monoids in a functor category. The authors go on further to describe a chain of Quillen equivalences between their model category and the model category of algebras over an E_∞-operad in the Joyal model category of simplicial sets. However they do not get a symmetric monoidal closed model category structure. Moreover in this paper we want to explicitly describe a pair of functors which give rise to a Quillen equivalence (in the case of ordinary categories).

In the paper [Seg74], Segal described a functor from (small) symmetric monoidal categories to the category of infinite loop spaces, or equivalently, the category of connective spectra. This functor is often called Segal's K-theory functor because when applied to the symmetric monoidal category of finite rank projective modules over a ring R, the resulting (connective) spectrum is Quillen's algebraic K-theory of R. This functor factors into a composite of two functors, first of which takes values in the category of (small) Γ- categories $\Gamma\mathbf{Cat}$, followed by a group completion functor. In this paper we will refer to this first factor as *Segal's Nerve* functor. We will construct an unnormalized version of the Segal's nerve functor and will denote it by $\mathcal{K}$. The main result of this paper is that the *unnormalized Segal's nerve functor* $\mathcal{K}$ is the right Quillen functor of a Quillen equivalence between the natural model category of permutative categories and the model category of coherently commutative monoidal categories. Unfortunately, the left adjoint to $\mathcal{K}$ does not have any simple description therefore in order to to prove our main result we will construct another Quillen equivalence, between the same two model categories, whose right adjoint is obtained by a *thickening* of $\mathcal{K}$. We will denote this by $\overline{\mathcal{K}}$ and refer to it as the *thickened Segal's nerve* functor. The (skeletal) category of finite (unbased) sets whose objects are ordinal numbers is an *enveloping category* of the commutative operad, see [Shaon]. In order to define a left adjoint to $\overline{\mathcal{K}}$ we will construct a *symmetric monoidal completion* of an oplax symmetric monoidal functor along the lines of Mandell [Man10, Prop 4.2]. In order to do so we define a permutative category $\mathfrak{L}$ equipped with an oplax symmetric monoidal inclusion functor $i : \mathcal{N} \longrightarrow \mathfrak{L}$, having the universal property that each oplax symmetric monoidal functor $X : \mathcal{N} \longrightarrow \mathbf{Cat}$ extends uniquely to a symmetric monoidal functor $\mathfrak{L}X : \mathfrak{L} \longrightarrow \mathbf{Cat}$ along the inclusion i. The category of oplax symmetric monoidal functors $[\mathcal{N}, \mathbf{Cat}]^{OL}$ is isomorphic to $\Gamma\mathbf{Cat}$ therefore this symmetric monoidal extension defines a functor $\mathfrak{L} : \Gamma\mathbf{Cat} \longrightarrow [\mathfrak{L}, \mathbf{Cat}]_{\otimes}$. Now the left adjoint to $\overline{\mathcal{K}}$, $\overline{\mathcal{L}}$, can be described as the following composite

$$\Gamma\mathbf{Cat} \xrightarrow{\mathfrak{L}(-)} [\mathfrak{L}, \mathbf{Cat}]_{\otimes} \xrightarrow{hocolim} \mathbf{Perm},$$

where *hocolim* is a homotopy colimit functor. The relation between permutative categories and connective spectra has been well explored in [Tho95], [Man10]. Thomason was the first one to show that every connective spectra is, up to equivalence, a K-theory of a permutative category. Mandell [Man10] used a different approach to establish a similar result based on the equivalence between Γ-spaces and connective spectra established in [BF78]. In the same paper Mandell proves a *non-group completed* version of Thomason's theorem [Man10, Theorem 1.4] by constructing an oplax version of Segal's nerve functor. This theorem states that the oplax version of Segal's nerve functor induces an equivalence of homotopy theories between a homotopy theory of permutative categories and a homotopy theory of coherently commutative monoidal categories where the weak equivalences of both homotopy theories are based on weak equivalences in the Thomason model category structure on **Cat** [Tho80]. We have based our theory on the natural model category structure on **Cat** wherein the notion of weak equivalence is much stronger. In a subsequent paper we plan to show that our main result implies the non-group completed version of Thomason's theorem [Man10, Theorem 1.4].

2. The Setup

In this section we will review the machinery needed for various constructions in this paper. We will begin with a review of symmetric monoidal categories and different types of functors between them functors between them. We will also review Γ- categories and collect some useful results about them. Most importantly we will be reviewing the notion of *Grothendieck construction* of functors taking values in **Cat** and use it to construct *Leinster's category* which will play a pivotal role in our theory. We will also review the *natural* model category structure on **Cat** and $\mathbf{Cat}_{\bullet}$.

2.1. Preliminaries. In this subsection we will briefly review the theory of permutative categories and monoidal and oplax functors between them. The definitions reviewed here and the notation specified here will be used throughout this paper.

2.2. Definition. *A symmetric monoidal category is consists of a 7-tuple*

$$(C, -\otimes -, 1_C, \alpha, \beta_l, \beta_r, \gamma)$$

where C is a category, $-\otimes - : C \times C \longrightarrow C$ is a bifunctor, 1_C is a distinguished object of C,

$$\alpha : (-\otimes -) \otimes - \Rightarrow - \otimes (-\otimes -)$$

is a natural isomorphism called the associativity natural transformation, $\beta_l : 1_C \otimes - \Rightarrow id_C$ *and* $\beta_r : - \otimes 1_C \Rightarrow id_C$ *are called the left and right unit natural isomorphisms and finally*

$$\gamma : (- \otimes -) \Rightarrow - \otimes - \circ \tau$$

is the symmetry natural isomorphism. This data is subject to some conditions which are well documented in [Mac71, Sec. VII.1, VII.7]

2.3. Definition. *A symmetric monoidal category* C *is called either a* permutative category *or a* strict *symmetric monoidal category if the natural isomorphisms* α*,* β_l *and* β_r *are the identity natural transformations.*

2.4. Definition. *An* oplax symmetric monoidal *functor* F *is a triple* $(F, \lambda_F, \epsilon_F)$*, where* $F : C \longrightarrow D$ *is a functor between symmetric monoidal categories* C *and* D*,*

$$\lambda_F : F \circ (- \underset{C}{\otimes} -) \Rightarrow (- \underset{D}{\otimes} -) \circ (F \times F)$$

is a natural transformation and $\epsilon_F : F(1_C) \longrightarrow 1_D$ *is a morphism in* D*, such that the following three conditions are satisfied*

OL.1 For each objects $c \in Ob(C)$*, the following diagram commutes*

$$\begin{tikzcd}
F(1_C \underset{C}{\otimes} c) \arrow[r, "\lambda_F(1_C, c)"] \arrow[d, "F(\beta_l^C(c))"'] & F(1_C) \underset{D}{\otimes} F(c) \arrow[d, "\epsilon_F \underset{D}{\otimes} id_{F(c)}"] \\
F(c) \arrow[r, "\beta_l^D(F(c))^{-1}"'] & 1_D \underset{D}{\otimes} F(c)
\end{tikzcd}$$

OL.2 For each pair of objects $c_1, c_2 \in Ob(C)$*, the following diagram commutes*

$$\begin{tikzcd}
F(c_1 \underset{C}{\otimes} c_2) \arrow[r, "\lambda_F(c_1, c_2)"] \arrow[d, "F(\gamma_C(c_1, c_2))"'] & F(c_1) \underset{D}{\otimes} F(c_2) \arrow[d, "\gamma_D(F(c_1), F(c_2))"] \\
F(c_2 \underset{C}{\otimes} c_1) \arrow[r, "\lambda_F(c_2, c_1)"'] & F(c_2) \underset{D}{\otimes} F(c_1)
\end{tikzcd}$$

OL.3 For each triple of objects $c_1, c_2, c_3 \in Ob(C)$, the following diagram commutes

$$\begin{tikzcd}
& F(c_1 \underset{C}{\otimes} c_2) \underset{D}{\otimes} F(c_3) \arrow[dr, "\lambda_F(c_1, c_2) \underset{D}{\otimes} id_{F(c_3)}"] & \\
F((c_1 \underset{C}{\otimes} c_2) \underset{C}{\otimes} c_3) \arrow[ur, "\lambda_F(c_1 \underset{C}{\otimes} c_2, c_3)"] \arrow[d, "F(\alpha_C(c_1, c_2, c_3))"'] & & (F(c_1) \underset{D}{\otimes} F(c_2)) \underset{D}{\otimes} F(c_3) \arrow[d, "\alpha_D(F(c_1), F(c_2), F(c_3))"'] \\
F(c_1 \underset{C}{\otimes} (c_2 \underset{C}{\otimes} c_3)) \arrow[dr, "\lambda_F(c_1, c_2 \underset{C}{\otimes} c_3)"'] & & F(c_1) \underset{D}{\otimes} (F(c_2) \underset{D}{\otimes} F(c_3)) \\
& F(c_1) \underset{D}{\otimes} (F(c_2 \underset{C}{\otimes} c_3)) \arrow[ur, "id_{F(c_1)} \underset{D}{\otimes} \lambda_F(c_2, c_3)"'] &
\end{tikzcd}$$

2.5. Definition. *An* oplax monoidal natural transformation η *between two oplax symmetric monoidal functors $F : C \longrightarrow D$ and $G : C \longrightarrow D$ is a natural transformation $\eta : F \Rightarrow G$ such that for each pair of objects c_1, c_2 of the symmetric monoidal category C, the following two diagrams commute:*

$$\begin{tikzcd}
F(c_1 \underset{C}{\otimes} c_2) \arrow[r, "\eta(c_1 \underset{C}{\otimes} c_2)"] \arrow[d, "\lambda_F(c_1, c_2)"'] & G(c_1 \underset{C}{\otimes} c_2) \arrow[d, "\lambda_G(c_1, c_2)"] \\
F(c_1) \underset{D}{\otimes} F(c_2) \arrow[r, "\eta(c_1) \underset{D}{\otimes} \eta(c_2)"'] & G(c_1) \underset{D}{\otimes} G(c_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(1_C) \arrow[rr, "\eta(1_C)"] \arrow[dr, "\epsilon_F"'] & & G(1_C) \arrow[dl, "\epsilon_G"] \\
& 1_D &
\end{tikzcd}$$

2.6. Notation. *We will say that a functor $F : C \longrightarrow D$ between two symmetric monoidal categories is* unital *or* normalized *if it preserves the unit of the symmetric monoidal structure* i.e. $F(1_C) = 1_D$. *In particular, we will say that an oplax symmetric monoidal functor is a unital (or normalized) oplax symmetric monoidal functor if the morphism ϵ_F is the identity.*

2.7. Proposition. *Let $F : C \longrightarrow D$ be a functor and*

$$\phi = \{\phi(c) : F(c) \xrightarrow{\cong} G(c)\}_{c \in Ob(C)}$$

is a family of isomorphisms in D indexed by the object set of C. Then there exists a unique functor $G : C \longrightarrow D$ such that the family ϕ glues together into a natural isomorphism $\phi : F \Rightarrow G$.

The following lemma is a useful property of unital symmetric monoidal functors:

In this paper we will frequently encounter oplax (and lax) symmetric monoidal functors. In particular we will be dealing with such functors taking values in **Cat**. Let $*$ denote the terminal category.

2.8. Definition. *We define a category* $\mathbf{Cat}_l$ *whose objects are pairs* (C, c)*, where* C *is a category and* $c : * \longrightarrow C$ *is a functor whose value is* $c \in C$*. A morphism from* (C, c) *to* (D, d) *in* $\mathbf{Cat}_l$ *is a pair* (F, α)*, where* $F : C \longrightarrow D$ *is a functor and* $\alpha : F(c) \longrightarrow d$ *is a map in* D*. The category* $\mathbf{Cat}_l$ *is equipped with an obvious projection functor*

$$p_l : \mathbf{Cat}_l \longrightarrow \mathbf{Cat}. \tag{1}$$

We will refer to the functor p_l *as the* universal left fibration over $\mathbf{Cat}$*.*

Let $(F, \alpha) : (C, c) \longrightarrow (D, d)$ and $(G, \beta) : (D, d) \longrightarrow (E, e)$ be a pair of composable arrows in $\mathbf{Cat}_l$. Then their composite is defined as follows:

$$(G, \beta) \circ (F, \alpha) := (G \circ F, \beta \cdot (id_G \circ \alpha)),$$

where $\cdot$ represents vertical composition and $\circ$ represents horizontal composition of 2-arrows in $\mathbf{Cat}$.

2.9. Definition. *The* category of elements *of a* $\mathbf{Cat}$ *valued functor* $F : C \longrightarrow \mathbf{Cat}$*, denoted by* $\int^{c \in C} F(c)$ *or* elF*, is a category which is defined by the following pullback square in* $\mathbf{Cat}$*:*

$$\begin{tikzcd} \int^{c \in C} F(c) \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathbf{Cat}_l \arrow[d, "p_l"] \\ C \arrow[r, "F"'] & \mathbf{Cat} \end{tikzcd}$$

The category $\int^{c \in C} F(c)$ has the following description:

The object set of $\int^{c \in C} F(c)$ consists of all pairs (c, d), where $c \in Ob(C)$ and $d : * \longrightarrow F(c)$ is a functor. A map $\phi : (c_1, d_1) \longrightarrow (c_2, d_2)$ is a pair (f, α), where $f : c_1 \longrightarrow c_2$ is a map in C and $\alpha : F(f) \circ d_1 \Rightarrow d_2$ is a natural transformation. The category of elements of F is equipped with an obvious projection functor $p : \int^{c \in C} F \longrightarrow C$.

2.10. Remark. *We observe that a functor* $d : * \longrightarrow F(c)$ *is the same as an object* $d \in F(c)$*. Similarly a natural transformation* $\alpha : F(f) \circ d \Rightarrow b$ *is the same as an arrow* $\alpha : F(f)(d) \longrightarrow b$ *in* $F(a)$*, where* $f : c \longrightarrow a$ *is an arrow in* C*. This observation leads to a simpler equivalent description of* $\int^{c \in C} F(c)$*. The objects of* $\int^{c \in C} F(c)$ *are pairs* (c, d)*, where* $c \in C$ *and* $d \in F(c)$*. A map from* (c, d) *to* (a, b) *in* $\int^{c \in C} F(c)$ *is a pair* (f, α)*, where* $f : c \longrightarrow a$ *is an arrow in* C *and* $\alpha : F(f)(d) \longrightarrow b$ *is an arrow in* $F(a)$*.*

Next we want to define a symmetric monoidal structure on the category $\int^{c \in C} F(c)$. In order to do so we will use two functors which we now define. The first is the following composite

$$p_1^{\otimes} : \int^{c \in C} F(c) \times \int^{c \in C} F(c) \xrightarrow{p_1 \times p_1} C \times C \xrightarrow{- \underset{C}{\otimes} -} C.$$

The second functor

$$p_2^{\otimes} : \int^{c \in C} F(c) \times \int^{c \in C} F(c) \longrightarrow \mathbf{Cat}_l$$

is defined on objects as follows:

$$p_2^{\otimes}((c_1, d_1), (c_2, d_2)) := d_1 \otimes d_2,$$

where the map on the right is defined by the following composite

$$* \xrightarrow{((d_1,d_2))} F(c_1) \times F(c_2) \xrightarrow{\lambda_F((c_1,c_2))} F(c_1 \underset{C}{\otimes} c_2).$$

Let

$$(f_1, \alpha_1) : (c_1, d_1) \longrightarrow (a_1, b_1) \text{ and } (f_2, \alpha_2) : (c_2, d_2) \longrightarrow (a_2, b_2)$$

be two maps in $\int^{c \in C} F(c)$. The functor is defined on arrows as follows:

$$p_2^{\otimes}((f_1, \alpha_1), (f_2, \alpha_2)) := (F(f_1 \underset{C}{\otimes} f_2), \alpha_1 \otimes \alpha_2),$$

where the second component $\alpha_1 \otimes \alpha_2$ is a natural transformation

$$\alpha_1 \otimes \alpha_2 : F(f_1 \underset{C}{\otimes} f_2) \circ \lambda_F((c_1, c_2)) \circ (d_1, d_2) \Rightarrow \lambda_F((a_1, a_2)) \circ (b_1, b_2).$$

In order to define this natural transformation, consider the following diagram:

$$\begin{tikzcd}
 & & F(c_1 \underset{C}{\otimes} c_2) \arrow[ddrr, "F(f_1 \underset{C}{\otimes} f_2)"] & & \\
 & F(c_1) \times F(c_2) \arrow[ur, "\lambda_F((c_1,c_2))"] \arrow[dr, "F(f_1) \times F(f_2)"] \arrow[d, phantom, "\Downarrow (\alpha_1,\alpha_2)"] & & & \\
* \arrow[ur, "((d_1,d_2))"] \arrow[rr, "((b_1,b_2))"'] & {} & F(a_1) \times F(a_2) \arrow[rr, "\lambda_F((a_1,a_2))"'] & & F(a_1 \underset{C}{\otimes} a_2)
\end{tikzcd}$$

Now we define

$$\alpha_1 \otimes \alpha_2 := id_{\lambda_F((a_1,a_2))} \circ (\alpha_1, \alpha_2).$$

The arrow $\alpha_1 \otimes \alpha_2(*)$ has domain

$$\lambda_F((a_1, a_2))(F(f_1)(d_1(*)), F(f_2)(d_2(*))) \in F(a_1 \underset{C}{\otimes} a_2).$$

The following diagram

$$\begin{tikzcd}
 & * \arrow[dl, "((d_1,d_2))"'] \arrow[dr, "((b_1,b_2))"] & \\
F(c_1) \times F(c_2) \arrow[rr, "F(f_1) \times F(f_2)"] \arrow[d, "\lambda_F((c_1,c_2))"'] & & F(a_1) \times F(a_2) \arrow[d, "\lambda_F((a_1,a_2))"] \\
F(c_1 \underset{C}{\otimes} c_2) \arrow[rr, "F(f_1 \underset{C}{\otimes} f_2)"'] & & F(a_1 \underset{C}{\otimes} a_2)
\end{tikzcd}$$

shows that

$$F(f_1 \underset{C}{\otimes} f_2)(\lambda_F((c_1, c_2))(d_1(*), d_2(*))) = \lambda_F(a_1, a_2)(F(f_1)(d_1(*)), F(f_2)(d_2(*))).$$

Now we have to verify that $p_2^{\otimes}$ is a bifunctor. Let $(g_1, \beta_1) : (a_1, b_1) \longrightarrow (x_1, z_1)$ and $(g_2, \beta_2) : (a_2, b_2) \longrightarrow (x_2, z_2)$ be another pair of maps in $\int^{c \in C} F(c)$. The following diagram will be useful in establishing the desired bifunctorality:

$$\begin{tikzcd}
 & & F(c_1 \underset{C}{\otimes} c_2) \arrow[ddr, "F(f_1 \underset{C}{\otimes} f_2)"] & & \\
 & F(c_1) \times F(c_2) \arrow[ur, "\lambda_F((c_1,c_2))"] \arrow[dr, "F(f_1)\times F(f_2)"] \arrow[d, phantom, "\Downarrow (\alpha_1,\alpha_2)"] & & & \\
* \arrow[ur, "(d_1,d_2)"] \arrow[rr, "(b_1,b_2)"'] \arrow[drrr, bend right, "(z_1,z_2)"'] & & F(a_1) \times F(a_2) \arrow[r, "\lambda_F((a_1,a_2))"'] \arrow[dr, "F(g_1)\times F(g_2)"] \arrow[d, phantom, "\Downarrow (\beta_1,\beta_2)"] & F(a_1 \underset{C}{\otimes} a_2) \arrow[dr, "F(g_1 \underset{C}{\otimes} g_2)"] & \\
 & & & F(x_1) \times F(x_2) \arrow[r, "\lambda_F((x_1,x_2))"'] & F(x_1 \underset{C}{\otimes} x_2)
\end{tikzcd}$$

Now consider the following chain of equalities:

$$\begin{aligned}
&p_2^{\otimes}((g_1, \beta_1), (g_2, \beta_2)) \circ p_2^{\otimes}((f_1, \alpha_1), (f_2, \alpha_2)) = \\
&\quad ((F(g_1 \underset{C}{\otimes} g_2), id_{\lambda_F((x_1,x_2))} \circ (\beta_1, \beta_2)) \circ ((F(f_1 \underset{C}{\otimes} f_2), id_{\lambda_F((a_1,a_2))} \circ (\alpha_1, \alpha_2)) = \\
&\quad (F((g_1 \underset{C}{\otimes} g_2) \circ (f_1 \underset{C}{\otimes} f_2)), (id_{\lambda_F((x_1,x_2))} \circ (\beta_1, \beta_2)) \cdot (id_{F(g_1 \underset{C}{\otimes} g_2)} \circ (\alpha_1, \alpha_2))) = \\
&(F((g_1 \underset{C}{\otimes} g_2) \circ (f_1 \underset{C}{\otimes} f_2)), (id_{\lambda_F((x_1,x_2))} \circ (\beta_1, \beta_2)) \cdot (id_{\lambda_F((x_1,x_2))} \circ (id_{F(g_1) \times F(g_2))} \circ (\alpha_1, \alpha_2))) = \\
&\quad (F(g_1 f_1 \underset{C}{\otimes} g_2 f_2), id_{\lambda_F((x_1,x_2))} \circ ((\beta_1, \beta_2) \cdot (id_{F(g_1) \times F(g_2))} \circ (\alpha_1, \alpha_2))) = \\
&\quad (F(g_1 f_1 \underset{C}{\otimes} g_2 f_2), id_{\lambda_F((x_1,x_2))} \circ ((\beta_1 \cdot (id_{F(g_1)} \circ \alpha_1), \beta_2 \cdot (id_{F(g_2)} \circ \alpha_1))) = \\
&\qquad\qquad p_2^{\otimes}((g_1, \beta_1) \circ (f_1, \alpha_1)), ((g_2, \beta_2) \circ (f_2, \alpha_2)).
\end{aligned}$$

The above chain of equalities prove that $p_2^{\otimes}$ is a bifunctor. The definitions of the functors $p_1^{\otimes}$ and $p_2^{\otimes}$ imply that the outer rectangle in the following diagram is commutative:

$$\begin{tikzcd}
\int^{c \in C} F(c) \times \int^{c \in C} F(c) \arrow[rr, "p_2^{\otimes}"] \arrow[dr, dashed, "-\boxtimes-"'] \arrow[dd, "p_1^{\otimes}"'] & & \mathbf{Cat}_l \arrow[dd, "p_l"] \\
 & \int^{c \in C} F(c) \arrow[ur] \arrow[dl] & \\
C \arrow[rr, "F"'] & & \mathbf{Cat}
\end{tikzcd}$$

Since $\int^{c\in C} F(c)$ is a pullback of p_l along F, therefore there exists a bifunctor

$$- \boxtimes - : \int^{c\in C} F(c) \times \int^{c\in C} F(c) \longrightarrow \int^{c\in C} F(c) \tag{2}$$

which makes the entire diagram commutative. We describe this bifunctor next. Let $((c_1, d_1), (c_2, d_2))$ be an object in $\int^{c\in C} F \times \int^{c\in C} F(c)$.

$$(c_1, d_1) \boxtimes (c_2, d_2) := (c_1 \underset{C}{\otimes} c_2, \lambda_F(c_1, c_2) \circ ((d_1, d_2))).$$

Let $(f_1, \alpha_1) : (c_1, d_1) \longrightarrow (a_1, b_1)$ and $(f_2, \alpha_2) : (c_2, d_2) \longrightarrow (a_2, b_2)$ be two maps in $\int^{c\in C} F(c)$.

$$(f_1, \alpha_1) \boxtimes (f_2, \alpha_2) := (f_1 \underset{C}{\otimes} f_2, id_{\lambda_F(a_1,a_2)} \circ (\alpha_1, \alpha_2)).$$

2.11. Theorem. *The category of elements of a* **Cat** *valued lax symmetric monoidal functor whose domain is a permutative category is a permutative category.*

Proof. Let $(F, \lambda_F) : C \longrightarrow \mathbf{Cat}$ be a lax symmetric monoidal functor. We begin by defining the symmetry natural isomorphism $\gamma_{\int^{c\in C} F(c)}$. Let $(c_1, d_1), (c_2, d_2)$ be a pair of objects in $\int^{c\in C} F$. We define

$$\gamma_{\int^{c\in C} F}(((c_1, d_1), (c_2, d_2))) := (\gamma_C(c_1, c_2), id).$$

The second component is identity because the lax symmetric monoidal structure of F implies that the following diagram commutes:

$$\begin{tikzcd}
* \arrow[dr, "((d_1,d_2))"] \arrow[drr, bend left, "((d_2,d_1))"] \arrow[ddr, bend right] & & \\
& F(c_1) \times F(c_2) \arrow[r, "\tau"] \arrow[d, "\lambda_F((c_1,c_2))"] & F(c_2) \times F(c_1) \arrow[d, "\lambda_F((c_2,c_1))"] \\
& F(c_1 \underset{C}{\otimes} c_2) \arrow[r, "\gamma_C((c_1,c_2))"'] & F(c_2 \underset{C}{\otimes} c_1)
\end{tikzcd}$$

It is easy to see that this defines a natural isomorphism. We claim that the proposed symmetric monoidal structure on $\int^{c\in C} F(c)$ is strictly associative. Given a third object (c_3, d_3) in $\int^{c\in C} F(c)$, we observe that

$$((c_1, d_1) \boxtimes (c_2, d_2)) \boxtimes (c_3, d_3) = \\ (c_1 \underset{C}{\otimes} c_2 \underset{C}{\otimes} c_3, (\lambda_F((c_1 \underset{C}{\otimes} c_2, c_3)) \circ (\lambda_{F((c_1,c_2))} \times id) \circ ((d_1, d_2), d_3)).$$

The following diagram, which is the lax version of (OL.3) for F,

$$\begin{tikzcd}
\ast \arrow[dr, "{((d_1,d_2),d_3)}"] \arrow[dddr, bend right, "{(d_1,(d_2,d_3))}"'] & & \\
& (F(c_1) \times F(c_2)) \times F(c_3) \arrow[r, "\lambda_{F((c_1,c_2))} \times id"] \arrow[dd, "\alpha_{F(c_1),F(c_2),F(c_3)}"] & F(c_1 \underset{C}{\otimes} c_2) \times F(c_3) \arrow[d, "\lambda_F((c_1 \underset{C}{\otimes} c_2, c_3))"] \\
& & F(c_1 \underset{C}{\otimes} c_2 \underset{C}{\otimes} c_3) \\
& F(c_1) \times (F(c_2) \times F(c_3)) \arrow[r, "id \times \lambda_{F((c_2,c_3))}"'] & F(c_1) \times F(c_2 \underset{C}{\otimes} c_3) \arrow[u, "\lambda_F(c_1, c_2 \underset{C}{\otimes} c_3)"']
\end{tikzcd}$$

tells us that

$$\begin{aligned}
&((c_1 \underset{C}{\otimes} c_2 \underset{C}{\otimes} c_3, \lambda_F((c_1 \underset{C}{\otimes} c_2, c_3) \circ (\lambda_{F((c_1,c_2))} \times id) \circ ((d_1,d_2),d_3)) = \\
&\qquad (c_1 \underset{C}{\otimes} c_2 \underset{C}{\otimes} c_3, \lambda_F((c_1, c_2 \underset{C}{\otimes} c_3) \circ (id \times \lambda_{F((c_2,c_3))})) \circ (d_1,(d_2,d_3)) = \\
&\qquad\qquad (c_1, d_1) \boxtimes ((c_2, d_2) \boxtimes (c_3, d_3).
\end{aligned}$$

Thus we have proved that the symmetric monoidal functor is strictly associative. It is easy to see that the symmetry isomorphism $\gamma_{\int^{c \in C} F(c)}$ satisfies the *hexagon diagram* because C is a permutative category by assumption. Thus we have proved that $\int^{c \in C} F(c)$ is a permutative category.

■

2.12. Review of Γ- categories. In this subsection we will briefly review the theory of Γ- categories. We begin by introducing some notations which will be used throughout the paper.

2.13. Notation. *We will denote by* $\underline{n}$ *the finite set* $\{1, 2, \ldots, n\}$ *and by* n^+ *the based set* $\{0, 1, 2, \ldots, n\}$ *whose basepoint is the element* 0.

2.14. Notation. *We will denote by* $\mathcal{N}$ *the skeletal category of finite unbased sets whose objects are* $\underline{n}$ *for all* $n \geq 0$ *and maps are functions of unbased sets. The category* $\mathcal{N}$ *is a (strict) symmetric monoidal category whose symmetric monoidal structure will be denoted by* $+$. *For to objects* $\underline{k}, \underline{l} \in \mathcal{N}$ *their* tensor product *is defined as follows:*

$$\underline{k} + \underline{l} := \underline{k + l}.$$

2.15. Notation. *We will denote by* Γ^{op} *the skeletal category of finite based sets whose objects are* n^+ *for all* $n \geq 0$ *and maps are functions of based sets.*

2.16. Definition. *A map $f : n^+ \longrightarrow m^+$ in Γ^{op} is called* inert *if its restriction to the set $\underline{n} -$ Supp(f) is a bijection, where Supp$(f) \subseteq \underline{n}$ is the support of f.*

2.17. Definition. *A morphism f in Γ^{op} is called* active *if $f^{-1}(\{0\}) = \{0\}$* i.e. *the pre-image of $\{0\}$ is the singleton set $\{0\}$.*

2.18. Notation. *We denote by* **Inrt** *the subcategory of Γ^{op} having the same set of objects as Γ^{op} and inert morphisms.*

2.19. Notation. *We denote by* **Act** *the subcategory of Γ^{op} having the same set of objects as Γ^{op} and active morphisms.*

2.20. Notation. *A map $f : \underline{n} \longrightarrow \underline{m}$ in the category $\mathcal{N}$ uniquely determines an active map in Γ^{op} which we will denote by $f^+ : n^+ \longrightarrow m^+$. This map agrees with f on non-zero elements of n^+.*

2.21. Notation. *Given a morphism $f : n^+ \longrightarrow m^+$ in Γ^{op}, we denote by Supp(f) the largest subset of $\underline{n}$ whose image under f does not contain the basepoint of m^+. The set Supp(f) inherits an order from $\underline{n}$ and therefore could be regarded as an object of $\mathcal{N}$. We denote by Supp$(f)^+$ the based set Supp$(f) \sqcup \{0\}$ regarded as an object of Γ^{op} with order inherited from $\underline{n}$.*

2.22. Proposition. *Each morphism in Γ^{op} can be uniquely factored into a composite of an inert map followed by an active map in Γ^{op}.*

Proof. Any map $f : n^+ \longrightarrow m^+$ in the category Γ^{op} can be factored as follows:

$$\begin{tikzcd} n^+ \arrow[rr, "f"] \arrow[dr, "f_{inrt}"'] & & m^+ \\ & \mathrm{Supp}(f)^+ \arrow[ur, "f_{act}"'] & \end{tikzcd} \tag{3}$$

where Supp$(f) \subseteq n$ is the *support* of the function f *i.e.* Supp(f) is the largest subset of n whose elements are mapped by f to a non zero element of m^+. The map f_{inrt} is the projection of n^+ onto the support of f and therefore f_{inrt} is an inert map. The map f_{act} is the restriction of f to Supp$(f) \subset \underline{n}$, therefore it is an *active* map in Γ^{op}.

■

The next lemma is special case of [Lei00, Prop. 3.1.1]

2.23. Lemma. *There is an isomorphism of categories between the category of oplax symmetric monoidal functors and oplax monoidal natural transformations $[\mathcal{N}, \mathbf{Cat}]^{OL}_{\otimes}$ and the category $\Gamma\mathbf{Cat}$.*

2.24. Natural model category structure on **Cat**. In this subsection we will review the *natural model category structure* on the category of all small categories **Cat**. The model category structures recalled this section have been published in [JT91]. The original content of this section is the explicit description of the generating cofibrations and a characterization of equivalence of categories. The weak equivalences in this model structure are *equivalences of categories*. We begin by reviewing this notion:

2.25. Definition. *A functor $F : C \longrightarrow D$ is called an equivalence of categories if there exists another functor $G : D \longrightarrow C$ and two natural isomorphisms*

$$\epsilon : FG \cong id_D \qquad and \qquad \eta : id_C \cong GF$$

The inclusion of the category of all (small) groupoids **Gpd** into **Cat** has a right adjoint which we denote by $J : \mathbf{Cat} \longrightarrow \mathbf{Gpd}$. For any category C, $J(C)$ is the largest groupoid contained in C. The following charaterization of an equivalence of categories will be useful throughout the paper:

2.26. Lemma. *A functor $F : C \longrightarrow D$ is an equivalence of categories if and only if the following two induced functor are equivalences of groupoids:*

$$J(F) : J(C) \longrightarrow J(D) \qquad and \qquad J([I, F]) : J([I, C]) \longrightarrow J([I, D])$$

Proof. ($\Rightarrow$) It is easy to see that J preserves equivalences of categories *i.e.* if $F : C \longrightarrow D$ is an equivalence of categories then $J(F) : J(C) \longrightarrow J(D)$ is also an equivalence. An equivalence of categories induces an equivalence on its category of arrows, thus the functor $J([I; F])$ is also an equivalence of categories.

($\Leftarrow$) Let us assume that the two conditions hold. Let $f : d \longrightarrow e$ be an arrow in D such that the domain and codomain objects d and e respectively are in the image of F. By assumption the functor $J([I, F])$ is essentially surjective therefore there exists an arrow $g : a \longrightarrow b$ in C such that the following diagram commutes:

$$\begin{tikzcd} F(a) \arrow[r, "\epsilon(d)"] \arrow[d, "F(g)"'] & d \arrow[d, "f"] \\ F(b) \arrow[r, "\epsilon(e)"'] & e \end{tikzcd}$$

where the pair $(\epsilon(d), \epsilon(e))$ is an isomorphism in the arrow category $[I; D]$. By the assumption that $J(F)$ is an equivalence of categories, there exist two unique (invertible) arrows h and k such that $F(h) = \epsilon(d)$ and $F(k) = \epsilon(e)$. This implies that there exists a unique arrow $k^{-1} \circ g \circ h$ such that $f = F(k \circ g \circ h^{-1})$. Thus we have proved that F is fully-faithful. Let p be an object of D which is NOT in the image of F then the assumption of equivalence of $J([I, F])$ guarantees the existence of an isomorphism $m : x \longrightarrow y$ in C such that the following diagram commutes:

$$\begin{tikzcd} F(x) \arrow[r, "\epsilon(d)"] \arrow[d, "F(m)"'] & d \arrow[d, equal, "id_d"] \\ F(y) \arrow[r, "\epsilon(e)"'] & d \end{tikzcd}$$

Thus we have shown that F is essentially surjective. ■

A significant part of this section will be devoted to review properties of fibrations in this model structure, namely *isofibrations*, which we now define:

2.27. Definition. *If C and D are categories, we shall say that a functor $F : C \longrightarrow D$ is an* isofibration *if for every object $c \in C$ and every isomorphism $v \in Mor(\mathcal{D})$ with source $F(c)$, there exists an isomorphism $u \in C$ with source c such that $F(u) = v$.*

2.28. Notation. *Let J be the groupoid generated by one isomorphism $0 \cong 1$. We shall denote the inclusion $\{0\} \subset J$ as a map $d_1 : 0 \longrightarrow J$ and the inclusion $\{1\} \subset J$ by the map $d_0 : 1 \longrightarrow J$.*

2.29. Notation. *Let A and B be two small categories, we will denote by $[A, B]$, the category of all functors from A to B and natural transformations between them.*

The next proposition and the lemma following it provide characterizations of isofibrations and acyclic isofibrations. These follow from the above definition and [Mac71, Ch. IV.4]

2.30. Proposition. *A functor $F : C \longrightarrow D$ is an isofibration if and only if it has the right lifting property with respect to the inclusion $i_0 : 0 \hookrightarrow J$ and therefore also with the inclusion $i_1 : 1 \hookrightarrow J$.*

We want to present a charaterization of *acyclic isofibrations*, *i.e.* those functors of categories which are both an isofibration and an equivalence of categories, similar to the characterization of isofibrations given by proposition 2.30. The following property of acyclic isofibrations will be useful in achieving this goal:

2.31. Lemma. *An equivalence of categories is an isofibration iff it is surjective on objects.*

2.32. Notation. *We will denote the category $0 \longrightarrow 1$ either by I or by $[1]$. We will denote the discrete category $\{0, 1\}$ either by ∂I or $\partial[1]$. We will denote the category $0 \xrightarrow{f_{01}} 1 \xrightarrow{f_{12}} 2$ by $[2]$.*

Now we define a category $\partial[2]$ which has the same object set as the category $[2]$, namely $\{0, 1, 2\}$. The Hom sets of this category are defined as follows:

$$Hom_{\partial[2]}(i, j) = \begin{cases} \{f_{01}\}, & \text{if } i = 0 \text{ and } j = 1 \\ \{f_{12}\}, & \text{if } i = 1 \text{ and } j = 2 \\ \{f_{02}, f_{12} \circ f_{01}\}, & \text{if } i = 0 \text{ and } j = 2 \\ \{id\}, & \text{otherwise.} \end{cases}$$

We have the following functor

$$\partial_2 : \partial[2] \hookrightarrow [2]$$

which is identity on objects. This functor sends the morphism f_{01}(resp. f_{12}) to the morphism $0 \longrightarrow 1$(resp. $1 \longrightarrow 2$) in the category [2]. Both morphisms $f_{02}, f_{12} \circ f_{01}$ are mapped to the composite morphism $0 \xrightarrow{f_{01}} 1 \xrightarrow{f_{12}} 2$. Similarly we have the map $\partial_1 : \partial[1] \longrightarrow [1]$ which is identity on objects. We have a third functor $\partial_0 : \emptyset \longrightarrow [0]$ which is obtained by the unique function $\emptyset \longrightarrow \{0\}$. We will refer to these three functors as the *boundary maps*.

2.33. Proposition. *A functor $F : C \longrightarrow D$ is an isofibration and an equivalence of categories if and only if it has the right lifting property with respect to the three boundary maps ∂_0, ∂_1 and ∂_2.*

Proof. Let us first assume that F is an isofibration as well as an equivalence of categories. Now Lemma 2.31 says that F is surjective on objects which is equivalent to F having the right lifting property with respect to the boundary map ∂_0. Now we observe that for any pair of objects $d, d' \in Ob(D)$, there exists a pair of objects $c, c' \in Ob(C)$ such that $F(c) = d$ and $F(c') = d'$ and the morphism

$$F_{c,c'} : Hom_C(c, c') \longrightarrow Hom_D(d, d')$$

is a bijection. This implies that F has the right lifting property with respect to the morphism ∂_1. Whenever we have the following (outer) commutative diagram

$$\begin{tikzcd} \partial[2] \arrow[r, "K"] \arrow[d, "\partial_2"'] & C \arrow[d, "F"] \\ {[2]} \arrow[r] \arrow[ur, dashed, "L"] & D \end{tikzcd}$$

we have the following equality

$$F(K(f_{02})) = F(K(f_{12})) \circ F(K(f_{01})),$$

where the maps f_{02}, f_{12} and f_{01} are defined above. The morphism

$$F_{K(0),K(2)} : Hom_C(K(0), K(2)) \longrightarrow Hom_D(F(K(0)), F(K(2)))$$

is a bijection, this implies that the morphism $K(f_{02}) : K(0) \longrightarrow K(2)$ is the same as the composite morphism $K(f_{12}) \circ K(f_{01}) : K(0) \longrightarrow K(2)$ Now we are ready to define the lifting (dotted) arrow L. We define the object function of the functor L to be the same as that of the functor K, *i.e.* $L_{Ob} = K_{Ob}$. We define $L(f_{01}) = K(f_{01})$ and $L(f_{12}) = K(f_{12})$. Now the discussion above implies that this definition makes the entire diagram commute.

Conversely, let us assume that the morphism F has the right lifting property with respect to the three boundary maps. The morphism F having the right lifting property with respect to ∂_0 is equivalent to F being surjective on objects. Now the right lifting property with respect to ∂_1 implies that for any map $g : d \longrightarrow d'$ in the category D, there exists a map $w : c \longrightarrow c'$ in C, such that $F(w) = g$, for each pair of objects $c, c' \in Ob(C)$ such that $F(c) = d$ and $F(c') = d'$. Let $c \in Ob(C)$ and $v : F(c) \longrightarrow d$ be an isomorphism in D. Now we can define a functor $A : [2] \longrightarrow D$, on objects by $A(0) = A(2) = F(c)$, $A(1) = d$ and on morphisms by $A(f_{01}) = v$ and $A(f_{12}) = v^{-1}$. As mentioned earlier, the right lifting property with respect to ∂_1 implies that there exist two maps $u : c \longrightarrow c'$ and $r : c' \longrightarrow c$ such that $F(u) = v$ and $F(r) = v^{-1}$. This allows us to define a functor $K : \partial[2] \longrightarrow C$, on objects by $K(0) = K(2) = c$ and $K(1) = c'$ and on morphisms by $K(f_{01}) = u$, $K(f_{12}) = r$ and $K(f_{02}) = r \circ u$. This definition gives us the following (outer) commutative diagram

$$\begin{tikzcd} \partial[2] \arrow[r, "K"] \arrow[d, "\partial_2"'] & C \arrow[d, "F"] \\ {[2]} \arrow[r, "A"'] \arrow[ur, dashed, "L"] & D \end{tikzcd}$$

Our assumption of right lifting property with respect to ∂_2 gives us a lift (dotted arrow) L which makes the entire diagram commute. This implies the $r \circ u = id_c$. A similar argument will show that $u \circ r = id_{c'}$. Thus we have shown that F is an isofibration which is surjective on objects. Lemma 2.31 says that F is both an equivalence of categories and an isofibration.

■

2.34. Definition. *We shall say that a functor $F : C \longrightarrow D$ is* monic (resp. surjective, bijective) on objects *if the object function of F, $F_{Ob} : Ob(C) \longrightarrow Ob(D)$, is injective (resp. surjective, bijective).*

2.35. Theorem. [Joy08] *There is a combinatorial model category structure on the category of all small categories* **Cat** *in which*

1. *A cofibration is a functor which is monic on objects.*
2. *A fibration is an isofibration and*
3. *A weak-equivalence is an equivalence of categories.*

Further, this model category structure is cartesian closed and proper. We will call this model category structure as the natural model category structure on **Cat**.

2.36. Notation. *We will denote by* 0 *the terminal category having one object* 0 *and just the identity map.*

2.37. Definition. *A small* pointed *category is a pair (C, ϕ) consisting of a small category C and a functor $0 \longrightarrow C$. A* basepoint preserving *functor between two pointed* *categories* (C, ϕ) *and* (D, ψ) *is a functor* $F : C \longrightarrow D$ *such that the following diagram commutes*

$$\begin{tikzcd} & 0 \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ C \arrow[rr, "F"'] & & D \end{tikzcd}$$

Every model category uniquely determines a model category structure on that category of its pointed objects, see [JT08, Proposition 4.1.1]. Thus we have the following theorem:

2.38. Theorem. *There is a combinatorial* **Cat**-*model category structure on the category of all pointed small categories and basepoint preserving functors* $\mathbf{Cat}_\bullet$ *in which*

1. *A cofibration is a basepoint preserving functor which is monic on objects.*
2. *A fibration is a basepoint preserving functor which is also an isofibration of (unbased) categories and*
3. *A weak-equivalence is a basepoint preserving functor which is also equivalence of (unbased) categories.*

We will call this model category structure as the natural model category structure on $\mathbf{Cat}_\bullet$.

Let C^+ denote the category $C \coprod *$ *i.e.* the category having two connected components C and the terminal category $*$. We will consider C^+ as a pointed category having basepoint $*$. The generating cofibrations and acyclic cofibrations in $\mathbf{Cat}_\bullet$ are obtained by adding an external basepoint to corresponding maps in **Cat**. The category $\mathbf{Cat}_\bullet$ is locally presentable follows from results in [AR94].

2.39. Leinster construction. In this section we will construct a permutative category which would help us in constructing the desired left adjoint to the Segal's Nerve functor. We will refer to this category as the *Leinster category* and we will denote it by $\mathfrak{L}$. The defining property of this permutative category is that for each permutative category P we get a following bijection of mapping sets:

$$\mathbf{Perm}(\mathfrak{L}, P) \cong \mathbf{OLSM}(\mathcal{N}, P)$$

where the mapping set $\mathbf{OLSM}(\mathcal{N}, P)$ is the set of all oplax symmetric monoidal functors from $\mathcal{N}$ to P. The existence of this category is ensured by [BKP89, Thm. 3.13][GJO17, Thm. 2.8]. An object in $\mathfrak{L}$ is an order preserving morphism of the category $\mathcal{N}$ namely an order preserving map of (finite) unbased sets $\vec{k} : \underline{k} \longrightarrow \underline{r}$. For another object $\vec{m} : \underline{m} \longrightarrow \underline{s}$ in $\mathfrak{L}$, a morphism between $\vec{k}$ and $\vec{m}$ is a pair (h, ϕ), where $h : \underline{s} \longrightarrow \underline{r}$ and $\phi : \underline{k} \longrightarrow \underline{m}$ are morphisms in $\mathcal{N}$ such that the following diagram commutes:

$$\begin{tikzcd} \underline{k} \arrow[r, "\phi"] \arrow[d, "\vec{k}"'] & \underline{m} \arrow[d, "\vec{m}"] \\ \underline{r} & \underline{s} \arrow[l, "h"] \end{tikzcd}$$

2.40. Notation. *For an object* $\vec{m} : \underline{m} \longrightarrow \underline{s}$ *in* $\mathfrak{L}$ *we will refer to the natural number* s *as the* length of $\vec{m}$.

2.41. Remark. *An object of* $\mathfrak{L}$, $\vec{m} : \underline{m} \longrightarrow \underline{s}$, *should be viewed as a finite sequence of objects of* $\mathcal{N}$ *namely* $(m_1, m_2, \dots, m_s)$ *for* $s > 0$, *with* $s = 0$ *corresponding to the empty sequence* $()$, *where* $m_i = \vec{m}^{-1}(i)$, *for* $1 \leq i \leq s$.

2.42. Remark. *An object* $\vec{m} : \underline{m} \longrightarrow \underline{s}$ *does not have to be a surjective map. In other words the corresponding sequence* $\vec{m} = (m_1, \dots, m_s)$ *can have components which are empty sets.*

2.43. Remark. *Let* $\vec{n}$ *and* $\vec{m}$ *be two objects in* $\mathfrak{L}$. *A morphism* $(h, \phi) : \vec{n} \longrightarrow \vec{m}$, *in* $\mathfrak{L}$, *should be viewed as a family of morphisms*

$$\phi(i) = n_i \longrightarrow \underset{h(j)=i}{+} m_j$$

for $1 \leq i \leq s$, *where* $+$ *represents the symmetric monoidal structure on* $\mathcal{N}$.

We want to recall from [Shaon] or appendix E how each Γ- category X can be extended to a symmetric monoidal functor $\mathfrak{L}(X) : \mathfrak{L} \longrightarrow \mathbf{Cat}$. This functor is defined on objects as follows:

$$\mathfrak{L}(X)(\vec{m}) := X(m_1^+) \times X(m_2^+) \times \cdots \times X(m_r^+)$$

where $() \neq \vec{m} = (m_1, m_2, \dots, m_r)$ is an object of $\mathfrak{L}$. $\mathfrak{L}(X)(()) = *$. For each map $F = (f, \phi) : \vec{m} \longrightarrow \vec{n}$ in $\mathfrak{L}$ we want to define a functor

$$\mathfrak{L}(X)(F) : \mathfrak{L}(X)(\vec{m}) \longrightarrow \mathfrak{L}(X)(\vec{n}).$$

Each map $\phi(i)$ in the family ϕ provides us with a composite functor

$$X(m_i^+) \xrightarrow{X(\phi(i))} X(\underset{f(j)=i}{+} n_j) \xrightarrow{K_i} \prod_{f(j)=i} X(n_j),$$

where $K_i = (X(\delta_{n_{j_1}}^{\underset{f(j)=i}{+} n_j}), \dots, X(\delta_{n_{j_r}}^{\underset{f(j)=i}{+} n_j}))$. For each pair n-fold product functor in $\mathbf{Cat}$, there is a canonical natural isomorphism between them which we denote by *can*. This gives us the following composite functor

$$\prod_{i=1}^{|\vec{m}|} X(m_i^+) \xrightarrow{\prod_{i=1}^{|\vec{m}|} X(\phi(i))} X(\underset{f(j)=i}{+} n_j) \xrightarrow{\prod_{i=1}^{|\vec{m}|} K_i} \prod_{i=1}^{|\vec{m}|} \prod_{f(j)=i} X(n_j) \xrightarrow{can} \prod_{k=1}^{|\vec{n}|} n_k$$

which is the definition of $\mathfrak{L}(X)(F)$. In other words

$$\mathfrak{L}(X)(F) := can \circ \prod_{i=1}^{|\vec{m}|} K_i \circ \prod_{i=1}^{|\vec{m}|} X(\phi(i))$$

2.44. PROPOSITION. *Let X be a* Γ- *category, there exists an extension of X to $\mathfrak{L}$, $\mathfrak{L}(X)$: $\mathfrak{L} \longrightarrow$* **Cat** *which is a symmetric monoidal functor.*

2.45. REMARK. *The symmetric monoidal extension described above is functorial in X. In other words we get a functor*

$$\mathfrak{L}(-) : \Gamma\mathbf{Cat} \longrightarrow [\mathfrak{L}, \mathbf{Cat}]_{\otimes} \tag{4}$$

2.46. DEFINITION. *For a* Γ- *category X we define*

$$\overline{\overline{\mathcal{L}}}(X) := \int^{\vec{n} \in \mathfrak{L}} \mathfrak{L}(X)(\vec{n}).$$

i.e. *the* Grothendieck construction *of* $\mathfrak{L}(X)$.

More concretely, an object in the category $\overline{\overline{\mathcal{L}}}(X)$ is a pair $(\vec{m}, \vec{x})$ where $\vec{m} : \underline{m} \longrightarrow \underline{s} \in Ob(\mathfrak{L})$ and

$$\vec{x} = (x_1, x_2, \ldots, x_s) \in Ob(X(m_1^+) \times X(m_2^+) \times \cdots \times X(m_s^+)).$$

A morphism from $(\vec{m}, \vec{x})$ to $(\vec{n}, \vec{y})$ in $\overline{\overline{\mathcal{L}}}(X)$ is a pair $((h, \phi), F)$ where $(h, \phi) : \vec{m} \longrightarrow \vec{n}$ is a map in $\mathfrak{L}$ and $F : \mathfrak{L}(X)((h, \phi))(\vec{x}) \longrightarrow \vec{y}$ is a map in the product category $X(n_1^+) \times X(n_2^+) \times \cdots \times X(n_r^+)$.

Now we define a *tensor product* on the category $\overline{\overline{\mathcal{L}}}(X)$. Let $(\vec{n}, \vec{x})$ and $(\vec{m}, \vec{y})$ be two objects of $\overline{\overline{\mathcal{L}}}(X)$, we define another object $(\vec{n}, \vec{x}) \underset{\overline{\overline{\mathcal{L}}}(X)}{\otimes} (\vec{m}, \vec{y})$ as follows:

$$(\vec{n}, \vec{x}) \underset{\overline{\overline{\mathcal{L}}}(X)}{\otimes} (\vec{m}, \vec{y}) := (\vec{n} \Box \vec{m}, \lambda_{\mathfrak{L}(X)}(\vec{n}, \vec{m})^{-1}((\vec{x}, \vec{y}))). \tag{5}$$

For a pair of morphisms $((h_1, \alpha), a) : (\vec{n}, \vec{x}) \longrightarrow (\vec{k}, \vec{s})$ and $((h_2, \beta), b) : (\vec{m}, \vec{y}) \longrightarrow (\vec{l}, \vec{t})$, in $\overline{\overline{\mathcal{L}}}(X)$, we define another morphism in $\overline{\overline{\mathcal{L}}}(X)$ as follows:

$$((h_1, \alpha), a) \underset{\overline{\overline{\mathcal{L}}}(X)}{\otimes} ((h_2, \beta), b) := ((h_1, \alpha) \Box (h_2, \beta), \lambda_{AX}((h_1, \alpha), (h_2, \beta))^{-1}((a, b))), \tag{6}$$

where $\lambda_{\mathfrak{L}(X)}((h_1, \alpha), (h_2, \beta))$ is the composite functor

$$(\mathfrak{L}(X)((h_1, \alpha)) \times \mathfrak{L}(X)((h_2, \beta))) \circ \lambda_{\mathfrak{L}(X)}(\vec{n}, \vec{m}) = \lambda_{\mathfrak{L}(X)}(\vec{k}, \vec{l}) \circ \mathfrak{L}(X)((h_1, \alpha) \Box (h_2, \beta)).$$

2.47. PROPOSITION. *The category $\overline{\overline{\mathcal{L}}}(X)$ is a permutative category with respect to the tensor product defined above.*

PROOF. The category $\mathfrak{L}$ is a permutative category. Now the proposition follows from theorem 2.11. ∎

2.48. Gabriel Factorization. In analogy with the way a functor can be factored as a fully faithful functor followed by an essentially surjective one, every strict symmetric monoidal $\Phi : E \longrightarrow F$ admits a factorization of the form

$$\begin{tikzcd} E \arrow[rr, "\Phi"] \arrow[dr, "\Gamma"'] & & F \\ & G \arrow[ur, "\Delta"'] & \end{tikzcd}$$

where Γ is essentially surjective and Δ is fully faithful. In fact we may suppose that Γ is identity on objects in which case we get the *Gabriel factorization* of Φ. In order to obtain a Gabriel factorization we define the symmetric monoidal category G as having the same objects as E and letting, for $c, d \in Ob(G)$,

$$G(c, d) := F(\Phi(c), \Phi(d)).$$

The composition in G is defined via the composition in F in the obvious way. The symmetric monoidal structure on G is defined on objects as follows:

$$e_1 \underset{G}{\otimes} e_2 := e_1 \underset{E}{\otimes} e_2$$

where $e_1, e_2 \in Ob(G) = Ob(E)$. For a pair of morphisms $f_1 : e_1 \longrightarrow h_1$ and $f_2 : e_2 \longrightarrow h_2$ we define

$$f_1 \underset{G}{\otimes} f_2 := f_1 \underset{F}{\otimes} f_2.$$

We recall that $\Pi_1 : \mathbf{Cat} \longrightarrow \mathbf{Gpd}$ is the $(2-)$ functor which assigns to each category C the groupoid obtained by inverting all maps in C. The following lemma is a consequence of the well known fact that Π_1 preserves finite products and therefore maps **Cat**-enriched adjunctions to **Gpd**-enriched adjunctions *i.e.*. equivalences:

2.49. Lemma. *Let $F : C \longrightarrow D$ be a functor which is either a left or a right adjoint, then the induced functor $\Pi_1(F) : \Pi_1(C) \longrightarrow \Pi_1(D)$ is an equivalence of categories.*

2.50. Proposition. *Let E be a symmetric monoidal category, F be a symmetric monoidal groupoid and $\Phi : E \longrightarrow F$ be a strict symmetric monoidal which is a composite of n strict symmetric monoidal functors* i.e. $\Phi = \phi_n \circ \cdots \circ \phi_1$ *such that each ϕ_i has either a left or a right adjoint for $1 \le i \le n$. Then the Gabriel category of Φ, G, is isomorphic to $\Pi_1(E)$.*

Proof. The functor Φ has a Gabriel factorization

$$\begin{tikzcd} E \arrow[rr, "\Phi"] \arrow[dr, "\Gamma"'] & & F \\ & G \arrow[ur, "\Delta"'] & \end{tikzcd}$$

see [GJ08, Sec. 1.1]. The above lemma 2.49 tells us that the functor

$$\Pi_1(\Phi) : \Pi_1(E) \longrightarrow \Pi_1(F) = F$$

is an equivalence of groupoids. In the above situation the Gabriel category G is a groupoid therefore $\Pi_1(G) = G$. We recall that $Ob(G) = Ob(E)$ and since $\Pi_1(\Phi)$ is an equivalence of categories therefore for each pair of objects $e_1, e_2 \in E$ we have the following

$$E(e_1, e_2) \cong F(\Phi(e_1), \Phi(e_2)) = G(e_1, e_2).$$

Thus the functor Γ is an isomorphism of categories.

∎

3. The model category of Permutative categories

In this section we will describe a model category structure on the category of all (small) *permutative categories* **Perm** and two model category structures on the category of all Γ- categories Γ**Cat**. The three desired Quillen adjunctions will be amongst the model categories described here. We begin with the category **Perm**. The desired model category structure on **Perm** is a restriction of the natural model category structure on $\mathbf{Cat}_\bullet$ which leads us to call it the *natural model category structure on* **Perm**. The model structure in this section is known to experts in the area but the original content of this section is the characterization of cofibrations in the aforementioned natural model category.

We begin by reviewing permutative categories. A *permutative category* is a symmetric monoidal category in which the associativity and unit natural isomorphisms are the identity natural transformations. A map in **Perm** is a *strict monoidal functor i.e.* a functor which strictly preserves the tensor product, the unit object and also the associativity, unit and symmetry isomorphisms. A permutative category can be equivalently described as an algebra over a categorical version of the Barratt-Eccles operad, see [Dun94, Proposition 2.8]. The objective of this section is to define a model category structure on **Perm** and explore its properties. The model category structure on **Perm** is well known, it is a special case of [Lac07, Thm. 4.5], it also follows from [BM07].

3.1. THEOREM. *There is a* **Cat**-*model category structure on the category of all small permutative categories and strict symmetric monoidal functors* **Perm** *in which*

1. *A fibration is a strict symmetric monoidal functor which is also an isofibration of (unbased) categories and*

2. *A weak-equivalence is a strict symmetric monoidal functor which is also an equivalence of (unbased) categories.*

3. *A cofibration is a strict symmetric monoidal functor having the left lifting property with respect to all maps which are both fibrations and weak equivalences.*

Further, this model category structure is combinatorial and proper.

3.2. Remark. *The* **Cat**-*enrichment of the above model category is given by the bifunctor*

$$[-,-]^{str}_{\otimes} : \mathbf{Perm}^{op} \times \mathbf{Perm} \longrightarrow \mathbf{Cat}.$$

For any two permutative categories C and D, $[C,D]^{str}_{\otimes}$ is the category whose objects are strict symmetric monoidal functors and maps are strict (unital) monoidal natural transformations. The cotensor product of a permutative category C with a category E is the functor category $[E,C]$ which inherits a (strict) pointwise symmetric monoidal structure from C.

A functor $F : C \longrightarrow D$ is an equivalence of categories if and only if there exists another functor $G : D \longrightarrow C$ and two natural isomorphisms $FG \cong id_C$ and $id_D \cong GF$. We would like to have a similar characterization for a weak equivalence in **Perm** but unfortunately this is only possible by relaxing the strictness condition on the functor G. The following theorem is a special case of [Kel74, Thm. 1.5]:

3.3. Theorem. *Let $F : C \longrightarrow D$ be a strict symmetric monoidal functor in* **Perm**. *Any adjunction (F, G, η, ϵ) consisting of a unital right adjoint functor $G : D \longrightarrow C$ and a pair of unital natural isomorphisms $\epsilon : FG \cong id_D$ and $\eta : id_C \cong GF$, enhances uniquely to a unital symmetric monoidal adjunction i.e. there exists a unique natural isomorphism*

$$\lambda_G := id \circ (\epsilon \times \epsilon) \cdot (\lambda_{GF} \circ id_{G \times G}) \cdot id \circ (\epsilon^{-1} \times \epsilon^{-1}). \tag{7}$$

enhancing $G = (G, \lambda_G)$, into a unital symmetric monoidal functor such that η and ϵ are unital monoidal natural isomorphims.

3.4. Corollary. *A strict symmetric monoidal functor $F : C \longrightarrow D$ is a weak equivalence in* **Perm** *if and only if there exists a symmetric monoidal functor $G : D \longrightarrow C$ and a pair of symmetric monoidal natural isomorphisms $\epsilon : FG \cong id_D$ and $\eta : id_C \cong GF$.*

Proof. The only if part of the statement of the corollary is obvious. Let us assume that F is a weak equivalence in **Perm**. By regarding the unit objects of C and D as basepoints, we may view F as a (pointed) functor in $\mathbf{Cat}_\bullet$ which is a weak equivalence in the natural model category of pointed categories $\mathbf{Cat}_\bullet$. Then, by definition, there exists a unital functor $G : D \longrightarrow C$ and two unital natural isomorphisms $\eta : id_C \Rightarrow GF$ and $\epsilon : FG \Rightarrow id_D$. Now the result follows from the theorem. ∎

Next we want to give a characterization of acyclic fibrations in **Perm**. Recall that a functor is an acyclic fibration in **Cat** if and only of it is an equivalence which is surjective on objects. The following corollary provides equivalent characterizations of acyclic fibrations in **Perm**

3.5. Corollary. *Given a strict symmetric monoidal functor $F : C \longrightarrow D$ between permutative categories, the following statements about F are equivalent:*

1. *F is an acyclic fibration in* **Perm**.
2. *F is an equivalence of categories and surjective on objects.*

3. There exist a unital symmetric monoidal functor $G : D \longrightarrow C$ such that $FG = id_D$ and a unital monoidal natural isomorphism $\eta : id_C \cong GF$.

Proof. (1) ⇒ (2) An acyclic fibration $F : C \longrightarrow D$ in **Perm** is an acyclic fibration in $\mathbf{Cat}_\bullet$, when the unit objects are regarded as basepoints of C and D. Every acyclic fibration in $\mathbf{Cat}_\bullet$ is an equivalence of categories and surjective on objects, see 2.31.

(2) ⇒ (3) Since every object is cofibrant in the natural model category structure on $\mathbf{Cat}_\bullet$, therefore there exist a unital functor $G : D \longrightarrow C$ such that $FG = id_D$. There also exists a unital natural isomorphism $\eta : id_C \cong GF$, see 2.38. Now the theorem tells us that there is a unique enhancement of G to a unital symmetric monoidal functor such that η is a unital monoidal natural isomorphisms.

(3) ⇒ (4) Conversely, if there exists a unital symmetric monoidal functor (G, λ_G) and a unital monoidal natural isomorphisms $\eta : id_C \cong GF$ such that $FG = id_D$ then F is an acyclic fibration in **Cat** and therefore it is an acyclic fibration in **Perm**. ∎

Every object in **Cat** is cofibrant in the natural model structure but this is not the case in **Perm**. The cofibrant objects satisfy a freeness condition, for example every free permutative category generated by a category is a cofibrant object in **Perm**. In general, the notion of cofibrations in **Perm** is stronger than that in **Cat** as the following lemma suggests:

3.6. Lemma. *A cofibration in* **Perm** *is monic on objects.*

Proof. We begin the proof by defining a permutative category EC whose set of objects is the same as that of C. The category EC has exactly one arrow between any pair of objects. This category gets a unique permutative category structure which agrees with the permutative category structure of C on objects. The category EC is equipped with a unique strict symmetric monoidal functor $\iota_C : C \longrightarrow EC$ which is identity on objects. It is easy to see that the category EC is a groupoid and the terminal map $EC \longrightarrow *$ is an acyclic fibration in **Perm**.

Let $i : C \longrightarrow D$ be a cofibration in **Perm**. Let us assume that the object function $Ob(i) : Ob(C) \longrightarrow Ob(D)$ is NOT a monomorphism. Now we have the following commutative diagram

$$\begin{tikzcd} C \arrow[r, "\iota_C"] \arrow[d, "i"'] & EC \arrow[d] \\ D \arrow[r] & * \end{tikzcd}$$

The above diagram has NO lift because $Ob(i)$ is NOT a monomorphism. Since the terminal map $EC \longrightarrow *$ is an acyclic fibration, we have a contradiction to our assumption that i is a cofibration in **Perm**. Thus a cofibration in **Perm** is always monic on objects. ∎

Frequently in this paper we would require a characterization of cofibrations in **Perm**. The object function of a strict symmetric monoidal functor, which is a homomorphisms of monoids, determines whether the functor is a cofibration in **Perm**. We now recall that the category of monoids has a (weak) factorization system:

3.7. Lemma. *There is a weak factorization system (L, R) on the category of monoids, where R is the class of surjective homomorphisms of monoids.*

Proof. We have to show that each homomorphism of monoids $f : X \longrightarrow Y$ admits a factorisation $f : X \xrightarrow{u} E \xrightarrow{p} Y$ with u lies in the class L and p lies in the class R. For this, let $q : F(Y) \longrightarrow Y$ be the homomorphism adjunct to the identity map on Y in the category of sets, where $F(Y)$ is the free monoid generated by the underlying set of Y. The homomorphism q is surjective. Let $E = X \coprod F(Y)$ be the coproduct of X and $F(Y)$ in the category of monoids, and let $u : X \longrightarrow E$ and $v : F(Y) \longrightarrow E$ be the inclusions. Then there is a unique map $p : E \longrightarrow Y$ such that $pu = f$ and $pv = q$. We claim that u is in L and p is in R. The homomorphism p is surjective because q is surjective. In order to show that u is in L we have to show that whenever we have a (outer) commutative diagram in the category of monoids, where s is in R

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "u"'] & C \arrow[d, two heads, "s"] \\
E = X \coprod F(Y) \arrow[r, "f"'] \arrow[ur, dashed, "L"] & D
\end{tikzcd}$$

there exists a diagonal filler L which makes the entire diagram commutative. The lower horizontal map f can be viewed as a pair of homomorphisms $f_1 : X \longrightarrow D$ and $f_2 : F(Y) \longrightarrow D$. Now, it would be sufficient to show that there exists a homomorphism $L_2 : F(Y) \longrightarrow C$ such that the following diagram commutes

$$\begin{tikzcd}
& C \arrow[d, two heads, "s"] \\
F(Y) \arrow[r, "f_2"'] \arrow[ur, dashed, "L_2"] & D
\end{tikzcd}$$

By adjointness, the existence of the homomorphism L_2 is equivalent to the existence of a morphism of sets, $T : Y \longrightarrow U(C)$, such that the following diagram commutes in the category of sets

$$\begin{tikzcd}
& U(C) \arrow[d, two heads, "U(s)"] \\
Y \arrow[r, "U(f_2)"'] \arrow[ur, dashed, "T"] & U(D)
\end{tikzcd}$$

where U is the forgetful functor from the category of monoids to the category of sets which is right adjoint to the free monoid functor F. Such a map T exists because s is a surjective map of sets. Thus we have shown that the homomorphism u lies in the class L. ∎

The next lemma provides the desired characterization of cofibrations.

3.8. Lemma. *A strict symmetric monoidal functor $F : C \longrightarrow D$ in* **Perm** *is a cofibration if and only if the object function of F lies in the class L* i.e. *it has the left lifting property with respect to surjective homomorphisms of monoids.*

Proof. Let us assume that $Ob(F)$ lies in the class L. Let $p : X \longrightarrow Y$ be an acyclic fibration in **Perm** then the object function $Ob(p) : Ob(X) \longrightarrow Ob(Y)$ is a surjective homomorphism. The assumption that $Ob(F)$ lies in L implies that whenever we have the following (outer) commutative diagram there exists a (dotted) diagonal filler $Ob(L)$ which makes the entire diagram commutative in the category of monoids

$$\begin{tikzcd} Ob(C) \arrow[r] \arrow[d, "Ob(F)"'] & Ob(X) \arrow[d, two heads, "Ob(p)"] \\ Ob(D) \arrow[r] \arrow[ur, dashed, "Ob(L)"] & Ob(Y) \end{tikzcd} \tag{8}$$

Now we want to show that whenever we have a (outer) commutative diagram, there exists a lift L which makes the following diagram commutative in **Perm**

$$\begin{tikzcd} C \arrow[r] \arrow[d, "F"'] & X \arrow[d, two heads, "p"] \\ D \arrow[r, "G"'] \arrow[ur, dashed, "L"] & Y \end{tikzcd}$$

We will present a construction of the strict symmetric monoidal functor L in the above diagram. We choose a lift $Ob(L)$ in the diagram (8) to be the object function of the functor L. Since p is an acyclic fibration therefore for each pair of objects $y, z \in Ob(X)$, each function

$$p_{y,z} : X(y, z) \longrightarrow Y(p(y), p(z))$$

is a bijection. For each pair of objects d_1, d_2 in D, we define a function

$$L_{d_1,d_2} : D(a, b) \longrightarrow X(L(d_1), L(d_2))$$

by the following composite diagram

$$D(d_1, d_2) \xrightarrow{G_{d_1,d_2}} Y(G(d_1), G(d_2)) \xrightarrow{p^{-1}_{L(d_1),L(d_1)}} X(L(d_1), L(d_2)).$$

In order to check that our definition respects composition, it would be sufficient to check that for another object $d_3 \in Ob(D)$, the following diagram commutes:

$$\begin{tikzcd} Y(F(d_1), F(d_2)) \times Y(F(d_2), F(d_3)) \arrow[r, "p^{-1}_{L(d_1),L(d_2)} \times p^{-1}_{L(d_2),L(d_3)}"] \arrow[d, "- \circ -"'] & X(L(d_1), L(d_2)) \times X(L(d_2), L(d_3)) \arrow[d, "- \circ -"] \\ Y(F(d_1), F(d_2)) \arrow[r, "p^{-1}_{L(d_1),L(d_3)}"'] & X(L(d_1), L(d_3)) \end{tikzcd}$$

The commutativity of the above diagram follows from the commutativity of the following diagram which is the result of the assumption that p is a functor

$$\begin{tikzcd}
Y(F(d_1), F(d_2)) \times Y(F(d_2), F(d_3)) \arrow[d, "- \circ -"'] & X(L(d_1), L(d_2)) \times X(L(d_2), L(d_3)) \arrow[l, "p_{L(d_1),L(d_2)} \times p_{L(d_2),L(d_3)}"'] \arrow[d, "- \circ -"] \\
Y(F(d_1), F(d_2)) & X(L(d_1), L(d_3)) \arrow[l, "p_{L(d_1),L(d_3)}"]
\end{tikzcd}$$

Thus we have shown that the family of functions $\{L_{d_1,d_2}\}_{d_1,d_2 \in Ob(D)}$ together with the object function $Ob(L)$ defines a functor L. Now we have to check that L is a strict symmetric monoidal functor. Clearly $L(d_1 \otimes d_2) = L(d_1) \otimes L(d_2)$ for each pair of objects $d_1, d_2 \in Ob(D)$ because the object function of L is a homomorphism of monoids. The same equality holds for each pair of maps in D. Finally we will show that L strictly preserves the symmetry isomorphism. By definition, $L(\gamma^D_{d_1,d_2}) = p^{-1}_{G(d_1 \otimes d_2), G(d_2 \otimes d_1)} \circ G(\gamma^D_{d_1,d_2})$. Since G is a strict symmetric monoidal functor, therefore $G(\gamma^D_{d_1,d_2}) = \gamma^Y_{G(d_1),G(d_2)} = \gamma^Y_{pL(d_1),pL(d_2)}$. Since p is also a strict symmetric monoidal functor therefore

$$p^{-1}_{G(d_1 \otimes d_2), G(d_2 \otimes d_1)}(\gamma^Y_{G(d_1),G(d_2)}) = p^{-1}_{G(d_1 \otimes d_2), G(d_2 \otimes d_1)}(\gamma^Y_{pL(d_1),pL(d_2)}) = \gamma^X_{L(d_1),L(d_2)}.$$

This means that $L(\gamma^D_{d_1,d_2}) = \gamma^X_{L(d_1),L(d_2)}$ for each pair of objects $d_1, d_2 \in Ob(D)$. Thus we have shown that L is a strict symmetric monoidal functor which makes the entire diagram (3) commutative *i.e.* F is an acyclic cofibration in **Perm**.

Conversely let us assume that F is an acyclic cofibration in **Perm**. We want to show that the object function of F lies in the class L. Let $f : M \longrightarrow N$ be a surjective homomorphism of monoids. The homomorphism f induces an acyclic fibration $E(f) : EM \longrightarrow EN$, where EM and EN are permutative categories whose monoid of objects are M and N respectively and there is exactly one map between each pair of objects. By assumption the functor F has the left lifting property with respect to all strict symmetric monoidal functors in the set

$$\{E(f) : EM \longrightarrow EN : f \in R\}$$

because every element of this set is an acyclic fibration in **Perm**. This implies that the object function of F has the left lifting property with respect to all maps in R.

∎

The next proposition provides three equivalent characterizations of acyclic cofibrations in **Perm**:

3.9. Proposition. *Let $F : C \longrightarrow D$ be a strict symmetric monoidal functor between permutative categories C and D, the following conditions on F are equivalent*

1. *The strict symmetric monoidal functor F is an acyclic cofibration in* **Perm**.

2. *There exist a strict symmetric monoidal functor $G : D \longrightarrow C$ such that $GF = id_C$ and a unital monoidal natural isomorphism $\eta : id_D \cong FG$ which is the unit of an adjunction $(G, F, \eta, id) : D \rightharpoonup C$.*

3. *There is a (permutative) subcategory S of D, an isomorphism $H : C \cong S$, in* **Perm**, *a strict symmetric monoidal functor $T : D \longrightarrow S$ and a unital monoidal natural isomorphism $\iota_S \circ T \cong id_D$, where $\iota_S : S \hookrightarrow D$ is the inclusion functor such that $T \circ \iota_S = id_S$ and $F = \iota_S \circ H$*

Proof. (1) $\Rightarrow$ (2) Since F is an acyclic cofibration in **Perm** therefore the (outer) commutative diagram has a diagonal filler G such that the entire diagram is commutative in **Perm**

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "F"'] & C \arrow[d, "p", two heads] \\ D \arrow[r] \arrow[ur, dashed, "G"] & * \end{tikzcd}$$

Now we construct the monoidal natural isomorphism $\eta : id_D \longrightarrow FG$. For each object $d \in Ob(D)$ which is in the image of F, there exists a unique $c \in Ob(C)$ such that $d = F(c)$. In this case we define $\eta(d) = id_d$. Let $d \in Ob(D)$ lie outside the image of F. Since F is an equivalence of categories, we may choose an object $c \in Ob(C)$ and an isomorphism $i^c_d : F(c) \cong d$ such that for each arrow $f : d \longrightarrow e$ in D, there exists a unique arrow $g : c \longrightarrow a$ in C which makes the following diagram commutative in D:

$$\begin{tikzcd} F(c) \arrow[r, "i_d"] \arrow[d, "F(g)"'] & d \arrow[d, "f"] \\ F(a) \arrow[r, "i_e"'] & e \end{tikzcd} \tag{9}$$

Whenever $d = d_1 \otimes d_2$, we may choose $c = c_1 \otimes c_2$ and $i_d = i_{d_1} \otimes i_{d_2}$. This gives us a composite isomorphism

$$\eta(d) := d \overset{(i_d)^{-1}}{\longrightarrow} FG(F(c)) \overset{FG(i_d)}{\longrightarrow} FG(d).$$

In light of the commutative diagram (9), it is easy to see that this isomorphism is natural. Thus we have defined a (unital) natural isomorphism $\eta : id_D \Rightarrow FG$. Our choice for each pair of objects $d_1, d_2 \in Ob(D)$ for $i_{d_1 \otimes d_2} = i_{d_1} \otimes i_{d_2}$ guarantees that η is a monoidal natural isomorphism *i.e.* $\eta(d_1 \otimes d_2) = \eta(d_1) \otimes \eta(d_2)$.

(2) $\Rightarrow$ (3) The permutative subcategory $S \subseteq D$ is the full subcategory of D whose objects lie in the image of F *i.e.* the object set of S is defined as follows:

$$Ob(S) := \{F(c) : c \in Ob(C)\}$$

If $F(c_1)$ and $F(c_2)$ lie in S then $F(c_1) \underset{D}{\otimes} F(c_2) = F(c_1 \underset{C}{\otimes} c_2)$ also lies in S. Thus S is a permutative subcategory. The isomorphism H is obtained by restricting the codomain of

F to S. The left adjoint of ι_S is the composite functor FG. The counit of the adjunction (FG, ι_S) is the identity natural isomorphism. The unit (monoidal) natural isomorphism is just η. Thus S is reflective.

(3) $\Rightarrow$ (1) If we assume (4) then any (outer) commutative square

$$\begin{tikzcd} C \arrow[r, "Q"] \arrow[d, "F"'] & X \arrow[d, two heads, "p"] \\ D \arrow[r, "R"'] \arrow[ur, dashed, "L"] & Y \end{tikzcd}$$

where p is a fibration in **Perm**, would have a diagonal filler L if and only if the lower square in the following (solid arrow) commutative diagram has a diagonal filler K

$$\begin{tikzcd} C \arrow[d, "H"'] \arrow[dr, "Q"] & \\ S \arrow[r] \arrow[d, hook, "\iota_S"'] & X \arrow[d, two heads, "p"] \\ D \arrow[r, "R"'] \arrow[ur, dashed, "K"'] & Y \end{tikzcd}$$

Since ι_S has a strict symmetric monoidal left adjoint T with an identity counit, therefore the composite $K = T \circ Q \circ H^{-1}$ is a diagonal filler of the lower square such that entire diagram commutes. This implies that F has the left lifting property with respect to fibrations in **Perm**. Thus F is an acyclic cofibration in **Perm**. ∎

4. The model category structures

A Γ- category is a functor from Γ^{op} to **Cat**. The category of functors from Γ^{op} to **Cat** and natural transformations between them $[\Gamma^{op}, \mathbf{Cat}]$ will be denoted by $\Gamma\mathbf{Cat}$. We begin by describing a model category structure on $\Gamma\mathbf{Cat}$ which is often referred to either as the *projective model category structure* or the *strict model category structure*. Following [Sch99] we will use the latter terminology.

4.1. Definition. *A morphism $F : X \longrightarrow Y$ of Γ- categories is called*

1. *a* strict equivalence *of Γ- categories if it is degreewise weak equivalence in the natural model category structure on* **Cat** i.e. $F(n^+) : X(n^+) \longrightarrow Y(n^+)$ *is an equivalence of categories.*
2. *a* strict fibration *of Γ- categories if it is degreewise a fibration in the natural model category structure on* **Cat** i.e. $F(n^+) : X(n^+) \longrightarrow Y(n^+)$ *is an isofibration.*
3. *a* Q-cofibration *of Γ- categories if it has the left lifting property with respect to all morphisms which are both strict weak equivalence and strict fibrations of Γ- categories.*

In light of proposition 2.33 we observe that a map of Γ- categories $F : X \longrightarrow Y$ is a strict acyclic fibration of Γ- categories if and only if it has the right lifting property with respect to all maps in the set

$$\mathcal{I} = \{\Gamma^n \times \partial_0, \Gamma^n \times \partial_1, \Gamma^n \times \partial_2 \mid \forall n \in Ob(\mathcal{N})\}. \tag{10}$$

We further observe, in light of proposition 2.30, that F is a strict fibration if and only it has the right lifting property with respect to all maps in the set

$$\mathcal{J} = \{\Gamma^n \times i_0, \Gamma^n \times i_1 \mid \forall n \in Ob(\mathcal{N})\}. \tag{11}$$

4.2. Theorem. *Strict equivalences, strict fibrations and Q-cofibrations of* Γ- *categories provide the category* $\Gamma\mathbf{Cat}$ *with a combinatorial model category structure.*

A proof of this proposition is given in [Lur09, Proposition A.3.3.2].

To each pair of objects $(X, C) \in Ob(\Gamma\mathbf{Cat}) \times Ob(\mathbf{Cat})$ we can assign a Γ- category $X \otimes C$ which is defined in degree n as follows:

$$(X \otimes C)(n^+) := X(n^+) \times C,$$

This assignment is functorial in both variables and therefore we have a bifunctor

$$- \otimes - : \Gamma\mathbf{Cat} \times \mathbf{Cat} \longrightarrow \Gamma\mathbf{Cat}.$$

Now we will define a couple of function objects for the category $\Gamma\mathbf{Cat}$. The first function object enriches the category $\Gamma\mathbf{Cat}$ over $\mathbf{Cat}$ *i.e.* there is a bifunctor

$$\mathcal{M}ap_{\Gamma\mathbf{Cat}}(-,-) : \Gamma\mathbf{Cat}^{op} \times \Gamma\mathbf{Cat} \longrightarrow \mathbf{Cat}$$

which assigns to any pair of objects

$$(X, Y) \in Ob(\Gamma\mathbf{Cat}) \times Ob(\Gamma\mathbf{Cat}),$$

a category $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, Y)$ whose set of objects is the following

$$Ob(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, Y)) := Hom_{\Gamma\mathbf{Cat}}(X, Y)$$

and the morphism set of this category are defined as follows:

$$Mor(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, Y)) := Hom_{\Gamma\mathbf{Cat}}(X \times I, Y)$$

For any Γ- category X, the functor $X \otimes - : \mathbf{Cat} \longrightarrow \Gamma\mathbf{Cat}$ is left adjoint to the functor $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, -) : \Gamma\mathbf{Cat} \longrightarrow \mathbf{Cat}$. The counit of this adjunction is the evaluation map $ev : X \otimes \mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, Y) \longrightarrow Y$ and the unit is the obvious functor $C \longrightarrow \mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, X \otimes$

C). To any pair of objects $(C, X) \in Ob(\mathbf{Cat}) \times Ob(\Gamma\mathbf{Cat})$ we can assign a Γ- category $\mathbf{hom}_{\Gamma\mathbf{Cat}}(C, X)$ which is defined in degree n as follows:

$$(\mathbf{hom}_{\Gamma\mathbf{Cat}}(C, X))(n^+) := [C, X(n^+)] \ .$$

This assignment is functorial in both variable and therefore we have a bifunctor

$$\mathbf{hom}_{\Gamma\mathbf{Cat}}(-,-) : \mathbf{Cat}^{op} \times \Gamma\mathbf{Cat} \longrightarrow \Gamma\mathbf{Cat}.$$

For any Γ- category X, the functor $\mathbf{hom}_{\Gamma\mathbf{Cat}}(-, X) : \mathbf{Cat} \longrightarrow \Gamma\mathbf{Cat}^{op}$ is left adjoint to the functor $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(-, X) : \Gamma\mathbf{Cat}^{op} \longrightarrow \mathbf{Cat}$. The following proposition summarizes the above discussion.

4.3. Proposition. *There is an adjunction of two variables*

$$(- \otimes -, \mathbf{hom}_{\Gamma\mathbf{Cat}}(-,-), \mathcal{M}ap_{\Gamma\mathbf{Cat}}(-,-)) : \Gamma\mathbf{Cat} \times \mathbf{Cat} \longrightarrow \Gamma\mathbf{Cat}. \tag{12}$$

4.4. Definition. *Given model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, an adjunction of two variables, $(\otimes, \mathbf{hom}_{\mathcal{C}}, \mathcal{M}ap_{\mathcal{C}}, \phi, \psi) : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$, is called a* Quillen adjunction of two variables*, if, given a cofibration $f : U \longrightarrow V$ in $\mathcal{C}$ and a cofibration $g : W \longrightarrow X$ in $\mathcal{D}$, the induced map*

$$f \square g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \longrightarrow V \otimes X$$

is a cofibration in $\mathcal{E}$ that is trivial if either f or g is. We will refer to the left adjoint of a Quillen adjunction of two variables as a Quillen bifunctor*.*

We recall that [Hov99, Lemma 4.2.2] provides three equivalent characterizations of the notion of a Quillen bifunctor. This lemma will be useful in this paper in establishing enriched model category structures.

4.5. Definition. *Let $\mathbf{S}$ be a monoidal model category. An* $\mathbf{S}$-enriched model category *or simply an $\mathbf{S}$-model category is an $\mathbf{S}$ enriched category $\mathbf{A}$ equipped with a model category structure (on its underlying category) such that there is a Quillen adjunction of two variables, see definition 4.4, $(\otimes, \mathbf{hom}_{\mathbf{A}}, \mathcal{M}ap_{\mathbf{A}}, \phi, \psi) : \mathbf{A} \times \mathbf{S} \longrightarrow \mathbf{A}$.*

The following theorem follows from [Lur09, Rem. A.3.3.4]

4.6. Theorem. *The strict model category of Γ- categories, $\Gamma\mathbf{Cat}$, is a $\mathbf{Cat}$-enriched model category.*

Let X and Y be two Γ- categories, the *Day convolution product* of X and Y denoted by $X * Y$ is defined as follows:

$$X * Y(n^+) := \int^{(k^+, l^+) \in \Gamma^{op}} \Gamma^{op}(k^+ \wedge l^+, n^+) \times X(k^+) \times Y(l^+). \tag{13}$$

Equivalently, one may define the Day convolution product of X and Y as the left Kan extension of their *external tensor product* $X\overline{\times}Y$ along the smash product functor

$$- \wedge - : \Gamma^{op} \times \Gamma^{op} \longrightarrow \Gamma^{op}.$$

we recall that the external tensor product $X\overline{\times}Y$ is a bifunctor

$$X\overline{\times}Y : \Gamma^{op} \times \Gamma^{op} \longrightarrow \mathbf{Cat}$$

which is defined on objects by

$$X\overline{\times}Y(m^+, n^+) = X(m^+) \times Y(n^+).$$

4.7. Proposition. *The category of all* Γ- *categories* $\Gamma\mathbf{Cat}$ *is a symmetric monoidal category under the Day convolution product* (13). *The unit of the symmetric monoidal structure is the representable* Γ- *category* Γ^1.

Next we define an internal function object of the category Γ- category which we will denote by

$$\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(-,-) : \Gamma\mathbf{Cat}^{op} \times \Gamma\mathbf{Cat} \longrightarrow \Gamma\mathbf{Cat}. \tag{14}$$

Let X and Y be two Γ- categories, we define the Γ- category $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(X,Y)$ as follows:

$$\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(X,Y)(n^+) := \mathcal{M}ap_{\Gamma\mathbf{Cat}}(X * \Gamma^n, Y).$$

4.8. Proposition. *The category* $\Gamma\mathbf{Cat}$ *is a closed symmetric monoidal category under the Day convolution product. The internal Hom is given by the bifunctor* (14) *defined above.*

The above proposition implies that for each $n \in \mathbb{N}$ the functor $- * \Gamma^n : \Gamma\mathbf{Cat} \longrightarrow \Gamma\mathbf{Cat}$ has a right adjoint $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^n, -) : \Gamma\mathbf{Cat} \longrightarrow \Gamma\mathbf{Cat}$. The functor $-*\Gamma^n$ has another right adjoint which we denote by $-(n^+ \wedge -) : \Gamma\mathbf{Cat} \longrightarrow \Gamma\mathbf{Cat}$. We will denote $-(n^+ \wedge -)(X)$ by $X(n^+ \wedge -)$, where X is a Γ- category. The Γ- category $X(n^+ \wedge -)$ is defined by the following composite:

$$\Gamma^{op} \xrightarrow{n^+ \wedge -} \Gamma^{op} \xrightarrow{X} \mathbf{Cat}. \tag{15}$$

The following proposition sums up this observation:

4.9. Proposition. *There is a natural isomorphism*

$$\phi : -(n^+ \wedge -) \cong \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^n, -).$$

In particular, for each Γ- *category* X *there is an isomorphism of* Γ- *categories*

$$\phi(X) : X(n^+ \wedge -) \cong \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^n, X).$$

The next theorem verifies the compatibility of the strict model category $\Gamma\mathbf{Cat}$ with the Day convolution product. This theorem can be proved by a straightforward verification of Lemma [Hov99, Lemma 4.2.2(3)] using proposition 4.8 along with adjointness:

4.10. Theorem. *The strict Q-model category* $\Gamma\mathbf{Cat}$ *is a symmetric monoidal closed model category under the Day convolution product.*

4.11. Coherently commutative monoidal categories. The objective of this subsection is to construct a new model category structure on the category $\Gamma\mathbf{Cat}$. This new model category is obtained by localizing the strict model category defined above and we call it the *The model category of coherently commutative monoidal categories*. We will refer to this new model category structure as the *model category structure of coherently commutative monoidal categories* on $\Gamma\mathbf{Cat}$. The aim of this new model structure is to endow its homotopy category with a semi-additive structure. In other words we want this new model category structure to have finite *homotopy biproducts*. We go on further to show that this new model category is symmetric monoidal with respect to the *Day convolution product*, see [Day70]. The proposed model category structure will be constructed using *left Bousfield localization* of model categories [Hir02, Definition 3.3.1]:

4.12. Remark. *The strict model category of all* Γ- *categories is a* $\mathbf{Cat}$-*enriched model category by theorem 4.6, this enrichment is equivalent to having a Quillen adjunction* $-\otimes\Gamma^1 : \mathbf{Cat} \rightleftharpoons \Gamma\mathbf{Cat} : \mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, -)$ *whose left adjoint preserves the tensor product, see [Bar07, Lemma 3.6]. Further the adjunction* $\tau_1 : \mathbf{sSets} \rightleftharpoons \mathbf{Cat} : N$*, see [Joy08], is a Quillen adjunction with respect to the Joyal model category structure on* $\mathbf{sSets}$ *and natural model category structure on* $\mathbf{Cat}$ *whose left adjoint* τ_1 *preserves finite products (and thus the tensor product in the cartesian closed Joyal model category of simplicial sets). Again lemma [Bar07, Lemma 3.6] implies that the strict model category of* Γ- *categories is a* $\mathbf{sSets}$-*enriched model category with respect to the Joyal model category structure on* $\mathbf{sSets}$*. The right Hom bifunctor of this enrichment*

$$Map(--) : \Gamma\mathbf{Cat}^{op} \times \Gamma\text{-}space \longrightarrow \mathbf{sSets}$$

assigns to a pair of objects (X, C)*, a simplicial sets* $Map(XC)$ *which is defined as follows:*

$$Map(X, C) := N(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, C)).$$

We want to construct a left Bousfield localization of the strict model category of Γ- categories. For each pair $k^+, l^+ \in \Gamma^{op}$, we have the obvious *projection maps* in $\Gamma\mathcal{S}$

$$\delta^{k+l}_k : (k+l)^+ \longrightarrow k^+ \quad and \quad \delta^{k+l}_l : (k+l)^+ \longrightarrow l^+.$$

The maps

$$\Gamma^{op}(\delta^{k+l}_k, -) : \Gamma^k \longrightarrow \Gamma^{k+l} \quad and \quad \Gamma^{op}(\delta^{k+l}_l, -) : \Gamma^l \longrightarrow \Gamma^{k+l}$$

induce a map of Γ-spaces on the coproduct which we denote as follows:

$$h^l_k : \Gamma^l \sqcup \Gamma^l \longrightarrow \Gamma^{l+k}.$$

We now define a class of maps $\mathcal{E}_\infty\mathcal{S}$ in $\Gamma\mathbf{Cat}$:

$$\mathcal{E}_\infty\mathcal{S} := \{h^l_k : \Gamma^l \sqcup \Gamma^l \longrightarrow \Gamma^{l+k} : l, k \in \mathbb{Z}^+\}$$

We recall that I is the category with two objects and one non-identity arrow between them. We define another class of maps in $\Gamma\mathbf{Cat}$:

$$I \times \mathcal{E}_\infty\mathcal{S} := \{I \times h^l_k : h^l_k \in \mathcal{E}_\infty\mathcal{S}\}$$

4.13. Definition. *We call a* Γ- *category* X *a* $(I \times \mathcal{E}_\infty\mathcal{S})$-local object *if, for each map* $h^l_k \in \mathcal{E}_\infty\mathcal{S}$*, the induced simplicial map*

$$\mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(\Delta[n] \times h^l_k, X) : \mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(\Delta[n] \times \Gamma^{k+l}, X) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(\Delta[n] \times (\Gamma^l \sqcup \Gamma^l), X),$$

is a homotopy equivalence of simplicial sets for all $n \geq 0$ *where* $\mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(-,-)$ *is the simplicial function object associated with the strict model category* $\Gamma\mathbf{Cat}$*, see [DK80a], [DK80c] and [DK80b].*

Remark (4.12) above and appendix D tell us that a model for $\mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(X, Y)$ is the Kan complex $J(N(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, Y)))$ which is the maximal kan complex contained in the quasicategory $N(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, Y))$.

The following proposition gives a characterization of $\mathcal{E}_\infty\mathcal{S}$-local objects

4.14. Proposition. *A* Γ- *category* X *is a* $(I \times \mathcal{E}_\infty\mathcal{S})$*-local object in* $\Gamma\mathbf{Cat}$ *if and only if it satisfies the Segal condition namely the functor*

$$(X(\delta^{k+l}_k), X(\delta^{k+l}_l)) : X(k + l^+) \longrightarrow X(k^+) \times X(l^+)$$

is an equivalence of categories for all $k^+, l^+ \in Ob(\Gamma^{op})$.

Proof. We begin the proof by observing that each element of the set $\mathcal{E}_\infty\mathcal{S}$ is a map of Γ- categories between cofibrant Γ- categories. Lemma D.8 implies that X is a $(I \times \mathcal{E}_\infty\mathcal{S})$-local object if and only if the following functor

$$\mathcal{M}ap_{\Gamma\mathbf{Cat}}(h^k_l, X) : \mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^{k+l}, X) \longrightarrow \mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^k \sqcup \Gamma^l, X)$$

is an equivalence of (ordinary) categories. We observe that we have the following commutative square in $\mathbf{Cat}$

$$\begin{tikzcd}
\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^{k+l}, X) \arrow[r, "\mathcal{M}ap_{\Gamma\mathbf{Cat}}(h^k_l, X)"] \arrow[d, "\cong"'] & \mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^k \sqcup \Gamma^l, X) \arrow[d, "\cong"] \\
X((k+l)^+) \arrow[r, "(X(\delta^{k+l}_k), X(\delta^{k+l}_l))"'] & X(k^+) \times X(l^+)
\end{tikzcd}$$

This implies that the functor $(X(\delta^{k+l}_k), X(\delta^{k+l}_l))$ is an equivalence of categories if and only if the functor $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(h^k_l, X)$ is an equivalence of categories. ∎

4.15. Definition. *We will refer to a* $(I \times \mathcal{E}_\infty \mathcal{S})$*-local object as a* coherently commutative monoidal category*.*

4.16. Definition. *A morphism of* Γ*- categories* $F : X \longrightarrow Y$ *is a* $(I \times \mathcal{E}_\infty \mathcal{S})$*-local equivalence if for each coherently commutative monoidal category* Z *the following simplicial map*

$$\mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(F, Z) : \mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(Y, Z) \longrightarrow \mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(X, Z)$$

is a homotopy equivalence of simplicial sets.

4.17. Proposition. *A morphism between two cofibrant* Γ*- categories* $F : X \longrightarrow Y$ *is an* $(I \times \mathcal{E}_\infty \mathcal{S})$*-local equivalence if and only if the functor*

$$\mathcal{M}ap_{\Gamma\mathbf{Cat}}(F, Z) : \mathcal{M}ap_{\Gamma\mathbf{Cat}}(Y, Z) \longrightarrow \mathcal{M}ap_{\Gamma\mathbf{Cat}}(X, Z)$$

is an equivalence of categories for each coherently commutative monoidal category Z*.*

4.18. Definition. *We will refer to a* $(I \times \mathcal{E}_\infty \mathcal{S})$*-local equivalence as an* equivalence of coherently commutative monoidal categories*.*

The main result of this section is about constructing a new model category structure on the category $\Gamma\mathbf{Cat}$, by localizing the strict model category of Γ- categories with respect to morphisms in the set $\mathcal{E}_\infty \mathcal{S}$. We recall the following theorem which will be the main tool in the construction of the desired model category. This theorem first appeared in an unpublished work [Smi] but a proof was later provided by Barwick in [Bar07].

4.19. Theorem. *[Bar07, Theorem 2.11] If* $\mathcal{M}$ *is a combinatorial model category and* $\mathcal{S}$ *is a small set of homotopy classes of morphisms of* $\mathcal{M}$*, the left Bousfield localization* $L_{\mathcal{S}}\mathcal{M}$ *of* $\mathcal{M}$ *along any set representing* $\mathcal{S}$ *exists and satisfies the following conditions.*

1. *The model category* $L_{\mathcal{S}}\mathcal{M}$ *is left proper and combinatorial.*
2. *As a category,* $L_{\mathcal{S}}\mathcal{M}$ *is simply* $\mathcal{M}$*.*
3. *The cofibrations of* $L_{\mathcal{S}}\mathcal{M}$ *are exactly those of* $\mathcal{M}$*.*
4. *The fibrant objects of* $L_{\mathcal{S}}\mathcal{M}$ *are the fibrant* $\mathcal{S}$*-local objects* Z *of* $\mathcal{M}$*.*
5. *The weak equivalences of* $L_{\mathcal{S}}\mathcal{M}$ *are the* $\mathcal{S}$*-local equivalences.*

4.20. Theorem. *There is a closed, left proper, combinatorial model category structure on the category of* Γ*- categories,* $\Gamma\mathbf{Cat}$*, in which*

1. *The class of cofibrations is the same as the class of Q-cofibrations of* Γ*- categories.*
2. *The weak equivalences are equivalence of coherently commutative monoidal categories.*

An object is fibrant in this model category if and only if it is a coherently commutative monoidal category.

PROOF. The strict model category of Γ- categories is a combinatorial model category therefore the existence of the model structure follows from theorem 4.19 stated above. ∎

4.21. NOTATION. *The model category constructed in theorem 4.20 will be called the model category of coherently commutative monoidal categories.*

The rest of this section is devoted to proving that the model category of coherently commutative monoidal categories is a symmetric monoidal closed model category. In order to do so we will need some general results which we state now.

The following proposition has been proved in [Joy08, Lemma E.2.13]

4.22. PROPOSITION. *A cofibration, $f : A \longrightarrow B$, between cofibrant objects in a model category $\mathcal{C}$ is a weak equivalence in $\mathcal{C}$ if and only if it has the right lifting property with respect to all fibrations between fibrant objects in $\mathcal{C}$.*

4.23. PROPOSITION. *Let X be a coherently commutative monoidal category, then for each $n \in Ob(\mathcal{N})$, the Γ- category $X(n^+ \wedge -)$ is also a coherently commutative monoidal category.*

PROOF. We begin by observing that $X(n^+ \wedge -)(1^+) = X(n^+)$ and since X is fibrant, the pointed category $X(n^+)$ is equivalent to $\prod_1^n X(1^+)$. Notice that the isomorphisms $(n^+ \wedge (k+l)^+) \cong \bigvee_1^n (k+l)^+ \cong (\bigvee_1^n k^+) \vee (\bigvee_1^n l^+) \cong ((\bigvee_1^n k^+) + (\bigvee_1^n l^+))$. The two projection maps $\delta_k^{k+l} : (k+l)^+ \longrightarrow k^+$ and $\delta_l^{k+l} : (k+l)^+ \longrightarrow l^+$ induce an equivalence of categories $X((\bigvee_1^n k^+) + (\bigvee_1^n l^+)) \longrightarrow X(\bigvee_1^n k^+) \times X(\bigvee_1^n l^+)$. Composing with the isomorphisms above, we get the following equivalence of pointed simplicial sets $X(n^+ \wedge -)((k+l)^+) \longrightarrow X(n^+ \wedge -)(k^+) \times X(n^+ \wedge -)(l^+)$. ∎

4.24. COROLLARY. *For each coherently commutative monoidal category X and $n \in \mathbb{N}$, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^n, X)$ is also a coherently commutative monoidal category.*

PROOF. The corollary follows from proposition 4.9. ∎

The category Γ^{op} is a symmetric monoidal category with respect to the smash product of pointed sets. In other words the smash product of pointed sets defines a bi-functor $- \wedge - : \Gamma^{op} \times \Gamma^{op} \longrightarrow \Gamma^{op}$. For each pair $k^+, l^+ \in Ob(\Gamma^{op})$, there are two natural transformations

$$\delta_k^{k+l} \wedge - : (k+l)^+ \wedge - \Rightarrow k^+ \wedge - \quad \text{and} \quad \delta_l^{k+l} \wedge - : (k+l)^+ \wedge - \Rightarrow l^+ \wedge -.$$

Horizontal composition of either of these two natural transformations with a Γ- category X determines a morphism of Γ- categories

$$id_X \circ (\delta_k^{k+l} \wedge -) =: X(\delta_k^{k+l} \wedge -) : X((k+l)^+ \wedge -) \longrightarrow X(k^+ \wedge -).$$

4.25. Proposition. *Let X be an coherently commutative monoidal category, then for each pair $(k,l) \in Ob(\mathcal{N}) \times Ob(\mathcal{N})$, the following morphism*

$$(X(\delta_k^{k+l} \wedge -), X(\delta_l^{k+l} \wedge -)) : X((k+l)^+ \wedge -) \longrightarrow X(k^+ \wedge -) \times X(l^+ \wedge -)$$

is a strict equivalence of Γ- *categories.*

Using the previous two propositions, we now show that the mapping space functor $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(-,-)$ provides the homotopically correct function object when the domain is cofibrant and codomain is fibrant.

4.26. Lemma. *Let W be a Q-cofibrant* Γ- *category and let X be a coherently commutative monoidal category. Then the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W,X)$ is also a coherently commutative monoidal category.*

Proof. We begin by recalling that

$$\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W,X)((k+l)^+) = \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, X((k+l)^+ \wedge -)).$$

Since X is a coherently commutative monoidal category, therefore $X((k+l)^+ \wedge -)$ is also a coherently commutative monoidal category, for all $k, l \geq 0$ according to proposition 4.23. The proposition 4.25 tells us that the map $(X(\delta_k^{k+l} \wedge -), X(\delta_l^{k+l} \wedge -))$ is a strict equivalence of Γ- categories. Now Theorem 4.10 implies that the following induced functor on the mapping (pointed) categories

$$\begin{aligned}(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, X(\delta_k^{k+l} \wedge -)), \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, X(\delta_l^{k+l} \wedge -))) &: \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, X((k+l)^+ \wedge -)) \\ &\longrightarrow \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, X((k)^+ \wedge -)) \times \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, X((l)^+ \wedge -))\end{aligned}$$

is an equivalence of categories. ∎

Finally we get to the main result of this section. All the lemmas proved above will be useful in proving the following theorem:

4.27. Theorem. *The model category of coherently commutative monoidal categories is a symmetric monoidal closed model category under the Day convolution product.*

Proof. Let $i : U \longrightarrow V$ be a Q-cofibration and $j : Y \longrightarrow Z$ be another Q-cofibration. We will prove the theorem by showing that the following *pushout product* morphism

$$i \square j : U * Z \coprod_{U*Y} V * Y \longrightarrow V * Z$$

is a Q-cofibration which is also an equivalence of coherently commutative monoidal categories whenever either i or j is an equivalence of coherently commutative monoidal categories. We first deal with the case of i being a generating Q-cofibraion. The closed symmetric monoidal model structure on the strict Q-model category, see theorem 4.10, implies that $i \square j$ is a Q-cofibration. Let us assume that j is an acyclic Q-cofibration *i.e.*

the Q-cofibration j is also an equivalence of coherently commutative monoidal categories. According to proposition 4.22 the Q-cofibration $i\Box j$ is an equivalence of coherently commutative monoidal categories if and only if it has the left lifting property with respect to all strict fibrations of Γ- categories between coherently commutative monoidal categories. Let $p : W \longrightarrow X$ be a strict fibration between two coherently commutative monoidal categories. A (dotted) lifting arrow would exists in the following diagram

$$\begin{tikzcd} U * Z \coprod\limits_{U*Y} V * Y \arrow[r] \arrow[d] & W \arrow[d, "p"] \\ V * Z \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

if and only if a (dotted) lifting arrow exists in the following adjoint commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(V, W) \arrow[d, "(j^*, p^*)"] \\ D \arrow[r] \arrow[ur, dotted] & \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(U, X) \underset{\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(U,Y)}{\times} \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(V, Y) \end{tikzcd}$$

The map (j^*, p^*) is a strict fibration of Γ- categories by lemma [Hov99, Lemma 4.2.2] and theorem 4.10. Further the observation that both V and U are Q-cofibrant and the above lemma 4.26 together imply that (j^*, p^*) is a strict fibration between coherently commutative monoidal categories and therefore a fibration in the model category of coherently commutative monoidal categories. Since j is an acyclic cofibration by assumption therefore the (dotted) lifting arrow exists in the above diagram. Thus we have shown that if i is a Q-cofibration and j is a Q-cofibration which is also a weak equivalence in the model category of coherently commutative monoidal categories then $i\Box j$ is an acyclic cofibration in the model category of coherently commutative monoidal categories. Now we deal with the general case of i being an arbitrary Q-cofibration. Consider the following set:

$$S = \{ i : U \longrightarrow V \mid \; i\Box j \quad \textit{is an acyclic cofibration in } \}$$

where $\Gamma\mathbf{Cat}$ is endowed with the model structure of coherently commutative monoidal categories. We have proved above that the set S contains all generating Q-cofibrations. We observe that the set S is closed under pushouts, transfinite compositions and retracts. Thus S contains all Q-cofibrations. Thus we have proved that $i\Box j$ is a cofibration which is acyclic if j is acyclic. The same argument as above when applied to the second argument of the Box product (*i.e.* in the variable j) shows that $i\Box j$ is an acyclic cofibration whenever i is an acyclic cofibration in the model category of coherently commutative monoids.

■

Finally we will provide a characterization of cofibrations in the model category of coherently commutative monoidal categories.

4.28. Notation. *For each morphism* $F : X \longrightarrow Y$ *in* $\Gamma\mathbf{Cat}$ *we get a collection of object functions* $\{Ob(F(k^+)) : Ob(X(k^+)) \longrightarrow Ob(Y(k^+)) : k^+ \in Ob(\Gamma^{op})\}$. *These functions glue together into a* Γ*-set, which we denote by* $Ob(F)$*, whose structure maps are just the object functions of the structure maps (functors) of* F*,* i.e.

$$Ob(F)(f) := Ob(F(f)),$$

for each $f \in Mor(\Gamma^{op})$.

4.29. Lemma. *A map* $i : A \longrightarrow B$ *in* $\Gamma\mathbf{Cat}$ *is a cofibration in the model category of coherently commutative monoidal categories if and only if the* Γ*-set* $Ob(i)$ *has the left lifting property with respect to every surjective function of* Γ*-sets.*

Proof. Let $p : X \longrightarrow Y$ be an acyclic fibration in the model category of coherently commutative monoidal categories. For any (outer) commutative diagram in $\Gamma\mathbf{Cat}$

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "G"'] \arrow[ur, dashed, "L"] & Y \end{tikzcd}$$

we will construct a dotted arrow L which will make the whole diagram commutative. The morphism $Ob(p)$ is a surjective map of Γ-sets. By assumption the (outer) commutative diagram

$$\begin{tikzcd} Ob(A) \arrow[r, "Ob(H)"] \arrow[d, "Ob(i)"'] & Ob(X) \arrow[d, "Ob(p)"] \\ Ob(B) \arrow[r, "Ob(G)"'] \arrow[ur, dashed, "Ob(L)"] & Ob(Y) \end{tikzcd}$$

has a (dotted) lifting arrow (of Γ-sets) $Ob(L)$ which makes the whole diagram commutative. For each $k^+ \in Ob(\Gamma^{op})$ and each pair of objects $a, b \in Ob(B(k^+))$ we want to define a function

$$L(k^+)_{a,b} : Mor_{B(k^+)}(a, b) \longrightarrow Mor_{X(k^+)}(Ob(L)(k^+)(a), Ob(L)(k^+)(b)).$$

By assumption, the map p is an acyclic fibration therefore for each $k^+ \in Ob(\Gamma^{op})$, the functor $p(k^+)$ is an acyclic fibration in the natural model category structure on $\mathbf{Cat}$. This implies that the function

$$p(k^+)_{v,w} : Mor_{X(k^+)}(Ob(L)(k^+)(a), Ob(L)(k^+)(b)) \longrightarrow Mor_{Y(k^+)}(G(k^+)(a), G(k^+)(b))$$

is a bijection. Now we define the function $L(k^+)_{a,b}$ to be the following composite

$$Mor_{B(k^+)}(a, b) \xrightarrow{G(k^+)_{a,b}} Mor_{Y(k^+)}(G(k^+)(a), G(k^+)(b))$$
$$\xrightarrow{p(k^+)^{-1}_{v,w}} Mor_{X(k^+)}(Ob(L)(k^+)(a), Ob(L)(k^+)(b)),$$

where $(v, w) = (Ob(L)(k^+)(a), Ob(L)(k^+)(b))$ An argument similar to the one in the proof of Lemma 3.8 shows that the above collection of maps $\{L(k^+)_{a,b} : a, b \in Ob(B(k^+))\}$ defines a functor $L(k^+)$ whose object function is the same as $Ob(L)(k^+)$. Now we want to check whether this collection of functors $\{L(k^+) : k^+ \in Ob(\Gamma^{op})\}$ glues together into a morphism of Γ- categories. It would be sufficient to show that for any $f : k^+ \longrightarrow m^+ \in Mor(\Gamma^{op})$, the following diagram commutes:

$$\begin{tikzcd}
Mor_{B(k^+)}(a,b) \arrow[r, "L(k^+)_{a,b}"] \arrow[d, "B(f)_{a,b}"'] & Mor_{X(k^+)}(v,w) \arrow[d, "X(f)_{v,w}"] \\
Mor_{B(m^+)}(B(f)(a), B(f)(b)) \arrow[r, "L(m^+)_{B(f)(a),B(f)(b)}"'] & Mor_{X(m^+)}(x,y)
\end{tikzcd} \tag{16}$$

where $(x, y) = (Ob(L)(m^+)(B(f)(a)), Ob(L)(m^+)(B(f)(b)))$. Since p and G are maps of Γ- categories therefore we have following (solid arrow) commutative diagram of mapping sets:

$$\begin{tikzcd}
 & Mor_{B(m^+)}(B(f)(a), B(f)(b)) \arrow[rr, "G(m^+)"] & & Mor_{Y(m^+)}(q,r) \arrow[dd, dashed, bend left, "p(m^+)^{-1}_{x,y}"] \\
Mor_{B(k^+)}(a,b) \arrow[ur, "B(f)_{a,b}"] \arrow[r, "G(k^+)_{a,b}"'] & Mor_{Y(k^+)}(G(k^+)(a), G(k^+)(b)) \arrow[urr, "Y(f)"'] \arrow[d, dashed, bend left, "p(k^+)^{-1}_{v,w}"] & & \\
 & Mor_{X(k^+)}(v,w) \arrow[u, "p(k^+)_{v,w}"] \arrow[rr, "X(f)_{v,w}"'] & & Mor_{X(m^+)}(x,y) \arrow[uu, "p(m^+)_{x,y}"]
\end{tikzcd}$$

where

$$(q, r) = (G(m^+)(B(f)(a)), G(m^+)(B(f)(b))) = (Y(f)(G(k^+)(a)), Y(f)(G(k^+)(b))).$$

Since the dotted arrows are the inverses to the associated solid arrows therefore the entire diagram is commutative. This commutativity implies that the diagram 16 is commutative.

■

5. Segal's Nerve functor

In the paper [Seg74], Segal described a construction of a Γ- category from a (small) symmetric monoidal category which we call the *Segal's nerve* of the symmetric monoidal category. His construction defined a functor which we call *Segal's nerve functor*. This functor was further studied in [SS79], [May78]. Oplax and lax variations of Segal's nerve functor were defined in [Man10], [EM06]. In this section we will review Segal's nerve functor and describe a new representation of Segal's nerve functor. The Segal's nerve functor is built on a family of discrete categories which carry a partial symmetric monoidal structure namely $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$, where $\mathcal{P}(n)$ denotes the *power set* of the finite set $\underline{n}$. The partial symmetric monoidal structure is just the union of disjoint subsets of $\underline{n}$. Segal's nerve of a symmetric monoidal category consists of, in degree n, the category of all functors which preserve this partial symmetric monoidal structure upto isomorphism. One of our goals in this section is to further clarify the situation by firstly defining an unnormalized version of Segal's nerve and secondly by *completing* the partial symmetric monoidal structures and thereby present a construction of Segal's nerve using (strict) symmetric monoidal functors. For each $n \in \mathbb{N}$ we construct a permutative category $\mathcal{L}(n)$ which is equipped with an inclusion functor $i : \mathcal{P}(n) \longrightarrow \mathcal{L}(n)$ and which satisfies the following universal property:

$$\begin{tikzcd} \mathcal{P}(n) \arrow[r, "F"] \arrow[d, "i"'] & C \\ \mathcal{L}(n) \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

where the functor F satisfies $F(S \sqcup T) \cong F(S) \underset{C}{\otimes} F(T)$ for all $S, T \in \mathcal{P}(n)$ and $S \cap T = \emptyset$. This allows us to define our *unnormalized Segal's nerve*, in degree n, as follows:

$$\mathcal{K}(C)(n^+) := [\mathcal{L}(n), C]^{str}_{\otimes}.$$

We will show the existence of a functor $\mathcal{L} : \Gamma\mathbf{Cat} \longrightarrow \mathbf{Perm}$ which is a left adjoint to the unnormalized Segal's Nerve functor $\mathcal{K}$. The main objective of this section is to show that the adjoint pair of functors $(\mathcal{L}, \mathcal{K})$ induces a Quillen equivalence between the natural model category **Perm** and the model category of coherently commutative monoidal categories $\Gamma\mathbf{Cat}$. We begin by reviewing Segal's construction.

5.1. Definition. *An nth* Segal bicycle *into a symmetric monoidal category C is a triple $\Psi = (\Psi, \sigma_\Psi, u_\Psi)$, where Ψ is a family of objects of C*

$$\Psi = \{\Psi(S) : \Psi(S) \in Ob(C)\}_{S \in \mathcal{P}(n)}$$

and σ_Ψ is a family of morphisms of C

$$\sigma_\Psi = \{\sigma_\Psi((S,T)) : \Psi(S \sqcup T) \xrightarrow{\cong} \Psi(S) \otimes \Psi(T) : f_{(S,T)} \in Mor(C)\}_{(S,T) \in \Lambda},$$

where the indexing set $\Lambda := \{(S,T) : S, T \subseteq n, S \cap T = \emptyset\}$. Finally $u_\Psi : \Psi(\emptyset) \xrightarrow{\cong} 1_C$ is an isomorphism in C. This triple is subject to the following conditions:

SB.1 For each $S \in \mathcal{P}(n)$, the following diagram commutes:

$$\begin{tikzcd} \Psi(\emptyset) \otimes \Psi(S) \arrow[d, "u_\psi"'] & \Psi(S) \arrow[l, "\sigma_\Psi((\emptyset, S))"'] \arrow[r, "\sigma_\Psi((S,\emptyset))"] \arrow[d, equal] & \Psi(S) \otimes \Psi(\emptyset) \arrow[d, "u_\Psi"] \\ 1_C \otimes \Psi(S) \arrow[r, "\beta_l"'] & \Psi(S) & \Psi(S) \otimes 1_C \arrow[l, "\beta_r"] \end{tikzcd}$$

SB.2 For each triple $S, T, U \in \mathcal{P}(n)$ of mutually disjoint subsets of $\underline{n}$, the following diagram commutes:

$$\begin{tikzcd}
\Psi(S \sqcup T \sqcup U) \arrow[rr, "\sigma_\Psi((S,T \coprod U))"] \arrow[d, "\sigma_\Psi((S \sqcup T, U))"'] & & \Psi(S) \otimes \Psi(T \sqcup U) \arrow[d, "id \otimes \sigma_\Psi((T,U))"] \\
\Psi(S \sqcup T) \otimes \Psi(U) \arrow[dr, "\sigma_\Psi((S,T)) \otimes id"'] & & \Psi(S) \otimes (\Psi(T) \otimes \Psi(U)) \arrow[dl, "\alpha_C(\Psi(S),\Psi(T),\Psi(U))"] \\
& (\Psi(S) \otimes \Psi(T)) \otimes \Psi(U)) &
\end{tikzcd}$$

SB.3 For each pair $S, T \in \mathcal{P}(n)$ of disjoint subsets of $\underline{n}$, the following diagram commutes:

$$\begin{tikzcd}
\Psi(S \sqcup T) \arrow[r, "\sigma_\Psi((S,T))"] \arrow[dr, "\sigma_\Psi((T,S))"'] & \Psi(S) \otimes \Psi(T) \arrow[d, "\gamma^C((S,T))"] \\
& \Psi(T) \otimes \Psi(S)
\end{tikzcd}$$

Next we define morphisms of Segal bicycles

5.2. Definition. *A morphism of nth Segal bicycles $\tau : (\Psi, \sigma_\Psi) \longrightarrow (\Omega, \sigma_\Omega)$ in a symmetric monoidal category C is a family of maps of C*

$$\tau = \{\tau(S) : \Psi(S) \longrightarrow \Omega(S)\}_{S \in \mathcal{P}(\underline{n})}$$

such that the following two diagram commutes

$$\begin{tikzcd}
\Psi(S \coprod T) \arrow[r, "\tau(S \coprod T)"] \arrow[d, "\sigma_\Psi((S,T))"'] & \Omega(S \coprod T) \arrow[d, "\sigma_\Omega((S,T))"] \\
\Psi(S) \otimes \Psi(T) \arrow[r, "\tau(S) \otimes \tau(T)"'] & \Omega(S) \otimes \Omega(T)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi(\emptyset) \arrow[rr, "\tau(\emptyset)"] \arrow[dr, "u_\Psi"'] & & \Omega(\emptyset) \arrow[dl, "u_\Omega"] \\
& 1_C &
\end{tikzcd}$$

Given a permutative category C, nth Segal bicycles and morphisms of *nth* Segal bicycles define a category which we denote by $\mathcal{K}(C)(n^+)$. Next we want to compare the notion of an *nth* Segal bicycle to that of a strict bicycle in the permutative category C:

5.3. Lemma. *Let C be a permutative category. For each $\underline{n}$, the category $\mathcal{K}(C)(n)$ is isomorphic to the category of all strict bicycles from Γ^n to C, $\boldsymbol{Bikes}^{Str}(\Gamma^n, C)$.*

Proof. We will prove the lemma by constructing a pair of inverse functors. We begin by defining a functor $F : \mathcal{K}(C)(n) \longrightarrow \mathbf{Bikes}^{Str}(\Gamma^n, C)$. Let $(\Psi, \sigma_\Psi, u_\Psi) \in Ob(\mathcal{K}(C)(n))$, then for each $k \in Ob(\mathcal{N})$ we define a functor $\phi(k) : \Gamma^n(k^+) \longrightarrow C$ as follows:

$$\phi(k)(f) := \Psi(Supp(f)),$$

where $Supp(f) \subset n$ is the *support of* $f : n^+ \longrightarrow k^+$. The collection $\{\phi(k)\}_{k \in Ob(\mathcal{N})}$ defines a (strict) cone $\mathcal{L}_\Phi$ because for any $h : k \longrightarrow l$ in $Mor(\mathcal{N})$, $Supp(f) = Supp(h \circ f)$. Similarly for each pair $(k, l) \in Ob(\mathcal{N}) \times Ob(\mathcal{N})$, we will define a natural isomorphism $\sigma_\Phi(k, l) : \phi(k + l) \Rightarrow \phi(k) \odot \phi(l)$. To each $g \in \Gamma^n((k + l)^+)$ we can associate a pair of functions (g_k, g_l) whose components are defined by the following two diagrams

$$\begin{tikzcd} n^+ \arrow[r, "g"] \arrow[dr, "g_k"'] & (k+l)^+ \arrow[d, "\delta^{k+l}_k"] \\ & k^+ \end{tikzcd} \qquad \begin{tikzcd} n^+ \arrow[r, "g"] \arrow[dr, "g_l"'] & (k+l)^+ \arrow[d, "\delta^{k+l}_l"] \\ & l^+ \end{tikzcd}$$

we define

$$\sigma_\Phi(k, l)(g) := \sigma_\Psi(Supp(g_k), Supp(g_l))$$

and $u_\Phi(0) := u_\Psi$. We define the object function of F as follows:

$$F((\Psi, \sigma_\Psi, u_\Psi)) := (\Phi, \sigma_\Phi, u_\Phi).$$

For each morphism $\tau : \Psi \longrightarrow \Upsilon$ in $\mathcal{K}(C)(n^+)$, we define a map of bicycles $F(\tau)$ as follows:

$$F(\tau) = \{F(\tau)(f) := \tau(Supp(f)) : \phi(k)(f) \longrightarrow \upsilon(k)(f) : f \in \bigsqcup_{k \in \mathbb{N}} \Gamma^n(k^+)\}$$

It follows from definition 5.2 that $F(\tau)$ is a morphism of strict bicycles.

Now we define the inverse functor G as follows: Let $(\mathcal{L}, \sigma, \tau) = \Phi : \Gamma^n \rightsquigarrow C$ be a strict bicycle where $\mathcal{L} = (\phi, \sigma)$ is its underlying strict cone, see appendix A, then we define an *n-th* unnormalized Segal bicycle $(G(\Phi), \sigma_{G(\Phi)}, u_{G(\Phi)})$ as follows:

$$G(\Phi) = \{G(\Phi)(S) : G(\Phi)(S) = \phi(f_S)\}_{S \in \mathcal{P}(\underline{n})}$$

where $f_S : n^+ \longrightarrow S^+$ is the projection map whose support is $S \subseteq \underline{n}$. Next we define the family of isomorphism $\sigma_{G(\Phi)}$. For each pair (S, T) of disjoint subsets of $\underline{n}$ we get a map $f_{S+T} : n^+ \longrightarrow (S + T)^+$ in the category $\Gamma^n((S + T)^+)$. We now define

$$\sigma_{G(\Phi)}((S, T)) := \sigma((S, T))(f_{S+T}).$$

Finally $u_{G(\Phi)} := \tau(id_{0^+})$.

A map of strict bicycles $F : \Phi \longrightarrow \Psi$ determines a collection of maps of C

$$G(F) := \{G(F)(S) : G(\Phi)(S) \longrightarrow G(\Psi)(S)\}_{S \in \mathcal{P}(\underline{n})}$$

This collection glues together to define a map of *n-th* unnormalized Segal bicycles. It is easy to see that the functors F and G are inverse of each other. ∎

5.4. Definition. *For each $n \in \mathbb{N}$ we will now define a permutative groupoid $\mathcal{L}(n)$. The objects of this groupoid are finite sequences of subsets of $\underline{n}$. We will denote an object of this groupoid by $(S_1, S_2, \dots, S_r)$, where $S_1, \dots S_r$ are subsets of $\underline{n}$. A morphism $(S_1, S_2, \dots, S_r) \longrightarrow (T_1, T_2, \dots, T_k)$ is an isomorphism of finite sets $F : S_1 \sqcup S_2 \sqcup \dots \sqcup S_r \xrightarrow{\cong} T_1 \sqcup T_2 \sqcup \dots \sqcup T_k$ such that the following diagram commutes*

$$\begin{tikzcd} S_1 \sqcup S_2 \sqcup \dots \sqcup S_r \arrow[rr, "F"] \arrow[dr] & & T_1 \sqcup T_2 \sqcup \dots \sqcup T_k \arrow[dl] \\ & \underline{n} & \end{tikzcd}$$

where the diagonal maps are the unique inclusions of the coproducts into $\underline{n}$.

We define a subcategory $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ of $\overline{\overline{\mathcal{L}}}(\Gamma^n)$ which will turn out to be a *coreflective* subcategory. An object of $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ is a finite sequence $S = (S_1, S_2, \dots, S_r)$, where S_i is a subset of $\underline{n}$ for $1 \le i \le r$.

5.5. Notation. *An object $S = (S_1, S_2, \dots, S_r) \in Ob(\mathcal{P}\overline{\overline{\mathcal{L}}}(n))$ uniquely determines a morphism (of unbased sets) $\sigma(S) : \bigsqcup_{i=1}^{r} S_i \longrightarrow \underline{n}$. We will refer to the map $\sigma(S)$ as the* canonical inclusion *of S in $\underline{n}$.*

5.6. Notation. *An object $S = (S_1, S_2, \dots, S_r) \in Ob(\mathcal{P}\overline{\overline{\mathcal{L}}}(n))$ uniquely determines a morphism (of unbased sets) $Ind(S) : \bigsqcup_{i=1}^{r} S_i \longrightarrow \underline{r}$. We will refer to the map $Ind(S)$ as the* canonical index *of S.*

Given another object $T = (T_1, T_2, \dots, T_s)$ in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$, where T_j is a subset of $\underline{n}$ for $1 \le j \le r$, a morphism $F : S \longrightarrow T$ in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ is a pair (h, p), where $h : \underline{s} \longrightarrow \underline{r}$ is a map of finite unbased sets and $p : \bigsqcup_{i=1}^{r} S_i \longrightarrow \bigsqcup_{j=1}^{s} T_j$ is a bijection. The pair is subject to the following condition:

1. The following diagram commutes:

$$\begin{tikzcd} & \underline{n} & \\ \bigsqcup_{i=1}^{r} S_i \arrow[ur, "\sigma(S)"] \arrow[rr, "p"] \arrow[d, "Ind(S)"'] & & \bigsqcup_{j=1}^{s} T_j \arrow[ul, "\sigma(T)"'] \arrow[d, "Ind(T)"] \\ \underline{r} & & \underline{s} \arrow[ll, "h"] \end{tikzcd}$$

5.7. Remark. *The construction above defines a contravariant functor*

$$\mathcal{P}\overline{\overline{\mathcal{L}}}(-) : \Gamma^{op} \longrightarrow \mathbf{Perm}.$$

A map $f : n^+ \longrightarrow m^+$ *in* Γ^{op} *defines a strict symmetric monoidal functor* $\mathcal{P}\overline{\overline{\mathcal{L}}}(f) : \mathcal{P}\overline{\overline{\mathcal{L}}}(m) \longrightarrow \mathcal{P}\overline{\overline{\mathcal{L}}}(n)$. *An object* $(S_1, S_2, \dots, S_r) \in \mathcal{P}\overline{\overline{\mathcal{L}}}(m)$ *is mapped by this functor to* $(f^{-1}(S_1), f^{-1}(S_2), \dots, f^{-1}(S_r)) \in \mathcal{P}\overline{\overline{\mathcal{L}}}(n)$.

5.8. Definition. *For each* $n \in \mathbb{N}$ *we define a permutative category* $\overline{\overline{\mathcal{L}}}(\Gamma^n)$ *as follows:*

$$\overline{\overline{\mathcal{L}}}(\Gamma^n) := \int^{\vec{k} \in \mathfrak{L}} \mathfrak{L}(\Gamma^n)(\vec{k}),$$

see 2.46. This construction defines a functor $\overline{\overline{\mathcal{L}}}(\Gamma^-)$ *which is the following composite*

$$\Gamma^{op} \xrightarrow{y} \Gamma\mathbf{Cat} \xrightarrow{\mathfrak{L}(-)} [\mathfrak{L}, \mathbf{Cat}]_{\otimes} \longrightarrow \mathbf{Perm}$$

where y *is the Yoneda functor.* $\mathfrak{L}(-)$ *is the functor defined 4.*

5.9. Proposition. *The category* $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ *is isomorphic to the full subcategory of* $\overline{\overline{\mathcal{L}}}(\Gamma^n)$ *whose objects are finite sequences of projection maps in* Γ^{op} *having domain* n^+.

Proof. We will define a functor $G : \mathcal{P}\overline{\overline{\mathcal{L}}}(n) \longrightarrow \overline{\overline{\mathcal{L}}}(\Gamma^n)$. This functor is defined on objects as follows:

$$G((S_1, S_2, \dots, S_r)) := (f_1, f_2, \dots, f_r),$$

where $S = (S_1, S_2, \dots, S_r)$ is an object in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ and each $f_i : n^+ \longrightarrow S_i^+$ is a projection map onto S_i. Let $T = (T_1, T_2, \dots, T_s)$ be another object in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$. A map $(h, p) : S \longrightarrow T$ in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ is also a map in $\mathfrak{L}$ such that $\mathfrak{L}(\Gamma^n)((h, p))(S) = T$. This defines the functor G which is fully faithful. ∎

5.10. Remark. *The category* $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ *is a coreflective subcategory of* $\overline{\overline{\mathcal{L}}}(\Gamma^n)$ *as a result of 5.19 and 5.21.*

5.11. Remark. *The functor* $G : \mathcal{P}\overline{\overline{\mathcal{L}}}(n) \longrightarrow \overline{\overline{\mathcal{L}}}(\Gamma^n)$ *defined in the proof above is a strict symmetric monoidal functor. This functor is a component of a natural transformation between two contravariant functors*

$$i : \mathcal{P}\overline{\overline{\mathcal{L}}}(-) \Rightarrow \overline{\overline{\mathcal{L}}}(\Gamma^-),$$

where $i(n^+) := G$ *and* $\overline{\overline{\mathcal{L}}}(\Gamma^-) : \Gamma^{op} \longrightarrow \mathbf{Perm}$ *is the functor that maps* n^+ *to* $\overline{\overline{\mathcal{L}}}(\Gamma^n)$.

5.12. Remark. *Composing the natural transformation* i *in the above remark with the functor* Π_1 *gives us a natural equivalence*

$$id_{\Pi_1} \circ i : \Pi_1 \circ \mathcal{P}\overline{\overline{\mathcal{L}}}(-) \Rightarrow \Pi_1 \circ \overline{\overline{\mathcal{L}}}(\Gamma^-)$$

i.e. *for each* $n^+ \in \Gamma^{op}$ *the functor*

$$id_{\Pi_1} \circ i(n^+) : \Pi_1(\mathcal{P}\overline{\overline{\mathcal{L}}}(n)) \longrightarrow \Pi_1(\overline{\overline{\mathcal{L}}}(\Gamma^n))$$

is an equivalence of categories.

5.13. NOTATION. *Let C be a strict symmetric monoidal category. Let us denote by $\underline{C}$ the underlying groupoid of C* i.e. *the groupoid obtained by discarding all non-invertible maps in C. We recall that $\underline{C}$ retains the strict symmetric monoidal structure of C.*

We recall that a strict bicycle $\Psi = (\psi, \sigma_\Psi, u_\Psi) : \Gamma^n \rightsquigarrow C$ defines an oplax symmetric monoidal functor $\underline{\Psi} : \mathcal{N} \longrightarrow \left(\underline{C}^{\Gamma^n}\right)^{Ps}$, see appendix B. For each $\underline{k} \in \mathcal{N}$, there is a functor $\underline{\Psi}(k) : \Gamma^n(k) \longrightarrow \underline{C}$ which is defined as follows:

$$\underline{\Psi}(k)(f) := \psi(k)(f)$$

for each $f \in \Gamma^n(k)$. For each morphism $h : k \longrightarrow l$ in $\mathcal{N}$, $\underline{\Psi}(h) := id$, *i.e.* the identity natural transformation. The family of natural isomorphisms σ_Φ and the unit natural isomorphism u_Ψ provide an oplax symmetric monoidal structure on $\underline{\Psi}$. The oplax symmetric monoidal inclusion functor $i : \left(\underline{C}^{\Gamma^n}\right)^{Ps} \longrightarrow \left(\underline{C}^{\mathfrak{L}(\Gamma^n)}\right)^{Ps}$ provides the following composite oplax symmetric monoidal functor

$$\mathcal{N} \xrightarrow{\Phi} \left(\underline{C}^{\Gamma^n}\right)^{Ps} \xrightarrow{i} \left(\underline{C}^{\mathfrak{L}(\Gamma^n)}\right)^{Ps}.$$

This composite oplax symmetric monoidal functor extends uniquely, along the inclusion $\mathcal{N} \hookrightarrow \mathfrak{L}$, into a strict symmetric monoidal functor $\mathfrak{L}(\underline{\Psi}) : \mathfrak{L} \longrightarrow \left(\underline{C}^{\mathfrak{L}(\Gamma^n)}\right)^{Ps}$, see appendix E. This functor uniquely determines another strict symmetric monoidal functor

$$\widetilde{\Phi} : \overline{\overline{\mathcal{L}}}(\Gamma^n) \longrightarrow \underline{C}. \tag{17}$$

Let $\phi : \mathcal{P}\overline{\overline{\mathcal{L}}}(n) \longrightarrow \underline{C}$ be a strict symmetric monoidal functor. The functor ϕ determines a strict Segal bicycle $(F(\phi), \sigma_{F(\phi)}, u_{F(\phi)})$ which we now define. For each $S \subseteq \underline{n}$, we define $F(\phi)(S) = \phi((S))$. The collection of isomorphism $\sigma_{F(\phi)}$ is defined as follows:

$$\sigma_{F(\phi)}((S,T)) := \phi((m, id)),$$

where $(m, id) : (S \sqcup T) \longrightarrow (S, T)$ is a map in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ whose first component is given by the multiplication map $m : \underline{2} \longrightarrow \underline{1}$. Finally, the isomorphism $u_{F(\phi)}$ is defined as follows:

$$u_{F(\phi)} := \phi((id, i)),$$

where $(id, i) : (\emptyset) \longrightarrow ()$ is the following map in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$:

$$\begin{tikzcd} \emptyset \arrow[r, equal] \arrow[d] & \emptyset \arrow[d] \\ \underline{1} & \underline{0} \arrow[l, "i"] \end{tikzcd}$$

The conditions $SB1, SB2$ and $SB3$ follow from the strict symmetric monoidal functor structure of ϕ. The above construction defines a functor

$$F : [\mathcal{P}\overline{\overline{\mathcal{L}}}(n), \underline{C}]_{\otimes}^{str} \longrightarrow \mathcal{K}(C)(n^+).$$

5.14. Lemma. *The functor F is an isomorphism of categories.*

Proof. We will define a functor $F^{-1} : K(C)(n^+) \longrightarrow [\mathcal{P}\overline{\overline{\mathcal{L}}}(n), \underline{C}]^{str}_{\otimes}$ which is the inverse of F. An object $\Phi \in K(C)(n^+)$ is an nth strict Segal bicycle. An nth strict Segal bicycle uniquely determines a strict symmetric monoidal functor $\widetilde{\Phi} : \overline{\overline{\mathcal{L}}}(\Gamma^n) \longrightarrow \underline{C}$, see (17). Now we define the strict symmetric monoidal functor $F^{-1}(\Phi)$ to be the following composite:

$$\mathcal{P}\overline{\overline{\mathcal{L}}}(n) \cong \overline{\overline{\mathcal{L}}}(\Gamma^n)^{proj} \hookrightarrow \overline{\overline{\mathcal{L}}}(\Gamma^n) \xrightarrow{\widetilde{\Phi}} \underline{C}.$$

∎

5.15. Remark. *In the statement of the above lemma the functor category $[\mathcal{P}\overline{\overline{\mathcal{L}}}(n), \underline{C}]^{str}_{\otimes}$ could be replaced by the isomorphic category $[\Pi_1(\mathcal{P}\overline{\overline{\mathcal{L}}}(n)), C]^{str}_{\otimes}$ where $\Pi_1(\mathcal{P}\overline{\overline{\mathcal{L}}}(n))$ is the groupoid obtained by inverting all morphisms in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$.*

5.16. Notation. *Let $f = (f_1, f_2, \dots, f_r)$ be a finite sequence, where $f_i : n^+ \longrightarrow k_i^+$ is a map of based sets for $1 \le i \le r$. We denote the finite sequence $(Supp(f_1), Supp(f_2), \dots, Supp(f_r))$ by $Supp(f)$.*

5.17. Notation. *Let $f = (f_1, f_2, \dots, f_r)$ be a finite sequence, where $f_i : n^+ \longrightarrow k_i^+$ is a map of based sets for $1 \le i \le r$. We denote the sum $\underset{i=1}{\overset{r}{\sqcup}} f_i|_{Supp(f_i)}$ by $tot(f)$, where the map $f_i|_{Supp(f_i)} : Supp(f_i) \longrightarrow k_i$ is the restriction of f_i.*

We now define another category $\overline{\overline{\mathcal{L}}}(n)$ which is equipped with an inclusion functor

$$\iota : \mathcal{P}\overline{\overline{\mathcal{L}}}(n) \hookrightarrow \overline{\overline{\mathcal{L}}}(n). \tag{18}$$

5.18. Definition. *An object in $\overline{\overline{\mathcal{L}}}(n)$ is a finite sequence $f = (f_1, f_2, \dots, f_r)$, where $f_i : n^+ \longrightarrow k_i^+$ is a map of based sets for $1 \le i \le r$. To each finite sequence f one can (uniquely) associate the following zig-zag*

$$\begin{tikzcd} \underset{i=1}{\overset{r}{\sqcup}} Supp(f_i) \arrow[r, "tot(f)"] \arrow[d, "\sigma(f)"'] & \underset{i=1}{\overset{r}{\sqcup}} k_i \arrow[r, "Ind(f)"] & \underline{r} \\ \underline{n} & & \end{tikzcd}$$

where $\sigma(f) := \sigma(Supp(f))$. A map from f to $g = (g_1, g_2, \dots, g_s)$ in $\overline{\overline{\mathcal{L}}}(n)$ is a triple is a triple (h, q, p), where $h : \underline{s} \longrightarrow \underline{r}$ is a map in $\mathcal{N}$, q is a map in $\mathcal{N}$ and p is a bijection in

$\mathcal{N}$ *such that the following diagram commutes*

$$\begin{tikzcd}
& \underline{n} & \\
\overset{r}{\underset{i=1}{\sqcup}} Supp(f_i) \arrow[ur, "\sigma(f)"] \arrow[rr, "p"] \arrow[d, "tot(f)"'] & & \overset{s}{\underset{i=1}{\sqcup}} Supp(g_i) \arrow[ul, "\sigma(g)"'] \arrow[d, "tot(g)"] \\
\overset{r}{\underset{i=1}{\sqcup}} k_i \arrow[rr, "q"] \arrow[d, "Ind(f)"'] & & \overset{r}{\underset{i=1}{\sqcup}} l_i \arrow[d, "Ind(g)"] \\
\underline{r} & & \underline{s} \arrow[ll, "h"']
\end{tikzcd}$$

Each subset $S \subseteq \underline{n}$ uniquely determines a projection map $f_S : n^+ \longrightarrow S^+$. The inclusion functor (18) is defined on objects as follows:

$$\iota((S_1, S_2, \ldots, S_r)) := f_S = (f_{S_1}, f_{S_2}, \ldots, f_{S_r}).$$

The functor is defined on morphisms as follows:

$$\iota((h, p)) := (h, p, p),$$

where $(h, p) : S = (S_1, S_2, \ldots, S_r) \longrightarrow T = (T_1, T_2, \ldots, T_s)$ is a map in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ and $(h, p, p) : f_S \longrightarrow f_T = (f_{T_1}, f_{T_2}, \ldots, f_{T_s})$ is a map in $\overline{\overline{\mathcal{L}}}(n)$ which is described by the following diagram:

$$\begin{tikzcd}
& \underline{n} & \\
\overset{r}{\underset{i=1}{\sqcup}} Supp(f_{S_i}) = \overset{r}{\underset{i=1}{\sqcup}} S_i \arrow[ur, "\sigma(f_S)"] \arrow[rr, "p"] \arrow[d, "id"'] & & \overset{s}{\underset{j=1}{\sqcup}} T_j = \overset{s}{\underset{j=1}{\sqcup}} Supp(f_{T_j}) \arrow[ul, "\sigma(f_T)"'] \arrow[d, "id"] \\
\overset{r}{\underset{i=1}{\sqcup}} S_i \arrow[rr, "p"] \arrow[d, "Ind(f)"'] & & \overset{s}{\underset{j=1}{\sqcup}} T_j \arrow[d, "Ind(g)"] \\
\underline{r} & & \underline{s} \arrow[ll, "h"']
\end{tikzcd}$$

5.19. Lemma. *For each $n \in \mathbb{N}$ the permutative category $\overline{\overline{\mathcal{L}}}(n)$ is isomorphic to the permutative category $\overline{\overline{\mathcal{L}}}(\Gamma^n)$.*

Proof. The objects of both categories are the same. We will show that each morphism in $\overline{\overline{\mathcal{L}}}(\Gamma^n)$ uniquely defines a morphism in $\overline{\overline{\mathcal{L}}}(n)$ and therefore there is an isomorphism of permutative categories $J(n) : \overline{\overline{\mathcal{L}}}(\Gamma^n) \longrightarrow \overline{\overline{\mathcal{L}}}(n)$ which is identity on objects.

Let $\vec{f} = (f_1, \dots, f_r)$ and $\vec{g} = (g_1, \dots, g_s)$ be two objects in $\overline{\overline{\mathcal{L}}}(\Gamma^n)$ where $f_i : n^+ \longrightarrow k_i^+$ for $1 \leq i \leq r$ and $g_i : n^+ \longrightarrow l_i^+$ for $1 \leq i \leq s$. We recall that a map $F : \vec{f} \longrightarrow \vec{g}$ in $\overline{\overline{\mathcal{L}}}(\Gamma^n)$ is a map $F = (h, p) : \vec{k} = (k_1, \dots, k_r) \longrightarrow \vec{l} = (l_1, \dots, l_s)$ in the category $\mathfrak{L}$ such that

$$\mathfrak{L}(\Gamma^n)((h, p))((f_1, \dots, f_r)) = (g_1, \dots, g_s). \tag{19}$$

We recall that the map $F = (h, p)$ in $\mathfrak{L}$ is a commutative square

$$\begin{tikzcd}
\underline{k} \arrow[r, "p"] \arrow[d, "\vec{k}"'] & \underline{l} \arrow[d, "\vec{l}"] \\
\underline{r} & \underline{s} \arrow[l, "h"]
\end{tikzcd}$$

This map (h, p) uniquely determines a (finite) sequence $(p_1, \dots, p_r)$ of maps in $\mathcal{N}$ where $p_i : k_i \longrightarrow \underset{h(j)=i}{+} l_j$ for $1 \leq i \leq r$. The condition (19) implies that the following diagram commutes for each $i \in \underline{r}$ and $q \in \underline{s}$ such that $h(q) = i$:

$$\begin{tikzcd}
n^+ \arrow[r, "f_i"] \arrow[dr, "g_q"'] & k_i^+ \arrow[r, "p_i^+"] & (\underset{h(j)=i}{+} l_j)^+ \arrow[dl, "u_q"] \\
& l_q^+ &
\end{tikzcd}$$

where $u_q : (\underset{h(j)=i}{+} l_j)^+ \longrightarrow l_q$ is the projection map onto l_q where $h(q) = i$. Since each p_i is a map in $\mathcal{N}$ and the supports of u_q are distinct non-intersecting sets for $1 \leq q \leq s$ therefore we have the following equality:

$$Supp(f_i) = Supp(p_i^+ \circ f_i) = \underset{h(j)=i}{\sqcup} Supp(g_j).$$

The r inclusion maps $\underset{h(j)=i}{\sqcup} Supp(g_j) \subseteq \underset{i \in \underline{s}}{\sqcup} Supp(g_i)$ for $1 \leq i \leq r$ determine a canonical bijection

$$\sigma_F : \underset{i \in \underline{r}}{\sqcup} \underset{h(j)=i}{\sqcup} Supp(g_j) \longrightarrow \underset{i \in \underline{s}}{\sqcup} Supp(g_i)$$

such that the following diagram commmutes:

$$\begin{tikzcd}
& \underline{n} & \\
\underset{i \in \underline{r}}{\sqcup} Supp(f_i) = \underset{i \in \underline{r}}{\sqcup} \underset{j \in h^{-1}(\{i\})}{\sqcup} Supp(g_j) \arrow[ur] \arrow[rr, "\sigma_F"] \arrow[d, "tot(f)"'] & & \underset{i \in \underline{s}}{\sqcup} Supp(g_i) \arrow[ul] \arrow[d, "tot(g)"] \\
\underset{i \in \underline{r}}{\sqcup} k_i \arrow[rr, "p"] \arrow[d, "Ind(f)"'] & & \underset{i \in \underline{s}}{\sqcup} l_i \arrow[d, "Ind(g)"] \\
\underline{r} & & \underline{s} \arrow[ll, "h"]
\end{tikzcd}$$

Thus we have uniquely associated to an arrow $F = (h, p)$ in $\overline{\overline{\mathcal{L}}}(\Gamma^n)$, an arrow in the category $\overline{\overline{\mathcal{L}}}(n)$. We define the morphism function of $J(n)$ as follows:

$$J(n)(F = (h, p)) := (h, p, \sigma_F).$$

As mentioned above the object function of $J(n)$ is the identity. This defines a functor which is an isomorphism of categories. ∎

5.20. Remark. *By proposition 2.7 there exists a unique functor* $\overline{\overline{\mathcal{L}}}(-) : \Gamma^{op} \longrightarrow \mathbf{Perm}$ *and the family isomorphisms* $J(n)$ *in the lemma above glue together to define a natural isomorphism* $J : \overline{\overline{\mathcal{L}}}(\Gamma^-) \Rightarrow \overline{\overline{\mathcal{L}}}(-)$. *This implies that we have a composite natural transformation*

$$\mathcal{P}\overline{\overline{\mathcal{L}}}(-) \overset{i}{\Rightarrow} \overline{\overline{\mathcal{L}}}(\Gamma^-) \overset{J}{\Rightarrow} \overline{\overline{\mathcal{L}}}(-)$$

where i *is the natural transformation obtained in remark 5.11. Further the natural equivalence of remark 5.12 extends to a natural equivalence* $\Pi \circ (J \circ i)$

$$id_{\Pi_1} \circ (J \circ i) : \Pi_1 \circ \mathcal{P}\overline{\overline{\mathcal{L}}}(-) \Rightarrow \Pi_1 \circ \overline{\overline{\mathcal{L}}}(\Gamma^-) \Rightarrow \Pi_1 \circ \overline{\overline{\mathcal{L}}}(-).$$

We define another functor $G : \overline{\overline{\mathcal{L}}}(n) \longrightarrow \mathcal{P}\overline{\overline{\mathcal{L}}}(n)$. This functor is defined on objects as follows:

$$G((f_1, f_2, \ldots, f_r)) := Supp(f) = (Supp(f_1), Supp(f_2), \ldots, Supp(f_r)).$$

This functor is defined on morphisms as follows:

$$G((h, q, p)) := (h, p).$$

5.21. Theorem. *The category* $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ *is a coreflective subcategory of* $\overline{\overline{\mathcal{L}}}(n)$.

Proof. We will show that the functor G defined above is a right adjoint to the inclusion functor ι. It is easy to see that $id_{\mathcal{P}\overline{\overline{\mathcal{L}}}(n)} = G\iota$. We define a natural transformation $\epsilon : \iota G \Rightarrow id_{L''(n)}$. Let $f = (f_1, f_2, \ldots, f_r)$ be an object in $\overline{\overline{\mathcal{L}}}(n)$. We define

$$\epsilon(f) := (id, tot(f), id) : (f_{Supp(f_1)}, f_{Supp(f_2)}, \ldots, f_{Supp(f_r)}) = f_{Supp(f)} = \iota G(f) \longrightarrow f$$

The following commutative diagram verifies that the triple on the right is a map in $\overline{\overline{\mathcal{L}}}(n)$:

$$\begin{tikzcd}
& \underline{n} & \\
\bigsqcup_{i=1}^{r} Supp(f_{Supp(f_i)}) \arrow[ur, "\sigma(f_{Supp(f)})"] \arrow[rr, equal, "id"] \arrow[d, equal, "id"'] & & \bigsqcup_{i=1}^{r} Supp(f_i) \arrow[ul, "\sigma(f)"'] \arrow[d, "tot(f)"] \\
\bigsqcup_{i=1}^{r} Supp(f_{Supp(f_i)}) \arrow[rr, "tot(f)"] \arrow[d, "Ind(f_S)"'] & & \bigsqcup_{i=1}^{r} k_i \arrow[d, "Ind(f)"] \\
\underline{r} \arrow[rr, equal, "id"] & & \underline{r}
\end{tikzcd}$$

The following chain of equalities verifies that ϵ is a natural transformation:

$$(h,q,p)\circ\epsilon(f) = (h,q,p)\circ(id, tot(f), id) = (h, q\circ tot(f), id) = \\ (h, tot(g)\circ p, p) = (id, tot(g), id)\circ(h,p,p) = \epsilon(g)\circ\iota G((h,q,p)).$$

■

Now we want to define another category $QL''(n)$ which is isomorphic to the full subcategory of $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$ whose objects are finite sequences of, not necessarily distinct, singleton subsets of $\underline{n}$. We will denote an object of $QL''(n)$ by $s = (s_1, s_2, \ldots, s_r)$. Equivalently we may describe this object S by a map

$$s : \underline{r} \longrightarrow \underline{n}$$

A map $p : (s_1, s_2, \ldots, s_r) \longrightarrow (t_1, t_2, \ldots, t_r) = t$ in $QL''(n)$ is a bijection $p : \underline{r} \longrightarrow \underline{r}$ such that the following diagram commutes:

$$\begin{tikzcd} \underline{r} \arrow[rr, "p"] \arrow[dr, "s"'] & & \underline{r} \arrow[dl, "t"] \\ & \underline{n} & \end{tikzcd}$$

We observe that the category $QL''(n)$ is in fact a groupoid. We define a functor $H : \mathcal{P}\overline{\overline{\mathcal{L}}}(n) \longrightarrow QL''(n)$. Let $S = (S_1, S_2, \ldots, S_r)$ be an object of $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$, we define $H(S)$ to be the following composite where the first map is the canonical bijection

$$H(S) : \overset{r}{\underset{i=1}{+}} S_i \xrightarrow{can^{-1}} \bigsqcup_{i=1}^{r} S_i \xrightarrow{\sigma(S)} \underline{n},$$

where $+$ denotes the tensor product in $\mathcal{N}$. Let $(h,p) : S \longrightarrow T = (T_1, T_2, \ldots, T_s)$ be a map in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$. We define the morphism function of the functor H as follows:

$$H((h,p)) := \mathcal{N}(p),$$

where $\mathcal{N}(p) : \overset{s}{\underset{j=1}{+}} T_j \longrightarrow \overset{r}{\underset{i=1}{+}} S_i$ is the bijection in $\mathcal{N}$ which makes the following diagram commutative

$$\begin{tikzcd} \bigsqcup_{i=1}^{r} S_i \arrow[r, "p"] \arrow[d, "can"'] & \bigsqcup_{j=1}^{s} T_j \arrow[d, "can"] \\ \overset{r}{\underset{i=1}{+}} S_i & \overset{s}{\underset{j=1}{+}} T_j \arrow[l, "\mathcal{N}(p)"] \end{tikzcd}$$

It is easy to check that $H : \mathcal{P}\overline{\overline{\mathcal{L}}}(n) \longrightarrow QL''(n)$ is a functor. The following commutative diagram indicates the naturality in our definition of the functor H:

$$
\begin{tikzcd}
 & & \underline{n} & \\
 & \bigsqcup\limits_{i=1}^{r} S_i \arrow[ur, "\sigma(S)"] \arrow[rr, equal, bend left] \arrow[dl, "p", "\cong"'] \arrow[dd, "Ind(S)"] & & \bigsqcup\limits_{i=1}^{r} S_i \arrow[ul, "\sigma(S)"'] \arrow[dl, "p", "\cong"'] \arrow[dd, "can"] \\
\bigsqcup\limits_{j=1}^{s} T_j \arrow[uurr, bend left, "\sigma(T)"] \arrow[rr, equal, bend left] \arrow[dd, "Ind(T)"'] & & \bigsqcup\limits_{j=1}^{s} T_j \arrow[uu, bend right, "\sigma(T)"'] \arrow[dd, "can"] & \\
 & \underline{r} & & \mathop{+}\limits_{i=1}^{r} S_i \arrow[ll, bend left, "Ind(S)\circ can^{-1}"] \\
\underline{s} \arrow[ur, "h"'] & & \mathop{+}\limits_{j=1}^{s} T_j \arrow[ur, "\mathcal{N}(p)"'] \arrow[ll, bend left, "Ind(T)\circ can^{-1}"] &
\end{tikzcd}
\tag{20}
$$

The above diagram will be useful in proving that H is a left-adjoint-inverse. Now we define another functor $\iota : QL''(n) \longrightarrow \mathcal{P}\overline{\overline{\mathcal{L}}}(n)$. Let $s : \underline{r} \longrightarrow \underline{n}$ be an object in $QL''(n)$. The canonical inclusion of s in $\underline{n}$ can be factored as follows:

$$
\begin{tikzcd}
 & \bigsqcup\limits_{i=1}^{r} s(i) \arrow[dl, "Ind(s)"'] \arrow[dr, "\sigma(s)"] & \\
\underline{r} \arrow[rr, "s"'] & & \underline{n}
\end{tikzcd}
\tag{21}
$$

where $Ind(s)$ is the bijection $s(i) \mapsto i$ and $\sigma(s)$ is the canonical inclusion map. The functor ι is defined on objects as follows:

$$\iota(s) := (s(1), s(2), \ldots, s(r)).$$

Let $p : s \longrightarrow t$ be a map in $QL''(n)$, the functor ι is defined on morphisms as follows:

$$\iota(p) := (p^{-1}, p'),$$

where p' is the unique bijection which makes the following diagram commute:

$$
\begin{tikzcd}
\bigsqcup\limits_{i=1}^{r} s(i) \arrow[r, "p'"] \arrow[d, "Ind(s)"'] & \bigsqcup\limits_{i=1}^{r} t(i) \arrow[d, "Ind(t)"] \\
\underline{r} \arrow[d, "s"'] & \underline{r} \arrow[l, "p^{-1}"] \arrow[dl, "t"] \\
\underline{n} &
\end{tikzcd}
$$

By the above commutative diagram and factorization (21) we get the following commutative diagram which shows that $\iota(p) = (p^{-1}, p')$ is indeed a morphism in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$:

$$\begin{tikzcd}
 & \underline{n} & \\
\bigsqcup_{i=1}^{r} s(i) \arrow[ur, "\sigma(s)"] \arrow[rr, "p'"] \arrow[d, "Ind(s)"] & & \bigsqcup_{i=1}^{r} t(i) \arrow[ul, "\sigma(t)"'] \arrow[d, "Ind(t)"'] \\
\underline{r} \arrow[uur, bend left, "s"] & & \underline{r} \arrow[ll, "p^{-1}"] \arrow[uul, bend right, "t"']
\end{tikzcd}$$

5.22. Theorem. *The category $QL''(n)$ is isomorphic to a reflective subcategory of $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$.*

Proof. We will show that the functor H defined above is a left-adjoint-inverse to ι which is also defined above. Clearly $H\iota = id_{QL''(n)}$. We now construct a natural transformation $\eta : id \to \iota H$. Let $S = (S_1, S_2, \dots, S_r)$ be an object of $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$. We define

$$\eta(S) := (Ind(S) \circ can^{-1}, id).$$

The following commutative diagram verifies that the pair on the right is a map in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$:

$$\begin{tikzcd}
 & \underline{n} & \\
\bigsqcup_{i=1}^{r} S_i \arrow[ur, "\sigma(S)"] \arrow[rr, equal] \arrow[d, "Ind(S)"'] & & \bigsqcup_{i=1}^{r} S_i \arrow[ul, "\sigma(S)"'] \arrow[d, "can"] \\
\underline{r} & & \underset{i=1}{\overset{r}{+}} S_i \arrow[ll, "Ind(S) \circ can^{-1}"]
\end{tikzcd}$$

We claim that η as defined above is a natural transformation. Let $(h, p) : S \longrightarrow T = (T_1, T_2, \dots, T_j)$ be a map in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$. In order to prove our claim we would like to show that the following diagram commutes in $\mathcal{P}\overline{\overline{\mathcal{L}}}(n)$:

$$\begin{tikzcd}
S \arrow[r, "\eta(S)"] \arrow[d, "(h,p)"'] & \iota H(S) \arrow[d, "\iota H((h,p))"] \\
T \arrow[r, "\eta(T)"'] & \iota H(T)
\end{tikzcd}$$

The following chain of equalities verifies that η is a natural transformation:

$$\begin{aligned}
\eta(T) \circ (h, p) &= (Ind(T) \circ can^{-1}, id) \circ (h, p) = (h \circ (Ind(T) \circ can^{-1}), p) = \\
&((Ind(S) \circ can^{-1}) \circ H(p), p) = (H(p), p) \circ (Ind(S) \circ can^{-1}, id) = \iota H((h, p)) \circ \eta(S).
\end{aligned}$$

we refer the reader to the commutative diagram (20) for an explanation of the middle equalities. The composite natural transformation

$$id_{\mathcal{P}\overline{\overline{\mathcal{L}}}(n)} \circ \iota \Rightarrow \iota H \iota \Rightarrow \iota \circ id_{QL''(n)}$$

is the identity, this follows from the observation that

$$\eta(\iota(s)) = (Ind(\iota(s)) \circ can^{-1}, id) = (can \circ can^{-1}, id) = \iota((id, id)) = id_{\iota(s)}.$$

Similarly we claim that the following composite natural transformation

$$H \circ id_{\mathcal{P}\overline{\overline{\mathcal{L}}}(n)} \Rightarrow H\iota H \Rightarrow id_{QL''(n)} \circ H$$

is the identity. Our claim follows from the observation that

$$H(\eta(S)) = H((Ind(S) \circ can^{-1}, id)) = (id, id) = id_S.$$

■

The above discussion can be summarized by the following diagram in which both pairs of functors are adjunctions

$$\begin{tikzcd} QL''(n) \arrow[r, hook] & \mathcal{P}\overline{\overline{\mathcal{L}}}(n) \arrow[l, bend right, "H"'] \arrow[r, bend right] & \overline{\overline{\mathcal{L}}}(n) \arrow[l, bend right, "G"'] \end{tikzcd} \tag{22}$$

We observe that the groupoid $\mathcal{L}(n)$, see definition 5.4, is just the Gabriel factorization of the functor H. Since the functor H has a right adjoint, proposition 2.50 implies that the groupoid $\mathcal{L}(n)$ is isomorphic to $\Pi_1(\mathcal{P}\overline{\overline{\mathcal{L}}}(n))$. Thus we have the following lemma:

5.23. Lemma. *For a permutative category C, the category $K(C)(n^+)$ is isomorphic to the category of strict symmetric monoidal functors $[\mathcal{L}(n), C]^{str}_{\otimes}$.*

Proof. The above discussion and lemma 5.14 give us the following chain of isomorphisms:

$$\mathcal{K}(C)(n^+) \cong [\Pi_1(\mathcal{P}\overline{\overline{\mathcal{L}}}(n)), C]^{str}_{\otimes} \cong [\mathcal{L}(n), C]^{str}_{\otimes} \tag{23}$$

■

5.24. Remark. *There is a functor $\mathcal{P}\overline{\overline{\mathcal{L}}}(-) : \Gamma^{op} \longrightarrow \mathbf{Perm}^{op}$, see 5.8, which gives us a composite functor $\Pi_1 \circ \mathcal{P}\overline{\overline{\mathcal{L}}}(-)$. For each $n \in \mathbb{N}$ we have an isomorphism of categories $I(n) : \Pi_1(\mathcal{P}\overline{\overline{\mathcal{L}}}(n)) \cong \mathcal{L}(n)$. Now proposition 2.7 implies that we have a functor $\mathcal{L}(-)$: $\Gamma^{op} \longrightarrow \mathbf{Perm}^{op}$ and a natural isomorphism $I : \Pi_1 \circ \mathcal{P}\overline{\overline{\mathcal{L}}}(-) \Rightarrow \mathcal{L}(-)$.*

5.25. Remark. *There is bifunctor defined by the following composite:*

$$[\mathcal{L}(-), -]^{str}_{\otimes} : \Gamma^{op} \times \mathbf{Perm} \overset{\mathcal{L}(-)\times id}{\longrightarrow} \mathbf{Perm}^{op} \times \mathbf{Perm} \overset{[-,-]^{str}_{\otimes}}{\longrightarrow} \mathbf{Cat}$$

where $\mathcal{L}(-)$ is the functor defined above and $[-, -]^{str}_{\otimes}$ is the function object defined in remark 3.2.

5.26. Remark. *The above lemma 5.23 and the above remark together imply that for each pair* $(n^+, C) \in \Gamma^{op} \times \mathbf{Perm}$ *there is an isomorphism of categories* $\eta(n) : [\mathcal{L}(n), C]^{str}_{\otimes} \cong \mathcal{K}(C)(n^+)$. *Now proposition 2.7 implies that there is a bifunctor*

$$\mathcal{K}(-,-) : \Gamma^{op} \times \mathbf{Perm} \longrightarrow \mathbf{Cat}$$

defined by $\mathcal{K}(n^+, C) = \mathcal{K}(C)(n^+)$ *which is equipped with a natural isomorphism* $\eta : [\mathcal{L}(-), -]^{str}_{\otimes} \cong \mathcal{K}(-,-)$. *This also implies that there is a functor*

$$\mathcal{K} : \mathbf{Perm} \longrightarrow \Gamma\mathbf{Cat}$$

defined by $\mathcal{K}(C) := \mathcal{K}(-, C)$ *for each permutative category* C.

5.27. Remark. *The above lemma 5.23 implies that for each permutative category* C, *there is a* Γ- *category*

$$[\mathcal{L}(-), C]^{str}_{\otimes} : \Gamma^{op} \longrightarrow \mathbf{Cat}$$

and it is isomorphic to $\mathcal{K}(C)$.

5.28. Remark. *The natural equivalence from remark 5.20 extends to the following composite natural equivalence:*

$$id_{\Pi_1} \circ (J \circ i) \circ I^{-1} : \mathcal{L}(n) \overset{I^{-1}}{\Rightarrow} \Pi_1 \circ \mathcal{P}\overline{\overline{\mathcal{L}}}(-) \Rightarrow \Pi_1 \circ \overline{\overline{\mathcal{L}}}(\Gamma^-) \Rightarrow \Pi_1 \circ \overline{\overline{\mathcal{L}}}(-).$$

where I *is the natural isomorphism from remark 5.24*

The above lemma 5.23 implies that the functor $\mathcal{K}$ preserves limits in **Perm** because degreewise it is isomorphic to a functor which preserves limits. The category Γ**Cat** is complete and cocomplete. Now the *formal criterion for existence of an adjoint* [Mac71, Thm. 2, Ch. X.7] implies that $\mathcal{K}$ has a left adjoint which we denote

$$\mathcal{L} : \Gamma\mathbf{Cat} \longrightarrow \mathbf{Perm}.$$

Each $n \in Ob(\mathcal{N})$ uniquely defines n *projection* maps of based sets $\delta^n_k : n^+ \longrightarrow 1^+$, $1 \leq k \leq n$. Each of these projection maps induce a strict symmetric monoidal functor $\mathcal{L}(\delta^n_k) : \mathcal{L}(1) \longrightarrow \mathcal{L}(n)$ which maps the object $1 \in Ob(\mathcal{L}(1))$ to the inverse image of 1 under the map δ^n_k *i.e.* $\mathcal{L}(\delta^n_k)(1) = (\delta^n_k)^{-1}(\{1\}) = \{k\} \subset n$ in the category $\mathcal{L}(n)$. These inclusion maps together induce a strict symmetric monoidal functor in **Perm**

$$\bigvee_{k=1}^{n} \mathcal{L}(\delta^k_n) : \bigvee_{k=1}^{n} \mathcal{L}(1) \longrightarrow \mathcal{L}(n),$$

where $\bigvee_{k=1}^{n} \mathcal{L}(1)$ is the coproduct of n copies of $\mathcal{L}(1)$ in **Perm**. We will now present a concrete construction of the coproduct $\bigvee_{k=1}^{n} \mathcal{L}(1)$ and also construct a strict symmetric monoidal functor $\bigvee_{k=1}^{n} \mathcal{L}(\iota^k_n)$. An (non-unit) object S of $\bigvee_{k=1}^{n} \mathcal{L}(1)$ is a (finite) sequence

$(s_1, s_2, \ldots, s_r)$ in which s_i is either the empty set or a singleton subset of $\underline{n}$ for $1 \leq i \leq r$. We observe that the object S is equipped with a (unique) morphism $\overset{r}{\underset{i=1}{\sqcup}} s_i \longrightarrow \underline{n}$. A morphism $f : S \longrightarrow T = (t_1, t_2, \ldots, t_q)$ is an isomorphism $f : \overset{r}{\underset{i=1}{\sqcup}} s_i \longrightarrow \overset{q}{\underset{i=1}{\sqcup}} t_i$ such that the following diagram commutes:

$$\begin{tikzcd}
\overset{r}{\underset{i=1}{\sqcup}} s_i \arrow[rr, "f", "\cong"'] \arrow[dr] & & \overset{q}{\underset{i=1}{\sqcup}} t_i \arrow[dl] \\
& \underline{n} &
\end{tikzcd}$$

The functor $\overset{n}{\underset{k=1}{\vee}} \mathcal{L}(\delta_k^n)$ is now the obvious inclusion functor.

5.29. Lemma. *The strict symmetric monoidal functor* $\overset{n}{\underset{k=1}{\vee}} \mathcal{L}(\delta_k^n)$ *is an acyclic cofibration in* **Perm**.

Proof. The functor $\overset{n}{\underset{k=1}{\vee}} \mathcal{L}(\delta_k^n)$ is a cofibration because its object function is a monomorphism of free monoids. The inclusion functor $\overset{n}{\underset{k=1}{\vee}} \mathcal{L}(\delta_k^n)$ is fully-faithful. Each object of $\mathcal{L}(n)$ is isomorphic to an object of $\overset{n}{\underset{k=1}{\vee}} \mathcal{L}(1)$. ∎

5.30. Corollary. *For each permutative category C, $\mathcal{K}(C)$ is a coherently commutative monoidal category.*

Proof. By Lemma 5.23, $\mathcal{K}(C)(n^+) \cong [\mathcal{L}(n), C]_{\otimes}^{str}$. Now we have the following commutative diagram in **Cat**

$$\begin{tikzcd}
\mathcal{K}(C)(n^+) \arrow[r] \arrow[d, "(\mathcal{K}(C)(\delta_1^n),\ldots,\mathcal{K}(C)(\delta_n^n))"'] & {[\mathcal{L}(n), C]_{\otimes}^{str}} \arrow[d, "({[\mathcal{L}(\delta_1^n),C]_{\otimes}^{str}},\ldots,{[\mathcal{L}(\delta_n^n),C]_{\otimes}^{str}})"] \\
\prod\limits_{i=1}^{n} \mathcal{K}(C)(1^+) \arrow[r] & \prod\limits_{i=1}^{n} {[\mathcal{L}(1), C]_{\otimes}^{str}}
\end{tikzcd}$$

According to the lemma 5.29, the strict symmetric monoidal functor $\overset{n}{\underset{k=1}{\vee}} \mathcal{L}(\delta_k^n)$ is an acyclic cofibration therefore it follows from remark 3.2 that the right vertical functor is an acyclic fibration in **Cat**. The two horizontal functors in this diagram are isomorphisms, therefore $(\mathcal{K}(C)(\delta_1^n), \ldots, \mathcal{K}(C)(\delta_n^n))$ is also an acyclic fibration in **Cat**. Thus we have proved that $\mathcal{K}(C)$ is a coherently commutative monoidal category for every $C \in Ob(\mathbf{Perm})$. ∎

The above corollary will be extremely useful in proving that the adjunction $(\mathcal{L}, \mathcal{K})$ is a Quillen adjunction. We recall that a map in Γ**Cat** between two coherently commutative monoidal categories is a weak equivalence (resp. fibration) if and only if it is degreewise a weak equivalence (resp. fibration) in **Cat**.

5.31. LEMMA. *The adjunction $(\mathcal{L}, \mathcal{K})$ is a Quillen adjunction between the natural model category* **Perm** *and the model category of coherently commutative monoidal categories* Γ**Cat**.

PROOF. We will prove the lemma by showing that the right adjoint functor $\mathcal{K}$ preserves fibrations and acyclic fibrations. Let $F : C \longrightarrow D$ be a fibration in **Perm**. In order to show that $\mathcal{K}(F)$ is a fibration in the model category of coherently commutative monoidal categories Γ**Cat**, it would be sufficient to show that $\mathcal{K}(F)(n^+)$ is a fibration in **Cat**, for all $n^+ \in Ob(\Gamma^{op})$. For each $n \in \mathbb{N}$ the groupoid $\mathcal{L}(n)$ is a cofibrant object in **Perm**. The natural model category **Perm** is a **Cat**-model category whose cotensor is given by the functor $[-,-]^{str}_{\otimes}$. This implies that the functor

$$[\mathcal{L}(n), F]^{str}_{\otimes} : [\mathcal{L}(n), C]^{str}_{\otimes} \longrightarrow [\mathcal{L}(n), D]^{str}_{\otimes}$$

is a fibration in **Cat** and it is an acyclic fibration in **Cat** whenever F is an acyclic fibration. ∎

6. The Thickened Nerve

In this section we will describe a *thickened* version of Segal's nerve functor which we will denote by $\overline{\mathcal{K}}$ and show that $\overline{\mathcal{K}}$ is the right Quillen functor of a Quillen equivalence. Unlike the left Quillen functor of the Quillen adjunction $(\mathcal{L}, \mathcal{K})$ mentioned in the previous section, whose mere existence was shown, we will explicitly describe a functor $\overline{\mathcal{L}} : \Gamma\mathbf{Cat} \longrightarrow \mathbf{Perm}$ and show that it is the left Quillen adjoint of $\overline{\mathcal{K}}$. The adjunction $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is proved in appendix C. The explicit description will play a vital role in proving that the Quillen pair $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen equivalence. In this section we will also present the main result of this paper which proves that the Quillen pair of functors $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence. The Quillen equivalence $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ will be used to prove the main result.

We begin by defining a functor $\overline{\mathcal{L}} : \Gamma\mathbf{Cat} \longrightarrow \mathbf{Perm}$ as follows:

$$\Gamma\mathbf{Cat} \xrightarrow{\mathfrak{L}(-)} [\mathfrak{L}, \mathbf{Cat}]_{\otimes} \xrightarrow{L_H \int^{\vec{n} \in \mathfrak{L}} -} \mathbf{Perm}$$

where $\mathfrak{L}(-)$ is the symmetric monoidal extension functor described in section 2.39. The second functor $L_H \int^{\vec{n} \in \mathfrak{L}} -$ first performs the Grothendieck construction on a functor $F \in [\mathfrak{L}, \mathbf{Cat}]_{\otimes}$ to obtain a permutative category $\int^{\vec{n} \in \mathfrak{L}} F$, see theorem 2.11. and then it localizes (or formally inverts) the *horizontal arrows* of the permutative category $\int^{\vec{n} \in \mathfrak{L}} F$. We recall that an arrow in the category $\int^{\vec{n} \in \mathfrak{L}} F$ is a pair (f, ϕ) where f is a map in $\mathfrak{L}$ and ϕ is an arrow in the category $F(codom(f))$. An arrow (f, ϕ) is called horizontal if ϕ is the identity morphism. Thus for a Γ- category X, $\overline{\mathcal{L}}(X) = L_H \int^{\vec{n} \in \mathfrak{L}} \mathfrak{L}(X)$ is the permutative category obtained by *localizing* with respect to the set of all *horizontal* morphisms in the (permutative) category $\int^{\vec{n} \in \mathfrak{L}} \mathfrak{L}(X)$, see [GZ67, Ch. 1] for a procedure of localization. The results of [Day73] imply that the category $\int^{\vec{n} \in \mathfrak{L}} \mathfrak{L}(X)$ has the universal property that any strict symmetric monoidal functor $F : \int^{\vec{n} \in \mathfrak{L}} \mathfrak{L}(X) \longrightarrow C$ which maps every horizontal morphism in $\int^{\vec{n} \in \mathfrak{L}} \mathfrak{L}(X)$ to an isomorphism in C extends uniquely to a strict symmetric monoidal functor $F_{Nat} : \overline{\mathcal{L}}(X) \longrightarrow C$ along the projection map $p : \int^{\vec{n} \in \mathfrak{L}} \mathfrak{L}(X) \longrightarrow \overline{\mathcal{L}}X$, *i.e.* the functor F_{Nat} makes the following diagram commute

$$\begin{tikzcd} \int^{\vec{n} \in \mathfrak{L}} \mathfrak{L}(X) \arrow[r, "F"] \arrow[d, "p"'] & C \\ \overline{\mathcal{L}}X \arrow[ur, dashed, "F_{Nat}"'] & \end{tikzcd} \tag{24}$$

The localization construction is functorial in X and therefore we get a functor $\overline{\mathcal{L}}(-) : \Gamma\mathbf{Cat} \longrightarrow \mathbf{Perm}$.

Now we define the thickened nerve functor $\overline{\mathcal{K}}$. We will first define this functor in the spirit of the papers [May78], [SS79], [Man10] and [EM06] and later we will provide a couple of new interpretation of this functor based on pseudo bicycles, see appendix A and strict symmetric monoidal functors.

6.1. Definition. *An n*th pseudo Segal bicycle *in a symmetric monoidal category C is a quadruple $(\Phi, \alpha_\Phi, \sigma_\phi, u_\Phi)$ of families of objects or morphisms of the symmetric monoidal category C, where*

1. $\Phi = \{c_f\}_{f \in A_n}$ is a family of objects of C, where the indexing set

$$A_n := \{f \in \Gamma^{op} : domain(f) = n^+\}.$$

2. $\alpha_\Phi = \{\alpha(h, f) : c_f \longrightarrow c_{h^+ \circ f}\}_{(h,f) \in D}$ is a family of isomorphisms in C, where the indexing set

$$D := \{(h, f) \in Mor(\mathcal{N}) \times A_n : dom(h)^+ = codom(f)\}$$

3. $\sigma_\Phi = \{\sigma(k, l, f) : c_f \longrightarrow c_{f_k} \otimes c_{f_l}\}_{(k,l,f) \in B}$ is a family of isomorphisms in C, where $f_k = \delta^{k+l}_k \circ f$ and $f_k = \delta^{k+l}_l \circ f$ and the indexing set

$$B := \{(k, l, f) \in \mathbb{N} \times \mathbb{N} \times A_n : codom(f) = (k + l)^+\}.$$

4. $u_\phi = \{u(f) : c_f \longrightarrow 1_C\}_{f \in A_n(0)}$ is a family of isomorphisms in C, where the indexing set is the following subset of A_n

$$A_n(0) := \{f \in A_n : codom(f) = 0^+\}.$$

The quadruple $(\Phi, \alpha_\Phi, \sigma_\phi, u_\Phi)$ is subject to the following conditions:

PSB.1 *For any (pointed) function $f : n^+ \longrightarrow m^+$ in the indexing set A_n, the map*

$$c_f \xrightarrow{\sigma(m,0,f)} c_{f_m} \underset{C}{\otimes} c_{f_0} \xrightarrow{id \underset{C}{\otimes} u(f_0)} c_{f_m} \underset{C}{\otimes} 1_C$$

is the inverse of the (right) unit isomorphism in C. Similarly the map

$$c_f \xrightarrow{\sigma(0,m,f)} c_{f_0} \underset{C}{\otimes} c_{f_m} \xrightarrow{u(f_0) \underset{C}{\otimes} id} 1_C \underset{C}{\otimes} c_{f_m}$$

is the inverse of the (left) unit isomorphism in C.

PSB.2 *For each triple $(k, l, f) \in B$, where the indexing set B is defined above, the following diagram commutes in the category C*

$$\begin{tikzcd} c_f \arrow[r, "\alpha(\gamma^{\mathcal{N}}_{k,l})_f"] \arrow[d, "\sigma(k,l,f)"'] & c_f \arrow[d, "\sigma(l,k,f)"] \\ c_{f_k} \underset{C}{\otimes} c_{f_l} \arrow[r, "\gamma^C_{c_{f_k}, c_{f_l}}"'] & c_{f_l} \underset{C}{\otimes} c_{f_k} \end{tikzcd}$$

PSB.3 *For any triple $k, l, m \in \mathbb{N}$, and each $f : n^+ \longrightarrow (k + l + m)^+$ in the set A_n, the following diagram commutes*

$$\begin{tikzcd} c_f \arrow[rr, "\sigma(k+l,m,f)"] \arrow[d, "\sigma(k,l+m,f)"'] & & c_{k+l} \underset{C}{\otimes} c_m \arrow[dd, "\sigma(k,l,f) \underset{C}{\otimes} id_{c_{f_m}}"] \\ c_{f_k} \underset{C}{\otimes} c_{f_{l+m}} \arrow[d, "id_{c_{f_k}} \underset{C}{\otimes} \sigma(l,m,f)"'] & & \\ c_{f_k} \underset{C}{\otimes} (c_{f_l} \underset{C}{\otimes} c_{f_m}) & & (c_k \underset{C}{\otimes} c_l) \underset{C}{\otimes} c_m \arrow[ll, "\alpha^C_{c_{f_k}, c_{f_l}, c_{f_m}}"] \end{tikzcd}$$

PSB.4 *For each triple $(k, l, h) \in B$, where the indexing set B is defined above, and each pair of active maps $f : k^+ \longrightarrow p^+$, $g : l^+ \longrightarrow q^+$ in Γ^{op}, the following diagram commutes in the category C*

$$\begin{tikzcd} c_h \arrow[r, "\sigma(k,l,h)"] \arrow[d, "\alpha(f+g,h)"'] & c_{h_k} \underset{C}{\otimes} c_{h_l} \arrow[d, "\alpha(f+g,h_k) \underset{C}{\otimes} \alpha(f+g,h_l)"] \\ c_{(f+g)\circ h} \arrow[r, "\sigma(k,l,(f+g)\circ h)"'] & c_{((f+g)\circ h)_k} \underset{C}{\otimes} c_{((f+g)\circ h)_l} \end{tikzcd}$$

Next we define the notion of a morphism of pseudo Segal bicycles:

6.2. Definition. *A morphism of nth unnormalized pseudo Segal bicycles*

$$F : (\Phi, \alpha_\Phi, \sigma_\phi, u_\Phi) \longrightarrow (\Psi, \alpha_\Psi, \sigma_\psi, u_\Psi)$$

is a family $F = \{F(f) : c_f^\Phi \longrightarrow c_f^\Psi\}_{f \in A_n}$ *of morphisms in* C *which satisfies the following conditions:*

1. *For each* $f \in A_n(0)$ $(codom(f) = 0^+)$*, the following diagram commutes:*

$$\begin{tikzcd} c_f^\Phi \arrow[rr, "F(f)"] \arrow[dr, "u_\Phi(f)"'] & & c_f^\Psi \arrow[dl, "u_\Psi(f)"] \\ & 1_C & \end{tikzcd}$$

2. *For each pair* $(f, h) \in A_n \times Mor(\mathcal{N})$ *such that the domain of* h^+*, namely* $dom(h)^+$*, is the same as the codomain of* f *, the following diagram commutes*

$$\begin{tikzcd} c_f^\Phi \arrow[r, "F(f)"] \arrow[d, "\alpha_\Phi(h,f)"'] & c_f^\Psi \arrow[d, "\alpha_\Psi(h,f)"] \\ c_{h\circ f}^\Phi \arrow[r, "F(h\circ f)"'] & c_{h\circ f}^\Psi \end{tikzcd}$$

3. *For each triple* $(k, l, f) \in B$*, where the index set* B *is defined above, the following diagram commutes*

$$\begin{tikzcd} c_f^\Phi \arrow[r, "F(f)"] \arrow[d, "\sigma^\Phi(k,l,f)"'] & c_f^\Psi \arrow[d, "\sigma^\Psi(k,l,f)"] \\ c_{f_k}^\Phi \underset{C}{\otimes} c_{f_l}^\Phi \arrow[r, "F(f_k)\underset{C}{\otimes}F(f_l)"'] & c_{f_k}^\Psi \underset{C}{\otimes} c_{f_l}^\Psi \end{tikzcd}$$

All nth unnormalized pseudo Segal bicycles in a symmetric monoidal category C and all morphisms of nth unnormalized pseudo Segal bicycles in C form a category which we denote by $\overline{\mathcal{K}}(C)(n^+)$.

6.3. Lemma. *Let* C *be a permutative category. For each* $\underline{n}$*, the category* $\overline{\mathcal{K}}(C)(n^+)$ *is isomorphic to the category of all pseudo bicycles from* Γ^n *to* C *namely* $\boldsymbol{Bikes}^{Ps}(\Gamma^n, C)$.

The proof of this lemma is just the adaptation of the argument of the proof lemma 5.3, which deals with the case of strict bicycles, to the present scenario of pseudo bicycles.

6.4. Definition. *For each $n \in Ob(\mathcal{N})$ we will now define a permutative groupoid $\overline{\mathcal{L}}(n)$. The objects of this groupoid are finite collections of morphisms in Γ^{op} having domain n^+, in other words the object monoid of the category $\overline{\mathcal{L}}(n)$ is the free monoid generated by the following set*

$$Ob(\overline{\mathcal{L}}(n)) := \underset{k \in Ob(\mathcal{N})}{\sqcup} \Gamma^n(k^+).$$

We will denote an object of this groupoid by $(f_1, f_2, \dots, f_r)$. A morphism $(f_1, f_2, \dots, f_r) \longrightarrow (g_1, g_2, \dots, g_k)$ is an isomorphism of finite sets

$$F : Supp(f_1) \sqcup Supp(f_2) \sqcup \cdots \sqcup Supp(f_r) \xrightarrow{\cong} Supp(g_1) \sqcup Supp(g_2) \sqcup \cdots \sqcup Supp(g_k)$$

such that the following diagram commutes

$$\begin{tikzcd}
Supp(f_1) \sqcup \cdots \sqcup Supp(f_r) \arrow[rr, "F"] \arrow[dr] & & Supp(g_1) \sqcup \cdots \sqcup Supp(g_k) \arrow[dl] \\
& n &
\end{tikzcd}$$

where the diagonal maps are the unique inclusions of the coproducts into n.

6.5. Remark. *The construction above defines a contravariant functor $\overline{\mathcal{L}}(-) : \Gamma^{op} \longrightarrow$* **Perm**. *A map $f : n^+ \longrightarrow m^+$ in Γ^{op} defines a strict symmetric monoidal functor $\overline{\mathcal{L}}(f) : \overline{\mathcal{L}}(m) \longrightarrow \overline{\mathcal{L}}(n)$. An object $(f_1, f_2, \dots, f_r) \in \overline{\mathcal{L}}(m)$ is mapped by this functor to $(f_1 \circ f, f_2 \circ f, \dots, f_r \circ f) \in \overline{\mathcal{L}}(n)$.*

6.6. Remark. *We observe that the category $\overline{\mathcal{L}}(n)$ defined above is a Gabriel factorization of the composite functor $G \circ H$, see equation (22), and therefore by proposition 2.50 it is isomorphic to $\Pi_1 \overline{\overline{\mathcal{L}}}(n)$, for each $n \in \mathbb{N}$. Further by proposition 2.7 these isomorphisms glue together to define a natural isomorphism $T : \Pi_1 \circ \overline{\overline{\mathcal{L}}}(-) \Rightarrow \overline{\mathcal{L}}(-)$.*

6.7. Remark. *The natural equivalence from remark 6.6 extends to the following composite natural equivalence:*

$$T \circ (id_{\Pi_1} \circ (J \circ i) \circ I^{-1}) : \mathcal{L}(n) \overset{I^{-1}}{\Rightarrow} \Pi_1 \circ \mathcal{P}\overline{\overline{\mathcal{L}}}(-) \Rightarrow \Pi_1 \circ \overline{\overline{\mathcal{L}}}(\Gamma^-) \Rightarrow \Pi_1 \circ \overline{\overline{\mathcal{L}}}(-) \overset{T}{\Rightarrow} \overline{\mathcal{L}}(-).$$

where T is the natural isomorphism from remark 6.6.

6.8. Proposition. *For each $n \in Ob(\mathcal{N})$, the permutative category $\overline{\mathcal{L}}(n)$ represents the functor $\boldsymbol{Bikes}^{Ps}(\Gamma^n, -)$. In other words there is a natural isomorphism*

$$\psi^n : [\overline{\mathcal{L}}(n), -]^{str}_{\otimes} \cong \boldsymbol{Bikes}^{Ps}(\Gamma^n, -).$$

It was proved in [AGV72, sec. 6] that, for any Γ- category X, the permutative category $\overline{\mathcal{L}}(X)$ is a *pseudo-colimt* of the functor $\mathfrak{L}(X)$. In other words $\overline{\mathcal{L}}(X)$ represents the category of *pseudo-cones i.e.* for any category C

$$Ps[\mathfrak{L}X, \Delta C] \cong [\overline{\mathcal{L}}(X), C].$$

This characterization provides the functor $\overline{\mathcal{L}}$ with some very desirable homotopical properties.

6.9. Lemma. *The functor $\overline{\mathcal{L}}$ preserves degreewise equivalences of* Γ*-categories.*

Proof. The functor $\overline{\mathcal{L}}$ is a composite of the functor $\mathfrak{L}$ followed by a pseudo-colimit functor. The functor $\mathfrak{L} : \Gamma\mathbf{Cat} \longrightarrow [\mathfrak{L}, \mathbf{Cat}]_{\otimes}^{str}$ preserves degreewise equivalences. The results of [Gam08] show that a pseudo colimit functor is a homotopy colimt functor and it preserves degreewise equivalences. Hence the functor $\overline{\mathcal{L}}$ preserves degreewise equivalences of Γ- categories.

■

Before moving on we would like to observe that for any object $\vec{n} \in Ob(\mathcal{L})$ there exists the following zig-zag of maps in $\mathcal{L}$

$$(1) \overset{(id,m_n)}{\longleftarrow} (n) \overset{(m_r,id)}{\longrightarrow} \vec{n} = (n_1, n_2, \dots, n_r) \tag{25}$$

where $n = n_1 + n_2 + \cdots + n_r$ and $m_n : n^+ \longrightarrow 1^+$ is the unique *multiplication* map from n^+ to 1^+ in Γ^{op}. To be more precise, the left map is given by the following commutative diagram

$$\begin{tikzcd} \underline{n} \arrow[r, "m_n"] \arrow[d] & \underline{1} \arrow[d] \\ \underline{1} \arrow[r, equal] & \underline{1} \end{tikzcd} \tag{26}$$

and the right map is given by the following commutative diagram

$$\begin{tikzcd} \underline{n} \arrow[r, equal] \arrow[d] & \underline{n} \arrow[d, "Ind(\vec{n})"] \\ \underline{1} & \underline{r} \arrow[l, "m_r"] \end{tikzcd} \tag{27}$$

The following corollary provides a useful insight into the structure of the localization of the category of elements of a coherently commutative monoidal category X, with respect to horizontal maps. It turns out that this localized category is a thickening of $X(1^+)$. This thickening is indicative of the fact that the homotopy colimit of a diagonal functor $\Delta(c)$ is equivalent to c. The category $\overline{\mathcal{L}}(X)$ is a further thickening of this localized category.

6.10. Corollary. *For each coherently commutative monoidal category X the inclusion functor $i : X(1^+) \longrightarrow \overline{\mathcal{L}}(X)$ is an equivalence of categories.*

Proof. The functor $i : X(1^+) \longrightarrow \overline{\mathcal{L}}(X)$ is an inclusion functor, it is defined on objects as follows:

$$i(x) := (id_{\underline{1}}, x)$$

and for a morphism $f : x \longrightarrow y$ in $X(1^+)$ it is defined as follows:

$$i(f) := ((id_{\underline{1}}, id_{\underline{1}}), f).$$

Clearly the functor i is fully faithful. Now we will show that i is also essentially surjective. In order to do so, we will use the maps (26) and (27) defined above. For each object $(\vec{n}, \vec{x}) \in \overline{\mathcal{L}}(X)$, the map (27) provides a functor

$$\mathfrak{L}(X)((m_r, id_{\underline{n}})) : X(m^+) \longrightarrow \prod_{i=1}^{r} X(m_i).$$

Since X is a coherently commutative monoidal category therefore the above functor is an equivalence of categories. Thus we may choose an object $x \in X(m^+)$ and an isomorphism $j : \mathfrak{L}(X)((m_r, id_{\underline{n}}))(x) \longrightarrow \vec{x}$ in $\prod_{i=1}^{r} X(m_i)$. We observe that the map

$$((m_r, id_{\underline{n}}), j) : ((n), x) \longrightarrow (\vec{n}, \vec{x})$$

is an isomorphism in $\overline{\mathcal{L}}(X)$ because $\overline{\mathcal{L}}(X)$ is obtained by inverting all horizontal maps in the category of elements of $\mathfrak{L}(X)$. The map (26) provides us with the following isomorphisms:

$$((id_{\underline{1}}, m_n), id_{\mathfrak{L}(X)((id_{\underline{1}}, m_n))((x))}) : ((n), (x)) \longrightarrow ((1), \mathfrak{L}(X)((id_{\underline{1}}, m_n))((x)))$$

The above two isomorphisms show that each object $(\vec{n}, \vec{x}) \in \overline{\mathcal{L}}(X)$ is isomorphic to an object in the image of the functor i namely $((1), \mathfrak{L}(X)((id_{\underline{1}}, m_n))((x)))$. The isomorphism is given by the composite

$$((m_r, id_{\underline{n}}), j) \circ ((id_{\underline{1}}, m_n), id_{\mathfrak{L}(X)((id_{\underline{1}}, m_n))((x))})^{-1}.$$

Thus we have proved that i is essentially surjective and therefore an equivalence.

∎

6.11. Remark. *There exists an inverse functor* $i^{-1} : \overline{\mathcal{L}}(X) \longrightarrow X(1^+)$ *such that* $i^{-1} \circ i = id_{X(1^+)}$.

Momentarily we will switch to the language of bicycles for the purpose of proving lemma 6.15. Each object of a permutative category C defines a *trivial* bicycle from Γ^1 to C which we denote by $\Phi_c = (\mathcal{L}_c, \sigma_c)$. We define this bicycle $\Phi_c : \Gamma^1 \rightsquigarrow C$ next. We begin by defining the underlying lax cone $\mathcal{L}_c = (\phi_c, \alpha_c)$. For each $k \in Ob(\mathcal{N})$, we define the functor $\phi_c(k) : \Gamma^1(k^+) \longrightarrow C$ as follows:

$$\phi_c(k)(f) = \begin{cases} c, & \text{if } f \neq 0 \\ 1_C, & \text{otherwise.} \end{cases} \tag{28}$$

For a map $h : k \longrightarrow l$ in the category $\mathcal{N}$, we define the map $\alpha_c(h)(f) : \phi_c(f) \longrightarrow \phi_c(h \circ f)$ as follows:

$$\alpha_c(h)(f) = \begin{cases} id_c, & \text{if } f \neq 0 \\ id_{1_C}, & \text{otherwise.} \end{cases} \tag{29}$$

It is easy to see that with the above definition, $\mathcal{L}_c = (\phi_c, \alpha_c)$ is a lax cone. This lax cone is given a bicycle structure by defining $\sigma_c(k,l) : \phi_c(k+l) \Rightarrow \phi_c(k) \odot \phi_c(l)$ to be the identity natural transformation. Thus we have defined a strict bicycle $(\mathcal{L}_c, \sigma_c) = \Phi_c : \Gamma^1 \rightsquigarrow C$. This construction defines a functor

$$\Phi_- : C \longrightarrow \mathbf{Bikes}^{Ps}(\Gamma^1, C) \tag{30}$$

6.12. Lemma. *Every bicycle $(\mathcal{L}, \sigma) = \Phi : \Gamma^1 \rightsquigarrow C$ is isomorphic to the trivial bicycle determined by the object $\phi(1)(id_{1^+}) \in Ob(C)$, namely $\Phi_{\phi(1)(id_{1^+})}$, where $\mathcal{L} = (\phi, \alpha)$ is the underlying lax symmetric monoidal cone of Φ.*

Proof. We will construct an isomorphism of bicycles

$$\eta(\Phi) : \Phi \longrightarrow \Phi_{\phi(1)(id_{1^+})}.$$

In order to do so, we will use the natural isomorphism

$$\alpha(m_k) : \phi(k^+) \Rightarrow \phi(1^+) \circ \Gamma^1(m_k)$$

provided by the bicycle Φ, where $m_k : k^+ \longrightarrow 1^+$ is the *multiplication* map. For each $k \in Ob(\mathcal{N})$ we define a natural isomorphism $\eta(\Phi)(k)$ as follows:

$$\eta(\Phi)(k)(f) := \alpha(m_k)(f) : \phi(k)(f) \longrightarrow \phi(1)(id_{1^+}),$$

where $f \in \Gamma^1(k^+)$. One can check that the natural isomorphisms in the collection $\{\eta(\Phi)(k)\}_{k \in Ob(\mathcal{N})}$ glue together into an isomorphism of bicycles $\eta(\Phi) : \Phi \longrightarrow \Phi_{\phi(1)(id_{1^+})}$. ∎

6.13. Corollary. *For any symmetric monoidal category C, the category $\overline{\mathcal{K}}C(1^+)$ is equivalent to C.*

Proof. For each permutative category C we define a functor $I(C) : \mathbf{Bikes}^{Ps}(\Gamma^1, C) \longrightarrow C$. On objects this functor is defined as follows:

$$I(C)(\Phi) = \phi(1)(id_{1^+}),$$

where $\mathcal{L} = (\phi, \alpha)$ is the underlying lax cone of Φ. For a morphism of (pseudo) bicycles $F : \Phi \longrightarrow \Psi$ we define

$$I(F) = F(1)(id_{1^+}).$$

This functor is inverse of the functor Φ_-.

∎

6.14. Lemma. *A strict symmetric monoidal functor in* **Perm** *is an equivalence of categories if and only if its image under the right adjoint functor* $\overline{\mathcal{K}}$ *is a strict equivalence in* Γ**Cat**.

Proof. Let $F : C \longrightarrow D$ be a strict symmetric monoidal functor in **Perm** which is an equivalence of categories. We consider the following diagram

$$\begin{tikzcd} \overline{\mathcal{K}}(C)(1^+) = \mathbf{Bikes}^{Ps}(\Gamma^1, C) \arrow[r, "\mathbf{Bikes}^{Ps}(\Gamma^1, F)"] \arrow[d, "I(C)"'] & \mathbf{Bikes}^{Ps}(\Gamma^1, D) = \overline{\mathcal{K}}(D)(1^+) \arrow[d, "I(D)"] \\ C \arrow[r, "F"'] & D \end{tikzcd} \tag{31}$$

where $I(C)$ and $I(D)$ are equivalences defined in the proof of corollary 6.13. If the functor F is an equivalence then the two out of three property of weak equivalences says that the map $\overline{\mathcal{K}}(F)(1^+) = \mathbf{Bikes}^{Ps}(\Gamma^1, F)$ is an equivalence of categories. Since $\overline{\mathcal{K}}(F)$ is a map between two coherently commutative monoidal Γ- categories, therefore it is a strict weak equivalence if and only if $\overline{\mathcal{K}}(F)(1^+)$ is an equivalence of categories.

Conversely, if the morphism of Γ- categories $\overline{\mathcal{K}}(F)$ is a (strict) weak equivalence then $\overline{\mathcal{K}}(F)(1^+)$ is an equivalence of categories. Another application of the two out of three property of weak equivalences to the commutative diagram (31) tells us that the functor F is an equivalence of categories.

■

It was shown by Leinster in [Lei00] that the degree one category of a coherently commutative monoidal category has a symmetric monoidal structure. We want to explore the homotopy properties of the unit natural transformation η of the adjunction $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$.

6.15. Lemma. *For each coherently commutative monoidal category X the unit map*

$$\eta(X) : X \longrightarrow \overline{\mathcal{K}}(\overline{\mathcal{L}}(X))$$

is a strict equivalence of Γ- *categories.*

Proof. The Γ- category $\overline{\mathcal{K}}(\overline{\mathcal{L}}(X))$ is a coherently commutative monoidal category, therefore $\eta(X)$ is a morphism between two coherently commutative monoidal categories. Now in light of Lemma 6.3 it would be sufficient to show that the degree one functor

$$\eta(X)(1^+) : X(1^+) \longrightarrow \mathbf{Bikes}^{Ps}(\Gamma^1, \overline{\mathcal{L}}(X))$$

is an equivalence of categories. We recall the definition of the functor $\eta(X)(1^+)$. For each $x \in X(1^+)$, the strict symmetric monoidal functor $\eta(X)(1^+)(x) = \Phi = (\mathcal{L}, \sigma) : \Gamma^1 \rightsquigarrow \overline{\mathcal{L}}(X)$ is defined as follows:

$$\phi(n)(f) := ((n), X(f)(x)),$$

where $\mathcal{L} = (\phi, \alpha)$ is the underlying lax cone of Φ. In light of corollaries 6.10 and 6.13 we have the following commutative diagram in **Cat**:

$$\begin{tikzcd} X(1^+) \arrow[r, "\eta(X)(1^+)"] \arrow[dr, "i"'] & \mathbf{Bikes}^{Ps}(\Gamma^1, \overline{\mathcal{L}}(X)) \arrow[d, "I"] \\ & \overline{\mathcal{L}}(X) \end{tikzcd}$$

where the vertical functor I in the diagram above is the functor from corollary 6.13 and the diagonal functor i in the above diagram is the functor from corollary 6.10. The two corollaries mentioned above say that i and I are equivalences of categories therefore by the two out of three property of the natural model category **Cat** the unit map in degree one $\eta(X)(1^+)$ is an equivalence of categories. Hence we have proved that the map of coherently commutative monoidal categories $\eta(X)$ is a (strict) equivalence of Γ- categories. ∎

The adjoint functors $\overline{\mathcal{L}}$ and $\overline{\mathcal{K}}$ have sufficiently good properties which ensure that the above lemma implies that for each Γ- category X the counit map $\eta(X)$ is a coherently commutative monoidal equivalence.

6.16. Corollary. *For a Γ- category X, the unit map $\eta(X) : X \longrightarrow \overline{\mathcal{K}}(\overline{\mathcal{L}}(X))$ is a coherently commutative monoidal equivalence.*

Proof. Let $r : X \longrightarrow X^f$ denote a fibrant replacement of X in the model category of coherently commutative monoidal categories. In other words X^f is a coherently commutative monoidal category and r is an acyclic cofibration in the model category of coherently commutative monoidal categories. Since η is a natural transformation therefore we have the following commutative diagram in Γ**Cat**

$$\begin{tikzcd} X^f \arrow[r, "\eta(X^f)"] & \overline{\mathcal{K}}(\overline{\mathcal{L}}(X^f)) \\ X \arrow[u, "r"] \arrow[r, "\eta(X)"'] & \overline{\mathcal{K}}(\overline{\mathcal{L}}(X)) \arrow[u, "\overline{\mathcal{K}}(\overline{\mathcal{L}}(r))"'] \end{tikzcd}$$

The above lemma tells us that the morphism $\eta(X^f)$ is a coherently commutative monoidal equivalence and so is r by assumption. Theorem 6.17 and lemma 6.14 together imply that $\overline{\mathcal{K}}(\overline{\mathcal{L}}(r))$ is a coherently commutative monoidal equivalence. Now the 2 out of 3 property of model categories implies that $\eta(X)$ is a coherently commutative monoidal equivalence. ∎

Finally we have developed enough machinery to provide a characterization of a coherently commutative monoidal equivalence.

6.17. Theorem. *A morphism of Γ- categories $F : X \longrightarrow Y$ is a coherently commutative monoidal equivalence if and only if the strict symmetric monoidal functor $\overline{\mathcal{L}}(F)$: $\overline{\mathcal{L}}(X) \longrightarrow \overline{\mathcal{L}}(Y)$ is an equivalence of (permutative) categories.*

Proof. Let us first assume that the morphism of Γ- categories F is a coherently commutative monoidal equivalence. Any choice of a cofibrant replacement functor Q for $\Gamma\mathbf{Cat}$ provides a commutative diagram

$$\begin{tikzcd} Q(X) \arrow[r, "Q(F)"] \arrow[d] & Q(Y) \arrow[d] \\ X \arrow[r, "F"'] & Y \end{tikzcd}$$

The vertical maps in this diagram are acyclic fibrations in the model category of coherently commutative monoidal categories which are strict equivalences of Γ- categories. Applying the functor $\overline{\mathcal{L}}$ to this commutative diagram we get the following commutative diagram in **Perm**

$$\begin{tikzcd} \overline{\mathcal{L}}(Q(X)) \arrow[r, "\overline{\mathcal{L}}(Q(F))"] \arrow[d] & \overline{\mathcal{L}}(Q(Y)) \arrow[d] \\ \overline{\mathcal{L}}(X) \arrow[r, "\overline{\mathcal{L}}(F)"'] & \overline{\mathcal{L}}(Y) \end{tikzcd}$$

The functor $\overline{\mathcal{L}}$ is a left Quillen functor therefore it preserves weak equivalences between cofibrant objects. This implies that the top horizontal arrow in the above diagram is a weak equivalence in **Perm**. The above lemma 6.9 implies that the vertical maps in the above diagram are weak equivalences in **Perm**. Now the two out of three property of weak equivalences in model categories implies that $\overline{\mathcal{L}}(F)$ is a weak equivalence in **Perm**.

Conversely, let us first assume that $\overline{\mathcal{L}}(F) : \overline{\mathcal{L}}(X) \longrightarrow \overline{\mathcal{L}}(Y)$ is a weak equivalence between coherently commutative monoidal categories in **Perm**. The functor $\overline{\mathcal{K}}$ preserves equivalences in **Perm**, therefore the morphism $\overline{\mathcal{K}}(\overline{\mathcal{L}}(F)) : \overline{\mathcal{K}}(\overline{\mathcal{L}}(X)) \longrightarrow \overline{\mathcal{K}}(\overline{\mathcal{L}}(Y))$ is a strict equivalence of Γ- categories. Now we have the following commutative diagram

$$\begin{tikzcd} \overline{\mathcal{K}}(\overline{\mathcal{L}}(X)) \arrow[r, "\overline{\mathcal{K}}(\overline{\mathcal{L}}(F))"] & \overline{\mathcal{K}}(\overline{\mathcal{L}}(Y)) \\ X \arrow[u, "\eta(X)"] \arrow[r, "F"'] & Y \arrow[u, "\eta(Y)"'] \end{tikzcd}$$

Lemma 6.9 implies that the two vertical arrows in the above diagram are strict equivalences of Γ- categories, therefore by the two-out-of-three property of model categories, F is also a coherently commutative monoidal equivalence. Now we tackle the general case. Let $F : X \longrightarrow Y$ be a morphism of Γ- categories such that $\overline{\mathcal{L}}(F)$ is an equivalence of categories. By a choice of a functorial factorization functor we get the following commutative diagram whose vertical arrows are acyclic cofibrations in the model category of coherently commutative monoidal categories and $R(X)$ and $R(Y)$ are coherently commutative monoidal categories:

$$\begin{tikzcd} R(X) \arrow[r, "R(F)"] & R(Y) \\ X \arrow[u, "\zeta(X)"] \arrow[r, "F"'] & Y \arrow[u, "\zeta(Y)"'] \end{tikzcd}$$

Applying the functor $\overline{\mathcal{L}}$ to the above diagram we get the following commutative diagram in **Perm**:

$$\begin{tikzcd} \overline{\mathcal{L}}(R(X)) \arrow[r, "\overline{\mathcal{L}}(R(F))"] & \overline{\mathcal{L}}(R(Y)) \\ \overline{\mathcal{L}}(X) \arrow[u, "\overline{\mathcal{L}}(\zeta(X))"] \arrow[r, "\overline{\mathcal{L}}(F)"'] & \overline{\mathcal{L}}(Y) \arrow[u, "\overline{\mathcal{L}}(\zeta(Y))"'] \end{tikzcd}$$

Since $\overline{\mathcal{L}}$ is a left Quillen functor therefore it preserves acyclic cofibrations. This implies that the two vertical morphisms in the above diagram are equivalences of categories. By assumption $\overline{\mathcal{L}}(F)$ is an equivalence of categories therefore the two out of three property implies that $\overline{\mathcal{L}}(R(F))$ is an equivalence of categories. The discussion earlier in this proof regarding strict equivalence between coherently commutative monoidal categories implies that $R(F)$ is a strict equivalence of Γ- categories. ■

The lemmas proved in this section and the results of section 3 together imply the main result of this paper which is the following:

6.18. THEOREM. *The adjunction* $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ *is a Quillen equivalence.*

PROOF. We observe that $\overline{\mathcal{L}}(n)$ is a cofibrant permutative category for all $n \in \mathbb{N}$. Since the permutative category $\overline{\mathcal{L}}(n)$ is cofibrant for all $n \geq 0$ therefore it is easy to see (remark 3.2) that the right adjoint functor $\overline{\mathcal{K}}$ preserves fibrations and trivial fibrations in **Perm** and therefore $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen adjunction. Let X be a cofibrant object in the model category of coherently commutative monoidal categories and let C be a permutative category. We will show that a map $F : \overline{\mathcal{L}}(X) \longrightarrow C$ is a coherently commutative monoidal equivalence if and only if its adjunct map $\phi(F) : X \longrightarrow \overline{\mathcal{K}}C$ is an equivalence of categories. Let us first assume that F is an equivalence in **Perm**. The adjunct map $\phi(F)$ is defined by the following commutative diagram:

$$\begin{tikzcd} \overline{\mathcal{K}}(\overline{\mathcal{L}}(X)) \arrow[r, "\overline{\mathcal{K}}(F)"] & \overline{\mathcal{K}}(C) \\ X \arrow[u, "\eta"] \arrow[ur, "\phi(F)"'] & \end{tikzcd}$$

The right adjoint functor $\overline{\mathcal{K}}$ preserves weak equivalences therefore the top horizontal arrow is a strict equivalence of Γ- categories. The unit map η is a coherently commutative monoidal equivalence by corollary 6.16. Now the 2 out of 3 property of model categories implies that $\phi(F)$ is also a coherently commutative monoidal equivalence.

Conversely, let us assume that $\phi(F)$ is a coherently commutative monoidal equivalence. The 2 out of 3 property of model categories implies that top horizontal arrow in the above commutative diagram, namely $\overline{\mathcal{K}}(F)$ is a coherently commutative monoidal equivalence and therefore a strict equivalence of Γ- categories. Now Lemma 6.14 implies that the strict symmetric monoidal functor F is an equivalence of categories. ∎

Now we are ready to state the main result of this paper which is a corollary of the above theorem:

6.19. Corollary. *The adjunction $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence.*

Proof. Remark 6.7 gives a natural equivalence of permutative categories

$$T \circ (id_{\Pi_1} \circ (J \circ i) \circ I^{-1}) : \mathcal{L}(n) \overset{I^{-1}}{\Rightarrow} \Pi_1 \circ \mathcal{P}\overline{\overline{\mathcal{L}}}(-) \Rightarrow \Pi_1 \circ \overline{\overline{\mathcal{L}}}(\Gamma^-) \Rightarrow \Pi_1 \circ \overline{\overline{\mathcal{L}}}(-) \overset{T}{\Rightarrow} \overline{\mathcal{L}}(-).$$

We observe that for all $n \in \mathbb{N}$ $T \circ (id_{\Pi_1} \circ (J \circ i) \circ I^{-1})(n)$ is a weak equivalence in **Perm** between cofibrant (and fibrant) permutative categories.

We recall from remark 3.2 that the bifunctor $[-,-]_{\otimes}^{str}$ is the Hom functor of the **Cat**-model category **Perm**. This implies that for each $n \in \mathbb{N}$ and each permutative category C, the functor

$$[T \circ (id_{\Pi_1} \circ (J \circ i) \circ I^{-1})(n), C]_{\otimes}^{str} : [\mathcal{L}(n), C]_{\otimes}^{str} \longrightarrow [\overline{\mathcal{L}}(n), C]_{\otimes}^{str}$$

is an equivalence of categories. In other words the natural transformation

$$[T \circ (id_{\Pi_1} \circ (J \circ i) \circ I^{-1})(-), C]_{\otimes}^{str} : [\mathcal{L}(-), C]_{\otimes}^{str} \longrightarrow [\overline{\mathcal{L}}(-), C]_{\otimes}^{str}$$

is a strict equivalence if Γ- categories. This morphism of Γ- categories uniquely determines a strict equivalence of Γ- categories $\eta(C) : \mathcal{K}(C) \Rightarrow \overline{\mathcal{K}}(C)$ for each permutative category C. The family $\{\eta(C)\}_{C \in Ob(\mathbf{Perm})}$ glues together to define a natural equivalence $\eta : \mathcal{K} \Rightarrow \overline{\mathcal{K}}$. The natural equivalence η induces a natural isomorphism between the derived functors of $\mathcal{K}$ and $\overline{\mathcal{K}}$. Now the corollary follows from the above theorem 6.18. ∎

A. The notion of a Bicycle

In the paper [Seg74], Segal described a functor from the category of all (small) symmetric monoidal categories to the category of Γ- category. The Γ- category assigned by this functor to a symmetric monoidal category was described by constructing a sequence of (pointed) categories whose objects are a pair of families of objects and maps in the symmetric monoidal category satisfying some coherence conditions, see [Man10], [SS79] for a complete definition. The objective of this section is to present a *thicker* version of Segal's pair of families as *pseudo cones* which satisfy some additional coherence conditions which are usually associated to oplax symmetric monoidal functors. We begin by providing a definition of a pseudo cone in the spirit of Segal's families:

A.1. Definition. *A* pseudo cone *from X to C is a pair (ϕ,α), where $\phi = \{\phi(\underline{n})\}_{\underline{n}\in Ob(\mathcal{N})}$ is a family of functors $\phi(\underline{n}) : X(n^+) \longrightarrow C$ and $\alpha = \{\alpha(f)\}_{f\in Mor\mathcal{N}}$ is a family of natural isomorphisms*

$$\alpha(f) : \phi(\underline{n}) \Rightarrow \phi(\underline{m}) \circ X(f),$$

where $f : \underline{n} \longrightarrow \underline{m}$ is a map in $\mathcal{N}$ which can also be regarded as an active map $f : n^+ \longrightarrow m^+$ in Γ^{op}. The pair (ϕ,α) is subject to the following conditions

1. *In the family α, the natural isomorphism indexed by the identity morphism of an object in $\mathcal{N}$ should be the identity natural transformation* i.e.

$$\alpha(id_{\underline{n}}) = id_{\phi(\underline{n})},$$

for all $\underline{n} \in Ob(\mathcal{N})$.

2. *For every pair $(f : \underline{n} \longrightarrow \underline{m}, g : \underline{m} \longrightarrow \underline{l})$ of composable arrows in $\mathcal{N}$, the following diagram commutes:*

$$\begin{tikzcd} & \phi(\underline{n}) \arrow[dl, Rightarrow, "\alpha(f)"'] \arrow[dr, Rightarrow, "\alpha(g\circ f)"] & \\ \phi(\underline{m}) \circ X(f) \arrow[rr, Rightarrow, "\alpha(g)\circ id_{X(f)}"'] & & \phi(\underline{m}) \circ X(g \circ f) \end{tikzcd}$$

A.2. Definition. *A a* strict cone *(ϕ,α) from X to C is a pseudo cone such that all natural isomorphisms in the family α are identity natural transformations.*

Now we define a morphism between two pseudo cones from X to C, $\mathcal{L}_1 = (\phi,\alpha)$ and $\mathcal{L}_2 = (\psi,\beta)$.

A.3. Definition. *A morphism of pseudo cones $F : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ consists of a family of natural transformations $F = \{F(\underline{n})\}_{\underline{n}\in Ob(\mathcal{N})}$, having domain $\phi(\underline{n})$ and codomain $\psi(\underline{n})$, which is compatible with the families α and β* i.e. *the following diagram commutes*

$$\begin{tikzcd} \phi(\underline{n}) \arrow[r, Rightarrow, "F(\underline{n})"] \arrow[d, Rightarrow, "\alpha(f)"'] & \psi(\underline{n}) \arrow[d, Rightarrow, "\beta(f)"] \\ \phi(\underline{m}) \circ X(f) \arrow[r, Rightarrow, "F(\underline{m})\circ id_{X(f)}"'] & \psi(\underline{m}) \circ X(f) \end{tikzcd}$$

Let $G : \mathcal{L}_2 \longrightarrow \mathcal{L}_3 = (\upsilon,\delta)$ be another morphism of pseudo cones from X to C, then their composition is defined degreewise *i.e.* $G \circ F$ consists of the collection $\{(G(\underline{n}) \cdot F(\underline{n}))\}_{\underline{n}\in Ob(\mathcal{N})}$. We observe that using the interchance law one gets the following equalities

$$\begin{aligned}(G(\underline{m}) \circ id_{X(f)}) \cdot (F(\underline{m}) \circ id_{X(f)}) &= (G(\underline{m}) \cdot F(\underline{m})) \circ (id_{X(f)} \cdot id_{X(f)}) \\ &= (G(\underline{m}) \cdot F(\underline{m})) \circ id_{X(f)}.\end{aligned}$$

In other words the following diagram commutes

$$\begin{tikzcd}
\phi(\underline{n}) \arrow[rr, Rightarrow, "G(\underline{n})\cdot F(\underline{n})"] \arrow[d, Rightarrow, "\alpha(f)"'] & & \upsilon(n) \arrow[d, Rightarrow, "\delta(f)"] \\
\phi(m) \circ X(f) \arrow[rr, Rightarrow, "(G(\underline{m})\cdot F(\underline{m}))\circ id_{X(f)}"'] & & \upsilon(\underline{m}) \circ X(f)
\end{tikzcd}$$

Thus the composite of two morphisms of pseudo cones, as defined above, is a morphism of pseudo cones. The associativity of vertical composition of natural transformations and the interchange law of natural transformations one can prove the following proposition:

A.4. Proposition. *The composition of morphisms of pseudo cones as defined above is strictly associative.*

Thus we have defined a category. We denote this category of pseudo cones from X to C by $\mathbf{PsCones}(X, C)$. In this paper we will be mainly concerned with pseudo cones having some additional structure which we describe next.

A.5. Definition. *A* Pseudo Bicycle Φ *from X to C, denoted $\Phi : X \rightsquigarrow C$, consists of a triple $\Phi = (\mathcal{L}, \sigma, \tau)$, where $\mathcal{L} = (\phi_{\mathcal{L}}, \alpha_{\mathcal{L}})$ is the underlying pseudo cone, $\tau : \phi_{\mathcal{L}}(\underline{0}) \Rightarrow \Delta(1_C)$ is a natural transformation to the constant functor on the category $X(0^+)$ taking value 1_C, and $\sigma = \{\sigma(\underline{k}, \underline{l})\}_{(\underline{k},\underline{l}) \in Ob(\mathcal{N}) \times Ob(\mathcal{N})}$ is a family of natural transformations*

$$\sigma(\underline{k}, \underline{l}) : \phi(\underline{k+l}) \Rightarrow \phi(\underline{k}) \odot \phi(\underline{l}).$$

The functor $\phi(\underline{k}) \odot \phi(\underline{l}) : X((k+l)^+) \longrightarrow C$ on the right is defined by the following composite

$$X((k+l)^+) \overset{X(\delta^{k+l}_k) \times (\delta^{k+l}_l)}{\longrightarrow} X(k^+) \times X(l^+) \overset{\phi(\underline{k}) \times \phi(\underline{l})}{\longrightarrow} C \times C \overset{- \otimes -}{\longrightarrow} C.$$

This triple is subject to the following conditions:

C.1 For any object $x \in X(m^+)$, the map

$$\sigma(\underline{m}, \underline{0})(x) : \phi(\underline{m+0})(x) \longrightarrow \phi(\underline{m})(x) \underset{C}{\otimes} \phi(0)(X(\delta^m_0)(x)) = (\phi(\underline{m}) \odot \phi(\underline{0}))(x)$$
$$\overset{id \underset{C}{\otimes} \tau(X(\delta^m_0)(x))}{\longrightarrow} \phi(\underline{m})(x) \underset{C}{\otimes} 1_C$$

is required to be the inverse of the (right) unit isomorphism in C. The map $\delta^m_0 : m^+ \longrightarrow 0^+$ in the arrow above is the unique map in Γ^{op} from m^+ to the terminal object. Similarly the map

$$\phi(\underline{0+m})(x) \overset{\sigma(\underline{0},\underline{m})(x)}{\longrightarrow} \phi(\underline{0})(X(\delta^m_0)(x)) \underset{C}{\otimes} \phi(\underline{m})(x) = (\phi(\underline{0}) \odot \phi(\underline{m}))(x)$$
$$\overset{\tau(X(\delta^m_0)(x)) \underset{C}{\otimes} id}{\longrightarrow} 1_C \underset{C}{\otimes} \phi(\underline{m})(x)$$

is the inverse of the (left) unit isomorphism in C.

C.2 *For each pair of objects $\underline{k}, \underline{l} \in Ob(\mathcal{N})$, we define a natural transformation $\gamma_{\phi(\underline{k}),\phi(\underline{l})} : \phi(\underline{k}) \odot \phi(\underline{l}) \Rightarrow \phi(\underline{l}) \odot \phi(\underline{k})$ as follows:*

$$\gamma_{\phi(\underline{k}),\phi(\underline{l})} := \gamma^C \circ id_{\phi(\underline{k})\times\phi(\underline{l})} \circ id_{(X(\delta_k^{k+l}),X(\delta_l^{k+l}))},$$

where γ^C is the symmetry natural isomorphism of C. We require that each natural transformation $\sigma(\underline{k},\underline{l})$ in the collection σ to satisfy the following symmetry condition

$$\begin{tikzcd}
\phi(\underline{k+l}) \arrow[r, Rightarrow, "\phi(\gamma^{\mathcal{N}}_{\underline{k},\underline{l}})"] \arrow[d, Rightarrow, "\sigma(\underline{k},\underline{l})"'] & \phi(\underline{l+k}) \arrow[d, Rightarrow, "\sigma(\underline{l},\underline{k})"] \\
\phi(\underline{k}) \odot \phi(\underline{l}) \arrow[r, Rightarrow, "\gamma_{\phi(\underline{k}),\phi(\underline{l})}"'] & \phi(\underline{l}) \odot \phi(\underline{k})
\end{tikzcd}$$

C.3 *For any triple of objects $\underline{k}, \underline{l}, \underline{m}$ in $\mathcal{N}$, the following diagram commutes*

$$\begin{tikzcd}
\phi((\underline{k+l+m})) \arrow[r, Rightarrow, "\sigma(\underline{k+l},\underline{m})"] \arrow[d, Rightarrow, "\sigma(\underline{k},\underline{l+m})"'] & \phi(\underline{k+l}) \odot \phi(\underline{m}) \arrow[dd, Rightarrow, "\sigma(\underline{k},\underline{l})\odot id_{\phi(\underline{m})}"] \\
\phi(\underline{k}) \odot \phi(\underline{l+m}) \arrow[d, Rightarrow, "id_{\phi(\underline{k})}\odot\sigma(\underline{l},\underline{m})"'] & \\
\phi(\underline{k}) \odot (\phi(l) \odot \phi(m)) & (\phi(\underline{k}) \odot \phi(\underline{l})) \odot \phi(\underline{m}) \arrow[l, Rightarrow, "\alpha_{\phi(\underline{k}),\phi(l),\phi(\underline{m})}"]
\end{tikzcd}$$

where the natural isomorphism $\alpha_{\phi(\underline{k}),\phi(\underline{l}),\phi(\underline{m})}$ is defined by the following diagram which, other than the bottom rectangle, is commutative:

$$\begin{tikzcd}
X((k+l+m)^+) \arrow[r, "F_1"] \arrow[dr, "F_2"'] & X((k+l)^+) \times X(m^+) \arrow[r, "F_3"] & (X(k^+) \times X(l^+)) \times X(m^+) \arrow[dd, equal] \\
& X(k^+) \times X((l+m)^+) \arrow[d, "id\times(X(\delta_l^{l+m}),X(\delta_m^{l+m}))"'] & \\
& X(k^+) \times (X(l^+) \times X(m^+)) \arrow[r, "\alpha"] \arrow[d, "\phi(\underline{k})\times(\phi(\underline{l})\times\phi(\underline{m}))"'] & (X(k^+) \times X(l^+)) \times X(m^+) \arrow[d, "(\phi(\underline{k})\times\phi(\underline{l}))\times\phi(\underline{m})"] \\
& C \times (C \times C) \arrow[r, "\alpha"] \arrow[d, "-\otimes(-\otimes-)"'] & (C \times C) \times C \arrow[d, "(-\otimes-)\otimes-"] \arrow[dl, Rightarrow, "\alpha^C"] \\
& C \arrow[r, equal] & C
\end{tikzcd}$$

where α^C is the associator of the symmetric monoidal category C, the arrow $F_3 = (X(\delta_k^{k+l}), X(\delta_l^{k+l})) \times id$, the arrow $F_1 = (X(\delta_{k+l}^{k+l+m}), X(\delta_m^{k+l+m}))$ and the arrow $F_2 = (X(\delta_k^{k+l+m}), X(\delta_{l+m}^{k+l+m}))$. We observe that the to and right vertical composite arrows ate just the functor $(\phi(\underline{k}) \odot \phi(\underline{l})) \odot \phi(\underline{m})$ and the diagonal and left vertical arrow are just the functor $\phi(\underline{k}) \odot (\phi(\underline{l}) \odot \phi(\underline{m}))$.

C.4 For each pair of maps $f : \underline{k} \longrightarrow \underline{p}$, $g : \underline{l} \longrightarrow \underline{q}$ in $\mathcal{N}$, the following diagram should commute

$$
\begin{tikzcd}
\phi(\underline{k+l}) \arrow[rr, Rightarrow, "\sigma(\underline{k},\underline{l})"] \arrow[d, Rightarrow, "\alpha(f+g)"'] & & \phi(\underline{k}) \odot \phi(\underline{l}) \arrow[d, Rightarrow, "\alpha(f)\odot\alpha(g)"] \\
\phi(\underline{p+q}) \circ X(f+g) \arrow[rr, Rightarrow, "\sigma(\underline{p},\underline{q})\circ id_{X(f+g)}"'] & & (\phi(\underline{p}) \circ X(f)) \odot (\phi(\underline{q}) \circ X(g))
\end{tikzcd}
$$

where $\alpha(f) \odot \alpha(g) = id_{-\otimes-} \circ (\alpha(f) \times \alpha(g)) \circ id_{X(f+g)}$. We observe that $(\phi(\underline{p}) \odot \phi(\underline{q})) \circ X(f+g) = (\phi(\underline{p}) \circ X(f)) \odot (\phi(\underline{q}) \circ X(g))$.

Let $\Psi : X \rightsquigarrow C$ be another bicycle which is composed of the pair $(\mathcal{K}, \delta)$

A.6. Definition. *A morphism of bicycles $F : \Phi \longrightarrow \Psi$ is a morphism of pseudo cones $F : \mathcal{L} \longrightarrow \mathcal{K}$ which is compatible with the additional structure of the two bicycles,* i.e. *for all pairs $(\underline{k}, \underline{l}) \in Ob(\mathcal{N}) \times Ob(\mathcal{N})$, the following diagram commutes*

$$
\begin{tikzcd}
\phi(\underline{k+l}) \arrow[r, Rightarrow, "F(\underline{k+l})"] \arrow[d, Rightarrow, "\sigma(\underline{k},\underline{l})"'] & \psi(\underline{k+l}) \arrow[d, Rightarrow, "\delta(\underline{k},\underline{l})"] \\
\phi(k) \odot \phi(l) \arrow[r, Rightarrow, "F(\underline{k})\odot F(\underline{l})"'] & \psi(\underline{k}) \odot \psi(\underline{l})
\end{tikzcd}
$$

For any pair $(\underline{k}, \underline{l}) \in Ob(\mathcal{N}) \times Ob(\mathcal{N})$, the natural transformations $F(\underline{k})$ and $F(\underline{l})$ determine another natural tranformation

$$F(\underline{k}) \times F(\underline{l}) : \phi(\underline{k}) \times \phi(\underline{l}) \Rightarrow \psi(\underline{k}) \times \psi(\underline{l})$$

which is defined on objects as follows:

$$(F(\underline{k}) \times F(\underline{l}))(x, y) := (F(k)(x), F(k)(y)),$$

where $(x, y) \in Ob(X(k^+)) \times Ob(X(l^+))$. It is defined similarly on morphisms of the product category $X(k^+) \times X(l^+)$. The natural transformation $F(\underline{k}) \odot F(\underline{l})$ in the diagram above is defined by the following composite

$$
\begin{tikzcd}
X((k+l)^+) \arrow[r, "L"] & X(k^+) \times X(l^+) \arrow[r, "\phi(\underline{k})\times\phi(\underline{l})"] \arrow[r, bend right=40, "\psi(\underline{k})\times\psi(\underline{l})"'] \arrow[r, phantom, shift right=3, "F(\underline{k})\times F(\underline{l})\Downarrow"] & C \times C \arrow[r, "-\otimes-"', "C"] & C
\end{tikzcd}
\tag{32}
$$

where $L = (X(\delta_k^{k+l}), X(\delta_l^{k+l}))$. Composition of morphisms of bicycles is done by treating them as morphisms of pseudo cones. We will use the following lemma to show that the composition of two composable morphisms of bicycles is always a morphism of bicycles.

A.7. Lemma. *Let $F : \Phi \longrightarrow \Psi$ and $G : \Psi \longrightarrow \Upsilon$ be two morphisms of bicycles, then for all pairs $(\underline{k}, \underline{l}) \in Ob(\mathcal{N}) \times Ob(\mathcal{N})$*

$$(G(\underline{k}) \odot G(\underline{l})) \cdot (F(\underline{k}) \odot F(l)) = (G(\underline{k}) \cdot F(\underline{k})) \odot (G(\underline{l}) \cdot F(\underline{l})).$$

Proof. The proof of the above lemma follows from the interchange law of compositions of natural transformations. ∎

A.8. Corollary. *Let $F : \Phi \longrightarrow \Psi$ and $G : \Psi \longrightarrow \Upsilon$ be two morphisms of bicycles, then their composite $G \circ F$ is a morphism of bicycles.*

Proof. We know that $G \circ F$ is a morphism of pseudo cones. All that we have to verify that the following diagram commutes

$$\begin{tikzcd} \phi(k+l) \arrow[r, Rightarrow, "G\cdot F(k+l)"] \arrow[d, Rightarrow, "\sigma(k,l)"'] & \upsilon(k+l) \arrow[d, Rightarrow, "\delta(k,l)"] \\ \phi(k) \odot \phi(l) \arrow[r, Rightarrow, "(G\cdot F(k))\odot(G\cdot F(l))"'] & \upsilon(k) \odot \upsilon(l) \end{tikzcd}$$

Since F and G are morphisms of bicycles, therefore the following equality always holds

$$\delta(k,l) \cdot (G \cdot F(k+l)) = ((G(k) \odot G(l)) \cdot (F(k) \odot F(l))) \cdot \sigma(k,l).$$

The lemma A.7 tells us that

$$(G(k) \odot G(l)) \cdot (F(k) \odot F(l)) = (G(k) \cdot F(k)) \odot (G(l) \cdot F(l)).$$

Thus the above diagram commutes. ■

A.9. Proposition. *The definition of the category of all pseudo bicycles from X to C determines a bifunctor*

$$\boldsymbol{Bikes}^{Ps}(-,-) : \Gamma\mathbf{Cat}^{op} \times \mathbf{Perm} \longrightarrow \mathbf{Cat}.$$

A.10. Definition. *A* strict bicycle $(\mathcal{L}, \sigma)$ *is a bicycle such that $\mathcal{L}$ is a strict cone and all natural transformations in the collection σ are natural isomorphisms.*

Strict bicycles from X to C constitute a full subcategory of the category of pseudo bicycles $\mathbf{Bikes}^{Ps}(X, C)$, which we denote by $\mathbf{Bikes}^{Str}(X, C)$. The definition of the category of all strict bicycles from X to C is functorial in both variables.

A.11. Proposition. *The definition of the category of all pseudo bicycles from X to C determines a bifunctor*

$$\boldsymbol{Bikes}^{Str}(-,-) : \Gamma\mathbf{Cat}^{op} \times \mathbf{Perm} \longrightarrow \mathbf{Cat}.$$

B. Bicycles as oplax sections

In this appendix we want to describe a (pseudo) bicycle as an oplax symmetric monoidal functor from the category $\mathcal{N}$. We will construct a symmetric monoidal category $\left(C^X\right)^{Ps}$. The objects of this category are all pairs $(\underline{n}, \phi)$ where $n \in Ob(\mathcal{N})$ and $\phi : X(n^+) \longrightarrow C$ is a functor. A map from $(\underline{n}, \phi)$ to $(\underline{m}, \psi)$ in $\left(C^X\right)^{Ps}$ is a pair (f, η) where $f : n \longrightarrow m$ is a map in the category $\mathcal{N}$ and $\eta : \phi(\underline{n}) \Rightarrow \psi(\underline{m}) \circ X(f)$ is a natural isomorphism. Let

$(g, \beta) : (\underline{m}, \psi) \longrightarrow (\underline{k}, \alpha)$ be another map in $\left(C^X\right)^{Ps}$, then we define their composition as follows:

$$(g, \beta) \circ (f, \eta) := (g \circ f, \beta * \eta),$$

where $\beta * \eta$ is the composite natural transformation $(\beta \circ X(f)) \cdot \eta$ in which $\phi \circ X(f)$ is the *horizontal* composition of the natural transformations $\phi(g)$ and $id_{X(f)}$ and $(\beta \circ X(f)) \cdot \eta$ is the *vertical* composition of the two natural transformations. Using the *interchange law* and the associativity of compositions one can show that the composition defined above is associative.

B.1. Proposition. *The composition law for the category $\left(C^X\right)^{Ps}$, as defined above, is strictly associative.*

Proof. Let $(f, \eta(f)) : (\underline{n}, \phi)$ to $(\underline{m}, \psi)$, $(g, \beta(g)) : (\underline{m}, \psi) \longrightarrow (\underline{k}, \alpha)$ and $(h, \theta(h)) : (\underline{k}, \alpha) \longrightarrow (\underline{j}, \delta)$ be three composable morphisms in $\left(C^X\right)^{Ps}$. We want to show that

$$((h, \theta) \circ (g, \beta) \circ (f, \eta = (h, \theta) \circ ((g, \beta \circ (f, \eta)).$$

In order to do so it would be sufficient to verify the associativity of the operation $*$, *i.e*, to verify $(\theta * \beta) * \eta) = \theta * (\beta * \eta)$. The situation is depicted in the following diagram

$$\begin{tikzcd}
X(n^+) \arrow[rrrrr, "\phi"] \arrow[d, equal] & & & & & C \arrow[d, equal] \\
X(n^+) \arrow[r, "X(f)"'] \arrow[d, equal] & X(m^+) \arrow[rrrr, "\psi"'] \arrow[d, equal] & & \Downarrow\eta & & C \arrow[d, equal] \\
X(n^+) \arrow[r, "X(f)"'] \arrow[d, equal] & X(m^+) \arrow[r, "X(g)"'] \arrow[d, equal] & X(k^+) \arrow[rrr, "\alpha"] \arrow[d, equal] & \Downarrow\beta & & C \arrow[d, equal] \\
X(n^+) \arrow[r, "X(f)"'] \arrow[u, phantom, "\Downarrow id_{X(f)}"] & X(m^+) \arrow[r, "X(g)"'] \arrow[u, phantom, "\Downarrow id_{X(g)}"] & X(k^+) \arrow[r, "X(h)"'] & X(j^+) \arrow[rr, "\delta"'] \arrow[u, phantom, "\Downarrow\theta"] & & C
\end{tikzcd}$$

We begin by considering the left hand side, namely

$$\theta * (\beta * \eta) = (\theta \circ id_{X(g)} \circ id_{X(f)}) \cdot ((\beta \circ id_{X(f)}) \cdot \eta),$$

where $\cdot$ represents vertical composition of natural transformations which is an associative operation. Therefore by rearranging we get

$$\begin{aligned} \theta * (\beta * \eta) = (\theta \circ id_{X(g)} \circ id_{X(f)}) \cdot ((\beta \circ id_{X(f)}) \cdot \eta) \\ ((\theta \circ id_{X(g)} \circ id_{X(f)}) \cdot (\beta \circ id_{X(f)})) \cdot \eta. \end{aligned}$$

Now the interchange law says that the vertical composite $((\theta \circ id_{X(g)} \circ id_{X(f)}) \cdot (\beta \circ id_{X(f)}))$ is the same as $((\theta \circ id_{X(g)}) \cdot \beta) \circ (id_{X(f)} \cdot id_{X(f)})) \cdot \eta$ which is the same as $(\theta * \beta) * \eta$. ∎

Thus we have defined the category $\left(C^X\right)^{Ps}$. Next we want to define a symmetric monoidal structure on the category $\left(C^X\right)^{Ps}$. Let $(\underline{n}, \phi)$ and $(\underline{m}, \psi)$ be two objects of $\left(C^X\right)^{Ps}$, we define

$$(\underline{n}, \phi) \otimes (\underline{m}, \psi) := (\underline{n} + \underline{m}, \phi \odot \psi),$$

where the second component on the right is defined as the following composite

$$X((n+m)^+) \overset{X(\delta_n^{n+m}) \times X(\delta_m^{n+m})}{\longrightarrow} X(n^+) \times X(m^+) \overset{\phi \times \psi}{\longrightarrow} C \times C \overset{- \underset{C}{\otimes} -}{\longrightarrow} C$$

Let $(f, \eta) : (\underline{n}, \phi) \longrightarrow (\underline{k}, \delta)$ and $(g, \beta) : (\underline{m}, \psi) \longrightarrow (l, \alpha)$ be two maps in $\left(C^X\right)^{Ps}$, then we define

$$(f, \eta) \otimes (g, \beta) := (f + g, \eta \odot \beta),$$

where $f + g : \underline{n} + \underline{m} \longrightarrow \underline{k} + \underline{l}$ is the map determined by the symmetric monoidal structure on $\mathcal{N}$ and the natural transformation $\eta \odot \beta$ is defined to be the following composite:

$$id_{- \underset{C}{\otimes} -} \circ (\eta \times \beta) \circ id_{X(\delta_n^{n+m}) \times X(\delta_m^{n+m})}.$$

In other words for any $x \in X((n+m)^+)$

$$(\eta \odot \beta)(x) := \eta(X(\delta_n^{n+m})(x)) \underset{C}{\otimes} \beta(X(\delta_m^{n+m})(x)),$$

where $x \in X((n+m)^+)$. It is easy to see that this defines a natural transformation between the functors

$$\phi \odot \psi : X((n+m)^+) \longrightarrow C$$

and the following composite functor

$$\begin{aligned} X((n+m)^+) \overset{X(\delta_n^{n+m}) \times X(\delta_m^{n+m})}{\longrightarrow} X(n^+) \times X(m^+) &\overset{X(f) \times X(g)}{\longrightarrow} X(k^+) \times X(l^+) \\ &\overset{\phi \times \psi}{\longrightarrow} C \times C \overset{- \underset{C}{\otimes} -}{\longrightarrow} C. \end{aligned}$$

We observe that for any two maps $f : \underline{n} \longrightarrow \underline{k}$ and $g : \underline{m} \longrightarrow \underline{l}$ in the category $\mathcal{N}$, the following diagram

$$\begin{tikzcd} X((n+m)^+) \arrow[r, "(X(\delta_n^{n+m}){,}X(\delta_m^{n+m}))"] \arrow[d, "X(f+g)"'] & X(n^+) \times X(m^+) \arrow[d, "X(f) \times X(g)"] \\ X((k+l)^+) \arrow[r, "(X(\delta_k^{k+l}){,}X(\delta_l^{k+l}))"'] & X(k^+) \times X(l^+) \end{tikzcd}$$

This shows that $\eta \odot \beta$ is a natural transformation between the functors $\phi \odot \psi$ and $(\phi \odot \psi) \circ X(f+g)$. A routine verification of axioms of a symmetric monoidal category gives us the following proposition:

B.2. Proposition. *The category* $\left(C^X\right)^{Ps}$ *is a symmetric monoidal category.*

Proof. The unit object of $\left(C^X\right)^{Ps}$ is the pair $(0, \phi(0))$, where $\phi(0) : X(0^+) \longrightarrow C$ is the constant functor assigning the value 1_C. We begin by verifying that the tensor product defined above defines a bifunctor

$$- \otimes - : \left(C^X\right)^{Ps} \times \left(C^X\right)^{Ps} \longrightarrow \left(C^X\right)^{Ps}.$$

Let $((f, \eta), (g, \beta)) : ((\underline{k}, \phi), (\underline{l}, \phi)) \longrightarrow ((\underline{m}, \phi), (\underline{n}, \phi))$ and $((p, \delta)(q, \theta)) : ((\underline{m}, \phi(m)), (\underline{n}, \phi)) \longrightarrow ((a, \phi(a)), (b, \phi(b)))$ be a pair of composable arrows in the product category $\left(C^X\right)^{Ps} \times \left(C^X\right)^{Ps}$. We will show that

$$(p \circ f, \eta(f) * \delta(p)) \otimes (q \circ g, \beta(f) * \theta(p)) = ((p, \delta(p)) \otimes (q, \theta(q))) \circ ((f, \eta(f)) \otimes (g, \beta(g))).$$

Throughout this proof we will refer to the following commutative diagram

$$\begin{tikzcd}
X((k+l)^+) \arrow[r, "(X(\delta_k^{k+l}){,}X(\delta_l^{k+l}))"] \arrow[d, "X(f+g)"'] & X(k^+) \times X(l^+) \arrow[r, "\phi(k) \times \phi(l)"] \arrow[d, "X(f) \times X(g)"'] & C \times C \arrow[d, equal] \arrow[dl, Rightarrow, "\eta(f) \times \beta(g)"] \\
X((m+n)^+) \arrow[r, "(X(\delta_m^{m+n}){,}X(\delta_n^{m+n}))"] \arrow[d, "X(p+q)"'] & X(m^+) \times X(n^+) \arrow[r, "\phi(m) \times \phi(n)"'] \arrow[d, "X(p) \times X(q)"'] & C \times C \arrow[d, equal] \arrow[dl, Rightarrow, "\delta(p) \times \theta(q)"] \\
X((a+b)^+) \arrow[r, "(X(\delta_a^{a+b}){,}X(\delta_b^{a+b}))"'] & X(a^+) \times X(b^+) \arrow[r, "\phi(a) \times \phi(b)"'] & C \times C
\end{tikzcd}$$

Since the addition operation, $+$, is the symmetric monoidal structure on $\mathcal{N}$, therefore $p \circ f + q \circ g = (p+q) \circ (f+g)$. We recall that

$$(p \circ f, \eta(f) * \delta(p)) \otimes (q \circ g, \beta(f) * \theta(p)) = (p \circ f + q \circ g, (\eta * \delta) \odot (\beta * \theta)(p \circ f + q \circ g)).$$

By definition, the natural transformation $(\eta * \delta) \odot (\beta * \theta)(p \circ f + q \circ g)$ is the following composite:

$$id_{-\otimes-} \circ (((\delta(p) \circ id_{X(f)}) \cdot \eta(f)) \times (((\theta(q) \circ id_{X(g)}) \cdot \beta(f))) \circ id_{X(\delta_k^{k+l}) \times X(\delta_l^{k+l})}.$$

We observe that the above composite is the same as the following composite:

$$id_{-\otimes-} \circ ((\delta(p) \times \theta(q)) \circ (id_{X(f)} \times id_{X(g)}) \cdot (\eta(f) \times \beta(g))) \circ id_{(X(\delta_k^{k+l}), X(\delta_l^{k+l}))}.$$

The composite natural transformation $((p, \delta) \otimes (q, \theta)) \circ ((f, \eta) \otimes (g, \beta(g)))$ is, by definition, the same as $(\theta \odot \delta \circ id_{f+g}) \cdot (\eta \odot \beta)$. Unwinding definitions gives us the following equality

$$\begin{aligned}
(\theta \odot \delta \circ id_{f+g}) \cdot (\eta \odot \beta(f+g)) = \\
(id_{-\otimes-} \circ (\delta \times \theta) \circ id_{(X(\delta_m^{m+n}), X(\delta_n^{m+n}))} \circ id_{X(f+g)}) \\
\cdot (id_{-\otimes-} \circ (\eta \times \beta) \circ id_{(X(\delta_k^{k+l}), X(\delta_l^{k+l}))}).
\end{aligned}$$

The above diagram tells us that $(X(\delta^{m+n}_m), X(\delta^{m+n}_n)) \circ X(f+g) = (X(f) \times X(g)) \circ (X(\delta^{k+l}_k), X(\delta^{k+l}_l))$. Now the interchange law of composition of natural transformations gives the following equalities

$$\begin{aligned}(\theta \odot \delta \circ id_{f+g}) \cdot (\eta \odot \beta(f+g)) = \\ (id_{-\otimes-} \circ (\delta \times \theta) \circ (id_{X(f)} \times id_{X(g)}) \circ id_{(X(\delta^{k+l}_k), X(\delta^{k+l}_l))} \\ \cdot (id_{-\otimes-} \circ (\eta \times \beta) \circ id_{(X(\delta^{k+l}_k), X(\delta^{k+l}_l))}) = \\ id_{-\otimes-} \circ ((\delta \times \theta) \circ (id_{X(f)} \times id_{X(g)}) \cdot (\eta \times \beta)) \circ id_{(X(\delta^{k+l}_k), X(\delta^{k+l}_l))}.\end{aligned}$$

■

We will refer to the category $\Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right)$ as the *category of elements of the exponential from X to C*.

B.3. Proposition. *The construction of the category of elements of the exponential described above defines a bifunctor*

$$\left(-^-\right)^{Ps} : \Gamma\mathbf{Cat}^{op} \times \mathbf{Perm} \longrightarrow \mathbf{Perm}. \tag{33}$$

The category $\left(C^X\right)^{Ps}$ has an associated projection functor $pr_{\mathcal{N}} : \left(C^X\right)^{Ps} \longrightarrow \mathcal{N}$ which projects the first coordinate. Now we are ready to define a bicycle

B.4. Definition. *A* oplax symmetric monoidal section of $\left(C^X\right)^{Ps}$ *is a unital oplax symmetric monoidal functor $\Phi : \mathcal{N} \longrightarrow \left(C^X\right)^{Ps}$ such that $pr_N \circ \Phi = id_{\mathcal{N}}$. A* morphism of oplax symmetric monoidal sections of $\left(C^X\right)^{Ps}$ *is an oplax natural transformation between two oplax symmetric monoidal section of $\left(C^X\right)^{Ps}$.*

We will denote the category of all oplax symmetric monoidal section of $\left(C^X\right)^{Ps}$ by $\Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right)$.

B.5. Proposition. *The category of all oplax symmetric monoidal section of $\left(C^X\right)^{Ps}$ is isomorphic to the category of all bicycles from X to C. We begin by defining I.*

Proof. We will define a pair of functors $I : \Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right) \longrightarrow \mathbf{Bikes}^{Ps}(X, C)$ and $J : \mathbf{Bikes}^{Ps}(X, C) \longrightarrow \Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right)$ and show that they are inverses of one another. For an object $\Phi \in Ob(\Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right))$ we define the bicycle $I(\Phi)$ to be the pair $(\mathcal{L}_\Phi, \sigma_\Phi)$ where $\mathcal{L}_\Phi$ is a pair (ϕ, α_Φ) consisting of a collection of functors ϕ which is composed of a functor $\phi(n) : X(n) \longrightarrow C$, for each $n \in Ob(\mathcal{N})$, which is defined as follows:

$$\phi(n) := \Phi(n).$$

and a collection of natural transformations α_Φ consisting of one natural transformation $\alpha_\Phi(f)$ for each $f \in Mor(\mathcal{N})$, which is defined as follows:

$$\alpha_\Phi(f) := \Phi(f).$$

Finally σ_Φ is a collection consisting of a natural transformation $\sigma_\Phi(k,l)$, for each pair of objects $(k,l) \in Ob(\mathcal{N}) \times Ob(\mathcal{N})$ which is defined as follows:

$$\sigma_\Phi(k,l) := \lambda_\Phi(k,l),$$

where λ_Φ is the natural transformation providing the oplax structure to the functor F. The pair $\mathcal{L}_\phi = (\phi, \alpha_\Phi)$ is a normalized lax cone because Φ is a functor from $\mathcal{N}$ to $\left(C^X\right)^{Ps}$. The conditions in the definition of a bicycle, namely C.1, C.2, C.3 follow from the oplax structure on Φ. Thus we have defined a bicycle $I(\Phi)$. A morphism $F : \Phi \longrightarrow \Theta$ in $\Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right)$ determines a collection of natural transformations C_F consisting of a natural transformation $F(n)$ for each $n \in Ob(\mathcal{N})$. This collection defines a morphism of bicycles because F is an oplax symmetric monoidal functor.

Now we define the functor J. Let $\Psi : X \rightsquigarrow C$ be a bicycle from X to C which is represented by a pair $(\mathcal{L}_\Psi, \sigma_\Psi)$ and whose underlying lax monoidal cone is given by a pair $\mathcal{L}_\Psi = (\psi, \alpha_\Psi)$. We define a oplax symmetric monoidal section of $\left(C^X\right)^{Ps}$, Φ, as follows:

$$\Phi(n) := \psi(n), \qquad \text{and} \qquad \Phi(f) := \alpha_\Psi(f).$$

This defines a functor Φ which is given the oplax symmetric monoidal structure by a natural transformation $\lambda_\Phi : \Phi \circ (- \underset{\mathcal{N}}{\otimes} -) \Rightarrow (- \underset{(C^X)^{Ps}}{\otimes} -) \circ (\Phi \times \Phi)$ which is defines as follows:

$$\lambda_\Phi(k,l) := \sigma_\Psi(k,l).$$

■

Along the lines of the symmetric monoidal category $\left(C^X\right)^{Ps}$, we want to define another symmetric monoidal category $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$ for every pair $(X, C) \in Ob(\Gamma^{op}) \times Ob(\mathbf{Perm})$. The objects of $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$ are all pairs $(\vec{n}, \phi(\vec{n}))$, where $\vec{n} \in Ob(\mathfrak{L})$ and $\phi(\vec{n}) : \mathfrak{L}(X)(\vec{n}) \longrightarrow C$ is a basepoint preserving functor. A map from $(\vec{n}, \phi(\vec{n}))$ to $(\vec{m}, \psi(\vec{n}))$ in $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$ is a pair $(f, \eta(f))$ where $f : \vec{n} \longrightarrow \vec{m}$ is a map in the category $\mathfrak{L}$ and $\eta(f) : \phi(\vec{n}) \Rightarrow \psi(\vec{m}) \circ \mathfrak{L}(X)(f)$ is a natural transformation. Let $(g, \beta(g)) : (\vec{m}, \psi(\vec{m})) \longrightarrow (\vec{k}, \alpha(\vec{k}))$ be another map in $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$, then we define their composition as follows:

$$(g, \beta(g)) \circ (f, \eta(f)) := (g \circ f, \beta(g) * \eta(f)),$$

where $\beta(g) * \eta(f)$ is the composite natural transformation $(\beta(g) \circ id_{X(f)}) \cdot \eta(f)$ in which $\phi(g) \circ id_{X(f)}$ is the *horizontal* composition of the natural transformations $\beta(g)$ and $id_{X(f)}$ and $(\beta(g) \circ X(f)) \cdot \eta(f)$ is the *vertical* composition of the two natural transformations.

Using the *interchange law* and the associativity of compositions, an argument similar to B.1 can be written which proves that the composition defined above is associative. The category $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$ is a symmetric monoidal category with the symmetric monoidal structure being an extension of the symmetric monoidal structure of $\left(C^{X}\right)^{Ps}$. Let $(\vec{n}, \phi(\vec{n}))$ and $(\vec{m}, \psi(\vec{m}))$ be two objects of $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$, we define

$$(\vec{n}, \phi(\vec{n})) \otimes (\vec{m}, \psi(\vec{m})) := (\vec{n}\Box\vec{m}, \phi(\vec{n}) \boxdot \psi(\vec{m})),$$

where the second component on the right is defined as the following composite

$$\mathfrak{L}(X)(\vec{n}\Box\vec{m}) \overset{\lambda_{\mathfrak{L}(X)}(n,m)}{\longrightarrow} \mathfrak{L}(X)(\vec{n}) \times \mathfrak{L}(X)(\vec{m}) \overset{\phi(\vec{n})\times\psi(\vec{m})}{\longrightarrow} C \times C \overset{-\underset{C}{\otimes}-}{\longrightarrow} C, \tag{34}$$

where $\lambda_{\mathfrak{L}(X)}(n, m)$ is the map given by the pseudo-functor structure of $\mathfrak{L}(X)$. Let $(f, \eta(f)) : (\vec{n}, \phi(\vec{n})) \longrightarrow (\vec{k}, \delta(k))$ and $(g, \beta(g)) : (\vec{m}, \psi(\vec{m})) \longrightarrow (\vec{l}, \alpha(\vec{l}))$ be two maps in $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$, then we define

$$(f, \eta(f)) \otimes (g, \beta(g)) := (f\Box g, \eta \boxdot \beta(f\Box g)),$$

where $f\Box g : n\Box m \longrightarrow k\Box l$ is the map determined by the symmetric monoidal structure on $\mathfrak{L}$ and the natural transformation $\eta \boxdot \beta(f\Box g)$ is defined as follows

$$(\eta \boxdot \beta(f\Box g)) := (id_{-\underset{C}{\otimes}-}) \circ (\eta(f) \times \beta(g)),$$

where $\eta(f) \times \beta(g) : \phi(\vec{n}) \times \psi(\vec{m}) \Rightarrow \delta(\vec{k}) \times \alpha(\vec{l})$ is the product of $\eta(f)$ and $\beta(f)$. An argument similar to Proposition B.2 shows that $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$ is a symmetric monoidal category which is permutative if C is permutative. We will refer to the category $\Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right)$ as the *symmetric monoidal completion of the category of elements of the exponential from* X *to* C.

B.6. Proposition. *The symmetric monoidal completion of the category of elements of the exponential described above defines a bifunctor*

$$\left(-^{\mathfrak{L}(-)}\right)^{Ps} : \Gamma\mathbf{Cat}^{op} \times \mathbf{Perm} \longrightarrow \mathbf{Perm}. \tag{35}$$

The category $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$ has an associated projection functor $pr_{\mathfrak{L}} : \left(C^{\mathfrak{L}(X)}\right)^{Ps} \longrightarrow \mathfrak{L}$ which to projects the first coordinate.

B.7. Definition. *A* strict symmetric monoidal section of $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$ *is a strict symmetric monoidal functor* $\Phi : \mathcal{N} \longrightarrow \left(C^{\mathfrak{L}(X)}\right)^{Ps}$ *such that* $pr_A \circ \Phi = id_{\mathfrak{L}}$. *A* morphism of strict symmetric monoidal sections of $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$ *is a symmetric monoidal natural transformation between two strict symmetric monoidal section of* $\left(C^{X}\right)^{Ps}$.

We will denote the (pointed) category of all strict symmetric monoidal sections of $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$ by $\Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right)$. There is an obvious inclusion functor $\mathcal{I} : \left(C^{X}\right)^{Ps} \hookrightarrow \left(C^{\mathfrak{L}(X)}\right)^{Ps}$ which is defined on objects as follows:

$$(\underline{n}, \phi) \mapsto ((n), \phi((n))).$$

B.8. PROPOSITION. *The inclusion functor* $\mathcal{I} : \left(C^X\right)^{Ps} \hookrightarrow \left(C^{\mathfrak{L}(X)}\right)^{Ps}$ *is a unital oplax symmetric monoidal functor.*

PROOF. Let $(k, \phi(k))$ and $(l, \psi(l))$ be two objects in the category $\left(C^X\right)^{Ps}$. Then

$$\mathcal{I}((k+l, \phi(k) \odot \psi(l))) = ((k+l), \phi(k) \odot \psi(l)).$$

There is a partition map $p_{k,l} : (k+l) \longrightarrow (k,l)$ in $\mathfrak{L}$ which makes the following diagram commutative:

$$\begin{tikzcd}
\mathfrak{L}(X)((k+l)) \arrow[rr, "\phi(k)\odot\psi(l)"] \arrow[dr, "\mathfrak{L}(X)(p_{k,l})"'] & & C \\
& \mathfrak{L}(X)((k,l)) \arrow[ur, "\phi((k))\boxdot\psi((l))"'] &
\end{tikzcd}$$

This diagram implies that the partition map $p_{k,l}$ defines a map

$$(p_{k,l}, id) : ((k+l), \phi((k)) \odot \psi((l)))) \longrightarrow ((k,l), \phi((k)) \boxdot \psi((l))))$$

in $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$. We denote this map by $\lambda_{\mathcal{I}}((k, \phi(k)), (l, \psi(l)))$. We observe that $\mathcal{I}(0, \phi(0)) = ((), \phi(()))$. Thus $\mathcal{I}$ strictly preserves the unit. Now we need to check the unit, symmetry and associativity conditions, we begin by checking the symmetry condition. We observe that the following diagram commutes

$$\begin{tikzcd}
(k+l, \phi(k) \odot \psi(l)) \arrow[r, "\mathcal{I}(\gamma)"] \arrow[d, "\lambda_{\mathcal{I}}((k,\phi(k)),(l,\psi(l)))"'] & (l+k, \phi(l) \odot \psi(k)) \arrow[d, "\lambda_{\mathcal{I}}((l,\phi(l)),(k,\psi(k)))"] \\
((k,l), \phi((k)) \boxdot \psi((l))) \arrow[r, "\gamma"'] & ((l,k), \phi((l)) \boxdot \psi((k)))
\end{tikzcd}$$

because $\gamma^{\left(C^X\right)^{Ps}}_{(k,\phi(k)),(l,\psi(l))} = \gamma^{\left(C^{\mathfrak{L}(X)}\right)^{Ps}}_{((k),\phi((k))),((l),\psi((l)))} \circ id_{\mathfrak{L}(X)(p_{k,l})}$. This equality follows from the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{L}(X)((k+l)) \arrow[r, "\mathfrak{L}(X)(p_{k,l})"] \arrow[d, "\mathfrak{L}(X)(\gamma^{\mathcal{N}}_{k,l})"'] & \mathfrak{L}(X)((k,l)) \arrow[rr, "\phi((k))\boxdot\psi((l))"] \arrow[d, "\mathfrak{L}(X)(\gamma^{\mathfrak{L}}_{(k),(l)})"'] & \arrow[d, phantom, "\Downarrow G"] & C \arrow[d, equal] \\
\mathfrak{L}(X)((l+k)) \arrow[r, "\mathfrak{L}(X)(p_{l,k})"'] & \mathfrak{L}(X)((l,k)) \arrow[rr, "\phi((l))\boxdot\psi((k))"'] & & C
\end{tikzcd}$$

where $(\gamma^{\mathfrak{L}}_{(k),(l)}, G) = \gamma^{\left(C^{\mathfrak{L}(X)}\right)^{Ps}}_{((k),\phi((k))),((l),\psi((l)))}$. A similar argument shows that the pair $(\mathcal{I}, \lambda_{\mathcal{I}})$ satisfies the associativity condition OL.3. Thus we have proved that $\mathcal{I}$ is a unital oplax symmetric monoidal functor. ∎

B.9. THEOREM. *For every pair* $(X, C) \in Ob(\Gamma\mathbf{Cat}) \times Ob(\mathbf{Perm})$*, the category* $\Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right)$ *is isomorphic to the category* $\Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right)$.

PROOF. We will define a functor $E : \Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right) \longrightarrow \Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right)$ which is the inverse of the functor $i^*_{\mathcal{N}} : \Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right) \longrightarrow \Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right)$. Let Φ be a oplax symmetric monoidal section of $\Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right)$, then composition with $\mathcal{I}$ gives us a unital oplax symmetric monoidal functor $\mathcal{I} \circ \Phi : \mathcal{N} \longrightarrow \Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right)$. Now proposition 2.44 and the isomorphism of categories $[\mathcal{N}, \Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right)]^{OL}_{\otimes} \cong [\Gamma^{op}, \Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right)]$ tells us that $\mathcal{I}\circ\Phi$ uniquely extends to a strict symmetric monoidal functor $\mathfrak{L}(\mathcal{I} \circ \Phi)$ along the inclusion map $i : \mathcal{N} \longrightarrow \mathfrak{L}$. Moreover this functor is a strict symmetric monoidal section of $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$. We define

$$E(\Phi) := \mathfrak{L}(\mathcal{I} \circ \Phi).$$

The uniqueness of the extension implies that the object function of the functor E is a bijection. A morphism $F : \Phi \longrightarrow \Psi$ in $\Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right)$ can be seen as an oplax symmetric monoidal functor $F : \mathcal{N} \longrightarrow [I; \left(C^X\right)^{Ps}]$, where I is the category having two objects 0 and 1 and exactly one non-identity morphism $0 \longrightarrow 1$, such that the following two diagrams commute

$$\begin{tikzcd} \mathcal{N} \arrow[r, "F"] \arrow[dr, "\Phi"'] & {[I; \left(C^X\right)^{Ps}]} \arrow[d, "{[i_0; \left(C^X\right)^{Ps}]}"] \\ & \left(C^X\right)^{Ps} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{N} \arrow[r, "F"] \arrow[dr, "\Phi"'] & {[I; \left(C^X\right)^{Ps}]} \arrow[d, "{[i_1; \left(C^X\right)^{Ps}]}"] \\ & \left(C^X\right)^{Ps} \end{tikzcd}$$

where $i_0 : 0 \longrightarrow I$ and $i_1 : 1 \longrightarrow I$ are the inclusion functors. We recall that the codomain functor category inherits a strict symmetric monoidal (permutative) structure from $\left(C^X\right)^{Ps}$. We can compose this functor with the oplax symmetric monoidal functor $[I, \mathcal{I}]$ to obtain a composite functor

$$\mathcal{N} \xrightarrow{F} [I; \left(C^X\right)^{Ps}] \xrightarrow{[I;\mathcal{I}]} [I; \left(C^{\mathfrak{L}(X)}\right)^{Ps}]$$

This composite oplax symmetric monoidal functor extends uniquely to a strict symmetric monoidal functor

$$\mathfrak{L}([I;\mathcal{I}] \circ F) : \mathfrak{L} \longrightarrow [I; \left(C^{\mathfrak{L}(X)}\right)^{Ps}]$$

along the inclusion map $i : \mathcal{N} \longrightarrow \mathfrak{L}$. This extended strict symmetric monoidal functor can be seen as a morphism in the category $\Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right)$. We define

$$E(F) := \mathfrak{L}([I;\mathcal{I}] \circ F).$$

One can check that $E(F \circ G) = E(F) \circ E(G)$. The uniqueness of the extension of $[I; \mathcal{I}] \circ F$ to $\mathfrak{L}([I; \mathcal{I}] \circ F)$ implies that the functor E is fully faithful. Thus we have proved that the functor E is an isomorphism of categories. ■

C. The adjunction $\overline{\mathcal{L}} \dashv \overline{\mathcal{K}}$

In this Appendix we will establish an adjunction $\overline{\mathcal{L}} \dashv \overline{\mathcal{K}}$, where $\overline{\mathcal{L}} : \Gamma\mathbf{Cat} \longrightarrow \mathbf{Perm}$ is the realization functor defined in section 6 and $\overline{\mathcal{K}} : \mathbf{Perm} \longrightarrow \Gamma\mathbf{Cat}$ is the functor which is also defined in section 6. We will establish the desired adjunction in two steps. In the first step we show that the mapping set $\mathbf{Perm}(\overline{\mathcal{L}}(X), C)$ is isomorphic to the set of all strict symmetric monoidal sections from $\mathfrak{L}$ to $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$, namely $Ob(\Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right))$. Throughout this section X will denote a Γ- category and C will denote a permutative category. In the second step we show that the Hom set $\Gamma\mathbf{Cat}(X, \overline{\mathcal{K}}(C))$ is isomorphic to the set of oplax symmetric monoidal sections $Ob(\Gamma^{OL}\left(\mathcal{N}, \left(C^{X}\right)^{Ps}\right))$. We begin by constructing a strict symmetric monoidal functor $i : \mathfrak{L} \longrightarrow \left(\overline{\mathcal{L}}(X)^{\mathfrak{L}(X)}\right)^{Ps}$. For each $\vec{n} \in Ob(\mathfrak{L})$ we will define a functor $i(\vec{n}) : \mathfrak{L}(X)(n^+) \longrightarrow \overline{\mathcal{L}}(X)$. For an $\vec{x} \in Ob(\mathfrak{L}(X)(n^+))$, we define

$$i(\vec{n})(\vec{x}) := (\vec{n}, \vec{x}).$$

For a morphism $a : \vec{x} \longrightarrow \vec{y}$ in $\mathfrak{L}(X)(n^+)$, we define

$$i(a) := (id_{\vec{n}}, a),$$

where $(id_{\vec{n}}, a) : (\vec{n}, \vec{x}) \longrightarrow (\vec{n}, \vec{y})$ is a morphism in $\mathfrak{L}(X)$. For each morphism $(h, \phi) : \vec{n} \longrightarrow \vec{m}$ in $\mathfrak{L}$ we will define a natural transformation $i((h, \phi) : i(\vec{n}) \Rightarrow i(\vec{m}) \circ \mathfrak{L}(X)((h, \phi))$. Let $\vec{x} \in Ob(\mathfrak{L}(X)(n^+))$, we define

$$i((h, \phi))(\vec{x}) := ((h, \phi), id_{\mathfrak{L}(X)((h,\phi))(\vec{x})}),$$

where $((h, \phi), id_{\mathfrak{L}(X)((h,\phi))(\vec{x})}) : (\vec{n}, \vec{x}) \longrightarrow (\vec{m}, \mathfrak{L}(X)((h, \phi)(\vec{x}))$ is a morphism in $\mathfrak{L}(X)$. It is easy to see that for any morphism $(h, \phi) : \vec{x} \longrightarrow \vec{y}$ in $\mathfrak{L}(X)(n^+)$ the following diagram commutes

$$\begin{tikzcd}
(\vec{n}, \vec{x}) \arrow[r, "i((h,\phi))(\vec{x})"] \arrow[d, "(id_{\vec{n}}, a)"'] & (\vec{m}, \mathfrak{L}(X)((h, \phi))(\vec{x})) \arrow[d, "(id_{\vec{m}}, \mathfrak{L}(X)((h,\phi))(a))"] \\
(\vec{n}, \vec{y}) \arrow[r, "i((h,\phi))(\vec{y})"'] & (\vec{m}, \mathfrak{L}(X)((h, \phi))(\vec{y}))
\end{tikzcd}$$

in the category $\mathfrak{L}(X)$. Thus we have defined a natural transformation $i((h, \phi))$.

C.1. PROPOSITION. *The collection of functors $\{i(\vec{n})\}_{\vec{n} \in \mathcal{N}}$ glue together to define a strict symmetric monoidal section of $\left(\overline{\mathcal{L}}(X)^{\mathfrak{L}(X)}\right)^{Ps}$.*

PROOF. Clearly $pr_{\mathfrak{L}} \circ i = id_{\mathfrak{L}}$. Further $i(\vec{n} \Box \vec{m}) = i(\vec{n}) \boxdot i(\vec{m})$. ■

A strict symmetric monoidal section of $\left(\overline{\mathcal{L}}(X)^{\mathfrak{L}(X)}\right)^{Ps}$, $\phi : \mathfrak{L} \longrightarrow \left(\overline{\mathcal{L}}(X)^{\mathfrak{L}(X)}\right)^{Ps}$, and a strict monoidal functor $\overline{\phi} : \overline{\mathcal{L}}(X) \longrightarrow C$ determine another strict symmetric monoidal section of $\left(C^{\mathfrak{L}(X)}\right)^{Ps}$, namely $\left(\overline{\phi}^{\mathfrak{L}(X)}\right)^{Ps} \circ \phi$. We want to show that i is a *universal strict symmetric monoidal section i.e.* for any strict symmetric monoidal section of $\phi : \mathfrak{L} \longrightarrow \left(C^{\mathfrak{L}(X)}\right)^{Ps}$, there exists a unique strict symmetric monoidal functor $\overline{\phi} : \overline{\mathcal{L}}(X) \longrightarrow C$. such that $\left(\overline{\phi}^{\mathfrak{L}(X)}\right)^{Ps} \circ i = \phi$.

C.2. Lemma. *The section i is a universal strict symmetric monoidal section.*

Proof. Let $\phi : \mathfrak{L} \longrightarrow \left(C^{\mathfrak{L}(X)}\right)^{Ps}$ be a lax symmetric monoidal section. We begin by constructing a strict monoidal functor $\overline{\phi} : \overline{\mathcal{L}}(X) \longrightarrow C$ such that $\left(\overline{\phi}^{\mathfrak{L}(X)}\right)^{Ps} \circ i = \phi$. On objects of $\overline{\mathcal{L}}(X)$, the functor $\overline{\phi}$ is defined as follows:

$$\overline{\phi}((\vec{n}, \vec{x})) := \phi(\vec{n})(\vec{x}).$$

The morphism function of $\overline{\phi}$ is defined as follows:

$$\overline{\phi}(((h, \psi), a)) := \phi(\vec{m})(a) \circ \phi((h, \psi))(\vec{x}).$$

One can easily check that $\overline{\phi}$ is a functor.

Let $(\vec{m}, \vec{y})$ be another object in $\overline{\mathcal{L}}(X)$, we consider

$$\overline{\phi}((\vec{n}, \vec{x}) \underset{\overline{\mathcal{L}}(X)}{\otimes} (\vec{m}, \vec{y})) = \overline{\phi}(\vec{n}\Box\vec{m}, \lambda(\vec{n}, \vec{m})^{-1}((\vec{x}, \vec{y}))) = \phi(\vec{n}\Box\vec{m})(\lambda(\vec{n}, \vec{m})^{-1}((\vec{x}, \vec{y}))),$$

where $(\vec{x}, \vec{y})$ is the concatenation of $\vec{x}$ and $\vec{y}$. Since ϕ is a symmetric monoidal functor, therefore $\phi(\vec{n}\Box\vec{m}) = \phi(\vec{n}) \boxdot \phi(\vec{m})$. Now we observe that

$$\phi(\vec{n}\Box\vec{m})(\lambda(\vec{n}, \vec{m})^{-1}((\vec{x}, \vec{y}))) = \phi(\vec{n}) \boxdot \phi(\vec{m})(\lambda(\vec{n}, \vec{m})^{-1}((\vec{x}, \vec{y}))) =$$
$$\phi(\vec{n})(\vec{x}) \underset{C}{\otimes} \phi(\vec{m})(\vec{y}) = \overline{\phi}(\vec{n}, \vec{x}) \underset{C}{\otimes} \overline{\phi}(\vec{m}, \vec{y}).$$

where the second equality follows from (34). Thus the functor $\overline{\phi}$ preserves the symmetric monoidal product strictly. Finally, we would like to show that this functor is uniquely defined. Let $G : \overline{\mathcal{L}}(X) \longrightarrow C$ be another a strict monoidal functor such that $\left(G^{\mathfrak{L}(X)}\right)^{Ps} \circ i = \phi$. Then for every object $(\vec{n}, \vec{x})$ in $\overline{\mathcal{L}}(X)$

$$G((\vec{n}, \vec{x})) = G \circ i(\vec{n})(\vec{x}) = \phi(\vec{n})(\vec{x}) = \overline{\phi}(\vec{n}, \vec{x}).$$

A similar argument for morphisms of $\overline{\mathcal{L}}(X)$ shows that G agrees with $\overline{\phi}$ on morphisms also. Thus we have proved that $\overline{\phi}$ is a universal normalized lax symmetric monoidal section. ∎

C.3. Notation. *As seen above we may* compose *a bicycle with a functor to obtain another bicycle. More precisely, let* $F : \overline{\mathcal{L}}(X) \longrightarrow C$ *be a strict symmetric monoidal functor, then we will denote by* $F \circ i$ *the composite strict symmetric monoidal functor*

$$\mathfrak{L} \xrightarrow{i} \Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(\overline{\mathcal{L}}(X)^{\mathfrak{L}(X)}\right)^{Ps}\right) \xrightarrow{\Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(F^{\mathfrak{L}(X)}\right)^{Ps}\right)} \Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right).$$

This composition *defines a functor which we will denote by*

$$i^* : [\overline{\mathcal{L}}(X), C]^{str}_{\otimes} \longrightarrow \Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right).$$

The above lemma and argument similar to the proof of theorem B.9 lead us to the following corollary:

C.4. Corollary. *The functor* $i^* : [\overline{\mathcal{L}}(X), C]^{str}_{\otimes} \longrightarrow \Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right)$ *is an isomorphism of categories which is natural in both* X *and* C*.*

The above corollary together with theorem B.9 and proposition B.5 provide us with the following chain of isomorphisms of categories:

$$\mathbf{Bikes}^{Ps}(X, C) \xrightarrow{J} \Gamma^{OL}\left(\mathcal{N}, \left(C^X\right)^{Ps}\right) \xrightarrow{E}$$
$$\Gamma^{str}_{\otimes}\left(\mathfrak{L}, \left(C^{\mathfrak{L}(X)}\right)^{Ps}\right) \xrightarrow{i^*} [\overline{\overline{\mathcal{L}}}(X), C]^{str}_{\otimes}. \quad (36)$$

Now we start the second step involved in establishing the adjunction $\overline{\mathcal{L}} \dashv \overline{\mathcal{K}}$. We want to define an oplax symmetric monoidal functor $\epsilon : \mathcal{N} \longrightarrow \left(C^{\overline{\mathcal{K}}(C)}\right)^{Ps}$. In order to do so we will define, for each $\underline{n} \in Ob(\mathcal{N})$, a functor $\epsilon(\underline{n}) : \overline{\mathcal{K}}(C)(n^+) \longrightarrow C$. On objects this functor is defined as follows:

$$\epsilon(\underline{n})(\Phi) := \Phi((id_{n^+}))$$

and on morphisms it is defined as follows:

$$\epsilon(\underline{n})(F) := F(\underline{n})((id_{n^+})),$$

where $F : \Phi \longrightarrow \Psi$ is a morphism in $\overline{\mathcal{K}}(C)$. It is easy to see that the above definition preserves composition and identity in $\overline{\mathcal{K}}(C)(n^+)$. We recall that for each map $f : \underline{n} \longrightarrow \underline{m}$ in $\mathcal{N}$ we get a functor $\overline{\mathcal{L}}(f) : \overline{\mathcal{L}}(m^+) \longrightarrow \overline{\mathcal{L}}(n^+)$ which maps an object $(f_1, f_2, \ldots, f_k) \in Ob(\overline{\mathcal{L}}(m))$ to $(f_1 \circ f, f_2 \circ f, \ldots, f_k \circ f) \in Ob(\overline{\mathcal{L}}(n))$. The functor $\overline{\mathcal{K}}(C)(f) : \overline{\mathcal{K}}(C)(n^+) \longrightarrow \overline{\mathcal{K}}(m^+)$ is defined by precomposition *i.e.* for each strict symmetric monoidal functor $\Phi : \overline{\mathcal{L}}(n) \longrightarrow \mathcal{C}$, $\overline{\mathcal{K}}(C)(f)(\Phi) := \Phi \circ \overline{\mathcal{L}}(f)$. For each morphism $f : \underline{n} \longrightarrow \underline{m}$ we will define a natural transformation $\epsilon(f) : \epsilon(\underline{n}) \Rightarrow \epsilon(\underline{m}) \circ \overline{\mathcal{K}}(f)$. We recall that the identity map of $\underline{n}$ determines a map $can : (id_{n^+}) \longrightarrow (f)$ in the category $\overline{\mathcal{L}}(n)$ *i.e.* the following diagram commutes

$$\begin{tikzcd}
Supp(id_{n^+}) = \underline{n} \arrow[rr, "id"] \arrow[dr, "id"'] & & \underline{n} = Supp(f) \arrow[dl, "id"] \\
& \underline{n} &
\end{tikzcd}$$

For an object $\Phi \in \overline{\mathcal{K}}(n^+)$ we define

$$\epsilon(f)(\Phi) := \Phi(can).$$

We observe that domain of $\Phi(can)$ is $\epsilon(\underline{n})(\Phi) = \Phi((id_{n^+}))$ and its codomain is

$$\epsilon(\underline{m})(\overline{\mathcal{K}}(C)(f)(\Phi)) = \overline{\mathcal{K}}(C)(f)(\Phi)(id_{m^+}) = \Phi((f)).$$

Let $F : \Phi \longrightarrow \Psi$ be a morphism in $\overline{\mathcal{K}}(C)(n^+)$, then we have the following commutative diagram

$$\begin{tikzcd} \Phi((id_{n^+})) \arrow[r, "\epsilon(f)(\Phi)"] \arrow[d, "F(\underline{n})((id_{n^+}))"'] & \Phi(f) \arrow[d, "F(\underline{m})(f)"] \\ \Psi((id_{n^+})) \arrow[r, "\epsilon(f)(\Psi)"'] & \Psi(f) \end{tikzcd} \tag{37}$$

where we observe that the map $F(\underline{n})((id_{n^+}))$ is the same as $\epsilon(\underline{n})(F)$ and the map $F(\underline{m})(f)$ is the same as $(\epsilon(m) \circ \overline{\mathcal{K}}(C)(f))(F)$. Thus we have defined a natural transformations $\epsilon(f)$ for all $f \in Mor(\mathcal{N})$. For another morphism $g : \underline{m} \longrightarrow \underline{k}$ in the category $\mathcal{N}$ one can check that $\epsilon(g \circ f) = \epsilon(g) \circ \epsilon(f)$.

C.5. Proposition. *The functor ϵ defined above is an oplax symmetric monoidal section of $\left(C^X\right)^{Ps}$.*

let X and Y be a Γ- categories and C be a permutative category. We will say that an oplax symmetric monoidal section $H : \mathcal{N} \longrightarrow \left(C^{\overline{\mathcal{L}}(X)}\right)^{Ps}$ is *co-universal* if for any other oplax symmetric monoidal section $M : \mathcal{N} \longrightarrow \left(C^{\overline{\mathcal{L}}(Y)}\right)^{Ps}$ there exists a unique morphism of Γ- categories $F : Y \longrightarrow X$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{N} \arrow[r, "H"] \arrow[dr, "M"'] & \left(C^{\overline{\mathcal{L}}(X)}\right)^{Ps} \arrow[d, "\left(C^{\overline{\mathcal{L}}(F)}\right)^{Ps}"] \\ & \left(C^{\overline{\mathcal{L}}(Y)}\right)^{Ps} \end{tikzcd}$$

In the above situation we get a bijection

$$Ob(\Gamma^{OL}_{\otimes}\left(\mathcal{N}, \left(C^Y\right)^{Ps}\right)) \cong Hom_{\Gamma\mathbf{Cat}}(Y, X).$$

C.6. Proposition. *The oplax symmetric monoidal functor $\epsilon : \mathcal{N} \longrightarrow \left(C^{\overline{\mathcal{K}}(C)}\right)^{Ps}$ defined above is co-universal.*

D. Local objects in **Cat**-model categories

D.1. Introduction. A model category E is enriched over categories if the category E is enriched(over **Cat**), tensored and cotensored, and the functor $[-;-] : E^{op} \times \mathrm{E} \longrightarrow \mathbf{Cat}$ is a Quillen functor of two variables, where $\mathbf{Cat} = (\mathbf{Cat}, Eq)$ is the natural model structure for categories. The purpose of this appendix is to introduce the notion of local object with respect to a map in a model category enriched over categories.

D.2. Preliminaries. Recall that a Quillen model structure on a category E is determined by its class of cofibrations together with its class of fibrant objects. For examples, the category of simplicial sets $\mathbf{sSets} = [\Delta^{op}, Set]$ admits two model structures in which the cofibrations are the monomorphisms: the fibrant objects are the Kan complexes in one, and they are the quasi-categories in the other. We call the former the model structure for Kan complexes and the latter the model structure for quasi-categories. We shall denote them respectively by $(\mathbf{sSets}, Kan)$ and $(\mathbf{sSets}, QCat)$. In this appendix we consider categories enriched over **Cat**. If $E = (E, [-;-])$ is a category enriched over **Cat**, then so is the category $\mathbf{CatFunc}(E; \mathbf{Cat})$ of enriched functors $E \longrightarrow \mathbf{Cat}$. An enriched functor $F : \mathrm{E} \longrightarrow \mathbf{Cat}$ isomorphic to the enriched functor $[A, -] : E \longrightarrow \mathbf{Cat}$ is said to be *representable*. The enriched functor F is said to be represented by A. We say that an enriched (over **Cat**) category $E = (E, [-,-])$ is *tensored by* I, where I is the category with two objects and one non-identity arrow, if the enriched functor

$$[I, [A, -]] : E \longrightarrow \mathbf{Cat}$$

is representable (by an object denoted $I \otimes A$) for every object $A \in E$. Dually, we say that an enriched category E is *cotensored by* I if the enriched functor

$$[I, [-, X]] : E^{op} \longrightarrow \mathbf{Cat}$$

is representable (by an object denoted X^I or $[I, X]$) for every object $X \in E$.

D.3. Definition. *We shall say that a model category E is* enriched over categories *if the category E is enriched over* **Cat**, *tensored and cotensored over I and the functor* $[-,-] : E^{op} \times E \longrightarrow \mathbf{Cat}$ *is a Quillen functor of two variables, where* $\mathbf{Cat} = (\mathbf{Cat}, Eq)$ i.e. **Cat** *is endowed with the natural model category structure.*

D.4. Notation. *We will denote the homotopy mapping spaces or the function complexes of a model category M, see [DK80b], [Hov99], [Hir02], by $\mathcal{M}ap^h_M(a, b)$, for each pair of objects $a, b \in M$.*

D.5. Function spaces for categories. If C is a category, we shall denote by $J(C)$ the sub-category of invertible arrows in C. The sub-category $J(C)$ is the largest sub-groupoid of C. More generally, if X is a quasi-category, we shall denote by $J(X)$ the largest sub- Kan complex of X. By construction, we have a pullback square

$$\begin{tikzcd} J(X) \arrow[r] \arrow[d] & X \arrow[d, "h"] \\ NJ(\tau_1(X)) \arrow[r] & N\tau_1(X) \end{tikzcd}$$

where $\tau_1(X)$ is the fundamental category of X and h is the canonical map. The function space X^A is a quasi-category for any simplicial set A. We shall denote by $X^{(A)}$ the full sub-simplicial set of X^A whose vertices are the maps $A \longrightarrow X$ that factor through the inclusion $J(X) \subseteq X$. The simplicial set $X^{(\Delta[1])}$ is a path-space for X. We recall that $\tau_1(X) \cong ho(X)$.

D.6. Lemma. *If C is a category, then the simplicial object $N(J(C)) \cong J(N(C))$, where $N : \mathbf{Cat} \longrightarrow \mathbf{sSets}$ is the nerve functor. Further this isomorphism is natural in C,* i.e. *there is a natural isomorphism between the two composite functors $NJ \cong JN$.*

Proof. We recall that the Kan complex $J(N(C))$ is defined by the following pullback square:

$$\begin{tikzcd} J(N(C)) \arrow[r] \arrow[d] & N(C) \arrow[d, equal] \\ NJ(C) \cong N(J(\tau_1(N(C)))) \arrow[r] & N(\tau_1(N(C))) \cong N(C) \end{tikzcd}$$

Since the above commutative diagram is a pullback diagram in which the right vertical arrow is the identity therefore we have the isomorphism $J(N(C)) \cong NJ(C)$. The second statement follows from the functorality of pullbacks. ∎

The category **Cat** can be enriched over simplicial sets by defining

$$\mathcal{M}ap_{\mathbf{Cat}}(C, D) := N([C, D]).$$

Thus making **Cat** a simplicial category. The adjunction $\tau_1 : \mathbf{sSets} \rightleftharpoons \mathbf{Cat} : N$ makes **Cat** a $(\mathbf{sSets}, QCat)$-model category, see [Joy08, Prop. 6.14]. In other words the functor $N([-;-]) : \mathbf{Cat}^{op} \times \mathbf{Cat} \longrightarrow \mathbf{sSets}$ is a Quillen functor of two variables with respect to the natural model category structure on **Cat** and the Joyal model category structure on **sSets**. We recall from [Sha, Appendix B.3] the mapping space:

$$\mathcal{M}ap^h_{\mathbf{Cat}}(C, D) = J(\mathcal{M}ap_{\mathbf{Cat}}(C, D)) = J(N([C, D])) \tag{38}$$

D.7. Local objects. Let Σ be a set of maps in a model category E. An object $X \in E$ is said to be Σ-local if the map

$$\mathcal{M}ap^h_E(u, X) : \mathcal{M}ap^h_E(A', X) \longrightarrow \mathcal{M}ap^h_E(A, X)$$

is a homotopy equivalence for every map $u : A \longrightarrow A'$ in Σ. Notice that if an object X is weakly equivalent to a Σ-local object, then X is Σ-local. If the model category

E is simplicial (=enriched over Kan complexes) and Σ is a set of maps between cofibrant objects, then a fibrant object $X \in E$ is Σ-local iff the map $\mathcal{M}ap_{\mathbf{sSets}}(u, X) : \mathcal{M}ap_{\mathbf{sSets}}(A', X) \longrightarrow \mathcal{M}ap_{\mathbf{sSets}}(A, X)$ is a homotopy equivalence for every map $u : A \longrightarrow A'$ in Σ, where $\mathcal{M}ap_{\mathbf{sSets}}(-,-) : E^{op} \times E \longrightarrow \mathbf{sSets}$ is the mapping space functor providing the simplicial enrichment of the category E. Now the argument used to prove [Sha, Lemma B.6] adapted to the present setting proves the following lemma:

D.8. Lemma. *Let E be a model category enriched over categories. If $u : A \longrightarrow B$ is a map between cofibrant objects, then the following conditions on a fibrant object $X \in E$ are equivalent*

1. *the map $[u, X] : [B, X] \longrightarrow [A, X]$ is an equivalence of categories;*
2. *the object X is local with respect to the pair of maps $\{u, I \otimes u\}$, where $I \otimes u : I \otimes A \longrightarrow I \otimes B$.*

E. From oplax to symmetric monoidal functors

Throughout this paper we have been using a universal characteristic property of the permutative category $\mathfrak{L}$ namely any oplax symmetric monoidal functor $F : \mathcal{N} \longrightarrow M$, where M is a permutative category extends uniquely to a symmetric monoidal functor $\mathfrak{L}(F) : \mathfrak{L} \longrightarrow M$. The objective of this section is to provide a proof of this universal property. We begin by understanding the Leinster's category $\mathfrak{L}$ abstractly.

The forgetful functor $U : \mathbf{Perm} \longrightarrow \mathbf{Cat}$ has a left adjoint $S : \mathbf{Cat} \longrightarrow \mathbf{Perm}$ which associates to a category C the permutative category $S(C)$ freely generated by C [GJ08, Sec. 3.1]. This permutative category is defined as the following coproduct:

$$S(C) := \bigsqcup_{n \in \mathbb{N}} S^n(C),$$

where $S^n(C)$ is the *symmetric n-power* of C. We observe that the nth symmetric group Σ_n acts naturally on C^n with the right action defined by

$$\overline{x} \cdot \sigma := (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}),$$

for $\overline{x} := (x_1, x_2, \dots, x_n) \in C^n$ and $\sigma \in \Sigma_n$. If we apply the Grothendieck construction to this right action, we obtain the symmetric n-power of C,

$$S^n(C) := \Sigma_n \int C^n,$$

Explicitly, an object of $S^n(C)$ is a finite sequence $\overline{x} = (x_1, x_2, \dots, x_n)$ of objects of C, and the hom set between $\overline{x}$ and $\overline{y}$ is defined as follows:

$$S^n(C)[\overline{x}, \overline{y}] := \bigsqcup_{\sigma \in \Sigma_n} C(x_1, y_{\sigma(1)}) \times C(x_2, y_{\sigma(2)}) \times \cdots \times C(x_n, y_{\sigma(n)}).$$

The tensor product on $S(C)$ is defined by concatenation and the symmetry natural isomorphism is given by the shuffle permutation. The unit of $S(C)$ is the empty sequence. There is an inclusion functor

$$\iota_C : C \longrightarrow S(C)$$

which takes an object $c \in Ob(C)$ to the one element sequence $(c) \in S(C)$. This functor exhibits $S(C)$ as the *free permutative category* generated by C. More precisely, for every permutative category $\mathcal{A}$, the restriction functor

$$\iota_C^* : \mathbf{Perm}(S(C), A) \longrightarrow \mathbf{Cat}(C, A)$$

is a bijection. We observe that the object set of the category $\mathfrak{L}$ is the following:

$$Ob(\mathfrak{L}) = Ob(S(\mathcal{N})).$$

Let P be a permutative category. For each pair of functors $F_1, F_2 : P \longrightarrow \mathbf{Sets}$ one can define another functor which is denoted by $F_1 * F_2 : P \longrightarrow \mathbf{Sets}$ and is called the *convolution product of* F_1 *with* F_2 as follows:

$$F_1 * F_2(m) := \int^{k \in \mathcal{N}} \int^{l \in P} F_1(k) \times F_2(l) \times P(k \underset{P}{\otimes} l, m).$$

The above construction defines a bifunctor

$$- * - : [P; \mathbf{Sets}] \times [P; \mathbf{Sets}] \longrightarrow [P; \mathbf{Sets}].$$

This bifunctor endows the functor category $[P; \mathbf{Sets}]$ with a monoidal structure whose unit object is the functor $P(0, -) : P \longrightarrow \mathbf{Sets}$. See [Day73], [Day70],[IK86].

The adjunction (S, U) discussed above has a counit map *i.e.* for each permutative category P there is a symmetric monoidal functor $\epsilon_P : S(P) \longrightarrow P$. This counit functor provides us with the following composite functor

$$P \xrightarrow{Y} [P^{op}; \mathbf{Sets}] \xrightarrow{\epsilon_P^*} [S(P)^{op}; \mathbf{Sets}]$$

where Y is the Yoneda's embedding functor. We denote this composite functor by $\rho_P : P \longrightarrow [S(P)^{op}; \mathbf{Sets}]$. For each $m \in Ob(P)$ we get a functor $\rho_P(m) : S(P)^{op} \longrightarrow \mathbf{Sets}$ which is defined as follows:

$$\rho_P(m)((k_1, k_2, \dots, k_r)) := P(\epsilon_P(\vec{k}); m) = P(k_1 \underset{P}{\otimes} k_2 \underset{P}{\otimes} \cdots \underset{P}{\otimes} k_r; m)$$

where $\vec{k} = (k_1, k_2, \dots, k_r)$ is an object of the permutative category $S(P)$. For another object $\vec{l} = (l_1, l_2, \dots, l_s)$ in $S(P)$ we define:

$$P^e(-, \vec{l}) := \rho_P(l_1) * \rho_P(l_2) * \cdots * \rho_P(l_s) = P(\epsilon_P(-); l_1) * P(\epsilon_{\mathcal{N}}(-); l_2) * \cdots * P(\epsilon_P(-); l_s). \quad (39)$$

In other words the mapping set $P^e(\vec{k}, \vec{l})$ is the following coend:

$$\int^{q_1 \in S(P)} \cdots \int^{q_s \in S(P)} P(\epsilon_P(\vec{q_1}); l_1) \times \cdots \times P(\epsilon_P(\vec{q_s}); l_s) \times S(P)(\vec{k}; q_1 \underset{S(P)}{\otimes} q_2 \underset{S(P)}{\otimes} \cdots \underset{S(P)}{\otimes} q_s).$$

The above iterated coend has a simple description: A map in the mapping set $P^e(\vec{k}, \vec{l})$ can be described as an $(s+1)$-tuple $(f_1, f_2, \ldots, f_s; h)$, where $|\vec{l}| = s$, $h : \underline{r} \longrightarrow \underline{s}$ is a map in $\mathcal{N}$ and $f_i : \underset{j \in h^{-1}(i)}{\otimes} k_j \longrightarrow l_i$ is a map in P for all $1 \leq i \leq s$. These formulas for mapping spaces for P^e could possibly be deduced from results in [MW15].

E.1. Lemma. *The collection of mapping sets $\{P^e(\vec{k}, \vec{l}) : \vec{k}, \vec{l} \in Ob(S(P))\}$ glue together to define a permutative category P^e whose object set is the same as that of $S(P)$.*

Proof. We begin the proof by defining composition in the category P^e. Let

$$(f_1, f_2, \ldots, f_s; h) : \vec{k} \longrightarrow \vec{l}, (g_1, g_2, \ldots, g_t; q) : \vec{l} \longrightarrow \vec{m} = (m_1, \ldots, m_t)$$

be two maps in P^e. We define their composite to be a map $(c_1, c_2, \ldots, c_t; q \circ h) : \vec{k} \longrightarrow \vec{m}$ where the arrow v_i is defined as follows:

$$\underset{z \in q \circ h^{-1}(i)}{\otimes} k_z \xrightarrow{can} \underset{j \in q^{-1}(i)}{\otimes} \underset{w \in h^{-1}(j)}{\otimes} k_w \xrightarrow{\underset{j \in q^{-1}(i)}{\otimes} f_j} \underset{j \in q^{-1}(i)}{\otimes} l_j \longrightarrow m_i$$

for $1 \leq i \leq t$. The isomorphism *can* in the above diagram is the unique isomorphism provided by the coherence theorem for symmetric monoidal categories between *words* of the same length. The associativity of this composition follows from the coherence theorem for symmetric monoidal categories and the associativity of composition in the permutative category P. The permutative structure is given by concatenation.

■

E.2. Remark. *The permutative category $\mathfrak{L}$ is isomorphic to the permutative category $((\mathcal{N}^{op})^e)^{op}$.*

As above, let $\vec{k}$ and $\vec{l}$ be two objects of P^e having lengths $|\vec{k}| = r$ and $|\vec{l}| = s$ respectively. We define a function

$$\lambda_{\vec{k}, \vec{l}} : P^e(\vec{k}, \vec{l}) \longrightarrow P(\overset{r}{\underset{i=1}{\otimes}} k_i; \overset{s}{\underset{j=1}{\otimes}} l_j).$$

as follows:

$$\lambda_{\vec{k}, \vec{l}}((f_1, f_2, \ldots, f_s; h)) := (\overset{s}{\underset{i=1}{\otimes}} \underset{j \in h^{-1}(i)}{\otimes} f_j) \circ can$$

E.3. Lemma. *The collection of functions $\{\lambda_{\vec{k}, \vec{l}} : \vec{k}, \vec{l} \in Ob(P^e)\}$ glue together to define a strict symmetric monoidal functor*

$$\lambda_P : P^e \longrightarrow P \tag{40}$$

whose objects function maps an object $\vec{k} = (k_1, k_2, \ldots, k_r)$ to $k_1 \underset{P}{\otimes} k_2 \underset{P}{\otimes} \cdots \underset{P}{\otimes} k_r$ in $Ob(P)$.

PROOF. We have to check that the function defined above respects composition in the category P^e. Let $(f_1, f_2, \dots, f_s; h) : \vec{k} \longrightarrow \vec{l}$ and $(g_1, g_2, \dots, g_t; q) : \vec{l} \longrightarrow \vec{m} = (m_1, m_2. \dots, m_t)$ be two composable arrows in the category P^e. We will show the following equality:

$$\lambda_{\vec{l},\vec{m}}((f_1, f_2, \dots, f_s; h)) \circ \lambda_{\vec{k},\vec{l}}((g_1, g_2, \dots, g_t; q)) = \lambda_{\vec{k},\vec{m}}((g_1, g_2, \dots, g_t; q) \circ (f_1, f_2, \dots, f_s; h)) \quad (41)$$

This is equivalent to showing that the following diagram commutes:

$$\begin{tikzcd}
\bigotimes_{i=1}^{t} \bigotimes_{z\in q\circ h^{-1}(i)} k_z \arrow[d, "can"'] & \bigotimes_{i=1}^{r} k_i \arrow[l, "can"'] \arrow[d, "can"] \\
\bigotimes_{i=1}^{t} \bigotimes_{j\in q^{-1}(i)} \bigotimes_{w\in h^{-1}(j)} k_w \arrow[d, "\bigotimes_{i=1}^{t}\bigotimes_{j\in q^{-1}(i)} f_j"'] & \bigotimes_{i=1}^{s} \bigotimes_{j\in h^{-1}(i)} k_j \arrow[l, "can"] \arrow[d, "\bigotimes_{i=1}^{s}\bigotimes_{j\in h^{-1}(i)} f_j"] \\
\bigotimes_{i=1}^{t} \bigotimes_{j\in q^{-1}(i)} l_j \arrow[d, "\bigotimes_{j\in q^{-1}(i)} g_i"'] \arrow[dr, equal] & \bigotimes_{i=1}^{s} l_i \arrow[l, "can"] \arrow[d, "can"] \\
\bigotimes_{i=1}^{t} m_i & \bigotimes_{i=1}^{t} \bigotimes_{j\in q^{-1}(i)} l_j \arrow[l, "\bigotimes_{j\in q^{-1}(i)} g_i"]
\end{tikzcd}$$

The composite of the top left arrow and the three left vertical arrows is the same as $\lambda_{\vec{l},\vec{m}}((f_1, f_2, \dots, f_s; h)) \circ \lambda_{\vec{k},\vec{l}}((g_1, g_2, \dots, g_t; q))$ and the composite of the right three vertical arrows and the bottom horizontal arrows is $\lambda_{\vec{k},\vec{m}}((g_1, g_2, \dots, g_t; q) \circ (f_1, f_2, \dots, f_s; h))$. The bottom square is obviously commutative because both composite arrows are composites of the same two arrows. The top rectangle is made up of canonical isomorphisms provided by the coherence theorem for symmetric monoidal categories. Now the commutativity of the top rectangle follows from [JS93, Corollary 1.6] which implies that any diagram made of canonical coherence maps commutes. The commutativity of the middle rectangle follows from the naturality of the canonical isomorphisms provided by the coherence theorem for symmetric monoidal categories.

■

We recall that the bifunctor $- \otimes - : P \times P \longrightarrow P$ providing the permutative structure of a permutative category P extends uniquely to a functor

$$- \overset{r}{\otimes} - : \prod_{1=1}^{r} P \longrightarrow P,$$

for all $r \in \mathbb{N}$. Similarly any natural transformation $\eta : (- \otimes -) \circ (F \times F) \Rightarrow F \circ (- \otimes -)$

extends uniquely to a natural transformation η_r as shown in the diagram below:

$$\begin{tikzcd}
\prod_{i=1}^{r} P \arrow[r, "- \overset{r}{\otimes} -"] \arrow[d, "\prod_{i=1}^{r} F"'] & P \arrow[d, "F"] \arrow[dl, Rightarrow, "\eta_r"] \\
\prod_{i=1}^{r} D \arrow[r, "- \overset{r}{\otimes} -"'] & D
\end{tikzcd}$$

E.4. Notation. *We will refer to the functor* $- \overset{r}{\otimes} -$ *as the* r*-fold extension of the permutative structure of* P *and refer to* η_r *as the* r*-fold extension of* η.

Any lax symmetric monoidal functor $(F, \mu, \eta) : P \longrightarrow D$ determines a strict symmetric monoidal functor

$$F^e : P^e \longrightarrow D^e \tag{42}$$

The object function of F^e is defined as follows:

$$\vec{k} = (k_1, k_2, \ldots, k_r) \mapsto (F(k_1), F(k_2), \ldots, F(k_r)).$$

Given another object $\vec{l} = (l_1, \ldots, l_s)$ in P we define a map

$$F_{\vec{k},\vec{l}} : P^e(\vec{k}, \vec{l}) \longrightarrow D^e(F(\vec{k}), F(\vec{l}))$$

as follows:

$$(f_1, f_2, \ldots, f_s; h) \mapsto (g_1, g_2, \ldots, g_s; h),$$

where the map $g_i : \underset{j \in h^{-1}(i)}{\otimes} F(k_j) \longrightarrow F(l_i)$ is the following composite:

$$\underset{j \in h^{-1}(i)}{\otimes} F(k_j) \xrightarrow{\mu_{r_i}} F(\underset{j \in h^{-1}(i)}{\otimes} k_j) \xrightarrow{F(f_i)} F(l_i),$$

where $r_i = h^{-1}(i)$. An application of the coherence theorem for symmetric monoidal categories shows that F^e is a functor and it is easy to see that it preserves the permutative structure.

The next theorem is about the universality of the above construction. The existence part of the unique symmetric monoidal functor mentioned in the theorem below follows from [BKP89, Thm. 3.13] which is a statement about algebras over 2-monads. We will provide a direct proof here.

E.5. Theorem. *The symmetric lax monoidal inclusion functor* $\iota : P \longrightarrow P^e$ *is universal: for any symmetric permutative category* D *and a symmetric lax monoidal functor* $\phi : P \longrightarrow D$ *there exists a unique strict symmetric monoidal functor* $\psi : P^e \longrightarrow D$ *such that* $\psi \circ \iota = \phi$*:*

$$\begin{tikzcd}
P \arrow[r, "\iota"] \arrow[dr, "\phi"'] & P^e \arrow[d, "\psi"] \\
& D
\end{tikzcd}$$

PROOF. We define the functor $\psi : P \longrightarrow D$ to be the following composite:

$$P^e \xrightarrow{\phi^e} D^e \xrightarrow{\lambda_D} D$$

The uniqueness of this functor is an easy consequence of the symmetric monoidal structure on the permutative category P^e and the strict symmetric monoidal nature of the functor ψ.

■

An oplax symmetric monoidal functor $F : P \longrightarrow D$ uniquely determines a Lax symmetric monoidal functor between the opposite categories namely $F^{op} : P^{op} \longrightarrow D^{op}$. This duality provides us with the following corollary:

E.6. COROLLARY. *Let P be a strict symmetric monoidal category, then there exists another strict symmetric monoidal category P_e which is equipped with an inclusion functor $\iota : P \longrightarrow P_e$ which is universal: For any strict symmetric monoidal category D and an oplax symmetric monoidal functor $F : P \longrightarrow D$ there exists a unique strict symmetric monoidal functor $\psi : P_e \longrightarrow D$ such that $\psi \circ \iota = F$* i.e. *the following diagram commutes:*

$$\begin{tikzcd} P \arrow[r, "\iota"] \arrow[dr, "F"'] & P_e \arrow[d, "\psi"] \\ & D \end{tikzcd}$$

The strict symmetric monoidal category P_e is isomorphic to $((P^{op})^e)^{op}$.

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