Theory and Applications of Categories, Vol. 36, No. 10, 2021, pp. 280–287.

MINIMAL ACCESSIBLE CATEGORIES

Dedicated to Robert Rosebrugh in gratitude for all his work for the journal

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Abstract. We give a purely category-theoretic proof of the result of Makkai and Paré saying that the category **Lin** of linearly ordered sets and order preserving injective mappings is a minimal finitely accessible category. We also discuss the existence of a minimal $\aleph_1$-accessible category.

1. Introduction

One of striking results of [8] is that the category **Lin** of linearly ordered sets and order preserving injective mappings is a minimal finitely accessible category. This means that for every large finitely accessible category $\mathcal{K}$ there is a faithful functor $\mathbf{Lin} \longrightarrow \mathcal{K}$ preserving directed colimits. [8] does not contain a proof of this result – Makkai and Paré just say that it essentially follows from the work of Morley [9]. Since there are many applications of this result (see, e.g., [5]), it might be useful to give an explicit proof of it. We do it by transferring the standard model-theoretic argument to the language of accessible categories. Another, more model-theoretic proof, of the theorem of Makkai and Paré was recently given by Boney [2].

Model-theoretically, the minimality of **Lin** means the existence of order indiscernibles and the proof uses the infinitary combinatorial argument called Erdös-Rado theorem. In order to apply it on a finitely accessible category $\mathcal{K}$, we need a faithful functor $U\colon \mathcal{K} \longrightarrow \mathbf{Set}$ preserving directed colimits such that every subset $Z \subseteq UK$ generates the smallest subobject $\langle Z \rangle$ of K and, moreover, any two morphisms $f, g\colon \langle Z \rangle \longrightarrow L$ such that Uf and Ug have the same restriction on Z are equal. We show that every finitely accessible category $\mathcal{K}$ admits a faithful directed colimits preserving functor $\mathcal{L} \longrightarrow \mathcal{K}$ from a finitely accessible category $\mathcal{L}$ with this property. Hence it suffices to construct $\mathbf{Lin} \longrightarrow \mathcal{L}$. The category $\mathcal{L}$ can be considered as a skolemization of $\mathcal{K}$.

The minimality of **Lin** among finitely accessible categories implies its minimality among (∞, ω)-elementary categories (see [8] 3.4.1) and, even, among accessible categories with directed colimits whose morphisms are monomorphisms ([5] 2.5). One cannot expect that **Lin** is a minimal accessible category because there is no faithful functor from **Lin** to

Supported by the Grant Agency of the Czech Republic under the grant 19-00902S

Received by the editors 2020-09-13 and, in final form, 2021-04-27.

Published on 2021-05-10 in the Rosebrugh Festschrift.

2020 Mathematics Subject Classification: 18C35, 03C48.

Key words and phrases: accessible category, indiscernibles, linear order.

the $\aleph_1$-accessible category of well ordered sets and order preserving injective mappings. The reason is that any well ordered set A is iso-rigid, it means that every isomorphism $A \longrightarrow A$ is the identity. Using [6], we give an example of a $\aleph_1$-accessible category $\mathcal{K}$ having every object K rigid, i.e., every morphism $K \longrightarrow K$ is the identity. This yields a candidate for a minimal $\aleph_1$-accessible category. Similarly, one gets a candidate for a minimal $\aleph_\alpha$-accesible category.

Acknowledgement. We are grateful to T. Beke and the referee for valuable comments to this paper.

2. Skolem cover

Let $\mathcal{K}$ be a finitely accessible category and $\mathcal{A}$ its representative small full subcategory of finitely presentable objects (i.e., any finitely presentable object of $\mathcal{K}$ is isomorphic to some $A \in \mathcal{A}$). Let

$$E\colon \mathcal{K} \longrightarrow \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

be the canonical embedding that takes each $K \in \mathcal{K}$ to the contravariant functor $\mathcal{K}(-, K)\colon\ \mathcal{A} \longrightarrow \mathbf{Set}$. We note that, by Proposition 2.8 in [1], this functor is fully faithful and preserves directed colimits and finitely presentable objects. Following Theorem 4.17 in [1], $\mathcal{K}$ is equivalent to a finitary-cone-injectivity class $\mathrm{Inj}(T)$ in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$; this means that there is a set T of cones $a = (a_i\colon X \longrightarrow EA_i)_{i\in I}$ where X is finitely presentable in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ and $A_i \in \mathcal{A}$, $i \in I$ such that $\mathrm{Inj}(T)$ consists of functors F injective to each cone $a \in T$. The latter means that for any morphism $f\colon X \longrightarrow F$ there is $i \in I$ and $g\colon EA_i \longrightarrow F$ with $ga_i = f$. Let $S(\mathcal{K})$ be the category whose objects are $(F, a_F)_{a\in T}$ consisting of $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ with a_F assigning to a cone a and $f\colon X \longrightarrow F$ a morphism $a_F(f)\colon EA_i \longrightarrow F$ for some $i \in I$ such that $a_F(f)a_i = f$. Morphisms $(F, a_F) \longrightarrow (F', a_{F'})$ are natural transformations $\varphi\colon F \longrightarrow F'$ such that $a_{F'}(\varphi f) = \varphi a_F(f)$. The forgetful functor $G\colon S(\mathcal{K}) \longrightarrow \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is faithful and has values in $\mathrm{Inj}(T)$. Its codomain restriction $S(\mathcal{K}) \longrightarrow \mathrm{Inj}(T)$ is surjective on objects. Since $E\colon \mathcal{K} \longrightarrow \mathrm{Inj}(T)$ is an equivalence, we get a faithful functor $H\colon S(\mathcal{K}) \longrightarrow \mathcal{K}$ which is essentially surjective on objects, i.e., any $K \in \mathcal{K}$ is isomorphic to some $H(F, a_F)$.

2.1. Lemma. *The category $S(\mathcal{K})$ is finitely accessible and $H\colon S(\mathcal{K}) \longrightarrow \mathcal{K}$ preserves directed colimits.*

Proof. Let $D\colon \mathcal{D} \longrightarrow S(\mathcal{K})$ be a directed diagram and consider the colimit $\delta\colon GD \longrightarrow F$ in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$. Let $a = (a_i\colon X \longrightarrow EA_i)_{i\in I}$ be a cone and $f\colon X \longrightarrow F$ a morphism. Since X is finitely presentable, there is $d \in \mathcal{D}$ and $g\colon X \longrightarrow GDd$ such that $f = \delta_d g$. We put $a_F = \delta_d a_{Dd}(g)$. Then $(F, a_F) = \operatorname{colim} D$. Thus $S(\mathcal{K})$ has directed colimits and G preserves them. Hence H preserves them too.

If F is finitely presentable in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ then any (F, a_F) is finitely presentable in $S(\mathcal{K})$. In order to show that any (F, a_F) is a directed colimit of finitely presentable objects in $S(\mathcal{K})$ it suffices to express F as a directed colimit of finitely presentable objects F_d in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$

and complete them to (F_d, a_{F_d}) using finite presentability of X again. Then (F, a_F) is a directed colimit of (F_d, a_{F_d}). Thus $S(\mathcal{K})$ is finitely accessible.

∎

In fact, we have shown that

$$S(\mathcal{K}) = S(\operatorname{Ind}\mathcal{A}) = \operatorname{Ind} S(\mathcal{A})$$

$S(\mathcal{K})$ will be called a *Skolem cover* of $\mathcal{K}$ because it is a skolemization of the $L_{\infty,\omega}$-theory corresponding to T. Let $U\colon \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}} \longrightarrow \mathbf{Set}$ assign to F the set $\coprod_{A\in\mathcal{A}} FA$. The functor U is faithful and preserves directed colimits. Thus $(\mathcal{K}, UE)$ and $(S(\mathcal{K}), UG)$ are concrete finitely accessible categories with concrete directed colimits and $H\colon S(\mathcal{K}) \longrightarrow \mathcal{K}$ is a concrete functor.

2.2. Lemma. *Let $(F, a_F) \in S(\mathcal{K})$ and $Z \subseteq UG(F, a_F)$. Then there is the smallest subobject (F_Z, a_{F_Z}) of (F, a_F) such that $Z \subseteq UGF_Z$.*

Moreover, for every $g, h\colon (F_Z, a_{F_Z}) \longrightarrow (F', a_{F'})$, we have $g = h$ provided that Ug and Uh have the same restriction on Z.

Proof. Let F_0 be the smallest subfunctor of F such that $Z \subseteq UF_0$; let $\sigma\colon F_0 \longrightarrow F$ denote the inclusion. Consider a cone $a\colon X \longrightarrow EA_i$ in T and a morphism $f\colon X \longrightarrow F_0$. Then $\sigma f = a_F(f)a_i$. Let F_1 be the smallest subfunctor of F containing $U(F_0)$ and the images of $U(a_F(f))$ for all cones a and all morphisms f. We iterate this construction by replacing F_0 with F_1, etc. In this way, we get the chain $F_0 \longrightarrow F_1 \longrightarrow \dots F_n \longrightarrow \dots$. Then $F_Z = \operatorname{colim} F_n$ carries the desired smallest subobject of (F, a_F). In fact, every $f\colon X \longrightarrow F_Z$ factorizes through some $F_n \longrightarrow F$ because X is finitely presentable.

Consider $g, h\colon (F_Z, a_{F_Z}) \longrightarrow (F', a_{F'})$ such that Ug and Uh have the same restriction on Z. Then Ug and Uh have the same restriction of UF_0. Consequently, they have the same restriction on UF_1, etc. Hence $g = h$.

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This is the virtue of the skolemization and reflects the fact that the skolemized theory is universal. We skolemized cone-injectivity while algebraic factorization systems (see [4]) skolemize injectivity. J. Bourke [3] came to the same point from a different motivation.

2.3. Remark. For any Z, there is only a set of non-isomorphic (F_Z, a_{F_Z}), $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

Indeed, let κ be greater than the number of morphisms of $\mathcal{A}$ and the cardinalities of UEA_i of cones a of T. Then, for every $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and $Z \subseteq UF$, the smallest subfunctor of F such that $Z \subseteq UF_0$ has UF_0 of cardinality smaller than κ. Consequently, every UF_Z has cardinality smaller than κ.

3. Minimal finitely accessible categories

3.1. Theorem. *For any large finitely accessible category $\mathcal{K}$ there is a faithful functor $\mathbf{Lin} \longrightarrow \mathcal{K}$ preserving directed colimits.*

PROOF. Following 2.2, we can assume that $\mathcal{K}$ is equipped with a faithful functor $U : \mathcal{K} \longrightarrow \mathbf{Set}$ preserving directed colimits and such that for any subset $Z \subseteq UK$ there is the smallest subobject K_Z of K such that $Z \subseteq UK_Z$. Let $\mathcal{L}$ be the category with objects (K, X) where $K \in \mathcal{K}$ and $X \subseteq UK$ is linearly ordered. Morphisms $(K_1, X_1) \longrightarrow (K_2, X_2)$ are morphisms $f\colon K_1 \longrightarrow K_2$ such that the restriction of Uf to X_1 is an order preserving injective mapping to X_2. The category $\mathcal{L}$ has directed colimits given as

$$\operatorname{colim}(K_i, X_i) = (\operatorname{colim} K_i, \operatorname{colim} X_i)$$

and any (K, X) with K finitely presentable in $\mathcal{K}$ and X finite is finitely presentable in $\mathcal{L}$. Thus $\mathcal{L}$ is finitely accessible and the forgetful functor $\mathcal{L} \longrightarrow \mathcal{K}$ preserves directed colimits.

For a $\mathcal{L}$-object (K, X), let $\rho_{(K,X)}$ be the greatest ordinal $0 < \rho_{(K,X)} \le \omega, |X|$ such that for any $n < \rho_{(K,X)}$ and any $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$ in X there is an isomorphism $s\colon K_{\{a_1,\dots,a_n\}} \longrightarrow K_{\{b_1,\dots,b_n\}}$ such that $Us(a_i) = b_i$ for $i = 1, \dots, n$. Assume that there is $(K, X) \in \mathcal{L}$ with $\rho_{(K,X)} = \omega$. Then X is infinite and thus it contains a chain $a_1 < a_2 < \cdots < a_n \dots$ or a chain $a_1 > a_2 > \cdots > a_n \dots$. Indeed, if X does not contain a strictly decreasing countable chain, it is well-ordered and hence it contains a strictly increasing countable chain.

Assume that X contains a chain $a_1 < a_2 < \cdots < a_n \dots$ (the other case is analogous). We will construct a functor $F\colon \mathbf{Lin} \longrightarrow \mathcal{K}$ as follows. Finitely presentable objects in $\mathbf{Lin}$ are finite chains C_n with elements $1 < 2 < \cdots < n$. Put $F_0(C_n) = K_{a_1,\dots,a_n}$. Given an injective order preserving mapping $h\colon C_m \longrightarrow C_n$, let $F_0 h$ be the composition

$$K_{a_1,\dots,a_m} \longrightarrow K_{a_{h(1)},\dots,a_{h(m)}} \longrightarrow K_{a_1,\dots,a_n}$$

where the first morphism is the isomorphism s above and the second morphism is the inclusion. Given $h_1\colon C_k \longrightarrow C_m$ and $h_2\colon C_m \longrightarrow C_n$ then, following 2.2, $F_0(h_2 h_1) = F_0(h_2)F_0(h_1)$. In fact, we always get the composition

$$K_{a_1,\dots,a_k} \longrightarrow K_{a_{h_2h_1(1)},\dots a_{h_2h_1(k)}}$$

followed by the inclusion $K_{a_{h_2h_1(1)},\dots a_{h_2h_1(k)}} \longrightarrow K_{a_1,\dots,a_n}$. Thus we get the functor $F_0\colon \mathbf{FinLin} \longrightarrow \mathcal{K}$ defined on finite linear orderings. This functor is faithful because if $h, h'\colon C_m \longrightarrow C_n$ have distinct values on i then $F_0(h)(a_i) = a_{hi} \ne a_{h'i} = F_0(h')(a_i)$. Since $\mathbf{Lin} = \operatorname{Ind} \mathbf{FinLin}$, F_0 extends to a functor $F\colon \mathbf{Lin} \longrightarrow \mathcal{K}$ preserving directed colimits. Since F_0 is faithful, F is faithful too.

Assume that $\rho_{(K,X)} < \omega$ for any $(K, X) \in \mathcal{L}$. We put $(K_1, X_1) < (K_2, X_2)$ provided that $\rho_{(K_2,X_2)} < \rho_{(K_1,X_1)}$ and $(K_1)_{\{a_1,\dots,a_{\rho_{(K_2,X_2)}}\}} \cong (K_2)_{\{b_1,\dots,b_{\rho_{(K_2,X_2)}}\}}$ for any $a_1 < \cdots < a_{\rho_{(K_2,X_2)}}$ in X_1 and any $b_1 < \cdots < b_{\rho_{(K_2,X_2)}}$ in X_2. Then $<$ partially orders objects of $\mathcal{L}$ and this order is well-founded in the sense that there is no decreasing chain

$$\cdots < (K_n, X_n) < (K_{n-1}, X_{n-1}) < \cdots < (K_1, X_1)\,.$$

Such chain would yield a diagram

$$(K_1)_{\{a_{11}\}} \longrightarrow (K_2)_{\{a_{21},a_{22}\}} \longrightarrow (K_n)_{\{a_{n1},\dots,a_{nn}\}} \longrightarrow \dots$$

whose colimit (K,X) in $\mathcal{L}$ has $\rho_{(K,X)} = \omega$. Thus we can assign an ordinal $\alpha(K,X)$ to each $(K,X) \in \mathcal{L}$ in such a way that

$$\alpha(K,X) = \sup_{(K',X')<(K,X)} \alpha(K',X') + 1\,.$$

Following 2.3, there is an infinite cardinal μ greater or equal to the number of non-isomorphic objects K_Z for Z finite and K arbitrary. For $(K,X) \in \mathcal{L}$, choose $a_1 < \dots < a_{\rho_{(K,X)}-1}$ in X and put

$$(K,X)^* = (K_{\{a_1,\dots,a_{\rho_{(K,X)}-1}\}}, X \cap UK_{\{a_1,\dots,a_{\rho_{(K,X)}-1}\}})\,.$$

We will prove that

$$|X| < \exp_{\omega(\alpha(K,X)^*+1)}(\mu)$$

for any $(K,X) \in \mathcal{L}$. Recall that $\exp_0(\mu) = \mu$, $\exp_{\xi+1}(\mu) = 2^{\exp_\xi(\mu)}$ and $\exp_\eta(\mu) = sup_{\xi<\eta} \exp_\xi(\mu)$. Since every set UK can be linearly ordered, $(K,UK) \in \mathcal{L}$ for any K in $\mathcal{K}$, this inequality implies that $\mathcal{K}$ is small.

The proof will use the recursion on $\alpha(K,X)^*$. Let $\alpha(K,X)^* = 0$ and assume that $|X| \geq \exp_\omega(\mu)$. For $n = \rho_{(K,X)}$, the set $[X]^n$ of the subsets of X of size n is decomposed into $\leq \mu$ parts following isomorphisms types of $K_{\{a_1,\dots,a_n\}}$. Following the Erdös-Rado partition theorem (see [7], Exercise 29.1), there is $X_0 \subseteq X$ such that $|X_0| > \mu$ and $K_{\{a_1,\dots,a_n\}} \cong K_{\{b_1,\dots,b_n\}}$ for any $a_1 < \dots < a_n$ and $b_1 < \dots < b_n$ in X_0. For $m < n$, the least subobjects associated to chains $a_1 < \dots a_m$ in X_0 are isomorphic because this is true for such chains in X. Thus

$$(K,X_0)^* = (K_{\{a_1,\dots,a_n\}}, X_0 \cap UK_{\{a_1,\dots,a_n\}}) < (K,X)^*,$$

which is impossible because $\alpha(K,X)^* = 0$.

Assume that the claim holds for any $(K,X) \in \mathcal{L}$ with $\alpha(K,X)^* < \beta$ and consider $(L,Y) \in \mathcal{L}$ with $\alpha(L,Y)^* = \beta$. Assume that $|Y| \geq \exp_{\omega(\alpha(L,Y)^*+1)}(\mu)$ and let $n = \rho_{(L,Y)}$. We have

$$|Y| \geq \exp_{\omega(\beta+1)}(\mu) > \exp_{\omega\beta+n-1}(\mu) = \exp_{n-1}(\exp_{\omega\beta}(\mu)).$$

Following the Erdös-Rado partition theorem, there is $Y_0 \subseteq Y$ such that $|Y_0| > \exp_{\omega\beta}(\mu)$ and $L_{\{b_1,\dots,b_n\}} \cong L_{\{c_1,\dots,c_n\}}$ for each $b_1 < \dots < b_n$ and $c_1 < \dots < c_n$ in Y_0. Then $\rho_{(L,Y_0)} > n$ and $(L,Y_0) < (L,Y)$. For $m < n$, the least subobjects associated to chains $a_1 < \dots a_m$ in Y_0 are isomorphic because this is true for such chains in Y. Thus $(L,Y_0)^* < (L,Y)^*$. Hence $\alpha(L,Y_0)^* < \beta$ and thus

$$|Y_0| < \exp_{\omega(\alpha(L,Y_0)^*+1)}(\mu) \leq \exp_{\omega\beta}(\mu),$$

which is a contradiction. ■

4. Towards minimal λ-accessible categories

4.1. Example. The category $\mathcal{W}$ of well-ordered sets is $\aleph_1$-accessible and any its object K is iso-rigid in the sense that the only isomorphism $K \longrightarrow K$ is the identity. Thus there is no faithful functor **Lin** $\longrightarrow \mathcal{W}$ and a prospective minimal $\aleph_1$-accessible category is iso-rigid.

4.2. Example. There is an $\aleph_1$-accessible category $\mathcal{L}$ having all objects K rigid in the sense that the only morphism $K \longrightarrow K$ is the identity. Thus there is no faithful functor $\mathcal{W} \longrightarrow \mathcal{L}$.

The construction of $\mathcal{L}$ is motivated by [6], II.3. Let $\mathcal{K}$ be the category of structures $(A, <, \sup, s, S, R)$ where $<$ is a well-ordering, sup is the countable join, s is the unary operation of taking the successor, S is an ω-ary relation choosing for every ordinal a of cofinality ω a countable increasing chain with the join a and R is a unary relation choosing ordinals of cofinality $> \omega$. This is achieved by taking the following set T of axioms:

1. $(\forall x_0, x_1, y_1, \dots, x_n, y_n, \dots)(S(x_0, x_1, \dots, x_n, \dots) \wedge S(x_0, y_1, \dots, y_n, \dots) \longrightarrow \bigwedge_{0<n}(x_n = y_n))$
2. $(\forall x_0, x_1, \dots, x_n, \dots)(S(x_0, x_1 \dots, x_n \dots) \longrightarrow (\bigwedge_{0<n} x_n < x_{n+1}) \wedge x_0 = \sup x_n)$
3. $(\forall x)(\exists y_1, \dots, y_n, \dots)((\bigwedge_{0<n}(y_n < y_{n+1}) \wedge x = \sup y_n) \longrightarrow (\exists x_1, \dots, x_n, \dots) S(x, x_1, \dots, x_n, \dots))$
4. $(\forall x)(R(x) \leftrightarrow \neg(\exists y)(x = s(y)) \wedge \neg(\exists x_1, \dots, x_n, \dots) S(x, x_1, \dots, x_n, \dots))$

Let A_2 be the set of isolated elements of A, A_0 be the set of all limit elements of $a \in A$ such that $S(a, a_1, \dots, a_n, \dots)$ for some $a_1, \dots, a_n, \dots \in A$ and $A_1 = A \setminus (A_0 \cup A_2)$. All the sets A_2, A_0 and A_1 are preserved by homomorphisms $f \colon A \longrightarrow B$ (due to the preservation of s, S and R resp.).

This category clearly has $\aleph_1$-directed colimits. Objects A of $\mathcal{L}$ generated by 0 are ordinals ω_1 with a choice of S for every $a \in A_0$. Thus there is $\aleph_0^{\aleph_1} = 2^{\aleph_1}$ such objects. These objects are $\aleph_1$-presentable and the same is true for objects $\omega_1 \cdot \alpha$ where $\alpha < \omega_1$. Clearly, every object of $\mathcal{L}$ is an $\aleph_1$-directed colimit of these objects $\omega_1 \cdot \alpha$, $\alpha < \omega_1$. Thus $\mathcal{L}$ is $\aleph_1$-accessible.

Assume that there exists a morphism $f \colon A \longrightarrow A$ in $\mathcal{L}$ which is not the identity. Let a be the least element in A such that $f(a) \neq a$. Since A is a well-ordered set and f is injective, $a < f(a)$. Hence

$$a < f(a) < f^2(a) < \dots < f^n(a) < \dots$$

Let $b = \sup f^n(a)$. Hence $f(b) = \sup f^{n+1}(a) = b$. There are $b_1 < b_2 < \dots < b_n < \dots$ such that $S(b, b_1, \dots, b_n, \dots)$. Since $S(f(b), f(b_1), \dots, f(b_n), \dots)$ and $f(b) = b$, we have $f(b_n) = b_n$ for each n. Since $a < b_m$ for some m, $f^n(a) < b_m$ for each n. Hence $b \leq b_m$, which is a contradiction.

4.3. Remark. (1) Let $\mathcal{L}_1$ be a full subcategory of $\mathcal{L}$ where we choose S for every $a \in A_0$ in every object generated by 0. This category does not depend of the choices of S and is also $\aleph_1$-accessible. In fact, it is $\mathrm{Ind}_{\aleph_1}(\mathcal{C}_1)$ where $\mathcal{C}_1$ is the category of ordinals $\omega_1 \cdot \alpha$, $\alpha < \omega_1$ with non-identity morphisms

$$\omega_1 \cdot f \colon \omega \cdot \alpha \longrightarrow \omega_1 \cdot \beta$$

where $f \colon \alpha \longrightarrow \beta$ is an order preserving injective mapping with $\alpha < \beta$. The category $\mathcal{C}_1$ is, in fact, the category of ordinals $\alpha < \omega_1$ where non-identity morphisms are order preserving injective mappings $\alpha \longrightarrow \beta$ for $\alpha < \beta < \omega_1$.

(2) The category **FinLin** is the category $\mathcal{C}_0$ of ordinals $\alpha < \omega$ where non-identity morphisms are order preserving injective mapping $\alpha \longrightarrow \beta$ for $\alpha < \beta < \omega$. Observe that **FinLin** is rigid, i.e., the only morphisms $\alpha \longrightarrow \alpha$ are the identities. Hence $\mathcal{L}_1 = \mathrm{Ind}_{\omega_1} \mathcal{C}_1$ is $\aleph_1$-modification of a minimal $\aleph_0$-accessible category **Lin**.

(3) Let $\mathcal{C}_\gamma$ be the category of ordinals $\alpha < \omega_\gamma$ where non-identity morphisms are order preserving injective mapping $\alpha \longrightarrow \beta$ for $\alpha < \beta < \omega_\gamma$. Then $\mathcal{L}_\gamma = \mathrm{Ind}_{\aleph_\gamma} \mathcal{C}_\gamma$ is an $\aleph_\gamma$-accessible category.

4.4. Problem. Is $\mathcal{L}_1$ a minimal $\aleph_1$-accessible category? This means that for every large $\aleph_1$-accessible category $\mathcal{K}$ there is a faithful functor $\mathcal{L}_1 \longrightarrow \mathcal{K}$ preserving $\aleph_1$-directed colimits.

Similarly, is $\mathcal{L}_\gamma$ a minimal $\aleph_\gamma$-accessible category for $0 < \gamma$?

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