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COHESIVE TOPOSES OF SHEAVES ON MONOIDS OF CONTINUOUS ENDOFUNCTIONS OF THE UNIT INTERVAL

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Abstract. We determine the largest submonoid of the monoid of continuous endomorphisms of the unit interval $[0, 1]$ on which the finite partitions form the basis of a Grothendieck topology, and thus determine a cohesive topos over sets [Lawvere, 2007]. We analyze some of the sheaf theoretic aspects of this topos in the spirit of [Johnstone, 1979]. Furthermore, we adapt the constructions in [Menni, 2014] to include another model of axiomatic cohesion, this one closer to [Lawvere, 1975]. We conclude the paper with a proof of the fact, shown in [Menni, 2014], that a sufficiently cohesive topos of presheaves does not satisfy the continuity axiom.

1. Introduction

Working within the context of Lawvere's Axiomatic Cohesion [Lawvere, 2007], in [Menni, 2014] it is showed that a pre-cohesive category of presheaves over sets satisfies the continuity axiom if and only if it is a quality type; that is, if the category of presheaves over sets is sufficiently cohesive, then it cannot satisfy the continuity axiom. Then [Menni, 2014] proceeds to construct a pre-cohesive and sufficiently cohesive topos over sets that satisfies the continuity axiom, showing that continuity and sufficient cohesion are compatible over sets. The manner in which this is achieved is by considering the unit interval $I = [0, 1]$ and drastically cutting down the monoid of continuous endomorphisms that one considers on I, linear for instance, or polynomial, etc. Then one considers the topology of finite partitions and takes the category of sheaves for this Grothendieck topology. This clearly begs the question: which is the biggest monoid of continuous endomorphisms of I for which this construction works and produces a cohesive topos over sets? This is the question we address in Section 2 of the present paper. It turns out that a continuous function $f : I \to I$ satisfies the stability condition for a Grothendieck topology for Menni's "topology" if and only if it is *unilateral* (see Definition 2.1); an idea of a class of functions that already appears in [Isbell, 1976]. Since the other two conditions for a Grothendieck topology are easily verified, we do obtain a topology and show, in Theorem 2.8, that Menni's topology works for any submonoid of unilateral endomorphisms of I that contains the linear endomorphisms.

In [Johnstone, 1979] it is described how Lawvere, in an attempt to construct a 'topos of topological spaces', "sought to make 'path' the primitive notion" by considering the

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monoid M of continuous endomorphisms of the unit interval $I = [0,1]$, and giving it the largest Grothendieck topology for which all the M-sets of continuous paths on any topological space are sheaves. Johnstone goes on to observe that this "topology on M fails to have enough covers" since, "In particular, Isbell showed that the sieve generated by the maps $x \mapsto \frac{1}{2}x$ and $x \mapsto \frac{1}{2}(x+1)\colon I \to I$ is not universally effective epimorphic" (see [Isbell, 1976]), thus it is not a cover in the topology. Thus [Johnstone, 1979] replaces 'continuous path' by 'convergent sequence' even though "this rather nullifies the philosophical justification for considering the Lawvere topos". Now, the construction in [Menni, 2014] can be construed as a return to Lawvere's idea of considering 'path' the primitive notion and taking the topology generated precisely by insisting that $x \mapsto \frac{1}{2}x$ and $x \mapsto \frac{1}{2}(x+1)$ should be a cover. The 'topological toposes' thus constructed with submonoids of unilateral endomorphisms contain at least the real manifolds and the CW-complexes. Section 3 of the present paper mirrors some of the sheaf theoretic results for the topological topos from [Johnstone, 1979] to show that they are also valid in these other 'topological toposes'. The proofs are very similar to those in [Johnstone, 1979].

In Section 4 we construct another topos, closer to Lawvere's idea of considering the full monoid of continuous endomorphisms of I. We show in 4.6 that this topos provides another model of cohesion over sets.

To close the paper, in Section 5 we give a simple construction that produces a presheaf that does not satisfy the Continuity axiom in any Sufficiently Cohesive topos of presheaves. This results appears in [Menni, 2014], but the proof there is a rather long proof by contradiction.

2. The biggest stable submonoid of the unit interval

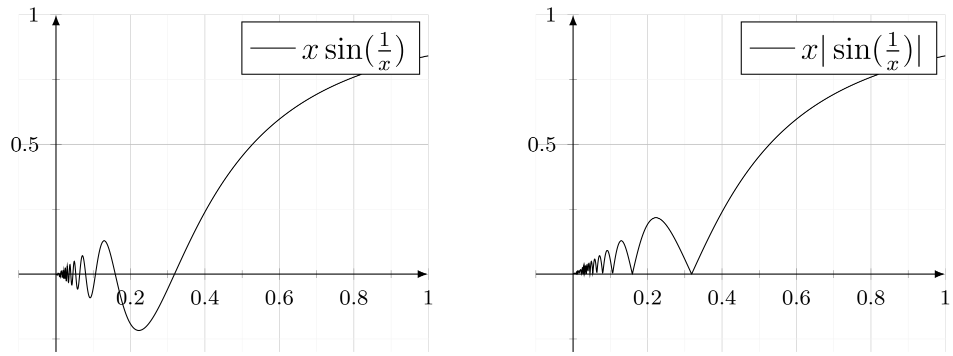
2.1. Definition. *Let $f\colon[0,1] \to [0,1]$ be a continuous function and let $t \in [0,1]$. We say that f is* unilateral at t *if there is an $\varepsilon > 0$ such that the following functions do not change sign:*

$$f - f(t)\colon[t, t+\varepsilon] \cap [0,1] \to [-1,1] \quad \textit{and} \quad f - f(t)\colon[t-\varepsilon, t] \cap [0,1] \to [-1,1].$$

We say that f is unilateral *if it is unilateral at every $t \in [0,1]$.*

2.2. Remark. In [Isbell, 1976] sheaves for the Lawvere topos are described as presheaves with two partial operations. One of them is the lifting of a path α, such lifting can be obtained as the amalgamation of those maps α_0 mapping a subinterval J of $[0,1]$ "lightly" into $[0,1]$ such that each interior point $p \in J$ has a neighborhood composed of two intervals on each of which $\alpha_0(x) - \alpha_0(p)$ does not change sign.

Note that the definition of unilateral can easily be written in terms of arbitrary closed intervals. With this observation we can give examples and non examples of unilateral functions. Given the following functions (adding its value at 0 in the only possible way)



is easy to see that $x\sin(\frac{1}{x})$ is not unilateral at the point 0. On the other hand, the function $x|\sin(\frac{1}{x})|$ is clearly unilateral at 0 and is not difficult to see that is unilateral at every point in $(0,1]$.

For any submonoid of the monoid of continuous endos of $[0,1]$ containing the linear functions we have an assignment K that sends the unique object $[0,1]$ of the submonoid to the set $K([0,1])$ of families

$$\{\, f_i : [0,1] \longrightarrow [0,1] \ \mid 1 \leq i \leq n \,\},$$

where $a = r_0 < r_1 < \cdots < r_{n-1} < r_n = b$ is a partition and f_i is the linear function such that $f_i(0) = r_{i-1}$ and $f_i(1) = r_i$. This assignment K satisfies two of the axioms defining bases for Grothendieck topologies but, in general, it does not satisfy the transitivity axiom. Menni identified a couple of submonoids such that the associated K satisfies transitivity and is therefore the basis for a Grothendieck topology. We identify the largest submonoid with this property.

2.3. PROPOSITION. *A continuous function $f : [0,1] \to [0,1]$ satisfies the stability axiom for the function K if and only if f is unilateral.*

PROOF. Suppose that f satisfies stability, and let $t \in [0,1]$. We do the case in which $f(t) \in (0,1)$ since the extreme cases are clear. Consider the partition $0 < f(t) < 1$. Since f satisfies the stability condition, there exists a partition $0 = r_0 < r_1 < \cdots < r_m = 1$ such that the image of each $[r_{i-1}, r_i]$ is contained in $[0, f(t)]$ or it is contained in $[f(t), 1]$. We consider the following cases: $t \in (r_{i-1}, r_i)$ for some i and t is one of $r_0, \ldots, r_m$. In the first case any $\varepsilon > 0$ such that $[t-\varepsilon, t+\varepsilon] \subseteq (r_{i-1}, r_i)$ will show that f is unilateral at the point t. In the second case if $t = r_i$ then is enough to take $\varepsilon > 0$ such that $[t-\varepsilon, t] \subseteq (r_{i-1}, r_i)$ and $[t, t+\varepsilon] \subseteq (r_i, r_{i+1})$ to show that f in unilateral at the point t.

For the converse, assume that f is unilateral and let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of $[0,1]$. For each $x \in [0,1]$ take $\delta_x > 0$ such that: If $f(x) \in (t_{j-1}, t_j)$ then $f([x-\delta_x, x+\delta_x] \cap [0,1]) \subseteq (t_{j-1}, t_j)$ (since f is continuous there is $\delta > 0$ such that $f((x-\delta, x+\delta) \cap [0,1]) \subseteq (t_{j-1}, t_j)$ and taking $0 < \delta_x < \delta$ we have the condition with closed intervals); and if $f(x) = t_j$ then $f([x-\delta_x, x+\delta_x] \cap [0,1]) \subseteq (t_{j-1}, t_{j+1})$, and $f - f(x)$ does not change sign in the intervals $[x, x+\delta_x] \cap [0,1]$ and $[x-\delta_x, x] \cap [0,1]$

(using continuity again and the fact that f is unilateral). We thus have the open cover $\bigcup_{x\in[0,1]}(x-\delta_x, x+\delta_x)$ of the compact space $[0,1]$, so there is a finite subcover

$$[0,1] \subseteq (x_1-\delta_1, x_1+\delta_1) \cup \cdots \cup (x_m-\delta_m, x_m+\delta_m),$$

with the obvious change of notation for the δ's. For those i such that $f(x_i) \in (t_{j-1}, t_j)$, it is clear that

$$[x_i-\delta_i, x_i+\delta_i] \cap [0,1] \hookrightarrow [0,1] \xrightarrow{f} [0,1]$$

factors through the inclusion $[t_{j-1}, t_j] \to [0,1]$, for some j. If, on the other hand, $f(x_i) = t_k$ for some k, then

$$[x_i-\delta_i, x_i] \cap [0,1] \hookrightarrow [0,1] \xrightarrow{f} [0,1]$$

factors through the inclusion $[t_{k-1}, t_k] \to [0,1]$ or through the inclusion $[t_k, t_{k+1}] \to [0,1]$ (depending on the sign of $f - t_k$). And similarly for $[x_i, x_i+\delta_i]$. It is then clear that the finite set of points

$$(\{0,1\} \cup \{x_1,\ldots,x_m\} \cup \{x_1+\delta_1,\ldots,x_m+\delta_m\} \cup \{x_1-\delta_1,\ldots,x_m-\delta_m\}) \cap [0,1]$$

generates a partition of $[0,1]$ such that f satisfies the stability condition. ∎

Since is easy to see that functions that satisfy stability are closed under composition, then we have the following result.

2.4. Corollary. *The composite of unilateral functions is a unilateral function.*

2.5. Remark. We have thus identified the biggest monoid of continuous endomorphisms of $[0,1]$ that contains the linear functions and for which Menni's topology of partitions is a Grothendieck topology as the endomorphisms of unilateral functions.

Since the identity on $[0,1]$ is clearly unilateral, we obtain a category **Uni** whose only object is $[0,1]$, and whose morphisms are the unilateral functions. Moreover, since K is a basis for a topology on **Uni**, we have the site $(\mathbf{Uni}, K)$.

Let M be any submonoid of the monoid of unilateral endomorphisms of the unit interval such that M contains the linear functions. Then M determines a subcategory $\mathbb{M}$ of **Uni**. By what we said above, we may endow $\mathbb{M}$ with Menni's topology J of partitions. We thus obtain a site $(\mathbb{M}, J)$. Denote $\mathcal{M} := \mathrm{Sh}(\mathbb{M}, J)$. We wish to show that the canonical $p\colon \mathcal{M} \to \mathbf{Set}$ is a model of cohesion as defined in [Lawvere, 2007].

We first construct, following the ideas in [Menni, 2014], a subcanonical site for this topos $\mathcal{M}$. As a first step we consider the category $\mathbf{M}_0$ whose objects are closed intervals $[a,b]$ with $a \le b$ in $\mathbb{R}$, and whose morphisms are those functions $f\colon [a,b] \to [c,d]$ that can be constructed as composites of the form

$$[a,b] \longrightarrow [0,1] \xrightarrow{m} [0,1] \longrightarrow [c,d]$$

where the unnamed arrows are linear functions and $m \in M$.

It is easy to see that the linear functions $[a, b] \to [c, d]$ are in $\mathbf{M}_0$, that all the morphisms in $\mathbf{M}_0$ are unilateral (with the obvious extension of Definition 2.1 to continuous functions between closed intervals), and that $\mathbb{M}$ is a full subcategory of $\mathbf{M}_0$.

We define an M-dissection of a continuous function $f : [a, b] \to [c, d]$ as a family

$$\{g_j : [r_{j-1}, r_j] \hookrightarrow [a, b] \mid 1 \leq j \leq n\}$$

such that $fg_j \in \mathbf{M}_0$. The function f is called piecewise-M if it has a M-dissection.

2.6. Proposition. *Piecewise-M functions are closed under composition.*

Proof. Let $f : A \to B$ and $g : B \to C$ be piecewise-M functions. Then we can take M-dissections $\{f_j : [r_{j-1}, r_j] \hookrightarrow A \mid 1 \leq j \leq n\}$ and $\{g_i : [t_{i-1}, t_i] \hookrightarrow B \mid 1 \leq i \leq m\}$ of f and g, respectively. Since unilateral is clearly a local property, it is not hard to see that f and g are unilateral. Using a similar argument to that in Lemma 2.3, there is a partition of A, say $x_0 < x_1 \cdots < x_k$, such that for each l there is an i that makes the following diagram commute

$$\begin{array}{ccc} [x_{l-1}, x_l] & \xrightarrow{f|_{[x_{l-1}, x_l]}} & [t_{i-1}, t_i] \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

With this we have obtained the partition $\{x_0, \ldots, x_k\} \cap [r_{j-1}, r_j]$ of each interval $[r_{j-1}, r_j]$ for which the composition gf is in $\mathbf{M}_0$. We have thus obtained an M-dissection of gf. ■

We now define the category $\mathbf{M}$ of real closed intervals $[a, b]$ $(a, b \in \mathbb{R},\ a \leq b)$ and piecewise-M functions between them. Since all the functions involved are unilateral, it is clear that Menni's topology of partitions defines a Grothendieck topology on $\mathbf{M}$; that is, a basis L for this topology has $L([a, a]) = \{1_{[a,a]}\}$ and for $a < b$, $L([a, b])$ as those families

$$\{[r_{j-1}, r_j] \hookrightarrow [a, b] \mid 1 \leq j \leq n\}$$

determined by partitions $a = r_0 < \cdots < r_n = b$. It is easy to see that $(\mathbf{M}, L)$ is a subcanonical site. Furthermore, an easy application of the comparison lemma [Johnstone, 2002, Theorem C.2.2.3] gives us the following result.

2.7. Proposition. *$\mathcal{M}$ is equivalent to* $\mathrm{Sh}(\mathbf{M}, L)$. ■

2.8. Theorem. *Let M be a submonoid of* $\mathbf{Uni}([0, 1], [0, 1])$ *such that M contains all the linear endomorphisms. Then M induces a cohesive topos over* **Set**.

PROOF. Since for all $x, y \in [a,b]$ there is a unique linear function $f : [0,1] \to [a,b]$ such that $f(0) = x$ and $f(1) = y$, then every object in $\mathbf{M}$ is arcwise connected with respect to the object $[0,1]$. Therefore, the bipointed object $[0,1]$ is a connector. Taking the points $0, 1 : 1 \to [0,1]$ and the linear functions $l, r : [0,1] \to [0,1]$ such that $l(x) = \frac{x}{2}$ and $r(x) = \frac{x+1}{2}$, is easy to see that $([0,1], 0, 1, l, r)$ is an abstract interval. Finally, note that the family $\{l, r\}$ generates the same sieve that the family

$$\{[0, \tfrac{1}{2}] \hookrightarrow [0,1], [\tfrac{1}{2}, 1] \hookrightarrow [0,1]\},$$

i.e., a covering sieve. Therefore, the topology is compatible with the abstract interval, and by Proposition 9.6 in [Menni, 2014] the canonical morphism $p : \mathrm{Sh}(\mathbf{M}, L) \to \mathbf{Set}$ is cohesive. ■

3. Monoids of unilateral functions and classical sheaf theory

Denote by **Top** the category of topological spaces. It is clear that we have a faithful functor $\mathbf{Top} \to \mathcal{M}$ given by $X \mapsto (\mathbf{Top}(-, X) : \mathcal{M}^{\mathrm{op}} \to \mathbf{Set})$.

3.1. LEMMA. *Let $f : X \to Y$ be a function between first countable and locally path connected spaces. If for every continuous path $\gamma : I \to X$ the composite $f\gamma$ is continuous, then f is continuous.*

PROOF. Assume that there is an f that satisfies the conditions but that it is not continuous. Let $x \in X$ and V a neighborhood of $f(x)$ showing that f is not continuous at the point x. Since X is locally path connected and first countable, we can construct a family of path connected neighborhoods $\{U_n \mid n \geq 1\}$ of x such that $n < m$ implies $U_m \subseteq U_n$. Now, using that f is not continuous, for each $n \geq 1$ there is $x_n \in U_n$ such that $f(x_n) \notin V$. Since all the neighborhoods U_n are path connected, then we can take a path $\gamma_n : I \to B_n$ connecting x_n with x_{n+1}. Finally, we paste all this paths to a path $\gamma : I \to X$ defined by $\gamma(t) = \gamma_n(t)$ if $t \in [\frac{1}{n}, \frac{1}{n+1}]$ and $\gamma(1) = x$. Note that $f\gamma(1 - \frac{1}{n}) \notin V$ for all $n \geq 1$, then the sequence $\{1 - \frac{1}{n}\}$ converges to 1 but when applying $f\gamma$ it is not a converging sequence to $f(x)$, contradicting the fact that $f\gamma$ is continuous. ■

This means that the faithful functor $\mathbf{Top} \to \mathrm{Sh}(\mathbf{M}, K)$ is full and faithful when restricted to the subcategory of first countable and locally path connected spaces. In particular this category contains the real manifolds and the CW-complexes.

To simplify the notation identify X and $\mathbf{Top}(-, X)$ as objects of $\mathcal{M}$.

We now define, as for the topological topos [Johnstone, 1979, Lemma 9.1], a geometric morphism $v : \mathcal{M}/X \to \mathrm{Sh}(X)$ for X a topological space. First, $v_* : \mathcal{M}/X \to \mathrm{Sh}(X)$ is given as follows: For $\psi : E \to X$ in $\mathcal{M}/X$, define the presheaf $v_*\psi$ in such a way that every open subset $U \subseteq X$ is sent to the set of natural transformations $\varphi : U \to E$ such that the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & E \\ & \searrow \quad \swarrow_{\psi} & \\ & X & \end{array} \tag{1}$$

Before continue mirroring [Johnstone, 1979] we give a technical lemma.

3.2. Lemma. *Let X be a topological space, $U \subseteq X$ be an open subset, $U = \bigcup_{i \in I} U_i$ an open covering and $f : [a, b] \to U$ be a continuous function. Then, there is a partition $a = r_0 < r_1 < \cdots < r_n = b$ such that for every $j \in \{1, \ldots, n\}$, $f_j := f|_{[r_{j-1}, r_j]} : [r_{j-1}, r_j] \to U_{i_j}$ for some i_j.*

Proof. For every $x \in [a, b]$ there is an $i_x \in I$ such that $f(x) \in U_{i_x}$ and by continuity a $\delta_x > 0$ such that $f([x - \delta_x, x + \delta_x] \cap [a, b]) \subseteq U_{i_x}$ (continuity gives $\delta > 0$ that satisfies the condition with open intervals, but taking $0 < \delta_x < \delta$ we have the condition with closed intervals). Then $\bigcup_{x \in [a,b]} (x - \delta_x, x + \delta_x)$ is an open covering of the compact space $[a, b]$ and by compactness we have a finite subcover

$$[a, b] \subseteq (x_1 - \delta_1, x_1 + \delta_1) \cup \cdots \cup (x_n - \delta_n, x_n + \delta_n)$$

with the obvious change of notation for the δ's. As in the proof of 2.3 the finite set of points

$$(\{a, b\} \cup \{x_{,} \ldots, x_n\} \cup \{x_1 - \delta_1, \ldots x_n - \delta_n\} \cup \{\{x_1 + \delta_1, \ldots x_n + \delta_n\}\}) \cap [a, b]$$

defines a partition of $[a, b]$ with the desired condition. ■

3.3. Lemma. *For every $\psi : E \to X$ in $\mathcal{M}/X$, the presheaf $v_*\psi$ is a sheaf over X.*

Proof. Let $U = \bigcup_i U_i$ be an open cover of an open $U \subseteq X$, and let

$$\{\varphi_i : U_i \to E \mid i \in I\} \tag{2}$$

be a compatible family for $v_*\psi$. We wish to define an amalgamation $\varphi : U \to E$ of the given family. So take a closed interval $[a, b]$ and $f : [a, b] \to U$ in $\mathbf{Top}([a, b], U)$. By the previous lemma there is a partition $a = r_0 < r_1 < \cdots < r_n = b$ such that for every $j \in \{1, \ldots, n\}$, $f_j := f|_{[r_{j-1}, r_j]} : [r_{j-1}, r_j] \to U_{i_j}$ for some i_j. We have that the family

$$\langle (\varphi_{i_j})_{[r_{j-1}, r_j]} (f_j) \in E[r_{j-1}, r_j] \rangle_{j=1}^{n} \tag{3}$$

is compatible since the only relevant intersections are of the form $[r_{j-1}, r_j] \cap [r_j, r_{j+1}]$. Using that E is a sheaf, the family (3) has a unique amalgamation, $\varphi_{[a,b]}(f)$. Observe that the elements of the family do not depend on the choices of the i_j's.

Naturality of φ follows from the fact that every $f : [a, b] \to [c, d]$ in M is unilateral, so for every partition $c = t_0 < t_1 < \cdots < t_n = d$ of $[c, d]$ there is a partition $a = r_0 < r_1 < \cdots < r_m = b$ of $[a, b]$ satisfying $f([r_{i-1}, r_i]) \subseteq [t_{j-1}, t_j]$. It follows that, in the construction of $\varphi(gf)$ with the aforementioned partitions, $Ef(\varphi(g))$ is an amalgamation.

Again, by the construction of φ, condition (1) is satisfied locally, so φ also satisfies this condition, and we have $v_*(\psi : E \to X) \in \mathrm{Sh}(X)$. ■

We also have a functor $v^*: \mathrm{Sh}(X) \to \mathcal{M}/X$ that takes a sheaf $F \in \mathrm{Sh}(X)$ to the usual local homemomorphism $\Lambda F \to X$ determined by F considered as an object of $\mathcal{M}/X$.

3.4. Theorem. *For every topological space X we have a geometric morphism*

$$v : \mathcal{M}/X \to \mathrm{Sh}(X).$$

Proof. The unit $\eta_F : F \to v_* v^*(F)$ at a sheaf $F \in \mathrm{Sh}(X)$ is defined, for an $s \in F(U)$, U an open of X, by $(\eta_F)_U(s) = \dot{s} : U \to \Lambda F$ (considered in $\mathcal{M}/X$), where $\dot{s}(y) = [s \in FV]_y$ is the class of s in $\varinjlim_{U \ni y} FU$. It is not hard to see that η_F is natural, and also, that η is natural (see Chapter II of [Mac Lane-Moerdijk, 1992], for instance, for the appropriate background to show these claims).

We now define the counit $\varepsilon : v^* v_* \to 1_{\mathcal{M}/X}$. Let $\psi : E \to X$ be an object of $\mathcal{M}/X$, and take $f : [a,b] \to \Lambda v_*(\psi)$ in $(v^* v_*(\psi))([a,b])$. Let f_0 be the composite of f with the projection $\Lambda v_*(\psi) \to X$. For $a < b$ we observe, as before, that there is a partition $a = r_0 < \cdots < r_n = b$ of $[a,b]$ such that each $f([r_{i-1}, r_i])$ is contained in a single $\dot{s}_i(U_i)$. Thus, for every $t \in [r_{i-1}, r_i]$ we have that $f(t) = [s_i \in (v_*(\psi))(U_i)]_{f_0(t)}$. It is not hard to see that the family

$$\langle (s_i)_{[r_{i-1}, r_i]}(f_0|_{[r_{i-1}, r_i]} \in E([r_{i-1}, r_i])) \rangle_{i=1}^n$$

is compatible; it thus defines a unique element $(\varepsilon_\psi)_{[a,b]}(f) \in E([a,b])$. Observe that this latter element does not depend on the choice of the partition or the choice of the s_i's. It is not hard to show that ε is indeed a natural transformation.

With all these assignments in place, it is routine verification that the triangular identities are satisfied. Furthermore, it is not hard to show that v^* preserves finite limits. ∎

3.5. Remark. In the same way as in [Menni, 2014, Proposition 10.6], the unit interval $[0,1]$ generates a total order with distinct endpoints, and since $\hat{\Delta}$ is the classifying topos of such orders, there is a geometric morphism $g : \mathrm{Sh}(\mathbf{M}, K) \to \hat{\Delta}$ whose inverse image that behaves like a geometric realization.

3.6. Remark. Since we have a cohesive morphism $p : \mathrm{Sh}(\mathbf{M}, K) \to \mathbf{Set}$, then we can follow the ideas in [Johnstone, 1979, Section 4] to conclude that the Dedekind's real object coincides with the usual real numbers in **Set**. More precisely, the discrete functor $p^* : \mathbf{Set} \to \mathrm{Sh}(\mathbf{M}, K)$ preserves the object $\mathbb{Q}$ of rational numbers, meaning that the rational numbers Q in $\mathrm{Sh}(\mathbf{M}, K)$ is simply the discrete space of rational numbers. Recall that a real number is a pair $r = (L, U)$ of subobjects of Q satisfying the following axioms:

1. $\forall q (q \in L \iff \exists q' > q \wedge q' \in L)$,
2. $\forall q (q \in U \iff \exists q' < q \wedge q' \in U)$,
3. $\forall q \forall q' (q \in L \wedge q' \in U \implies q < q')$,
4. $\forall n \exists q \exists q' (q \in L \wedge q' \in U \wedge q' - q < \frac{1}{n})$.

Since the subobjects of Q in $\mathrm{Sh}(\mathbf{M}, K)$ are the subobjects of $\mathbb{Q}$ in **Set** as discrete spaces and $p_* : \mathrm{Sh}(\mathbf{M}, K) \to \mathbf{Set}$ preserve the truth of the previous axioms, then the object R of Dedekind reals in $\mathrm{Sh}(\mathbf{M}, K)$ satisfies $R(1) = \mathbb{R}$. Finally, it is clear that $R([a, b])$ is the set of continuous paths $\gamma : [a, b] \to \mathbb{R}$.

There is also a sufficiently simple description of the subobject classifier as the set of closed sieves. Informally, a unilateral function is in a closed sieve R if and only if the unilateral function is piecewise in R. A sieve R on $[a, b]$ is closed if for any unilateral function $f : [c, d] \to [a, b]$ there is a partition $c = r_0 < \cdots < r_n = b$ such that:

$$\{[r_{i-1}, r_i] \hookrightarrow [c, d] \xrightarrow{f} [a, b] \mid 1 \le i \le n\} \subseteq R \implies f \in R.$$

4. The topos of closed intervals and overlapping covers

This section will follow the ideas in [Menni, 2014] to construct a topos closer to the topos given in [Lawvere, 1975]. Let $\mathbf{C}$ be the category whose objects are closed intervals of the real line, and where the arrows are continuous functions between them.

A basis for a Grothendieck topology on $\mathbf{C}$ is given as follows: $K([a, a])$ is the trivial family $\{\mathrm{id}\}$ and, for $a < b$, $K([a, b])$ consists of the families of the form $\langle [r_i, s_i] \hookrightarrow [a, b] \rangle_{i=1}^{n}$ such that $r_i < s_i$ for all i, and

$$(a, b) \subseteq \bigcup_{i=1}^{n} (r_i, s_i). \tag{4}$$

We observe that, given that the family is finite, a is r_i for some i, and b is s_j for some j.

4.1. Lemma. *K is a basis for a topology on $\mathbf{C}$.*

Proof. The only axiom that is not immediate is stability. Let $\langle [r_i, s_i] \hookrightarrow [c, d] \rangle_{i=1}^{n}$ be a covering family and let $f : [a, b] \to [c, d]$ be a continuous function. Assume $c = r_1$ and $d = s_n$. Let $t \in [a, b]$. If $f(t) = c$ or $f(t) = d$ we have, by continuity (and the same argument in Proposition 2.3 for the condition with closed intervals), that there is $\varepsilon_t > 0$ such that $f([t - \varepsilon_t, t + \varepsilon_t] \cap [a, b]) \subseteq [r_1, s_1)$ or $f([t - \varepsilon_t, t + \varepsilon_t] \cap [a, b]) \subseteq (r_n, s_n]$, respectively. Otherwise there is an i such that $f(t) \in (r_i, s_i)$, thus we take $\varepsilon_t > 0$ such that $f([t - \varepsilon_t, t + \varepsilon_t] \cap [a, b]) \subseteq (r_i, s_i)$. Then, we have an open cover $[a, b] \subseteq \bigcup_{t \in [a,b]} (t - \varepsilon_t, t + \varepsilon_t)$, and by compactness, a finite subcover

$$[a, b] \subseteq \bigcup_{i=1}^{m} (t_i - \varepsilon_i, t_i + \varepsilon_i)$$

(with the obvious change of notation); where we may assume $t_1 = a$ and $t_m = b$. Now, it is not hard to see that the family $\{[t_j - \varepsilon_j, t_j + \varepsilon_j] \cap [a, b] \hookrightarrow [a, b] \mid 1 \le j \le m\}$ is a covering for $[a, b]$ such that the composite

$$[t_j - \varepsilon_j, t_j + \varepsilon_j] \cap [a, b] \hookrightarrow [a, b] \xrightarrow{f} [c, d]$$

is factored through some inclusion $[r_i, s_i] \hookrightarrow [c, d]$, proving that K is indeed a basis for a Grothendieck topology. ∎

Since the basis K satisfies (4), a is r_i for some i and b is s_j for some j, then the covering $\langle [r_i, s_i] \hookrightarrow [a, b] \rangle_{i=1}^{n}$ is an actual covering in the topological sense. Therefore it is not hard to see that the site $(\mathbf{C}, K)$ is subcanonical. Moreover $(\mathbf{C}, K)$ is a connected and locally connected site and $\mathbf{C}$ is a category with terminal object in which every object has a point. Then, there is a pre-cohesive $p \colon \mathrm{Sh}(\mathbf{C}, K) \to \mathbf{Set}$ (see section 1 of [Johnstone, 2011]).

Note that the category $\mathbf{C}$ has a bipointed object $I = [0, 1]$ and every pair of points $s, t \in [a, b]$ are connected by a linear parametrization of I. So, by lemmas 8.7 and 8.13 of [Menni, 2014] I is a connector for $\mathrm{Sh}(\mathbf{C}, K)$.

To conclude that $p \colon \mathrm{Sh}(\mathbf{C}, K) \to \mathbf{Set}$ is cohesive we need a slight modification of the concept of abstract interval (Definition 9.2 in [Menni, 2014]) to be compatible with our topology.

4.2. Definition. *An* overlapping abstract interval *is a bipointed object* $1 \to I \leftarrow 1$ *with two monomorphisms* $l, m, r : I \to I$ *such that*

1. *The cospan* $0 : 1 \to I \leftarrow 1 : 1$ *is disjoint.*
2. *The following diagrams commute*

$$\begin{array}{ccc} 0 & \xrightarrow{0} & I \\ & \searrow^{0} & \downarrow{\scriptstyle l} \\ & & I \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{1} & I \\ & \searrow^{1} & \downarrow{\scriptstyle r} \\ & & I \end{array} \qquad \begin{array}{ccc} I & \longrightarrow & I \\ \downarrow & \searrow^{m} & \downarrow{\scriptstyle r} \\ I & \xrightarrow[l]{} & I \end{array}$$

and the square is a pullback.

Note that m captures the intersection of l and r and the unnamed arrows are linear parametrizations of that intersection.

For our topos of overlapping covers, the unit interval with the points $1 \xrightarrow{0} [0, 1] \xleftarrow{1} 1$ and the functions $l, m, r : [0, 1] \to [0, 1]$ defined as

$$l(x) = \frac{2}{3}x, \qquad m(x) = \frac{1}{3}x + \frac{1}{3}, \qquad r(x) = \frac{2}{3}x + \frac{1}{3}$$

is an overlapping abstract interval. Based on this example we will denote linear parametrizations with their intended image. For example, l can be denoted by $[0, \frac{2}{3}]$, m by $[\frac{1}{3}, \frac{2}{3}]$ and r by $[\frac{1}{3}, 1]$.

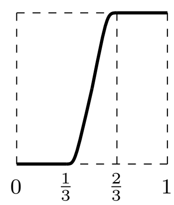
Moreover, to be able to amalgamate two paths such that the end point of one coincides with the starting point of the other, we will follow [Isbell, 1976] where a method to paste such paths is shown. If π_1 and π_2 are paths that coincide in endpoints, then Isbell shows that we need to use a constant path π_0 and form the path $\pi_1 \pi_0 \pi_2$ suggesting that to be able to paste paths that agree on endpoint we need to "spend" some time in the transition.

4.3. Definition. *We say that an overlapping abstract interval admits splices if there are arrows* $\mathtt{s}: I \to I$, $[0, \frac{1}{3}], [\frac{2}{3}, 1]: I \to I$ *such that the following diagrams commute:*

$$\begin{array}{ccc} I & \xrightarrow{[0,\frac{1}{3}]} & I \\ \downarrow & & \downarrow \mathtt{s} \\ 1 & \xrightarrow[0]{} & I \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{[\frac{2}{3},1]} & I \\ \downarrow & & \downarrow \mathtt{s} \\ 1 & \xrightarrow[1]{} & I \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{0} & I \\ & {\scriptstyle 0}\searrow & \downarrow [0,\frac{1}{3}] \\ & & I \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{1} & I \\ & {\scriptstyle 1}\searrow & \downarrow [\frac{2}{3},1] \\ & & I \end{array}$$

Following [Menni, 2014], we will say that a coverage on $\mathbf{C}$ is compatible with the overlapping abstract interval if $\{l, r\}$ covers I.

In our example, the interval $I = [0, 1]$ is an overlapping abstract interval that admits splices in $\mathbf{C}$, where the graph of $\mathtt{s}$ is



and it is clearly compatible with K.

Note that if we apply $\mathtt{s}$ to an actual path $\gamma: [0, 1] \to [a, b]$, i.e., when we consider the function $\mathbf{C}(\mathtt{s}, [a, b]): \mathbf{C}([0, 1], [a, b]) \to \mathbf{C}([0, 1], [a, b])$ applied to γ, we obtain a path $\gamma \cdot \mathtt{s}$ that is constant $\gamma(0)$ in the first third, then γ in a third of the time and ends up being constant $\gamma(1)$. Therefore, the result is a path that is homotopically equivalent to the original, as mentioned in [Isbell, 1976]. Furthermore, this example satisfies an additional axiom that was not listed in definition 4.3:

$$\begin{array}{ccc} I & \xrightarrow{m} & I \\ & \diagdown\!\!\diagdown & \downarrow \mathtt{s} \\ & & I \end{array}$$

This axiom ensures that the result of splicing a path is homotopically equivalent to the original. Because we do not use this axiom we do not write it as part of the definition 4.3.

As mentioned above, the purpose of $\mathtt{s}$ is to amalgamate paths that coincide only in endpoints. We want to amalgamate such paths to be able to prove a result like the Lemma 9.5 of [Menni, 2014] in our case. But $\mathtt{s}$ alone cannot amalgamate paths, it needs the arrows l and r of Definition 4.2 and the covering K to be compatible with the overlapping abstract interval. Finally, the initial and final points of the amalgamation must be given by the original paths. The triangles in Definitions 4.2 and 4.3 should be used for this purpose.

4.4. Lemma. *Let* $\mathbf{C}$ *be a category with an overlapping abstract interval* I *which admits splices and let* K *be a covering on* $\mathbf{C}$ *compatible with the overlapping abstract interval. For each* $X \in \mathrm{Sh}(\mathbf{C}, K)$ *and* $z_1, z_2 \in XI$ *such that* $z_1 \cdot 1 = z_2 \cdot 0$ *there is* $z \in XI$ *such that* $z \cdot 0 = z_1 \cdot 0$ *and* $z \cdot 1 = z_2 \cdot 1$.

PROOF. Given $z_1, z_2 \in XI$ such that $z_1 \cdot 1 = z_2 \cdot 0$, we can apply $\mathtt{s}$ to the paths z_1 and z_2 to obtain $z_1 \cdot \mathtt{s}, z_2 \cdot \mathtt{s} \in XI$. Since I is an abstract interval and K is compatible with it, then $l, r : I \to I$ is a covering whose intersection is parameterized by $[0,1] \xrightarrow{[\frac{2}{3},1]} [0,1] \xleftarrow{[0,\frac{1}{3}]} [0,1]$, and the two squares in Definition 4.3 we have that the paths $z_1 \cdot \mathtt{s}, z_2 \cdot \mathtt{s}$ agree in the intersection of the covering family, so they define a compatible family. Since X is a sheaf there is an amalgamation $z \in XI$ such that (by the last two triangles in 4.3 and the triangles in 4.2) $z \cdot 0 = z_1 \cdot 0$ and $z \cdot 1 = z_2 \cdot 1$. ∎

Now we can prove the corresponding Lemma 9.5 in [Menni, 2014] for our case.

4.5. LEMMA. *Let* $\mathbf{C}$, K *and* $X \in \mathrm{Sh}(\mathbf{C}, K)$ *be as in the previous lemma. If* $x, y \in X1$ *are connectable, then they can be connected by a cospan* $0 : 1 \to I \leftarrow 1 : 1$.

PROOF. We will proceed as in [Menni, 2014], so assume that x and y are connected by a combinatorial arc of length 2 as follows.

$$x \longleftarrow\!\shortmid\; z_1 \longmapsto x_1 \longleftarrow\!\shortmid\; z_2 \longmapsto y$$

$$1 \xrightarrow[0]{} I \xleftarrow[1]{} 1 \xrightarrow[0]{} I \xleftarrow[1]{} 1$$

By Lemma 4.4 there is $z \in XI$ such that:

$$x \longleftarrow\!\shortmid\; z \longmapsto y$$

$$1 \xrightarrow[0]{} I \xleftarrow[1]{} 1$$

Therefore, x and y can be connected by a single cospan. ∎

This lemma together with proof of Proposition 9.6 in [Menni, 2014] give the following result.

4.6. THEOREM. $p : \mathrm{Sh}(\mathbf{C}, K) \to \mathbf{Set}$ *is cohesive.*

5. Continuity and toposes of presheaves

As we mentioned in the introduction, the proof in [Menni, 2014] of the fact that a Sufficiently Cohesive topos of presheaves (over $\mathbf{Set}$) does not satisfy the Continuity axiom is a bit long and ends up in a rather odd proof by contradiction. So, in this section, we give a simple, elementary construction that produces a presheaf that does not satisfy the Continuity axiom in any Sufficiently Cohesive topos of presheaves.

So let $\mathcal{E}$ be a topos of presheaves and denote by $p : \mathcal{E} \to \mathbf{Set}$ the canonical geometric morphism, and assume that p is pre-cohesive and sufficiently cohesive. So $\mathcal{E} = \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ for a small category $\mathbf{C}$; and we may assume that idempotents split in $\mathbf{C}$. According to Proposition 2.10 in [Menni, 2014], $\mathbf{C}$ has a terminal object and every object in $\mathbf{C}$ has a point. Furthermore, by Corollary 2.11 in [Menni, 2014], we have that there exists an object $C \in \mathbf{C}$ that has two distinct points $a, b : 1 \to C$.

Take the presheaf $P = \coprod_{n\in\mathbb{N}} \mathbf{C}(-,C)$. For every $D \in \mathbf{C}$ denote the unique arrow to 1 by $!_D : D \to 1$. Since D has a point, we have that $!_D$ is a split epi. Since a and b are different, we must have $a{\cdot}!_D \neq b{\cdot}!_D$. We induce on PD the equivalence relation $\simeq$ that identifies $(2n, b{\cdot}!_D)$ with $(2n+1, b{\cdot}!_D)$ and also $(2n+1, a{\cdot}!_D)$ with $(2n+2, a{\cdot}!_D)$, for $n \in \mathbb{N}$. It is clear that $\simeq$ is a congruence; and we take $Q = P/\simeq$. A picture of the relations in Q is

$$\begin{array}{ccccccccc} 0 & & 1 & & 2 & & 3 & & \dots \\ a{\cdot}!_D & & a{\cdot}!_D & \overset{\simeq}{\text{———}} & a{\cdot}!_D & & a{\cdot}!_D & \overset{\simeq}{\text{———}} & \dots \\ b{\cdot}!_D & \overset{\simeq}{\text{———}} & b{\cdot}!_D & & b{\cdot}!_D & \overset{\simeq}{\text{———}} & b{\cdot}!_D & & \dots \end{array}$$

where the columns represent the elements in the coproduct. Since $\mathbf{C}(-,C)$ is connected, then congruence makes Q connected. However, the connection between different elements, for example $[(0,a)]$ with $[(n,a)]$, is bigger and bigger. With this idea we obtain Proposition 7.3 of [Menni, 2014].

5.1. PROPOSITION. *Let $\mathcal{E}$ be a topos of presheaves such that the canonical $p : \mathcal{E} \to \mathbf{Set}$ is pre-cohesive. If p is sufficiently cohesive, then it does not satisfy the Continuity axiom.*

PROOF. Define Q as above. It is not hard to see that, since $p_!\mathbf{C}(-,C) = 1$, then $p_!Q = 1$. Note that the classes $[(0,a)]$ and $[(n,a)]$ represent different elements in Q, but they are connected by a finite path. Nevertheless if we vary n, the connections form $[(0,a)]$ to $[(n,a)]$ are not bounded. Then, any connection from $\langle[(0,a)]\rangle_{n\in\mathbb{N}}$ to $\langle[(n,a)]\rangle_{n\in\mathbb{N}}$ is infinite, which means that they define different elements of $p_!(Q^{p^*\mathbb{N}})$. Thus $p_!(Q^{p^*\mathbb{N}}) \to (p_!Q)^{\mathbb{N}} = 1$ is not an isomorphism. ∎

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