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EXPONENTIABILITY IN DOUBLE CATEGORIES AND THE GLUEING CONSTRUCTION

SUSAN NIEFIELD

ABSTRACT. We consider pre-exponentiable objects of a pre-cartesian double category $\mathbb{D}$, i.e., objects Y such that the lax functor $- \times Y : \mathbb{D} \longrightarrow \mathbb{D}$ has a right adjoint in the 2-category **LxDbl** of double categories and lax functors. When $\mathbb{D}$ has 2-glueing (in the sense of [N12a]), we show that Y is pre-exponentiable in $\mathbb{D}$ if and only if Y is exponentiable in $\mathbb{D}_0$ and $- \times Y$ is an oplax functor. Thus, such a $\mathbb{D}$ is pre-cartesian closed as a double category if and only if $\mathbb{D}_0$ is a cartesian closed category. Applications include the double categories $\mathbb{C}\text{at}$, $\mathbb{P}\text{os}$, $\mathbb{T}\text{op}$, $\mathbb{L}\text{oc}$, and $\mathbb{T}\text{opos}$, whose objects are small categories, posets, topological space, locales, and toposes, respectively.

1. Introduction

Adjunctions between double categories have been considered by Grandis and Paré [GP04] in several categories. Since left adjoints are oplax and right adjoints are lax, the pair is an orthogonal adjunction in the double category $\mathbb{D}\text{bl}$ whose horizontal and vertical morphisms are lax and oplax double functors, respectively. If the left adjoint is lax, then it is an adjunction in the 2-category **LxDbl** whose morphisms are lax functors. If both adjoints are pseudo functors, then it is an adjunction in the 2-category **PsDbl** whose morphisms are pseudo functors. Since the left adjoints we consider are lax functors, the setting for what follows will be **LxDbl**.

In [A18], Aleiferi defines a pre-cartesian double category $\mathbb{D}$ as one for which the diagonal $\Delta : \mathbb{D} \longrightarrow \mathbb{D} \times \mathbb{D}$ has a right adjoint in the 2-category **LxDbl**. Then $\mathbb{D}$ is called cartesian if the right adjoint is a pseudo functor, i.e., a right adjoint in the 2-category **PsDbl**.

In this paper, we consider pre-exponentiable objects in pre-cartesian double categories $\mathbb{D}$, i.e., Y such that the lax functor $- \times Y : \mathbb{D} \longrightarrow \mathbb{D}$ has a right adjoint in the 2-category **LxDbl**. We restrict to right adjoints in **LxDbl** rather than **PsDbl** because, in some of our examples (e.g., $\mathbb{C}\text{at}$ and $\mathbb{P}\text{os}$), the right adjoints that exist are not pseudo even though $\mathbb{D}$ is cartesian, i.e., $- \times Y$ is pseudo.

The five examples of interest here are double categories $\mathbb{D}$ with 2-glueing, in the sense of [N12a, N12b], which is a common generalization of Artin-Wraith glueing for toposes [J77] and a special case of Bénabou's equivalence $\text{Lax}_{\text{N}}(B, \mathbf{Prof}) \simeq \mathbf{Cat}/B$ used by Street

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in [St01]. In particular, $\mathbb{D}$ is a fibrant double category [Sh08] with cotabulators [GP99], and there is an object $\mathbb{2}$ together with an equivalence of categories $\mathbb{D}_1 \simeq \mathbb{D}_0/\mathbb{2}$. Using this equivalence (and a little more), we obtain a general proof that Y is pre-exponentiable in $\mathbb{D}$ if an only if it is exponentiable in $\mathbb{D}_0$. Thus, $\mathbb{D}$ is pre-cartesian closed if and only if $\mathbb{D}_0$ is cartesian closed.

It turns out that some of the structure in the definition of $\mathbb{2}$-glueing is not needed to prove our general theorems here, but is only used to construct the pseudo inverse to the functor $\mathbb{D}_1 \longrightarrow \mathbb{D}_0/\mathbb{2}$ in the examples of interest. Consequently, rather than work in the context of double categories with $\mathbb{2}$-glueing introduced in [N12a], our approach here will be to first introduce the properties we need for our theorems, and only use the extra conditions to show that our examples satisfy these properties. In particular, we need not assume that $\mathbb{D}$ is fibrant.

We proceed as follows. In Section 2, we isolate the properties of $\mathbb{2}$-glueing that will be used for our general results. We then recall the definitions of pre-cartesian and cartesian double categories in Section 3, and show that our "glueing categories" are pre-cartesian. In Section 4, we introduce pre-exponentiable objects and pre-cartesian closed double categories, and prove a theorem which applies to the five double categories of interest, as well as their double slices. In Section 5, we recall the definition of a fibrant double category, and show that the $\mathbb{2}$-glueing double categories of [N12a] are glueing categories in the sense considered here. We conclude, in Section 6, with examples showing that $\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$ are not cartesian closed double categories, and that $\mathbb{D}_1$ need not be cartesian closed when $\mathbb{D}$ is pre-cartesian closed as a double category.

2. Glueing Categories

In this section, we introduce the notion of a glueing category, present the five relevant examples, and show that if $\mathbb{D}$ is a glueing category, then so are its double slices $\mathbb{D}/\!/B$.

2.1. Definition. *A double category is an internal pseudo category*

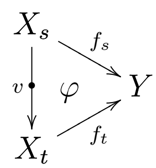
$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\ \odot\ } \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0 \quad (\text{with } \mathrm{id}^\bullet \colon \mathbb{D}_0 \to \mathbb{D}_1)$$

in the 2-category **CAT** *of locally small categories.*

Unpacking this definition, $\mathbb{D}$ consists of *objects* and *horizontal morphisms* (those of $\mathbb{D}_0$), *vertical morphisms* (objects of $\mathbb{D}_1$ with domain and codomain given by s and t), and *cells* (morphisms of $\mathbb{D}_1$) denoted by

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\downarrow\!\!\bullet & \varphi & \bullet\!\!\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array} \tag{2.1}$$

Horizontal composition is denoted by $\circ$ and vertical composition by $\odot$, both of which are sometimes elided. Note that when w is the vertical $\mathrm{id}^{\bullet}_{Y}$, we often denote the cell (2.1) by



There are five double categories of interest here.

2.2. Example. $\mathbb{C}$at has small categories as objects, functors and profunctors as horizontal and vertical morphisms, respectively, and natural transformations $v \longrightarrow w(f_s-, f_t-)$ as cells of the form (2.1). Note that our profunctors $v\colon X_s \longrightarrow X_t$ are **Set**-valued functors on $X_s^{op} \times X_t$.

2.3. Example. $\mathbb{P}$os has partially-ordered sets as objects and order-preserving maps as horizontal morphisms. Vertical morphisms $v\colon X_s \longrightarrow X_t$ are order ideals $v \subseteq X_s^{op} \times X_t$ (i.e., up-sets), and there is a cell of the form (2.1) if and only if

$$(x_s, x_t) \in v \Rightarrow (f_s(x_s), f_t(x_t)) \in w$$

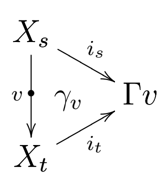
2.4. Example. $\mathbb{T}$op has topological spaces as objects and continuous maps as horizontal morphisms. Vertical morphisms $v\colon X_s \longrightarrow X_t$ are finite intersection-preserving maps $v\colon \mathcal{O}(X_s) \longrightarrow \mathcal{O}(X_t)$ on the open set lattices, and there is a cell of the form (2.1) if and only if $f_t^{-1} \odot w \subseteq v \odot f_s^{-1}$.

2.5. Example. $\mathbb{L}$oc has locales as objects, locale morphisms (in the sense of [J82]) as horizontal morphisms, and finite meet-preserving maps as vertical morphisms. There is a cell of the form (2.1) if and only if $f_t^* \odot w \leq v \odot f_s^*$.

2.6. Example. $\mathbb{T}$opos has Grothendieck toposes as objects, geometric morphisms (in the sense of [J77]) as horizontal morphisms, and left exact functors as vertical morphisms, and natural transformations $f_t^* \odot w \longrightarrow v \odot f_s^*$, or equivalently, $w \odot f_{s*} \longrightarrow f_{t*} \odot v$, as cells of the form (2.1) .

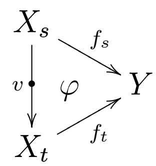
We will see that these double categories have "cotabulators" in the following sense.

2.7. Definition. *A cotabulator of a vertical morphism $v\colon X_s \longrightarrow X_t$ consists an object Γv and a cell*



(2.7)

such that for any other cell



there exists a unique horizontal morphism $f\colon \Gamma v \longrightarrow Y$ *such that* $\mathrm{id}^\bullet_f \gamma_v = \varphi$.

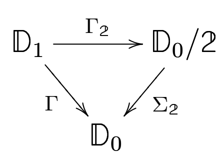
One can show that the cotabulator Γv exists, for all v, if and only if $\mathrm{id}^\bullet\colon \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ has a left adjoint [GP99]. Thus, when $\mathbb{D}$ has cotabulators, we get a functor $\Gamma\colon \mathbb{D}_1 \longrightarrow \mathbb{D}_0$ which is left adjoint to $\mathrm{id}^\bullet$.

2.8. Example. Cotabulators in $\mathbb{C}$at are given by "collages." Recall that the *collage* Γv of a profunctor $v\colon X_s^{op} \times X_t \longrightarrow \mathbf{Set}$ is defined as follows. Objects of Γv are pairs (x_a, a), where $a \in \{s, t\}$ and $x_a \in X_a$. Morphisms $(x_a, a) \longrightarrow (y_b, b)$ are those of X_n, if $a = b$, and elements of $v(x_s, x_t)$, if $a = s$ and $b = t$, modulo an equivalence relation. There are no morphisms $(y_t, t) \longrightarrow (x_s, s)$. Collages in $\mathbb{P}$os are defined similarly.

2.9. Example. Given $v\colon X_s \longrightarrow\!\!\!\!\bullet\;\, X_t$ in $\mathbb{T}$op, i.e., a finite intersection preserving map $v\colon \mathcal{O}(X_s) \longrightarrow \mathcal{O}(X_t)$, the space Γv is given by "glueing along v," i.e., the disjoint union $X_s \sqcup X_t$ with U open if U_s is open in X_s, U_t is open in X_t, and $U_t \subseteq v(U_s)$, where $U_a = U \cap X_a$. The maps $i_a\colon X_a \longrightarrow \Gamma v$ are the usual inclusions.

2.10. Example. Cotabulators in $\mathbb{L}$oc and $\mathbb{T}$opos are also given by the glueing construction used in [N81] which is also know as "Artin-Wraith glueing" in the topos case [J77].

Suppose $\mathbb{D}$ has cotabulators and $\mathbb{D}_0$ has a terminal object 1. Then one can show that 1 is a horizontal terminal object, in the sense of [GP99]. Consider $\mathbb{2} = \Gamma(\mathrm{id}^\bullet_1)$, the Sierpinski object. Then $\Gamma\colon \mathbb{D}_1 \longrightarrow \mathbb{D}_0$ induces a functor $\Gamma_{\mathbb{2}}\colon \mathbb{D}_1 \longrightarrow \mathbb{D}_0/\mathbb{2}$ such that the diagram



commutes, where $\Sigma_{\mathbb{2}}$ is the forgetful functor. If, in addition, $\mathbb{D}_0$ has pullbacks, then $\Gamma_{\mathbb{2}}$ has a right adjoint (denoted by Φ_2), since a functor to a finitely complete slice category has a right adjoint if and only if its composition with the forgetful functor does [N82a].

Each of our five double categories has a terminal object. One can show that $\mathbb{2}$ is the non-discrete 2-element poset in $\mathbb{P}$os, the category $\mathbb{2}$ in $\mathbb{C}$at, the Sierpinski space $\mathbb{2}$ in $\mathbb{T}$op, the Sierpinski locale $\mathcal{O}(\mathbb{2})$ in $\mathbb{L}$oc, and the Sierpinski topos $\mathbf{Set}^{\mathbb{2}}$ in $\mathbb{T}$opos.

2.11. Definition. *Suppose* $\mathbb{D}$ *is a double category with cotabulators and* $\mathbb{D}_0$ *has finite limits. Then* $\mathbb{D}$ *is called a glueing category if the functor* $\Gamma_{\mathbb{2}}\colon \mathbb{D}_1 \longrightarrow \mathbb{D}_0/\mathbb{2}$ *is an equivalence* *of categories, and the diagrams*

$$\begin{array}{ccc} X_s & \xrightarrow{i_s} & \Gamma v \\ \downarrow & & \downarrow{\scriptstyle \Gamma_2(v)} \\ 1 & \xrightarrow[i_s]{} & 2 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X_t & \xrightarrow{i_t} & \Gamma v \\ \downarrow & & \downarrow{\scriptstyle \Gamma_2(v)} \\ 1 & \xrightarrow[i_t]{} & 2 \end{array} \tag{2.11}$$

are pullbacks in $\mathbb{D}_0$, for all $v\colon X_s \dashrightarrow X_t$, where i_s and i_t are the morphisms in diagram (2.7) of Definition 2.7.

The double categories in Examples 2.2-2.6 are all glueing categories. We know each has cotabulators and the required finite limits, and the pullback condition follows directly from the descriptions of cotablulators cited above. For an alternate proof, in the Section 5, we will show that Definition 2.11 agrees with the notion of 2-glueing considered in [N12a] when $\mathbb{D}$ is a fibrant double category, in the sense of [Sh08], but first we show that glueing is preserved by slicing.

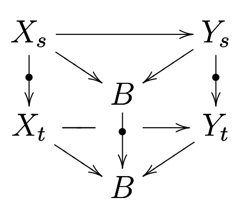
Recall that Paré [P11] extends the Bénabou equivalence [St01]

$$\mathrm{Lax}_{\mathrm{N}}(B, \mathbf{Prof}) \simeq \mathbf{Cat}/B$$

to double categories by replacing $\mathbf{Cat}/B$ with the slice double category $\mathbf{Cat}//B$ defined, in general, as follows. Objects of $\mathbb{D}//B$ are horizontal morphisms $X \longrightarrow B$, horizontal arrows are commutative triangles, vertical arrows are cells

$$\begin{array}{ccc} X_s & \xrightarrow{p_s} & B \\ {\scriptstyle v}\downarrow\!\bullet & \pi & \bullet\!\downarrow{\scriptstyle \mathrm{id}^\bullet_B} \\ X_t & \xrightarrow[p_t]{} & B \end{array}$$

and cells are commutative diagrams of cells



with the induced horizontal and vertical composition. Thus,

$$(\mathbb{D}//B)_0 = \mathbb{D}/B \qquad \text{and} \qquad (\mathbb{D}_0//B)_1 = \mathbb{D}_1/\,\mathrm{id}^\bullet_B$$

2.12. Proposition. *If $\mathbb{D}$ is a glueing categoryy, then $\Gamma_2(\,\mathrm{id}^\bullet_B)$ is given by the projection $\pi_2\colon B \times 2 \longrightarrow 2$.*

PROOF. To show that $\Gamma_2 \,\mathrm{id}^\bullet \cong 2^*$, consider the diagram

$$\mathbb{D}_1 \underset{\Phi_2}{\overset{\Gamma_2}{\rightleftarrows}} \mathbb{D}_0/2 \underset{2^*}{\overset{\Sigma_2}{\rightleftarrows}} \mathbb{D}_0$$

where Φ_2 is a right adjoint pseudo inverse of Γ_2 and 2^* is right adjoint to the forgetful functor Σ_2. It suffices to show that $\Phi_2\Gamma_2 \,\mathrm{id}^\bullet \cong \Phi_2 2^*$, or equivalently, $\mathrm{id}^\bullet \cong \Phi_2 2^*$. Since $\Sigma_2\Gamma_2 \dashv \Phi_2 2^*$, $\Sigma_2\Gamma_2 = \Gamma$, and $\Gamma \dashv \mathrm{id}^\bullet$, the desired result follows. ∎

Verification of the following proposition is straightforward. Since the result will only be used as a source of examples when we compare exponentiability in $\mathbb{D}$ and $\mathbb{D}_1$ in Section 6, we omit the details.

2.13. PROPOSITION. *If $\mathbb{D}$ is a glueing category, then so is $\mathbb{D}//B$, for every B. Moreover,*

$$(\mathbb{D}//B)_1 \simeq \mathbb{D}_0/(B \times 2)$$

PROOF. The first part is a direct consequence of the assumption that $\mathbb{D}$ is a glueing category. Since $\Gamma_2(\mathrm{id}^\bullet_B)$ is isomorphic to the projection by Proposition 2.12, it follows that

$$(\mathbb{D}//B)_1 = \mathbb{D}_1/\,\mathrm{id}^\bullet_B \simeq (\mathbb{D}_0/2)/(B \times 2 \xrightarrow{\pi_2} 2) \cong \mathbb{D}_0/(B \times 2)$$

as desired. ∎

3. Cartesian Double Categories

In this section, we recall the definitions of oplax/lax adjunction [GP04] and cartesian double category [A18], and apply these to our five examples.

3.1. DEFINITION. *A lax functor $F\colon \mathbb{D} \to \mathbb{E}$ consists of functors $F_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1\colon \mathbb{D}_1 \to \mathbb{E}_1$, compatible with s and t, together with comparison cells*

$$\mathrm{id}^\bullet_{FX} \longrightarrow F(\mathrm{id}^\bullet_X) \quad \text{and} \quad Fw \odot Fv \longrightarrow F(w \odot v)$$

for every object X and every composition $w \odot v$ in $\mathbb{D}$, and satisfying naturality and coherence conditions. An oplax (or colax) functor is defined with the comparison cells in the opposite direction.

Note that we drop the subscripts on F when the context is clear. There is a double category $\mathbb{D}$bl, introduced in [GP04], whose objects are double category, horizontal morphisms are lax functors, and vertical morphisms are oplax functors, with suitable cells.

3.2. DEFINITION. *An oplax/lax adjunction is an orthogonal adjunction in $\mathbb{D}bl$.*

These adjunctions are characterized in Theorem 3.6 of [GP04] as follows.

3.3. Proposition. *An oplax functor* $F\colon \mathbb{D} \to \mathbb{D}'$ *has a lax right adjoint* G *if and only if (1) for every object* X' *in* $\mathbb{D}'$*, there is a universal arrow* $\varepsilon_{X'}\colon FGX' \to X'$ *from the functor* F_0 *to the object* X'*, and (2) for every vertical morphism* $v'\colon X'_s \multimap X'_t$ *in* $\mathbb{D}'$*, there is a universal cell*

$$\begin{array}{ccc} FGX'_s & \xrightarrow{\varepsilon_{X'_s}} & X'_s \\ {\scriptstyle v'}\big\downarrow & \varepsilon_{v'} & \big\downarrow{\scriptstyle FGv'} \\ FGX'_t & \xrightarrow[\varepsilon_{X'_t}]{} & X'_t \end{array}$$

from the functor F_1 *to the object* v *in* $\mathbb{D}'_1$.

Note that this universality says that $F_1 \dashv G_1$ and $F_0 \dashv G_0$. Moreover, one can show that, given such adjoints, F is oplax if and only if G is lax, and so we get:

3.4. Corollary. *The following are equivalent for functors* $F_n\colon \mathbb{D}_n \to \mathbb{E}_n$ *and* $G_n\colon \mathbb{E}_n \to \mathbb{D}_n$*, where* $n = 0, 1$*, compatible with* s *and* t.

(a) $F\colon \mathbb{D} \to \mathbb{E}$ *is oplax and* $G\colon \mathbb{E} \to \mathbb{D}$ *is a lax right adjoint.*

(b) $F_0 \dashv G_0$, $F_1 \dashv G_1$, *and* F *is oplax.*

(c) $F_0 \dashv G_0$, $F_1 \dashv G_1$, *and* G *is lax.*

Recall [GP04] that if the left adjoint F above is also lax, then it is an adjunction in the 2-category **LxDbl** whose morphisms are lax functors. If both adjoints are pseudo functors, then it is an adjunction in the 2-category **PsDbl** whose morphisms are pseudo functors.

3.5. Definition. *A double category* $\mathbb{D}$ *is called pre-cartesian if the functors* $\Delta\colon \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ *and* $!\colon \mathbb{D} \to \mathbb{1}$ *have right adjoints, denoted by* $\times$ *and* 1*, respectively. If* $\times$ *and* 1 *are pseudo functors, we say* $\mathbb{D}$ *is a cartesian double category.*

We will see that our five examples are pre-cartesian. We know that the first two are cartesian (see [A18] for $\mathbb{C}\mathrm{at}$, $\mathbb{P}\mathrm{os}$ is similar). Although we do not know if our other three examples are cartesian, we will only need to know $- \times Y\colon \mathbb{D} \to \mathbb{D}$ is pseudo (actually, just oplax) when Y is exponentiable in $\mathbb{D}_0$, and that will follow from a result in [N12b].

Recall [GP99] that $\mathbb{D}$ is called *horizontally invariant* if, for all vertical morphisms $w\colon Y_s \multimap Y_t$ and isomorphism $f_s\colon X_s \to Y_s$ and $f_t\colon X_t \to Y_t$

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\big\downarrow & \varphi & \big\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

there exists a vertical morphism $v\colon X_s \multimap X_t$ and an invertible cell $\varphi\colon v \to w$ such that $s(\varphi) = f_s$ and $t(\varphi) = f_t$.

3.6. Theorem. *If* $\mathbb{D}$ *is a horizontally invariant glueing category, then* $\mathbb{D}$ *is pre-cartesian.*

Proof. By Corollary 3.4, $!: \mathbb{D} \to \mathbb{1}$ has a lax right adjoint since $\mathbb{D}$ has a horizontal terminal object 1.

To see that $\Delta: \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ has a lax right adjoint, by Corollary 3.4, it suffices to show that $\Delta_1: \mathbb{D}_1 \to \mathbb{D}_1 \times \mathbb{D}_1$ does, since Δ is colax and $\Delta_0: \mathbb{D}_0 \to \mathbb{D}_0 \times \mathbb{D}_0$ has a right adjoint. Given $v: X_s \to X_t$ and $w: Y_s \to Y_t$, let $X = \Gamma v$ and $Y = \Gamma w$. We know $X \times_2 Y$, with the usual projection to 2, is the product of $\Gamma_2(v)$ and $\Gamma_2(w)$ in $\mathbb{D}_0/2$ and $(X \times_2 Y)_n \cong X_n \times Y_n$ (for $n = s, t$) since pullback along $i_n: 1 \to 2$ preserves equalizers (being right adjoint composition with i_n) and there is an equalizer

$$X \times_2 Y \longrightarrow X \times Y \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} 2$$

Since $\mathbb{D}$ is horizontally invariant, there is a vertical morphism $v \times w: X_s \times Y_s \to X_t \times Y_t$ and an invertible cell

$$\begin{array}{ccc} X_s \times Y_s & \xrightarrow{\cong} & (X \times_2 Y)_s \\ {\scriptstyle v\times w}\big\downarrow & \theta & \big\downarrow{\scriptstyle \Phi_2(X\times_2 Y \to 2)} \\ X_t \times Y_t & \xrightarrow{\cong} & (X \times_2 Y)_t \end{array}$$

and so $v \times w$ is the product of v and w in $\mathbb{D}_1$. ■

3.7. Corollary. $\mathbb{P}$os*,* $\mathbb{C}$at*,* $\mathbb{T}$op*,* $\mathbb{L}$oc*, and* $\mathbb{T}$opos *are pre-cartesian double categories.*

4. Pre-cartesian Closed Double Categories

In this section, we introduce the notion of an pre-exponentiable object in a double category. We consider pre-exponentiability in pre-cartesian, rather than cartesian, double categories since (even when $- \times Y$ is pseudo) the exponentials that exist in $\mathbb{D}$ are not always pseudo. For certain glueing categories $\mathbb{D}$ such that $\mathbb{D}_0$ is finitely cocomplete, we show that Y is pre-exponentiable in $\mathbb{D}$ if and only if it is exponentiable in $\mathbb{D}_0$.

We begin by recalling, from [N82a], some properties of exponentiability in $\mathcal{D}/B$, where $\mathcal{D}$ is a category with finite limits.

4.1. Remark. When $q: Y \to B$ is exponentiable in $\mathcal{D}/B$ and $r: Z \to B$, we write the exponential as $q^r: Z^Y \to B$. Given $b: 1 \to B$, let $(\ )_b: \mathcal{D}/B \to \mathcal{D}$ denote the functor defined by pullback along b. Note that $(\ )_b$ is usually denoted by b^*, but we avoid this notation here due to its use in the definition of conjoin in 5.1 below. It is well known that $(\ )_b$ has a left adjoint denoted by Σ_b and defined by composing with the morphism b. Since pullback along any morphism preserves exponentiability, we know Y_b is exponentiable, whenever

$q\colon Y \longrightarrow B$ is. In this case, $(Z^Y)_b \cong Z_b^{Y_b}$, since the diagram of left adjoints

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{-\times Y_b} & \mathcal{D} \\ {\scriptstyle \Sigma_b}\downarrow & & \downarrow{\scriptstyle \Sigma_b} \\ \mathcal{D}/B & \xrightarrow[-\times_B Y]{} & \mathcal{D}/B \end{array}$$

commutes up to isomorphism.

4.2. Definition. *Suppose $\mathbb{D}$ is a pre-cartesian double category. An object Y is called pre-exponentiable in $\mathbb{D}$ if the lax functor $- \times Y\colon \mathbb{D} \longrightarrow \mathbb{D}$ has a right adjoint in the 2-category* **LxDbl**. *If every object is pre-exponentiable, then $\mathbb{D}$ is called pre-cartesian closed.*

Note that, by Proposition 2.12, the functor $(- \times Y)_1\colon \mathbb{D}_1 \longrightarrow \mathbb{D}_1$ is actually $- \times \mathrm{id}_Y^\bullet$ on $\mathbb{D}_1$ which corresponds to $- \times_2 (Y \times 2)$ on $\mathbb{D}_0/2$ via the equivalence $\mathbb{D}_1 \simeq \mathbb{D}_0/2$.

4.3. Theorem. *Suppose $\mathbb{D}$ is a horizontally invariant pre-cartesian double category. If Y is pre-exponentiable in $\mathbb{D}$, then $- \times Y\colon \mathbb{D} \longrightarrow \mathbb{D}$ is oplax and Y is exponentiable in $\mathbb{D}_0$. Moreover, the converse holds when $\mathbb{D}$ is a glueing category.*

Proof. The first part clearly follows from Corollary 3.4. For the converse, suppose $\mathbb{D}$ is a glueing category. Applying Corollary 3.4 again, since $- \times Y\colon \mathbb{D} \longrightarrow \mathbb{D}$ is oplax, it suffices to show that the right adjoint $(\ )^Y$ to $- \times Y\colon \mathbb{D}_0 \longrightarrow \mathbb{D}_0$ extends to an endofunctor of $\mathbb{D}_1$. Since $\mathbb{D}$ is horizontally invariant, using the equivalence $\mathbb{D}_1 \cong \mathbb{D}_0/2$ as in the proof of Theorem 3.6, we need only show that, for $r\colon Z \longrightarrow 2$, the exponential $r^{\pi_2}\colon Z^{Y\times 2} \longrightarrow 2$ satisfies

$$(Z^{Y\times 2})_s \cong (Z_s)^Y \qquad \text{and} \qquad (Z^{Y\times 2})_t \cong (Z_t)^Y$$

but that follows from Remark 4.1. ∎

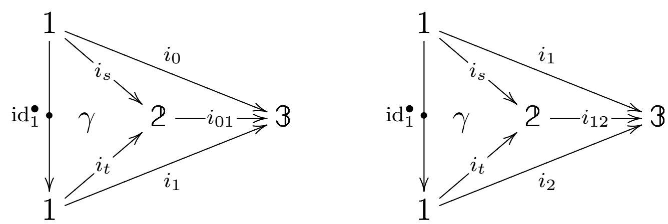
4.4. Corollary. *Suppose $\mathbb{D}$ is a horizontally invariant pre-cartesian double category and $- \times Y\colon \mathbb{D} \longrightarrow \mathbb{D}$ is oplax. Then Y is pre-exponentiable in $\mathbb{D}$ if and only if it is exponentiable in $\mathbb{D}_0$.* ∎

To apply this theorem, we will see that $- \times Y\colon \mathbb{D} \longrightarrow \mathbb{D}$ is a pseudo functor in each of our examples, whenever Y is exponentiable in $\mathbb{D}_0$. There are several ways to see this using results from [N12a] and [N12b]. We present an approach here which, unlike [N12a] and [N12b], avoids the assumption that $\mathbb{D}$ is fibrant.

Suppose $\mathbb{D}$ is glueing category such that $\mathbb{D}_0$ has pushouts. Let 3 denote the pushout

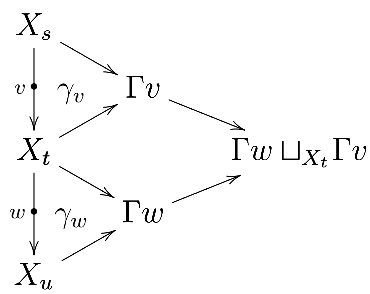
$$\begin{array}{ccc} 1 & \xrightarrow{i_s} & 2 \\ {\scriptstyle i_t}\downarrow & & \downarrow{\scriptstyle i_{12}} \\ 2 & \xrightarrow[i_{01}]{} & 3 \end{array}$$

where i_s and i_t are the morphism defining $\Gamma(\mathrm{id}_1^\bullet)$ in 2.7. There is also a morphism $i_{02}\colon 2 \longrightarrow 3$ induced by vertically pasting the following diagrams along $i_1 = i_{12} i_s = i_{01} i_t$.



where $i_0 = i_{01} i_s$ and $i_2 = i_{12} i_t$.

Given $X_s \xrightarrow{v} X_t \xrightarrow{w} X_u$, let $j\colon \Gamma(w \odot v) \longrightarrow \Gamma v \sqcup_{X_t} \Gamma w$ denote the morphism induced by the diagram of cells



Then we have a commutative diagram

$$\begin{array}{ccc} \Gamma(w \odot v) & \xrightarrow{\ j\ } & \Gamma w \sqcup_{X_t} \Gamma v \\ {\scriptstyle \Gamma_2(w\odot v)}\downarrow & & \downarrow \\ 2 & \xrightarrow[i_{02}]{} & 3 \end{array} \tag{4.5}$$

where the unlabelled morphism is induced by the coproducts.

4.5. Definition. *Suppose $\mathbb{D}$ is a glueing category. Then $\mathbb{D}$ has the 02-pullback condition if $\mathbb{D}_0$ has pushouts and the diagram (4.5) is a pullback, for every $X_s \xrightarrow{v} X_t \xrightarrow{w} X_u$ in $\mathbb{D}$.*

4.6. Lemma. *Suppose $\mathbb{D}$ is a glueing category satisfying the 02-pullback condition. If Y is exponentiable in $\mathbb{D}_0$, then $- \times Y\colon \mathbb{D} \longrightarrow \mathbb{D}$ is a pseudo functor.*

Proof. Given $X_s \xrightarrow{v} X_t \xrightarrow{w} X_u$, we will show that

$$\theta\colon (w \times Y) \odot (v \times Y) \longrightarrow (w \odot v) \times Y$$

is invertible in $\mathbb{D}_1$, by showing that $\Gamma\theta$ is invertible in $\mathbb{D}_0$ so that $\Gamma_2\theta$ is invertible in $\mathbb{D}_0/2$.

Since $\Gamma(v \times Y) \cong \Gamma v \times Y$, for all v, applying the 02-pullback condition, we see that

$$\begin{array}{ccc} \Gamma((w \times Y) \odot (v \times Y)) & \longrightarrow & (\Gamma w \times Y) \sqcup_{X_t \times Y} (\Gamma v \times Y) \\ \downarrow & & \downarrow \\ \mathbb{2} & \xrightarrow[i_{02}]{} & \mathbb{3} \end{array}$$

is a pullback, and

$$\begin{array}{ccc} \Gamma(w \odot v) \times Y & \longrightarrow & (\Gamma w \sqcup_{X_t} \Gamma v) \times Y \\ \downarrow & & \downarrow \\ \mathbb{2} & \xrightarrow[i_{02}]{} & \mathbb{3} \end{array}$$

is also a pullback being the vertical composite of two pullbacks. Since Y is exponentiable, we know that the induced morphism

$$(\Gamma w \times Y) \sqcup_{X_t} (\Gamma v \times Y) \longrightarrow (\Gamma w \sqcup_{X_t} \Gamma v) \times Y$$

is invertible, it follows that the corresponding morphism

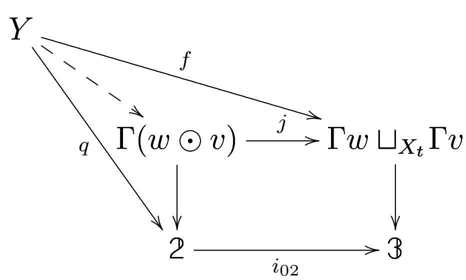
$$\Gamma\theta \colon \Gamma((w \times Y) \odot (v \times Y)) \longrightarrow \Gamma(w \odot v) \times Y$$

is invertible, as desired. ∎

Applying Theorem 4.3 and Lemma 4.6, we get:

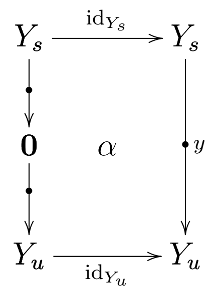
4.7. Corollary. *Suppose $\mathbb{D}$ is a glueing category satisfying the 02-pullback condition. Then Y is pre-exponentiable in $\mathbb{D}$ if and only if Y is exponentiable in $\mathbb{D}_0$.*

To prove the 02-pullback condition for our five examples, given $X_s \xrightarrow{v} X_t \xrightarrow{w} X_u$, consider the diagram

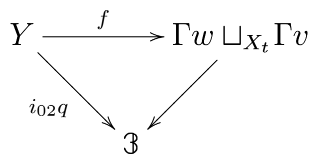


In each case, $\mathbb{D}$ has an object $\mathbf{0}$ which is horizontally initial, and vertically both initial and terminal. It is the empty object for $\mathbb{P}$os, $\mathbb{C}$at, and $\mathbb{T}$op, the one element locale for $\mathbb{L}$oc, and the one object topos for $\mathbb{T}$opos.

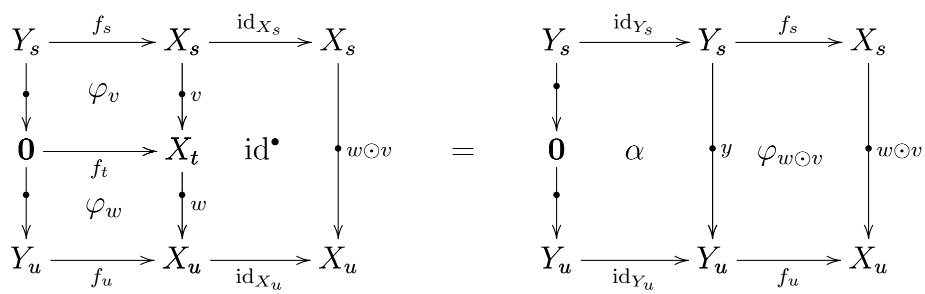
We know $q = \Gamma_2(y)$, for some $y\colon Y_s \to\!\!\bullet\!\!\to Y_u$. One can show that there is a unique cell



and Y is the lax colimit of this diagram. Moreover,



is induced by taking lax colimits of the diagram of cells



Thus, f is completely determined by the cell $\varphi_{w \odot v}$, and so there exists a unique morphism $\bar{f}\colon Y \longrightarrow \Gamma(w \odot v)$ making the pullback diagram commute. Therefore, the double categories in Examples 2.2–2.6 satisfy the 02-pullback condition.

Since $\mathbb{C}\mathrm{at}_0$ and $\mathbb{P}\mathrm{os}_0$ are cartesian closed, we get:

4.8. Corollary. $\mathbb{C}\mathrm{at}$ *and* $\mathbb{P}\mathrm{os}$ *are pre-cartesian closed double categories.*

4.9. Corollary. *A space Y is pre-exponentiable in* $\mathbb{T}\mathrm{op}$ *if and only if the set $\mathcal{O}(Y)$ of open subsets is a continuous lattice (in the sense of Scott [Sc72]), or equivalently, Y is locally compact, if Y is sober.*

Proof. This follows from Corollary 4.7 and the Day/Kelly characterization [DK70] of exponentiable spaces, and the Hoffman/Lawson characterization [HL78] of locally compact sober spaces. ∎

4.10. Corollary. *A locale Y is pre-exponentiable in* $\mathbb{L}\mathrm{oc}$ *if and only if Y locally compact, or equivalently, Y is a continuous lattice.*

Proof. This follows from Corollary 4.7 and Hyland characterization [H81] of exponentiable locales. ∎

4.11. Corollary. *A Grothendieck topos Y is pre-exponentiable in* $\mathbb{T}$opos *if and only if Y is a continuous category.*

Proof. This follows from Corollary 4.7 and the Johnstone/Joyal characterization [JJ82] of exponentiable toposes. ∎

4.12. Remark. Suppose $\mathbb{D}$ satisfies the 02-pullback condition and B is an object of $\mathbb{D}$. Given $X_s \overset{v}{-\!\!\bullet\!\!\rightarrow} X_t \overset{w}{-\!\!\bullet\!\!\rightarrow} X_u$, in $\mathbb{D}//B$, consider the commutative diagram

$$\begin{array}{ccc} \Gamma(w \odot v) & \longrightarrow & \Gamma w \sqcup_{X_t} \Gamma v \\ \downarrow & & \downarrow \\ B \times \mathbb{2} & \longrightarrow & B \times \mathbb{3} \\ \downarrow & & \downarrow \\ \mathbb{2} & \longrightarrow & \mathbb{3} \end{array}$$

By the 02-pullback condition, we know that the rectangle is a pullback. Since the bottom square is a pullback, in any case, it follows that the top square is a pullback, and so $\mathbb{D}//B$ satisfies the 02-pullback condition. Thus, we get:

4.13. Corollary. *Suppose $\mathbb{D}$ is a glueing category satisfying the 02-pullback condition. Then $Y \longrightarrow B$ is pre-exponentiable in $\mathbb{D}//B$ if and only if it is exponentiable in $\mathbb{D}_0/B$.*

Applying Corollary 4.13 to the five examples, we get the following results. A morphism $q\colon Y \longrightarrow B$ is pre-exponentiable in $\mathbb{C}$at if and it is a Giraud/Conduché fibration, i.e., it satisfies a factorization lifting property introduced independently in [G64] and [C72], and one is pre-exponentiable in $\mathbb{P}$os if and only if it is an interpolation-lifting map [N01], i.e., it satisfies a weak factorization lifting condition. By [N82a], pre-exponentiable morphisms in $\mathbb{T}$op are those satisfying a generalization of the Day/Kelly result [DK70]. For a sober space B with locally closed points, this is equivalent to the corresponding internal locale being locally compact in the topos Sh(B) of sheaves on B [N82b]. Since Hyland's theorem [H81] is constructive, one gets the analogous latter characterization for $\mathbb{L}$oc. Similarly, for a Grothendieck topos B, the pre-exponentiables in $\mathbb{T}$opos/B are the B-continuous categories.

For each of these double categories $\mathbb{D}$, the pre-exponentiable inclusions in $\mathbb{D}_0$ are precisely the "locally closed" ones (see [N81], [BN00], [N01]). These are the usual "intersections" of open and closed subspaces, sublocales, and toposes, and pullbacks of discrete fibrations and discrete opfibrations for posets and categories. In [N12a], we defined a locally closed inclusion in a double category as follows, and used cotabulators to give a general construction of their exponentials in $\mathbb{D}_0$ which applied to each of the five examples.

4.14. Definition. *Morphism* $i_s\colon X_s \to B$ *and* $i_t\colon X_t \to B$ *are called open inclusion and closed inclusions, respectively, if there is a cotabulator diagram in* $\mathbb{D}$ *of the form*

$$\begin{array}{ccc} X_s & & \\ {\scriptstyle v}\big\downarrow\!\bullet & \gamma_v & \overset{i_s}{\searrow}\; B \\ X_t & \overset{i_t}{\nearrow} & \end{array}$$

A morphism is locally closed if it is the pullback of open and closed inclusions.

5. Fibrant Glueing Categories

In this section, we recall the definition of a fibrant double category (also known as a framed bicategory), and show that our definition of glueing agrees with that of [N12a] when $\mathbb{D}$ is fibrant. Since our five main examples of double categories are fibrant, we thus obtain another justification that they satisfy our definition of glueing category.

5.1. Definition. [GP04] *A companion for a horizontal morphism* $f\colon X \to Y$ *is a vertical morphism* $f_*\colon X \to Y$ *together with cells*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}^\bullet_X}\big\downarrow & \alpha & \big\downarrow{\scriptstyle f_*} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f_*}\big\downarrow & \beta & \big\downarrow{\scriptstyle \mathrm{id}^\bullet_Y} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

whose horizontal and vertical compositions are identity cells.
A conjoint for f *is a vertical morphism* $f^*\colon Y \to X$ *together with cells*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}^\bullet_X}\big\downarrow & \rho & \big\downarrow{\scriptstyle f^*} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ {\scriptstyle f^*}\big\downarrow & \sigma & \big\downarrow{\scriptstyle \mathrm{id}^\bullet_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

whose horizontal and vertical compositions are identity cells.

5.2. Definition. [Sh08] *A double category* $\mathbb{D}$ *is called fibrant if every horizontal morphism has a companion and a conjoint.*

Our five examples are fibrant double categories. For $f\colon X \to Y$ in $\mathbb{C}\mathrm{at}$

$$f_* = Y(f-,-) \quad \text{and} \quad f^* = Y(-,f-)$$

and in $\mathbb{P}\mathrm{os}$

$$f_* = \{(x,y) | fx \le y\} \quad \text{and} \quad f^* = \{(y,x) | y \le fx\}$$

Companions in $\mathbb{L}$oc and $\mathbb{T}$opos are given by direct and inverse images, respectively. For $f\colon X \longrightarrow Y$ in $\mathbb{T}$op, f_* and f^* agree with those of $\mathcal{O}(f)\colon \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$ in $\mathbb{L}$oc.

Suppose $\mathbb{D}$ is a fibrant double category, and let $\Phi_2\colon \mathbb{D}_0/2 \longrightarrow \mathbb{D}_1$ be the functor defined on objects by $\Phi_2(X \xrightarrow{p} 2) = (X_s \xrightarrow{j_{s*}} X \xrightarrow{j_t{}^*} X_t)$, where j_s and j_t are defined by the pullbacks

$$\begin{array}{ccc} X_s & \xrightarrow{j_s} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[i_s]{} & 2 \end{array} \qquad\qquad \begin{array}{ccc} X_t & \xrightarrow{j_t} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[i_t]{} & 2 \end{array}$$

Given a morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & 2 & \end{array}$$

$\Phi_2(f)$ is the vertical composition of the cells

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle j_{s*}}\downarrow & \varphi_s & \downarrow{\scriptstyle j_{s*}} \\ X & \xrightarrow[f]{} & Y \\ {\scriptstyle j_t{}^*}\downarrow & \varphi_t & \downarrow{\scriptstyle j_t{}^*} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

where f_s and f_t are induced by the pullbacks defining $\Phi_2(p)$ and $\Phi_2(q)$. Note that the left square of

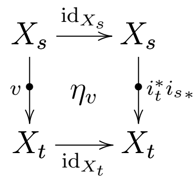
$$\begin{array}{ccccc} X_s & \xrightarrow{f_s} & Y_s & \longrightarrow & 1 \\ {\scriptstyle j_s}\downarrow & & \downarrow{\scriptstyle j_s} & & \downarrow{\scriptstyle i_s} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[q]{} & 2 \end{array}$$

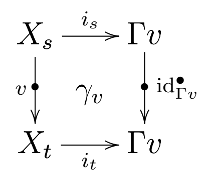
is a pullback, since the right square and the outer rectangle are pullbacks, by definition of Φ_2. The square defining f_t is similarly seen to be a pullback.

In [N12a], fibrant double categories with 2-glueing are defined to be those for which the functor Φ_2, defined above, is an equivalence of categories. Thus, this agrees with the definition of glueing category given in Section 2 when $\mathbb{D}$ is fibrant, if we can show that $\Gamma_2 \dashv \Phi_2$, since any right adjoint to Γ_2 will be an equivalence of categories when Γ_2 is.

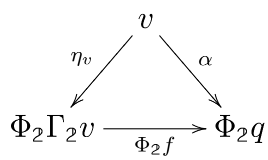
5.3. Proposition. *If $\mathbb{D}$ is a fibrant double category with cotabulators satisfying the pullback condition (2.11), then the functor Φ_2, defined above, is right adjoint to Γ_2.*

Proof. Let $\eta_v\colon v \longrightarrow \Phi_2\Gamma_2 v$ denote that cell given by the diagram on the left

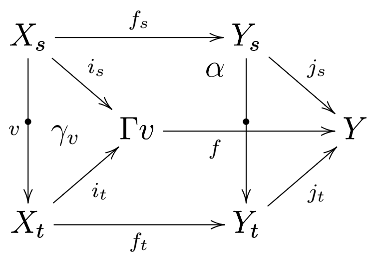




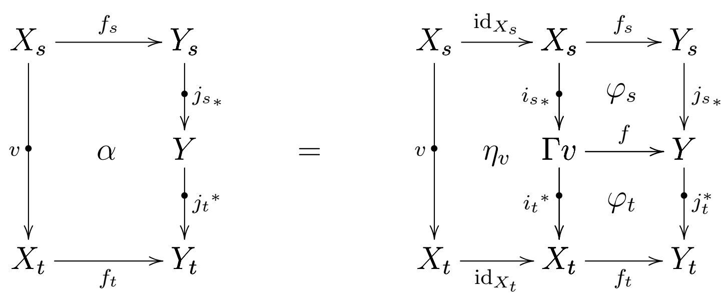
To show that $\Gamma_2 \dashv \Phi_2$, we will show that for all $q\colon Y \longrightarrow 2$ and all cells $\alpha\colon v \longrightarrow \Phi_2 q$, there exists a unique $f\colon \Gamma_2 v \longrightarrow q$ such that



Since there exists f such that the diagram



of cells commutes if and only if there exists f such that



the desired result follows. ■

Since our five examples are fibrant double categories, we get the following:

5.4. Corollary. *The double categories in Examples 2.2-2.6 are glueing categories.* ■

6. Counterexamples

In this section, we show that $(\mathbb{P}\mathrm{os}//2)_1$ is not cartesian closed. Thus, $\mathbb{D}$ pre-cartesian closed does not imply that $\mathbb{D}_1$ is cartesian closed. The only objects of $\mathbb{D}_1$ that need to be exponentiable are the vertical identities $\mathrm{id}^\bullet_Y$. We conclude with an example to show why we restricted to right adjoints in **LxDbl** rather than **PsDbl**. In particular, we will see that the right adjoints that exist in $\mathbb{P}\mathrm{os}$ are not pseudo even though $\mathbb{P}\mathrm{os}$ is cartesian, i.e., $- \times Y$ is pseudo. Thus, $\mathbb{P}\mathrm{os}$ (and $\mathbb{C}\mathrm{at}$) are pre-cartesian closed but not cartesian closed.

The first counterexample is straighorward. By Proposition 2.13, we know

$$(\mathbb{P}\mathrm{os}/\!/2)_1 \simeq \mathbb{P}\mathrm{os}/(2 \times 2)$$

and the latter is not cartesian closed, since one easily shows that the weak factorization lifting property [N01] does not hold.

The second one is elementary, but technical. To see that the right adjoints are not pseudo in $\mathbb{P}\mathrm{os}$, we will show that $(\ )^Y$ does not preserve vertical composition when $Y = 2$, i.e., $(J \odot I)^2 \neq J^2 \odot I^2$, for some $Z_s \overset{I}{\longrightarrow\!\!\!\!\!\bullet\ } Z_t \overset{J}{\longrightarrow\!\!\!\!\!\bullet\ } Z_u$.

Recall that vertical morphisms $I\colon Z_s \longrightarrow\!\!\!\!\!\bullet\ Z_t$ in $\mathbb{P}\mathrm{os}$ are order ideals, i.e., $I \subseteq Z_s \times Z_t$ satisfying

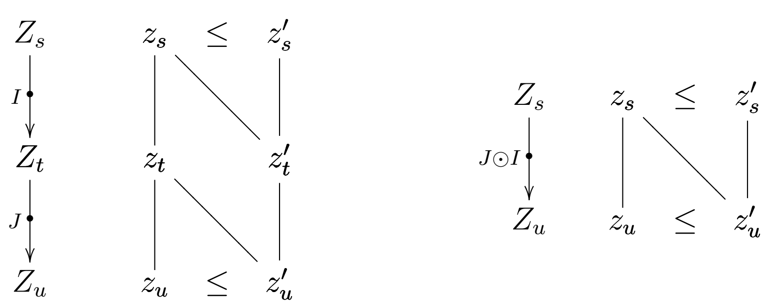
$$(z_s, z_t) \in I,\ z'_s \leq z_s,\ z_t \leq z'_t \implies (z'_s, z'_t) \in I$$

with the usual composition of relations. Using the construction of exponentials in $\mathbb{P}\mathrm{os}/\!/2$ given in [N01] and the equivalence $\mathbb{P}\mathrm{os}_1 \simeq \mathbb{P}\mathrm{os}/\!/2$, we see that

$$I^Y = \{(\sigma_s, \sigma_t) \in Z_s^Y \times Z_t^Y \mid (\sigma_s(y), \sigma_t(y)) \in I, \forall y \in Y\}$$

where Z^Y is the usual exponential of posets, i.e., the set of order preserving maps $Y \longrightarrow Z$.

Consider



where $Z_s = \{z_s \leq z'_s\}$, $Z_t = \{z_t, z'_t\}$, and $Z_u = \{z_u \leq z'_u\}$, where Z_t is discrete, and let $I = \{(z_s, z_t), (z_s, z'_t), (z'_s, z'_t)\}$ and $J = \{(z_t, z_u), (z_t, z'_u), (z'_t, z'_u)\}$. Then $J \odot I = \{(z_s, z_u), (z_s, z'_u), (z'_s, z'_u)\}$. One can visualize these ideals as:

Define $\sigma_s\colon 2 \longrightarrow Z_s$ and $\sigma_u\colon 2 \longrightarrow Z_u$ by $\sigma_s(0) = z_s$, $\sigma_s(1) = z'_s$, $\sigma_u(0) = z_u$, and $\sigma_u(1) = z'_u$. Then $(\sigma_s, \sigma_u) \in (J \odot I)^2$. To see that $(\sigma_s, \sigma_u) \notin J^2 \odot I^2$, suppose $(\sigma_s, \sigma_t) \in I^2$ and $(\sigma_t, \sigma_u) \in J^2$, for some $\sigma_t \in Z_t^2$. Then $\sigma_t(0) \leq \sigma_t(1)$, but $\sigma_t(0) = z_t$, since $(\sigma_t(0), z_u) = (\sigma_t(0), \sigma_u(0)) \in J$, and $\sigma_t(1) = z'_t$, since $(z'_s, \sigma_t(1)) = (\sigma_s(1), \sigma_t(1)) \in I$, contradicting that Z_t is discrete.

Therefore, $\mathbb{P}\mathrm{os}$, and similarly, $\mathbb{C}\mathrm{at}$, are not cartesian closed.

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Union College
Department of Mathematics
Schenectady, NY 12308
Email: niefiels@union.edu

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