

## MINIMAL ACCESSIBLE CATEGORIES

*Dedicated to Robert Rosebrugh in gratitude for all his work for the journal*

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ABSTRACT. We give a purely category-theoretic proof of the result of Makkai and Paré saying that the category **Lin** of linearly ordered sets and order preserving injective mappings is a minimal finitely accessible category. We also discuss the existence of a minimal  $\aleph_1$ -accessible category.

### 1. Introduction

One of striking results of [8] is that the category **Lin** of linearly ordered sets and order preserving injective mappings is a minimal finitely accessible category. This means that for every large finitely accessible category  $\mathcal{K}$  there is a faithful functor  $\mathbf{Lin} \rightarrow \mathcal{K}$  preserving directed colimits. [8] does not contain a proof of this result – Makkai and Paré just say that it essentially follows from the work of Morley [9]. Since there are many applications of this result (see, e.g., [5]), it might be useful to give an explicit proof of it. We do it by transferring the standard model-theoretic argument to the language of accessible categories. Another, more model-theoretic proof, of the theorem of Makkai and Paré was recently given by Boney [2].

Model-theoretically, the minimality of **Lin** means the existence of order indiscernibles and the proof uses the infinitary combinatorial argument called Erdős-Rado theorem. In order to apply it on a finitely accessible category  $\mathcal{K}$ , we need a faithful functor  $U: \mathcal{K} \rightarrow \mathbf{Set}$  preserving directed colimits such that every subset  $Z \subseteq UK$  generates the smallest sub-object  $\langle Z \rangle$  of  $K$  and, moreover, any two morphisms  $f, g: \langle Z \rangle \rightarrow L$  such that  $Uf$  and  $Ug$  have the same restriction on  $Z$  are equal. We show that every finitely accessible category  $\mathcal{K}$  admits a faithful directed colimits preserving functor  $\mathcal{L} \rightarrow \mathcal{K}$  from a finitely accessible category  $\mathcal{L}$  with this property. Hence it suffices to construct  $\mathbf{Lin} \rightarrow \mathcal{L}$ . The category  $\mathcal{L}$  can be considered as a skolemization of  $\mathcal{K}$ .

The minimality of **Lin** among finitely accessible categories implies its minimality among  $(\infty, \omega)$ -elementary categories (see [8] 3.4.1) and, even, among accessible categories with directed colimits whose morphisms are monomorphisms ([5] 2.5). One cannot expect that **Lin** is a minimal accessible category because there is no faithful functor from **Lin** to

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the  $\aleph_1$ -accessible category of well ordered sets and order preserving injective mappings. The reason is that any well ordered set  $A$  is iso-rigid, it means that every isomorphism  $A \rightarrow A$  is the identity. Using [6], we give an example of a  $\aleph_1$ -accessible category  $\mathcal{K}$  having every object  $K$  rigid, i.e., every morphism  $K \rightarrow K$  is the identity. This yields a candidate for a minimal  $\aleph_1$ -accessible category. Similarly, one gets a candidate for a minimal  $\aleph_\alpha$ -accessible category.

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## 2. Skolem cover

Let  $\mathcal{K}$  be a finitely accessible category and  $\mathcal{A}$  its representative small full subcategory of finitely presentable objects (i.e., any finitely presentable object of  $\mathcal{K}$  is isomorphic to some  $A \in \mathcal{A}$ ). Let

$$E: \mathcal{K} \rightarrow \mathbf{Set}^{\mathcal{A}^{\text{op}}}$$

be the canonical embedding that takes each  $K \in \mathcal{K}$  to the contravariant functor  $\mathcal{K}(-, K): \mathcal{A} \rightarrow \mathbf{Set}$ . We note that, by Proposition 2.8 in [1], this functor is fully faithful and preserves directed colimits and finitely presentable objects. Following Theorem 4.17 in [1],  $\mathcal{K}$  is equivalent to a finitary-cone-injectivity class  $\text{Inj}(T)$  in  $\mathbf{Set}^{\mathcal{A}^{\text{op}}}$ ; this means that there is a set  $T$  of cones  $a = (a_i: X \rightarrow EA_i)_{i \in I}$  where  $X$  is finitely presentable in  $\mathbf{Set}^{\mathcal{A}^{\text{op}}}$  and  $A_i \in \mathcal{A}$ ,  $i \in I$  such that  $\text{Inj}(T)$  consists of functors  $F$  injective to each cone  $a \in T$ . The latter means that for any morphism  $f: X \rightarrow F$  there is  $i \in I$  and  $g: EA_i \rightarrow F$  with  $ga_i = f$ . Let  $S(\mathcal{K})$  be the category whose objects are  $(F, a_F)_{a \in T}$  consisting of  $F: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Set}$  with  $a_F$  assigning to a cone  $a$  and  $f: X \rightarrow F$  a morphism  $a_F(f): EA_i \rightarrow F$  for some  $i \in I$  such that  $a_F(f)a_i = f$ . Morphisms  $(F, a_F) \rightarrow (F', a_{F'})$  are natural transformations  $\varphi: F \rightarrow F'$  such that  $a_{F'}(\varphi f) = \varphi a_F(f)$ . The forgetful functor  $G: S(\mathcal{K}) \rightarrow \mathbf{Set}^{\mathcal{A}^{\text{op}}}$  is faithful and has values in  $\text{Inj}(T)$ . Its codomain restriction  $S(\mathcal{K}) \rightarrow \text{Inj}(T)$  is surjective on objects. Since  $E: \mathcal{K} \rightarrow \text{Inj}(T)$  is an equivalence, we get a faithful functor  $H: S(\mathcal{K}) \rightarrow \mathcal{K}$  which is essentially surjective on objects, i.e., any  $K \in \mathcal{K}$  is isomorphic to some  $H(F, a_F)$ .

2.1. LEMMA. *The category  $S(\mathcal{K})$  is finitely accessible and  $H: S(\mathcal{K}) \rightarrow \mathcal{K}$  preserves directed colimits.*

PROOF. Let  $D: \mathcal{D} \rightarrow S(\mathcal{K})$  be a directed diagram and consider the colimit  $\delta: GD \rightarrow F$  in  $\mathbf{Set}^{\mathcal{A}^{\text{op}}}$ . Let  $a = (a_i: X \rightarrow EA_i)_{i \in I}$  be a cone and  $f: X \rightarrow F$  a morphism. Since  $X$  is finitely presentable, there is  $d \in \mathcal{D}$  and  $g: X \rightarrow GDd$  such that  $f = \delta_d g$ . We put  $a_F = \delta_d a_{Dd}(g)$ . Then  $(F, a_F) = \text{colim } D$ . Thus  $S(\mathcal{K})$  has directed colimits and  $G$  preserves them. Hence  $H$  preserves them too.

If  $F$  is finitely presentable in  $\mathbf{Set}^{\mathcal{A}^{\text{op}}}$  then any  $(F, a_F)$  is finitely presentable in  $S(\mathcal{K})$ . In order to show that any  $(F, a_F)$  is a directed colimit of finitely presentable objects in  $S(\mathcal{K})$  it suffices to express  $F$  as a directed colimit of finitely presentable objects  $F_d$  in  $\mathbf{Set}^{\mathcal{A}^{\text{op}}}$

and complete them to  $(F_d, a_{F_d})$  using finite presentability of  $X$  again. Then  $(F, a_F)$  is a directed colimit of  $(F_d, a_{F_d})$ . Thus  $S(\mathcal{K})$  is finitely accessible. ■

In fact, we have shown that

$$S(\mathcal{K}) = S(\text{Ind } \mathcal{A}) = \text{Ind } S(\mathcal{A})$$

$S(\mathcal{K})$  will be called a *Skolem cover* of  $\mathcal{K}$  because it is a skolemization of the  $L_{\infty, \omega}$ -theory corresponding to  $T$ . Let  $U: \mathbf{Set}^{\mathcal{A}^{\text{op}}} \rightarrow \mathbf{Set}$  assign to  $F$  the set  $\coprod_{A \in \mathcal{A}} FA$ . The functor  $U$  is faithful and preserves directed colimits. Thus  $(\mathcal{K}, UE)$  and  $(S(\mathcal{K}), UG)$  are concrete finitely accessible categories with concrete directed colimits and  $H: S(\mathcal{K}) \rightarrow \mathcal{K}$  is a concrete functor.

**2.2. LEMMA.** *Let  $(F, a_F) \in S(\mathcal{K})$  and  $Z \subseteq UG(F, a_F)$ . Then there is the smallest subobject  $(F_Z, a_{F_Z})$  of  $(F, a_F)$  such that  $Z \subseteq UGF_Z$ .*

*Moreover, for every  $g, h: (F_Z, a_{F_Z}) \rightarrow (F', a_{F'})$ , we have  $g = h$  provided that  $Ug$  and  $Uh$  have the same restriction on  $Z$ .*

**PROOF.** Let  $F_0$  be the smallest subfunctor of  $F$  such that  $Z \subseteq UF_0$ ; let  $\sigma: F_0 \rightarrow F$  denote the inclusion. Consider a cone  $a: X \rightarrow EA_i$  in  $T$  and a morphism  $f: X \rightarrow F_0$ . Then  $\sigma f = a_F(f)a_i$ . Let  $F_1$  be the smallest subfunctor of  $F$  containing  $U(F_0)$  and the images of  $U(a_F(f))$  for all cones  $a$  and all morphisms  $f$ . We iterate this construction by replacing  $F_0$  with  $F_1$ , etc. In this way, we get the chain  $F_0 \rightarrow F_1 \rightarrow \dots \rightarrow F_n \rightarrow \dots$ . Then  $F_Z = \text{colim } F_n$  carries the desired smallest subobject of  $(F, a_F)$ . In fact, every  $f: X \rightarrow F_Z$  factorizes through some  $F_n \rightarrow F$  because  $X$  is finitely presentable.

Consider  $g, h: (F_Z, a_{F_Z}) \rightarrow (F', a_{F'})$  such that  $Ug$  and  $Uh$  have the same restriction on  $Z$ . Then  $Ug$  and  $Uh$  have the same restriction of  $UF_0$ . Consequently, they have the same restriction on  $UF_1$ , etc. Hence  $g = h$ . ■

This is the virtue of the skolemization and reflects the fact that the skolemized theory is universal. We skolemized cone-injectivity while algebraic factorization systems (see [4]) skolemize injectivity. J. Bourke [3] came to the same point from a different motivation.

**2.3. REMARK.** For any  $Z$ , there is only a set of non-isomorphic  $(F_Z, a_{F_Z})$ ,  $F: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Set}$ .

Indeed, let  $\kappa$  be greater than the number of morphisms of  $\mathcal{A}$  and the cardinalities of  $UEA_i$  of cones  $a$  of  $T$ . Then, for every  $F: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Set}$  and  $Z \subseteq UF$ , the smallest subfunctor of  $F$  such that  $Z \subseteq UF_0$  has  $UF_0$  of cardinality smaller than  $\kappa$ . Consequently, every  $UF_Z$  has cardinality smaller than  $\kappa$ .

### 3. Minimal finitely accessible categories

**3.1. THEOREM.** *For any large finitely accessible category  $\mathcal{K}$  there is a faithful functor  $\mathbf{Lin} \rightarrow \mathcal{K}$  preserving directed colimits.*

PROOF. Following 2.2, we can assume that  $\mathcal{K}$  is equipped with a faithful functor  $U : \mathcal{K} \rightarrow \mathbf{Set}$  preserving directed colimits and such that for any subset  $Z \subseteq UK$  there is the smallest subobject  $K_Z$  of  $K$  such that  $Z \subseteq UK_Z$ . Let  $\mathcal{L}$  be the category with objects  $(K, X)$  where  $K \in \mathcal{K}$  and  $X \subseteq UK$  is linearly ordered. Morphisms  $(K_1, X_1) \rightarrow (K_2, X_2)$  are morphisms  $f: K_1 \rightarrow K_2$  such that the restriction of  $Uf$  to  $X_1$  is an order preserving injective mapping to  $X_2$ . The category  $\mathcal{L}$  has directed colimits given as

$$\text{colim}(K_i, X_i) = (\text{colim } K_i, \text{colim } X_i)$$

and any  $(K, X)$  with  $K$  finitely presentable in  $\mathcal{K}$  and  $X$  finite is finitely presentable in  $\mathcal{L}$ . Thus  $\mathcal{L}$  is finitely accessible and the forgetful functor  $\mathcal{L} \rightarrow \mathcal{K}$  preserves directed colimits.

For a  $\mathcal{L}$ -object  $(K, X)$ , let  $\rho_{(K,X)}$  be the greatest ordinal  $0 < \rho_{(K,X)} \leq \omega, |X|$  such that for any  $n < \rho_{(K,X)}$  and any  $a_1 < a_2 < \dots < a_n$  and  $b_1 < b_2 < \dots < b_n$  in  $X$  there is an isomorphism  $s: K_{\{a_1, \dots, a_n\}} \rightarrow K_{\{b_1, \dots, b_n\}}$  such that  $Us(a_i) = b_i$  for  $i = 1, \dots, n$ . Assume that there is  $(K, X) \in \mathcal{L}$  with  $\rho_{(K,X)} = \omega$ . Then  $X$  is infinite and thus it contains a chain  $a_1 < a_2 < \dots < a_n \dots$  or a chain  $a_1 > a_2 > \dots > a_n \dots$ . Indeed, if  $X$  does not contain a strictly decreasing countable chain, it is well-ordered and hence it contains a strictly increasing countable chain.

Assume that  $X$  contains a chain  $a_1 < a_2 < \dots < a_n \dots$  (the other case is analogous). We will construct a functor  $F: \mathbf{Lin} \rightarrow \mathcal{K}$  as follows. Finitely presentable objects in  $\mathbf{Lin}$  are finite chains  $C_n$  with elements  $1 < 2 < \dots < n$ . Put  $F_0(C_n) = K_{a_1, \dots, a_n}$ . Given an injective order preserving mapping  $h: C_m \rightarrow C_n$ , let  $F_0h$  be the composition

$$K_{a_1, \dots, a_m} \rightarrow K_{a_{h(1)}, \dots, a_{h(m)}} \rightarrow K_{a_1, \dots, a_n}$$

where the first morphism is the isomorphism  $s$  above and the second morphism is the inclusion. Given  $h_1: C_k \rightarrow C_m$  and  $h_2: C_m \rightarrow C_n$  then, following 2.2,  $F_0(h_2h_1) = F_0(h_2)F_0(h_1)$ . In fact, we always get the composition

$$K_{a_1, \dots, a_k} \rightarrow K_{a_{h_2h_1(1)}, \dots, a_{h_2h_1(k)}}$$

followed by the inclusion  $K_{a_{h_2h_1(1)}, \dots, a_{h_2h_1(k)}} \rightarrow K_{a_1, \dots, a_n}$ . Thus we get the functor  $F_0: \mathbf{FinLin} \rightarrow \mathcal{K}$  defined on finite linear orderings. This functor is faithful because if  $h, h': C_m \rightarrow C_n$  have distinct values on  $i$  then  $F_0(h)(a_i) = a_{hi} \neq a_{h'i} = F_0(h')(a_i)$ . Since  $\mathbf{Lin} = \text{Ind } \mathbf{FinLin}$ ,  $F_0$  extends to a functor  $F: \mathbf{Lin} \rightarrow \mathcal{K}$  preserving directed colimits. Since  $F_0$  is faithful,  $F$  is faithful too.

Assume that  $\rho_{(K,X)} < \omega$  for any  $(K, X) \in \mathcal{L}$ . We put  $(K_1, X_1) < (K_2, X_2)$  provided that  $\rho_{(K_2,X_2)} < \rho_{(K_1,X_1)}$  and  $(K_1)_{\{a_1, \dots, a_{\rho_{(K_2,X_2)}}\}} \cong (K_2)_{\{b_1, \dots, b_{\rho_{(K_2,X_2)}}\}}$  for any  $a_1 < \dots < a_{\rho_{(K_2,X_2)}}$  in  $X_1$  and any  $b_1 < \dots < b_{\rho_{(K_2,X_2)}}$  in  $X_2$ . Then  $<$  partially orders objects of  $\mathcal{L}$  and this order is well-founded in the sense that there is no decreasing chain

$$\dots < (K_n, X_n) < (K_{n-1}, X_{n-1}) < \dots < (K_1, X_1).$$

Such chain would yield a diagram

$$(K_1)_{\{a_{11}\}} \longrightarrow (K_2)_{\{a_{21}, a_{22}\}} \longrightarrow (K_n)_{\{a_{n1}, \dots, a_{nn}\}} \longrightarrow \dots$$

whose colimit  $(K, X)$  in  $\mathcal{L}$  has  $\rho_{(K, X)} = \omega$ . Thus we can assign an ordinal  $\alpha(K, X)$  to each  $(K, X) \in \mathcal{L}$  in such a way that

$$\alpha(K, X) = \sup_{(K', X') < (K, X)} \alpha(K', X') + 1.$$

Following 2.3, there is an infinite cardinal  $\mu$  greater or equal to the number of non-isomorphic objects  $K_Z$  for  $Z$  finite and  $K$  arbitrary. For  $(K, X) \in \mathcal{L}$ , choose  $a_1 < \dots < a_{\rho_{(K, X)} - 1}$  in  $X$  and put

$$(K, X)^* = (K_{\{a_1, \dots, a_{\rho_{(K, X)} - 1}\}}, X \cap UK_{\{a_1, \dots, a_{\rho_{(K, X)} - 1}\}}).$$

We will prove that

$$|X| < \exp_{\omega(\alpha(K, X)^* + 1)}(\mu)$$

for any  $(K, X) \in \mathcal{L}$ . Recall that  $\exp_0(\mu) = \mu$ ,  $\exp_{\xi+1}(\mu) = 2^{\exp_{\xi}(\mu)}$  and  $\exp_{\eta}(\mu) = \sup_{\xi < \eta} \exp_{\xi}(\mu)$ . Since every set  $UK$  can be linearly ordered,  $(K, UK) \in \mathcal{L}$  for any  $K$  in  $\mathcal{K}$ , this inequality implies that  $\mathcal{K}$  is small.

The proof will use the recursion on  $\alpha(K, X)^*$ . Let  $\alpha(K, X)^* = 0$  and assume that  $|X| \geq \exp_{\omega}(\mu)$ . For  $n = \rho_{(K, X)}$ , the set  $[X]^n$  of the subsets of  $X$  of size  $n$  is decomposed into  $\leq \mu$  parts following isomorphisms types of  $K_{\{a_1, \dots, a_n\}}$ . Following the Erdős-Rado partition theorem (see [7], Exercise 29.1), there is  $X_0 \subseteq X$  such that  $|X_0| > \mu$  and  $K_{\{a_1, \dots, a_n\}} \cong K_{\{b_1, \dots, b_n\}}$  for any  $a_1 < \dots < a_n$  and  $b_1 < \dots < b_n$  in  $X_0$ . For  $m < n$ , the least subobjects associated to chains  $a_1 < \dots < a_m$  in  $X_0$  are isomorphic because this is true for such chains in  $X$ . Thus

$$(K, X_0)^* = (K_{\{a_1, \dots, a_n\}}, X_0 \cap UK_{\{a_1, \dots, a_n\}}) < (K, X)^*,$$

which is impossible because  $\alpha(K, X)^* = 0$ .

Assume that the claim holds for any  $(K, X) \in \mathcal{L}$  with  $\alpha(K, X)^* < \beta$  and consider  $(L, Y) \in \mathcal{L}$  with  $\alpha(L, Y)^* = \beta$ . Assume that  $|Y| \geq \exp_{\omega(\alpha(L, Y)^* + 1)}(\mu)$  and let  $n = \rho_{(L, Y)}$ . We have

$$|Y| \geq \exp_{\omega(\beta+1)}(\mu) > \exp_{\omega\beta+n-1}(\mu) = \exp_{n-1}(\exp_{\omega\beta}(\mu)).$$

Following the Erdős-Rado partition theorem, there is  $Y_0 \subseteq Y$  such that  $|Y_0| > \exp_{\omega\beta}(\mu)$  and  $L_{\{b_1, \dots, b_n\}} \cong L_{\{c_1, \dots, c_n\}}$  for each  $b_1 < \dots < b_n$  and  $c_1 < \dots < c_n$  in  $Y_0$ . Then  $\rho_{(L, Y_0)} > n$  and  $(L, Y_0) < (L, Y)$ . For  $m < n$ , the least subobjects associated to chains  $a_1 < \dots < a_m$  in  $Y_0$  are isomorphic because this is true for such chains in  $Y$ . Thus  $(L, Y_0)^* < (L, Y)^*$ . Hence  $\alpha(L, Y_0)^* < \beta$  and thus

$$|Y_0| < \exp_{\omega(\alpha(L, Y_0)^* + 1)}(\mu) \leq \exp_{\omega\beta}(\mu),$$

which is a contradiction. ■

#### 4. Towards minimal $\lambda$ -accessible categories

4.1. **EXAMPLE.** The category  $\mathcal{W}$  of well-ordered sets is  $\aleph_1$ -accessible and any its object  $K$  is iso-rigid in the sense that the only isomorphism  $K \rightarrow K$  is the identity. Thus there is no faithful functor  $\mathbf{Lin} \rightarrow \mathcal{W}$  and a prospective minimal  $\aleph_1$ -accessible category is iso-rigid.

4.2. **EXAMPLE.** There is an  $\aleph_1$ -accessible category  $\mathcal{L}$  having all objects  $K$  rigid in the sense that the only morphism  $K \rightarrow K$  is the identity. Thus there is no faithful functor  $\mathcal{W} \rightarrow \mathcal{L}$ .

The construction of  $\mathcal{L}$  is motivated by [6], II.3. Let  $\mathcal{K}$  be the category of structures  $(A, <, \sup, s, S, R)$  where  $<$  is a well-ordering,  $\sup$  is the countable join,  $s$  is the unary operation of taking the successor,  $S$  is an  $\omega$ -ary relation choosing for every ordinal  $a$  of cofinality  $\omega$  a countable increasing chain with the join  $a$  and  $R$  is a unary relation choosing ordinals of cofinality  $> \omega$ . This is achieved by taking the following set  $T$  of axioms:

1.  $(\forall x_0, x_1, y_1, \dots, x_n, y_n, \dots)(S(x_0, x_1, \dots, x_n, \dots) \wedge S(x_0, y_1, \dots, y_n, \dots) \rightarrow \bigwedge_{0 < n} (x_n = y_n))$
2.  $(\forall x_0, x_1, \dots, x_n, \dots)(S(x_0, x_1, \dots, x_n, \dots) \rightarrow (\bigwedge_{0 < n} x_n < x_{n+1}) \wedge x_0 = \sup x_n)$
3.  $(\forall x)(\exists y_1, \dots, y_n, \dots)((\bigwedge_{0 < n} (y_n < y_{n+1}) \wedge x = \sup y_n) \rightarrow (\exists x_1, \dots, x_n, \dots)S(x, x_1, \dots, x_n, \dots))$
4.  $(\forall x)(R(x) \leftrightarrow \neg(\exists y)(x = s(y)) \wedge \neg(\exists x_1, \dots, x_n, \dots)S(x, x_1, \dots, x_n, \dots))$

Let  $A_2$  be the set of isolated elements of  $A$ ,  $A_0$  be the set of all limit elements of  $a \in A$  such that  $S(a, a_1, \dots, a_n, \dots)$  for some  $a_1, \dots, a_n, \dots \in A$  and  $A_1 = A \setminus (A_0 \cup A_2)$ . All the sets  $A_2$ ,  $A_0$  and  $A_1$  are preserved by homomorphisms  $f: A \rightarrow B$  (due to the preservation of  $s$ ,  $S$  and  $R$  resp.).

This category clearly has  $\aleph_1$ -directed colimits. Objects  $A$  of  $\mathcal{L}$  generated by 0 are ordinals  $\omega_1$  with a choice of  $S$  for every  $a \in A_0$ . Thus there is  $\aleph_0^{\aleph_1} = 2^{\aleph_1}$  such objects. These objects are  $\aleph_1$ -presentable and the same is true for objects  $\omega_1 \cdot \alpha$  where  $\alpha < \omega_1$ . Clearly, every object of  $\mathcal{L}$  is an  $\aleph_1$ -directed colimit of these objects  $\omega_1 \cdot \alpha$ ,  $\alpha < \omega_1$ . Thus  $\mathcal{L}$  is  $\aleph_1$ -accessible.

Assume that there exists a morphism  $f: A \rightarrow A$  in  $\mathcal{L}$  which is not the identity. Let  $a$  be the least element in  $A$  such that  $f(a) \neq a$ . Since  $A$  is a well-ordered set and  $f$  is injective,  $a < f(a)$ . Hence

$$a < f(a) < f^2(a) < \dots < f^n(a) < \dots$$

Let  $b = \sup f^n(a)$ . Hence  $f(b) = \sup f^{n+1}(a) = b$ . There are  $b_1 < b_2 < \dots < b_n < \dots$  such that  $S(b, b_1, \dots, b_n, \dots)$ . Since  $S(f(b), f(b_1), \dots, f(b_n), \dots)$  and  $f(b) = b$ , we have  $f(b_n) = b_n$  for each  $n$ . Since  $a < b_m$  for some  $m$ ,  $f^n(a) < b_m$  for each  $n$ . Hence  $b \leq b_m$ , which is a contradiction.

4.3. **REMARK.** (1) Let  $\mathcal{L}_1$  be a full subcategory of  $\mathcal{L}$  where we choose  $S$  for every  $a \in A_0$  in every object generated by 0. This category does not depend of the choices of  $S$  and is also  $\aleph_1$ -accessible. In fact, it is  $\text{Ind}_{\aleph_1}(\mathcal{C}_1)$  where  $\mathcal{C}_1$  is the category of ordinals  $\omega_1 \cdot \alpha$ ,  $\alpha < \omega_1$  with non-identity morphisms

$$\omega_1 \cdot f: \omega \cdot \alpha \longrightarrow \omega_1 \cdot \beta$$

where  $f: \alpha \longrightarrow \beta$  is an order preserving injective mapping with  $\alpha < \beta$ . The category  $\mathcal{C}_1$  is, in fact, the category of ordinals  $\alpha < \omega_1$  where non-identity morphisms are order preserving injective mappings  $\alpha \longrightarrow \beta$  for  $\alpha < \beta < \omega_1$ .

(2) The category **FinLin** is the category  $\mathcal{C}_0$  of ordinals  $\alpha < \omega$  where non-identity morphisms are order preserving injective mapping  $\alpha \longrightarrow \beta$  for  $\alpha < \beta < \omega$ . Observe that **FinLin** is rigid, i.e., the only morphisms  $\alpha \longrightarrow \alpha$  are the identities. Hence  $\mathcal{L}_1 = \text{Ind}_{\omega_1} \mathcal{C}_1$  is  $\aleph_1$ -modification of a minimal  $\aleph_0$ -accessible category **Lin**.

(3) Let  $\mathcal{C}_\gamma$  be the category of ordinals  $\alpha < \omega_\gamma$  where non-identity morphisms are order preserving injective mapping  $\alpha \longrightarrow \beta$  for  $\alpha < \beta < \omega_\gamma$ . Then  $\mathcal{L}_\gamma = \text{Ind}_{\aleph_\gamma} \mathcal{C}_\gamma$  is an  $\aleph_\gamma$ -accessible category.

4.4. **PROBLEM.** Is  $\mathcal{L}_1$  a minimal  $\aleph_1$ -accessible category? This means that for every large  $\aleph_1$ -accessible category  $\mathcal{K}$  there is a faithful functor  $\mathcal{L}_1 \longrightarrow \mathcal{K}$  preserving  $\aleph_1$ -directed colimits.

Similarly, is  $\mathcal{L}_\gamma$  a minimal  $\aleph_\gamma$ -accessible category for  $0 < \gamma$ ?

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