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EXTENSIONS AND GLUEING IN DOUBLE CATEGORIES

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Abstract. Let $\mathbb{D}$ be a double category with an initial object. Any cotabulator Γv of a vertical morphism $v\colon X_0 \xrightarrow{\bullet} X_1$ gives rise to an extension (i.e., short exact sequence) $X_0 \xrightarrow{\bullet} \Gamma v \xrightarrow{\bullet} X_1$ in the vertical bicategory $\mathcal{V}\mathbb{D}$. If $\mathbb{D}$ has "open cokernels," then every extension is equivalent in $\mathcal{V}\mathbb{D}$ to one of this form. Examples include the double categories $\mathbb{L}\mathrm{oc}$, $\mathbb{T}\mathrm{opos}$, $\mathbb{P}\mathrm{os}$, and $\mathbb{C}\mathrm{at}$, whose objects are locales, toposes, posets, and small categories, respectively, and Γv is given by glueing along v in the first two cases, and by the collage of v in the others.

1. Introduction

In [FM20], Faul and Manuell characterize extensions in the category $\mathrm{Frm}_\wedge$ whose objects are frames (i.e., locales) and morphisms are finite meet preserving maps. They show that

$$X_1 \xrightarrow{\pi_{1*}} \mathrm{Gl}(v) \xrightarrow{\pi_0^*} X_0$$

is an extension of X_0 by X_1, where $\mathrm{Gl}(v)$ denotes the frame obtained by Artin-Wraith glueing along $v\colon X_0 \to X_1$ in $\mathrm{Frm}_\wedge$ with π_0 and π_1 the usual projections, and every extension is isomorphic in $\mathrm{Frm}_\wedge$ to one of this form. Thus, $\mathrm{Gl}(v)$ plays the role for frames that the semidirect product does for groups.

$\mathrm{Frm}_\wedge^{co}$ is the vertical bicategory of the double category $\mathbb{L}\mathrm{oc}$ of locales (c.f., [N12a]),

$$\begin{array}{ccc} X_0 & \xrightarrow{i_0} & \\ {\scriptstyle v}\big\downarrow\bullet & \geq & \mathrm{Gl}(v) \\ X_1 & \xrightarrow{i_1} & \end{array}$$

is a cotabulator (in the sense of [P11]), and the characterization in [FM20] essentially uses the fact that $\mathbb{L}\mathrm{oc}$ is a double category $\mathbb{D}$ satisfying

- $\mathbb{D}$ has a horizontal initial object (Definition 2.3)
- $\mathbb{D}$ has cotabulators which "restrict to the vertical bicategory" (Definition 3.5)
- $\mathbb{D}$ has "open" cokernels (Definition 4.1)

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With an appropriate definition of an extension in a bicategory, we will see that if $\mathbb{D}$ is a pseudo double category satisfying the three bulleted conditions, then every extension in the vertical bicategory $\mathcal{V}\mathbb{D}$ is equivalent to one arising from a cotabulator of a vertical morphism of $\mathbb{D}$. Moreover, these three properties hold in our four main examples $\mathbb{L}$oc, $\mathbb{T}$opos, $\mathbb{P}$os, and $\mathbb{C}$at; and cotabulators are given by Artin-Wraith glueing in the first two and collages in the others. Thus, these constructions are analogous to the semidirect product vis a vis extensions.

We begin, in Section 2, by recalling the relevant properties of double categories and the four main examples of interest. The next two sections introduce the notion of extensions in $\mathcal{V}\mathbb{D}$ and give their characterization in terms of cotabulators in $\mathbb{D}$. In Section 5, we show that the set $\mathrm{Ext}(X_0, X_1)$ of extensions of X_0 by X_1 defines a bifunctor $(\mathcal{V}\mathbb{D})^{op} \times \mathcal{V}\mathbb{D} \to \mathrm{Set}$. We conclude, in Section 6, with the introduction and characterization of a category of adjoint extensions of X_0 by X_1 in $\mathbb{D}$.

This paper is dedicated to Bob Rosebrugh for his leadership in digital communication over the past thirty years, from the development of the category theory mailing list and home page to the conception and implementation of Theory and Applications of Categories, one of the first refereed electronic mathematics journals.

2. Double Categories and Cotabulators

Introduced by Ehresmann [E63] in the 1960's, double categories provided a setting to transfer results from one category to others that admit analogous structures. This section is a review of the definitions and examples used in this paper. For details and more information, we refer the reader to the work of Grandis and Paré [GP99, GP04, P11], as well as Shulman [Sh08].

2.1. Definition. *A double category is an internal pseudo category*

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathbb{D}_0 \quad (\mathrm{id}^\bullet \colon \mathbb{D}_0 \to \mathbb{D}_1)$$

in the 2-category **CAT** *of locally small categories.*

The objects X and morphisms $f\colon X \to Y$ of $\mathbb{D}_0$ are called the *objects* and *horizontal morphisms* of $\mathbb{D}$. The objects of $\mathbb{D}_1$ are called *vertical morphisms*, and denoted by $v\colon X_0 \to X_1$, and their morphisms

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle v}\downarrow & \varphi & \downarrow{\scriptstyle w} \\ X_1 & \xrightarrow[f_1]{} & Y_t \end{array} \tag{2.1}$$

are called *cells* of $\mathbb{D}$. Horizontal composition is denoted by $\circ$ and vertical composition by $\odot$, both of which are sometimes elided. Note that when w is the vertical identity $\mathrm{id}^\bullet_Y$ on

Y, we often denote the cell (2.1) by

$$\begin{array}{ccc} X_0 & & \\ {\scriptstyle v}\big\downarrow\!\bullet & \varphi & \overset{f_0}{\searrow} \; Y \\ & & \underset{f_1}{\nearrow} \\ X_1 & & \end{array}$$

2.2. Definition. *The objects, morphisms, and special cells*

$$\begin{array}{ccc} X_0 & \xrightarrow{\mathrm{id}_{X_0}} & X_0 \\ {\scriptstyle v}\big\downarrow\!\bullet & \varphi & \bullet\!\big\downarrow{\scriptstyle w} \\ X_1 & \xrightarrow[\mathrm{id}_{X_1}]{} & X_1 \end{array}$$

form a bicategory $\mathcal{V}\mathbb{D}$*, called the vertical bicategory of* $\mathbb{D}$*. The horizontal* 2*-category* $\mathbf{H}\mathbb{D}$ *is defined dually.*

2.3. Definition. *A horizontal initial object of* $\mathbb{D}$ *is an object* O *such that there is a unique horizontal morphism* $u_X\colon O \longrightarrow X$*, for every object* X*, and a unique cell*

$$\begin{array}{ccc} O & \xrightarrow{u_{X_0}} & X_0 \\ {\scriptstyle \mathrm{id}^\bullet_O}\big\downarrow\!\bullet & u_v & \bullet\!\big\downarrow{\scriptstyle v} \\ O & \xrightarrow[u_{X_1}]{} & X_1 \end{array} \tag{2.3}$$

for every vertical morphism v*.*

2.4. Definition. *A companion for a horizontal morphism* $f\colon X \longrightarrow Y$ *is a vertical morphism* $f_*\colon X \longrightarrow\!\!\bullet\!\!\rightarrow Y$ *together with cells*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}^\bullet_X}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle f_*} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f_*}\big\downarrow\!\bullet & \beta & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}^\bullet_Y} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

whose horizontal and vertical compositions are identity cells. A conjoint for f *is a vertical morphism* $f^*\colon Y \longrightarrow\!\!\bullet\!\!\rightarrow X$ *together with cells*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}^\bullet_X}\big\downarrow\!\bullet & \rho & \bullet\!\big\downarrow{\scriptstyle f^*} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ {\scriptstyle f^*}\big\downarrow\!\bullet & \sigma & \bullet\!\big\downarrow{\scriptstyle \mathrm{id}^\bullet_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

whose horizontal and vertical compositions are identity cells.

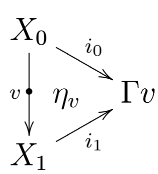
2.5. Example. $\mathbb{L}$oc has locales as objects, locale morphisms $f\colon X \to Y$, in the sense of [J82], as horizontal morphisms, and finite meet preserving maps $v\colon X_0 \mathrel{\bullet\!\!\!\!\to} X_1$ as vertical morphisms. There is a cell of the form (2.1) if and only if $w \odot f_{0*} \le f_{1*} \odot v$ or equivalently, $f_1^* \odot w \le v \odot f_0^*$. The initial locale O is the horizontal initial object, and the companion and conjoint of f are the direct and inverse images f_* and f^*, respectively. The vertical bicategory $\mathcal{V}\mathbb{L}\text{oc}$ is the partially ordered bicategory $\text{Frm}_\wedge^{co}$.

2.6. Example. $\mathbb{T}$opos has (elementary) toposes $\mathcal{X}$ as objects, geometric morphisms $f\colon \mathcal{X} \to \mathcal{Y}$ (in the sense of [J77]) as horizontal morphisms, finite limit preserving functors $v\colon \mathcal{X}_0 \mathrel{\bullet\!\!\!\!\to} \mathcal{X}_1$ as vertical morphisms, natural transformations $\varphi\colon w \odot f_{0*} \Rightarrow f_{1*} \odot v$ or equivalently, $\hat{\varphi}\colon f_1^* \odot w \Rightarrow v \odot f_0^*$, as cells of the form (2.1). The horizontal initial object $\mathcal{O}$ is the one object topos, and the companion and conjoint of f are the direct and inverse images f_* and f^*, respectively.

2.7. Example. $\mathbb{P}$os has posets as objects, order-preserving maps $f\colon X \to Y$ as horizontal morphisms and order ideals $v\colon X_0 \mathrel{\bullet\!\!\!\!\to} X_1$ (i.e., upward closed sets $v \subseteq X_0^{op} \times X_1$) as vertical morphisms. There is a cell of the form (2.1) if and only if $(f_0(x_0), f_1(x_1)) \in w$, for all $(x_0, x_1) \in v$. The empty poset O is the horizontal initial object, and the companion $f_*\colon X \mathrel{\bullet\!\!\!\!\to} Y$ and conjoint $f^*\colon Y \mathrel{\bullet\!\!\!\!\to} X$ of $f\colon X \to Y$ are defined by $f_* = \{(x,y) \mid fx \le y\}$ and $f^* = \{(y,x) \mid y \le fx\}$, respectively. The vertical bicategory $\mathcal{V}\mathbb{P}\text{os}$ is the partially ordered bicategory of posets and ideals.

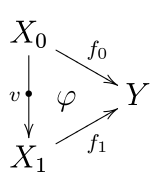
2.8. Example. $\mathbb{C}$at has small categories as objects, functors $f\colon X \to Y$ as horizontal morphisms, profunctors $v\colon X_0 \mathrel{\bullet\!\!\!\!\to} X_1$ (i.e., functors $v\colon X_0^{op} \times X_1 \to \mathbf{Set}$) as vertical morphisms, and natural transformations $\varphi\colon f_{1*} \odot v \Rightarrow w \odot f_{0*}$ or equivalently, $\hat{\varphi}\colon v \odot f_0^* \Rightarrow f_1^* \odot w$, as cells of the form (2.1). The empty category O is the horizontal initial object, and the companion $f_*\colon X \mathrel{\bullet\!\!\!\!\to} Y$ and conjoint $f^*\colon Y \mathrel{\bullet\!\!\!\!\to} X$ of $f\colon X \to Y$ are defined by $f_*(x,y) = Y(fx,y)$ and $f^*(y,x) = Y(y,fx)$, respectively. The vertical bicategory $\mathcal{V}\mathbb{C}\text{at}$ is isomorphic to Prof.

2.9. Definition. *A cotabulator of a vertical morphism $v\colon X_0 \mathrel{\bullet\!\!\!\!\to} X_1$ consists of an object Γv and a cell*



(2.9)

such that for any cell



there exists a unique horizontal morphism $f\colon \Gamma v \to Y$ such that $fi_0 = f_0$, $fi_1 = f_1$, and $\mathrm{id}_f^\bullet \eta_v = \varphi$.

One can show that the cotabulator Γv exists, for all v, if and only if $\mathrm{id}^\bullet\colon \mathbb{D}_0 \to \mathbb{D}_1$ has a left adjoint [P11].

2.10. Example. Cotabulators in $\mathbb{L}$oc and $\mathbb{T}$opos are given by the glueing construction used in [N81] which is also know as Artin-Wraith glueing in the topos case [J77]. The topos $\mathrm{Gl}(v)$ obtained by glueing along $v\colon \mathcal{X}_0 \xrightarrow{\bullet} \mathcal{X}_1$ has objects $(X_0, X_1, \alpha\colon X_1 \to vX_0)$, where $X_0 \in |\mathcal{X}_0|$ and $X_1 \in |\mathcal{X}_1|$, and morphisms pairs (f_0, f_1) such that

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha} & vX_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle vf_0} \\ Y_1 & \xrightarrow[\beta]{} & vY_0 \end{array}$$

commutes, with $i_0^*(X_0, X_1, \alpha) = X_0$, $i_1^*(X_0, X_1, \alpha) = X_1$, $i_{0*}(X_0) = (X_0, vX_0, \mathrm{id}_{vX_0})$, $i_{1*}(X_1) = (1, X_1, !)$, and $\eta_v\colon i_{0*} \Rightarrow i_{1*}v$ at X_0 given by $(!, \mathrm{id}_{vX_0})$. The cotabulator in $\mathbb{L}$oc is defined similarly as $\mathrm{Gl}(v) = \{(x_0, x_1) | x_1 \le vx_0\}$ with $i_{0*}(x_0) = (x_0, vx_0)$ and $i_{1*}(x_1) = (\top, X_1)$.

2.11. Example. Cotabulators in $\mathbb{C}$at are given by "collages." Recall that the *collage* Γv of a profunctor $v\colon X_0^{op} \times X_1 \to \mathbf{Set}$, also denoted by $X_0 \sqcup_v X_1$, is constructed by taking the disjoint union of the categories X_0 and X_1 together with a set $\Gamma v(x_0, x_1) = v(x_0, x_1)$ of morphisms, for each $x_0 \in X_0$ and $x_1 \in X_1$. The functors i_0 and i_1 are given by the inclusion. Collages in $\mathbb{P}$os are defined similarly.

3. Extensions as Cotabulators

In this section, we introduce the notion of an extension in a bicategory $\mathcal{B}$ and prove a general theorem relating cotabulators in $\mathbb{D}$ to extensions in the vertical bicategory $\mathcal{V}\mathbb{D}$. Thus, relating Artin-Wraith glueing in $\mathbb{T}$opos and $\mathbb{L}$oc, and collages in $\mathbb{C}$at and $\mathbb{P}$os, to extensions in the corresponding vertical bicategories.

3.1. Definition. *We say $\mathcal{B}$ has initial morphisms if each category $\mathcal{B}(X, Y)$ has an initial object, denoted by 0_{XY}, and the unique cells $0_{WY} \Rightarrow 0_{XY} f$ and $0_{XZ} \Rightarrow g\, 0_{XY}$ are invertible, for all $f\colon W \to X$ and $g\colon Y \to Z$.*

3.2. Proposition. *If $\mathbb{D}$ has a horizontal initial object O and the unique morphism $u_X\colon O \to X$ has a companion u_{X*} and conjoint u_X^*, for all X, then $\mathcal{B}(X, Y)$ has an initial object, for all X, Y and 0_{XY} is given by*

$$X \xrightarrow{u_X^*} O \xrightarrow{u_{Y*}} Y$$

Proof. This follows from Definition 2.3, since for all $v\colon X \to\!\!\!\!\!\bullet\;\, Y$, there is a unique cell

$$\begin{array}{ccc} O & \xrightarrow{u_X} & X \\ {\scriptstyle \mathrm{id}^\bullet_O}\big\downarrow\bullet & u_v & \bullet\big\downarrow{\scriptstyle v} \\ O & \xrightarrow[u_Y]{} & Y \end{array}$$

■

Note that for the double categories $\mathbb{D}$ in Examples 2.5–2.8, the unique cells $0_{WY}{\Rightarrow}0_{XY}f$ and $0_{XZ}{\Rightarrow}g\,0_{XY}$ are invertible, for all $f\colon W \longrightarrow X$ and $g\colon Y \longrightarrow Z$, and so $\mathcal{V}\mathbb{D}$ has initial morphisms.

3.3. Definition. *Suppose $\mathcal{B}$ has initial morphisms. A kernel of a morphism $f\colon X \longrightarrow Y$ is an inverter (in the sense of [CJSV94]) of the unique cell $u_f\colon 0_{XY} \Rightarrow f$, i.e., a morphism $k\colon K \longrightarrow X$ such that, for all B, the functor $\mathcal{B}(B,k)\colon \mathcal{B}(B,K) \longrightarrow \mathcal{B}(B,X)$ induces an equivalence of categories between $\mathcal{B}(B,K)$ and the full subcategory of $\mathcal{B}(B,X)$ consisting of morphisms $b\colon B \longrightarrow X$ such that $u_f b\colon 0_{XY}b \Rightarrow fb$ is invertible. The dual of a kernel is called a cokernel.*

Unpacking the definition of kernel, we see that $u_f k$ is an invertible cell satisfying the following universal property. For all $b\colon B \longrightarrow X$ such that $u_f b$ is invertible, there exists $\bar{b}\colon B \longrightarrow K$ and an invertible cell $\theta\colon k\bar{b} \Rightarrow b$, and for any $\beta\colon b_1 \Rightarrow b_2$ with $u_f b_1$ and $u_f b_2$ invertible and invertible cells $\theta_1\colon k\bar{b}_1 \Rightarrow b_1$ and $\theta_2\colon k\bar{b}_2 \Rightarrow b_2$, there exists a unique $\bar{\beta}\colon \bar{b}_1 {\Rightarrow} \bar{b}_2$ such that

$$\begin{array}{ccc} k\bar{b}_1 & \xRightarrow{\theta_1} & b_1 \\ {\scriptstyle k\bar{\beta}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \beta} \\ k\bar{b}_2 & \xRightarrow[\theta_2]{} & b_2 \end{array}$$

commutes. Moreover, the uniqueness of $\bar{\beta}$ implies that for any b, the pair $(\bar{b}, \theta)$ is unique up to isomorphism.

Note that if $\mathbb{D}$ is flat, in the sense of [GP99], and the only invertible cells are identities, then this agrees with the usual (strict) definition of kernel and cokernel in $\mathcal{V}\mathbb{D}$.

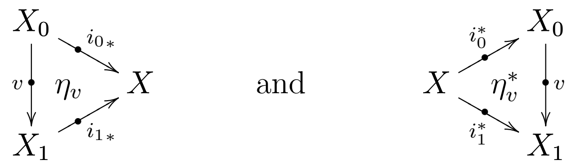
3.4. Definition. *A diagram $K \xrightarrow{k} X \xrightarrow{e} Q$ is called an extension of Q by K in a bicategory $\mathcal{B}$ if k is a kernel of e and e is a cokernel of k.*

To show that cotabulators in $\mathbb{D}$ give rise to extensions in $\mathcal{V}\mathbb{D}$, we use the following property of cotabulators in our main examples.

3.5. Definition. *A cotabulator*

$$\begin{array}{ccc} X_0 & & \\ {\scriptstyle v}\big\downarrow\bullet & \eta_v & X \\ X_1 & & \end{array} \qquad i_0\colon X_0 \to X,\quad i_1\colon X_1 \to X$$

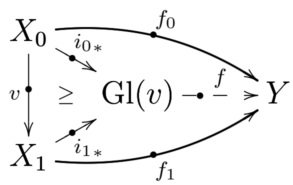
in $\mathbb{D}$ restricts to $\mathcal{V}\mathbb{D}$ and $\mathcal{V}\mathbb{D}^{co}$ if i_0 and i_1 have companions and cotabulators, and



become a pseudo cotabulator and a pseudo tabulator in the bicategories $\mathcal{V}\mathbb{D}$ and $\mathcal{V}\mathbb{D}^{co}$, respectively.

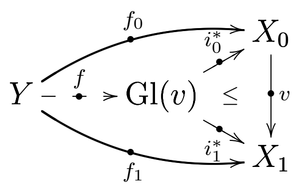
3.6. Example. Cotabulators in $\mathbb{L}$oc restrict to $\mathcal{V}\mathbb{L}$oc and $\mathcal{V}\mathbb{L}\text{oc}^{co}$, as follows.

For $\mathcal{V}\mathbb{L}$oc, consider the diagram



where $f_0 \leq f_1 v$, and define $f(x_0, x_1) = f_0(x_0) \wedge f_1(x_1)$. Then f is a finite meet preserving map making the diagram commute, since $f_0 \leq f_1 v$ implies $f(i_{0*}(x_0)) = f(x_0, vx_0) = f_0(x_0) \wedge f_1(vx_0) = f_0(x_0)$ and $f(i_{1*}(x_1)) = f(\top, x_1) = f_0(\top) \wedge f_1(x_1) = f_1(x_1)$. Functoriality (and uniqueness) of f holds since $(x_0, x_1) = (x_0, vx_0) \wedge (\top, x_1)$, for all $(x_0, x_1) \in \text{Gl}(v)$.

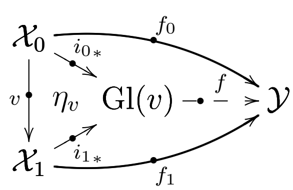
For $\mathcal{V}\mathbb{L}\text{oc}^{co}$, consider the diagram



where $f_1 \leq vf_0$, and define $f(y) = (f_0(y), f_1(y))$. Then $f(y) \in \text{Gl}(v)$, since $f_1 \leq vf_0$, and f is the unique finite meet preserving map making the diagram commute, since i_0^* and i_1^* are the projections onto X_0 and X_1, respectively.

3.7. Example. Cotabulators in $\mathbb{T}$opos restrict to $\mathcal{V}\mathbb{T}$opos and $\mathcal{V}\mathbb{T}\text{opos}^{co}$, as follows.

Consider the diagram



(3.7)

and $\varphi\colon f_0 \Rightarrow f_1 v$. Define $f(X_0, X_1, \alpha)$ by the pullback

$$\begin{array}{ccc} f(X_0, X_1, \alpha) & \longrightarrow & f_1 X_1 \\ \downarrow & & \downarrow {\scriptstyle f_1\alpha} \\ f_0 X_0 & \xrightarrow[\varphi_{X_0}]{} & f_1 v X_0 \end{array}$$

Then f preserves finite limits since f_0 and f_1 do, a straightforward argument shows that the diagram (3.7) commutes up to isomorphism, and functoriality of f follows since

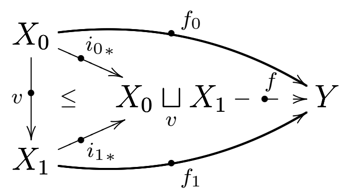
$$\begin{array}{ccc} (X_0, X_1, \alpha) & \xrightarrow{(!,\mathrm{id})} & (1, X_1, !) \\ {\scriptstyle(\mathrm{id},\alpha)}\downarrow & & \downarrow{\scriptstyle(!,\mathrm{id})} \\ (X_0, vX_0, \mathrm{id}) & \xrightarrow[(!,\mathrm{id})]{} & (1, vX_0, !) \end{array}$$

is a pullback for all (X_0, X_1, α) in $\mathrm{Gl}(v)$.

The proof for $\mathcal{V}\mathbb{T}\mathrm{opos}^{co}$ is similar to that of $\mathcal{V}\mathbb{L}\mathrm{oc}^{co}$.

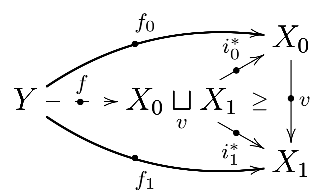
3.8. Example. Cotabulators in $\mathbb{P}\mathrm{os}$ restrict to $\mathcal{V}\mathbb{P}\mathrm{os}$ and $\mathcal{V}\mathbb{P}\mathrm{os}^{co}$, as follows.

For $\mathcal{V}\mathbb{P}\mathrm{os}$, consider the diagram



where $f_1 v \leq f_0$, and let f denote the disjoint union of f_0 and f_1. Then f is an ideal making the diagram commute, since f_0 and f_1 are ideals, and $f_1 v \leq f_0$. Functoriality (and uniqueness) easily follow.

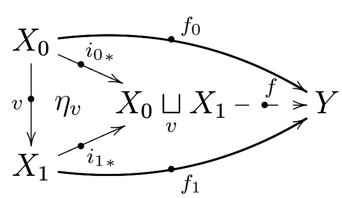
For $\mathcal{V}\mathbb{P}\mathrm{os}^{co}$, consider the diagram



where $vf_0 \leq f_1$, and let f denote the disjoint union of f_0 and f_1. Then f is an ideal making the diagram commute, since f_0 and f_1 are ideals, and $f_1 v \leq f_0$. Functoriality (and uniqueness) easily follow.

3.9. Example. Cotabulators in $\mathbb{C}\mathrm{at}$ restrict to $\mathcal{V}\mathbb{C}\mathrm{at}$ and $\mathcal{V}\mathbb{C}\mathrm{at}^{co}$, as follows.

For $\mathcal{V}\mathbb{C}\mathrm{at}$, consider the diagram

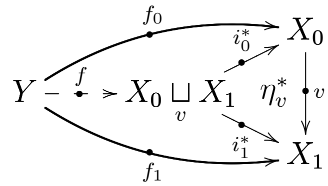


with $\varphi\colon f_1 v \Rightarrow f_0$. Define f by

$$f(x, y) = \begin{cases} f_0(x, y) & \text{if } x \in X_0 \\ f_1(x, y) & \text{if } x \in X_1 \end{cases}$$

Since there is a cell $\varphi\colon f_1 v \Rightarrow f_0$, we know f is a profunctor making the diagram commute up to an invertible cell, and functoriality (and uniqueness) easily follow.

For $\mathcal{V}\mathbb{C}\mathrm{at}^{co}$, consider the diagram

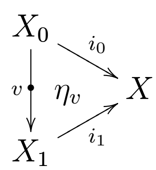


with $\varphi\colon v f_0 \Rightarrow f_1$. Define f by

$$f(y,x) = \begin{cases} f_0(y,x) & \text{if } x \in X_0 \\ f_1(y,x) & \text{if } x \in X_1 \end{cases}$$

Then f is a profunctor making the diagram commute up to an invertible cell, since there is a cell $\varphi\colon v f_0 \Rightarrow f_1$, and functoriality (and uniqueness) easily follow.

3.10. Proposition. *If*

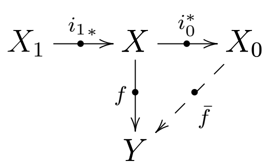


is a cotabulator in $\mathbb{D}$ *which restricts to* $\mathcal{V}\mathbb{D}$ *and* $\mathcal{V}\mathbb{D}^{co}$*, and* $u_{i_0^* i_{1*}}\colon 0_{X_1 X_0} \Rightarrow i_0^* i_{1*}$ *is invertible, then*

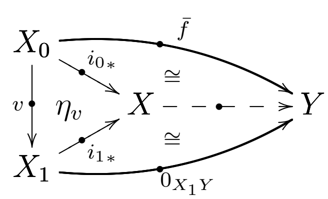
$$X_1 \xrightarrow{i_{1*}} X \xrightarrow{i_0^*} X_0$$

is an extension of X_0 *by* X_1 *in* $\mathcal{V}\mathbb{D}$*.*

Proof. To see that i_0^* is a cokernel of i_{1*}, given $f\colon X \to Y$ and an invertible cell $0_{X_1 Y} \Rightarrow f i_{1*}$, consider



where $\bar{f} = f i_{0*}$. Both $\bar{f} i_0^*$ and f complete the diagram



since $(\bar{f} i_0^*) i_{0*} \cong \bar{f}$ and $(\bar{f} i_0^*) i_{1*} \cong 0_{X_1 Y}$, and so there is an invertible cell $\bar{f} i_0^* \cong f$, compatible with the given cells.

Given $\varphi\colon f_1 \Rightarrow f_2$ and invertible cells $\theta_1\colon \bar{f}_1 i_0^* \Rightarrow f_1$ and $\theta_2\colon \bar{f}_2 i_0^* \Rightarrow f_2$, define $\bar{\varphi}\colon \bar{f}_1 \Rightarrow \bar{f}_2$ by $\bar{\varphi} = \varphi i_{0*}$. Then

$$\begin{array}{ccc} \bar{f}_1 i_0^* & \xRightarrow{\theta_1} & f_1 \\ {\scriptstyle \bar{\varphi} i_0^*}\Downarrow & & \Downarrow{\scriptstyle \varphi} \\ \bar{f}_2 i_0^* & \xRightarrow[\theta_2]{} & f_2 \end{array}$$

commutes by uniqueness of the cell α such that $\alpha i_{0*} = \bar{\varphi}$ and $\alpha i_{1*} = \mathrm{id}^\bullet_{0_{X_1 Y}}$. Therefore, i_0^* is the cokernel of i_{1*}.

The proof that i_{1*} is the kernel of i_0^* is dual. ∎

Applying Proposition 3.10 to the examples, we get:

3.11. Corollary. *If $v\colon X_0 \longrightarrow\!\!\!\!\!\bullet\; X_1$ is a profunctor (respectively, order ideal), then*

$$X_1 \xrightarrow{i_{1*}} X_0 \sqcup_v X_1 \xrightarrow{i_0^*} X_0$$

is an extension in $\mathcal{V}\mathbb{C}\mathrm{at}$ *(respectively,* $\mathcal{V}\mathbb{P}\mathrm{os}$*).* ∎

3.12. Corollary. (Faul and Manuell [FM20]) *If $v\colon X_0 \longrightarrow\!\!\!\!\!\bullet\; X_1$ is a finite meet preserving map of frames (i.e., locales), then*

$$X_1 \xrightarrow{i_{1*}} \mathrm{Gl}(v) \xrightarrow{i_0^*} X_0$$

is an extension in $\mathrm{Frm}_\wedge$ *(i.e.,* $\mathcal{V}\mathbb{L}\mathrm{oc}$*).* ∎

3.13. Corollary. *If $v\colon \mathcal{X}_0 \longrightarrow\!\!\!\!\!\bullet\; \mathcal{X}_1$ is a finite limit preserving map of toposes, then*

$$\mathcal{X}_1 \xrightarrow{i_{1*}} \mathrm{Gl}(v) \xrightarrow{i_0^*} \mathcal{X}_0$$

is an extension in $\mathcal{V}\mathbb{T}\mathrm{opos}$. ∎

4. Extensions and Open Cokernels

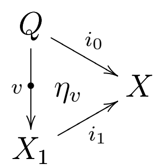
In this section, we show that if cokernels exist in $\mathcal{V}\mathbb{D}$ and satisfy a property shared by $\mathbb{L}\mathrm{oc}$, $\mathbb{T}\mathrm{opos}$, $\mathbb{P}\mathrm{os}$, and $\mathbb{C}\mathrm{at}$, then every extension is equivalent to one arising from a cotabulator as in Proposition 3.10.

To obtain such an equivalence in [FM20], the authors show that every morphism $f\colon X \longrightarrow Y$ in $\mathrm{Frm}_\wedge$ has a cokernel which they construct as $i_o^*\colon X \longrightarrow {\downarrow} u$, where $u = f(\bot)$ and i_o is the inclusion of the open sublocale ${\downarrow} u$ of X, in the sense of [J82]. Since X can be reconstructed, up to isomorphism, from ${\downarrow} u$ and its closed complement $i_c\colon {\uparrow} u \longrightarrow X$ via Artin-Wraith glueing along $i_c^* i_{o*}$, they show that every extension is isomorphic to one of the form

$${\uparrow} u \xrightarrow{i_{c*}} \mathrm{Gl}(i_c^* i_{o*}) \xrightarrow{i_o^*} {\downarrow} u$$

The notion of open/closed sublocales and this reconstruction is generalized to double categories called "glueing categories" in [N12a], [N12b] where examples include the double categories of interest here. Moreover, the open/closed subobjects agree with the definitions for $\mathbb{L}\mathrm{oc}_0$ [J82], $\mathbb{T}\mathrm{opos}_0$ [J77], $\mathbb{C}\mathrm{at}_0$ [BN00], and $\mathbb{P}\mathrm{os}_0$ [N01]. The assumption that $\mathbb{D}$ is a glueing category is unnecessary here since we only use the following definition of open morphism in $\mathcal{V}\mathbb{D}$.

4.1. Definition. *Suppose $\mathcal{V}\mathbb{D}$ has initial morphisms. A vertical morphism $e\colon X \to Q$ is called open in $\mathbb{D}$ if $e \cong i_0^*$, for some cotabulator*



(4.1)

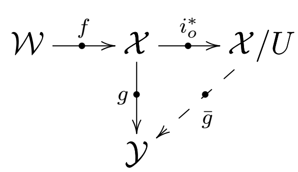
in $\mathbb{D}$ which restricts to $\mathcal{V}\mathbb{D}$ and $\mathcal{V}\mathbb{D}^{co}$ and satisfies $u_{i_0^* i_{1*}}\colon 0_{X_1Q} \Rightarrow i_0^* i_{1*}$ and $\hat{\eta}_v\colon v \Rightarrow i_1^* i_{0*}$ are invertible. We say $\mathbb{D}$ has open cokernels if every vertical morphism has a cokernel which is open in $\mathbb{D}$.

From [FM20], we know $\mathbb{L}\mathrm{oc}$ has open cokernels, and their construction generalizes to $\mathbb{T}\mathrm{opos}$ as follows.

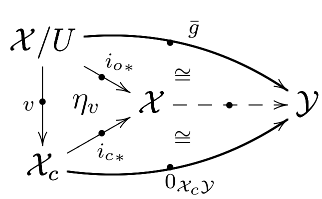
4.2. Proposition. $\mathbb{T}\mathrm{opos}$ *has open cokernels.*

Proof. Given $f\colon \mathcal{W} \to \mathcal{X}$, let $U = f(0)$, and consider $i_o\colon \mathcal{X}/U \to \mathcal{X}$, where $i_o^* = U^*$ and $i_{o*} = \Pi_U$. Then $i_o^* f \cong 0_{\mathcal{W}\mathcal{X}/U}$, the constant functor at the terminal object of $\mathcal{X}/U$, since $f(W) \times U = f(W) \times f(0) \cong f(W \times 0) \cong f(0) = U$, and so $i_o^* f(W) \cong \mathrm{id}_U$.

Suppose $g\colon \mathcal{X} \to \mathcal{Y}$ satisfies $gf \cong 0_{\mathcal{W}\mathcal{Y}}$, i.e., $(gf)(W) = 1$, for all W in $\mathcal{W}$,



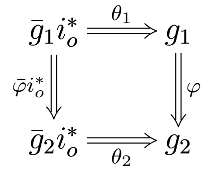
and define $\bar{g} = g i_{o*}$. To see that $\bar{g} i_o^* \cong g$, it suffices to show that both morphisms make the diagram



commute up to compatible isomorphisms, where $v = i_c^* i_{o*}$ and $i_c\colon \mathcal{X}_c \to \mathcal{X}$ is the inclusion of the closed complement of $\mathcal{X}/U$ in $\mathcal{X}$. Since $i_o^* i_{o*} \cong \mathrm{id}_{\mathcal{X}/U}$, we know $\bar{g} i_o^* i_{o*} \cong \bar{g} = g i_{o*}$. Also, $\bar{g} i_o^* i_{c*} \cong \bar{g} 0_{\mathcal{X}_c \mathcal{X}/U} \cong 0_{\mathcal{X}_c\mathcal{Y}}$. To see that $g i_{c*} \cong 0_{\mathcal{X}_c\mathcal{Y}}$, recall [J77] that $X \in \mathcal{X}_c$, i.e., is a j_c-sheaf, if and only if $X \times U \cong U$. But, $g(f(W)) = 1$, for all W, and so

$g(X) \cong g(X) \times 1 \cong g(X) \times g(f(0)) \cong g(X \times f(0)) \cong g(X \times U) \cong g(U) \cong g(f(0)) \cong 1$, and it follows that $gi_{c*} \cong 0_{\mathcal{X}_c\mathcal{Y}}$.

Given $\varphi: g_1 \Rightarrow g_2$ and invertible cells $\theta_1: \bar{g}_1 i_o^* \Rightarrow g_1$ and $\theta_2: \bar{g}_2 i_o^* \Rightarrow g_2$, define $\bar{\varphi}: \bar{g}_1 \Rightarrow \bar{g}_2$ by $\bar{\varphi} = \varphi i_{o*}$. Then



commutes by uniqueness of the cell α such that $\alpha i_{o*} = \bar{\varphi}$ and $\alpha i_{c*} = \mathrm{id}^\bullet_{0_{\mathcal{X}_c\mathcal{Y}}}$.

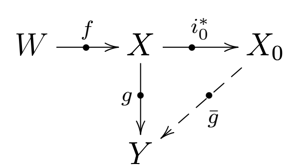
Therefore, i_o^* is the cokernel of i_{c*}. ■

By the description in [N12a] of cotabulators as collages in $\mathbb{C}$at (and $\mathbb{P}$os), we know $e: X \to X_0$ is open if and only if $e \cong i_0^*$, for some open inclusion $i_0: X_0 \to X$, i.e., $x_0 \in X_0$ and $\alpha: x \to x_0$ in X implies $x \in X_0$.

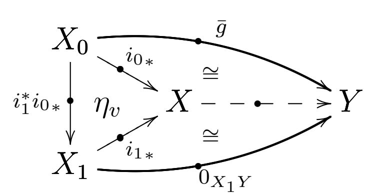
4.3. Proposition. *$\mathbb{C}$at and $\mathbb{P}$os have open cokernels.*

Proof. Given $f: W \to X$, let $i_0: X_0 \to X$ denote the inclusion of the full subcategory consisting of objects x_o such that $f(w, x_0) = \emptyset$, for all $w \in W$. Then X_0 is open, for if $f(w, x_0) = \emptyset$ and $\alpha: x \to x_0$, then so is $f(w, x)$, since $f(w, \alpha): f(w, x) \to f(w, x_0)$.

Now, $i_0^* f = 0_{WX_0}$, since $i_0^*(x, x_0) \times f(w, x) \to f(w, x_0)$, for all $x \in X$, implies $i_0^* f(w, x_0) = \emptyset$, for all $w \in W$ and $x_0 \in X_0$. Thus, to see that i_0^* is a cokernel of f, suppose $g: X \to Y$ satisfies $gf \cong 0_{WY}$ in



and define $\bar{g} = gi_{0*}$. For $\bar{g}i_0^* \cong g$, it suffices to show that both morphisms make the diagram



commute up to compatible isomorphisms, where $i_1: X_1 \to X$ is the inclusion of the closed complement of X_0 in X. Note that $X_1 = \{x \in X | f(w, x) \neq \emptyset, \text{for some } w \in W\}$. Thus, as in the proof of Proposition 4.2, we know that $\bar{g}i_0^* i_{0*} \cong gi_{0*}$, so it remains to show that $(\bar{g}i_0^*)i_{1*} \cong gi_{1*}$, or equivalently, $gi_{1*} \cong 0_{X_1Y}$, since $(\bar{g}i_0^*)i_{1*} \cong \bar{g}(i_0^* i_{1*}) \cong \bar{g}0_{X_1X_0} \cong 0_{X_1Y}$.

Consider $g(x, y) \times i_{1*}(x_1, x) \to g(x_1, y)$. Now, $gf \cong 0_{WY}$ implies $g(x_1, y) \times f(w, x_1) = \emptyset$, for all $w \in W$. Since $f(w, x_1) \neq \emptyset$, for some $w \in W$, by definition of X_1, it follows that $g(x_1, y) = \emptyset$. Thus, $g(x, y) \times i_{1*}(x_1, x) = \emptyset$, and so $gi_{1*} \cong 0_{X_1Y}$, as desired.

Given $\varphi\colon g_1 \Rightarrow g_2$ and invertible cells $\theta_1\colon \bar{g}_1 i_0^* \Rightarrow g_1$ and $\theta_2\colon \bar{g}_2 i_0^* \Rightarrow g_2$, define $\bar{\varphi}\colon \bar{g}_1 \Rightarrow \bar{g}_2$ by $\bar{\varphi} = \varphi i_{0*}$. Then

$$\begin{array}{ccc} \bar{g}_1 i_0^* & \overset{\theta_1}{\Longrightarrow} & g_1 \\ {\scriptstyle \bar{\varphi} i_0^*}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \varphi} \\ \bar{g}_2 i_0^* & \underset{\theta_2}{\Longrightarrow} & g_2 \end{array}$$

commutes by uniqueness of the cell α such that $\alpha i_{0*} = \bar{\varphi}$ and $\alpha i_{1*} = \mathrm{id}^\bullet_{0_{\mathcal{X}_1\mathcal{Y}}}$. Therefore, i_0^* is the cokernel of i_{c*}.

The proof for $\mathbb{P}\mathrm{os}$ is analogous with ideals in place of profunctors. ∎

4.4. Definition. *A morphism of extensions in a bicategory* $\mathcal{B}$ *is a diagram*

$$\begin{array}{ccccc} K_1 & \xrightarrow{k_1} & X_1 & \xrightarrow{e_1} & Q_1 \\ {\scriptstyle f}\big\downarrow & \cong & \big\downarrow{\scriptstyle g} & \cong & \big\downarrow{\scriptstyle h} \\ K_2 & \xrightarrow[k_2]{} & X_2 & \xrightarrow[e_2]{} & Q_2 \end{array}$$

It is an equivalence of extensions if f, g, and h are equivalences in $\mathcal{B}$. The set of equivalence classes of extensions of Q by K is denoted by $\mathrm{Ext}(Q, K)$*.*

4.5. Theorem. *Suppose $\mathbb{D}$ has open cokernels. Then every extension* $K \xrightarrow{k} X \xrightarrow{e} Q$ *is equivalent to:*

(1) one of the form $X_1 \xrightarrow{j_{1*}} X \xrightarrow{j_0^*} Q$ *, for some cotabulator*

$$\begin{array}{ccc} Q & & \\ & \searrow^{j_0} & \\ {\scriptstyle v}\big\downarrow & \eta_v & X \\ & \nearrow_{j_1} & \\ X_1 & & \end{array} \qquad (4.5)$$

(2) one of the form $K \xrightarrow{i_{1*}} \Gamma(k^* e_*) \xrightarrow{i_0^*} Q$ *, where k^* and e_* are right and left adjoint to k and e.*

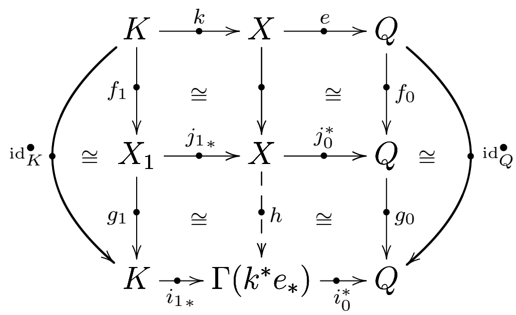
Proof. Since $\mathbb{D}$ has open cokernels, there is a cotabulator of the form (4.5) in $\mathbb{D}$ which restricts to $\mathcal{V}\mathbb{D}$ and $\mathcal{V}\mathbb{D}^{co}$ and satisfies $u_{j_0^* j_{1*}}\colon 0_{X_1 Q} \Rightarrow j_0^* j_{1*}$ and $\hat{\eta}_v\colon v \Rightarrow j_1^* j_{0*}$ are invertible.

Consider the diagram

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{e} & Q \\ {\scriptstyle f_1}\big\downarrow & \varphi_1 & \big\downarrow{\scriptstyle \mathrm{id}^\bullet_X} & \varphi_0 & \big\downarrow{\scriptstyle f_0} \\ X_1 & \xrightarrow[j_{1*}]{} & X & \xrightarrow[j_0^*]{} & Q \end{array}$$

where (f_0, φ_0) exists and is an equivalence since both e and j_0^* are kernels of k, and then we get (f_1, φ_1), since $j_0^* k \cong 0_{KQ}$. Now, $f_0 e j_{1*} \cong j_0^* j_{1*} \cong 0_{X_1 Q}$ and f_0 is an equivalence, and so $e j_{1*} \cong 0_{X_1 Q}$ and we get a pseudo inverse of (f_1, φ_1), as desired, proving (1).

For (2), let g_0 and g_1 denote pseudo inverses of f_0 and f_1, respectively, and consider



where $k^* = g_1 j_1^*$ and $e_* = j_{0*} f_0$, Now, k^* is a right pseudo adjoint for k, since $kk^* \cong (j_{1*} f_1)(g_1 j_1^*) \cong j_{1*}(f_1 g_1) j_1^* \cong j_{1*} j_1^* \Rightarrow \mathrm{id}^\bullet_X$ and $\mathrm{id}^\bullet_K \cong g_1 f_1 \Rightarrow g_1(j_1^* j_{1*}) f_1 \cong (g_1 j_1^*)(j_{1*} f_1) \cong k^* k$; and e_* is a left pseudo adjoint for e via $e_* e \cong (j_{0*} f_0)(g_0 j_0^*) \cong j_{0*}(f_0 g_0) j_0^* \cong j_{0*} j_0^* \Rightarrow \mathrm{id}^\bullet_X$ and $\mathrm{id}^\bullet_Q \cong g_0 f_0 \Rightarrow g_0(j_0^* j_{0*}) f_0 \cong (g_0 j_0^*)(j_{0*} f_0) \cong e e_*$. One can show that there is an equivalence $h\colon X \to \Gamma(k^* e_*)$ making the bottom two squares commute up to invertible cells, since $k^* e_* \cong (g_1 j_1^*)(j_{0*} f_0) \cong g_1(j_1^* j_{0*}) f_0$, $X \simeq \Gamma(j_1^* j_{0*})$, and g_1 and f_0 are equivalences, and so (2) follows. ∎

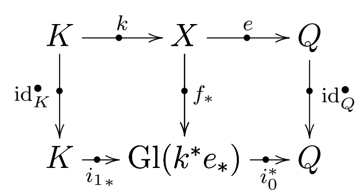
Thus, we get:

4.6. Corollary. *Every extension* $\mathcal{K} \xrightarrow{k} \mathcal{X} \xrightarrow{e} \mathcal{Q}$ *in* $\mathcal{V}\mathbb{T}\mathrm{opos}$ *is equivalent to one of the form* $\mathcal{K} \xrightarrow{i_{1*}} \mathrm{Gl}(k^* e_*) \xrightarrow{i_0^*} \mathcal{Q}$ *, for some left adjoint* k^* *of* k *and right adjoint* e_* *of* e. ∎

4.7. Corollary. *Every extension* $K \xrightarrow{k} X \xrightarrow{e} Q$ *in* $\mathcal{V}\mathbb{C}\mathrm{at}$ *is equivalent to one of the form* $K \xrightarrow{i_{1*}} Q \sqcup_{k^* e_*} K \xrightarrow{i_0^*} Q$ *, for some right adjoint* k^* *of* k *and left adjoint* e_* *of* e. ∎

Note that equivalences in $\mathcal{V}\mathbb{L}\mathrm{oc}$ and $\mathcal{V}\mathbb{P}\mathrm{os}$ are isomorphisms in $\mathbb{L}\mathrm{oc}$ and $\mathbb{P}\mathrm{os}$, since both double categories are flat and the only invertible cells are identities.

4.8. Corollary. *If* $K \xrightarrow{k} X \xrightarrow{e} Q$ *is an extension in* $\mathrm{Frm}_\wedge$*, then* k *has a left adjoint* k^* *and* e *has a right adjoint* e_* *such that the diagram*



commutes in $\mathrm{Frm}_\wedge$*, for some isomorphism* f *in* Frm.

Proof. Since cells in $\mathbb{L}$oc are of the form

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle v}\downarrow & \geq & \downarrow{\scriptstyle w} \\ X_1 & \xrightarrow[f_1]{} & Y_1 \end{array}$$

this follows from Theorem 4.5. ∎

4.9. Corollary. *If $K \xrightarrow{k} X \xrightarrow{e} Q$ is an extension in $\mathcal{V}\mathbb{P}$os, then k has a right adjoint k^* and e has a left adjoint e_* such that the diagram*

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{e} & Q \\ {\scriptstyle \mathrm{id}_K^\bullet}\downarrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \mathrm{id}_Q^\bullet} \\ K & \xrightarrow[i_{1*}]{} & Q \sqcup_{k^*e_*} K & \xrightarrow[i_0^*]{} & Q \end{array}$$

commutes in $\mathcal{V}\mathbb{P}$os, for some isomorphism f in $\mathbb{P}$os.

Proof. Since cells in $\mathbb{P}$os are of the form

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle v}\downarrow & \leq & \downarrow{\scriptstyle w} \\ X_1 & \xrightarrow[f_1]{} & Y_1 \end{array}$$

this follows from Theorem 4.5. ∎

For the other two examples, one can ask whether the equivalences in $\mathcal{V}\mathbb{T}$opos and $\mathcal{V}\mathbb{C}$at are equivalences in the 2-categories H$\mathbb{T}$opos and H$\mathbb{C}$at. We will see that they are in the first case but not necessarily in the second.

4.10. Proposition. *If $f: \mathcal{X} \to \mathcal{Y}$ is an equivalence in $\mathcal{V}\mathbb{T}$opos, then it is the right adjoint of an equivalence in $\mathcal{V}\mathbb{T}$opos, and so the equivalence $\mathcal{X} \simeq \mathrm{Gl}(k^*e_*)$ in Corollary 4.6 is an equivalence in H$\mathbb{T}$opos.*

Proof. It suffices to show that f has a finite limit preserving left adjoint. Suppose $g: \mathcal{Y} \to \mathcal{X}$ is a pseudo inverse of f in $\mathcal{V}\mathbb{T}$opos, and consider $f^* = gfg$. Then f^* is a left adjoint pseudoinverse of f [M71], and f^* preserves finite limits, since $f \cong g$ and g does. ∎

The following example shows that there is not an analogous proposition for $\mathbb{C}\mathrm{at}$.

4.11. Example. Suppose Y is a non-Cauchy complete category, in the sense of [W81], and $f\colon X \longrightarrow Y$ is its Cauchy completion (for example, a one object category Y with a single non-identity morphism e such that $e^2 = e$ and its two object Cauchy completion X). Then there is an equivalence of extensions

$$\begin{array}{ccccc} X & \xrightarrow{\langle \mathrm{id},1\rangle_*} & X \times 2 & \xrightarrow{\langle \mathrm{id},0\rangle^*} & X \\ {\scriptstyle f_*}\downarrow & \cong & \downarrow{\scriptstyle (f\times\mathrm{id})_*} & \cong & \downarrow{\scriptstyle f_*} \\ Y & \xrightarrow[\langle \mathrm{id},1\rangle_*]{} & Y \times 2 & \xrightarrow[\langle \mathrm{id},0\rangle^*]{} & Y \end{array}$$

but X and Y are not equivalent in $\mathbb{H}\mathbb{C}\mathrm{at}$.

5. Functoriality

In [FM20], they show that $(X_0, X_1) \mapsto \mathrm{Ext}(X_0, X_1)$ defines a functor $\mathrm{Frm}_\wedge^{op} \times \mathrm{Frm}_\wedge \longrightarrow \mathrm{Set}$ defined by pullback in X_0 and pushout in X_1. To do so, they show that $W_0 \xrightarrow{u} X_0 \xrightarrow{v} X_1 \xrightarrow{w} Y_1$ gives rise to pushout and pullback diagrams

$$\begin{array}{ccc} X_1 & \xrightarrow{i_{1*}} & \mathrm{Gl}(v) \\ {\scriptstyle w}\downarrow & & \downarrow \\ Y_1 & \xrightarrow[j_{1*}]{} & \mathrm{Gl}(wv) \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Gl}(vu) & \xrightarrow{j_0^*} & W_0 \\ \downarrow & & \downarrow{\scriptstyle u} \\ \mathrm{Gl}(v) & \xrightarrow[i_0^*]{} & X_0 \end{array}$$

in $\mathrm{Frm}_\wedge$. We will see that the latter extends to $\mathbb{P}\mathrm{os}$, $\mathbb{C}\mathrm{at}$, and $\mathbb{T}\mathrm{opos}$ for pseudo pushouts and pullbacks, and so we get a pseudo functor $\mathcal{V}\mathbb{D}^{op} \times \mathcal{V}\mathbb{D} \longrightarrow \mathrm{Set}$.

5.1. Lemma. *Suppose $\mathbb{D}$ has cotabulators that restrict to $\mathcal{V}\mathbb{D}$ and $\mathcal{V}\mathbb{D}^{co}$.*

(1) Given $X_0 \xrightarrow{v} X_1 \xrightarrow{w} Y_1$ in $\mathbb{D}$, the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{i_{1*}} & \Gamma(v) \\ {\scriptstyle w}\downarrow & \cong & \downarrow{\scriptstyle \bar{w}} \\ Y_1 & \xrightarrow[j_{1*}]{} & \Gamma(wv) \end{array}$$

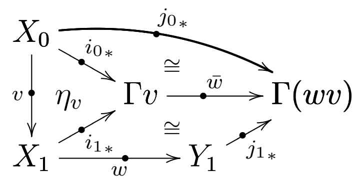
is a pseudo pushout in $\mathcal{V}\mathbb{D}$, where $\bar{w}$ is induced by $\eta_{wv}\colon j_{1*}(wv) \Rightarrow j_{0*}$.

(2) Given $W_0 \xrightarrow{u} X_0 \xrightarrow{v} X_1$ in $\mathbb{D}$, the diagram

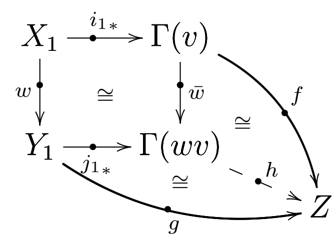
$$\begin{array}{ccc} \Gamma(vu) & \xrightarrow{j_0^*} & W_0 \\ {\scriptstyle \bar{u}}\downarrow & \cong & \downarrow{\scriptstyle u} \\ \Gamma(v) & \xrightarrow[i_0^*]{} & X_0 \end{array}$$

is a pseudo pullback in $\mathcal{V}\mathbb{D}$*, where* $\bar{u}$ *is induced by* $\eta_{vu}\colon (vu)j_0^* \Rightarrow j_1^*$.

Proof. For (1), note that the cell $\eta_{wv}\colon j_{1*}(wv) \Rightarrow j_{0*}$ factors as



Given f an g such that $gw \cong fi_{1*}$,



Since $f\eta_v\colon f(i_{1*}v) \Rightarrow fi_{0*}$ and $g(wv) \cong (gw)v \cong (fi_{1*})v \cong f(i_{1*}v)$, we get a cell $g(wv) \Rightarrow fi_{0*}$, and so for the universal property of $\Gamma(wv)$, there exists $h\colon \Gamma(wv) \to Z$ with invertible cells $hj_{1*} \cong g$ and $hj_{0*} \cong fi_{0*}$. Since $fi_{0*} \cong hj_{0*} \cong h(\bar{w}i_{0*}) \cong (h\bar{w})i_{0*}$ and $fi_{1*} \cong gw \cong (hj_{1*})w \cong h(j_{1*}w) \cong h(\bar{w}i_{1*}) \cong (h\bar{w})i_{1*}$, the universal property of Γv implies that $f \cong h\bar{w}$. Moreover, these are all compatible cells since they arise from the universal property of cotabulators.

As in Proposition 3.10, given cells from a pair of f's to g's, there exists a cell from the induced h's which is compatible with the invertible cells.

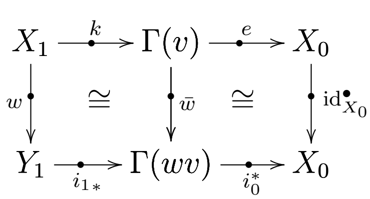
The proof of (2) is dual. ■

Recall that $\mathrm{Ext}(Q, K)$ denotes the set of equivalence classes of extensions of Q by K, introduced in Definition 4.4.

5.2. Theorem. *Suppose* $\mathbb{D}$ *has open cokernels and cotabulators which restrict to* $\mathcal{V}\mathbb{D}$ *and* $\mathcal{V}\mathbb{D}^{co}$*. If* $u\colon Y_0 \to X_0$ *and* $w\colon X_1 \to Y_1$ *are vertical morphisms in* $\mathbb{D}$*, then pseudo pushout along* w *and pseudo pullback along* v *induce a function* $\mathrm{Ext}(X_0, X_1) \to \mathrm{Ext}(Y_0, Y_1)$*, and hence, a functor* $\mathrm{Ext}\colon \mathcal{V}\mathbb{D}^{op} \times \mathcal{V}\mathbb{D} \to \mathrm{Set}$.

Proof. By Theorem 4.5, every extension of X_0 by X_1 in $\mathcal{V}\mathbb{D}$ is equivalent to one of the form $X_0 \xrightarrow{k} \Gamma(v) \xrightarrow{e} X_1$, for some $v\colon X_0 \to X_1$.

Given $w\colon X_1 \to Y_1$, consider the diagram



where the left square is a pseudo pushout by Lemma 5.1, and there is an invertible cell in the right square, since e is a cokernel of k and $(i_0^* \bar{w})k \cong 0_{X_1 X_0}$. Thus, we get a function $\mathrm{Ext}(X_0, w)\colon \mathrm{Ext}(X_0, X_1) \longrightarrow \mathrm{Ext}(X_0, Y_1)$, and hence, a functor $\mathrm{Ext}(X_0, -)\colon \mathcal{V}\mathbb{D} \longrightarrow \mathrm{Set}$, since this $\mathrm{Ext}(X_0, w)$ is defined by universal properties.

Dually, pseudo pullback in the first variable induces a functor $\mathrm{Ext}(-, X_1)\colon \mathcal{V}\mathbb{D}^{op} \longrightarrow \mathrm{Set}$. ∎

5.3. Examples. There is a functor $\mathrm{Ext}\colon \mathcal{V}\mathbb{D}^{op} \times \mathcal{V}\mathbb{D} \longrightarrow \mathrm{Set}$ defined by pullback in the first variable and pushout in the second, for $\mathbb{D} = \mathbb{L}\mathrm{oc}, \mathbb{T}\mathrm{opos}, \mathbb{P}\mathrm{os}, \mathbb{C}\mathrm{at}$.

6. Adjoint Extensions

We conclude with a characterization of a category of adjoint extension analogous to that of $\overline{\mathrm{Adj}}$ in [FM20].

6.1. Definition. *An adjoint extension of Q by K is one of the form* $K \xrightarrow{k_*} X \xrightarrow{e^*} Q$ *for some cotabulator of the form*

$$\begin{array}{ccc} Q & \xrightarrow{\ e\ } & \\ {\scriptstyle k^*e_*}\big\downarrow & \eta_{k^*e_*} & X \\ K & \xrightarrow[\ k\ ]{} & \end{array}$$

in $\mathbb{D}$. Let $\mathrm{Adj}(Q, K)$ denote the category whose objects are adjoint extensions of Q by K and morphisms are diagrams of the form

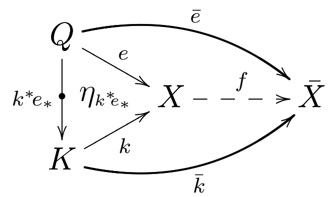
$$\begin{array}{ccccc} K & \xrightarrow{k_*} & X & \xrightarrow{e^*} & Q \\ {\scriptstyle \mathrm{id}_K^\bullet}\big\downarrow & \cong & \big\downarrow{\scriptstyle f_*} & \cong & \big\downarrow{\scriptstyle \mathrm{id}_Q^\bullet} \\ K & \xrightarrow[\bar{k}_*]{} & \bar{X} & \xrightarrow[\bar{e}^*]{} & Q \end{array} \tag{6.1}$$

for some $f\colon X \longrightarrow \bar{X}$ in $\mathbb{D}_0$ such that $fe = \bar{e}$

Every morphism $f_*\colon (k_*, e^*) \longrightarrow (\bar{k}_*, \bar{e}^*)$ induces a cell $\varphi\colon k^* e_* \Rightarrow \bar{k}^* \bar{e}_*$ via

$$\begin{array}{ccc} Q & \xrightarrow{\mathrm{id}_Q^\bullet} & Q \\ {\scriptstyle e_*}\big\downarrow & \cong & \big\downarrow{\scriptstyle \bar{e}_*} \\ X & \xrightarrow{f_*} & \bar{X} \\ {\scriptstyle k^*}\big\downarrow & \hat{\varphi}_1 & \big\downarrow{\scriptstyle \bar{k}^*} \\ K & \xrightarrow[\mathrm{id}_K^\bullet]{} & K \end{array}$$

where $\hat{\varphi}_1$ is the transpose of the invertible cell $\varphi_1 : \bar{k}_* \Rightarrow f_* k_*$. Conversely, every special cell $\varphi : k^* e_* \Rightarrow \bar{k}^* \bar{e}_*$ induces $f : X \longrightarrow \bar{X}$



such that $fe = \bar{e}$, $fk = \bar{k}$, and $f\eta_{k^*e_*} = \varphi$, and hence, a morphism $(k_*, e^*) \longrightarrow (\bar{k}_*, \bar{e}^*)$ in $\mathrm{Adj}(Q, K)$. Moreover, this correspondence is a bijection by the universal property of the cotabulator. Thus, we get:

6.2. Proposition. $\mathrm{Adj}(Q, K)$ *is equivalent to a full subcategory of* $\mathcal{V}\mathbb{D}(Q, K)$*, for all objects* Q, K *in* $\mathbb{D}$*.* ∎

6.3. Corollary. *If* $\mathbb{D}$ *has cotabulators whose cells* $\hat{\eta}_v : v \Rightarrow i_1^* i_{0*}$ *(as in 4.1) are invertible, then* $\mathrm{Adj}(Q, K)$ *is equivalent to* $\mathcal{V}\mathbb{D}(Q, K)$*, for all* Q, K *in* $\mathbb{D}$*.* ∎

6.4. Examples. For $\mathbb{D}$ equal $\mathbb{L}\mathrm{oc}, \mathbb{T}\mathrm{opos}, \mathbb{P}\mathrm{os}$ and $\mathbb{C}\mathrm{at}$, the requisite cells are invertible, and so $\mathrm{Adj}(Q, K)$ is equivalent to $\mathcal{V}\mathbb{D}(Q, K)$, for all Q, K in $\mathbb{D}$.

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