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BRAIDED SKEW MONOIDAL CATEGORIES

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Abstract. We introduce the notion of a braiding on a skew monoidal category, whose curious feature is that the defining isomorphisms involve three objects rather than two. Examples are shown to arise from 2-category theory and from bialgebras. In order to describe the 2-categorical examples, we take a multicategorical approach. We explain how certain braided skew monoidal structures in the 2-categorical setting give rise to braided monoidal bicategories. For the bialgebraic examples, we show that, for a skew monoidal category arising from a bialgebra, braidings on the skew monoidal category are in bijection with cobraidings (also known as coquasitriangular structures) on the bialgebra.

1. Introduction

A skew monoidal category is a category $\mathcal{C}$ equipped with a functor $\mathcal{C}^2 \to \mathcal{C} \colon (X, Y) \mapsto XY$, an object $I \in \mathcal{C}$, and natural transformations

$$(XY)Z \xrightarrow{a} X(YZ)$$
$$IX \xrightarrow{\ell} X$$
$$X \xrightarrow{r} XI$$

satisfying five coherence conditions [16]. When the maps a, ℓ, and r are invertible, we recover the usual notion of monoidal category.

The generalisation allows for new examples. For instance, if B is a bialgebra we obtain a new skew monoidal structure $\mathbf{Vect}[B]$ on the category $\mathbf{Vect}$ of vector spaces, with product $X \star Y = X \otimes B \otimes Y$ and I the ground field K. In this case the associativity map a is defined using the "fusion map" of B, and is invertible just when the bialgebra is Hopf; on the other hand the unit maps ℓ and r are never invertible unless $B = I$. More generally bialgebroids give rise to, and can by characterised by, certain skew monoidal categories [16].

Another class of examples [2] arises if one attempts to study 2-categorical structures as strictly as possible. For instance, there is a skew monoidal structure on the 2-category $\mathbf{FProd}_{\mathrm{s}}$ of categories equipped a choice of finite products, and functors which *strictly* preserve them (not just in the usual up to isomorphism sense). The tensor product AB

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has the universal property that maps $AB \to C$ correspond to functors $A \times B \to C$ preserving products strictly in the first variable but up to isomorphism in the second. Although this example may seem slightly bizarre, there is in fact good reason to study it. What one really cares about is the 2-category **FProd** of categories with finite products, finite-product-preserving functors (in the usual up-to-isomorphism sense), and natural transformations. But this is harder to work with — for example, it has only bicategorical colimits. In particular it has the structure of a monoidal closed bicategory, but the verification of this is technically rather challenging. The skew monoidal structure on $\mathbf{FProd}_{\mathrm{s}}$ is much easier to construct, and in fact — as explained in Section 6.4.3 of [2] — contains the monoidal bicategory structure on **FProd** within it.

Instead of the categories with finite products appearing in the previous paragraph, one can do much the same thing with structures such as symmetric monoidal categories, or permutative categories, leading the way open to possible applications to K-theory, using [5].

A natural question to ask is whether there exists a sensible notion of *braiding* for skew monoidal categories, generalising the classical theory of braided monoidal categories [9]. A naive approach would be to ask for an invertible natural transformation

$$s\colon AB \to BA$$

interacting suitably with the skew monoidal structure. However we would like our notion of braiding to capture the example of $\mathbf{FProd}_{\mathrm{s}}$ and, in that case, the objects AB and BA are not isomorphic. Instead, what we find is that $(AB)C$ and $(AC)B$ both classify functors preserving products strictly in A and up to isomorphism in B and C, and so are isomorphic.

In the present paper we introduce a notion of braiding on a skew monoidal category which is given by an invertible natural transformation

$$s\colon (AB)C \to (AC)B$$

satisfying certain axioms. Apart from capturing the above example and others like it, the definition is justified in various ways. For example in Theorem 4.10 we establish that braidings on the skew monoidal category $\mathbf{Vect}[B]$ are in bijection with *cobraidings* (also known as coquasitriangular structures) [10, 15] on the bialgebra B.

Let us now give a brief outline of the paper. In Section 2 we define braidings and describe various consequences of the axioms — in particular, showing that if the underlying skew monoidal structure is monoidal then our definition restricts to the classical one. In Section 3 we introduce the notion, perhaps more intuitive, of a braided skew closed category. In this setting the braiding is specified by an isomorphism

$$[A,[B,C]] \to [B,[A,C]]$$

just as in the classical setting of symmetric closed categories. Sections 4 and 6 are driven by our two leading classes of examples. Motivated by bialgebras, in Section 4 we study skew cowarpings and monoidal comonads on monoidal categories. The main result of Section 4 is Theorem 4.7, which asserts that, given a monoidal comonad G on a monoidal category $\mathcal{C}$ satisfying a mild hypothesis, there is a bijection between braidings on the monoidal category $\mathcal{C}^G$ of coalgebras and braidings on the cowarped skew monoidal category $\mathcal{C}[G]$. This is then specialised to the bialgebra setting in Theorem 4.10. In Section 5 we introduce braided skew multicategories and show how to pass from these, assuming a representability condition, to braided skew monoidal categories. We then use this in Section 6 to exhibit braidings on the 2-categorical examples such as $\mathbf{FProd}_{\mathrm{s}}$. The more technical parts of the proof in Section 5 are treated in an appendix.

2. Braided skew monoidal categories

Let $\mathcal{C}$ be a skew monoidal category with structure maps $a\colon (AB)C \to A(BC)$, $\ell\colon IA \to A$, and $r\colon A \to AI$.

2.1. Remark. There is a variant of the notion of skew monoidal category in which the directions of a, ℓ, and r are all reversed. We call this a *right skew monoidal category*. (Our skew monoidal categories are also called *left skew*.) If $\mathcal{C}$ is skew monoidal then there are induced right skew monoidal structures on the opposite category $\mathcal{C}$, and also on $\mathcal{C}$ with reverse multiplication; we call the latter $\mathcal{C}^{\mathrm{rev}}$. On the other hand if we use the reverse multiplication on $\mathcal{C}^{\mathrm{op}}$ we get another (left) skew monoidal category, called $\mathcal{C}^{\mathrm{oprev}}$.

2.2. Definition. A *braiding* on $\mathcal{C}$ consists of natural isomorphisms $s\colon (XA)B \to (XB)A$ making the following diagrams commute:

$$\begin{tikzcd}
& ((XA)C)B \arrow[r, "s1"] & ((XC)A)B \arrow[dr, "s"] & \\
((XA)B)C \arrow[ur, "s"] \arrow[dr, "s1"'] & & & ((XC)B)A \\
& ((XB)A)C \arrow[r, "s"'] & ((XB)C)A \arrow[ur, "s1"'] &
\end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd}
((XA)B)C \arrow[r, "s1"] \arrow[d, "a"'] & ((XB)A)C \arrow[r, "s"] & ((XB)C)A \arrow[d, "a1"] \\
(XA)(BC) \arrow[rr, "s"'] & & (X(BC))A
\end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd}
((XA)B)C \arrow[r, "s"] \arrow[d, "a1"'] & ((XA)C)B \arrow[r, "s1"] & ((XC)A)B \arrow[d, "a"] \\
(X(AB))C \arrow[rr, "s"'] & & (XC)(AB)
\end{tikzcd} \tag{2.3}$$

$$\begin{tikzcd}
((XA)B)C \arrow[r, "a1"] \arrow[d, "s"'] & (X(AB))C \arrow[r, "a"] & X((AB)C) \arrow[d, "1s"] \\
((XA)C)B \arrow[r, "a1"'] & (X(AC))B \arrow[r, "a"'] & X((AC)B)
\end{tikzcd} \tag{2.4}$$

The braiding is a *symmetry* if the diagram

$$\begin{tikzcd}
& (XB)A \arrow[dr, "s"] & \\
(XA)B \arrow[ur, "s"] \arrow[rr, equal] & & (XA)B
\end{tikzcd} \tag{2.5}$$

commutes, wherein the equality symbol represents the identity map.

In later sections we shall study in detail the various examples of braided skew monoidal categories described in the introduction; here we content ourselves with giving a rather simple class of examples, including a symmetric skew monoidal structure on the category of pointed sets.

2.3. Example. Let $\mathcal{V}$ be a monoidal category and $M = (M, m, i)$ a monoid in $\mathcal{V}$; we write as if $\mathcal{V}$ were strict. Write $\mathcal{V}^M$ for the category of left M-modules; these consist of an object $X \in \mathcal{V}$ equipped with an associative and unital action $x\colon MX \to X$. The category $\mathcal{V}^M$ has a skew monoidal structure with product $(X, x) \otimes (Y, y) = (XY, xY\colon MXY \to XY)$, unit (M, m), and associativity inherited from $\mathcal{V}$. The left unit map $(M, m)\otimes(X, x) \to (X, x)$ is given by $x\colon MX \to X$ and the right unit map $(X, x) \to (M, m)\otimes(X, x)$ by $Xi\colon X \to XM$. If $\mathcal{V}$ has a braiding c, then $\mathcal{V}^M$ becomes braided via the isomorphisms

$$\begin{tikzcd}
(X,x) \otimes (Y,y) \otimes (Z,z) \arrow[r, "s"] \arrow[d, equal] & (X,x) \otimes (Y,y) \otimes (Z,z) \arrow[d, equal] \\
(XYZ, xYZ) \arrow[r, "1c"] & (XZY, xZY)
\end{tikzcd}$$

and this will be a symmetry if c is one.

2.4. Example. In particular, we may take $\mathcal{V}$ to be the category of sets, with symmetric monoidal structure given by coproduct, and take M to be the terminal monoid. Then $\mathcal{V}^M$ is the category of pointed sets. The product of pointed sets (X, x_0) and (Y, y_0) is the disjoint union $X + Y$ with point $x_0 \in X \subseteq X + Y$ and the unit object is the singleton pointed set.

We shall see in the proof of Proposition 2.6 below that (2.2) and (2.3) are analogues of the braid equations and (2.1) of the Yang-Baxter equation, while (2.4) like (2.1) is automatic in the classical setting of braided monoidal categories.

2.5. Remark. Observe that (2.3) for s is precisely (2.2) for s^{-1}. On the other hand, (2.1) holds for s^{-1} if and only if it does so for s; and the same is true of (2.4). We write $\mathcal{C}^{\mathrm{inv}}$ for the skew monoidal category $\mathcal{C}$ equipped with the natural isomorphism s^{-1}. If s is a symmetry, so that $s^{-1} = s$, then (2.3) is equivalent to (2.2), and $\mathcal{C}^{\mathrm{inv}} = \mathcal{C}$.

There is no explicit compatibility requirement between the braiding and the left and right unit maps, but see Propositions 2.8 and 2.9 below.

2.6. Proposition. *If $\mathcal{C}$ is a monoidal category, braidings and symmetries in the sense of Definition 2.2 are in bijection with braidings and symmetries in the usual sense.*

Proof. There is a unique natural isomorphism $c\colon BC \to CB$ making the diagram

$$\begin{tikzcd}
(IB)C \arrow[r, "\ell 1"] \arrow[d, "s"'] & BC \arrow[d, "c"] \\
(IC)B \arrow[r, "\ell 1"'] & CB
\end{tikzcd}$$

commute; now use (2.4) with $A = I$

$$\begin{tikzcd}
& & & (XB)C \arrow[dr, "a"] & \\
(XB)C \arrow[urrr, bend left, "1"] \arrow[r, "(r1)1"'] \arrow[d, "s"'] & ((XI)B)C \arrow[r, "a1"'] \arrow[d, "s"'] & (X(IB))C \arrow[r, "a"'] \arrow[ur, "(1\ell)1"] & X((IB)C) \arrow[r, "1(\ell 1)"'] \arrow[d, "1s"] & X(BC) \arrow[d, "1c"] \\
(XC)B \arrow[drrr, bend right, "1"'] \arrow[r, "(r1)1"] & ((XI)C)B \arrow[r, "a1"] & (X(IC))B \arrow[r, "a"] \arrow[dr, "(1\ell)1"'] & X((IC)B) \arrow[r, "1(\ell 1)"] & X(CB) \\
& & & (XC)B \arrow[ur, "a"'] &
\end{tikzcd}$$

to deduce that s necessarily has the form

$$(XB)C \xrightarrow{a} X(BC) \xrightarrow{1c} X(CB) \xrightarrow{a^{-1}} (XC)B.$$

Then (2.2) and (2.3) are equivalent to the usual two axioms [9] for a braiding, and (2.1) is a consequence by [9, Proposition 1.2]. (2.4) is automatic by

$$\begin{tikzcd}
((XA)B)C \arrow[r, "a1"] \arrow[d, "a"] \arrow[ddd, bend right=60, "s"'] & (X(AB))C \arrow[r, "a"] & X((AB)C) \arrow[d, "1a"'] \arrow[ddd, bend left=60, "1s"] \\
(XA)(BC) \arrow[rr, "a"] \arrow[d, "1c"] & & X(A(BC)) \arrow[d, "1(1c)"'] \\
(XA)(CB) \arrow[rr, "a"] \arrow[d, "a^{-1}"] & & X(A(CB)) \arrow[d, "1a^{-1}"'] \\
((XA)C)B \arrow[r, "a1"'] & (X(AC))B \arrow[r, "a"'] & X((AC)B)
\end{tikzcd}$$

while finally (2.5) is clearly equivalent to the usual symmetry axiom for c. ∎

Thus if a, ℓ, and r are all invertible, then we recover the usual notion of braided or symmetric monoidal category. But in fact it is enough just to suppose that ℓ is invertible: see Proposition 2.12 below.

Consequences of the axioms. Let s be a braiding on the skew monoidal category $\mathcal{C}$.

2.7. Lemma. *If* (2.2) *holds then the composite*

$$(WB)A \xrightarrow{r} ((WB)A)I \xrightarrow{s} ((WB)I)A \xrightarrow{a1} (W(BI))A$$

is equal to $(1r)1$.

Proof. This holds by commutativity of

$$\begin{tikzcd}
 & (WB)A \arrow[dr, "r"] & & \\
(WA)B \arrow[ur, "s"] \arrow[r, "r"'] \arrow[d, "s"'] \arrow[dr, "1r"'] & ((WA)B)I \arrow[r, "s1"'] \arrow[d, "a"] & ((WB)A)I \arrow[r, "s"'] & ((WB)I)A \arrow[d, "a1"] \\
(WB)A \arrow[rrr, bend right, "(1r)1"'] & (WA)(BI) \arrow[rr, "s"] & & (W(BI))A
\end{tikzcd}$$

and invertibility of s. ∎

2.8. Proposition. *If* (2.2) *holds then the diagram*

$$\begin{tikzcd}
WA \arrow[r, "r"] \arrow[dr, "r1"'] & (WA)I \arrow[d, "s"] \\
 & (WI)A
\end{tikzcd}$$

commutes.

Proof. Use the previous lemma in

$$\begin{tikzcd}
WA \arrow[r, "r"] \arrow[d, "r1"'] & (WA)I \arrow[r, "s"] \arrow[d, "(r1)1"'] & (WI)A \arrow[d, "(r1)1"'] \arrow[ddd, bend left, "1"] \\
(WI)A \arrow[r, "r"] \arrow[drr, "(1r)1"] \arrow[ddrr, "1"'] & ((WI)A)I \arrow[r, "s"] & ((WI)I)A \arrow[d, "a1"'] \\
 & & (W(II))A \arrow[d, "(1\ell)1"'] \\
 & & (WI)A.
\end{tikzcd}$$

∎

This easily implies:

2.9. Proposition. *If* (2.2) *holds then the diagram*

$$\begin{tikzcd}
(XA)I \arrow[r, "s"] & (XI)A \arrow[r, "a"] & X(IA) \arrow[d, "1\ell"] \\
XA \arrow[u, "r"] \arrow[rr, equal] & & XA
\end{tikzcd} \tag{2.6}$$

commutes.

2.10. Proposition. *If* (2.2) *holds then the diagram*

$$\begin{tikzcd}
(XA)B \arrow[r, "r1"] \arrow[d, "s"'] & ((XA)I)B \arrow[r, "a"] & (XA)(IB) \arrow[d, "s"] \\
(XB)A \arrow[r, "(r1)1"'] & ((XI)B)A \arrow[r, "a1"'] & (X(IB))A
\end{tikzcd}$$

commutes.

Proof.

$$\begin{tikzcd}
(XA)B \arrow[r, "r1"] \arrow[dd, "s"'] \arrow[dr, "(r1)1"'] & ((XA)I)B \arrow[r, "a"] \arrow[d, "s1"] & (XA)(IB) \arrow[dd, "s"] \\
& ((XI)A)B \arrow[d, "s"] & \\
(XB)A \arrow[r, "(r1)1"'] & ((XI)B)A \arrow[r, "a1"'] & (X(IB))A
\end{tikzcd}$$

■

Dually, we have

2.11. Proposition. *If* (2.3) *holds then the diagrams*

$$\begin{tikzcd}
WA \arrow[r, "r1"] \arrow[dr, "r"'] & (WI)A \arrow[d, "s"] \\
& (WA)I
\end{tikzcd}$$

$$\begin{tikzcd}
(XA)B \arrow[r, "(r1)1"] \arrow[d, "s"'] & ((XI)A)B \arrow[r, "a1"] & (X(IA))B \arrow[d, "s"] \\
(XB)A \arrow[r, "r1"'] & ((XB)I)A \arrow[r, "a"'] & (XB)(IA)
\end{tikzcd}$$

commute.

Proof. Apply Propositions 2.8 and 2.10 to $\mathcal{C}^{\mathrm{inv}}$. ■

Recall that a skew monoidal category is *left normal* when the left unit maps $\ell\colon IA \to A$ are invertible.

2.12. Proposition. *If $\mathcal{C}$ is a braided skew monoidal category which is left normal, then $\mathcal{C}$ is in fact monoidal.*

Proof. As in Proposition 2.6, there is a unique natural isomorphism $c\colon AB \to BA$ making the diagram

$$\begin{tikzcd}
(IA)B \arrow[r, "s"] \arrow[d, "\ell 1"'] & (IB)A \arrow[d, "\ell 1"] \\
AB \arrow[r, "c"'] & BA
\end{tikzcd}$$

commute, and then by Proposition 2.9, the diagram

$$\begin{tikzcd}
IB \arrow[r, "r"'] \arrow[d, "\ell"'] \arrow[rrrr, bend left, "1"] & (IB)I \arrow[r, "s"'] \arrow[d, "\ell 1"] & (II)B \arrow[r, "a"'] \arrow[d, "\ell 1"'] & I(IB) \arrow[r, "1\ell"'] \arrow[dl, "\ell"] & IB \arrow[d, "\ell"] \\
B \arrow[r, "r"'] & BI \arrow[r, "c"'] & IB \arrow[rr, "\ell"'] & & B
\end{tikzcd}$$

commutes. Since $c\colon BI \to IB$ and the various instances of ℓ are invertible, it follows that $r\colon B \to BI$ is also invertible; thus the skew monoidal category $\mathcal{C}$ is also right normal.

By (2.2) and one of the skew monoidal category axioms, the diagram

$$\begin{tikzcd}
((IA)B)C \arrow[r, "s1"] \arrow[d, "a"'] & ((IB)A)C \arrow[r, "s"] & ((IB)C)A \arrow[d, "a1"'] \arrow[dr, "(\ell 1)1"] & \\
(IA)(BC) \arrow[rr, "s"'] & & (I(BC))A \arrow[r, "\ell 1"'] & (BC)A
\end{tikzcd}$$

commutes, and so the left vertical is invertible. But now by naturality the diagram

$$\begin{tikzcd}
((IA)B)C \arrow[r, "(\ell 1)1"] \arrow[d, "a"'] & (AB)C \arrow[d, "a"] \\
(IA)(BC) \arrow[r, "\ell 1"'] & A(BC)
\end{tikzcd}$$

commutes, and so the right vertical is invertible. This proves that the skew monoidal category $\mathcal{C}$ is actually monoidal. ∎

We close this section with something which is *not* a consequence of the axioms. In a braided skew monoidal category the diagram

$$\begin{tikzcd} ((XI)A)B \arrow[rr, "s1"] \arrow[d, "a1"'] & & ((XA)I)B \arrow[d, "a"] \\ (X(IA))B \arrow[dr, "(1\ell)1"'] & & (XA)(IB) \arrow[dl, "1\ell"] \\ & (XA)B & \end{tikzcd}$$

need not commute. In particular, it does not commute in the braided skew monoidal category of pointed sets described in Example 2.4. Note, however, that composing either composite with $(r1)1\colon (XA)B \to ((XI)A)B$ gives the identity.

3. Braided skew closed categories

Let $\mathcal{C}$ be a skew closed category in the sense of [14] with structure maps

$$[B,C] \xrightarrow{L} [[A,B],[A,C]]$$
$$[I,A] \xrightarrow{i} A$$
$$I \xrightarrow{j} [A,A].$$

3.1. Definition. A *braiding* on $\mathcal{C}$ consists of natural isomorphisms

$$s'\colon [B,[A,Y]] \to [A,[B,Y]]$$

making the following diagrams commute.

$$\begin{tikzcd} {[C,[B,[A,Y]]]} \arrow[r, "s'"] \arrow[d, "{[1,s']}"'] & {[B,[C,[A,Y]]]} \arrow[r, "{[1,s']}"] & {[B,[A,[C,Y]]]} \arrow[d, "s'"] \\ {[C,[A,[B,Y]]]} \arrow[r, "s'"'] & {[A,[C,[B,Y]]]} \arrow[r, "{[1,s']}"'] & {[A,[B,[C,Y]]]} \end{tikzcd} \tag{3.1}$$

$$\begin{tikzcd} {[B,[A,Y]]} \arrow[rr, "s'"] \arrow[d, "L"'] & & {[A,[B,Y]]} \arrow[d, "{[1,L]}"] \\ {[[C,B],[C,[A,Y]]]} \arrow[r, "{[1,s']}"'] & {[[C,B],[A,[C,Y]]]} \arrow[r, "s'"'] & {[A,[[C,B],[C,Y]]]} \end{tikzcd} \tag{3.2}$$

$$\begin{tikzcd} {[B,[A,Y]]} \arrow[rr, "s'"] \arrow[d, "{[1,L]}"'] & & {[A,[B,Y]]} \arrow[d, "L"] \\ {[B,[[C,A],[C,Y]]]} \arrow[r, "s'"'] & {[[C,A],[B,[C,Y]]]} \arrow[r, "{[1,s']}"'] & {[[C,A],[C,[B,Y]]]} \end{tikzcd} \tag{3.3}$$

$$\begin{tikzcd}
{[X,Y]} \arrow[r, "L"] \arrow[d, "L"'] & {[[C,X],[C,Y]]} \arrow[r, "L"] & {[[B,[C,X]],[B,[C,Y]]]} \arrow[d, "{[1,s']}"] \\
{[[B,X],[B,Y]]} \arrow[r, "L"'] & {[[C,[B,X]],[C,[B,Y]]]} \arrow[r, "{[s',1]}"'] & {[[B,[C,X]],[C,[B,Y]]]}
\end{tikzcd} \tag{3.4}$$

The braiding is a *symmetry* if the diagram

$$\begin{tikzcd}
{[B,[A,Y]]} \arrow[r, "s'"] \arrow[dr, "1"'] & {[A,[B,Y]]} \arrow[d, "s'"] \\
& {[B,[A,Y]]}
\end{tikzcd} \tag{3.5}$$

commutes.

3.2. Remark. Condition (3.3) holds for s' just when (3.2) does for the inverse of s'; thus in the symmetric case (3.3) is not needed. A definition of symmetric skew closed category was proposed in [2] — a skew closed category equipped with a natural isomorphism s' satisfying (3.1), (3.2), (3.5), and an axiom concerning the unit I. We prove in Corollary 3.6 that the unit axiom is redundant, and so our definition implies that of [2]. In fact the implication is strict: we shall see in Remark 4.9 that (3.4) does not follow from the other axioms.

3.3. Remark. Just as in the classical case, a skew-closed category $\mathcal{C}$ admits an enrichment over itself, and the representable functors $[B,-]$ are $\mathcal{C}$-enriched. The condition (3.4) states that the isomorphisms

$$s' \colon [B,[C,-]] \to [C,[B,-]]$$

are not just natural, but $\mathcal{C}$-natural.

Suppose that $\mathcal{C}$ is a skew monoidal category which is *closed*, by which we mean that each functor $-\otimes A \colon \mathcal{C} \to \mathcal{C}$ has a right adjoint $[A,-]$, so that there are natural isomorphisms $\mathcal{C}(XA,B) \cong \mathcal{C}(X,[A,B])$. The skew monoidal structure gives rise to a skew closed structure [14], with maps

$$[B,C] \xrightarrow{L} [[A,B],[A,C]]$$
$$[I,A] \xrightarrow{i} A$$
$$I \xrightarrow{j} [A,A].$$

The associativity map a determines a map $t \colon [AB,Y] \to [A,[B,Y]]$, which may be constructed from L as the composite

$$[AB,C] \xrightarrow{L} [[B,AB],[B,C]] \xrightarrow{[u,1]} [A,[B,C]]$$

where u is the unit of the tensor-hom adjunction. Conversely, L can be constructed from t as the composite

$$[B,C] \xrightarrow{[\varepsilon,1]} [[A,B]A,C] \xrightarrow{t} [[A,B],[A,C]]$$

where $\varepsilon\colon [A,B]A \to B$ is the counit of the tensor-hom adjunction.

There is a bijection between natural isomorphisms

$$s\colon (XA)B \to (XB)A \tag{3.6}$$

and natural isomorphisms

$$s'\colon [A,[B,Y]] \to [B,[A,Y]] \tag{3.7}$$

as related by the commutative square

$$\begin{tikzcd}
\mathcal{C}((XA)B,Y) \arrow[r, "\mathcal{C}(s,1)"] \arrow[d] & \mathcal{C}((XB)A,Y) \arrow[d] \\
\mathcal{C}(X,[A,[B,Y]]) \arrow[r, "\mathcal{C}(1,s')"] & \mathcal{C}(X,[B,[A,Y]])
\end{tikzcd}$$

in which the vertical maps are composites of adjointness isomorphisms. A useful way to think of this correspondence is to write $T_A\colon \mathcal{C} \to \mathcal{C}$ for the functor sending X to XA, and H_A for its right adjoint. Then, for given A and B, the $s\colon (XA)B \to (XB)A$ can be seen as the components of a natural transformation $T_BT_A \to T_AT_B$. Since $T_A \dashv H_A$ and $T_B \dashv H_B$, we may compose adjunctions to obtain $T_BT_A \dashv H_AH_B$, and now $s'\colon H_AH_B \to H_BH_A$ is simply the mate of $s\colon T_AT_B \to T_BT_A$.

3.4. Theorem. *Let $\mathcal{C}$ be closed skew monoidal. The equations (2.1)–(2.5) for a natural isomorphism $s\colon (XA)B \to (XB)A$ correspond, in turn, to the equations (3.1)–(3.5) for $s'\colon [A,[B,Y]] \to [B,[A,Y]]$. In particular, there is a bijection between braidings or symmetries on $\mathcal{C}$ as a skew monoidal category and those on $\mathcal{C}$ as a skew closed category.*

Proof. Routine calculation shows that (2.1) and (3.1) are equivalent, and likewise (2.5) and (3.5). The remaining cases require a little more work. First we establish the correspondence between (2.2) and (3.2). The equation (2.2) asserts the commutativity of

$$\begin{tikzcd}
T_CT_BT_A \arrow[r, "1s"] \arrow[d, "a1"'] & T_CT_AT_B \arrow[r, "s1"] & T_AT_CT_B \arrow[d, "1a"] \\
T_{BC}T_A \arrow[rr, "s"'] & & T_AT_{BC}
\end{tikzcd}$$

and, on taking mates, we see that this is equivalent to

$$\begin{tikzcd}
H_{BC}H_A \arrow[rr, "s'"] \arrow[d, "t1"'] & & H_AH_{BC} \arrow[d, "1t"] \\
H_BH_CH_A \arrow[r, "1s'"'] & H_BH_AH_C \arrow[r, "s'1"'] & H_AH_BH_C.
\end{tikzcd}$$

There is a contravariant functor P sending $X \in \mathcal{C}$ to $H_X H_A$, and we may regard the domain $H_{BC}H_A$ of the above displayed equation as this contravariant functor applied to $X = T_C B$. Similarly there is a contravariant functor Q sending X to $H_A H_X H_C$, and the codomain of the displayed equation is QB. The equation asserts the equality of two natural maps $P(T_C B) \to Q(B)$. Taking mates once again, this time with respect to the adjunction $T_C \dashv H_C$, and noting the contravariance of P and Q, we see that this is equivalent to an equation between two induced maps $P(B) \to Q(H_C B)$; specifically, to commutativity of

$$\begin{tikzcd}
H_B H_A \arrow[rr, "s'"] \arrow[d, "L1"'] & & H_A H_B \arrow[d, "1L"] \\
H_{H_C B} H_C H_A \arrow[r, "1s'"'] & H_{H_C B} H_A H_C \arrow[r, "s'1"'] & H_A H_{H_C B} H_C
\end{tikzcd}$$

which is the displayed equation (3.2) of the proposition.

Since (2.3) for s is (2.2) for its inverse, and similarly (3.3) for s' is (3.2) for its inverse, the argument above shows that (2.3) is equivalent to (3.3).

Finally we establish the correspondence between (2.4) and (3.4). First observe that a morphism $f \colon A \to B$ in $\mathcal{C}$ induces a natural transformation $T_f \colon T_A \to T_B$, whose component at an object X is $1f \colon XA \to XB$; this in turn has a mate $H_f \colon H_B \to H_A$. Then we may express (2.4) as

$$\begin{tikzcd}
T_C T_B T_A \arrow[r, "1a"] \arrow[d, "s1"'] & T_C T_{AB} \arrow[r, "a"] & T_{(AB)C} \arrow[d, "T_s"] \\
T_B T_C T_A \arrow[r, "1a"'] & T_B T_{AC} \arrow[r, "a"'] & T_{(AC)B}
\end{tikzcd}$$

which, on taking mates, becomes

$$\begin{tikzcd}
H_{(AC)B} \arrow[r, "t"] \arrow[d, "H_s"'] & H_{AC} H_B \arrow[r, "t1"] & H_A H_C H_B \arrow[d, "1s'"] \\
H_{(AB)C} \arrow[r, "t"'] & H_{AB} H_C \arrow[r, "t1"'] & H_A H_B H_C .
\end{tikzcd}$$

We can regard this as an equality of maps $P(T_B T_c A) \to Q(A)$ for contravariant functors P and Q, and so on taking mates as an equality of maps $P(A) \to Q(H_C H_B A)$; specifically, the equality of the upper composites, and hence also of the lower composites, in the following two diagrams.

$$\begin{tikzcd}
H_A \arrow[r, "H_\varepsilon"] \arrow[dr, "H_\varepsilon"] \arrow[ddr, "L"'] & H_{(H_BA)B} \arrow[r, "H_{\varepsilon 1}"] & H_{((H_CH_BA)C)B} \arrow[dr, "H_s"] & \\
& H_{(H_CA)C} \arrow[r, "H_{\varepsilon 1}"] \arrow[d, "t"] & H_{((H_BH_CA)B)C} \arrow[r, "H_{(s'1)1}"] \arrow[d, "t"] & H_{((H_CH_BA)B)C} \arrow[d, "t"] \\
& H_{H_CA}H_C \arrow[r, "H_\varepsilon 1"] \arrow[dr, "L1"'] & H_{(H_BH_CA)B}H_C \arrow[r, "H_{s'1}1"] \arrow[d, "t1"] & H_{(H_CH_BA)B}H_C \arrow[d, "t1"] \\
& & H_{H_BH_CA}H_BH_C \arrow[r, "H_{s'}11"'] & H_{H_CH_BA}H_BH_C
\end{tikzcd}$$

$$\begin{tikzcd}
H_A \arrow[r, "H_\varepsilon"] \arrow[dr, "L"'] & H_{(H_BA)B} \arrow[r, "H_{\varepsilon 1}"] \arrow[d, "t"] & H_{((H_CH_BA)C)B} \arrow[d, "t"] & \\
& H_{H_BA}H_B \arrow[r, "H_\varepsilon 1"] \arrow[dr, "L"'] & H_{(H_CH_BA)C}H_B \arrow[d, "t1"] & \\
& & H_{H_CH_BA}H_CH_B \arrow[r, "1s'"] & H_{H_CH_BA}H_BH_C
\end{tikzcd}$$

The equality of the lower composites is the condition (3.4) stated in the proposition. ∎

We conclude the section by showing that the unit axiom (S4) of [2] is redundant.

3.5. Proposition. *If* (3.2) *commutes, then so too does*

$$\begin{tikzcd}
{[I,[B,C]]} \arrow[r, "s'"] \arrow[dr, "i"'] & {[B,[I,C]]} \arrow[d, "{[1,i]}"] \\
& {[B,C]}.
\end{tikzcd} \tag{3.8}$$

Proof. By (3.2), the large rectangular region in

$$\begin{tikzcd}
{[A,[B,C]]} \arrow[rr, "s'"'] \arrow[rrr, bend left, "{[i,1]}"] \arrow[d, "L"'] & & {[B,[A,C]]} \arrow[d, "{[1,L]}"'] \arrow[ddr, "{[1,[i,1]]}"] & {[[I,A],[B,C]]} \arrow[dd, "s'"] \\
{[[I,A],[I,[B,C]]]} \arrow[r, "{[1,s']}"'] & {[[I,A],[B,[I,C]]]} \arrow[r, "s'"] \arrow[dr, "{[1,[1,i]]}"'] & {[B,[[I,A],[I,C]]]} \arrow[dr, "{[1,[1,i]]}"'] & \\
& & {[[I,A],[B,C]]} \arrow[r, "s'"'] & {[B,[[I,A],C]]}
\end{tikzcd}$$

commutes, while the other two quadrilaterals commute by naturality of s', and the triangular region by one of the skew closed category axioms. Thus the exterior commutes. Cancel the isomorphism s' at the end of each composite and set $A = I$ to deduce commutativity of the upper region of the diagram

$$\begin{tikzcd}
{[I,[B,C]]} \arrow[rr, "1"] \arrow[d, "L"] \arrow[ddd, bend right=60, "1"'] & & {[I,[B,C]]} \arrow[d, "{[i,1]}"'] \arrow[dd, bend left=60, "1"] \\
{[[I,I],[I,[B,C]]]} \arrow[r, "{[1,s']}"] \arrow[d, "{[j,1]}"] & {[[I,I],[B,[I,C]]]} \arrow[r, "{[1,[1,i]]}"] \arrow[d, "{[j,1]}"'] & {[[I,I],[B,C]]} \arrow[d, "{[j,1]}"'] \\
{[I,[I,[B,C]]]} \arrow[r, "{[1,s']}"'] \arrow[d, "i"] & {[I,[B,[I,C]]]} \arrow[r, "{[1,[1,i]]}"'] & {[I,[B,C]]} \arrow[d, "i"'] \\
{[I,[B,C]]} \arrow[r, "s'"'] & {[B,[I,C]]} \arrow[r, "{[1,i]}"'] & {[B,C]}
\end{tikzcd}$$

in which the central regions commute by functoriality of the internal hom, the lower region by naturality of i, and the left and right regions by skew closed category axioms. ∎

3.6. Corollary. *If* (3.2) *commutes then axiom (S4) of [2] holds; that is, the composite*

$$[B,C] \xrightarrow{L} [[B,B],[B,C]] \xrightarrow{[j,1]} [I,[B,C]] \xrightarrow{s'} [B,[I,C]] \xrightarrow{[1,i]} [B,C]$$

is the identity. In particular, axiom (S4) of [2] is redundant.

Proof. Use (3.2) to replace the last two factors in the displayed composite by the single map $i\colon [I,[B,C]] \to [B,C]$, then use one of the skew closed category axioms to deduce that the resulting composite is the identity.

Since (3.2) is (S3) of [2], it follows that (S4) is redundant as claimed. ∎

4. Braided cowarpings and bialgebras

For this section, we suppose that $\mathcal{C}$ is in fact a monoidal category, and often write as if it were strict, and write $X.Y$ for the product of X and Y. Some aspects would work more generally for a skew monoidal category.

4.1. Skew cowarpings. A *skew cowarping* on $\mathcal{C}$ is a skew warping [11] on $\mathcal{C}^{\mathrm{oprev}}$. Explicitly, this involves the following data

- a functor $Q\colon \mathcal{C} \to \mathcal{C}$
- an object $K \in \mathcal{C}$
- maps $v\colon QX.QY \to Q(X.QY)$

- $v_0 \colon I \to QK$
- $k \colon K.QX \to X$

subject to five axioms. The "cowarped" tensor product is given by $X * Y = X.QY$ with unit K. The structure maps are

$$\begin{array}{c}(X*Y)*Z = X.QY.QZ \xrightarrow{1.v} X.Q(Y.QZ) = X*(Y*Z)\\ K*X = K.QX \xrightarrow{k} X \\ X \xrightarrow{1.v_0} X.QK = X*K\end{array}$$

and this defines a skew monoidal category $\mathcal{C}[Q]$.

4.2. Proposition. *Let $\mathcal{C}$ be a monoidal category, Q a skew cowarping, and*

$$y \colon QX.QY \to QY.QX$$

a natural isomorphism. The induced maps

$$(X*Y)*Z = X.QY.QZ \xrightarrow{1.y} X.QZ.QY = (X*Z)*Y$$

equip $\mathcal{C}[Q]$ with the structure of a braided skew monoidal category if and only if the following diagrams commute.

$$\begin{tikzcd} QX.QY.QZ \arrow[r, "y.1"] \arrow[d, "1.y"'] & QY.QX.QZ \arrow[r, "1.y"] & QY.QZ.QX \arrow[d, "y.1"] \\ QX.QZ.QY \arrow[r, "y.1"'] & QZ.QX.QY \arrow[r, "1.y"'] & QZ.QY.QX \end{tikzcd} \tag{4.1}$$

$$\begin{tikzcd} QX.QY.QZ \arrow[r, "y.1"] \arrow[d, "1.v"'] & QY.QX.QZ \arrow[r, "1.y"] & QY.QZ.QX \arrow[d, "v.1"] \\ QX.Q(Y.QZ) \arrow[rr, "y"'] & & Q(Y.QZ).QX \end{tikzcd} \tag{4.2}$$

$$\begin{tikzcd} QX.QY.QZ \arrow[r, "1.y"] \arrow[d, "v.1"'] & QX.QZ.QY \arrow[r, "y.1"] & QZ.QX.QY \arrow[d, "1.v"] \\ Q(X.QY).QZ \arrow[rr, "y"'] & & QZ.Q(X.QY) \end{tikzcd} \tag{4.3}$$

$$\begin{tikzcd} QX.QY.QZ \arrow[r, "v.1"] \arrow[d, "1.y"'] & Q(X.QY).QZ \arrow[r, "v"] & Q(X.QY.QZ) \arrow[d, "Q(1.y)"] \\ QX.QZ.QY \arrow[r, "v.1"'] & Q(X.QZ).QY \arrow[r, "v"'] & Q(X.QZ.QY) \end{tikzcd} \tag{4.4}$$

Once again, (4.3) is just (4.2) for y^{-1}.

Proof. It is straightforward to see that the above four equations imply, in turn, the four equations (2.1) to (2.4) for a braiding on $\mathcal{C}[Q]$. In the opposite direction one obtains the above four equations above by taking the first variable in (2.1) to (2.4) to be I. ∎

We shall refer to a natural isomorphism $y\colon QX.QY \to QY.QX$ satisfying the above four axioms as a *braiding* on the skew cowarping Q.

4.3. Remark. Many monoidal categories $\mathcal{C}$ have the following property: for any two objects X and A, the maps

$$IA \xrightarrow{x1} XA,$$

where $x\colon I \to X$, are jointly epimorphic. This means that two morphisms $f, g\colon XA \to B$ are equal provided that their composites with $x1$ are equal for any x. In particular this is true if $\mathcal{C}$ is right-closed and $\mathcal{C}(I,-)$ is faithful. Examples include the categories of sets, of R-modules for a commutative ring R, or of topological spaces; non-examples include the category of categories and the category of chain complexes. In this context we can strengthen the preceding result.

4.4. Proposition. *Let $\mathcal{C}$ be a monoidal category having the property that:*

- *for any two objects X and A the maps $x1\colon IA \to XA$, where $x\colon I \to X$, are jointly epimorphic.*

Let Q be a skew cowarping on $\mathcal{C}$. The construction of Proposition 4.2 defines a bijection between braidings on $\mathcal{C}[Q]$ and braidings on Q.

Proof. Without the assumption we may obtain a braiding on Q from a braiding s on $\mathcal{C}[Q]$ using $s\colon (I * Y) * Z \to (I * Z) * Y$. Applying this to a braiding on $\mathcal{C}[Q]$ arising, as in Proposition 4.2, from one on Q returns the original braiding on Q. But to see that every braiding on $\mathcal{C}[Q]$ arises in this way, we rely on the assumption. ∎

4.5. Monoidal comonads. A monoidal comonad $G = (G, \delta, \varepsilon, G_2, G_0)$ on $\mathcal{C}$ determines a skew cowarping with $QX = GX$ and $K = I$, and with v given by

$$GX.GY \xrightarrow{1.\delta} GX.G^2Y \xrightarrow{G_2} G(X.GY)$$

with v_0 and k defined using $G_0\colon GI \to I$ and ε respectively; conversely, any skew cowarping with $K = I$ arises in this way from a monoidal comonad [11, Proposition 3.5].

By a *braiding* on the monoidal comonad G we simply mean a braiding on the associated skew cowarping.

Given a monoidal comonad G, in addition to the cowarped skew monoidal category $\mathcal{C}[G]$, we can form the lifted monoidal structure on the Eilenberg-Moore category $\mathcal{C}^G$ of coalgebras.

4.6. Theorem. *For a monoidal comonad G on a monoidal category $\mathcal{C}$, there is a bijection between braidings on the monoidal category $\mathcal{C}^G$ and braidings on G.*

Proof. First suppose that c is a braiding on $\mathcal{C}^G$. In particular, for any cofree algebras GX and GY there is an isomorphism $c\colon GX.GY \to GY.GX$ in $\mathcal{C}^G$, and this is natural in X and Y. By [9, Proposition 2.1] the Yang-Baxter equation (4.1) holds. Now (4.2) holds by commutativity of the diagram

$$\begin{tikzcd}
GX.GY.GZ \arrow[r, "c.1"] \arrow[d, "1.v"'] \arrow[rr, bend right=10, "c_{GX,GY.GZ}"'] & GY.GX.GZ \arrow[r, "1.c"] & GY.GZ.GX \arrow[d, "v.1"] \\
GX.G(Y.GZ) \arrow[rr, "c"'] & & G(Y.GZ).GX
\end{tikzcd}$$

in which the upper region commutes by one of the braid axioms for $\mathcal{C}^G$, and the lower one by naturality of the braiding with respect to the morphism $v\colon GY.GZ \to G(Y.GZ)$ in $\mathcal{C}^G$. Equation (4.3) holds by a similar, dual, argument. To see that (4.4) holds, first observe that the horizontal composites have the form

$$\begin{tikzcd}
GX.GY.GZ \arrow[r, "1.\delta.1"] \arrow[dr, "1.\delta.\delta"'] & GX.G^2Y.GZ \arrow[r, "G_2.1"] \arrow[d, "1.1.\delta"] & G(X.GY).GZ \arrow[d, "1.\delta"] \\
& GX.G^2Y.G^2Z \arrow[r, "G_2.1"] \arrow[d, "1.G_2"'] & G(X.GY).G^2Z \arrow[d, "G_2"] \\
& GX.G(GY.GZ) \arrow[r, "G_2"'] & G(X.GY.GZ)
\end{tikzcd}$$

and now (4.4) takes the form

$$\begin{tikzcd}
GX.GY.GZ \arrow[r, "1.\delta.\delta"] \arrow[d, "1.c"'] & GX.G^2Y.G^2Z \arrow[r, "1.G_2"] & GX.G(GY.GZ) \arrow[r, "G_2"] \arrow[d, "1.Gc"] & G(X.GY.GZ) \arrow[d, "G(1.c)"] \\
GX.GZ.GY \arrow[r, "1.\delta.\delta"'] & GX.G^2Z.G^2Y \arrow[r, "1.G_2"'] & GX.G(GZ.GY) \arrow[r, "G_2"'] & G(X.GZ.GY)
\end{tikzcd}$$

where the left region commutes because c is a G-coalgebra homomorphism, and the right region by naturality of G_2.

Suppose conversely that $y\colon GX.GY \to GY.GX$ is a braiding on G. First take $X = I$ in (4.4), to deduce commutativity of

$$\begin{tikzcd}
GY.GZ \arrow[r, "v_0.1.1"'] \arrow[d, "y"'] \arrow[rr, bend left=15, "\delta.1"] & GI.GY.GZ \arrow[r, "v.1"'] \arrow[d, "1.y"'] & G^2Y.GZ \arrow[r, "v"'] & G(GY.GZ) \arrow[d, "Gy"] \\
GZ.GY \arrow[r, "v_0.1.1"] \arrow[rr, bend right=15, "\delta.1"'] & GI.GZ.GY \arrow[r, "v.1"] & G^2Z.GY \arrow[r, "v"] & G(GZ.GY)
\end{tikzcd}$$

in which the horizontal composites are the coalgebra structure maps; thus y is a coalgebra homomorphism.

Then (4.2) and (4.3) imply (2.2) and (2.3), and so by Propositions 2.8 and 2.11, the diagrams

$$\begin{tikzcd} GX \arrow[r, "1.G_0"] \arrow[dr, "G_0.1"'] & GX.GI \arrow[d, "y"] \\ & GI.GX \end{tikzcd} \qquad \begin{tikzcd} GX \arrow[r, "G_0.1"] \arrow[dr, "1.G_0"'] & GI.GX \arrow[d, "y"] \\ & GX.GI \end{tikzcd}$$

commute. Then

$$\begin{tikzcd} GX.GZ \arrow[rr, "y"] \arrow[d, "1.G_0.1"] \arrow[dr, "G_0.1.1"] \arrow[ddd, bend right, "1.\delta"'] & & GZ.GX \arrow[d, "G_0.1.1"'] \arrow[ddd, bend left, "\delta.1"] \\ GX.GI.GZ \arrow[r, "y.1"] \arrow[d, "1.1.\delta"] & GI.GX.GZ \arrow[r, "1.y"] & GI.GZ.GX \arrow[d, "1.\delta.1"'] \\ GX.GI.G^2Z \arrow[d, "1.G_2"] & & GI.G^2Z.GX \arrow[d, "G_2.1"'] \\ GX.G^2Z \arrow[rr, "y"'] & & G^2Z.GX \end{tikzcd}$$

commutes by (4.2), and similarly

$$\begin{tikzcd} GX.GZ \arrow[r, "y"] \arrow[d, "\delta.1"'] & GZ.GX \arrow[d, "1.\delta"] \\ G^2X.GZ \arrow[r, "y"'] & GZ.G^2X \end{tikzcd} \tag{4.5}$$

commutes by (4.3). Combining these, we see that

$$\begin{tikzcd} GX.GZ \arrow[r, "y"] \arrow[d, "\delta.\delta"'] & GZ.GX \arrow[d, "\delta.\delta"] \\ G^2X.G^2Z \arrow[r, "y"'] & G^2Z.G^2X \end{tikzcd} \tag{4.6}$$

commutes.

Let (A, α) and (B, β) be G-coalgebras. The rows of

$$\begin{tikzcd} A.B \arrow[r, "\alpha.\beta"] \arrow[d, dotted, "c"'] & GA.GB \arrow[r, shift left, "G\alpha.G\beta"] \arrow[r, shift right, "\delta.\delta"'] \arrow[d, "y"] & G^2A.G^2B \arrow[d, "y"] \\ B.A \arrow[r, "\beta.\alpha"] & GB.GA \arrow[r, shift left, "G\beta.G\alpha"] \arrow[r, shift right, "\delta.\delta"'] & G^2B.G^2A \end{tikzcd}$$

are split equalizers in $\mathcal{C}$ and so are equalizers in $\mathcal{C}^G$. The solid vertical ys commute with the rows by naturality of y and commutativity of (4.6), thus there is a unique induced invertible $c\colon A.B \to B.A$ making the left square commute.

It follows from (4.6) that for cofree coalgebras GX and GY, the map $c\colon GX.GY \to GY.GX$ is just y. The braid axioms will hold for all coalgebras if and only if they hold for cofree coalgebras. One of these holds by commutativity of

$$\begin{tikzcd}
GX.GY.GZ \arrow[r, "\delta.1.\delta"] \arrow[d, "c_{GX.GY,GZ}"'] \arrow[dr, "\delta.\delta.\delta"] & G^2X.GY.G^2Z \arrow[dr, "v.1"] & \\
GZ.GX.GY \arrow[dr, "\delta.\delta.\delta"'] & G^2X.G^2Y.G^2Z \arrow[r, "G_2.1"] \arrow[d, "c_{G^2X.G^2Y,G^2Z}"] & G(GX.GY).G^2Z \arrow[d, "c"] \\
& G^2Z.G^2X.G^2Y \arrow[r, "1.G_2"'] & G^2Z.G(GX.GY)
\end{tikzcd}$$

$$\begin{tikzcd}
GX.GY.GZ \arrow[r, "\delta.1.\delta"] \arrow[d, "1.c"'] & G^2X.GY.G^2Z \arrow[d, "1.c"'] \arrow[dr, "v.1"] & \\
GX.GZ.GY \arrow[d, "c.1"'] & G^2.G^2Z.GY \arrow[d, "c.1"'] & G(GX.GY).G^2Z \arrow[dd, "c"] \\
GZ.GX.GY \arrow[r, "\delta.\delta.1"'] \arrow[dr, "\delta.\delta.\delta"'] & G^2Z.G^2X.GY \arrow[dr, "1.v"] & \\
& G^2Z.G^2X.G^2Y \arrow[r, "1.G_2"'] & G^2Z.G(GX.GY)
\end{tikzcd}$$

and the other is similar. ∎

Combining the above result with Proposition 4.4 we obtain

4.7. Theorem. *Let G be a monoidal comonad on a monoidal category $\mathcal{C}$ having the property of Remark 4.3:*

- *for any two objects X and A the maps $x1\colon IA \to XA$, where $x\colon I \to X$, are jointly epimorphic.*

Then there is a bijection between

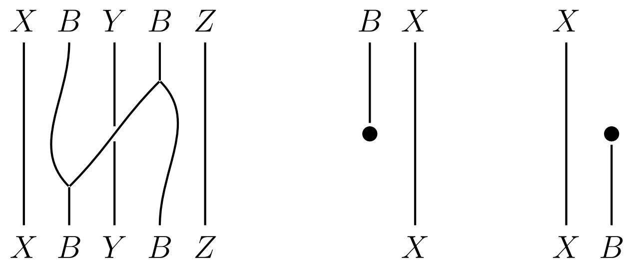
1. *braidings on G,*
2. *braidings on the monoidal category $\mathcal{C}^G$ of coalgebras, and*
3. *braidings on the cowarped skew monoidal category $\mathcal{C}[G]$.*

4.8. The case of bialgebras. Let $\mathcal{V}$ be a symmetric monoidal category, and B a bialgebra in $\mathcal{V}$. The coalgebra structure of B induces a comonad G on $\mathcal{V}$ given by tensoring on the left with B; the algebra structure comprising $\mu\colon BB \to B$ and $\eta\colon I \to B$ makes this into a monoidal comonad with structure maps

$$GX.GY = BXBY \xrightarrow{1c1} BBXY \xrightarrow{\mu 11} BXY = G(XY)$$
$$I \xrightarrow{\eta 1} BI$$

where $c\colon XB \to BX$ is the symmetry isomorphism.

In this setting we write $\mathcal{V}[B]$ for the cowarped skew monoidal structure which has tensor product $X \star Y = XBY$ and unit I. The associator and unit maps for $\mathcal{V}[B]$ are given in the string diagrams below



when read moving down the page, with the multiplication represented by the merging of two strings, the comultiplication by the splitting of two strings, and the unit and counit by the appearance or disappearance of a string at a dot.

If $\mathcal{V}$ is closed then the isomorphisms $\mathcal{V}(XBY, Z) \cong \mathcal{V}(X, [BY, Z])$ ensure that $\mathcal{V}[B]$ is too. This is the case, for instance, if $\mathcal{V} = \text{R-}\mathbf{Mod}$ for a commutative ring R.

4.9. Remark. There is an evident natural isomorphism $GX.GY \cong GY.GX$ with components the symmetry isomorphisms $c\colon (BX)(BY) \cong (BY)(BX)$. The diagrams (4.1), (4.2), and (4.3) always commute, as does the symmetry axiom, but (4.4) commutes if and only if the algebra B is commutative. For the warped skew monoidal category $\mathcal{V}[B]$ the natural isomorphisms $1.c\colon X.BY.BZ \cong X.BY.BZ$ of Proposition 4.2 satisfy all of the equations for a braided skew monoidal category except for (2.4), which commutes just when the algebra B is commutative.

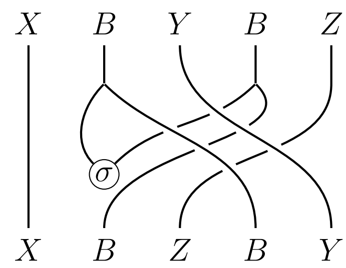
Of course there are many bialgebras whose underlying algebra is not commutative, for instance the group ring of $\mathbb{Z}[G]$ of a non-abelian group G. Therefore the cowarped skew monoidal structure $\mathrm{Ab}[\mathbb{Z}[G]]$ exhibits the independence of (2.4) from the other axioms for a braiding. Since this skew monoidal structure is closed, it follows from Theorem 3.4 that (3.4) is independent of the other axioms for a braided/symmetric skew closed structure.

4.10. Theorem. *Let R be a commutative ring and B an R-bialgebra. There are bijections between*

1. *Cobraidings (coquasitriangular structures) on the bialgebra B;*
2. *Braidings on the skew monoidal category* R-$\mathbf{Mod}[B]$*;*
3. *Braidings on the monoidal category* R-$\mathbf{Mod}^B$ *of B-comodules.*

Proof. The monoidal category R-$\mathbf{Mod}$ satisfies the assumptions of Remark 4.3, and so we may use Theorem 4.7 to deduce the bijection between (2) and (3). The bijection between (1) and (3) is well-known, and can be found for example in [15, Proposition 15.2], on taking $\mathcal{V} = \text{R-}\mathbf{Mod}^{\mathrm{op}}$. ∎

Explicitly, given a braiding s on the skew monoidal category R-**Mod**$[B]$, the isomorphism $s\colon (I*I)*I \cong (I*I)*I$ amounts (modulo unit isomorphisms) to an isomorphism $BB \cong BB$, and composing with $\varepsilon\varepsilon\colon BB \to I$ gives the corresponding coquasitriangular structure. Conversely, given coquasitriangular structure $\sigma\colon BB \to I$, the braiding $(X*Y)*Z \cong (X*Z)*Y$ corresponds to the map $XBYBZ \to XBZBY$ given in the string diagram below.



4.11. Duality. One can now consider what our results give under the various duality principles described in Remark 2.1. Of particular interest is $\mathcal{C}^{\mathrm{op}}$: skew cowarpings on $\mathcal{C}^{\mathrm{op}}$ are a reverse version of the skew warpings of [11], and any opmonoidal monad (T, μ, η) on $\mathcal{C}$ gives rise to one. There is a resulting "warped" right skew monoidal category $\mathcal{C}[T]$ with tensor $X * Y = X.TY$.

It follows formally from Theorem 4.6 that braidings on the Eilenberg-Moore category $\mathcal{C}^T$, with the lifted monoidal structure, are in bijection with braidings on the opmonoidal monad T.

We could also apply Theorem 4.7 to $\mathcal{C}^{\mathrm{op}}$, but this is not so interesting, since for the typical choices of $\mathcal{C}$ the property of Remark 4.3 is less likely to hold for $\mathcal{C}^{\mathrm{op}}$. But in fact it is not hard to see that Proposition 4.4 and Theorem 4.7 hold for $\mathcal{C}$ provided that *either* $\mathcal{C}$ *or* $\mathcal{C}^{op}$ has the property of Remark 4.3. Thus if $\mathcal{C}$ has the property of Remark 4.3, then braidings on $\mathcal{C}^T$ correspond to braidings on the right skew monoidal category $\mathcal{C}[T]$.

The analogue of Theorem 4.10 then says that for a commutative ring R and R-bialgebra B there are bijections between:

1. braidings (quasitriangular structures) on B;
2. braidings on the right skew monoidal category R-**Mod**$[B]$;
3. braidings on the monoidal category B-**Mod** of B-modules

and so in particular there is a "trivial" braiding, lifted from the base braided monoidal category, whenever the bialgebra B is cocommutative.

More generally, consider a bialgebroid B over a (not necessarily commutative) ring R; this amounts to a cocontinuous opmonoidal monad T on the monoidal category R-**Mod**-R of R-bimodules. The opmonoidal structure gives rise to a lifted monoidal structure on the Eilenberg-Moore category; this is just the category of B-modules. The base monoidal category R-**Mod**-R is of course not braided. Nonetheless, we have a bijection between braidings on the Eilenberg-Moore category and braidings on T, giving rise to a notion of braided bialgebroid; compare this with the quasitriangular structures of [4].

4.12. Skew semiwarpings. For $\mathcal{C}$ skew monoidal, the tensor product $A * B = (IA)B$ does not, in general, form part of a skew monoidal structure on $\mathcal{C}$ — but is *skew semimonoidal* in the sense of [12, Section 7]. A braiding on $\mathcal{C}$ then yields natural isomorphisms $A * B \to B * A$ satisfying the classical braid axioms. We establish these results in the present section, and will employ them in Section 6.4 to construct symmetric monoidal bicategories.

We start by considering a generalization of the skew warpings of [11] in which all structure involving the unit is omitted. A *skew semiwarping* on a skew monoidal category $\mathcal{C}$ will be a functor $T\colon \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $v\colon T(TA.B) \to TA.TB$ making the diagram

$$\begin{tikzcd}
T(T(TA.B).C) \arrow[r, "T(v.1)"] \arrow[d, "v"'] & T((TA.TB).C) \arrow[r, "Ta"] & T(TA.(TB.C)) \arrow[r, "v"] & TA.T(TB.C) \arrow[d, "1.v"] \\
T(TA.B).TC \arrow[r, "v.1"'] & (TA.TB).TC \arrow[rr, "a"'] & & TA.(TB.TC)
\end{tikzcd} \tag{4.7}$$

commute. Just as in [11, Section 3], we may define a “warped” tensor product $A * B = TA.B$ and the composite

$$T(TA.B).C \xrightarrow{v.1} (TA.TB).C \xrightarrow{a} TA.(TB.C)$$

defines a morphism $\alpha\colon (A * B) * C \to A * (B * C)$ satisfying the pentagon equation. Thus $\mathcal{C}$ becomes a *skew semimonoidal category* in the sense of [12, Section 7] with respect to this warped structure.

We define an *augmentation* on such a skew semiwarping to be a natural transformation $\varepsilon\colon T \to 1$ making the following diagram commute

$$\begin{tikzcd}
T(TA.B) \arrow[r, "v"] \arrow[dr, "\varepsilon"'] & TA.TB \arrow[d, "1.\varepsilon"] \\
& TA.B
\end{tikzcd}$$

For such an augmentation ε, the diagram

$$\begin{tikzcd}
(A * B) * C \arrow[r, equal] \arrow[dd, "\alpha"'] & T(TA.B).C \arrow[r, "T(\varepsilon.1).1"] \arrow[d, "v.1"] \arrow[dr, "\varepsilon.1"] & T(A.B).C \arrow[r, "\varepsilon.1"] & (A.B).C \arrow[dd, "a"] \\
& (TA.TB).C \arrow[r, "(1.\varepsilon).1"] \arrow[d, "a"] & (TA.B).C \arrow[ur, "(\varepsilon.1).1"'] \arrow[d, "a"] & \\
A * (B * C) \arrow[r, equal] & TA.(TB.C) \arrow[r, "1.(\varepsilon.1)"'] & TA.(B.C) \arrow[r, "\varepsilon.1"'] & A.(B.C)
\end{tikzcd}$$

commutes, and so $\varepsilon.1\colon TA.B \to A.B$ is compatible with the associativity maps α and a; in other words, it makes the identity functor $1\colon \mathcal{C} \to \mathcal{C}$ into a semimonoidal functor from underlying skew semimonoidal category of the original skew monoidal $\mathcal{C}$, to the warped skew semimonoidal category.

4.13. Proposition. *For a skew monoidal $\mathcal{C}$ with $T\colon \mathcal{C} \to \mathcal{C}$ defined by tensoring on the left with I, the maps*

$$I(IA.B) \xrightarrow{\ell} IA.B \xrightarrow{r.1} (IA)I.B \xrightarrow{a} IA.IB$$

define a skew semiwarping v, for which $\ell\colon IA \to A$ is an augmentation.

Proof. First we verify that v is a skew semiwarping. Observe that v is the instance $X = IA$ of the natural transformation $w\colon I(X.B) \to X.IB$ given by

$$I(X.B) \xrightarrow{\ell} X.B \xrightarrow{r.1} XI.B \xrightarrow{a} X.IB$$

and therefore that the left hand square in

$$\begin{tikzcd}
I(I(IA.B).C) \arrow[r, "I(v.1)"] \arrow[d, "v"'] & I((IA.IB).C) \arrow[r, "1a"] \arrow[d, "v"] & I(IA.(IB.C)) \arrow[r, "v"] & IA.I(IB.C) \arrow[d, "1.v"] \\
I(IA.B).IC \arrow[r, "v.1"'] & (IA.IB).IC \arrow[rr, "a"'] & & IA.(IB.IC)
\end{tikzcd}$$

commutes by naturality of u; thus it remains to show the commutativity of the region on the right. This in turn follows by commutativity of

$$\begin{tikzcd}
I(IA.(IB.C)) \arrow[r, "\ell"'] \arrow[rrr, bend left, "v"] & IA.(IB.C) \arrow[r, "r1"] \arrow[rrd, "1"] & (IA)I.(IB.C) \arrow[r, "a"'] & IA.I(IB.C) \arrow[d, "1.\ell"'] \arrow[ddd, bend left, "1.v"] \\
I((IA.IB).C) \arrow[u, "1a"] \arrow[r, "\ell"] \arrow[rrrd, "v"'] & (IA.IB).C \arrow[rr, "a"] & & IA.(IB.C) \arrow[d, "1.(r.1)"'] \\
 & & & IA.((IB)I.C) \arrow[d, "1.a"'] \\
 & & & IA.(IB.IC).
\end{tikzcd}$$

As for the augmentation property, this holds by commutativity of

$$\begin{tikzcd}
I(IA.B) \arrow[r, "\ell"'] \arrow[rrr, bend left, "v"] & IA.B \arrow[r, "r.1"'] \arrow[rrd, bend right, "1"'] & (IA)I.B \arrow[r, "a"'] & IA.IB \arrow[d, "1.\ell"] \\
 & & & IA.B.
\end{tikzcd}$$

■

If now $\mathcal{C}$ is braided, the natural isomorphisms $s\colon (IA)B \to (IB)A$ can be seen as natural isomorphisms $c\colon A * B \to B * A$. We close this section with a result indicating the sense in which these c deserve to be thought of as a braiding.

4.14. PROPOSITION. *The natural isomorphism $c\colon A * B \to B * A$ and its inverse both satisfy the axiom for a braiding asserting the commutativity of*

$$\begin{tikzcd}
(A*B)*C \arrow[r, "c*1"] \arrow[d, "\alpha"'] & (B*A)*C \arrow[r, "\alpha"] & B*(A*C) \arrow[d, "1*c"] \\
A*(B*C) \arrow[r, "c"'] & (B*C)*A \arrow[r, "\alpha"'] & B*(C*A).
\end{tikzcd}$$

If c is a symmetry then the equation

$$\begin{tikzcd}
A*B \arrow[r, "c_{A,B}"] \arrow[rr, bend right, "1"'] & B*A \arrow[r, "c_{A,B}"] & A*B
\end{tikzcd}$$

also holds.

PROOF. The second sentence is immediate. The first holds by commutativity of the following diagram

$$\begin{tikzcd}
I(IX.Y).Z \arrow[r, "\ell.1"] \arrow[ddd, "1s.1"'] & (IX.Y).Z \arrow[r, "(r1)1"] \arrow[dr, "(r1.1)1"] \arrow[dd, "1"] & ((IX)I.Y).Z \arrow[r, "a1"] \arrow[d, "(s1)1"] & (IX.IY).Z \arrow[r, "a"] \arrow[dd, "s1"] & IX.(IY.Z) \arrow[ddd, "s"] \\
 & & ((II)X.Y).Z \arrow[dl, "(\ell1.1)1"'] \arrow[d, "s1"] & & \\
 & (IX.Y)Z \arrow[d, "s1"] & ((II)Y.X).Z \arrow[r, "a1.1"] \arrow[dl, "(\ell1.1)1"'] & (I(IY).X).Z \arrow[dll, "\ell1.1"] \arrow[d, "s"] & \\
I(IY.X).Z \arrow[r, "\ell.1"] & (IY.X)Z \arrow[drr, "s"] \arrow[d, "(r1)1"] & & (I(IY).Z).X \arrow[r, "a1"] \arrow[d, "\ell1.1"] & I(IY.Z).X \arrow[dl, "\ell1"] \\
 & ((IY)I.X)Z \arrow[r, "s"] \arrow[d, "a1"] & ((IY)I.Z).X \arrow[d, "a1"] & (IY.Z).X \arrow[l, "(r1)1"] & \\
 & (IY.IX)Z \arrow[d, "a"] & (IY.IZ).X \arrow[d, "a"] & & \\
 & IY.(IX)Z \arrow[r, "1s"'] & IY.(IZ.X) & &
\end{tikzcd}$$

and the fact that s^{-1} is also a braiding on the skew monoidal category $\mathcal{C}$. ∎

5. Braided skew multicategories

In the earlier paper [3], we defined a notion of *skew multicategory*, and showed that skew monoidal categories could be characterized as skew multicategories satisfying a condition we called *left representability*. This is a skew analogue of the relationship between monoidal categories and representable multicategories.

In this section we build on the ideas of [3], defining a notion of braiding on a skew multicategory, and showing that, in the left representable case, this is equivalent to a braiding on the corresponding skew monoidal category.

We begin by revisiting the notion of skew multicategory defined in [3], to which we refer for further detail.

5.1. SKEW MULTICATEGORIES. A *skew multicategory* $\mathbb{A}$ consists of a multicategory $\mathbb{A}_\ell$ together with extra structure which we shall shortly describe. We generally write A for the set of objects, and $\overline{a}$ to denote a list $a_1, \ldots, a_n$ of objects, so that we may write $\mathbb{A}_\ell(\overline{a}; b)$ for the multihom. The elements of such a multihom are called "loose multimaps". We now turn to the extra structure. This consists of:

- for each *non-empty* such list $\overline{a}$ and each $b \in A$, there is a set $\mathbb{A}_t(\overline{a}; b)$ "of tight multimaps", with a function $j \colon \mathbb{A}_t(\overline{a}; b) \to \mathbb{A}_\ell(\overline{a}; b)$
- for each $a \in A$ there is an element $1_a \in \mathbb{A}_t(a; a)$ which is sent by j to the corresponding identity of $\mathbb{A}_\ell$
- substitution maps

$$\mathbb{A}_t(b_1, \ldots, b_n; c) \times \mathbb{A}_t(\overline{a}_1; b_1) \times \prod_{i=2}^{n} \mathbb{A}_\ell(\overline{a}_i; b_i) \to \mathbb{A}_t(\overline{a}_1, \ldots, \overline{a}_n; c)$$

  whose effect we denote by $(g, f_1, \ldots, f_n) \mapsto g(f_1, \ldots, f_n)$

subject to the evident associativity and identity axioms, along with the requirement that j respect substitution. There is an induced category $\mathcal{A}$ of tight unary maps with the same objects as $\mathbb{A}$ and with $\mathcal{A}(a, b) = \mathbb{A}_t(a; b)$.

For a discussion of various possible reformulations of the notion of skew multicategory, see [3, Sections 3–4].

5.2. REMARK. In practice, many examples of skew multicategories have the property that the functions $j \colon \mathbb{A}_t(\overline{a}; b) \to \mathbb{A}_\ell(\overline{a}; b)$ are subset inclusions. Such skew multicategories amount to *ordinary multicategories* equipped with a *subcollection of tight multimaps* which are non-nullary, contain the identities, and with $g(f_1, \ldots, f_n)$ tight just when both g and f_1 are.

For the moment we content ourselves with a single example; more will be given in Section 6 below.

5.3. Example. There is a multicategory $\mathbb{FP}$ of categories equipped with a choice of finite products and whose multimaps are functors $F\colon \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathcal{B}$ preserving products in each variable in the usual up to isomorphism sense. A nullary map, an element of $\mathbb{FP}(\,;\mathcal{A})$, is an object of $\mathcal{A}$. Declaring a multimap to be tight just when it preserves products *strictly in the first variable* equips $\mathbb{FP}$ with the structure of a skew multicategory.

5.4. Skew monoidal categories arising from left representable skew multicategories. For a skew multicategory $\mathbb{A}$ and a list $\overline{a}$ of objects, there are induced functors $\mathbb{A}_t(\overline{a};-)$ and $\mathbb{A}_\ell(\overline{a};-)$ from $\mathcal{A}$ to **Set** (with the former defined only if $\overline{a}$ is non-empty). We say that $\mathbb{A}$ is *weakly representable* if these functors $\mathbb{A}_\ell(\overline{a};-)$ and $\mathbb{A}_t(\overline{a};-)$ are representable. This says that, for $x \in \{t,\ell\}$, there are objects $m_x\overline{a}$ and multimaps $\theta_x(\overline{a}) \in \mathbb{A}_x(\overline{a};m_x\overline{a})$ with the universal property that the induced functions

$$- \circ_1 \theta_x(\overline{a})\colon \mathbb{A}_t(m_x\overline{a};b) \to \mathbb{A}_x(\overline{a};b)$$

are bijections for all $b \in A$.

If now $\overline{b}$ is a list and c an object, the multimap $\theta_x(\overline{a})$ induces a function

$$- \circ_1 \theta_x(\overline{a})\colon \mathbb{A}_t(m_x\overline{a},\overline{b};c) \to \mathbb{A}_x(\overline{a},\overline{b};c) \tag{5.1}$$

and the weakly representable $\mathbb{A}$ is said to be *left representable* if the function (5.1) is a bijection for all $x, \overline{a}, \overline{b}$ and c, and all universal multimaps $\theta_x(\overline{a})$.

Theorem 6.1 of [3] asserts that there is a 2-equivalence between the 2-categories of left representable skew multicategories and of skew monoidal categories. We now describe the skew monoidal structure on $\mathcal{A}$ associated to the left representable $\mathbb{A}$.

Setting $AB = m_t(A,B)$ gives the defining representation

$$\mathcal{A}(AB,C) \cong \mathbb{A}_t(A,B;C) \tag{5.2}$$

with universal multimap denoted $e_{A,B} \in \mathbb{A}_t(A,B;AB)$. We sometimes write it as

$$e\colon A,B \to AB$$

omitting the subscript.

Given $f\colon A \to C$ and $g\colon B \to D$ in $\mathcal{A}$, the morphism $fg\colon AB \to CD$ is the unique one such that

$$fg \circ e_{A,B} = e_{C,D}(f,g) \tag{5.3}$$

which condition we also write as

$$\begin{tikzcd} A,B \arrow[r, "{f,g}"] \arrow[d, "e"'] & C,D \arrow[d, "e"] \\ AB \arrow[r, "fg"'] & CD. \end{tikzcd} \tag{5.4}$$

Functoriality follows from the universal property.

By left representability we have natural isomorphisms

$$\mathcal{A}((AB)C, D) \cong \mathbb{A}_t(AB, C; D) \cong \mathbb{A}_t(A, B, C; D).$$

The universal multimap is the composite $e_{AB,C} \circ_1 e_{A,B} \in \mathbb{A}_t(A, B, C; (AB)C)$, which we may represent as

$$A, B, C \xrightarrow{e,1} AB, C \xrightarrow{e} (AB)C. \tag{5.5}$$

By its universal property we obtain the associator $a_{A,B,C} \colon (AB)C \to A(BC)$ as the unique map such that

$$a_{A,B,C} \circ_1 e_{AB,C} \circ_1 e_{A,B} = e_{A,BC} \circ_1 e_{B,C} \tag{5.6}$$

or equally

$$\begin{tikzcd} A, B, C \arrow[r, "{e,1}"] \arrow[d, "e"'] & AB, C \arrow[r, "e"] & (AB)C \arrow[d, "a"] \\ A, BC \arrow[rr, "e"] & & A(BC). \end{tikzcd} \tag{5.7}$$

We define the unit I as the representing object for $\mathbb{A}_l(;-) \colon \mathcal{A} \to \mathbf{Set}$. We write $u \in \mathbb{A}_l(;-)$ for the universal multimap, and depict it as $u \colon (-) \to I$. By left representability we have $\mathcal{A}(IA, B) \cong \mathbb{A}_t(I, A; B) \cong \mathbb{A}_l(A; B)$. Taking $B = A$ and the image of the identity 1_a under $\mathbb{A}_t(A; A) \to \mathbb{A}_l(A; A)$ yields $\ell \colon IA \to A$, the unique map such that

$$\begin{tikzcd} A \arrow[r, "{u,1}"] \arrow[rrr, bend right, "j1_A"'] & I, A \arrow[r, "e"] & IA \arrow[r, "l"] & A \end{tikzcd}$$

commutes. The right unit map $r \colon A \to AI$ is the composite below:

$$A \xrightarrow{1,u} A, I \xrightarrow{e} AI.$$

5.5. Braided skew multicategories. We begin by recalling braided multicategories. These differ from the usual notion of symmetric multicategory in that they involve actions of the braid groups $\mathcal{B}_n$ rather than the symmetric groups $\mathcal{S}_n$.

Braid groups and symmetric groups. Recall that the Artin braid group $\mathcal{B}_n$ has presentation

$$\langle \beta_1, \ldots, \beta_{n-1} | \beta_i\beta_j = \beta_j\beta_i \text{ for } j < i-1, \beta_i\beta_{i+1}\beta_{i+1} = \beta_i\beta_{i+1}\beta_i \rangle.$$

There is an evident homomorphism $|-|_n \colon \mathcal{B}_n \to \mathcal{S}_n$ sending β_i to the transposition $(i, i+1)$ so that, in particular, $\mathcal{B}_n$ acts on $\{1, \ldots, n\}$.

In addition to the group operation, one can form the tensor product of braids. Combining this with the group operations, the sets $\mathcal{B}_n$ admit an evident substitution

$$\mathcal{B}_n \times \mathcal{B}_{m_1} \times \ldots \mathcal{B}_{m_n} \to \mathcal{B}_{m_1+\ldots+m_n} \colon (s, (t_1, \ldots, t_n)) \mapsto s(t_1, \ldots, t_n)$$

which, indeed, form the substitution for an operad $\mathcal{B}$, and the functions $|-|_n \colon \mathcal{B}_n \to \mathcal{S}_n$ define an operad morphism from $\mathcal{B}$ to the corresponding operad $\mathcal{S}$.

Braided multicategories. A *braiding* on a multicategory $\mathbb{A}$ consists of

1. for each $s \in \mathcal{B}_n$ a function

$$s^*\colon \mathbb{A}(a_1,\ldots,a_n;b) \to \mathbb{A}(a_{s1},\ldots,a_{sn};b)\colon f \mapsto fs$$

satisfying the action equations $(fs)t = f(st)$ and $f1_{\mathcal{B}_n} = f$ as well as

2. the equivariance equation

$$(f(g_1,\ldots,g_n))s(t_1,\ldots,t_n) = fs(g_{s_1}t_{s_1},\ldots,g_{s_n}t_{s_n})$$

for all $f \in \mathbb{A}(b_1,\ldots,b_n;c)$ and $s \in \mathcal{B}_n$, together with $g_i \in \mathbb{A}(\overline{a_i};b_i)$ and $t_i \in \mathcal{B}_{|\overline{a_i}|}$ for $i \in \{1,\ldots,n\}$.

The braiding is a symmetry if the actions satisfy $s^* = t^*$ whenever $|s| = |t|$ Alternatively, and more simply, modify the definition above by replacing each occurrence of $\mathcal{B}$ by $\mathcal{S}$.

For concrete calculations it may be useful to reformulate this structure in terms of the generating braids. To give bijections satisfying (1) is to give bijections $f \mapsto f\beta_i$ for each i, subject to the braid relations $(f\beta_i)\beta_j = (f\beta_j)\beta_i$ for $j < i-1$ and $((f\beta_i)\beta_{i+1})\beta_i = ((f\beta_{i+1})\beta_i)\beta_{i+1}$. The equivariance conditions become more complicated. Given $f \in \mathbb{A}(a_1,\ldots,a_n;b_i)$ and $g \in \mathbb{A}(b_1,\ldots,b_m;c)$ they say

$$g \circ_i f\beta_j = (g \circ_i f)\beta_{i+j-1} \tag{5.8}$$

and

$$g\beta_j \circ_i f = \begin{cases} (g \circ_i f)\beta_j & \text{if } j < i-1 \\ (g \circ_{i-1} f)\beta_{j+n-1}\ldots\beta_j & \text{if } j = i-1 \\ (g \circ_{i+1} f)\beta_j\ldots\beta_{j+n-1} & \text{if } j = i \\ (g \circ_i f)\beta_{j+n-1} & \text{if } j > i. \end{cases} \tag{5.9}$$

Braided skew multicategories. Now let $\mathbb{A}$ be a skew multicategory. For a braiding on $\mathbb{A}$ we require, to begin with, that the ordinary multicategory $\mathbb{A}_l$ of loose multimaps be equipped with actions

$$s^*\colon \mathbb{A}_l(a_1,\ldots,a_n;b) \to \mathbb{A}_l(a_{s1},\ldots,a_{sn};b)$$

exhibiting it as a braided multicategory.

Consider the subgroup $\mathcal{B}^1_n = \langle\beta_2,\ldots,\beta_n\rangle \leq \mathcal{B}_n$; that is, we omit the single generator having a non-trivial action on $1 \in \{1,\ldots,n\}$. Of course $\mathcal{B}^1_n \cong \mathcal{B}_{n-1}$. Observe also that $s(t_1,\ldots,t_n) \in \mathcal{B}^1_{m_1+\ldots m_n}$ whenever $s \in \mathcal{B}^1_n$ and $t_1 \in \mathcal{B}^1_{m_1}$. In a braided skew multicategory we also require:

(1*) for each $s \in \mathcal{B}^1_n$ a function $s^*\colon \mathbb{A}_t(a_1,\ldots,a_n;b) \to \mathbb{A}_t(a_{s1},\ldots,a_{sn};b)$ such that these satisfy the action equations $(fs)t = f(st)$, $f1 = f$ as well as the compatibility $j(fs) = j(f)s$.

(2*) given $f \in \mathbb{A}_t(b_1, \ldots, b_n; c)$, $s \in \mathcal{B}^1_n$, $g_1 \in \mathbb{A}_t(\overline{a_1}; b_1)$ and $t_1 \in \mathcal{B}^1_{m_1}$, plus $g_i \in \mathbb{A}_l(\overline{a_i}; b_i)$, $t_i \in \mathcal{B}_{m_i}$ for $i \in \{2, \ldots, n\}$, we require the equivariance equation

$$(f(g_1, \ldots, g_n))s(t_1, \ldots, t_n) = fs(g_{s_1}t_{s_1}, \ldots, g_{s_n}t_{s_n}).$$

Once again this can be reformulated in terms of the generators. There are assignments $g \mapsto g\beta_i$ for g tight n-ary and $1 < i < n$; and $g \mapsto g\beta_i$ for g loose n-ary and $1 \leq i < n$. These satisfy the braid relations as well as equations (5.8) and (5.9) insofar as these make sense. Finally the two actions should be compatible in the sense that $j(g\beta_i) = j(g)\beta_i$ for g tight n-ary and $1 < i < n$.

For a symmetric skew multicategory we also require that $s^* = t^*$ whenever $|s| = |t|$ and $s, t \in \mathcal{B}^1_n$.

Alternatively, letting $\mathcal{S}^1_n \subseteq \mathcal{S}_n$ denote the subgroup of permutations fixing $1 \in \{1, \ldots, n\}$, we obtain a simpler definition of symmetric skew multicategory, by replacing each appearance of $\mathcal{B}$ by $\mathcal{S}$.

5.6. Remark. Recall from Remark 5.2 that a skew multicategory $\mathbb{A}$ for which the comparison functions $j_{\overline{a},b} \colon \mathbb{A}_t(a_1, \ldots, a_n; b) \to \mathbb{A}_l(a_1, \ldots, a_n; b)$ are inclusions amounts to an ordinary multicategory equipped with a subcollection of tight morphisms which are not nullary, contain the identities and have the property that $f(g_1, \ldots, g_n)$ is tight just when f and g_1 are. Under this correspondence a braiding on the skew multicategory simply amounts to a braiding on the associated multicategory with the property that if $s \in \mathcal{B}^1_n$ and f is a tight multimap of arity n then fs is tight too. There is a corresponding result for symmetries with $\mathcal{B}^1_n$ replaced by $\mathcal{S}^1_n$.

5.7. Example. The multicategory $\mathbb{FP}$ of Example 5.3 admits a symmetry lifted directly from the cartesian multicategory $\mathbb{C}\mathrm{at}$. For if $s \in \mathcal{S}_n$ then $Fs \colon \mathcal{A}_{s1} \times \ldots \times \mathcal{A}_{sn} \to \mathcal{B}$ will preserve products in the ith variable just when F preserves products in the s_ith variable. Since the tight multimaps in $\mathbb{FP}$ are defined to be those preserving products strictly in the first variable Fs will be tight so long as F is and $s \in \mathcal{S}^1_n$. Accordingly $\mathbb{FP}$ is a symmetric skew multicategory.

5.8. Braidings on left representable skew multicategories. Suppose that $\mathbb{A}$ is a left representable skew multicategory corresponding to a skew monoidal category $\mathcal{A}$. Given a braiding on $\mathbb{A}$, the generator $\beta_2 \in \mathcal{B}^1_3$ induces a bijection $\beta_2^* \colon \mathbb{A}_t(A, C, B; D) \to \mathbb{A}_t(A, B, C; D)$. By equivariance these isomorphisms are natural in $\mathcal{A}$, as are the vertical isomorphisms below.

$$\begin{tikzcd}
\mathcal{A}((AC)B; D) \arrow[r, "\mathcal{A}(s_{A,B,C}, D)"] \arrow[d, "\cong"'] & \mathcal{A}((AB)C, D) \arrow[d, "\cong"] \\
\mathbb{A}_t(A, C, B; D) \arrow[r, "\beta_2^*"] & \mathbb{A}_t(A, B, C; D).
\end{tikzcd} \tag{5.10}$$

By the Yoneda Lemma there is a unique natural isomorphism

$$s_{A,B,C} \colon (AB)C \to (AC)B$$

rendering the diagram above commutative.

In the appendix, we prove:

5.9. Theorem. *Let $\mathcal{A}$ be a skew monoidal category, and $\mathbb{A}$ the corresponding left representable skew multicategory. The above assignment defines a bijection between braidings on $\mathbb{A}$ and braidings on $\mathcal{A}$, which restricts to a bijection between symmetries on $\mathbb{A}$ and symmetries on $\mathcal{A}$.*

6. Symmetric skew monoidal 2-categories and symmetric monoidal bicategories

This last section is geared towards understanding the braidings on skew monoidal 2-categories, like $\mathbf{FProd}_{\mathrm{s}}$ as well as (bicategorical) braidings on the induced monoidal bicategories. We do this using the corresponding skew multicategories, and the results of the previous section.

We have not, thus far, mentioned 2-categorical (that is, **Cat**-enriched) structure. The various structures that we have been dealing with — skew monoidal categories, skew multicategories and their braided and symmetric variants — each admit **Cat**-enriched analogues. For instance a skew monoidal 2-category involves a tensor product 2-functor, as well as 2-natural transformations a, ℓ, r all satisfying the usual five equations. For a braided (or symmetric) skew monoidal 2-category we of course require that the braiding be 2-natural as well. In a skew 2-multicategory $\mathbb{A}$ one has *categories* of tight and loose multimaps connected by a *functor* $j\colon \mathbb{A}_t(\overline{a}; b) \to \mathbb{A}_\ell(\overline{a}; b)$ and, again, the substitution maps themselves become functors rather than just functions. In the braided/symmetric variants the actions s^* on the multihoms are themselves functors. Again we can speak of **Cat**-enriched left representability, which now involves isomorphisms of categories rather than mere bijections of sets.

Just as in the unenriched setting, skew 2-multicategories in which each $j\colon \mathbb{A}_t(\overline{a}; b) \to \mathbb{A}_\ell(\overline{a}; b)$ is the inclusion of a full subcategory (rather than a mere subset) can be identified with 2-multicategories equipped with a subcollection of non-nullary tight multimorphisms having the same closure properties described in 5.2. Again, a braiding/symmetry in this context simply amounts to a braiding/symmetry on the 2-multicategory which respects tight multimaps in the sense described in 5.6.

$\mathbb{FP}$ is a simple example of such a symmetric skew 2-multicategory: for each of $x = t, l$ the morphisms of $\mathbb{FP}_x(\mathcal{A}_1, \ldots, \mathcal{A}_n; \mathcal{B})$ are the natural transformations.

6.1. Examples. More generally any pseudocommutative 2-monad T on **Cat** [7] gives rise to a skew 2-multicategory $\mathbb{T}$-Alg. When T is the 2-monad for categories with finite products we obtain the skew 2-multicategory described in Example 5.3. But there are many more examples — the 2-monads for permutative categories, symmetric monoidal categories and categories equipped with a given class of limits (or colimits) are all pseudocommutative [7, 13].

For such a T an object of the skew 2-multicategory $\mathbb{T}$-Alg is a strict T-algebra $\mathbf{A}$; we write $\mathcal{A}$ for the underlying category of the T-algebra $\mathbf{A}$. A multimorphism $F\colon (\mathbf{A}_1, \ldots, \mathbf{A}_n) \to \mathbf{B}$ is a functor $F\colon \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathcal{B}$ equipped with the structure of an algebra pseudomorphism in each variable separately, with these n pseudomorphism structures commuting with each other in the sense explained in [7]. Nullary morphisms $a\colon (-) \to \mathbf{A}$ are just objects a of $\mathcal{A}$. There are also *transformations* between the multimorphisms: these are natural transformations which are T-algebra transformations in each variable separately; once again, see [7] for the details.

We obtain a skew 2-multicategory $\mathbb{T}$-Alg by defining a multimap F as above to be tight if it is a strict algebra morphism in the first variable — that is, if for all $a_2 \in \mathcal{A}_2, \ldots, a_n \in \mathcal{A}_n$ the pseudomorphism $F(-, a_2, \ldots, a_n)\colon \mathbf{A}_1 \to \mathbf{B}$ is strict. See Section 4.2 of [3] and the references therein for more on this example.

6.2. Proposition. *Let T be an accessible symmetric pseudo-commutative 2-monad on* **Cat***. Then the skew 2-multicategory $\mathbb{T}$-Alg of Example 6.1 is symmetric. It is also left representable and closed.*

Proof. Left representability and closedness of the underlying skew multicategories are established in Examples 4.8 of [3]. The additional 2-categorical aspects that must be verified are straightforward. The underlying 2-multicategory of loose maps is symmetric by Proposition 18 of [7]. A multimap $F\colon (\mathbf{A}_1, \ldots, \mathbf{A}_n) \to \mathbf{B}$ involves a functor $\mathcal{A}_1 \times \ldots \mathcal{A}_n \to \mathcal{B}$ equipped with a pseudomap structure on the functor

$$F(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n)\colon \mathcal{A}_i \to \mathcal{B}$$

for each $i \in \{1, \ldots, n\}$ and tuple $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. The different pseudomap structures are required to satisfy compatibility axioms. The symmetry $s \in \mathcal{S}_n$ permutes the variables and the pseudomap structures — in particular, if $s \in \mathcal{S}_n^1$ the multimap $Fs\colon (\mathbf{A}_1, \mathbf{A}_{s2} \ldots, \mathbf{A}_{sn}) \to \mathbf{B}$ has pseudomap $(Fs)(-, a_2, \ldots, a_n)\colon \mathbf{A}_1 \to \mathbf{B}$ given by $F(-, a_{s^{-1}2}, \ldots, a_{s^{-1}n})\colon \mathbf{A}_1 \to \mathbf{B}$ which is strict whenever F is strict in the first variable. Hence Fs is tight if F is tight and $s \in \mathcal{S}_n^1$, as required. ∎

6.3. Corollary. *Let T be an accessible symmetric pseudo-commutative 2-monad on* **Cat***. Then the 2-category* $\mathbf{T\text{-}Alg}_s$ *of strict algebras and strict morphisms admits a closed symmetric skew monoidal structure.*

Proof. The construction of a symmetric skew monoidal category from a left representable symmetric skew multicategory given in Theorem 5.9 admits an evident **Cat**-enriched version. Applying this, together with Proposition 6.2, we obtain the desired symmetric skew monoidal structure on the 2-category $\mathbf{T\text{-}Alg}_s$. For closedness we combine Proposition 6.2 above and Theorem 6.4 of [3] (in its **Cat**-enriched form). ∎

All of the pseudo-commutative 2-monads described in Examples 6.1 are accessible symmetric pseudo-commutative [7, 13]. Accordingly it follows from Corollary 6.3 that the categories of permutative categories, of symmetric monoidal categories, and of categories with a given class of limits or a given class of colimits all admit closed symmetric skew monoidal structures.

As a special case of this we obtain the symmetric skew structure on $\mathbf{FProd}_{\mathrm{s}}$ described in the introduction, in which maps $\mathcal{A}\mathcal{B} \to \mathcal{C}$ correspond to functors $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ preserving products strictly in the first variable and up to isomorphism in the second. The internal hom $[\mathcal{A}, \mathcal{B}]$ is the usual category of finite product preserving functors (in the up to isomorphism sense) with products pointwise as in $\mathcal{B}$.

6.4. Braided monoidal bicategories. Finally, we explain how our 2-categorical examples give rise to symmetric monoidal bicategories [1, 6].

To begin with we observe that if $\mathcal{C}$ is a skew monoidal 2-category whose coherence constraints are equivalences, then $\mathcal{C}$ is, in particular, a monoidal bicategory. (This is the approach of Section 6.4.3 of [2], on which we now build.) To equip our monoidal bicategory with a symmetry, we need to provide pseudonatural equivalences $AB \to BA$ together with, replacing the symmetry equations, certain invertible modifications satisfying a host of coherence axioms. Now the 2-variable morphisms $AB \to BA$ are not part of the structure that we are, in the skew setting, presented with. However since the components $\ell_A B : A * B = (IA)B \to AB$ are equivalences, we can obtain a symmetry of the required form as the composite

$$AB \xrightarrow{\ell^*_A B} (IA)B \xrightarrow{s_{I,A,B}} (IB)A \xrightarrow{\ell_B A} BA \tag{6.1}$$

in which $\ell^*_A : A \to IA$ is the equivalence inverse of ℓ_A. The composite equivalence will be the desired component of the braiding on our monoidal bicategory.

6.5. Proposition. *Let $\mathcal{C}$ be a symmetric skew monoidal 2-category whose coherence constraints are equivalences. Then $\mathcal{C}$ is a symmetric monoidal bicategory with braiding components as in* (6.1)*.*

Proof. Let us firstly note that the symmetry axioms for a symmetric monoidal bicategory do *not* refer to the unit. Using the prefix *semi*, as usual, to specify that part of a structure not mentioning the unit, it follows that a symmetry on a monoidal bicategory amounts to a symmetry on its underlying semimonoidal bicategory.

Now the analysis of Section 4.12 applies equally to this **Cat**-enriched context, and so we obtain a skew semimonoidal 2-category with product $A*B = IA.B$ and a semimonoidal 2-functor $(1, \ell 1) : (\mathcal{C}, .) \to (\mathcal{C}, *)$. Since the associators for . are equivalences and since the equivalences $A * B \to AB$ commute with the respective associators it follows, by 2 out of 3 for equivalences, that the associators for $(\mathcal{C}, *)$ are equivalences — thus $(\mathcal{C}, *)$ is a semimonoidal bicategory too. Furthermore, since by Proposition 4.14 the classical symmetry axioms holds on the nose, $(\mathcal{C}, *)$ is in fact a symmetric semimonoidal bicategory.

Finally we transport the braiding across the componentwise equivalence

$$(1, l1)\colon (\mathcal{C}, .) \to (\mathcal{C}, *)$$

of semimonoidal bicategories to obtain the structure of a symmetric semimonoidal bicategory on $(\mathcal{C}, .)$. The resulting symmetry, obtained by transport of structure, is that described in (6.1). ∎

Now adapting the approach of [2, Section 6.4.3] let $(\mathbf{T}\text{-}\mathbf{Alg}_s)_c$ denote the full sub 2-category of $\mathbf{T}\text{-}\mathbf{Alg}_s$ containing the flexible (cofibrant) T-algebras. By [2, Proposition 6.5] if T is an accessible pseudo-commutative 2-monad on $\mathbf{Cat}$ then the skew monoidal structure on $\mathbf{T}\text{-}\mathbf{Alg}_s$ restricts to $(\mathbf{T}\text{-}\mathbf{Alg}_s)_c$ where the coherence constraints become equivalences. If moreover T is symmetric pseudo-commutative then, by Corollary 6.3, $(\mathbf{T}\text{-}\mathbf{Alg}_s)_c$ is a symmetric skew monoidal 2-category whose coherence constraints are equivalences, and so, by Proposition 6.5 above, it is a symmetric monoidal bicategory. Finally we observe that, as in [2, Lemma 6.6], the composite inclusion $(\mathbf{T}\text{-}\mathbf{Alg}_s)_c \to \mathbf{T}\text{-}\mathbf{Alg}_s \to \mathbf{T}\text{-}\mathbf{Alg}$ of the flexible algebras and strict morphisms into the 2-category of strict algebras and pseudomorphisms is a biequivalence of 2-categories; transporting the structure of a symmetric monoidal bicategory along the biequivalence we obtain the symmetric monoidal bicategory structure on $\mathbf{T}\text{-}\mathbf{Alg}$. That is:

6.6. Theorem. *Let T be an accessible symmetric pseudo-commutative 2-monad on* $\mathbf{Cat}$. *Then the 2-category* $\mathbf{T}\text{-}\mathbf{Alg}$ *admits the structure of a symmetric monoidal bicategory.*

A. Proof of Theorem 5.9

In this appendix we give the remaining details in the comparison between braidings on skew monoidal categories and braidings on the corresponding left representable skew multicategory.

Suppose that $\mathbb{A}$ is a left representable skew multicategory corresponding to a skew monoidal category $\mathcal{A}$. Commutativity of the diagram

$$\begin{tikzcd}
\mathcal{A}((ac)b; d) \arrow[r, "\mathcal{A}(s_{a,b,c}, d)"] \arrow[d, "\cong"'] & \mathcal{A}((ab)c, d) \arrow[d, "\cong"] \\
\mathbb{A}_t(a, c, b; d) \arrow[r, "\beta_2^*"] & \mathbb{A}_t(a, b, c; d)
\end{tikzcd}$$

determines a bijection between isomorphisms $s_{a,b,c} : (ab)c \to (ac)b$ and invertible actions $g \mapsto g\beta_2$ for each tight ternary g, subject to the requirement that $f \circ_1 g\beta_2 = (f \circ_1 g)\beta_2$ for all tight unary f. The first step is to understand braidings on $\mathcal{A}$ in terms of tight maps in $\mathbb{A}$.

A.1. Theorem. *There is a bijection between braidings on the skew monoidal category $\mathcal{A}$ and invertible actions $g \mapsto g\beta_j$ for tight m-ary g and $1 < j < m$, subject to the braid equations and satisfying the conditions*

$$f \circ_i g\beta_j = (f \circ_i g)\beta_{i+j-1} \tag{A.1}$$
$$g\beta_j \circ_i f = (g \circ_i f)\beta_j \qquad \text{if } j < i-1 \tag{A.2}$$
$$g\beta_j \circ_i f = (g \circ_{i-1} f)\beta_{j+n-1} \dots \beta_j \quad \text{if } j = i-1 \tag{A.3}$$
$$g\beta_j \circ_i f = (g \circ_{i+1} f)\beta_j \dots \beta_{j+n-1} \quad \text{if } j = i \tag{A.4}$$
$$g\beta_j \circ_i f = (g \circ_i f)\beta_{j+n-1} \qquad \text{if } j > i \tag{A.5}$$

for tight n-ary f. Under this bijection the braiding s corresponds as above to the action $g \mapsto g\beta_2$ for tight ternary g.

Suppose given isomorphisms $s_{a,b,c} \colon (ab)c \to (ac)b$ and the corresponding actions $g \mapsto g\beta_2$ for tight ternary g. Then (A.1) holds for $n = 1$ and $m = 3$ (in which case necessarily $i = 1$ and $j = 2$). Naturality of s is equivalent to (A.3), (A.4), and (A.5) for $n = 1$ and $m = 3$ (in which case necessarily $j = 2$). Note also that when $n = 1$ and $m = 3$ condition (A.2) is empty.

Given objects $a, b, c \in \mathbb{A}$, let $\theta_3^{abc} \in \mathbb{A}_t(a, b, c; (ab)c)$ be the universal tight ternary map. Postcomposing this by $s_{a,b,c} \colon (ab)c \to (ac)b$ equally gives the map $\theta_3^{acb}\beta_2$. An arbitrary tight ternary map $g \in \mathbb{A}_t(a, b, c; x)$ has the form $g' \circ_1 \theta_3^{abc}$ for a unique tight unary g', and we then have $g\beta_2 = g' \circ_1 \theta_3^{abc}\beta_2$. In the following we shall usually omit the superscripts in the maps θ_3^{abc}.

Now for $n \geq 3$ each tight n-ary g has the form $g = g' \circ_1 \theta_3$ for a unique tight $n-2$ -ary g'. We may now define $g\beta_2$ to be $g' \circ_1 \theta_3\beta_2$, and deduce that $h \circ_1 \theta_3\beta_2 = (h \circ_1 \theta_3)\beta_2$ for all tight h.

A.2. Proposition. *The special case*

$$f \circ_1 g\beta_2 = (f \circ_1 g)\beta_2 \tag{A.6}$$

of (A.1) holds for all tight f and g. Furthermore the action is uniquely determined by that property together with the values of $\theta_3\beta_2$.

Proof. Uniqueness follows on taking $g = \theta_3$. To verify (A.6) in general, write $g = g' \circ_1 \theta_3$; then

$$f \circ_1 g\beta_2 = f \circ_1 (g' \circ_1 \theta_3\beta_2) = (f \circ_1 g') \circ_1 \theta_3\beta_2 = ((f \circ_1 g') \circ_1 \theta_3)\beta_2 = (f \circ_1 g)\beta_2.$$

■

A.3. PROPOSITION. *Suppose that the equations* (A.3)– (A.5) *hold for all tight ternary* g *and tight unary* f*. Then they hold for all tight* g *and tight unary* f *when* $j = 2$*.*

PROOF. This follows by

$$\begin{aligned}
g\beta_2 \circ_1 f &= (g' \circ_1 \theta_3\beta_2) \circ_1 f \\
&= g' \circ_1 (\theta_3\beta_2 \circ_1 f) \\
&= g' \circ_1 (\theta_3 \circ_1 f)\beta_2 && \text{(by (A.5))} \\
&= (g' \circ_1 (\theta_3 \circ_1 f))\beta_2 && \text{(by (A.6))} \\
&= (g \circ_1 f)\beta_2 \\
g\beta_2 \circ_2 f &= (g' \circ_1 \theta_3\beta_2) \circ_2 f \\
&= g' \circ_1 (\theta_3\beta_2 \circ_2 f) \\
&= g' \circ_1 (\theta_3 \circ_3 f)\beta_2 && \text{(by (A.4))} \\
&= (g' \circ_1 (\theta_3 \circ_3 f))\beta_2 && \text{(by (A.6))} \\
&= (g \circ_3 f)\beta_2 \\
g\beta_2 \circ_3 f &= (g' \circ_1 \theta_3\beta_2) \circ_3 f \\
&= g' \circ_1 (\theta_3\beta_2 \circ_3 f) \\
&= g' \circ_1 (\theta_3 \circ_2 f)\beta_2 && \text{(by (A.3))} \\
&= (g' \circ_1 (\theta_3 \circ_2 f)\beta_2 && \text{(by (A.6))} \\
&= (g \circ_2 f)\beta_2.
\end{aligned}$$

In what follows we let $\theta_m \in \mathbb{A}_t(a_1, \ldots, a_m; a_1 \ldots a_m)$ be a universal tight m-ary map with $m > 3$. We can now define further actions $g \mapsto g\beta_j$ as follows.

A.4. PROPOSITION. *There is a unique assignment* $g \mapsto g\beta_j$ *for a tight* m*-ary* g *and* $2 \leq j < m$*, such that* (A.5) *holds for all tight* f*.*

PROOF. For the uniqueness, take $f = \theta_{j-1}$, to see that

$$(g'' \circ_1 \theta_{j-1})\beta_j = g''\beta_2 \circ_1 \theta_{j-1} \tag{A.7}$$

and use the fact that any tight g can be written as $g'' \circ_1 \theta_{j-1}$ for a unique tight g''.

If f is an arbitrary tight n-ary map, write $\theta_{j-1} \circ_i f = f' \circ_1 \theta_{n+j-2}$, where f' is tight unary, and now

$$\begin{aligned}
g\beta_j \circ_i f &= (g''\beta_2 \circ_1 \theta_{j-1}) \circ_i f \\
&= g''\beta_2 \circ_1 (\theta_{j-1} \circ_i f) \\
&= g''\beta_2 \circ_1 (f' \circ_1 \theta_{n+j-2}) \\
&= (g''\beta_2 \circ_1 f') \circ_1 \theta_{n+j-2} \\
&= (g'' \circ_1 f')\beta_2 \circ_1 \theta_{n+j-2} && \text{(Proposition A.3)} \\
&= ((g'' \circ_1 f') \circ_1 \theta_{n+j-2})\beta_{n+j-1} && \text{(by definition)} \\
&= (g'' \circ_1 (f' \circ_1 \theta_{n+j-2})\beta_{n+j-1} \\
&= (g'' \circ_1 (\theta_{j-1} \circ_i f))\beta_{n+j-1} \\
&= (g \circ_1 f)\beta_{n+j-1}
\end{aligned}$$

gives the result. ∎

A.5. Proposition. *There is a bijection between natural isomorphisms s and invertible assignments $g \mapsto g\beta_j$ for tight m-ary g and for $2 \le j < m$, subject to equation (A.1) for $i = 1$ and all tight g and f; (A.2) and (A.5) for all tight g and f; (A.3) and (A.4) for all tight g and tight unary f.*

Proof. We have proved (A.5) in Proposition A.4. The relevant parts of (A.1) hold by the calculation

$$\begin{aligned}
f \circ_1 g\beta_j &= f \circ_1 (g''\beta_2 \circ_1 \theta_{j-1}) \\
&= (f \circ_1 g''\beta_2) \circ_1 \theta_{j-1} \\
&= (f \circ_1 g'')\beta_2 \circ_1 \theta_{j-1} && \text{(by (A.6))} \\
&= ((f \circ_1 g'') \circ_1 \theta_{j-1})\beta_j && \text{(by (A.7))} \\
&= (f \circ_1 g)\beta_j.
\end{aligned}$$

For (A.4) with tight unary f we have

$$\begin{aligned}
g\beta_j \circ_j f &= (g''\beta_2 \circ_1 \theta_{j-1}) \circ_j f \\
&= (g''\beta_2 \circ_2 f) \circ_1 \theta_{j-1} \\
&= (g'' \circ_3 f)\beta_2 \circ_1 \theta_{j-1} && \text{(Proposition A.3)} \\
&= ((g'' \circ_3 f) \circ_1 \theta_{j-1})\beta_j && \text{(by (A.7))} \\
&= (g'' \circ_1 \theta_{j-1}) \circ_{j+1} f)\beta_j \\
&= (g \circ_{j+1} f)\beta_j.
\end{aligned}$$

A similar argument gives (A.3) for tight unary f. Finally for (A.2) with g tight m-ary, f tight n-ary, and $j+1 < i \leq m$, write $g = g' \circ_1 \theta_{j+1}$; then

$$\begin{aligned}
g\beta_j \circ_i f &= (g' \circ_1 \theta_{j+1})\beta_j \circ_i f \\
&= (g' \circ_1 \theta_{j+1}\beta_j) \circ_i f && \text{(Proposition A.4)} \\
&= (g' \circ_{i-j} f) \circ_1 \theta_{j+1}\beta_j \\
&= ((g' \circ_{i-j} f) \circ_1 \theta_{j+1})\beta_j && \text{(Proposition A.4)} \\
&= ((g' \circ_1 \theta_{j+1}) \circ_i f)\beta_j \\
&= (g \circ_i f)\beta_j.
\end{aligned}$$

Next we investigate the various conditions on s in terms of the corresponding assignments $g \mapsto g\beta_j$. First, we observe:

A.6. Proposition. *If g is tight m-ary and $2 \leq j < j+1 < i < m$ then $g\beta_i\beta_j = g\beta_j\beta_i$.*

Proof. Write $g = g' \circ_1 \theta_{i-1}$. Then

$$\begin{aligned}
(g\beta_i)\beta_j &= (g'\beta_2 \circ_1 \theta_{i-1})\beta_j && \text{(defn)} \\
&= g'\beta_2 \circ_1 \theta_{i-1}\beta_j && \text{(by (A.1) with } i=1) \\
&= g'\beta_2 \circ_1 (\theta_{i-j+1} \circ_1 \theta_{j-1})\beta_j \\
&= g'\beta_2 \circ_1 (\theta_{i-j+1}\beta_2 \circ_1 \theta_{j-1}) && \text{(defn)} \\
&= (g'\beta_2 \circ_1 \theta_{i-j+1}\beta_2) \circ_1 \theta_{j-1} \\
&= (g' \circ_1 \theta_{i-j+1}\beta_2)\beta_{i-j+2} \circ_1 \theta_{j-1} && \text{(by (A.5))} \\
&= ((g' \circ_1 \theta_{i-j+1}\beta_2) \circ_1 \theta_{j-1})\beta_i && \text{(by (A.5))} \\
&= (g' \circ_1 (\theta_{i-j+1}\beta_2 \circ_1 \theta_{j-1}))\beta_i \\
&= (g' \circ_1 (\theta_{i-j+1} \circ_1 \theta_{j-1})\beta_j)\beta_i && \text{(defn)} \\
&= (g' \circ_1 \theta_{i-1}\beta_j)\beta_i \\
&= ((g' \circ_1 \theta_{i-1})\beta_j)\beta_i && \text{(by (A.1) with } i=1) \\
&= (g\beta_j)\beta_i.
\end{aligned}$$

A.7. Proposition. *Let $s\colon (ab)c \to (ac)b$ be natural isomorphisms in $\mathcal{A}$, and $g \mapsto g\beta_j$ the corresponding actions on tight morphisms in $\mathbb{A}$. The following conditions are equivalent:*

(a) s satisfies (2.4);

(b) $\theta_2 \circ_2 \theta_3\beta_2 = (\theta_2 \circ_2 \theta_3)\beta_3$;

(c) (A.1) holds for all tight f and g.

Proof. The equivalence of (a) and (b) is a straightforward translation using universality of the θs, while (b) is a special case of (c). So we just need to check that the general case follows from the special one.

First we show that $f \circ_2 \theta_3\beta_2 = (f \circ_2 \theta_3)\beta_3$ for all tight f. Write $f = f' \circ_1 \theta_2$ with f' tight, then:

$$\begin{aligned}
f \circ_2 \theta_3\beta_2 &= (f' \circ_1 \theta_2) \circ_2 \theta_3\beta_2 \\
&= f' \circ_1 (\theta_2 \circ_2 \theta_3\beta_2) \\
&= f' \circ_1 (\theta_2 \circ_2 \theta_3)\beta_3 && \text{(by (b))} \\
&= (f' \circ_1 (\theta_2 \circ_2 \theta_3))\beta_3 && \text{(by (A.1) with } i = 1) \\
&= ((f' \circ_1 \theta_2) \circ_2 \theta_3)\beta_3 \\
&= (f \circ_2 \theta_3)\beta_3.
\end{aligned}$$

For $i > 2$ write $f = f' \circ_1 \theta_{i-1}$, and observe that

$$\begin{aligned}
f \circ_i \theta_3\beta_2 &= (f' \circ_1 \theta_{i-1}) \circ_i \theta_3\beta_2 \\
&= (f' \circ_2 \theta_3\beta_2) \circ_1 \theta_{i-1} \\
&= (f' \circ_2 \theta_3)\beta_3 \circ_1 \theta_{i-1} && \text{(by the preceding calculation)} \\
&= ((f' \circ_2 \theta_3) \circ_1 \theta_{i-1})\beta_{i+1} && \text{(by (A.5))} \\
&= ((f' \circ_1 \theta_{i-1}) \circ_i \theta_3)\beta_{i+1} \\
&= (f \circ_i \theta_3)\beta_{i+1}.
\end{aligned}$$

Thus (A.1) holds for $g = \theta_3$. For tight g and $j = 2$ write $g = g' \circ_1 \theta_3$, so that

$$\begin{aligned}
f \circ_i g\beta_2 &= f \circ_i (g' \circ_1 \theta_3\beta_2) \\
&= (f \circ_i g') \circ_i \theta_3\beta_2 \\
&= ((f \circ_i g') \circ_i \theta_3)\beta_{i+1} && \text{(by the preceding calculation)} \\
&= (f \circ_i (g' \circ_1 \theta_3))\beta_{i+1} \\
&= (f \circ_i g)\beta_{i+1}.
\end{aligned}$$

Finally for $j > 2$ write $g = g'' \circ_1 \theta_{j-1}$, so that

$$\begin{aligned}
f \circ_i g\beta_j &= f \circ_i (g''\beta_2 \circ_1 \theta_{j-1}) \\
&= (f \circ_i g''\beta_2) \circ_i \theta_{j-1} \\
&= (f \circ_i g'')\beta_{i+1} \circ_i \theta_{j-1} && \text{(by the preceding calculation)} \\
&= ((f \circ_i g'') \circ_i \theta_{j-1})\beta_{i+j-1} && \text{(by (A.5))} \\
&= (f \circ_i (g'' \circ_1 \theta_{j-1}))\beta_{i+j-1} \\
&= (f \circ_i g)\beta_{i+j-1}
\end{aligned}$$

as required. ∎

A.8. PROPOSITION. *Let $s\colon (ab)c \to (ac)b$ be natural isomorphisms in $\mathcal{A}$, and $g \mapsto g\beta_j$ the corresponding actions on tight morphisms in $\mathbb{A}$. The following conditions are equivalent:*

(a) s satisfies (2.2);

(b) $\theta_3\beta_2 \circ_3 \theta_2 = (\theta_3 \circ_2 \theta_2)\beta_3\beta_2$;

(c) (A.3) holds for all tight f and g.

PROOF. The equivalence of (a) and (b) is a straightforward translation using universality of the θs, while (b) is a special case of (c). So we just need to check that the general case follows from the special one.

First we show that $g\beta_2 \circ_3 \theta_2 = (g \circ_2 \theta_2)\beta_3\beta_2$ for all tight g (of arity at least 3). Write $g = g' \circ_1 \theta_3$; then

$$\begin{aligned}
g\beta_2 \circ_3 \theta_2 &= (g' \circ_1 \theta_3)\beta_2 \circ_3 \theta_2 \\
&= (g' \circ_1 \theta_3\beta_2) \circ_3 \theta_2 && \text{(by (A.5))} \\
&= g' \circ_1 (\theta_3\beta_2 \circ_3 \theta_2) \\
&= g' \circ_1 (\theta_3 \circ_2 \theta_2)\beta_3\beta_2 && \text{(by (b))} \\
&= (g' \circ_1 (\theta_3 \circ_2 \theta_2)\beta_3)\beta_2 && \text{(by (A.1) with } i = 1) \\
&= (g' \circ_1 (\theta_3 \circ_2 \theta_2))\beta_3\beta_2 && \text{(by (A.1) with } i = 1) \\
&= (g \circ_2 \theta_2)\beta_3\beta_2.
\end{aligned}$$

For $j > 2$ write $g = g'' \circ_1 \theta_{j-1}$, and observe that

$$\begin{aligned}
g\beta_j \circ_{j+1} \theta_2 &= (g'' \circ_1 \theta_{j-1})\beta_j \circ_{j+1} \theta_2 \\
&= (g''\beta_2 \circ_1 \theta_{j-1}) \circ_{j+1} \theta_2 && \text{(by (A.5))} \\
&= (g''\beta_2 \circ_3 \theta_2) \circ_1 \theta_{j-1} \\
&= (g'' \circ_2 \theta_2)\beta_3\beta_2 \circ_1 \theta_{j-1} && \text{(by the preceding calculation)} \\
&= ((g'' \circ_2 \theta_2)\beta_3 \circ_1 \theta_{j-1})\beta_j && \text{(by (A.5))} \\
&= ((g'' \circ_2 \theta_2) \circ_1 \theta_{j-1})\beta_{j+1}\beta_j && \text{(by (A.5))} \\
&= (g \circ_j \theta_2)\beta_{j+1}\beta_j.
\end{aligned}$$

Finally prove the general case by induction on the arity n of f. The base case $n = 1$ holds by Proposition A.5. Write $f = f' \circ_1 \theta_2$; then f' has arity strictly less than that of f, allowing the induction, and

$$\begin{aligned}
g\beta_j \circ_{j+1} f &= g\beta_j \circ_{j+1} (f' \circ_1 \theta_2) \\
&= (g\beta_j \circ_{j+1} f') \circ_{j+1} \theta_2 \\
&= (g \circ_j f')\beta_{j+n-2} \dots \beta_j \circ_{j+1} \theta_2 && \text{(ind hyp)} \\
&= ((g \circ_j f')\beta_{j+n-2} \dots \beta_{j+1} \circ_j \theta_2)\beta_{j+1}\beta_j && \text{(previous case)} \\
&= ((g \circ_j f') \circ_j \theta_2)\beta_{j+n-1} \dots \beta_j && \text{(by (A.5))} \\
&= (g \circ_j f)\beta_{j+n-1} \dots \beta_j
\end{aligned}$$

as required. ∎

Similarly we have:

A.9. Proposition. *Let* $s\colon (ab)c \to (ac)b$ *be natural isomorphisms in* $\mathcal{A}$*, and* $g \mapsto g\beta_j$ *the corresponding actions on tight morphisms in* $\mathbb{A}$*. The following conditions are equivalent:*

(a) s *satisfies* (2.3)*;*

(b) $\theta_3\beta_2 \circ_2 \theta_2 = (\theta_3 \circ_3 \theta_2)\beta_2\beta_3$*;*

(c) (A.4) *holds for all tight* f *and* g*.*

The next result completes the proof of Theorem A.1.

A.10. Proposition. *Let* $s\colon (ab)c \to (ac)b$ *be natural isomorphisms in* $\mathcal{A}$*, and* $g \mapsto g\beta_j$ *the corresponding actions on tight morphisms in* $\mathbb{A}$*. If* (A.1) *holds for all tight* g *and* f *then the following conditions are equivalent:*

(a) s *satisfies* (2.1)*;*

(b) $\theta_4\beta_2\beta_3\beta_2 = \theta_4\beta_3\beta_2\beta_3$*;*

(c) $g\beta_j\beta_{j+1}\beta_j = g\beta_{j+1}\beta_j\beta_{j+1}$ *for all tight* n*-ary* g*.*

Proof. Once again (a) and (b) are clearly equivalent and (b) is a special case of (c). Thus it remains to prove that (b) implies (c). First observe that if $g = g' \circ_1 \theta_4$ then

$$\begin{aligned}
g\beta_2\beta_3\beta_2 &= (g' \circ_1 \theta_4)\beta_2\beta_3\beta_2 \\
&= g' \circ_1 \theta_4\beta_2\beta_3\beta_2 && \text{(by (A.1))} \\
&= g' \circ_1 \theta_4\beta_3\beta_2\beta_3 && \text{(by (b))} \\
&= (g' \circ_1 \theta_4)\beta_3\beta_2\beta_3 && \text{(by (A.1))} \\
&= g\beta_3\beta_2\beta_3
\end{aligned}$$

and now if $g = g'' \circ_1 \theta_{j-1}$ then

$$\begin{aligned}
g\beta_j\beta_{j+1}\beta_j &= (g'' \circ_1 \theta_{j-1})\beta_j\beta_{j+1}\beta_j \\
&= g''\beta_2\beta_3\beta_2 \circ_1 \theta_{j-1} && \text{(by (A.5))} \\
&= g''\beta_3\beta_2\beta_3 \circ_1 \theta_{j-1} && \text{(by preceding calculation)} \\
&= (g'' \circ_1 \theta_{j-1})\beta_{j+1}\beta_j\beta_{j+1} && \text{(by (A.5))} \\
&= g\beta_{j+1}\beta_j\beta_{j+1}.
\end{aligned}$$

∎

This completes the proof of Theorem A.1. We now turn to the loose maps. A loose m-ary morphism g can be written as $g' \circ_1 u$ for a unique tight $(m+1)$-ary morphism, where u is the (loose) nullary morphism classifier. If $1 \leq j < m$ we define $g\beta_j = g'\beta_{j+1} \circ_1 u$. This clearly gives an action of the braid groups on loose morphisms and moreover this definition is forced by the requirement that the equation (A.5) hold for loose morphisms, as it must in a braided skew multicategory; furthermore, the actions on tight and loose maps are mutually compatible.

A.11. Lemma. *Equations (A.2)–(A.5) hold for all tight g with $f = u$.*

Proof. For (A.5), first consider the case that $i = 1$. This says that $g\beta_j \circ_1 u = (g \circ_1 u)\beta_{j-1}$ and holds by definition of the right hand side. For $i > 1$ write $g = h \circ_1 \theta_i$ with h tight. Then

$$\begin{aligned}
g\beta_j \circ_i u &= (h \circ_1 \theta_i)\beta_j \circ_i u \\
&= (h\beta_{j-i+1} \circ_1 \theta_i) \circ_i u && \text{(by (A.5))} \\
&= h\beta_{j-i+1} \circ_1 (\theta_i \circ_i u) \\
&= (h \circ_1 (\theta_i \circ_i u))\beta_{j-1} && \text{(by (A.5) and fact that } \theta_i \circ_i u \text{ is tight)} \\
&= ((h \circ_1 \theta_i) \circ_i u)\beta_{j-1} \\
&= (g \circ_i u)\beta_{j-1}.
\end{aligned}$$

For (A.3) we first observe that the special case $\theta_3\beta_2 \circ_3 u = \theta_3 \circ_2 u$ is equivalent to the equality in Proposition 2.8. Thus it remains to show that the general case follows from this special case. For this let us write $g = g' \circ \theta_{j+1}$.

$$\begin{aligned}
g\beta_j \circ_{j+1} u &= (g' \circ_1 \theta_{j+1})\beta_j \circ_{j+1} u \\
&= ((g' \circ_1 \theta_3) \circ_1 \theta_{j-1})\beta_j \circ_{j+1} u \\
&= ((g' \circ_1 \theta_3)\beta_2 \circ_1 \theta_{j-1}) \circ_{j+1} u && \text{(by (A.5))} \\
&= ((g' \circ_1 \theta_3\beta_2) \circ_1 \theta_{j-1}) \circ_{j+1} u && \text{(by (A.1))} \\
&= (g' \circ_1 (\theta_3\beta_2 \circ_1 \theta_{j-1})) \circ_{j+1} u \\
&= g' \circ_1 ((\theta_3\beta_2 \circ_1 \theta_{j-1}) \circ_{j+1} u) \\
&= g' \circ_1 ((\theta_3\beta_2 \circ_3 u) \circ_1 \theta_{j-1}) \\
&= g' \circ_1 ((\theta_3 \circ_2 u) \circ_1 \theta_{j-1}) && \text{(by special case)} \\
&= g' \circ_1 ((\theta_3 \circ_1 \theta_{j-1}) \circ_j u) \\
&= (g' \circ_1 \theta_{j+1}) \circ_j u \\
&= g \circ_j u.
\end{aligned}$$

For (A.4) observe that the special case $\theta_3\beta_2 \circ_2 u = \theta_3 \circ_3 u$ is equivalent to the equality in Proposition 2.11, and use a similar argument to show that the general case follows from this special case.

Finally for (A.2) write $g = h \circ_1 \theta_{j+1}$ with h tight. Then

$$\begin{aligned}
g\beta_j \circ_i u &= (h \circ_1 \theta_{j+1})\beta_j \circ_i u \\
&= (h \circ_1 \theta_{j+1}\beta_j) \circ_i u && \text{(by (A.1))} \\
&= (h \circ_{i-j+1} u) \circ_1 \theta_{j+1}\beta_j \\
&= ((h \circ_{i-j+1} u) \circ_1 \theta_{j+1})\beta_j && \text{(by (A.1) and fact that } h \circ_{i-j+1} u \text{ is tight)} \\
&= ((h \circ_1 \theta_{j+1}) \circ_i u)\beta_j \\
&= (g \circ_i u)\beta_j.
\end{aligned}$$

■

To complete the proof of Theorem 5.9 it remains to show:

A.12. Proposition. *The remaining conditions* (A.1)–(A.5) *also hold when f or g are loose.*

Proof. Consider (A.5). If g is tight, but f is loose, write $f = f' \circ_1 u$ with f' tight of arity $n+1$; then

$$\begin{aligned}
g\beta_j \circ_i f &= g\beta_j \circ_i (f' \circ_1 u) \\
&= (g\beta_j \circ_i f') \circ_i u \\
&= (g \circ_i f')\beta_{j+n} \circ_i u && \text{((A.5) for tight morphisms)} \\
&= ((g \circ_i f') \circ_i u)\beta_{j+n-1} && \text{(by Lemma A.11)} \\
&= (g \circ_i (f' \circ_1 u))\beta_{j+n-1} \\
&= (g \circ_i f)\beta_{j+n-1}.
\end{aligned}$$

If g is loose, write $g = g' \circ_1 u$ with g' tight, and then observe that

$$\begin{aligned}
g\beta_j \circ_1 u &= (g' \circ_1 u)\beta_j \circ_1 u \\
&= (g'\beta_{j+1} \circ_1 u) \circ_1 u && \text{(by definition)} \\
&= (g'\beta_{j+1} \circ_2 u) \circ_1 u \\
&= (g' \circ_2 u)\beta_j \circ_1 u && \text{(previous case)} \\
&= ((g' \circ_2 u) \circ_1 u)\beta_{j-1} && \text{(by definition)} \\
&= ((g' \circ_1 u) \circ_1 u)\beta_{j-1} \\
&= (g \circ_1 u)\beta_{j-1} \\
g\beta_j \circ_i f &= (g' \circ_1 u)\beta_j \circ_i f \\
&= (g'\beta_{j+1} \circ_1 u) \circ_i f && \text{(by definition)} \\
&= ((g'\beta_{j+1} \circ_{i+1} f) \circ_1 u \\
&= ((g' \circ_{i+1} f)\beta_{j+n} \circ_1 u && \text{(previous case)} \\
&= ((g' \circ_{i+1} f) \circ_1 u)\beta_{j+n-1} && \text{(previous case)} \\
&= ((g' \circ_1 u) \circ_i f)\beta_{j+n-1} \\
&= (g \circ_i f)\beta_{j+n-1}
\end{aligned}$$

which completes all cases of (A.5).

Next we do (A.1). If f is tight but g loose, write $g = g' \circ_1 u$ with g' tight. Then

$$\begin{aligned}
f \circ_i g\beta_j &= f \circ_i (g' \circ_1 u)\beta_j \\
&= f \circ_i (g'\beta_{j+1} \circ_1 u) && \text{(by definition)} \\
&= (f \circ_i g'\beta_{j+1}) \circ_i u \\
&= (f \circ_i g')\beta_{i+j} \circ_i u && \text{(by (A.5))} \\
&= ((f \circ_i g') \circ_i u)\beta_{i+j-1} && \text{(by (A.5))} \\
&= (f \circ_i g)\beta_{i+j-1}.
\end{aligned}$$

If f is loose, write $f = f' \circ_1 u$ with f' tight. Then

$$\begin{aligned}
f \circ_i g\beta_j &= ((f' \circ_1 u) \circ_i g\beta_j \\
&= (f' \circ_{i+1} g\beta_j) \circ_1 u \\
&= (f' \circ_{i+1} g)\beta_{i+j} \circ_1 u && \text{(previous case)} \\
&= ((f' \circ_{i+1} g) \circ_1 u)\beta_{i+j-1} && \text{(by (A.5))} \\
&= (f \circ_i g)\beta_{i+j-1}
\end{aligned}$$

which completes all cases of (A.1).

Now consider (A.2). If g is tight but f loose, write $f = f' \circ_1 u$ with f' tight; then

$$\begin{aligned}
g\beta_j \circ_i f &= g\beta_j \circ_i (f' \circ_1 u) \\
&= (g\beta_j \circ_i f') \circ_i u \\
&= (g \circ_i f')\beta_j \circ_i u && \text{(by (A.2) for tight morphisms)} \\
&= ((g \circ_i f') \circ_i u)\beta_j && \text{(by Lemma A.11)} \\
&= (g \circ_i f)\beta_j.
\end{aligned}$$

If g is loose write $g = g' \circ_1 u$ with g' tight:

$$\begin{aligned}
g\beta_j \circ_i f &= (g' \circ_1 u)\beta_j \circ_i f \\
&= (g'\beta_{j+1} \circ_1 u) \circ_i f && \text{(by Lemma A.11)} \\
&= (g'\beta_{j+1} \circ_{i+1} f) \circ_1 u \\
&= (g' \circ_{i+1} f)\beta_{j+1} \circ_1 u && \text{(previous case)} \\
&= ((g' \circ_{i+1} f) \circ_1 u)\beta_j && \text{(by (A.5))} \\
&= (g \circ_i f)\beta_j.
\end{aligned}$$

This completes (A.2).

Next we do (A.3), so that $j = i - 1$. If g is tight and f loose, write $f = f' \circ_1 u$ with

f' tight; then:

$$\begin{aligned} g\beta_j \circ_{j+1} f &= g\beta_j \circ_{j+1} (f' \circ_1 u) \\ &= (g\beta_j \circ_{j+1} f') \circ_{j+1} u \\ &= (g \circ_j f')\beta_{j+n} \dots \beta_j \circ_{j+1} u && \text{(by (A.3) for tight morphisms)} \\ &= (g \circ_{i-1} f')\beta_{j+n} \dots \beta_{j+1} \circ_j u && \text{(by Lemma A.11)} \\ &= ((g \circ_{i-1} f') \circ_{i-1} u)\beta_{j+n-1} \dots \beta_j && \text{(by repeated use of (A.5))} \\ &= (g \circ_{i-1} f)\beta_{j+n-1} \dots \beta_j \end{aligned}$$

while if g is loose we may write $g = g' \circ_1 u$ and

$$\begin{aligned} g\beta_j \circ_{j+1} f &= (g' \circ_1 u)\beta_j \circ_{j+1} f \\ &= (g'\beta_{j+1} \circ_1 u) \circ_{j+1} f && \text{(by definition)} \\ &= (g'\beta_{j+1} \circ_{j+2} f) \circ_1 u \\ &= (g' \circ_i f)\beta_{j+n} \dots \beta_{j+1} \circ_1 u && \text{(previous case)} \\ &= ((g' \circ_i f) \circ_1 u)\beta_{j+n-1} \dots \beta_j && \text{(by repeated use of (A.5))} \\ &= (g \circ_{i-1} f)\beta_{j+n-1} \dots \beta_j \end{aligned}$$

completing the proof of (A.3).

That leaves only (A.4), where $j = i$. This is entirely analogous to the case of (A.3), and is left to the reader. ■

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